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Second order finite volume IMEX Runge-Kutta schemes for two dimensional parabolic PDEs in finance

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Abstract We present a novel and general methodology for building second order finite volume implicit-explicit Runge-Kutta numerical schemes for solving two dimensional financial parabolic PDEs with mixed derivatives. The methods achieve second order convergence even in the presence of non-regular initial conditions. The IMEX time integrator allows to overcome the tiny time-step induced by the diffusive term in the explicit schemes, also providing accurate and non-oscillatory approximations of the Greeks.

1 Introduction

The goal of this article is to design general second order numerical schemes for solving financial parabolic PDEs with mixed derivatives. The technique is based in the combination of finite volume (FV) methods, and implicit-explicit (IMEX) Runge-Kutta (RK) time integrators. To our knowledge, this is the first time that these numerical schemes have been successfully applied to the numerical solution of parabolic financial PDEs.

The main mathematical challenge in the numerical solution of financial advection-diffusion PDEs arises when the advection term becomes large when compared to the diffusion one. Under this situation, instabilities show up, because the problem becomes more hyperbolic. In order to face these problems, several techniques have been introduced. One way to overcome these instability phenomena is to avoid centered schemes and to consider upwind discretizations of advection terms, thus taking information upstream and not downstream. One technique to properly solve convection-dominated diffusion problems is to consider finite volume discretization methods. They are very well stablished, allowing to build schemes with arbitrary order of convergence, even in the presence of large convective terms and non-regular initial or boundary conditions, which are well-documented difficulties in

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the financial literature. Seminal works on applying finite volume method to option pricing problems are due to Forsyth and Zvan, see [20, 21], where vertex type finite volume methods were applied for problems in two spatial dimensions. More recently, in [3] the authors propose a Kurganov-Tadmor scheme [13, 16] for solving option pricing problems, written in conservative form, with appropriate time integration methods and slope limiters. Finally in [2] the authors apply a third order Kurganov-Levy scheme presented in [12] based in the CWENO reconstructions presented in [14]. All these works only deal with Black-Scholes PDEs in dimension one or Asian PDE problems. Therefore, they do not solve general two-dimensional pricing problems with mixed derivatives.

In all the previous works the authors use explicit-schemes in time to solve the stiff systems of ordinary differential equations (ODEs) obtained after performing the spatial semidiscretization. The main disadvantage of these approaches is that tiny time-steps must be used in order to maintain stability. Typically, Von Neumann stability analysis demands $\Delta t \leq (\Delta x)^2/(2\eta)$, being η the diffusion velocity. One technique to overcome this restriction is to use implicit-explicit time integrators, see [1, 4, 6, 11, 17]. The idea is to apply an implicit discretization to the stiff term related to the relaxed source terms, and an explicit one to the nonstiff term. This idea can also be extended to convection diffusion equations, see [5]. Following these works, our goal is to first semi-discretize in space 2D Black-Scholes equations (having mixed derivatives) by means of second-order finite volume methods, thus properly treating convection terms and non-smooth payoffs. Later, we propose to integrate in time the resulting system of stiff ODEs by means of the second-order IMEX Runge-Kutta time marching scheme. In this way, we will apply an implicit discretization to the diffusion (stiff) part and an explicit one to the convection and source terms (non stiff). As a result, Δt will only depend on the stability of the convection.

2 Option pricing models

In this work we will focus in two important PDE models in finance: baskets of two assets and the Heston stochastic volatility model. Both models can be formulated in terms of two dimensional parabolic PDEs.

We consider a basket of two assets, with prices given by s_1 and s_2. Under the Black-Scholes model, these prices follow the system of stochastic differential equations:

$$\frac{\mathrm{d}s_{it}}{s_{it}} = (r - q_i)\mathrm{d}t + \sigma_i \mathrm{d}W_{it}, \quad s_{i0} \text{ known}, \ i = 1, 2, \tag{1}$$

where W is a two dimensional correlated Brownian motion. We set $W_{1t} = \bar{W}_{1t}$, $W_{2t} = \rho\bar{W}_{1t} + \sqrt{1-\rho^2}\bar{W}_{2t}$, where $(\bar{W}_1, \bar{W}_2)$ is a two dimensional standard Brownian motion and $\rho \in (-1, 1)$ is the constant correlation parameter. Besides, $\sigma_1, \sigma_2 \in \mathbb{R}_+$ are the market volatilities of the assets s_1, s_2, respectively. Finally, r is the interest rate, and q_1, q_2 are the dividend yields of the assets. The price of an option with maturity T and payoff function $u_T(s_1, s_2)$ is $u(s_{1t}, s_{2t}, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^Q\left[u_T(s_{1T}, s_{2T})\right]$, where Q is the selected martingale measure. By applying the two-dimensional Itô formula, we can obtain a backward PDE for the price of the option $u(s_1, s_2, t)$. Doing a time reversal $\tau = T - t$ change of variable, the following forward PDE is obtained:

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 u}{\partial s_1^2} - \frac{1}{2}\sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2} - \rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} \\
&\qquad - (r - q_1)s_1 \frac{\partial u}{\partial s_1} - (r - q_2)s_2 \frac{\partial u}{\partial s_2} + ru = 0, \qquad \tau \in (0, T], \\
&u(s_1, s_2, 0) = u_0(s_1, s_2), \qquad s_1, s_2 > 0.
\end{aligned} \tag{2}$$

In the sequel, the τ notation is dropped for simplicity and the forward time is again written as t. In this work we consider the payoff of the arithmetic basket call option, $u_0(s_1, s_2) = \max\left(\frac{1}{2}(s_1 + s_2) - K, 0\right)$, where K is the fixed strike price.

In the Heston model [10], there is an underlying asset s whose volatility is a stochastic process driven by a second Brownian motion:

$$\begin{aligned}
\mathrm{d}s_t &= (r - q)s_t \mathrm{d}t + \sqrt{v_t} s_t \mathrm{d}W_{1t}, \quad s_0 \text{ known}, \\
\mathrm{d}v_t &= \kappa(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_{2t}, \quad v_0 \text{ known},
\end{aligned} \tag{3}$$

where $r, q, \kappa, \theta, \sigma, s_0, v_0$ are constant parameters in $\mathbb{R}_+$ and W is again a two dimensional correlated Brownian motion. Moreover, r is the fixed interest rate, q is the constant dividend yield, κ is the mean-reversion speed for the variance, θ is the mean reversion level for the variance, σ is the volatility of the variance (the so-called volatility of the volatility) and v_0 is the initial level of the variance. The process v represents the variance of s and its stochastic differential equation is a version of the square root process described by Cox, Ingersoll and Ross. The price of a derivative with payoff function u_T is the solution of a backward PDE. Once more, performing a time reversal $\tau = T - t$ change of variable, the forward PDE:

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} - \frac{1}{2}s^2 v \frac{\partial^2 u}{\partial s^2} - \rho\sigma s v \frac{\partial^2 u}{\partial v \partial s} - \frac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} \\
&\qquad - (r - q)s \frac{\partial u}{\partial s} - \kappa(\theta - v)\frac{\partial u}{\partial v} + ru = 0, \qquad \tau \in (0, T], \\
&u(s, v, 0) = u_0(s), \qquad s > 0,
\end{aligned} \tag{4}$$

is obtained. As before, the forward time τ will be written as t. The payoff function of an European call option is $u_0(s) = \max(s - K, 0)$, where K is the strike.

3 Finite volume IMEX Runge-Kutta numerical method

Equation (2) and (4) can be written in the compact conservative form:

$$\frac{\partial}{\partial t}u(x, y, t) + \operatorname{div}\mathbf{F}(u) = \operatorname{div}\mathbf{G}(\nabla u) + h(u), \tag{5}$$

with $\mathbf{F}(u) = (f_1(u), f_2(u))$ and $\mathbf{G}(\nabla u) = (g_1(\nabla u), g_2(\nabla u))$. The numerical solution of equation (5) using a finite volume scheme is difficult due to the diffusive terms. The obtained semidiscrete scheme is a stiff ODE system:

$$\frac{\partial U}{\partial t} + E(U) = I(U), \tag{6}$$

where $U = U(t) \in \mathbb{R}^N$ and $E, I : \mathbb{R}^N \to \mathbb{R}^N$, being E the non-stiff term and I the stiff part. IMEX time-marching schemes play a major rule in the treatment of these stiff systems. In this article we have considered the implicit-explicit (IMEX) Runge-Kutta time discretization numerical scheme proposed in [17].

3.1 Finite volume space discretization

Let Δx, Δy be the mesh length in each spatial direction. We define the grid points $x_i = i\Delta x$, $y_j = j\Delta y$, $i, j \in \mathbb{Z}$. Let $x_{i+1/2} = x_i + \frac{1}{2}\Delta x$ and $y_{j+1/2} = y_j + \frac{1}{2}\Delta y$. We consider rectangular finite volumes $V_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, where (x_i, y_j) is the center of the finite volume V_{ij}. The area of V_{ij} will be denoted as $|V_{ij}|$, i.e. $|V_{ij}| = \Delta x\Delta y$. Besides, Γ_{ij} represents the boundary of V_{ij}. Let $\bar{u}_{ij} = \frac{1}{|V_{ij}|}\int_{V_{ij}} u(x, y, t)\mathrm{d}x\mathrm{d}y$ be the volume averages that will be the unknowns of the problem . Integrating equation (5) in space on V_{ij} and dividing by $|V_{ij}|$ we obtain the semi-discrete equation

$$\frac{\mathrm{d}\bar{u}_{ij}}{\mathrm{d}t} = -\frac{1}{|V_{ij}|}\int_{V_{ij}} \operatorname{div}\mathbf{F}(u)\,\mathrm{d}x\mathrm{d}y + \frac{1}{|V_{ij}|}\int_{V_{ij}} \operatorname{div}\mathbf{G}(u_x, u_y)\,\mathrm{d}x\mathrm{d}y + \frac{1}{|V_{ij}|}\int_{V_{ij}} h(u)\,\mathrm{d}x\mathrm{d}y. \tag{7}$$

Applying the divergence theorem:

$$\frac{\mathrm{d}\bar{u}_{ij}}{\mathrm{d}t} = -\frac{1}{|V_{ij}|}\oint_{\Gamma_{ij}} \mathbf{F}(u)\cdot\mathbf{n}\,d\gamma + \frac{1}{|V_{ij}|}\oint_{\Gamma_{ij}} \mathbf{G}(u_x, u_y)\cdot\mathbf{n}\,d\gamma + \frac{1}{|V_{ij}|}\int_{V_{ij}} h(u)\,\mathrm{d}x\mathrm{d}y. \tag{8}$$

Therefore, in order to convert (8) into a numerical scheme, we have to approximate on the right hand side of this equation with functions of $\{\bar{u}_{ij}(t)\}$. The source and convective terms will be treated explicitly in time, while the diffusion part will be managed implicitly. For the advective part we get

$$\oint_{\Gamma_{ij}} \mathbf{F}(u)\mathbf{n} = \int_{\Gamma_{i+1/2,j}} f_1(u)\mathrm{d}\gamma_j + \int_{\Gamma_{i-1/2,j}} f_1(u)\mathrm{d}\gamma_j + \int_{\Gamma_{i,j+1/2}} f_2(u)\mathrm{d}\gamma_i + \int_{\Gamma_{i,j-1/2}} f_2(u)\mathrm{d}\gamma_i, \tag{9}$$

where for second order schemes each line integral can be approximated by means of midpoint quadrature rule as $\int_{\Gamma_{i\pm1/2,j}} f_1(u)\mathrm{d}\gamma_j \approx \pm\Delta y f_1(u(x_{i\pm1/2}, y_j))$, and analogously for $\int_{\Gamma_{i,j\pm1/2}} f_2(u)\mathrm{d}\gamma_i$. At this point the unknown function u is reconstructed by a piecewise polynomial using the volume averages $\{\bar{u}_{ij}(t)\}$. More precisely, starting from $\{\bar{u}_{ij}\}$, we compute a piecewise polynomial reconstruction $\mathcal{R}(x, y) = \sum_{i,j} P_{ij}(x, y)\mathbf{1}_{ij}(x, y)$, where P_{ij} is a suitable polynomial satisfying some accuracy and non oscillatory property, and $\mathbf{1}_{ij}$ is the indicator function of the volume V_{ij}. Second order numerical schemes can be obtained by means of piecewise linear polynomials, although higher order schemes can be obtained by polynomials of higher order. Here we consider the natural extension of MUSCL reconstruction to 2D Cartesian grids (see [19]). The flux functions at the midpoints of the boundaries of the volumes can be computed by using a suitable numerical flux function, consistent with the analytical flux. Hereafter we detail the approximation of $f_1(u(x_{i+1/2}, y_j))$, being the other fluxes approximated in the same way: $f_1(u(x_{i+1/2}, y_j)) \approx \mathcal{F}_1(u^-_{i+1/2,j}, u^+_{i+1/2,j})$. The values $u^{\pm}_{i+1/2,j}$ are obtained from the reconstruction as $u^{\pm}_{i+1/2,j} = \lim_{x \to x^{\pm}_{i+1/2}} \mathcal{R}(x, j\Delta y)$, with x in a normal line to the boundary $\Gamma_{i+1/2,j}$. For example the left reconstructed value at the edge $\Gamma_{i+1/2,j}$ is $u^-_{i+1/2,j} = \bar{u}_{i,j} + \Delta x/2\, u'_{i,j}$, where the slope $u'_{i,j}$ is a first order approximation of the space derivative in the X direction of $u(x, y, t)$ at point (x_i, y_j) at every time t. This slope must satisfy the TVD property and thus we must use slope limiters. In our case we use the minmod limiter. All the calculations performed in this article have been carried out using the Local Lax Friedrich numerical flux. Finally, the volume integral of the source term is discretized using the midpoint quadrature rule.

The diffusion part is discretized implicitly:

$$\oint_{\Gamma_{ij}} \mathbf{G}(u_x, u_y) \cdot \mathbf{n}\, d\gamma = \int_{\Gamma_{i+1/2,j}} g_1(u_x, u_y) \mathrm{d}\gamma_j + \int_{\Gamma_{i-1/2,j}} g_1(u_x, u_y) \mathrm{d}\gamma_j + \int_{\Gamma_{i,j+1/2}} g_2(u_x, u_y) \mathrm{d}\gamma_i + \int_{\Gamma_{i,j-1/2}} g_2(u_x, u_y) \mathrm{d}\gamma_i. \quad (10)$$

Before approximating each one of these line integrals, a suitable approximation of the partial derivatives u_x and u_y has to be built for the volume V_{ij}. With this aim, we build the second order Lagrange interpolating polynomial of u centered in the volume V_{ij}. Let L^u_{ij} be that polynomial. Considering the nine nodes of the dual mesh $\{x_{i+k}, y_{j+l}\},\ k, l = -1, 0, 1$, and the averaged values $\{\bar{u}_{i+k,j+l}\}$ of the solution at each volume $V_{i+k\, j+l}$, this polynomial is given by $L^{\bar{u}}_{ij}(x, y) = \sum_{k,l=-1}^{1} \bar{u}_{i+k,j+l}\, \ell_{i+k}(x) \ell_{j+l}(y)$, where ℓ_{i+k} and ℓ_{j+l} are the one dimensional Lagrange polynomial basis, i.e:

$$\ell_{i+k}(x) = \prod_{\substack{p=i-1 \\ p \neq i+k}}^{i+1} \frac{x - x_p}{x_{i+k} - x_p}, \qquad \ell_{j+l}(y) = \prod_{\substack{q=j-1 \\ q \neq j+l}}^{j+1} \frac{y - y_q}{y_{j+l} - y_q}.$$

Therefore, we use the approximations $u_x \approx \partial_x L^{\bar{u}}_{ij}$ and $u_y \approx \partial_y L^{\bar{u}}_{ij}$. For example, the line integrals in equation (10) are approximated, in the X direction as

$$\int_{\Gamma_{i\pm1/2,j}} g_1\Big(\partial_x L^{\bar{u}}_{ij}, \partial_y L^{\bar{u}}_{ij}\Big) \mathrm{d}\gamma_j \approx \pm\, \Delta y\, g_1\Big(\partial_x L^{\bar{u}}_{ij}(x_{i\pm1/2}, y_j), \partial_y L^{\bar{u}}_{ij}(x_{i\pm1/2}, y_j)\Big).$$

3.2 IMEX Runge-Kutta time discretization

An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the diffusion terms (stiff terms) and an explicit one to the convective and source terms (non stiff terms), see [1, 4, 6, 11, 17]. When applied to system (6) it takes the form

$$U^{(k)} = U^n - \Delta t \sum_{l=1}^{k-1} \tilde{a}_{kl} E(U^{(l)}) + \Delta t \sum_{l=1}^{s} a_{kl} I(U^{(l)}), \quad (11)$$

$$U^{n+1} = U^n - \Delta t \sum_{k=1}^{s} \tilde{\omega}_k E(U^{(k)}) + \Delta t \sum_{k=1}^{s} \omega_k I(U^{(k)}), \quad (12)$$

where $U^n = (\bar{u}^n_{ij})$, $U^{n+1} = (\bar{u}^{n+1}_{ij})$ are the vectors of unknowns volume averages at times t^n and t^{n+1}, and $U^{(k)}$ and $U^{(l)}$ are the vector of unknowns at the stages k, l

of the IMEX Runge-Kutta scheme. The matrices $\bar{A} = (\tilde{a}_{ij})$, $\tilde{a}_{kl} = 0$ for $l \geq k$ and $A = (a_{kl})$ are $s \times s$ matrices such that the scheme is explicit in E and implicit in I. The coefficient vectors $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_s)$ and $w = (w_1, \ldots, w_s)$ complete the IMEX Runge-Kutta scheme. One should consider diagonally implicit Runge-Kutta (DIRK) schemes for the implicit terms ($a_{kl} = 0$, for $l > k$) in order to solve efficiently the algebraic equations corresponding to the implicit part of the discretization at each time step. IMEX Runge-Kutta schemes can be represented by a double *tableau* in the usual Butcher notation,

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{\omega} \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & \omega \end{array},$$

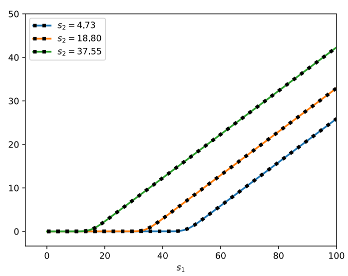
where the coefficients vectors $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_s)^\mathsf{T}$ and $c = (c_1, \ldots, c_s)^\mathsf{T}$ used for the treatment of non autonomous systems, are given by the usual relation $\tilde{c}_k = \sum_{l=1}^{k-1} \tilde{a}_{kl}$, $c_k = \sum_{l=1}^{k} a_{kl}$. The PDEs (2) and (4) we will be dealing with in this article, after the space discretization, give rise to a system of non-autonomous ODEs. Therefore, the constants $\tilde{c}_k$ and c_k were omitted in the time marching scheme (11)-(12). In this work we will consider the second order IMEX-SSP2(2,2,2) (L-stable scheme, see [17]), whose *tableaus* for the explicit (left) and implicit (right) parts are

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} \gamma & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & 1/2 & 1/2 \end{array} \qquad \gamma = 1 - \frac{1}{\sqrt{2}}.$$

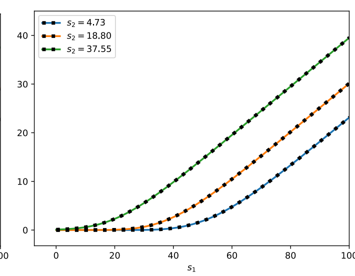
The selection of this second-order IMEX scheme is only to match it to the finite volume framework and to show that globally the scheme works well in the context of parabolic PDEs finance. Nevertheless, many other second-order IMEX schemes with $s = 2$, and with the same properties (stiff problems, L-stable) could be applied in this context as well, see [1] for example.

4 Numerical experiments

In this section, several numerical experiments are developed to assess the accuracy and performance of the new numerical methods proposed in previous sections for the discussed two dimensional problems. Experiments are carried out on several option pricing problems. In subsection 4.1 basket call options over two underlyings following Black-Scholes model are priced. Vanilla call options under Heston stochastic volatility model are priced in subsection 4.2. Each subsection is organized as follows. We start by writing the PDE model in conservative form. Then, we solve the PDE problems using the discussed finite volume IMEX Runge-Kutta numerical method. Both convection-dominated and diffusion-dominated scenarios are considered. All pictures, tables and errors in this whole section will be computed considering the numerical solution at the last time step where $t = T$, i.e, we approximate numerically the option prices “today”. Later, reference solutions for the studied pricing problems are accurately computed at $t = T$ by means of the COS Fourier method [8]. This allow us to compute L_1 and L_∞ error norms, together with the relative and mean absolute errors. On top of that, orders of convergence are validated in-depth. Both explicit and IMEX Runge-Kutta time integrators were implemented over the same explained space discretization. In the explicit case second order Heun's ODE solver was considered. Finite volume with both time marching schemes achieves the announced second order accuracy in all regimes. IMEX and explicit methods yield similar results in terms of accuracy and convergence order. Nevertheless, the IMEX solver can take advantage of much larger times steps, thus offering much better performance in terms of execution times. The explicit method dramatically slows down the computation due to the presence of diffusive terms. The designed algorithms were implemented using C++ (GNU compiler 9.3.0) and ran in a machine with an AMD Zen3 5950X processor. All codes were compiled using double precision. In this article a CFL of 0.5 is taken into account in the stability conditions.



(a) Test 1



(b) Test 2

Fig. 1 Cuts of price surfaces. Numerical solution, continuous; COS solution with squares.

4.1 Options on a basket of two assets

In this experiment we solve the basket European option pricing problem considering two underlyings. The PDE model (2) can be written in the conservative form (5):

$$\frac{\partial u}{\partial t} + \frac{\partial f_1}{\partial s_1}(u) + \frac{\partial f_2}{\partial s_2}(u) = \frac{\partial g_1}{\partial s_1}(u_{s_1}, u_{s_2}) + \frac{\partial g_2}{\partial s_2}(u_{s_1}, u_{s_2}) + h(u), \tag{13}$$

where the functions f_1, f_2, g_1, g_2 and h are given by:

$$\begin{aligned}
f_i(u) &= (\sigma_i^2 - r + q_i)s_i u(s_1, s_2, t) + \frac{\rho}{2}\sigma_1\sigma_2 s_i u(s_1, s_2, t),\\
g_1(u_{s_1}, u_{s_2}) &= \frac{1}{2}\sigma_1^2 s_1^2 u_{s_1}(s_1, s_2, t) + \frac{\rho}{2}\sigma_1\sigma_2 s_1 s_2 u_{s_2}(s_1, s_2, t),\\
g_2(u_{s_1}, u_{s_2}) &= \frac{1}{2}\sigma_2^2 s_2^2 u_{s_2}(s_1, s_2, t) + \frac{\rho}{2}\sigma_1\sigma_2 s_1 s_2 u_{s_1}(s_1, s_2, t),\\
h(u) &= (\sigma_2^2 + \rho\sigma_1\sigma_2 + \sigma_1^2 + q_1 + q_2 - 3r)u(s_1, s_2, t).
\end{aligned}$$

We perform two basket option pricing tests with the market parameters given by $q_1 = 0$, $q_2 = 0$, $\rho = 0.5$, $T = 0.25$, $K = 30$, and $\sigma_1 = \sigma_2 = 0.1$, $r = 0.5$ for Test 1, and $\sigma_1 = \sigma_2 = 0.5$, $r = 0.1$ for Test 2. The first set of parameters, denoted as Test 1, stands for a convection-dominated regime. Although nowadays this setup with an interest rate $r = 0.5$ is financially unrealistic, it is useful as a stress-test. The second group of parameters for Test 2, represents a diffusion-dominated setting. In both scenarios, in order to avoid noise coming from the artificial boundary conditions, the computational space domain was considered as $(s_1, s_2) \in [0, 5K] \times [0, 5K]$.

Cuts of the surfaces, with planes parallel to $s_2 = 0$, computed with the finite volume IMEX Runge-Kutta solver, for basket option prices at $t = T$ are shown in Figure 1, along with COS solutions. The finite volume IMEX Runge-Kutta numerical scheme offers high resolution approximation of basket option values.

At this point we compute the reference option prices by using the described COS method. For a mesh of size $N_1 \times N_2 = 1600 \times 1600$, i.e., a mesh with 1600 discretization points in each space direction. The here developed finite volumes (explicit or IMEX) schemes offer high-resolution approximations even at regions of discontinuities and non-smoothness in the initial condition. Table 1 records L_1 errors and orders of convergence for Test 2. The errors and orders of convergence are shown for both the IMEX and explicit finite volume numerical schemes. Both schemes achieve second-order accuracy in the L_1 norm. Additionally, these tables present the time steps and the execution times for the two time integrators. As said before, IMEX method is able to converge using much larger time steps than the explicit one. The stability condition of the explicit scheme requires extremely small time steps, thus making the method useless in practice for refined grids in space. Therefore, IMEX offers much better performance in terms of execution times, and allows us to solve the PDE problems with refined meshes in space. In fact, IMEX method is able to run for grids finer than 800×800 grid, while the explicit method is not in reasonable computational times. IMEX clearly outperforms the explicit method in this scenario with diffusion dominance, which is the typical situation in finance. Although for convective dominated problems in coarse grids both time marching schemes perform similarly, as soon as the mesh is refined IMEX is the only practical choice.

	IMEX				Explicit			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)	L_1 error	Order	Δt	Time (s)
25×25	9.6620×10^{1}	$--$	4.71×10^{-2}	4.5×10^{-3}	9.0224×10^{1}	$--$	1.42×10^{-3}	1.7×10^{-2}
50×50	2.5178×10^{1}	1.94	2.35×10^{-2}	3.3×10^{-2}	2.3440×10^{1}	1.94	3.56×10^{-4}	1.2×10^{-1}
100×100	6.4828×10^{0}	1.95	1.18×10^{-2}	1.7×10^{-1}	5.9498×10^{0}	1.97	8.89×10^{-5}	1.8×10^{0}
200×200	1.6209×10^{0}	2.00	5.89×10^{-3}	1.2×10^{0}	1.4834×10^{0}	2.00	2.22×10^{-5}	3.0×10^{1}
400×400	3.9419×10^{-1}	2.03	2.94×10^{-3}	9.8×10^{0}	3.5473×10^{-1}	2.06	5.56×10^{-6}	4.9×10^{2}
800×800	7.9229×10^{-2}	2.31	1.47×10^{-3}	8.5×10^{1}	7.1095×10^{-2}	2.31	1.39×10^{-6}	7.9×10^{3}

Table 1 L_1 errors and orders of convergence of the IMEX and explicit methods for Test 2.

4.2 Heston model

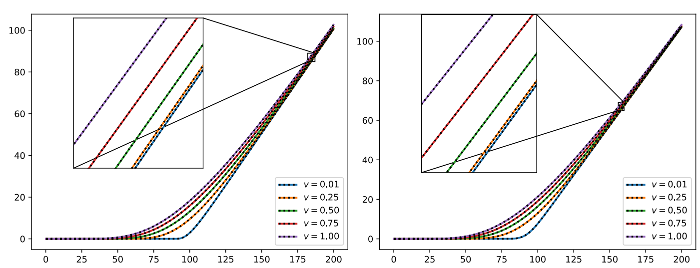
In this experiment we price a vanilla call option under the Heston stochastic volatility model. The PDE model (4) can be written in the conservative form (5) as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial f_1}{\partial s}(u) + \frac{\partial f_2}{\partial v}(u) = \frac{\partial g_1}{\partial s}(u_s, u_v) + \frac{\partial g_2}{\partial v}(u_s, u_v) + h(u), \tag{14}$$

where the functions f_1, f_2, g_1, g_2 and h are given by:

$$
\begin{aligned}
f_1(u) &= (v - r + q)su(s,v,t),\\
f_2(u) &= \left(\rho\sigma v - \kappa(\theta - v) + \frac{1}{2}\sigma^2\right)u(s,v,t),\\
g_1(u_s,u_v) &= \frac{1}{2}s^2 v u_s(s,v,t) + \rho\sigma s v u_v(s,v,t),\\
g_2(u_s,u_v) &= \frac{1}{2}\sigma^2 v u_v(s,v,t),\\
h(u) &= (v - 2r + q + \kappa + \rho\sigma)u(s,v,t).
\end{aligned}
$$

In this section we perform two vanilla call option pricing tests with the market parameters given by $q = 0.0$, $\kappa = 1.5$, $\theta = 0.04$, $\rho = -0.9$, $T = 0.25$, $K = 100$ and $\sigma = 0.3, r = 0.025$ for the Test 3, and $\sigma = 0.025, r = 0.3$ for Test 4. The second group of parameters, denoted as Test 4, is a variation of Test 3, swapping the interest rate r and the volatility of the volatility σ. In both tests, in order to minimize the numerical errors coming from the artificial boundary conditions, the computational space domain was considered as $(s,v) \in [0,800] \times [0,4]$. Sections of the computed price surfaces, with planes parallel to $v = 0$, are shown in Figure 2 in combination with the semi-analytical COS solutions. In Table 2, the L_1 errors and the L_1 orders of the explicit and IMEX methods for the Test 4 are displayed, respectively. Again, both numerical schemes achieve second-order accuracy in the L_1 norm. IMEX is able to converge using much larger time steps than the explicit method, thus it consumes much less computing time.



(a) Test 3 (b) Test 4

Fig. 2 Cuts of prices surfaces. Numerical solution, continuous; COS solution, squares.

	IMEX				Explicit			
$N_1 \times N_2$	L_1 error	Order	Δt	Time (s)	L_1 error	Order	Δt	Time (s)
25×25	8.9656×10^1	$--$	3.87×10^{-3}	1.5×10^{-2}	9.0576×10^1	$--$	1.99×10^{-4}	1.8×10^{-1}
50×50	2.4203×10^1	1.89	1.94×10^{-3}	9.0×10^{-2}	2.4440×10^1	1.89	4.99×10^{-5}	2.7×10^0
100×100	9.3022×10^0	1.38	9.69×10^{-4}	7.0×10^{-1}	9.3648×10^0	1.38	1.25×10^{-5}	4.4×10^2
200×200	1.2578×10^0	2.89	4.84×10^{-4}	6.9×10^0	1.2671×10^0	2.89	3.12×10^{-6}	7.1×10^2
400×400	2.9883×10^{-1}	2.07	2.42×10^{-4}	6.8×10^1	3.0089×10^{-1}	2.07	7.79×10^{-7}	1.2×10^4
800×800	5.9846×10^{-2}	2.32	1.21×10^{-4}	6.3×10^2	6.0411×10^{-2}	2.32	1.95×10^{-7}	1.9×10^5

Table 2 L_1 errors and orders of convergence of the IMEX and explicit methods,Test 4.

5 Conclusions

We have shown that finite volume IMEX Runge-Kutta numerical schemes are suitable for solving convection-diffusion PDE option pricing problems. This opens the door to the application of these schemes to numerous models in finance, even those giving rise to non-linear PDEs. The obtained numerical schemes are highly efficient. On the one hand, large time steps can be used, avoiding the need to use small time steps enforced by the diffusion stability condition that appears when explicit schemes are considered. On the other hand, the schemes are second order accurate. This fact is of paramount importance in order to obtain accurate approximations of the Greeks without oscillations. We have shown that second order accuracy is preserved even when non smooth initial conditions (payoffs) are considered, which is the usual situation in finance. Thus, additional smoothing techniques for the initial condition do not need to be taken into account. Moreover, the here developed option price calculators can be extended to build numerical solvers with higher order.

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References

1. U.M. Ascher, S.J. Ruut, R.J. Spiteri. Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations. *Applied Numerical Mathematics*, 25(2):151-167, 1997.
2. O. Bhatoo, A.A. Peer, E. Tadmor, D. Tangman, A.A. El Faidal Saib. Conservative Third-Order Central-Upwind Schemes for Option Pricing Problems. *Vietnam Journal of Mathematics*, 47(4):813–8332, 2019.
3. O. Bhatoo, A.A. Peer, E. Tadmor, D. Tangman, A.A. El Faidal Saib. Efficient conservative second-order central-upwind schemes for option-pricing problems. *Journal of computational Finance*, 22(5):39–78, 2019.
4. S. Boscarino, L. Pareschi, G. Russo. Implicit-Explicit Runge–Kutta Schemes for Hyperbolic Systems and Kinetic Equations in the Diffusion Limit. *SIAM Journal on Scientific Computing*, 25(1):A22-A51, 2013.
5. M. Briani, R. Natalini, G. Russo. Implicit–explicit numerical schemes for jump–diffusion processes. *Calcolo*, 44(1):33–57, 2007.
6. M.P. Calvo, J. de Frutos, J. Novo. Linearly implicit Runge-Kutta methods for advection-reaction-diffusion equations. *Appl. Numer. Math.*, 37:535–549, 2001.
7. Y. d'Halluin, P. A. Forsyth, G. Labahn. A semi-Lagrangian approach for American Asian options under jump diffusion. *SIAM Journal on Scientific Computing*, 27(1):315–345, 2006.
8. F. Fang, C. W. Oosterlee. A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM Journal on Scientific Computing*, 31(2):826–848, 2008.
9. S. Heston, G. Zhou. On the Rate of Convergence of Discrete-Time Contingent Claims. *Mathematical Finance*, 10:53–75, 2000.

10. S.L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6(2):327–343, 1993.
11. C.A. Kennedy, M.H. Carpenter. Additive Runge–Kutta schemes for convection- diffusion-reaction equations. *Appl. Numer. Math.*, 44:139–181, 2003.
12. A. Kurganov, D. Levy. A Third-Order Semidiscrete Central Scheme for Conservation Laws and Convection-Diffusion Equations. *SIAM Journal on Scientific Computing*, 22(4):1461–1488, 2000.
13. A. Kurganov, E. Tadmor. New high-resolution central schemes for nonlinear conservation laws and convection–diffusion equations. *Journal of Computational Physics*, 160(1):241 – 282, 2000.
14. D. Levy, G. Puppo, G. Russo. Compact central WENO schemes for multidimensional conservation laws. *SIAM Journal on Scientific Computing*, 22(2):656–672, 2000.
15. R. C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.
16. H. Nessyahu, E. Tadmor. Non-oscillatory central differencing for hyperbolic conservation laws. *Journal of Computational Physics*, 87(2):408–463, 1990.
17. L. Pareschi, G. Russo. Implicit–Explicit Runge–Kutta Schemes and Applications to Hyperbolic Systems with Relaxation. *Journal of Scientific Computing*, 25:129–155, 2005.
18. D. M. Pooley, K. R. Vetzal, P. A. Forsyth. Convergence remedies for non-smooth payoffs in option pricing. *Journal of Computational Finance*, 6(4), 2002.
19. B. van Leer. Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method. *Journal of Computational Physics*, 32(1):101–136, 1979.
20. R. Zvan, P. A. Forsyth, K. R. Vetzal. Robust numerical methods for PDE models of Asian options. *Journal of computational Finance*, 1(2):39–78, 1997.
21. R. Zvan, P. A. Forsyth, K. R. Vetzal. A Finite Volume Approach For Contingent Claims Valuation. *IMA Journal of Numerical Analysis*, 21(3):703–731, 07 2001.