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Tesis Doctoral

Modelos de aguas poco profundas obtenidos mediante la técnica de desarrollos asintóticos

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A mis padres y a Julio

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Introducción

Objetivos

El primer objetivo de esta tesis doctoral es obtener las ecuaciones que rigen el movimiento de las aguas poco profundas (conocidas por ”shallow waters”en inglés o “aguas someras” en castellano) a partir de las más generales de Euler o Navier-Stokes, utilizando para ello una técnica matemática rigurosa.

Clásicamente las ecuaciones de aguas poco profundas se obtienen a partir de las de Euler o Navier-Stokes, que nos permiten modelizar el comportamiento de un fluido (no viscoso o con viscosidad, respectivamente), mediante ciertas hipótesis simplificadoras. Dichas hipótesis no siempre están debidamente justificadas, lo que conduce a gran variedad de modelos, sin que resulte claro cuál de ellos es el “mejor”. Las ecuaciones de aguas poco profundas constituyen una buena aproximación de la realidad cuando la profundidad del dominio estudiado es pequeña si se compara con el tamaño global del dominio, lo que hace que dichas ecuaciones describan especialmente bien el flujo en canales, ríos y lagos, flujos de marea, corrientes marinas, avance de un frente de onda, arrastre de sedimentos, variación de concentración salina, dispersión de contaminantes, rotura de presas, flujos atmosféricos, tsunamis, etc. (véase por ejemplo [14, 18, 23, 25, 26, 39, 44, 45, 47, 48, 54, 58, 66, 67, 71, 72, 75, 74, 95, 105, 109, 110, 112, 113]). En definitiva, son unas ecuaciones ampliamente utilizadas, pero que adolecen de una justificación rigurosa.

En esta tesis pretendemos justificar dichas ecuaciones rigurosamente mediante la técnica de desarrollos asintóticos donde el pequeño parámetro estará relacionado con la profundidad. En general, cuando se usa el análisis asintótico para analizar fluidos, se hace en el dominio original (véase, por ejemplo, [43, 114]), que en este caso depende del parámetro ε y del tiempo t, o la superficie se supone que es constante (véase, por ejemplo, [6, 7, 12, 13, 16, 17]). Si se trabaja en un dominio independiente de ε, toda la “dependencia” de este parámetro aparece explícita en las ecuaciones, mientras que cuando se trabaja en dominios dependientes de ε, parte de esta dependencia puede permanecer “oculta” en el dominio. Es por ello que preferimos seguir las técnicas habituales en análisis asintótico aplicado a sólidos (véase [29, 30, 33, 81, 82, 104] y las referencias en ellos señaladas) en nuestro estudio de las aguas someras. Así, si la profundidad viene dada en cada punto por $\varepsilon h(t, x, y)$, donde ε es un pequeño parámetro que haremos tender a cero, previamente realizamos un cambio de variable a un dominio independiente del parámetro y del tiempo, de modo que la dependencia de ε pase a estar en las ecuaciones y no en el dominio. Es ahora cuando estudiamos lo que pasa al hacer tender ε a cero. Mediante este método obtendremos varios modelos de aguas poco profundas, constituyendo el propio método utilizado una justificación del modelo obtenido y de algunas de las hipótesis que se utilizan habitualmente.

Una vez obtenidos los nuevos modelos de aguas someras, el segundo objetivo de la tesis consiste en analizar numéricamente las ecuaciones halladas, y compararlas con los modelos que se pueden encontrar en la literatura, para comprobar así que los modelos que proponemos "mejoran" los usados actualmente.

Estructura

Comenzamos esta tesis con un capítulo 1 en el que resumimos los modelos de aguas someras que se encuentran en la literatura. Para ello empezamos por recordar los modelos tridimensionales de Euler y Navier-Stokes (sección 1.1) y, a continuación (sección 1.2), presentamos algunos ejemplos ilustrativos de las diferentes deducciones de las ecuaciones de aguas someras que, a partir de las ecuaciones generales de Euler y de Navier-Stokes, autores como Stoker [96] (1948), [97] (1958), Friedrichs [40] (1948), Keller [52] (1948), Chow [28] (1959), Dronkers [34] (1964), Henderson [46] (1966), Strelkoff [98] (1969), Yen [111] (1973), Whitham [108] (1974), Liggett [59] (1975), Cunge et al [31] (1980), Lai [55] (1986), Abbot y Basco [1] (1990), Tan [101] (1992), Chaudhry [26] (1993), Fe [36] (2005), Cea [22] (2005) han realizado. Haremos especial hincapié en las hipótesis y simplificaciones realizadas durante la deducción de los modelos, pues uno de los objetivos de este trabajo es obtener y justificar los modelos de aguas someras sin necesidad de realizar dichas hipótesis y simplificaciones.

En el capítulo 2 obtenemos un modelo unidimensional de aguas someras sin viscosidad. Para ello partimos de las ecuaciones de Euler bidimensionales y, como deseamos obtener un modelo de aguas someras, la profundidad debe ser pequeña comparada con la longitud del dominio, aunque el calado no tiene porqué ser pequeño en términos absolutos. Con este propósito se introduce un pequeño parámetro adimensional, ε, del orden del cociente entre la profundidad media y la longitud del canal. Tanto el dominio como las variables y funciones que aparecen en las ecuaciones dependen de este parámetro. Realizamos un cambio de variable a un dominio de referencia independiente del parámetro ε y del tiempo (es decir, la dependencia del parámetro pasa del dominio a las funciones). Suponemos ahora que la solución del problema que deseamos resolver admite un desarrollo en serie de potencias de ε. Se sustituyen estos desarrollos en serie de potencias en las ecuaciones. El paso siguiente consiste en identificar los términos multiplicados por la misma potencia de ε, obteniendo un polinomio en ε igualado a cero, por lo que sus coeficientes han de ser nulos. De este modo se logra una serie de ecuaciones que nos permitirán determinar los distintos términos a partir de los que se construyen aproximaciones del calado, la presión y la velocidad horizontal y vertical. Deshacemos el cambio de variable, volviendo al dominio original, donde consideramos aproximaciones de orden cero, uno y dos. Este mismo proceso se sigue en este capítulo dos veces, distinguiendo si la vorticidad inicial es nula o no. El modelo propuesto en el primer caso recupera y generaliza el modelo clásico de aguas someras. Cuando la vorticidad no se considera nula en el modelo obtenido la velocidad horizontal depende de forma explícita de la variable z a través de la vorticidad.

En el capítulo 3 obtenemos un modelo unidimensional de aguas someras con un nuevo término de viscosidad diferente de los que se pueden encontrar en la literatura. Para ello actuaremos de forma similar a como lo hicimos en el capítulo 2, pero partiendo en esta ocasión de las ecuaciones de Navier-Stokes bidimensionales.

Se dedican los capítulos 4 y 5 a la obtención de modelos bidimensionales de aguas someras sin viscosidad y con ella, respectivamente. En ambos casos, el dominio en el que vamos a trabajar se caracteriza porque la altura es pequeña comparada con sus otras dimensiones. Un río, una ría o una región del mar son ejemplos de este tipo de dominio. En el primero de estos capítulos, seguimos los pasos dados en el capítulo 2, pero partiendo ahora de las ecuaciones de Euler tridimensionales. Los modelos que se obtienen generalizan los obtenidos en dimensión uno en el capítulo 2. Además, se estudia el caso en el que se aceptan las hipótesis usuales en oceanografía dinámica, para ver cómo influyen en el modelo obtenido, e intentar comprender su significado. El punto de partida en el capítulo 5 son las ecuaciones de Navier-Stokes tridimensionales, y el objetivo que se persigue en este capítulo es obtener un modelo bidimensional de aguas someras con viscosidad. Para ello actuaremos de forma similar a como lo hicimos en los capítulos 3 y 4. Al igual que en el capítulo 3, el término de viscosidad propuesto no se encuentra en la literatura.

En el capítulo 6 comparamos el modelo de aguas someras con viscosidad propuesto en el capítulo 3 para dimensión uno y en el capítulo 5 para dimensión dos con los modelos de aguas someras que se pueden encontrar en la literatura. En primer lugar compararemos los distintos modelos analíticamente, observando en qué términos se diferencian las ecuaciones de los distintos modelos, y en segundo lugar los compararemos numéricamente, resolviendo los modelos para soluciones exactas conocidas de las ecuaciones de Navier-Stokes y también para otros ejemplos. La comparación de los resultados obtenidos por los distintos modelos al tratar de aproximar las soluciones exactas nos muestra que el nuevo término de viscosidad que incorpora el modelo que proponemos supone una mejora respecto a los otros modelos. Los otros test realizados nos permiten estudiar el comportamiento cualitativo de los modelos obtenidos frente a los encontrados en la literatura.

En el capítulo 7 comparamos el nuevo modelo de aguas someras sin viscosidad obtenido en el capítulo 4 (y que generaliza el de dimensión uno obtenido en el capítulo 2) con el modelo clásico de aguas someras sin viscosidad. El nuevo modelo añade una dependencia explícita de las velocidades horizontales respecto a la profundidad que le permite ser más preciso que el clásico, como se comprueba al compararlos analítica y numéricamente.

Finalmente, en el capítulo 8, se presentan las conclusiones de este trabajo, en especial los nuevos modelos (con y sin viscosidad) obtenidos.

Capítulo 1

Modelos clásicos de aguas someras. Deducción y clasificación

En este capítulo intentaremos resumir los modelos de aguas someras que se encuentran en la literatura. Para ello comenzaremos por recordar los modelos tridimensionales de Euler y Navier-Stokes (sección 1.1) y, a continuación (sección 1.2), presentaremos algunos de los modelos de aguas someras más habituales de entre los que aparecen en la literatura.

1.1. Ecuaciones que rigen el movimiento de un fluido

1.1.1. Conservación de la masa

En este trabajo, consideraremos únicamente fluidos incompresibles, por lo que la densidad del fluido será independiente de la presión. Por ello la ecuación de conservación de la masa resulta:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.1.1}$$

donde u y v son las componentes horizontales de la velocidad en la dirección de los ejes X e Y, respectivamente, y w es la velocidad vertical.

1.1.2. Las ecuaciones de Euler

El movimiento de un fluido ideal (no viscoso) viene descrito por las ecuaciones de Euler.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + F_z \qquad (1.1.2)$$

donde ρ_0 es la densidad del fluido (que vamos a suponer constante), p la presión y F_x, F_y y F_z las tres componentes de las fuerzas externas por unidad de masa que actúan sobre el fluido.

1.1.3. Las ecuaciones de Navier-Stokes

Como es bien conocido, las ecuaciones que rigen el comportamiento de un fluido viscoso son las ecuaciones de Navier-Stokes deducidas por Claude Navier (1821) y George Stokes (1845) de forma independiente. Se trata de un sistema de tres ecuaciones en derivadas parciales no lineales para la conservación del momento:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x + \nu\Delta u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_y + \nu\Delta v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + F_z + \nu\Delta w \qquad (1.1.3)$$

donde ν representa la viscosidad cinemática ($\nu = \mu/\rho_0$ donde μ es la viscosidad dinámica).

1.1.4. Fuerzas exteriores

En este trabajo consideraremos que las únicas fuerzas exteriores que actúan son las debidas a la aceleración de Coriolis y a la de la gravedad.

Se toma junto al sistema de ejes fijos a la Tierra un sistema inercial fijo. La relación entre la aceleración expresada en unos y otros ejes es la siguiente:

$$\vec{\mathbf{a}}_f = \left(\frac{d\vec{\mathbf{u}}}{dt}\right)_T + 2\vec{\phi}\times\vec{\mathbf{u}} + \vec{\phi}\times\left(\vec{\phi}\times\vec{\mathbf{R}}\right)$$

con:

- $\vec{\mathbf{u}} = (u, v, w)$ es la velocidad del fluido
- $\vec{\mathbf{a}}_f$ aceleración relativa al sistema de ejes fijos ideales
- $\left(\frac{d\vec{\mathbf{u}}}{dt}\right)_T$ aceleración relativa al sistema de ejes fijos sobre la Tierra

- $\vec{\phi}$ velocidad angular de rotación de la Tierra ($\phi = 7{,}29 \times 10^{-5}$ rad/s)
- $\vec{\mathbf{R}}$ vector distancia del cuerpo al centro de la Tierra
- $2\vec{\phi} \times \vec{\mathbf{u}}$ acelaración de Coriolis
- $\vec{\phi} \times \left(\vec{\phi} \times \vec{\mathbf{R}}\right)$ aceleración centrípeta

La acelaración de un cuerpo en caída libre cerca de la Tierra y sin fricción será

$$\vec{\mathbf{g}} = \vec{\mathbf{g}}_f - \vec{\phi} \times \left(\vec{\phi} \times \vec{\mathbf{R}}\right) \tag{1.1.4}$$

Depende sólo de la posición geográfica pero como la variación (la aceleración centrípeta) es de aproximadamente un 0.5 %, se puede despreciar (ver págs. 94-95 de [21]) resultando

$$\vec{\mathbf{g}} = (0, 0, -g) \tag{1.1.5}$$

donde g se suele considerar constante (su valor depende de la latitud y de la altura, pero en este trabajo tomaremos $g = 9.8$ m/s^2).

En cuanto a la aceleración de Coriolis, se puede ver en [21] (págs. 95-96), en [114] (pág. 24), en [115] (págs. 1-3) o en [116] (pág. 17) que:

$$\vec{\phi} = \phi\,\vec{\mathbf{e}} \qquad \text{con } \vec{\mathbf{e}} = (\operatorname{sen}\varphi)\,\vec{k} + (\cos\varphi)\,\vec{\jmath}$$

donde $\vec{\imath}$ apunta al Este, $\vec{\jmath}$ al Norte y $\vec{k}$ al Cénit y φ es la latitud Norte. Por tanto:

$$\begin{aligned} -2\vec{\phi} \times \vec{\mathbf{u}} &= -2 \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & \phi\,(\cos\varphi) & \phi\,(\operatorname{sen}\varphi) \\ u & v & w \end{vmatrix} \\ &= -2\phi\,((\cos\varphi)w - (\operatorname{sen}\varphi)v, (\operatorname{sen}\varphi)u, -(\cos\varphi)u) \end{aligned} \tag{1.1.6}$$

Como se considera que las fuerzas de exteriores que actúan sobre el fluido son la gravedad y la de Coriolis, las ecuaciones de Euler ((1.1.2)) resultan

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - 2\phi\,((\cos\varphi)w - (\operatorname{sen}\varphi)v) \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi(\operatorname{sen}\varphi)u \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + 2\phi(\cos\varphi)u \end{aligned} \tag{1.1.7}$$

y análogamente las ecuaciones de Navier-Stokes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - 2\phi\left((\cos\varphi)w - (\operatorname{sen}\varphi)v\right) + \nu\Delta u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi(\operatorname{sen}\varphi)u + \nu\Delta v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + 2\phi(\cos\varphi)u + \nu\Delta w \tag{1.1.8}$$

Observación 1.1 *En [21] (págs. 95-96) se justifica que se puede prescindir del término* $(\cos\varphi)w$ *por ser la velocidad vertical* w *muy pequeña. Además, el término* $2\phi(\cos\varphi)u$ *se desprecia usualmente en oceanografía dinámica tomando* $g = g - 2\phi(\cos\varphi)u$*, con lo que clásicamente la aceleración de Coriolis se toma como*

$$-2\vec{\phi}\times\vec{\mathbf{u}} = -2\phi\left(-(\operatorname{sen}\varphi)v, (\operatorname{sen}\varphi)u, 0\right) \tag{1.1.9}$$

Bajo estas hipótesis, las ecuaciones de Euler ((1.1.2) *) resultan*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi(\operatorname{sen}\varphi)v$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi(\operatorname{sen}\varphi)u$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g \tag{1.1.10}$$

y las ecuaciones de Navier-Stokes ((1.1.3) *):*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi(\operatorname{sen}\varphi)v + \nu\Delta u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi(\operatorname{sen}\varphi)u + \nu\Delta v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \nu\Delta w \tag{1.1.11}$$

En este trabajo, como se verá más adelante, no se han realizado estas simplificaciones sino que se ha considerado que la aceleración de Coriolis viene dada por (1.1.6)*, aunque se ha analizado cómo influiría esta hipótesis en la metodología empleada (véanse las secciones 4.10 y 5.8).*

1.2. Deducción de modelos clásicos de aguas someras

Las ecuaciones de Euler y de Navier-Stokes nos permiten modelizar el comportamiento de un fluido (no viscoso o con viscosidad, respectivamente). Estas ecuaciones se pueden plantear en tres dimensiones (como en (1.1.2) o en (1.1.3)) o en dos dimensiones (ignorando la segunda ecuación en cada uno de los sistemas (1.1.2) o (1.1.3)). Existen, sin embargo, gran cantidad de casos en los que, siendo el movimiento tridimensional (respectivamente, bidimensional), éste se produce en dos (respectivamente, una) dimensiones, debido a que la dimensión restante es muy pequeña si la comparamos con las demás. Si a ello sumamos la presencia de una superficie libre, la resolución numérica de las ecuaciones de Euler o de Navier-Stokes es demasiado complicada en estos casos, y se suele recurrir a modelos como los de aguas someras (bidimensionales o unidimensionales), que permiten una aproximación razonable con un gran ahorro de tiempo de cálculo. Los modelos de aguas someras se emplean de forma usual en el estudio del flujo en canales, ríos ([72]) y lagos, flujos de marea ([45, 95]), corrientes marinas ([109]), avance de un frente de onda, arrastre de sedimentos ([75, 74, 110]), variación de concentración salina ([71]), dispersión de contaminantes ([105]), rotura de presas ([18, 39]), flujos atmosféricos ([25, 44]), tsunamis ([113]), etc. Véase también por ejemplo [14, 23, 26, 47, 48, 54, 58, 66, 67, 112].

Se considera que se puede emplear un modelo de aguas someras para obtener el flujo cuando la profundidad característica es mucho menor que la dimensión característica del dominio.

Diversos autores (Stoker [96] (1948), [97] (1958), Friedrichs [40] (1948), Keller [52] (1948), Chow [28] (1959), Dronkers [34] (1964), Henderson [46] (1966), Strelkoff [98] (1969), Yen [111] (1973), Whitham [108] (1974), Liggett [59] (1975), Cunge et al [31] (1980), Lai [55] (1986), Abbot y Basco [1] (1990), Tan [101] (1992), Chaudhry [26] (1993), Fe [36] (2005), Cea [22] (2005)) han deducido las ecuaciones de aguas someras usando diferentes procedimientos. A continuación presentaremos algunos ejemplos ilustrativos de estas deducciones a partir de las ecuaciones generales de Euler y de Navier-Stokes. Haremos especial hincapié en las hipótesis y simplificaciones realizadas durante la deducción de los modelos, pues uno de los objetivos de este trabajo es obtener y justificar los modelos de aguas someras sin necesidad de realizar dichas hipótesis y simplificaciones.

1.2.1. Obtención de la superficie libre

Cuando se desea calcular un flujo en superficie libre la posición de dicha superficie libre se convierte en una incógnita. Para calcular esta nueva incógnita, la profundidad del agua, los modelos de aguas someras clásicos suelen hacerlo integrando verticalmente la ecuación de conservación de la masa ((1.1.1)) entre el fondo ($z = H(x, y)$) y la superficie del agua ($z = s(t, x, y)$) (véase por ejemplo [101] páginas

22-23 ó [26] páginas 347-348):

$$\int_H^s \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0 \tag{1.2.1}$$

aplicando la regla de Leibnitz a los dos primeros sumandos

$$\frac{\partial}{\partial x} \int_H^s u \, dz - u|_{z=s} \frac{\partial s}{\partial x} + u|_{z=H} \frac{\partial H}{\partial x} + \frac{\partial}{\partial y} \int_H^s v \, dz - v|_{z=s} \frac{\partial s}{\partial y} + v|_{z=H} \frac{\partial H}{\partial y}$$
$$+ w|_{z=s} - w|_{z=H} = 0 \tag{1.2.2}$$

e imponiendo las condiciones cinemáticas de frontera libre (en $z = s$) y fondo (en $z = H$):

$$w|_{z=s} = \frac{\partial s}{\partial t} + u|_{z=s} \frac{\partial s}{\partial x} + v|_{z=s} \frac{\partial s}{\partial y} \tag{1.2.3}$$

$$w|_{z=H} = u|_{z=s} \frac{\partial H}{\partial x} + v|_{z=s} \frac{\partial H}{\partial y} \tag{1.2.4}$$

donde se ha tenido en cuenta que el fondo (dado por H) no varía con el tiempo.

Insertando las igualdades (1.2.3) y (1.2.4) en la ecuación (1.2.2) y denotando las velocidades horizontales medias por:

$$\bar{u} = \frac{1}{h} \int_H^s u \, dz, \qquad \bar{v} = \frac{1}{h} \int_H^s v \, dz \tag{1.2.5}$$

resulta (siendo $h = s - H$ el calado)

$$\frac{\partial s}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0 \tag{1.2.6}$$

que por ser el fondo constante en el tiempo se puede escribir:

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0 \tag{1.2.7}$$

1.2.2. Presión hidrostática

Clásicamente, una de las hipótesis clave para la obtención de los modelos de aguas someras consiste en suponer que la presión que actúa sobre el fluido es la hidrostática. Algunos autores directamente asumen esta hipótesis sin justificación alguna (véase [101, 108]), otros, como por ejemplo [26] (página 348) o [36], la justifican basándose en que el movimiento principal del flujo ocurre en planos horizontales y que la aceleración vertical es despreciable frente a la aceleración de la gravedad, hipótesis que permiten considerar despreciable la aceleración vertical del fluido, es decir, el miembro de la izquierda de la tercera ecuación de Navier-Stokes (1.1.11.c) (donde, implícitamente, se está suponiendo la hipótesis oceanográfica (1.1.9)), así como el laplaciano de la componente vertical de la velocidad, de modo que esta ecuación queda reducida a:

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g = 0 \tag{1.2.8}$$

Integrando esta ecuación entre la superficie libre ($z = s$) y un z cualquiera resulta

$$p = p_s + \rho_0 g(s - z) \tag{1.2.9}$$

donde p_s es la presión en la superficie, que se supone conocida.

Otro modo de justificar esta hipótesis (véase [22]) parte de la ecuación estacionaria de Navier-Stokes para la velocidad vertical:

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \nu\Delta w \tag{1.2.10}$$

La presión total (p) se descompone en la presión hidrostática (p_h) y la presión dinámica (p_d):

$$p = p_h + p_d, \quad p_h = p_s + \rho_0 g(s - z) \tag{1.2.11}$$

Se introducen L_c, H_c, U_c y W_c, la longitud, el calado, la velocidad horizontal y vertical característicos. Al tratarse de aguas someras, L_c es mucho mayor que H_c. De la ecuación de conservación de la masa (1.1.1) se obtiene la siguiente relación entre la velocidad vertical y horizontal características:

$$\frac{U_c}{L_c} \sim \frac{W_c}{H_c} \tag{1.2.12}$$

donde $\sim$ indica una relación de proporcionalidad. El análisis de las escalas características de los distintos términos de la ecuación (1.2.10) nos proporciona:

$$\begin{gathered} u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p_d}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \\ \frac{U_c W_c}{L_c} \sim \frac{U_c W_c}{L_c} \sim \frac{W_c^2}{H_c} \sim \frac{U_c^2}{H_c} \sim \nu\frac{W_c}{L_c^2} \sim \nu\frac{W_c}{L_c^2} \sim \nu\frac{W_c}{H_c^2} \end{gathered} \tag{1.2.13}$$

Multiplicando la ecuación anterior por H_c, dividiendo por U_c^2 y teniendo en cuenta (1.2.12) se obtiene la siguiente relación entre las escalas características:

$$\frac{H_c^2}{L_c^2} \sim 1 \sim \nu\frac{H_c^2}{L_c^3 U_c} \sim \nu\frac{1}{L_c U_c} \tag{1.2.14}$$

donde el orden de magnitud 1 corresponde al término de la presión dinámica. Como se ha supuesto que la escala horizontal es mucho mayor que la vertical,

$$\frac{H_c^2}{L_c^2} << 1, \quad \nu\frac{H_c^2}{L_c^3 U_c} << 1 \tag{1.2.15}$$

además se supone que el número de Reynolds horizontal es mucho mayor que 1:

$$\frac{\nu}{L_c U_c} << 1 \tag{1.2.16}$$

Bajo estas hipótesis, el orden de magnitud de los términos convectivos y viscosos es menor que el del término de la presión, de modo que se puede considerar que la ecuación (1.2.10) se reduce a

$$\frac{\partial p_d}{\partial z} \approx 0$$

Como la presión en la superficie es igual a la presión atmosférica, sustituyendo en (1.2.11) se tiene que la presión total se puede aproximar por la hidrostática:

$$p \approx p_s + \rho_0 g(s - z) \tag{1.2.17}$$

1.2.3. Obtención de las ecuaciones dinámicas del modelo de aguas someras suponiendo u y v independientes de z (caso sin viscosidad)

En [108] (páginas 454-456) se obtiene un modelo de aguas someras considerando un fluido no viscoso e incompresible en un campo gravitacional constante, asumiendo que la densidad permanece constante y que la única fuerza externa que actúa sobre el fluido es la gravedad. Las ecuaciones que rigen el flujo en esta situación son (1.1.1)-(1.1.2).

Según este autor los modelos de aguas someras tienen sentido cuando $\kappa H \to 0$ (siendo κ el número de onda), es decir, para casos con gran longitud de onda. En este caso, la relación de dispersión es aproximadamente $\omega^2 \sim gH\kappa^2$ (ω es la frecuencia) y la velocidad de fase ($c_0 = \sqrt{gH}$) es independiente de κ.

El paso clave para la obtención del modelo de aguas someras, según se indica en [108], es aproximar la componente vertical de las ecuaciones del momento (tercera ecuación de Euler):

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g$$

por

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g = 0 \tag{1.2.18}$$

ya que entonces

$$p - p_s = \rho_0 g(s - z) \tag{1.2.19}$$

es decir, se está suponiendo que la presión es la hidrostática.

Las componentes horizontales de las ecuaciones de Euler se convierten así en (se supone p_s constante)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -g\frac{\partial s}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -g\frac{\partial s}{\partial y} \tag{1.2.20}$$

El autor razona que, como los términos de la derecha no dependen de z, la variación de u y v siguiendo a una partícula es independiente de z. Por tanto, si u y v son independientes de z inicialmente, permanecen así. Considera que éste es el caso, de modo que (1.2.20) se reescriben:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial s}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial s}{\partial y} = 0 \tag{1.2.21}$$

(con u, v y s independientes de z).

Aunque la aceleración vertical se desprecia en (1.2.18) comparada con los términos que se conservan, no hay ninguna razón para despreciar $\dfrac{\partial w}{\partial z}$ en (1.1.1). Precisamente, en [108] se utiliza una forma integral de (1.1.1) para obtener la ecuación de conservación

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \tag{1.2.22}$$

El razonamiento y las herramientas empleadas son los mismos que en el apartado 1.2.1, pero como en este caso se está considerando que u y v son independientes de z, $\bar{u} = u$ y $\bar{v} = v$.

Así, el modelo de aguas someras que se propone es

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial s}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial s}{\partial y} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \tag{1.2.23}$$

que es el mismo modelo que se puede encontrar, por ejemplo, en [5] (página 3).

Observación 1.2 *La versión unidimensional de* (1.2.23) *es*

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial s}{\partial x} = 0 \qquad (1.2.24)$$

modelo de aguas someras unidimensional clásico que, en su versión más sencilla ($H = 0 \Rightarrow s = h$) puede verse en [56] página 581.

En este caso las hipótesis que se han utilizado para la obtención del modelo han sido las siguientes:

- La presión es hidrostática.
- La aceleración vertical es despreciable frente a la aceleración de la gravedad.
- Se desprecia la aceleración de Coriolis.
- Las componentes horizontales de la velocidad son independientes de z.

La estimación del orden de magnitud de la aproximación que realiza el autor es la siguiente: el error para p en (1.2.9) es de orden $\rho_0 H \frac{\partial w}{\partial t}$ y, como de (1.1.1) se deduce $w \approx -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$, entonces, el error relativo en (1.2.21) es de orden (véase [108], página 456)

$$-\frac{\frac{\partial p}{\partial x}}{\rho_0 \frac{\partial u}{\partial t}} \approx \frac{H^2 \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}{\frac{\partial u}{\partial t}} \approx \frac{H^2}{L^2}$$

donde L es el largo del dominio en la dirección x (de forma similar se obtiene el error en la dirección y).

1.2.4. Obtención de las ecuaciones dinámicas del modelo de aguas someras considerando que el flujo tridimensional es plano y compresible

En [101] (páginas 23-25) se obtiene un modelo de aguas someras considerando para ello que la presión es la hidrostática y el flujo tridimensional es el siguiente flujo plano compresible. En el plano XY, se tiene un fluido virtual con altura unidad en la dirección z que fluye a la velocidad original promediada $(\bar{u}, \bar{v})$. Tiene una densidad virtual $\rho_0 h$, tal que la masa en una columna rectangular de fluido con ambos lados

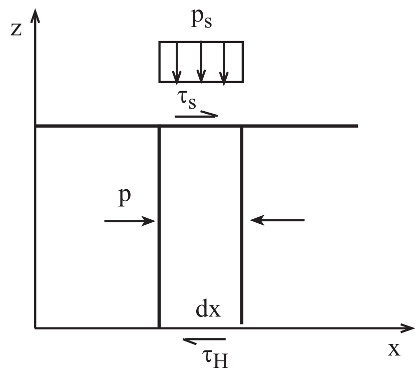


Figura 1.1: Modelo 2D del flujo en aguas someras

de longitud unidad, permanece constante. La presión ejercida en una cara lateral de la columna es igual a la presión total sobre toda la altura de agua $\rho_0 g \frac{h^2}{2}$ (bajo la hipótesis de que la presión es la hidrostática).

El autor deduce las ecuaciones del movimiento de la columna (véase la figura 1.1) de la forma siguiente. El momento de la columna de fluido varía según la ecuación:

$$\rho_0 h \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right)$$

La diferencia de presión entre las dos caras laterales a una distancia dx es:

$$-h \frac{\partial p_s}{\partial x} - \rho_0 g h \frac{\partial s}{\partial x}$$

La diferencia entre las tensiones cortantes en la cara de la superficie y la cara del fondo viene dada por

$$\tau_{s_x} - \tau_{H_x}$$

siendo τ_{s_x} la componente en la dirección x de la tensión del viento en la superficie del agua y τ_{H_x} la componente en la dirección x de la fricción en el fondo. La diferencia entre las tensiones cortantes en las dos caras verticales a una distancia dy es

$$\frac{\partial}{\partial x} (h \tau_{xx}) + \frac{\partial}{\partial y} (h \tau_{yx})$$

donde τ_{xx}, τ_{yx} son las tensiones transversales promediadas.

De la segunda ley de Newton se deduce

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = - \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + g \frac{\partial s}{\partial x} \right) + \frac{\tau_{s_x} - \tau_{H_x}}{\rho_0 h} + F_{Bx}$$
$$+ \frac{1}{\rho_0 h} \left[\frac{\partial}{\partial x} (h \tau_{xx}) + \frac{\partial}{\partial y} (h \tau_{yx}) \right] \qquad (1.2.25)$$

siendo F_{Bx} la fuerza aplicada al cuerpo por unidad de masa. Entonces por analogía con la deducción de las ecuaciones de Navier-Stokes, en [101] se aproxima el último término a la derecha de la expresión anterior como sigue

$$\frac{1}{h}\left[\frac{\partial}{\partial x}(h\tau_{xx})+\frac{\partial}{\partial y}(h\tau_{yx})\right]\approx\bar{\mu}_t\left[\Delta\bar{u}+\frac{1}{3}\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{v}}{\partial x\partial y}\right)\right] \tag{1.2.26}$$

donde $\bar{\mu}_t$ (la viscosidad dinámica turbulenta promediada) se ha supuesto constante y bajo la hipótesis adicional $\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}=0$, finalmente resulta

$$\frac{1}{h}\left[\frac{\partial}{\partial x}(h\tau_{xx})+\frac{\partial}{\partial y}(h\tau_{yx})\right]\approx\bar{\mu}_t\Delta\bar{u} \tag{1.2.27}$$

Para terminar, los símbolos de promedio en altura se suprimen, es decir, se utiliza u, v y μ_t en lugar de $\bar{u}$, $\bar{v}$ y $\bar{\mu}_t$ (se está suponiendo que $u\simeq\bar{u}$, $v\simeq\bar{v}$ y $\mu_t\simeq\bar{\mu}_t$), y el modelo que se propone es el siguiente:

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}+g\frac{\partial s}{\partial x}\right)+\frac{\tau_{s_x}-\tau_{H_x}}{\rho_0 h}+F_{Bx}+\nu_t\Delta u\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+g\frac{\partial s}{\partial y}\right)+\frac{\tau_{s_y}-\tau_{H_y}}{\rho_0 h}+F_{By}+\nu_t\Delta v\end{aligned} \tag{1.2.28}$$

junto con la ecuación (1.2.7) que ahora se escribe:

$$\frac{\partial h}{\partial t}+\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}=0 \tag{1.2.29}$$

En estas ecuaciones ν_t representa la viscosidad turbulenta horizontal promediada en altura, que es diferente de la viscosidad local en el caso tridimensional.

De nuevo se ha supuesto para la obtención de este modelo que la presión es hidrostática, la velocidad vertical no se ha tenido en cuenta en absoluto y además, se considera que la variación en vertical de las componentes horizontales de la velocidad es pequeña, lo que permite sustituirlas por su valor promedio en la vertical.

1.2.5. Obtención de las ecuaciones dinámicas del modelo de aguas someras con viscosidad

Siguiendo y adaptando a Chaudhry [26] (páginas 346-354), Fe [36] (páginas 46-52) y Lai [55] (páginas 171-179) vamos a obtener ahora algunos modelos de aguas someras con viscosidad.

Como punto de partida se toman las ecuaciones de Navier-Stokes ((1.1.11), es decir, se acepta la hipótesis oceanográfica (1.1.9)) y la ecuación de la conservación de la masa ((1.1.1)):

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+2\phi(\operatorname{sen}\varphi)v+\nu\Delta u \tag{1.2.30}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}-2\phi(\operatorname{sen}\varphi)u+\nu\Delta v \tag{1.2.31}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g+\nu\Delta w \tag{1.2.32}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.2.33}$$

y se procede como en la sección 1.2.2 para obtener (1.2.9):

$$p=p_s+\rho_0 g(s-z) \tag{1.2.34}$$

Se suma ahora a la primera ecuación de Navier-Stokes ((1.2.30)) la ecuación de conservación de la masa ((1.2.33)) multiplicada por u y, teniendo en cuenta (1.2.34), se obtiene:

$$\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-g\frac{\partial s}{\partial x}+2\phi(\operatorname{sen}\varphi)v+\nu\Delta u \tag{1.2.35}$$

De manera análoga, multiplicando (1.2.33) por v, sumándola a (1.2.31) y teniendo en cuenta (1.2.34) se llega a:

$$\frac{\partial v}{\partial t}+\frac{\partial(uv)}{\partial x}+\frac{\partial(v^2)}{\partial y}+\frac{\partial(vw)}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}-g\frac{\partial s}{\partial y}-2\phi(\operatorname{sen}\varphi)u+\nu\Delta v \tag{1.2.36}$$

Se integran, a continuación, las ecuaciones (1.2.35) y (1.2.36) en la vertical

$$\int_H^s\left(\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\right)dz$$
$$=-h\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}+g\frac{\partial s}{\partial x}\right)+2\phi\,(\operatorname{sen}\varphi)\int_H^s v dz+\nu\int_H^s\Delta u\,dz \tag{1.2.37}$$

$$\int_H^s\left(\frac{\partial v}{\partial t}+\frac{\partial(uv)}{\partial x}+\frac{\partial(v^2)}{\partial y}+\frac{\partial(vw)}{\partial z}\right)dz$$
$$=-h\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+g\frac{\partial s}{\partial y}\right)-2\phi\,(\operatorname{sen}\varphi)\int_H^s u dz+\nu\int_H^s\Delta v\,dz \tag{1.2.38}$$

y se aplica la regla de Leibnitz a los miembros a la izquierda teniendo en cuenta que el fondo no varía con el tiempo:

$$\int_H^s \left(\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} \right) dz$$
$$= \frac{\partial}{\partial t} \int_H^s u dz - u|_{z=s} \frac{\partial s}{\partial t} + \frac{\partial}{\partial x} \int_H^s u^2 dz - u^2\big|_{z=s} \frac{\partial s}{\partial x} + u^2\big|_{z=H} \frac{\partial H}{\partial x}$$
$$+ \frac{\partial}{\partial y} \int_H^s (uv) dz - (uv)|_{z=s} \frac{\partial s}{\partial y} + (uv)|_{z=H} \frac{\partial H}{\partial y}$$
$$+ (uw)|_{z=s} - (uw)|_{z=H} \tag{1.2.39}$$

Imponiendo las condiciones cinemáticas de frontera libre y fondo (1.2.3) y (1.2.4) resulta

$$\int_H^s \left(\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} \right) dz$$
$$= \frac{\partial}{\partial t} \int_H^s u dz + \frac{\partial}{\partial x} \int_H^s u^2 dz + \frac{\partial}{\partial y} \int_H^s (uv) dz \tag{1.2.40}$$

Para integrar u^2 y uv se sustituyen u y v por sus valores promedio $\bar{u}$ y $\bar{v}$. Esta sustitución se justifica porque se supone que la variación vertical de las componentes horizontales de la velocidad es pequeña. Así, (1.2.40) se puede escribir

$$\int_H^s \left(\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} \right) dz$$
$$\simeq \frac{\partial (\bar{u}h)}{\partial t} + \frac{\partial (\bar{u}^2 h)}{\partial x} + \frac{\partial (\bar{u}\bar{v}h)}{\partial y} \tag{1.2.41}$$

Para el miembro de la izquierda de (1.2.38), análogamente, se tiene

$$\int_H^s \left(\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} \right) dz$$
$$\simeq \frac{\partial (\bar{v}h)}{\partial t} + \frac{\partial (\bar{u}\bar{v}h)}{\partial x} + \frac{\partial (\bar{v}^2 h)}{\partial y} \tag{1.2.42}$$

Veamos ahora cómo se integran los términos $\nu \int_H^s \Delta u \, dz$ y $\nu \int_H^s \Delta v \, dz$ de los que se van a obtener los términos de viscosidad. Se comienza por reescribir estos términos del modo siguiente

$$\nu \int_H^s \Delta u \, dz = \nu \int_H^s \Delta_{xy} u \, dz + \nu \int_H^s \frac{\partial^2 u}{\partial z^2} \, dz \tag{1.2.43}$$

$$\nu \int_H^s \Delta v \, dz = \nu \int_H^s \Delta_{xy} v \, dz + \nu \int_H^s \frac{\partial^2 v}{\partial z^2} \, dz \tag{1.2.44}$$

donde $\Delta_{xy} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. Se podría suponer que $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial^2 v}{\partial z^2} = 0$ (como hacen algunos de los autores antes mencionados), y las expresiones anteriores se reducen a:

$$\nu \int_H^s \Delta u \, dz = \nu \int_H^s \Delta_{xy} u \, dz \tag{1.2.45}$$

$$\nu \int_H^s \Delta v \, dz = \nu \int_H^s \Delta_{xy} v \, dz \tag{1.2.46}$$

En caso contrario, se escribe la viscosidad cinemática en función de la dinámica ($\nu = \mu/\rho_0$) y se obtiene

$$\nu \int_H^s \frac{\partial^2 u}{\partial z^2} \, dz = \nu \left(\left.\frac{\partial u}{\partial z}\right|_{z=s} - \left.\frac{\partial u}{\partial z}\right|_{z=H} \right) = \frac{1}{\rho_0} \left(\tau_{s_x} - \tau_{H_x} \right) \tag{1.2.47}$$

$$\nu \int_H^s \frac{\partial^2 v}{\partial z^2} \, dz = \nu \left(\left.\frac{\partial v}{\partial z}\right|_{z=s} - \left.\frac{\partial v}{\partial z}\right|_{z=H} \right) = \frac{1}{\rho_0} \left(\tau_{s_y} - \tau_{H_y} \right) \tag{1.2.48}$$

donde τ_{s_x}, τ_{s_y}, τ_{H_x} y τ_{H_y} son las componentes en la dirección x e y de la tensión tangencial que actúa sobre la superficie libre y el fondo respectivamente. En este caso (1.2.43) y (1.2.44) resultan:

$$\nu \int_H^s \Delta u \, dz = \nu \int_H^s \Delta_{xy} u \, dz + \frac{1}{\rho_0} \left(\tau_{s_x} - \tau_{H_x} \right) \tag{1.2.49}$$

$$\nu \int_H^s \Delta v \, dz = \nu \int_H^s \Delta_{xy} v \, dz + \frac{1}{\rho_0} \left(\tau_{s_y} - \tau_{H_y} \right) \tag{1.2.50}$$

También hay varias posibilidades para el cálculo de $\nu \int_H^s \Delta_{xy} u \, dz$ y $\nu \int_H^s \Delta_{xy} v \, dz$. En primer lugar, si se reemplaza directamente u y v por $\bar{u}$ y $\bar{v}$, estos términos se reducen a:

$$\nu \int_H^s \Delta_{xy} u \, dz \approx \nu h \Delta_{xy} \bar{u} \tag{1.2.51}$$

$$\nu \int_H^s \Delta_{xy} v \, dz \approx \nu h \Delta_{xy} \bar{v} \tag{1.2.52}$$

También se puede comenzar por aplicar la regla de Leibnitz:

$$\int_H^s \Delta_{xy} u\,dz = \int_H^s \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dz = \frac{\partial}{\partial x}\int_H^s \frac{\partial u}{\partial x}\,dz - \left.\frac{\partial u}{\partial x}\right|_{z=s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right|_{z=H}\frac{\partial H}{\partial x}$$

$$+ \frac{\partial}{\partial y}\int_H^s \frac{\partial u}{\partial y}\,dz - \left.\frac{\partial u}{\partial y}\right|_{z=s}\frac{\partial s}{\partial y} + \left.\frac{\partial u}{\partial y}\right|_{z=H}\frac{\partial H}{\partial y} \qquad (1.2.53)$$

y suponiendo que la lámina de agua es aproximadamente paralela al fondo,

$$\frac{\partial H}{\partial x} \approx \frac{\partial s}{\partial x}, \quad \frac{\partial H}{\partial y} \approx \frac{\partial s}{\partial y} \qquad (1.2.54)$$

sustituir u por su valor promedio $\bar{u}$, de modo que se anulan los sumandos siguientes

$$-\left.\frac{\partial u}{\partial x}\right|_{z=s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right|_{z=H}\frac{\partial H}{\partial x} - \left.\frac{\partial u}{\partial y}\right|_{z=s}\frac{\partial s}{\partial y} + \left.\frac{\partial u}{\partial y}\right|_{z=H}\frac{\partial H}{\partial y}$$

$$\approx -\frac{\partial \bar{u}}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial \bar{u}}{\partial x}\frac{\partial s}{\partial x} - \frac{\partial \bar{u}}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial \bar{u}}{\partial y}\frac{\partial s}{\partial y} = 0$$

Así (1.2.53) se reduce a:

$$\nu \int_H^s \Delta_{xy} u\,dz \approx \nu \left[\frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y}\right)\right] \qquad (1.2.55)$$

y, análogamente, también se tiene

$$\nu \int_H^s \Delta_{xy} v\,dz \approx \nu \left[\frac{\partial}{\partial x}\left(h\frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right)\right] \qquad (1.2.56)$$

La tercera posibilidad para el término de viscosidad se obtiene aplicando dos veces, en lugar de una, la regla de Leibnitz y realizando la hipótesis (1.2.54). El resultado es el siguiente:

$$\nu \int_H^s \Delta_{xy} u\,dz \approx \nu \Delta_{xy}(h\bar{u}) \qquad (1.2.57)$$

$$\nu \int_H^s \Delta_{xy} v\,dz \approx \nu \Delta_{xy}(h\bar{v}) \qquad (1.2.58)$$

Tendríamos ahora la posibilidad de plantear seis modelos diferentes de aguas someras según se escojan para sustituir $\nu \int_H^s \Delta u\,dz$ y $\nu \int_H^s \Delta v\,dz$ (1.2.45)-(1.2.46) o (1.2.49)-(1.2.50) y para sustituir en éstos, (1.2.51)-(1.2.52), (1.2.55)-(1.2.56) o

(1.2.57)-(1.2.58):

$$\nu \int_H^s \Delta u \, dz = \begin{cases} \nu \displaystyle\int_H^s \Delta_{xy} u \, dz \\ \text{ó} \\ \nu \displaystyle\int_H^s \Delta_{xy} u \, dz + \dfrac{1}{\rho_0} \left(\tau_{s_x} - \tau_{H_x}\right) \end{cases} \tag{1.2.59}$$

$$\nu \int_H^s \Delta_{xy} u \, dz \approx \begin{cases} \nu h \Delta_{xy} \bar{u} \\ \text{ó} \\ \nu \left[\dfrac{\partial}{\partial x} \left(h \dfrac{\partial \bar{u}}{\partial x} \right) + \dfrac{\partial}{\partial y} \left(h \dfrac{\partial \bar{u}}{\partial y} \right) \right] \\ \text{ó} \\ \nu \Delta_{xy} (h \bar{u}) \end{cases} \tag{1.2.60}$$

(análogo para $\nu \displaystyle\int_H^s \Delta v \, dz$).

Entonces, en las ecuaciones (1.2.37)-(1.2.38), las integrales del término de la izquierda se sustituyen según lo visto en (1.2.41)-(1.2.42) y, las de la derecha, como en (1.2.49)-(1.2.50) (elegimos esta posibilidad frente a (1.2.45)-(1.2.46), pues esta última es un caso particular de (1.2.49)-(1.2.50) si se supone que $\tau_{s_x} = \tau_{s_y} = \tau_{H_x} = \tau_{H_y} = 0$):

$$\frac{\partial (\bar{u}h)}{\partial t} + \frac{\partial (\bar{u}^2 h)}{\partial x} + \frac{\partial (\bar{u}\bar{v}h)}{\partial y} = -h \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + g \frac{\partial s}{\partial x} \right)$$
$$+ 2\phi (\operatorname{sen} \varphi) h \bar{v} + \nu \int_H^s \Delta_{xy} u \, dz + \frac{1}{\rho_0} \left(\tau_{s_x} - \tau_{H_x} \right) \tag{1.2.61}$$

$$\frac{\partial (\bar{v}h)}{\partial t} + \frac{\partial (\bar{u}\bar{v}h)}{\partial x} + \frac{\partial (\bar{v}^2 h)}{\partial y} = -h \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + g \frac{\partial s}{\partial y} \right)$$
$$- 2\phi (\operatorname{sen} \varphi) h \bar{u} + \nu \int_H^s \Delta_{xy} v \, dz + \frac{1}{\rho_0} \left(\tau_{s_y} - \tau_{H_y} \right) \tag{1.2.62}$$

Realizando las derivadas de los productos y simplificando se obtiene:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = - \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + g \frac{\partial s}{\partial x} \right)$$

$$+ 2\phi(\operatorname{sen}\varphi)\bar{v} + \frac{\nu}{h}\int_H^s \Delta_{xy} u\, dz + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \tag{1.2.63}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right)$$

$$- 2\phi(\operatorname{sen}\varphi)\bar{u} + \frac{\nu}{h}\int_H^s \Delta_{xy} v\, dz + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \tag{1.2.64}$$

A partir de las ecuaciones (1.2.63)-(1.2.64) se deducen tres modelos de aguas someras con distintos términos de viscosidad. Si se realiza la sustitución (1.2.60.a), el término de viscosidad es similar al de las ecuaciones de Navier-Stokes (por simplificar, denotaremos las velocidades medias $\bar{u}$ y $\bar{v}$ por u y v, que es lo que se suele hacer en la literatura):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right)$$

$$+ 2\phi(\operatorname{sen}\varphi)v + \nu\Delta_{xy} u + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \tag{1.2.65}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right)$$

$$- 2\phi(\operatorname{sen}\varphi)u + \nu\Delta_{xy} v + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \tag{1.2.66}$$

Al escoger (1.2.55)-(1.2.56), se obtiene un modelo de aguas someras que incluye en el término de viscosidad la variación del calado:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right)$$

$$+ 2\phi(\operatorname{sen}\varphi)v + \nu\left[\Delta_{xy} u + \frac{1}{h}\left(\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial u}{\partial y}\right)\right] + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \tag{1.2.67}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right)$$

$$- 2\phi(\operatorname{sen}\varphi)u + \nu\left[\Delta_{xy} v + \frac{1}{h}\left(\frac{\partial h}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial v}{\partial y}\right)\right] + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \tag{1.2.68}$$

Se puede encontrar modelos con estos términos de viscosidad en [53] (página 302) y [100] (página 1137).

Por último, utilizando (1.2.57)-(1.2.58), en el término de viscosidad aparece el laplaciano del flujo en lugar del de la velocidad:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right)$$

$$+ 2\phi(\operatorname{sen}\varphi)v + \frac{\nu}{h}\Delta_{xy}(hu) + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \qquad (1.2.69)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right)$$

$$- 2\phi(\operatorname{sen}\varphi)u + \frac{\nu}{h}\Delta_{xy}(hv) + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \qquad (1.2.70)$$

En [11] (páginas 60-61), se proponen precisamente las tres posibilidades ((1.2.65)-(1.2.66), (1.2.67)-(1.2.68), (1.2.69)-(1.2.70)) que hemos visto para los términos de viscosidad.

Observación 1.3 *Los otros tres modelos que se obtienen al suponer* (1.2.45)-(1.2.46) *en lugar de* (1.2.49)-(1.2.50) *se diferencian de los tres que acabamos de proponer únicamente en que no aparecen los términos*

$$\frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right), \quad \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right).$$

Las hipótesis que se han realizado para derivar estos modelos de aguas someras son las siguientes

1. El movimiento principal ocurre en planos horizontales
2. La aceleración vertical es despreciable frente a la aceleración de la gravedad y la aceleración de Coriolis se toma según la hipótesis oceanográfica.
3. La presión es la hidrostática
4. La variación en vertical de las componentes horizontales de la velocidad es pequeña, lo que permite sustituirlas por su valor promedio al integrar en la vertical.
5. La lámina de agua es aproximadamente paralela al fondo

Debido a estas hipótesis la precisión de los resultados que se obtienen con estos modelos dependen del caso particular que se estudie.

Algunos autores obtienen un modelo de aguas someras a partir de las ecuaciones de Reynolds que se deducen mediante el promedio temporal de las de Navier-Stokes (Reynolds Averaged Navier-Stokes). Estas ecuaciones (conocidas por RANS) se deducen tras descomponer los valores instantáneos de las variables en un valor medio y un valor de fluctuación. También al obtener las ecuaciones de aguas someras de este modo son distintos los términos viscosos que se proponen. Peraire et al. ([76]) incluye los siguientes términos:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right) + 2\phi(\operatorname{sen}\varphi)v$$

$$+ \frac{\partial}{\partial x}\left(2\nu_t h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\nu_t h\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \quad (1.2.71)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right) - 2\phi(\operatorname{sen}\varphi)u$$

$$+ \frac{\partial}{\partial x}\left[\nu_t h\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(2\nu_t h\frac{\partial v}{\partial y}\right) + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \quad (1.2.72)$$

donde ν_t es la viscosidad cinemática turbulenta. La formulación que utiliza Anastasiou et al. ([2]) desprecia alguno de los términos anteriores, y en lugar de (1.2.71)-(1.2.72) propone:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right) + 2\phi(\operatorname{sen}\varphi)v$$

$$+ \frac{\partial}{\partial x}\left(\nu_t h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_t h\frac{\partial u}{\partial y}\right) + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \quad (1.2.73)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right) - 2\phi(\operatorname{sen}\varphi)u$$

$$+ \frac{\partial}{\partial x}\left(\nu_t h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_t h\frac{\partial v}{\partial y}\right) + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \quad (1.2.74)$$

Rodríguez-Vellando ([94]) considera despreciable la variación de $\nu_t h$ obteniendo

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + g\frac{\partial s}{\partial x}\right) + 2\phi(\operatorname{sen}\varphi)v$$

$$+ \nu_t h\Delta_{xy}u + \frac{1}{\rho_0 h}\left(\tau_{s_x} - \tau_{H_x}\right) \quad (1.2.75)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + g\frac{\partial s}{\partial y}\right) - 2\phi(\operatorname{sen}\varphi)u$$

$$+ \nu_t h\Delta_{xy}v + \frac{1}{\rho_0 h}\left(\tau_{s_y} - \tau_{H_y}\right) \quad (1.2.76)$$

en lugar de (1.2.71)-(1.2.72).

Capítulo 2

Modelo unidimensional de aguas someras obtenido a partir de las ecuaciones de Euler

2.1. Formulación del problema

En este capítulo pretendemos obtener un modelo unidimensional de aguas someras sin viscosidad. Para ello partiremos de las ecuaciones de Euler bidimensionales, en un dominio cuyas dos variables representarán las coordenadas horizontal y vertical, respectivamente.Un ejemplo de este tipo de dominios son los canales. Para fijar ideas, nosotros consideraremos que nuestro dominio de trabajo es un canal, aunque la misma descripción (véase (2.1.1)) es válida para otros casos.

2.1.1. Ecuaciones de partida

Comenzamos por considerar un canal que representamos mediante el dominio Ω (Figura 2.1) definido por:

$$\Omega = \{(x, z)/x \in [0, L], z \in [H(x), H(x) + h(t, x)]\} \tag{2.1.1}$$

donde x es la coordenada horizontal, z la coordenada vertical, $z = H(x)$ es la ecuación del fondo del canal que suponemos conocido y $z = s(t, x) = H(x) + h(t, x)$ es la ecuación de la superficie (desconocida), siendo $h(t, x)$ la altura de agua sobre el fondo.

Consideramos que el flujo se rige por las ecuaciones bidimensionales de Euler en Ω y que la única fuerza externa actuando sobre el fluido es la debida a la gravedad,

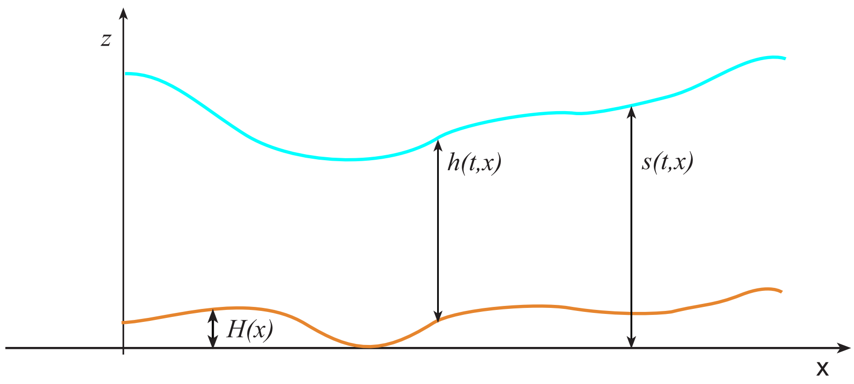


Figura 2.1: Canal

esto es, se verifica

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{2.1.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g \tag{2.1.3}$$

donde se ha empleado la siguiente notación:

$(u(t,x,z), w(t,x,z))$	vector velocidad
$p(t,x,z)$	presión
g	aceleración de la gravedad (constante)
ρ_0	densidad del fluido

El fluido se supone incompresible por lo que verifica:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.1.4}$$

Hemos de imponer las condiciones de contorno al sistema (2.1.2)-(2.1.4). Así, suponemos que la presión es la atmosférica en la superficie

$$p = p_s \quad \text{en } z = s(t,x) \tag{2.1.5}$$

($p_s = p_s(t,x)$ es la presión atmosférica en la superficie, que se supone conocida), y que el fluido no atraviesa el fondo

$$(u,w)\cdot\vec{\mathbf{n}} = 0 \quad \text{en } z = H(x) \tag{2.1.6}$$

donde $\vec{\mathbf{n}} = (n_1, n_2)$ representa al vector normal exterior unitario en la frontera del dominio.

Suponemos, además, que el caudal de entrada (uh en $x = 0$) y el de salida (uh en $x = L$) son conocidos en cada instante.

Para cerrar el problema se deben fijar las condiciones iniciales:

$$\begin{aligned} u(0,x,z) &= u_0(x,z) \\ w(0,x,z) &= w_0(x,z) \end{aligned} \tag{2.1.7}$$

Como veremos en lo que sigue, para aplicar el método de desarrollos asintóticos a las ecuaciones de Euler, necesitaremos de una ecuación más. Para ello supondremos que la vorticidad en el instante inicial es cero $\left(\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]_{t=0} = 0\right)$, de lo que se deduce que (véase [27]) la vorticidad es cero en todo momento:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{2.1.8}$$

Observación 2.1 *Obsérvese que esta suposición no permite cualquier elección de* u_0 *y* w_0 *en* (2.1.7)*, ya que implica que* $\frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x} = 0$*. En la sección 2.10 estudiaremos cómo evitar este problema permitiendo que la vorticidad inicial no sea nula.*

2.1.2. Cambio de notación

Deseamos obtener un modelo de aguas someras, por lo que la profundidad debe ser pequeña comparada con la longitud del dominio, aunque la profundidad del agua no tiene porqué ser pequeña en términos absolutos. Con este propósito se introduce un pequeño parámetro adimensional, ε, del orden del cociente entre la profundidad media y la longitud del canal. Tanto el dominio como las variables y funciones mencionadas antes dependen de este parámetro. Indicaremos con el superíndice ε dicha dependencia. Las ecuaciones se reescriben de la siguiente forma:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + w^\varepsilon \frac{\partial u^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial x^\varepsilon} \tag{2.1.9}$$

$$\frac{\partial w^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial w^\varepsilon}{\partial x^\varepsilon} + w^\varepsilon \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial z^\varepsilon} - g \tag{2.1.10}$$

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = 0 \tag{2.1.11}$$

en $[0,T] \times \Omega^\varepsilon$, siendo

$$\Omega^\varepsilon = \{(x^\varepsilon, z^\varepsilon)/x^\varepsilon \in [0,L], z^\varepsilon \in [H^\varepsilon(x^\varepsilon), H^\varepsilon(x^\varepsilon) + h^\varepsilon(t^\varepsilon, x^\varepsilon)]\}$$

donde ahora, $(u^\varepsilon, w^\varepsilon) = (u^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon), w^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon))$ es el vector velocidad y $p^\varepsilon = p^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$ es la presión.

Las condiciones de contorno se escriben

$$p^\varepsilon = p_s^\varepsilon \qquad \text{en } z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon) = H^\varepsilon(x^\varepsilon) + h^\varepsilon(t^\varepsilon, x^\varepsilon), \tag{2.1.12}$$

$$(u^\varepsilon, w^\varepsilon)\cdot \vec{\mathbf{n}}^\varepsilon = 0 \qquad \text{en } z^\varepsilon = H^\varepsilon(x^\varepsilon), \tag{2.1.13}$$

las condiciones iniciales

$$u^\varepsilon(0, x^\varepsilon, z^\varepsilon) = u_0^\varepsilon(x^\varepsilon, z^\varepsilon), \tag{2.1.14}$$

$$w^\varepsilon(0, x^\varepsilon, z^\varepsilon) = w_0^\varepsilon(x^\varepsilon, z^\varepsilon), \tag{2.1.15}$$

y suponemos que la vorticidad es nula para $t = 0$ y, por tanto, para todo t:

$$\frac{\partial u^\varepsilon}{\partial z^\varepsilon} - \frac{\partial w^\varepsilon}{\partial x^\varepsilon} = 0 \tag{2.1.16}$$

Deberíamos añadir también que el caudal es conocido en $x^\varepsilon = 0$ y $x^\varepsilon = L$ en cada instante, pero como estas condiciones son impuestas de varias formas en la literatura y no es necesario explicitarlas en lo que sigue, preferimos no incluirlas de momento, aunque éstas u otras condiciones similares serán necesarias en la resolución del modelo finalmente obtenido.

2.2. Determinación de la altura de agua

La función $H(x)$ que nos da el perfil del fondo puede considerarse conocida, pero $h(t, x)$ no, por lo que se debe introducir alguna ecuación para determinarla.

Normalmente, para obtener dicha ecuación, se usa que $z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon)$ es una superficie libre. En nuestro caso lograremos otra formulación equivalente a partir de la ley de la conservación de la masa (del volumen en este caso, ya que la densidad es contante).

Para ello se puede utilizar el hecho de que el fluido es incompresible:

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = 0$$

por lo que toda variación de volumen del fluido entre dos puntos x_1^ε y x_2^ε ha de venir dada por la diferencia entre el fluido entrante y el saliente:

$$\int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x_1^\varepsilon)}^{H^\varepsilon(x_1^\varepsilon)+h^\varepsilon(t^\varepsilon,x_1^\varepsilon)} u^\varepsilon(t^\varepsilon, x_1^\varepsilon) dz^\varepsilon dt^\varepsilon - \int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x_2^\varepsilon)}^{H^\varepsilon(x_2^\varepsilon)+h^\varepsilon(t^\varepsilon,x_2^\varepsilon)} u^\varepsilon(t^\varepsilon, x_2^\varepsilon) dz^\varepsilon dt^\varepsilon$$

$$= \int_{x_1^\varepsilon}^{x_2^\varepsilon} h^\varepsilon(t_2^\varepsilon, x^\varepsilon) dx^\varepsilon - \int_{x_1^\varepsilon}^{x_2^\varepsilon} h^\varepsilon(t_1^\varepsilon, x^\varepsilon) dx^\varepsilon \tag{2.2.1}$$

Usando que $F(b) - F(a) = \int_a^b \frac{\partial F}{\partial x}\, dx$, la igualdad anterior resulta:

$$\int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left[-\frac{\partial}{\partial x^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon)}^{H^\varepsilon(x^\varepsilon)+h^\varepsilon(t^\varepsilon,x^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon) dz^\varepsilon \right) \right] dx^\varepsilon dt^\varepsilon$$

$$= \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left(h^\varepsilon(t_2^\varepsilon, x^\varepsilon) - h^\varepsilon(t_1^\varepsilon, x^\varepsilon)\right) dx^\varepsilon \qquad (2.2.2)$$

Ahora, dividiendo por $(t_2^\varepsilon - t_1^\varepsilon)$, se llega a:

$$\frac{1}{t_2^\varepsilon - t_1^\varepsilon} \int_{t_1^\varepsilon}^{t_2^\varepsilon} \left[\int_{x_1^\varepsilon}^{x_2^\varepsilon} -\frac{\partial}{\partial x^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon)}^{H^\varepsilon(x^\varepsilon)+h^\varepsilon(t^\varepsilon,x^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon) dz^\varepsilon \right) dx^\varepsilon \right] dt^\varepsilon$$

$$= \frac{1}{t_2^\varepsilon - t_1^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left(h^\varepsilon(t_2^\varepsilon, x^\varepsilon) - h^\varepsilon(t_1^\varepsilon, x^\varepsilon)\right) dx^\varepsilon \qquad (2.2.3)$$

y tomando el límite $t_2^\varepsilon \to t_1^\varepsilon = t^\varepsilon$ se obtiene:

$$-\int_{x_1^\varepsilon}^{x_2^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon)}^{H^\varepsilon(x^\varepsilon)+h^\varepsilon(t^\varepsilon,x^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon) dz^\varepsilon \right) dx^\varepsilon = \int_{x_1^\varepsilon}^{x_2^\varepsilon} \frac{\partial h^\varepsilon}{\partial t^\varepsilon}(t^\varepsilon, x^\varepsilon) dx^\varepsilon \quad (2.2.4)$$

que es equivalente, por el teorema del valor medio, a:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}(t^\varepsilon, x^\varepsilon) + \frac{\partial}{\partial x^\varepsilon} \int_{H^\varepsilon(x^\varepsilon)}^{s^\varepsilon(t^\varepsilon,x^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon) dz^\varepsilon = 0 \qquad (2.2.5)$$

Observación 2.2 *Si, como en [56], se asume que* $\frac{\partial u^\varepsilon}{\partial z^\varepsilon} = 0$*, a partir de (2.2.5) se obtiene:*

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (u^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0$$

que es la ecuación que se propone en [56].

2.3. Construcción del dominio de referencia

En general, cuando se usa el análisis asintótico para analizar fluidos, se hace en el dominio original (véase, por ejemplo, [43, 114]), que en este caso depende del parámetro ε y del tiempo t^ε, o la superficie se supone que es constante (véase, por ejemplo, [6, 7, 12, 13, 16, 17]). Si se trabaja en un dominio independiente de ε, toda la "dependencia" de este parámetro aparece explícita en las ecuaciones, mientras que cuando se trabaja en dominios dependientes de ε, parte de esta dependencia puede permanecer "oculta" en el dominio. Es por ello que preferimos seguir las técnicas habituales en análisis asintótico aplicado a sólidos (véase [29, 30, 33, 81, 82, 104] y las referencias en ellos señaladas) en nuestro estudio de las aguas someras. Esto es, comenzamos por realizar un cambio de variable a un dominio de referencia independiente del parámetro ε y del tiempo (es decir, la dependencia del parámetro pasa del dominio a las funciones).

Sea $\Omega = [0, L] \times [0, 1]$ el dominio de referencia. Se supone:

$$h^\varepsilon(t^\varepsilon, x^\varepsilon) = \varepsilon h(t, x) \tag{2.3.1}$$

$$H^\varepsilon(x^\varepsilon) = \varepsilon H(x) \tag{2.3.2}$$

(por tanto $s^\varepsilon(t^\varepsilon, x^\varepsilon) = \varepsilon s(t, x)$) y se define el siguiente cambio de variable, de Ω a Ω^ε

$$\begin{aligned} t^\varepsilon &= t \\ x^\varepsilon &= x \\ z^\varepsilon &= \varepsilon[H(x) + zh(t, x)] \end{aligned} \tag{2.3.3}$$

Así el jacobiano del cambio de variable es:

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial t}{\partial t^\varepsilon} & \dfrac{\partial x}{\partial t^\varepsilon} & \dfrac{\partial z}{\partial t^\varepsilon} \\ \dfrac{\partial t}{\partial x^\varepsilon} & \dfrac{\partial x}{\partial x^\varepsilon} & \dfrac{\partial z}{\partial x^\varepsilon} \\ \dfrac{\partial t}{\partial z^\varepsilon} & \dfrac{\partial x}{\partial z^\varepsilon} & \dfrac{\partial z}{\partial z^\varepsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\dfrac{z}{h}\dfrac{\partial h}{\partial t} \\ 0 & 1 & -\dfrac{H' + z\dfrac{\partial h}{\partial x}}{h} \\ 0 & 0 & \dfrac{1}{\varepsilon h} \end{pmatrix}$$

y el jacobiano del cambio de variable inverso:

$$\mathbf{J^{-1}} = \begin{pmatrix} \dfrac{\partial t^\varepsilon}{\partial t} & \dfrac{\partial x^\varepsilon}{\partial t} & \dfrac{\partial z^\varepsilon}{\partial t} \\ \dfrac{\partial t^\varepsilon}{\partial x} & \dfrac{\partial x^\varepsilon}{\partial x} & \dfrac{\partial z^\varepsilon}{\partial x} \\ \dfrac{\partial t^\varepsilon}{\partial z} & \dfrac{\partial x^\varepsilon}{\partial z} & \dfrac{\partial z^\varepsilon}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \varepsilon z\dfrac{\partial h}{\partial t} \\ 0 & 1 & \varepsilon[H' + z\dfrac{\partial h}{\partial x}] \\ 0 & 0 & \varepsilon h \end{pmatrix}$$

cuyos determinantes son: $|J| = \dfrac{1}{\varepsilon h}$ y $|J^{-1}| = \varepsilon h$.

Las hipótesis (2.3.1) y (2.3.2) únicamente explicitan que h^ε y H^ε son de orden ε, es decir, que son pequeñas comparadas con la longitud del dominio.

Dada una función F^ε cualquiera definida en $[0,T]\times\bar{\Omega}^\varepsilon$, se puede construir a partir de ella otra función $F(\varepsilon)$ definida en $[0,T]\times\bar{\Omega}$ utilizando para ello el cambio de variable del modo natural: $F(\varepsilon)(t,x,z) = F^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon)$. La relación entre las derivadas parciales de una y otra función es:

$$\begin{aligned}
\frac{\partial F^\varepsilon}{\partial t^\varepsilon} &= \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial t^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial t^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial t^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial F(\varepsilon)}{\partial z} \\
&= D_t F(\varepsilon) \\
\frac{\partial F^\varepsilon}{\partial x^\varepsilon} &= \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial x^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial x^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial x^\varepsilon} \\
&= \frac{\partial F(\varepsilon)}{\partial x} - \frac{H' + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_x F(\varepsilon) \\
\frac{\partial F^\varepsilon}{\partial z^\varepsilon} &= \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial z^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial z^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial z^\varepsilon} = \frac{1}{\varepsilon h}\frac{\partial F(\varepsilon)}{\partial z} \\
&= \frac{1}{\varepsilon} D_z F(\varepsilon)
\end{aligned}$$

donde hemos introducido la siguiente notación:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial}{\partial z} \\
D_x &= \frac{\partial}{\partial x} - \frac{H' + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial}{\partial z} \\
D_z &= \frac{1}{h}\frac{\partial}{\partial z}
\end{aligned} \tag{2.3.4}$$

Si ahora definimos,

$$\begin{aligned}
u(\varepsilon)(t,x,z) &= u^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon) \\
w(\varepsilon)(t,x,z) &= w^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon) \\
p(\varepsilon)(t,x,z) &= p^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon)
\end{aligned}$$

el problema (2.1.9)-(2.1.11) se puede escribir en el dominio de referencia Ω de la forma siguiente:

$$D_t u(\varepsilon) + u(\varepsilon)D_x u(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z u(\varepsilon) = -\frac{1}{\rho_0}D_x p(\varepsilon) \tag{2.3.5}$$

$$D_t w(\varepsilon) + u(\varepsilon)D_x w(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z w(\varepsilon) = -\frac{1}{\rho_0}\frac{1}{\varepsilon}D_z p(\varepsilon) - g \tag{2.3.6}$$

$$D_x u(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \tag{2.3.7}$$

La condición de contorno (2.1.12) tras el cambio de variable se escribe:

$$p^\varepsilon = p_s^\varepsilon \text{ en } z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon) \Rightarrow p(\varepsilon) = p_s \text{ en } z = 1 \tag{2.3.8}$$

(donde implícitamente hemos supuesto que $p_s^\varepsilon(t^\varepsilon, x^\varepsilon) = p_s(t, x)$, es decir, la presión atmosférica superficial es independiente de ε, lo que nos parece una hipótesis lógica dado que no depende ni de z^ε ni de la profundidad).

Para aplicar el cambio de variable a la condición (2.1.13) tendremos en cuenta que $\vec{\mathbf{n}}^\varepsilon$ es la normal exterior unitaria en $z^\varepsilon = H^\varepsilon$, y como

$$\frac{\partial H^\varepsilon}{\partial x^\varepsilon} = \varepsilon \frac{\partial H}{\partial x} = \varepsilon H'$$

$\vec{\mathbf{n}}^\varepsilon$ debe ser paralelo al vector $(\varepsilon H', -1)$, y por ser unitario:

$$\vec{\mathbf{n}}^\varepsilon = \left(\frac{\varepsilon H'}{\sqrt{\varepsilon^2 (H')^2 + 1}}, \frac{-1}{\sqrt{\varepsilon^2 (H')^2 + 1}} \right)$$

Ahora, si aplicamos el cambio de variable a (2.1.13),

$$(u^\varepsilon, w^\varepsilon) \cdot \vec{\mathbf{n}}^\varepsilon = 0 \quad \text{en } z^\varepsilon = H^\varepsilon(x^\varepsilon)$$

resulta

$$\frac{u(\varepsilon)\varepsilon H'}{\sqrt{\varepsilon^2 (H')^2 + 1}} + \frac{w(\varepsilon)(-1)}{\sqrt{\varepsilon^2 (H')^2 + 1}} = 0 \text{ en } z = 0$$

que escribimos de forma equivalente

$$w(\varepsilon) = \varepsilon u(\varepsilon) H' \text{ en } z = 0$$

Las condiciones iniciales (2.1.14)-(2.1.15) resultan:

$$u(\varepsilon)(0, x, z) = u_0(\varepsilon)(x, z) \tag{2.3.9}$$

$$w(\varepsilon)(0, x, z) = w_0(\varepsilon)(x, z) \tag{2.3.10}$$

De (2.1.16) deducimos que

$$\frac{1}{\varepsilon} D_z u(\varepsilon) - D_x w(\varepsilon) = 0 \text{ en } \Omega \tag{2.3.11}$$

Finalmente, si se aplica el cambio de variable a la ecuación obtenida para el cálculo del calado ((2.2.5)) teniendo en cuenta que h^ε no depende de z^ε, obtenemos:

$$\frac{\partial h}{\partial t} + \int_0^1 \frac{\partial (u(\varepsilon) h)}{\partial x} dz = 0 \tag{2.3.12}$$

2.4. Ecuaciones en el dominio de referencia

Se resumen a continuación las ecuaciones vistas hasta el momento, que determinan el problema a resolver en el dominio de referencia Ω.

Las ecuaciones de Euler:

$$D_t u(\varepsilon) + u(\varepsilon) D_x u(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z u(\varepsilon) = -\frac{1}{\rho_0} D_x p(\varepsilon) \tag{2.4.1}$$

$$D_t w(\varepsilon) + u(\varepsilon) D_x w(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon) = -\frac{1}{\rho_0} \frac{1}{\varepsilon} D_z p(\varepsilon) - g \tag{2.4.2}$$

la condición de incompresibilidad:

$$D_x u(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \tag{2.4.3}$$

las condiciones de contorno:

$$p(\varepsilon) = p_s \qquad \text{en } z = 1, \tag{2.4.4}$$

$$w(\varepsilon) = \varepsilon u(\varepsilon) H' \qquad \text{en } z = 0, \tag{2.4.5}$$

(a las que habría que añadir las de caudal conocido en $x = 0$ y en $x = L$), las condiciones iniciales:

$$u(\varepsilon)(0, x, z) = u_0(\varepsilon)(x, z), \tag{2.4.6}$$

$$w(\varepsilon)(0, x, z) = w_0(\varepsilon)(x, z), \tag{2.4.7}$$

la condición de vorticidad nula:

$$\frac{1}{\varepsilon} D_z u(\varepsilon) - D_x w(\varepsilon) = 0 \text{ en } \Omega \tag{2.4.8}$$

y la ecuación que determina la función h:

$$\frac{\partial h}{\partial t} + \int_0^1 \frac{\partial (u(\varepsilon) h)}{\partial x} dz = 0 \tag{2.4.9}$$

2.5. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (2.4.1)-(2.4.9) admite un desarrollo en serie de potencias de ε en la forma:

$$u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$$

$$w(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots \tag{2.5.1}$$

$$p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots$$

Se sustituyen estos desarrollos en serie de potencias en las ecuaciones (2.4.1)-(2.4.9). Realizando esta sustitución en la primera ecuación de Euler ((2.4.1)) se obtiene:

$$D_t u^0 + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + \cdots$$
$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots) \left[D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots \right]$$
$$+ (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots) \frac{1}{\varepsilon} \left[D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots \right]$$
$$= -\frac{1}{\rho_0} \left(D_x p^0 + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 + \cdots \right)$$

El paso siguiente consiste en identificar los términos multiplicados por la misma potencia de ε. En este caso se tiene:

$$\varepsilon^{-1} w^0 D_z u^0 + \varepsilon^0 \left(D_t u^0 + u^0 D_x u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0 \right)$$
$$+ \varepsilon \left(D_t u^1 + u^0 D_x u^1 + u^1 D_x u^0 + w^0 D_z u^2 + w^1 D_z u^1 + w^2 D_z u^0 + \frac{1}{\rho_0} D_x p^1 \right)$$
$$+ \varepsilon^2 \left(D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 D_x u^0 + w^0 D_z u^3 + w^1 D_z u^2 \right.$$
$$\left. + w^2 D_z u^1 + w^3 D_z u^0 + \frac{1}{\rho_0} D_x p^2 \right) + O(\varepsilon^3) = 0 \qquad (2.5.2)$$

Reemplazando $u(\varepsilon)$, $w(\varepsilon)$ y $p(\varepsilon)$ por sus desarrollos en serie de potencias de ε, (2.5.1), en la segunda ecuación de Euler ((2.4.2)), ésta resulta:

$$D_t w^0 + \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + \varepsilon^3 D_t w^3 + \cdots$$
$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots) \left[D_x w^0 + \varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \varepsilon^3 D_x w^3 + \cdots \right]$$
$$+ (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots) \frac{1}{\varepsilon} \left[D_z w^0 + \varepsilon D_z w^1 + \varepsilon^2 D_z w^2 \right.$$
$$\left. + \varepsilon^3 D_z w^3 + \cdots \right] = -\frac{1}{\rho_0} \frac{1}{\varepsilon} \left(D_z p^0 + \varepsilon D_z p^1 + \varepsilon^2 D_z p^2 + \varepsilon^3 D_z p^3 + \cdots \right) - g$$

Agrupando en potencias de ε se obtiene:

$$\varepsilon^{-1} \left(w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 \right)$$
$$+ \varepsilon^0 \left(D_t w^0 + u^0 D_x w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g \right)$$

$$+\varepsilon\left(D_t w^1 + u^0 D_x w^1 + u^1 D_x w^0 + w^0 D_z w^2 + w^1 D_z w^1 + w^2 D_z w^0 + \frac{1}{\rho_0} D_z p^2\right) + \varepsilon^2\left(D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + u^2 D_x w^0 + w^0 D_z w^3 + w^1 D_z w^2 + w^2 D_z w^1 + w^3 D_z w^0 + \frac{1}{\rho_0} D_z p^3\right) + O(\varepsilon^3) = 0 \quad (2.5.3)$$

Repetimos el proceso para la ecuación de la incompresibilidad ((2.4.3)). En primer lugar se realiza la sustitución:

$$D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots + \frac{1}{\varepsilon}\left(D_z w^0 + \varepsilon D_z w^1 + \varepsilon^2 D_z w^2 + \varepsilon^3 D_z w^3 + \cdots\right) = 0$$

y después la identificación de los términos multiplicados por cada potencia de ε

$$\varepsilon^{-1} D_z w^0 + D_x u^0 + D_z w^1 + \varepsilon\left(D_x u^1 + D_z w^2\right) + \varepsilon^2\left(D_x u^2 + D_z w^3\right) + O(\varepsilon^3) = 0 \quad (2.5.4)$$

De la condición de contorno (2.4.4) se tiene

$$p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots = p_s \text{ en } z = 1 \quad (2.5.5)$$

Al sustituir los desarrollos (2.5.1) en la condición (2.4.5) resulta

$$w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots = \varepsilon(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)H' \text{ en } z = 0$$

y agrupando

$$w^0 + \varepsilon(w^1 - u^0 H') + \varepsilon^2(w^2 - u^1 H') + \varepsilon^3(w^3 - u^2 H') + \cdots = 0 \text{ en } z = 0 \quad (2.5.6)$$

A partir de la ecuación (2.4.9) necesaria para la determinación del calado, sustituyendo $u(\varepsilon)$ por su desarrollo en serie de potencias de ε se obtiene:

$$\frac{\partial h}{\partial t} + \int_0^1 \left(\frac{\partial(hu^0)}{\partial x} + \varepsilon\frac{\partial(hu^1)}{\partial x} + \varepsilon^2\frac{\partial(hu^2)}{\partial x} + \cdots\right) dz = 0 \quad (2.5.7)$$

De la vorticidad nula (2.4.8), sustituyendo $u(\varepsilon)$ y $w(\varepsilon)$ por sus expresiones en serie de potencias de ε, se tiene:

$$\frac{1}{\varepsilon}\left(D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right) - \left(D_x w^0 + \varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \cdots\right) = 0$$

Agrupando en potencias de ε:

$$\varepsilon^{-1} D_z u^0 + \varepsilon^0\left(D_z u^1 - D_x w^0\right) + \varepsilon\left(D_z u^2 - D_x w^1\right) + \cdots = 0 \quad (2.5.8)$$

Puesto que u^0, w^0, p^0, u^1, w^1, etc. son independientes de ε, una vez agrupados los términos que multiplican a una misma potencia de ε, en las ecuaciones anteriores obtenemos un polinomio en ε igualado a cero, por lo que sus coeficientes han de ser nulos. De este modo se logra una serie de ecuaciones que nos permitirán determinar u^0, w^0, p^0, u^1, w^1, etc.

Comenzamos por los coeficientes de ε^{-1} que aparecen en (2.5.2)-(2.5.4) y (2.5.8):

$$w^0 D_z u^0 = 0 \tag{2.5.9}$$

$$w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 = 0 \tag{2.5.10}$$

$$D_z w^0 = 0 \tag{2.5.11}$$

$$D_z u^0 = 0 \tag{2.5.12}$$

De las igualdades (2.5.11) y (2.5.12) deducimos que los términos de orden 0 de ambas componentes de la velocidad son independientes de z.

Igualando a cero los coeficientes de ε^0 que aparecen en (2.5.2)-(2.5.8) tenemos las siguientes igualdades:

$$D_t u^0 + u^0 D_x u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0 = 0 \tag{2.5.13}$$

$$D_t w^0 + u^0 D_x w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g = 0 \tag{2.5.14}$$

$$D_x u^0 + D_z w^1 = 0 \tag{2.5.15}$$

$$p^0 = p_s \text{ en } z = 1 \tag{2.5.16}$$

$$w^0 = 0 \text{ en } z = 0 \tag{2.5.17}$$

$$\frac{\partial h}{\partial t} + \int_0^1 \frac{\partial (h u^0)}{\partial x}\, dz = 0 \tag{2.5.18}$$

$$D_z u^1 - D_x w^0 = 0 \tag{2.5.19}$$

Como consecuencia de las igualdades (2.5.11) y (2.5.17):

$$w^0 = 0, \tag{2.5.20}$$

con lo que:

$$w(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots .$$

Ahora, usando (2.5.20), la igualdad (2.5.10) se reduce a $D_z p^0 = 0$ que, junto con (2.5.16), nos permite obtener el término de orden 0 de la presión:

$$p^0 = p_s(t, x) \tag{2.5.21}$$

Teniendo en cuenta las igualdades (2.5.12), (2.5.20) y (2.5.21), así como (2.3.4), se pueden reescribir las ecuaciones (2.5.13)-(2.5.15), (2.5.18) y (2.5.19) de la forma siguiente:

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} = 0 \tag{2.5.22}$$

$$\frac{1}{\rho_0} D_z p^1 + g = 0 \tag{2.5.23}$$

$$\frac{\partial u^0}{\partial x} + D_z w^1 = 0 \tag{2.5.24}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (h u^0)}{\partial x} = 0 \tag{2.5.25}$$

$$D_z u^1 = 0 \tag{2.5.26}$$

Las ecuaciones (2.5.22) y (2.5.25) se utilizarán para calcular u^0 y h.

Continuamos igualando a cero los términos que multiplican a ε en las ecuaciones (2.5.2)-(2.5.8). Tenemos en cuenta a la hora de reescribir estos términos que $D_z u^0 = D_z u^1 = 0$ ((2.5.12),(2.5.26)) y $w^0 = 0$ (2.5.20), obtenemos:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} D_x p^1 = 0 \tag{2.5.27}$$

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 = 0 \tag{2.5.28}$$

$$\frac{\partial u^1}{\partial x} + D_z w^2 = 0 \tag{2.5.29}$$

$$p^1 = 0 \text{ en } z = 1 \tag{2.5.30}$$

$$w^1 - u^0 H' = 0 \text{ en } z = 0 \tag{2.5.31}$$

$$\frac{\partial (h u^1)}{\partial x} = 0 \tag{2.5.32}$$

$$D_z u^2 - D_x w^1 = 0 \tag{2.5.33}$$

La integración de la ecuación (2.5.23) respecto a z, imponiendo la condición (2.5.30), nos proporciona la siguiente expresión para el término de orden 1 de la presión:

$$p^1 = \rho_0 g h (1 - z) \tag{2.5.34}$$

lo que nos permite calcular

$$D_x p^1 = \rho_0 g \frac{\partial s}{\partial x} \tag{2.5.35}$$

Sustituyendo $D_x p^1$ por la expresión anterior, la ecuación (2.5.27) (que utilizaremos para calcular u^1) resulta:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g \frac{\partial s}{\partial x} = 0 \tag{2.5.36}$$

Integramos también (2.5.24) respecto a z teniendo en cuenta que u^0 no depende de z e imponiendo la condición (2.5.31). Encontramos la siguiente expresión para w^1 en términos de u^0 y H:

$$w^1 = u^0 H' - h \frac{\partial u^0}{\partial x} z \tag{2.5.37}$$

Una vez más, repetimos el proceso e igualamos, ahora, a cero los coeficientes de ε^2 que aparecen en (2.5.2)-(2.5.8). Se tiene en cuenta que $D_z u^0 = D_z u^1 = 0$ ((2.5.12),(2.5.26)) y $w^0 = 0$ (2.5.20) para simplificar las igualdades:

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + \frac{1}{\rho_0} D_x p^2 = 0 \tag{2.5.38}$$

$$D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1 + \frac{1}{\rho_0} D_z p^3 = 0 \tag{2.5.39}$$

$$D_x u^2 + D_z w^3 = 0 \tag{2.5.40}$$

$$p^2 = 0 \text{ en } z = 1 \tag{2.5.41}$$

$$w^2 - u^1 H' = 0 \text{ en } z = 0 \tag{2.5.42}$$

$$\int_0^1 \frac{\partial (h u^2)}{\partial x} \, dz = 0 \tag{2.5.43}$$

$$D_z u^3 - D_x w^2 = 0 \tag{2.5.44}$$

Procediendo del mismo modo que al deducir la expresión (2.5.37) obtenemos también w^2 en función de u^1 y H' (en este caso se integra (2.5.29) y se impone la condición (2.5.42)):

$$w^2 = u^1 H' - h \frac{\partial u^1}{\partial x} z \tag{2.5.45}$$

En resumen, tenemos las siguientes ecuaciones, igualdades y condiciones para el cálculo de h, u^k, w^k y p^k $(k = 0, 1, 2, \cdots)$ que nos permitirán construir una aproximación de la solución del problema (2.4.1)-(2.4.9):

$$\frac{\partial u^0}{\partial z} = 0 \tag{2.5.46}$$

$$w^0 = 0 \tag{2.5.47}$$

$$p^0 = p_s(t,x) \tag{2.5.48}$$

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} \tag{2.5.49}$$

$$p^1 = \rho_0 g h(1-z) \tag{2.5.50}$$

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x} z \tag{2.5.51}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} = 0 \tag{2.5.52}$$

$$\frac{\partial u^1}{\partial z} = 0 \tag{2.5.53}$$

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} = -g\frac{\partial s}{\partial x} \tag{2.5.54}$$

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 = 0 \tag{2.5.55}$$

$$w^2 = u^1 H' - h\frac{\partial u^1}{\partial x} z \tag{2.5.56}$$

$$\frac{\partial (hu^1)}{\partial x} = 0 \tag{2.5.57}$$

$$D_z u^2 - D_x w^1 = 0 \tag{2.5.58}$$

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + \frac{1}{\rho_0} D_x p^2 = 0 \tag{2.5.59}$$

$$D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1 + \frac{1}{\rho_0} D_z p^3 = 0 \tag{2.5.60}$$

$$D_x u^2 + D_z w^3 = 0 \tag{2.5.61}$$

$$p^2 = 0 \text{ en } z = 1 \tag{2.5.62}$$

$$\int_0^1 \frac{\partial(hu^2)}{\partial x}\,dz = 0 \tag{2.5.63}$$

$$D_z u^3 - D_x w^2 = 0 \tag{2.5.64}$$

Debe observarse que algunas de las ecuaciones pueden ser incompatibles entre sí. Por ejemplo, si utilizamos (2.5.54) para calcular u^1, no tiene porqué verificarse (2.5.57). Este aparente problema lo resolveremos en las secciones que siguen escogiendo qué ecuaciones debe satisfacer nuestra aproximación y observando qué aproximación se obtiene con dicha elección.

2.6. Aproximación de orden cero

Se considera la aproximación de orden cero en ε:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

donde w^0 y p^0 son conocidos ((2.5.47) y (2.5.48)).

Como ya se dijo anteriormente, u^0 se calcula a partir de (2.5.49)

$$\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}$$

y el calado, resolviendo la ecuación (2.5.52)

$$\frac{\partial h}{\partial t} + \frac{\partial(hu^0)}{\partial x} = 0$$

(suponemos conocidos $u^0(0,x)$, $h(0,x)$ y, por fijar ideas, $u^0(t,0)$ y $h(t,0)$, aunque otras elecciones son posibles).

Una vez conocidos u^0 y h, w^1 viene dado por (2.5.51)

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x}z$$

Si deshacemos el cambio de variable, volviendo al dominio original, la aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,z) = u^0(t,x)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,z) = \varepsilon w^1(t,x,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,z) = p^0(t,x)$$

verifica,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{2.6.1}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{2.6.2}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{2.6.3}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{2.6.4}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} \tag{2.6.5}$$

Nos interesa ahora estudiar el error que cometemos al sustituir la solución exacta de las ecuaciones de Euler por la aproximación que acabamos de construir. Un método para valorar dicho error (con una validez meramente orientativa) es sustituir la solución aproximada en las ecuaciones de Euler y observar con qué orden de ε se verifican.

Si sustituimos la aproximación de orden cero en la primera de las ecuaciones de Euler, obtenemos:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = \frac{\partial \tilde{u}(\varepsilon)}{\partial t} + \tilde{u}(\varepsilon)\frac{\partial \tilde{u}(\varepsilon)}{\partial x} = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}$$

Por tanto, la primera ecuación se verifica exactamente.

Para la segunda ecuación de Euler se tiene que:

$$\begin{aligned}
&\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g \\
&\quad = D_t \tilde{w}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon) + \frac{1}{\rho_0}\frac{1}{\varepsilon} D_z \tilde{p}(\varepsilon) + g \\
&\quad = \varepsilon D_t w^1 + \varepsilon u^0 D_x w^1 + \varepsilon w^1 \frac{1}{\varepsilon}\varepsilon D_z w^1 + g = O(1)
\end{aligned}$$

Es decir, esta ecuación se cumple con un error de orden ε^0, o lo que es lo mismo, no se verifica ni tan siquiera aproximadamente.

La ecuación de la incompresibilidad se cumple de forma exacta, como se deduce de (2.6.2):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = 0$$

Es inmediato comprobar que las condiciones de contorno (2.1.12) y (2.1.13) se verifican exactamente (teniendo en cuenta (2.6.2) y (2.6.3)).

Por último, la ecuación (2.1.16) (que impone que la vorticidad es nula), se verifica con un error de orden $O(\varepsilon)$, ya que

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} - \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} = -D_x \tilde{w}(\varepsilon) = -\varepsilon D_x w^1 = O(\varepsilon).$$

Se puede proponer un modelo de orden 0 en ε (al menos formalmente) que viene dado por las ecuaciones (2.6.1)-(2.6.5):

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \\
&\tilde{p}^\varepsilon = p_s^\varepsilon
\end{aligned} \tag{2.6.6}$$

donde es necesario conocer $\tilde{u}^\varepsilon(0, x^\varepsilon)$, $h^\varepsilon(0, x^\varepsilon)$ y, por ejemplo, $\tilde{u}^\varepsilon(t^\varepsilon, 0)$ y $h^\varepsilon(t^\varepsilon, 0)$.

Este modelo no es especialmente interesante, pero nos permite señalar que, como observaremos en las secciones siguientes, para cada aproximación construida (de distinto orden en ε) se puede proponer un modelo distinto.

2.7. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación de orden 1 en ε:

$$\begin{aligned}
&\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 \\
&\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 \\
&\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1
\end{aligned}$$

Recordemos que w^0 y p^0 son conocidos ((2.5.47) y (2.5.48)), u^0 y h se calculan resolviendo (2.5.49) y (2.5.52), respectivamente, y w^1 está determinado por (2.5.51) en función de u^0 y h.

También tenemos una expresión para p^1 en la que sólo es necesario conocer la profundidad del agua ((2.5.50)):

$$p^1 = \rho_0 g h (1 - z)$$

Para obtener u^1 resolvemos (2.5.54):

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} = -g\frac{\partial s}{\partial x}$$

(son necesarios $u^1(0,x)$ y $u^1(t,0)$).

A continuación, w^2 viene dado por (2.5.56):

$$w^2 = u^1 H' - h\frac{\partial u^1}{\partial x} z$$

Ahora, usando (2.5.48) y (2.5.50) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon \rho_0 g h (1 - z) \tag{2.7.1}$$

De igual modo, por (2.5.47), (2.5.51) y (2.5.56), sabemos que:

$$\begin{aligned} \tilde{w}(\varepsilon) &= \varepsilon w^1 + \varepsilon^2 w^2 = \varepsilon \left(u^0 H' - h\frac{\partial u^0}{\partial x} z \right) + \varepsilon^2 \left(u^1 H' - h\frac{\partial u^1}{\partial x} z \right) \\ &= (u^0 + \varepsilon u^1)\varepsilon H' - \frac{\partial (u^0 + \varepsilon u^1)}{\partial x} \varepsilon h z = \tilde{u}(\varepsilon)\varepsilon H' - \frac{\partial u(\varepsilon)}{\partial x} \varepsilon h z \end{aligned} \tag{2.7.2}$$

Se deshace el cambio de variable y se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\begin{aligned} \tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{u}(\varepsilon)(t,x,z) = u^0(t,x) + \varepsilon u^1(t,x) \\ \tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{w}(\varepsilon)(t,x,z) = \varepsilon w^1(t,x,z) + \varepsilon^2 w^2(t,x,z) \\ \tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{p}(\varepsilon)(t,x,z) = p^0(t,x) + \varepsilon p^1(t,x,z) \end{aligned}$$

que verifica, según lo visto en (2.5.46) y (2.5.53),

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{2.7.3}$$

Además, si se deshace el cambio de variable en (2.7.1), obtenemos la aproximación de la presión en Ω^ε:

$$\tilde{p}^\varepsilon = p_s + \rho_0 g (s^\varepsilon - z^\varepsilon) \tag{2.7.4}$$

Análogamente, deshaciendo el cambio de variable en (2.7.2), se tiene que la aproximación de la componente vertical de la velocidad verifica:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{2.7.5}$$

Teniendo en cuenta (2.5.49), (2.5.52) y (2.5.54), obtenemos que las ecuaciones que verifican $\tilde{u}^\varepsilon$ y h^ε y que permiten su cálculo son las siguientes:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x}\right)$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \varepsilon \left(\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x}\right) + \varepsilon^2 u^1 \frac{\partial u^1}{\partial x}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \varepsilon \frac{\partial s}{\partial x} g + \varepsilon^2 u^1 \frac{\partial u^1}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + O(\varepsilon^2)$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = \varepsilon \frac{\partial h}{\partial t} + \frac{\partial(u^0 + \varepsilon u^1)}{\partial x}\varepsilon h + (u^0 + \varepsilon u^1)\varepsilon \frac{\partial h}{\partial x}$$

$$= \varepsilon \left[\frac{\partial h}{\partial t} + \frac{\partial u^0}{\partial x} h + u^0 \frac{\partial h}{\partial x} + \varepsilon \left(\frac{\partial u^1}{\partial x} h + u^1 \frac{\partial h}{\partial x}\right)\right] = O(\varepsilon^2)$$

lo que implica que,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2) \tag{2.7.6}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{2.7.7}$$

Observación 2.3 *Si se hubiese empleado la igualdad (2.5.57):*

$$\frac{\partial(h u^1)}{\partial x} = 0$$

la ecuación (2.7.7) resultaría:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0$$

pero la igualdad (2.5.57) no se emplea ya que u^1 no se ha calculado a partir de ella, y resultaría incompatible con (2.5.54) que es la escogida para el cálculo de u^1.

Veamos en qué medida verifica la aproximación de primer orden las ecuaciones de Euler de partida:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} = \frac{\partial \tilde{u}(\varepsilon)}{\partial t} + \tilde{u}(\varepsilon)\frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{1}{\rho_0} D_x \tilde{p}(\varepsilon)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x}\right) + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \varepsilon g \frac{\partial s}{\partial x}$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \varepsilon \left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + g \frac{\partial s}{\partial x}\right] + \varepsilon^2 u^1 \frac{\partial u^1}{\partial x}$$

Usando (2.5.49) y (2.5.54) se puede escribir:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2)$$

Así, la primera ecuación de Euler se verifica con un error de orden ε^2.

Para la segunda ecuación de Euler se tiene, usando (2.5.50), que:

$$\begin{aligned}
&\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g \\
&\quad = D_t \tilde{w}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{w}(\varepsilon) + \tilde{w}(\varepsilon) \frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon) + \frac{1}{\rho_0} \frac{1}{\varepsilon} D_z \tilde{p}(\varepsilon) + g \\
&\quad = \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + (u^0 + \varepsilon u^1) \left[\varepsilon D_x w^1 + \varepsilon^2 D_x w^2 \right] \\
&\quad + (\varepsilon w^1 + \varepsilon^2 w^2) \frac{1}{\varepsilon} \left[\varepsilon D_z w^1 + \varepsilon^2 D_z w^2 \right] + \frac{1}{\rho_0} \frac{1}{\varepsilon} \varepsilon D_z p^1 + g = O(\varepsilon)
\end{aligned}$$

La aproximación de primer orden verifica la segunda ecuación de Euler con un error $O(\varepsilon)$.

La ecuación de la incompresibilidad se verifica de forma exacta como se ve utilizando (2.5.24) y (2.5.29)

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = \frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon) = \frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + D_z w^1 + \varepsilon D_z w^2 = 0,$$

o directamente por (2.7.5).

Lo mismo sucede con las condiciones de contorno ((2.1.12) y (2.1.13)), teniendo en cuenta (2.7.4) y (2.7.5).

Si ahora comprobamos la ecuación (2.1.16), obtenemos

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} - \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} = -D_x \tilde{w}(\varepsilon) = -\varepsilon D_x w^1 - \varepsilon^2 D_x w^2 = O(\varepsilon)$$

es decir, la condición de vorticidad nula se verifica con un error del orden de ε.

Si en (2.7.6) y (2.7.7) se desprecian los términos de orden $O(\varepsilon^2)$ se obtiene el siguiente modelo de aguas someras cuyo orden de precisión, al menos formalmente, es $O(\varepsilon)$:

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0} g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} \\
&\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon) \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \qquad (2.7.8)
\end{aligned}$$

donde $\tilde{u}^\varepsilon$ no depende de z^ε y se suponen conocidos $\tilde{u}^\varepsilon(0,x^\varepsilon)$, $h^\varepsilon(0,x^\varepsilon)$ y, por ejemplo, $\tilde{u}^\varepsilon(t^\varepsilon,0)$ y $h^\varepsilon(t^\varepsilon,0)$.

Si se compara el modelo clásico de "shallow waters", (1.2.24) (por ejemplo [56]) con éste se puede apreciar que son el mismo. En nuestro caso se obtiene, además, una velocidad horizontal (w^ε) no nula.

2.8. Aproximación de segundo orden

Se considera la aproximación de segundo orden en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

Los términos w^0, p^0, u^0, h, w^1, p^1, u^1 y w^2 se calculan del mismo modo que en la sección anterior para la aproximación de primer orden a partir de (2.5.46)- (2.5.54) y (2.5.56).

Buscamos, ahora, p^2, para ello partimos de la ecuación (2.5.55):

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 = -\frac{1}{\rho_0} D_z p^2$$

Utilizamos la expresión de w^1 ((2.5.51)) donde la dependencia de z es explícita y reescribimos (2.5.55)

$$-\frac{1}{\rho_0 h}\frac{\partial p^2}{\partial z} = \frac{\partial u^0}{\partial t}H' - \left(\frac{\partial h}{\partial t}\frac{\partial u^0}{\partial x} + h\frac{\partial^2 u^0}{\partial t \partial x}\right) z - \frac{z}{h}\frac{\partial h}{\partial t}\left(-h\frac{\partial u^0}{\partial x}\right)$$

$$+ u^0 \left[\frac{\partial u^0}{\partial x}H' + u^0 H'' - \left(\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} + h\frac{\partial^2 u^0}{\partial x^2}\right) z - \frac{H' + z\frac{\partial h}{\partial x}}{h}\left(-h\frac{\partial u^0}{\partial x}\right)\right]$$

$$+ \left(u^0 H' - h\frac{\partial u^0}{\partial x} z\right)\frac{1}{h}\left(-h\frac{\partial u^0}{\partial x}\right)$$

$$= \frac{\partial u^0}{\partial t}H' + \left(u^0\right)^2 H'' + u^0\frac{\partial u^0}{\partial x}H' + hz\left[\left(\frac{\partial u^0}{\partial x}\right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0\frac{\partial^2 u^0}{\partial x^2}\right]$$

Integrando respecto a z e imponiendo la condición (2.5.62) ($p^2 = 0$ en $z = 1$) se obtiene:

$$p^2 = \rho_0 h(1-z)\left[\frac{\partial u^0}{\partial t}H' + \left(u^0\right)^2 H'' + u^0\frac{\partial u^0}{\partial x}H'\right]$$

$$+\frac{\rho_0}{2}h^2(1-z^2)\left[\left(\frac{\partial u^0}{\partial x}\right)^2 - \frac{\partial^2 u^0}{\partial t\partial x} - u^0\frac{\partial^2 u^0}{\partial x^2}\right] \tag{2.8.1}$$

A continuación calculamos u^2 a partir de (2.5.58):

$$D_z u^2 - D_x w^1 = 0$$

En primer lugar sustituimos w^1 por su expresión (2.5.51), y la ecuación anterior resulta:

$$\frac{1}{h}\frac{\partial u^2}{\partial z} = \frac{\partial u^0}{\partial x}H' + u^0 H'' - \left(\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} + h\frac{\partial^2 u^0}{\partial x^2}\right)z - \frac{H' + z\frac{\partial h}{\partial x}}{h}\left(-h\frac{\partial u^0}{\partial x}\right)$$

$$= 2\frac{\partial u^0}{\partial x}H' + u^0 H'' - h\frac{\partial^2 u^0}{\partial x^2}z$$

Integrando respecto a z se obtiene la siguiente expresión para u^2:

$$u^2 = u_0^2 + zh\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) - \frac{1}{2}z^2 h^2\frac{\partial^2 u^0}{\partial x^2} \tag{2.8.2}$$

donde $u_0^2(t,x) = u^2(t,x,0)$ está determinado por (2.5.59)

$$D_t u^2 + u^0 D_x u^2 + u^1\frac{\partial u^1}{\partial x} + u^2\frac{\partial u^0}{\partial x} + w^1 D_z u^2 = -\frac{1}{\rho_0}D_x p^2$$

Si se sustituye en esta ecuación u^2, w^1 y p^2 por las expresiones (2.8.2), (2.5.51) y (2.8.1), respectivamente, resulta:

$$\frac{\partial u_0^2}{\partial t} + z\left[\frac{\partial h}{\partial t}\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) + h\left(2\frac{\partial^2 u^0}{\partial x\partial t}H' + \frac{\partial u^0}{\partial t}H''\right)\right]$$

$$-z^2\left(h\frac{\partial h}{\partial t}\frac{\partial^2 u^0}{\partial x^2} + \frac{1}{2}h^2\frac{\partial^3 u^0}{\partial x^2\partial t}\right) - z\frac{\partial h}{\partial t}\left(2\frac{\partial u^0}{\partial x}H' + u^0 H'' - zh\frac{\partial^2 u^0}{\partial x^2}\right)$$

$$+u^0\left\{\frac{\partial u_0^2}{\partial x} + z\left[\frac{\partial h}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) + h\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right)\right]\right.$$

$$\left.-z^2\left(h\frac{\partial h}{\partial x}\frac{\partial^2 u^0}{\partial x^2} + \frac{1}{2}h^2\frac{\partial^3 u^0}{\partial x^3}\right) - \left(H' + z\frac{\partial h}{\partial x}\right)\left(2\frac{\partial u^0}{\partial x}H' + u^0 H'' - zh\frac{\partial^2 u^0}{\partial x^2}\right)\right\}$$

$$+ u^1 \frac{\partial u^1}{\partial x} + \left[u_0^2 + zh \left(2\frac{\partial u^0}{\partial x} H' + u^0 H'' \right) - \frac{1}{2} z^2 h^2 \frac{\partial^2 u^0}{\partial x^2} \right] \frac{\partial u^0}{\partial x}$$

$$+ \left(u^0 H' - hz \frac{\partial u^0}{\partial x} \right) \left(2\frac{\partial u^0}{\partial x} H' + u^0 H'' - zh \frac{\partial^2 u^0}{\partial x^2} \right)$$

$$= - \left\{ (1-z) \frac{\partial}{\partial x} \left[h \left(\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right) \right] \right.$$

$$+ \frac{1}{2}(1-z^2) \frac{\partial}{\partial x} \left[h^2 \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right]$$

$$+ \left(H' + z \frac{\partial h}{\partial x} \right) \left[\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right.$$

$$\left. \left. + hz \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right] \right\}$$

que se simplifica de modo que todos los términos en z y z^2 desaparecen y la ecuación para el cálculo de u_0^2 que obtenemos es:

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} = - \left\{ \frac{\partial}{\partial x} \left[h \left(\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right) \right] \right.$$

$$+ \frac{1}{2} \frac{\partial}{\partial x} \left[h^2 \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right]$$

$$\left. + H' \left[\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right] \right\} = - \frac{1}{\rho_0} \left. D_x p^2 \right|_{z=0} \tag{2.8.3}$$

Una vez calculado u^2, w^3 se calcula integrando respecto a z (2.5.61)

$$D_x u^2 + D_z w^3 = 0$$

para ello, en primer lugar se sustituye la expresión obtenida para u^2 en (2.8.2):

$$\frac{\partial u_0^2}{\partial x} + z \frac{\partial h}{\partial x} \left(2\frac{\partial u^0}{\partial x} H' + u^0 H'' \right) + zh \frac{\partial}{\partial x} \left(2\frac{\partial u^0}{\partial x} H' + u^0 H'' \right) - z^2 h \frac{\partial h}{\partial x} \frac{\partial^2 u^0}{\partial x^2} -$$

$$- \frac{1}{2} z^2 h^2 \frac{\partial^3 u^0}{\partial x^3} - \left(H' + z \frac{\partial h}{\partial x} \right) \left(2\frac{\partial u^0}{\partial x} H' + u^0 H'' - zh \frac{\partial^2 u^0}{\partial x^2} \right) + \frac{1}{h} \frac{\partial w^3}{\partial z} = 0$$

A continuación simplificamos, integramos respecto a z e imponemos la condición (derivada de (2.5.6)), $w^3 = u_0^2 H'$ en $z = 0$, para llegar a la expresión siguiente de

w^3 en función de u_0^2 y u^0

$$w^3 = u_0^2 H' + zh\left[H'\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) - \frac{\partial u_0^2}{\partial x}\right] -$$

$$- \frac{1}{2}z^2h^2\left[\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) + H'\frac{\partial^2 u^0}{\partial x^2}\right] + \frac{1}{6}z^3h^3\frac{\partial^3 u^0}{\partial x^3} \quad (2.8.4)$$

Usando las expresiones encontradas para p^0, p^1 y p^2, (2.5.48), (2.5.50) y (2.8.1) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia

$$\begin{aligned}\tilde{p}(\varepsilon) &= p^0 + \varepsilon p^1 + \varepsilon^2 p^2 \\ &= p_s + \varepsilon\rho_0 h(1-z)\left[g + \varepsilon\left(\frac{\partial u^0}{\partial t}H' + \left(u^0\right)^2 H'' + u^0\frac{\partial u^0}{\partial x}H'\right)\right] \\ &+ \varepsilon^2\frac{\rho_0 h^2}{2}(1-z^2)\left[\left(\frac{\partial u^0}{\partial x}\right)^2 - \frac{\partial^2 u^0}{\partial t\partial x} - u^0\frac{\partial^2 u^0}{\partial x^2}\right] \qquad (2.8.5)\end{aligned}$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0,1,2,3)$ ((2.5.47), (2.5.51), (2.5.56) y (2.8.4)) obtenemos una aproximación de la velocidad vertical en Ω

$$\begin{aligned}\tilde{w}(\varepsilon) &= \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 \\ &= \varepsilon\left(u^0 H' - h\frac{\partial u^0}{\partial x}z\right) + \varepsilon^2\left(u^1 H' - h\frac{\partial u^1}{\partial x}z\right) \\ &+ \varepsilon^3\left\{u_0^2 H' + zh\left[H'\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) - \frac{\partial u_0^2}{\partial x}\right]\right. \\ &\left. - \frac{1}{2}z^2h^2\left[\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0H''\right) + H'\frac{\partial^2 u^0}{\partial x^2}\right] + \frac{1}{6}z^3h^3\frac{\partial^3 u^0}{\partial x^3}\right\} \\ &= \left\{u^0 + \varepsilon u^1 + \varepsilon^2\left[u_0^2 + zh\left(2\frac{\partial u^0}{\partial x}H' + u^0H''\right) - \frac{1}{2}z^2h^2\frac{\partial^2 u^0}{\partial x^2}\right]\right\}\varepsilon H' \\ &- \varepsilon zh\left\{\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\left[\frac{\partial u_0^2}{\partial x} + z\frac{\partial h}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0H''\right)\right.\right. \\ &+ zh\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x}H' + u^0H''\right) - z^2h\frac{\partial h}{\partial x}\frac{\partial^2 u^0}{\partial x^2} - \frac{1}{2}z^2h^2\frac{\partial^3 u^0}{\partial x^3} \\ &\left.\left. - \left(H' + z\frac{\partial h}{\partial x}\right)\left(2\frac{\partial u^0}{\partial x}H' + u^0H'' - zh\frac{\partial^2 u^0}{\partial x^2}\right)\right]\right\}\end{aligned}$$

$$+\varepsilon^3\left[\frac{1}{2}z^2h^2\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x}H'+u^0H''\right)-\frac{1}{3}z^3h^3\frac{\partial^3 u^0}{\partial x^3}\right.$$
$$\left.-\,zhH'\left(2\frac{\partial u^0}{\partial x}H'+u^0H''-zh\frac{\partial^2 u^0}{\partial x^2}\right)\right]=$$
$$=\left(u^0+\varepsilon u^1+\varepsilon^2u^2\right)\varepsilon H'-\varepsilon zh\left[\frac{\partial u^0}{\partial x}+\varepsilon\frac{\partial u^1}{\partial x}+\varepsilon^2 D_x u^2\right]+O(\varepsilon^3)$$

es decir

$$\tilde{w}(\varepsilon)=\tilde{u}(\varepsilon)\varepsilon H'-\varepsilon zhD_x\tilde{u}(\varepsilon)+O(\varepsilon^3) \tag{2.8.6}$$

Si ahora se deshace el cambio de variable, volviendo al dominio original, obtenemos la siguiente aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon)=\tilde{u}(\varepsilon)(t,x,z)=u^0(t,x)+\varepsilon u^1(t,x)+\varepsilon^2u^2(t,x,z) \tag{2.8.7}$$

$$\tilde{w}^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon)=\tilde{w}(\varepsilon)(t,x,z)=\varepsilon w^1(t,x,z)+\varepsilon^2w^2(t,x,z)+\varepsilon^3w^3(t,x,z) \tag{2.8.8}$$

$$\tilde{p}^\varepsilon(t^\varepsilon,x^\varepsilon,z^\varepsilon)=\tilde{p}(\varepsilon)(t,x,z)=p^0(t,x)+\varepsilon p^1(t,x,z)+\varepsilon^2p^2(t,x,z) \tag{2.8.9}$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (2.8.5):

$$\tilde{p}^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)\left[g+\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\left(u^{0,\varepsilon}\right)^2\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}+u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right]$$
$$+\frac{\rho_0}{2}\left[(h^\varepsilon)^2-(z^\varepsilon-H^\varepsilon)^2\right]\left[\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2-\frac{\partial^2 u^{0,\varepsilon}}{\partial t^\varepsilon\partial x^\varepsilon}-u^0\frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2}\right] \tag{2.8.10}$$

donde $u^{0,\varepsilon}$ representa a u^0 tras el cambio de variable (2.3.3).

Y de forma análoga, realizando el cambio de variable de Ω a Ω^ε en (2.8.6), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon=\tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+(H^\varepsilon-z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+O(\varepsilon^3) \tag{2.8.11}$$

Podemos ahora proponer varios modelos (asintóticamente equivalentes) a partir de lo que hemos visto en esta sección. En primer lugar, se tiene el modelo que resulta de forma natural de aplicar el método de desarrollos asintóticos y que consiste en calcular u^0, h, u^1 y u_0^2 (en este orden pues, u^0 es necesario para el cálculo de los restantes, h para el cálculo de u^1 y u_0^2 y finalmente u^1 para el cálculo de u_0^2)

resolviendo las ecuaciones (2.5.49), (2.5.52), (2.5.54) y (2.8.3):

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} = 0$$

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} = -g\frac{\partial s}{\partial x}$$

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} = -\left\{ \frac{\partial}{\partial x}\left[h \left(\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right) \right] \right.$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\left[h^2 \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right]$$

$$\left. + H' \left[\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right] \right\}$$

A continuación se construye u^2 según la expresión (2.8.2):

$$u^2 = u_0^2 + zh \left(2 \frac{\partial u^0}{\partial x} H' + u^0 H'' \right) - \frac{1}{2} z^2 h^2 \frac{\partial^2 u^0}{\partial x^2}$$

y finalmente la aproximación de la velocidad horizontal en el dominio original resulta:

$$\tilde{u}^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

Mientras que la profundidad del agua se obtiene deshaciendo el cambio de variable como $h^\varepsilon = \varepsilon h$.

Este modelo tiene el grave inconveniente de que, al calcular por separado u^0, u^1 y u_0^2, supone el triple de esfuerzo de cálculo que el modelo (2.7.8). Sería conveniente poder obtener u^0, u^1 y u_0^2 resolviendo una única ecuación. Para ello denotamos por

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \hat{u}(\varepsilon)(t, x) = u^0(t, x) + \varepsilon u^1(t, x) + \varepsilon^2 u_0^2(t, x)$$

y observamos que:

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = \frac{\partial \hat{u}(\varepsilon)}{\partial t} + \hat{u}(\varepsilon) \frac{\partial \hat{u}(\varepsilon)}{\partial x} = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 \frac{\partial u_0^2}{\partial t}$$

$$+ \left(u^0 + \varepsilon u^1 + \varepsilon_0^2 u^2\right) \left[\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \frac{\partial u_0^2}{\partial x} \right]$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \varepsilon \left(\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} \right)$$

$$+ \varepsilon^2 \left(\frac{\partial u_0^2}{\partial t} + u_0^2 \frac{\partial u^0}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u^0 \frac{\partial u_0^2}{\partial x} \right) + O(\varepsilon^3)$$

y teniendo en cuenta (2.5.49), (2.5.54) y (2.8.3) resulta:

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \varepsilon g \frac{\partial s}{\partial x} - \varepsilon^2 \frac{1}{\rho_0} D_x p^2\big|_{z=0} + O(\varepsilon^3)$$

$$= -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + O(\varepsilon^3) \tag{2.8.12}$$

Del mismo modo, tenemos que:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\hat{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = \varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial(\hat{u}(\varepsilon) h)}{\partial x}$$

$$= \varepsilon \frac{\partial h}{\partial t} + \varepsilon \left[\frac{\partial(u^0 h)}{\partial x} + \varepsilon \frac{\partial(u^1 h)}{\partial x} + \varepsilon^2 \frac{\partial(u_0^2 h)}{\partial x}\right] = \varepsilon\left(\frac{\partial h}{\partial t} + \frac{\partial(u^0 h)}{\partial x}\right) + O(\varepsilon^2)$$

y por (2.5.52):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\hat{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{2.8.13}$$

La presión también se puede dar en términos de $\hat{u}^\varepsilon$ en lugar de $u^{0\varepsilon}$:

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (\hat{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \hat{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right]$$

$$+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 \hat{u}^\varepsilon}{\partial t^\varepsilon \partial x^\varepsilon} - \hat{u}^\varepsilon \frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right] + O(\varepsilon^3) \tag{2.8.14}$$

Una vez conocido $\hat{u}^\varepsilon$ veamos cómo se calcula $\tilde{u}^\varepsilon$ utilizando (2.8.7) y (2.8.2):

$$\tilde{u}^\varepsilon = \tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$= u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2 + \varepsilon^2\left[zh\left(2\frac{\partial u^0}{\partial x}H' + u^0 H''\right) - \frac{1}{2}z^2 h^2 \frac{\partial^2 u^0}{\partial x^2}\right]$$

$$= \hat{u}(\varepsilon) + \varepsilon^2\left[zh\left(2\frac{\partial \hat{u}(\varepsilon)}{\partial x}H' + \hat{u}(\varepsilon)H''\right) - \frac{1}{2}z^2 h^2 \frac{\partial^2 \hat{u}(\varepsilon)}{\partial x^2}\right] + O(\varepsilon^3)$$

$$= \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left(2\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \hat{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right)$$

$$- \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} + O(\varepsilon^3) \tag{2.8.15}$$

A continuación proponemos un modelo de aguas someras resultado de despreciar los términos en $O(\varepsilon^2)$ en la ecuación (2.8.13) y los términos de orden ε^3 de (2.8.11), (2.8.12), (2.8.14) y (2.8.15), obteniéndose:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\hat{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{2.8.16}$$

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \tag{2.8.17}$$

$$\begin{aligned}\tilde{p}^\varepsilon = \tilde{p}^\varepsilon_s + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (\hat{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \hat{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right] \\ + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 \hat{u}^\varepsilon}{\partial t^\varepsilon \partial x^\varepsilon} - \hat{u}^\varepsilon \frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right]\end{aligned} \tag{2.8.18}$$

$$\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left(2\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \hat{u}(\varepsilon)\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right) - \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} \tag{2.8.19}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{2.8.20}$$

donde es necesario conocer $\hat{u}^\varepsilon(0, x^\varepsilon)$, $h^\varepsilon(0, x^\varepsilon)$ y, por ejemplo, $\hat{u}^\varepsilon(t^\varepsilon, 0)$ y $h^\varepsilon(t^\varepsilon, 0)$.

En el modelo que acabamos de proponer las ecuaciones (2.8.16)-(2.8.18) están acopladas, lo que dificulta su resolución. Si utilizamos (2.8.10) para calcular $\tilde{p}^\varepsilon$ evitamos ese problema, pero debemos calcular previamente u^0 y h^ε, por lo que el modelo sería:

$$\begin{aligned}&\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x^\varepsilon} \\ &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^{0,\varepsilon})}{\partial x^\varepsilon} = 0 \\ &\tilde{p}^\varepsilon = p^\varepsilon_s + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \left(u^{0,\varepsilon}\right)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right] \\ &\qquad + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 u^{0,\varepsilon}}{\partial t^\varepsilon \partial x^\varepsilon} - u^0 \frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2}\right] \\ &\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \\ &\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left(2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right) - \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} \\ &\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\end{aligned} \tag{2.8.21}$$

Los modelos (2.8.16)-(2.8.20) y (2.8.21) son más precisos (al menos formalmente) que el modelo (2.7.8), pues los términos despreciados son del orden de $O(\varepsilon^3)$ en lugar de $O(\varepsilon^2)$, salvo en el caso de la ecuación para el cálculo de h^ε, que sigue siendo de orden $O(\varepsilon^2)$. Pero, como ya mencionamos más arriba, el sistema (2.8.16)-(2.8.20) está fuertemente acoplado, lo que dificulta su resolución enormemente, y el modelo (2.8.21) dobla en esfuerzo de cálculo al modelo (2.7.8).

Nuestra opinión es que la supuesta mejora en el orden de precisión que introducen estos modelos no justifica la complejidad que presenta su resolución. En todo caso, podría utilizarse la expresión de $\tilde{p}^\varepsilon$ escrita en términos de $\tilde{u}^\varepsilon$ como una mejora de la obtenida en la sección anterior (véase (2.7.4)).

Por último, veamos en qué medida verifica la aproximación de segundo orden las ecuaciones de Euler. Comencemos por la primera ecuación de Euler:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = D_t \tilde{u}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{u}(\varepsilon) + \tilde{w}(\varepsilon) \frac{1}{\varepsilon} D_z \tilde{u}(\varepsilon)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 D_t u^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2) \left[\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2 \right]$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3) \varepsilon^2 \frac{1}{\varepsilon} D_z u^2$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \varepsilon \left(\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} \right)$$

$$+ \varepsilon^2 \left(D_t u^2 + u^2 \frac{\partial u^0}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u^0 D_x u^2 + w^1 D_z u^2 \right) + O(\varepsilon^3)$$

Usando (2.5.49), (2.5.54) y (2.5.59) resulta:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \varepsilon \frac{\partial s}{\partial x} g - \varepsilon^2 \frac{1}{\rho_0} D_x p^2 + O(\varepsilon^3) = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^3)$$

Es decir, la primera ecuación de Euler se verifica con un error $O(\varepsilon^3)$.

Para la segunda ecuación de Euler se tiene que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = D_t \tilde{w}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{w}(\varepsilon) + \tilde{w}(\varepsilon) \frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon)$$

$$= \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + \varepsilon^3 D_t w^3$$

$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2) \left[\varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \varepsilon^3 D_x w^3 \right]$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3) \left[D_z w^1 + \varepsilon D_z w^2 + \varepsilon^2 D_z w^3 \right]$$

$$= \varepsilon \left(D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 \right)$$

$$+\,\varepsilon^2\left(D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1\right)$$
$$+\,\varepsilon^3\left(D_t w^3 + u^0 D_x w^3 + u^1 D_x w^2 + u^2 D_x w^1 + w^1 D_z w^3 + w^2 D_z w^2 + w^3 D_z w^1\right)$$
$$+\,\varepsilon^4\left(u^1 D_x w^3 + u^2 D_x w^2 + w^2 D_z w^3 + w^3 D_z w^2\right) + \varepsilon^5\left(u^2 D_x w^3 + w^3 D_z w^3\right)$$

Usando (2.5.55) se puede escribir:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = -\varepsilon \frac{1}{\rho_0} D_z p^2 + O(\varepsilon^2) \tag{2.8.22}$$

Como

$$D_z p(\varepsilon) = \varepsilon D_z p^1 + \varepsilon^2 D_z p^2 = -\varepsilon \rho_0 g + \varepsilon^2 D_z p^2$$

podemos reescribir la igualdad (2.8.22) del modo siguiente:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{1}{\varepsilon} D_z p(\varepsilon) - g + O(\varepsilon^2) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} - g + O(\varepsilon^2)$$

La aproximación de segundo orden verifica la segunda ecuación de Euler con un error del orden de ε^2.

Para la ecuación de la incompresibilidad, se tiene por la ecuación (2.8.11):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = O(\varepsilon^3)$$

Por construcción de $\tilde{p}^\varepsilon$ y $\tilde{w}^\varepsilon$ (véase (2.8.18) y (2.8.20)) las condiciones de contorno (2.1.12) y (2.1.13) se verifican exactamente.

Veamos ahora lo que sucede con la vorticidad:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} - \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} = \frac{1}{\varepsilon} D_z \tilde{u}(\varepsilon) - D_x \tilde{w}(\varepsilon) = \varepsilon D_z u^2 - \varepsilon D_x w^1 - \varepsilon^2 D_x w^2 - \varepsilon^3 D_x w^3$$

Empleando la igualdad (2.5.58) la ecuación de la vorticidad resulta:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} - \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} = O(\varepsilon^2)$$

es decir, la vorticidad se anula salvo un error de $O(\varepsilon^2)$.

2.9. Modelo propuesto

Una vez analizados las ventajas e inconvenientes de los modelos (2.8.16)-(2.8.20) y (2.8.21), parece justificado que el modelo propuesto sea (2.7.8) (suprimiendo ˜

para simplificar la notación):

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon}=0\\
&\frac{\partial u^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-\frac{1}{\rho_0}g\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\\
&p^\varepsilon=p_s+\rho_0 g(s^\varepsilon-z^\varepsilon)\\
&w^\varepsilon=u^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+(H^\varepsilon-z^\varepsilon)\frac{\partial u^\varepsilon}{\partial x^\varepsilon}
\end{aligned}\tag{2.9.1}$$

Podríamos utilizar en lugar de la expresión (2.7.4) para la presión, la mejora obtenida en la aproximación de orden dos escrita en función de $\tilde{u}^\varepsilon$ en lugar de $u^{0,\varepsilon}$ o $\hat{u}^\varepsilon$ pues no es necesario conocer la velocidad de segundo orden para ello:

$$\begin{aligned}
p^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)&\left[g+\frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+(\tilde{u}^\varepsilon)^2\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}+\tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}\right]\\
+\frac{\rho_0}{2}\left[(h^\varepsilon)^2-(z^\varepsilon-H^\varepsilon)^2\right]&\left[\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)^2-\frac{\partial^2\tilde{u}^\varepsilon}{\partial t^\varepsilon\partial x^\varepsilon}-\tilde{u}^\varepsilon\frac{\partial^2\tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right]
\end{aligned}\tag{2.9.2}$$

2.10. Caso en el que la vorticidad inicial es no nula

Partimos ahora de las mismas ecuaciones (2.1.9)-(2.1.15) pero en lugar de considerar que la vorticidad es nula en el instante inicial, lo que nos asegura que la vorticidad se anula en todo instante, incluimos entre las ecuaciones de partida la siguiente que verifica la vorticidad (véase [102]):

$$\frac{\partial\gamma^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial\gamma^\varepsilon}{\partial x^\varepsilon}+w^\varepsilon\frac{\partial\gamma^\varepsilon}{\partial z^\varepsilon}=0\tag{2.10.1}$$

donde

$$\gamma^\varepsilon=\frac{\partial u^\varepsilon}{\partial z^\varepsilon}-\frac{\partial w^\varepsilon}{\partial x^\varepsilon}\tag{2.10.2}$$

La ecuación de la vorticidad, tras el cambio de variable descrito en (2.3.3), con la notación (2.3.4), se escribe:

$$D_t\gamma(\varepsilon)+u(\varepsilon)D_x\gamma(\varepsilon)+w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma(\varepsilon)=0\tag{2.10.3}$$

y la vorticidad en función de las componentes de la velocidad resulta:

$$\gamma(\varepsilon)=\frac{1}{\varepsilon}D_z u(\varepsilon)-D_x w(\varepsilon)\tag{2.10.4}$$

Se supone que $\gamma(\varepsilon)$ admite un desarrollo en serie de potencias de ε del mismo modo que se ha supuesto para $u(\varepsilon)$, $w(\varepsilon)$ y $p(\varepsilon)$:

$$\gamma(\varepsilon) = \varepsilon^{-1}\gamma^{-1} + \gamma^0 + \varepsilon\gamma^1 + \varepsilon^2\gamma^2 + \cdots \tag{2.10.5}$$

En (2.10.5) suponemos que el desarrollo en serie de potencias de $\gamma(\varepsilon)$ comienza en el término de orden -1 en ε. Esta hipótesis resulta ser la natural si sustituimos (2.5.1) en (2.10.4) (véase (2.10.10)).

Sustituyendo los desarrollos en serie de potencias (2.5.1) y (2.10.5) en la ecuación (2.10.3), obtenemos:

$$\begin{aligned}&\varepsilon^{-1}D_t\gamma^{-1} + D_t\gamma^0 + \varepsilon D_t\gamma^1 + \varepsilon^2 D_t\gamma^2 + \cdots\\ &\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\left[\varepsilon^{-1}D_x\gamma^{-1} + D_x\gamma^0 + \varepsilon D_x\gamma^1 + \varepsilon^2 D_x\gamma^2 + \cdots\right]\\ &\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \cdots)\frac{1}{\varepsilon}\left[\varepsilon^{-1}D_z\gamma^{-1} + D_z\gamma^0 + \varepsilon D_z\gamma^1 + \varepsilon^2 D_z\gamma^2 + \cdots\right] = 0\end{aligned}$$

Agrupando en potencias de ε e igualando a cero los coeficientes (teniendo en cuenta que $w^0 = 0$):

$$D_t\gamma^{-1} + u^0 D_x\gamma^{-1} + w^1 D_x\gamma^{-1} = 0 \tag{2.10.6}$$

$$D_t\gamma^0 + u^0 D_x\gamma^0 + u^1 D_x\gamma^{-1} + w^1 D_z\gamma^0 + w^2 D_z\gamma^{-1} = 0 \tag{2.10.7}$$

$$\begin{aligned}&D_t\gamma^1 + u^0 D_x\gamma^1 + u^1 D_x\gamma^0 + u^2 D_x\gamma^{-1} + w^1 D_z\gamma^1 + w^2 D_z\gamma^0\\ &\quad + w^3 D_z\gamma^{-1} = 0\end{aligned} \tag{2.10.8}$$

$$\begin{aligned}&D_t\gamma^2 + u^0 D_x\gamma^2 + u^1 D_x\gamma^1 + u^2 D_x\gamma^0 + u^3 D_x\gamma^{-1} + w^1 D_z\gamma^2\\ &\quad + w^2 D_z\gamma^1 + w^3 D_z\gamma^0 + w^4 D_z\gamma^{-1} = 0\end{aligned} \tag{2.10.9}$$

Repetimos el proceso a partir de (2.10.4):

$$\begin{aligned}\varepsilon^{-1}\gamma^{-1} + \gamma^0 + \varepsilon\gamma^1 + \varepsilon^2\gamma^2 + \cdots &= \frac{1}{\varepsilon}\left(D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right)\\ &\quad - \left(\varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \cdots\right)\end{aligned} \tag{2.10.10}$$

Agrupando en potencias de ε:

$$\begin{cases}\gamma^k &= D_z u^{k+1} \qquad\qquad k = -1, 0\\ \gamma^k &= D_z u^{k+1} - D_x w^k \quad k \geq 1\end{cases} \tag{2.10.11}$$

Observación 2.4 *Los términos* $\gamma^k \quad (k = -1, 0, 1, 2)$ *van a anularse o no dependiendo de las condiciones iniciales y de contorno.*

La ecuación (2.10.6), si la condiciones de contorno e iniciales son las adecuadas, tiene por solución $\gamma^{-1} = 0$. *Como, además, no parece razonable que* $\gamma^{-1} \neq 0$ *pues entonces, por (2.10.11.a),* u^0 *depende de* z *y eso contradice el hecho de estemos trabajando en aguas poco profundas, se asume que*

$$\gamma^{-1} = 0 \tag{2.10.12}$$

y dependiendo de que el resto de $\gamma^k \quad (k = 0, 1, 2)$ *se anulen o no se obtendrán diferentes modelos.*

Obsérvese que (2.10.12) *se obtendría directamente si en* (2.10.5) *suponemos que el desarrollo comienza en el término de orden 0 en* ε.

Ahora, teniendo en cuenta esta hipótesis, de la igualdad (2.10.11.a) se deduce que:

$$\frac{\partial u^0}{\partial z} = 0 \tag{2.10.13}$$

igualdad que, ya se había deducido cuando suponíamos nula la vorticidad inicial (véase (2.5.46)), precisamente a partir de esta hipótesis.

Dado que $\gamma^{-1} = 0$ la ecuaciones obtenidas a partir de la ecuación de vorticidad resultan:

$$D_t\gamma^0 + u^0 D_x\gamma^0 + w^1 D_z\gamma^0 = 0 \tag{2.10.14}$$

$$D_t\gamma^1 + u^0 D_x\gamma^1 + u^1 D_x\gamma^0 + w^1 D_z\gamma^1 + w^2 D_z\gamma^0 = 0 \tag{2.10.15}$$

$$\begin{aligned} &D_t\gamma^2 + u^0 D_x\gamma^2 + u^1 D_x\gamma^1 + u^2 D_x\gamma^0 \\ &\qquad + w^1 D_z\gamma^2 + w^2 D_z\gamma^1 + w^3 D_z\gamma^0 = 0 \end{aligned} \tag{2.10.16}$$

En este caso para el cálculo de h, u^k, w^k y p^k $(k = 0, 1, 2, \cdots)$ tenemos las ecuaciones (2.10.14)-(2.10.16) y las igualdades (2.10.11)-(2.10.13) en lugar de (2.5.46), (2.5.53), (2.5.58) y (2.5.64) que se deducían de la condición de vorticidad nula, además de las siguientes ecuaciones, igualdades y condiciones que se siguen verificando:

$$w^0 = 0 \tag{2.10.17}$$

$$p^0 = p_s(t, x) \tag{2.10.18}$$

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} \tag{2.10.19}$$

$$p^1 = \rho_0 g h(1 - z) \tag{2.10.20}$$

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x}z \quad (2.10.21)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} = 0 \quad (2.10.22)$$

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 = 0 \quad (2.10.23)$$

$$D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1 + \frac{1}{\rho_0} D_z p^3 = 0 \quad (2.10.24)$$

$$D_x u^2 + D_z w^3 = 0 \quad (2.10.25)$$

$$p^2 = 0 \text{ en } z = 1 \quad (2.10.26)$$

$$\int_0^1 \frac{\partial (hu^2)}{\partial x}\, dz = 0 \quad (2.10.27)$$

Algunas han variado debido a que $D_z u^1 \neq 0$:

$$D_t u^1 + u^0 D_x u^1 + u^1 \frac{\partial u^0}{\partial x} + w^1 D_z u^1 = -g\frac{\partial s}{\partial x} \quad (2.10.28)$$

$$D_x u^1 + D_z w^2 = 0 \quad (2.10.29)$$

$$\int_0^1 \frac{\partial (hu^1)}{\partial x}\, dz = 0 \quad (2.10.30)$$

$$w^2 = u^1 H' \text{ en } z = 0 \quad (2.10.31)$$

$$D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + w^2 D_z u^1 + \frac{1}{\rho_0} D_x p^2 = 0 \quad (2.10.32)$$

2.10.1. Aproximación de orden cero

En este caso se considera la aproximación de orden cero:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

$$\tilde{\gamma}(\varepsilon) = \varepsilon^{-1}\gamma^{-1} + \gamma^0$$

El modelo que se obtiene es el mismo que en el caso en el que se supuso la vorticidad nula en el instante inicial ((2.6.6)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon$$

(donde es necesario conocer $\tilde{u}^\varepsilon(0, x^\varepsilon)$, $h^\varepsilon(0, x^\varepsilon)$ y, por ejemplo, $\tilde{u}^\varepsilon(t^\varepsilon, 0)$ y $h^\varepsilon(t^\varepsilon, 0)$), al que se añade una ecuación para el cálculo de la vorticidad que se obtiene a partir de (2.10.12) y (2.10.14):

$$\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} = 0$$

2.10.2. Aproximación de primer orden

Se considera ahora la siguiente aproximación:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{\gamma}(\varepsilon) = \varepsilon^{-1}\gamma^{-1} + \gamma^0 + \varepsilon \gamma^1$$

donde w^0, p^0 y γ^{-1} son conocidos ((2.10.17), (2.10.18), (2.10.12)) y u^0, h, w^1 y p^1 se obtienen a partir de las mismas ecuaciones o igualdades que antes ((2.10.19), (2.10.22), (2.10.21), (2.10.20)).

En las secciones anteriores, como consecuencia de suponer que la vorticidad es nula en el instante inicial, obteníamos que u^1 no dependía de z (véase (2.5.53)). En esta sección, sin embargo, no se realiza dicha hipótesis, y tenemos que (véase (2.10.11))

$$\frac{\partial u^1}{\partial z} = h\gamma^0$$

Por tanto, para que u^1 sea independiente de la coordenada vertical debemos suponer que $\gamma^0 = 0$. Estudiaremos primero qué ocurre si se trabaja bajo esta hipótesis y después si $\gamma^0 \neq 0$.

2.10.2.1. Caso I: $\gamma^0 = 0$

Como acabamos de mencionar, bajo esta hipótesis se recupera la independencia de z de u^1 y, repitiendo los pasos de la sección 2.7 volvemos a obtener el modelo (2.7.8), al que ahora podemos añadir una ecuación para el cálculo de la vorticidad:

$$\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} = D_t \tilde{\gamma}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{\gamma}(\varepsilon) + \tilde{w}(\varepsilon) D_z \tilde{\gamma}(\varepsilon)$$

$$= \varepsilon D_t \gamma^1 + (u^0 + \varepsilon u^1)\varepsilon D_x \gamma^1 + (\varepsilon w^1 + \varepsilon^2 w^2) D_z \gamma^1$$

$$= \varepsilon \left(D_t \gamma^1 + u^0 D_x \gamma^1 + w^1 D_z \gamma^1 \right) + O(\varepsilon^2) = O(\varepsilon^2)$$

Si se desprecian los términos en $O(\varepsilon^2)$ de esta ecuación, al igual que se hizo en las ecuaciones (2.7.6) y (2.7.7) para obtener (2.7.8), la ecuación para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo es:

$$\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{2.10.33}$$

2.10.2.2. Caso II: $\gamma^0 \neq 0$

Si ahora suponemos $\gamma^0 \neq 0$, la dependencia de z de u^1 viene dada por

$$\frac{\partial u^1}{\partial z} = h\gamma^0 \tag{2.10.34}$$

Para calcular u^1 a partir de (2.10.34) es necesario conocer la dependencia de z de γ^0. A partir de (2.10.14) (y de (2.10.21)) se deduce que, si las condiciones iniciales y de contorno sobre γ^0 son polinómicas en z, entonces γ^0 es polinómico en z. Teniendo en cuenta que buscamos un modelo de aguas someras, no sería descabellado suponer que las condiciones iniciales y de contorno sobre γ^0 son incluso independientes de z. Deseamos generalizar este caso e incluir la dependencia lineal y también cuadrática, para lo que supondremos:

$$\gamma^0 = \gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2} \tag{2.10.35}$$

Sustituyendo en (2.10.14) tanto γ^0 como w^1 (utilizando (2.10.35) y (2.10.21), respectivamente), de modo que la dependencia de z sea explícita, obtenemos:

$$D_t(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2}) + u^0 D_x(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2})$$

$$+ w^1 D_z(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2})$$

$$= \frac{\partial \gamma^{0,0}}{\partial t} + z\frac{\partial \gamma^{0,1}}{\partial t} + z^2 \frac{\partial \gamma^{0,2}}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}(\gamma^{0,1} + 2z\gamma^{0,2})$$

$$+ u^0 \left(\frac{\partial \gamma^{0,0}}{\partial x} + z\frac{\partial \gamma^{0,1}}{\partial x} + z^2 \frac{\partial \gamma^{0,2}}{\partial x} - \frac{H' + z\dfrac{\partial h}{\partial x}}{h}(\gamma^{0,1} + 2z\gamma^{0,2}) \right)$$

$$+ \frac{1}{h}\left(u^0 H' - h\frac{\partial u^0}{\partial x} z \right)(\gamma^{0,1} + 2z\gamma^{0,2}) = 0$$

Agrupando según las potencias de z:

$$\frac{\partial \gamma^{0,0}}{\partial t} + u^0 \frac{\partial \gamma^{0,0}}{\partial x} = 0 \tag{2.10.36}$$

$$\frac{\partial \gamma^{0,1}}{\partial t} - \frac{1}{h}\frac{\partial h}{\partial t}\gamma^{0,1} + u^0 \left(\frac{\partial \gamma^{0,1}}{\partial x} - \frac{1}{h}\frac{\partial h}{\partial x}\gamma^{0,1} \right) - \frac{\partial u^0}{\partial x}\gamma^{0,1} = 0 \tag{2.10.37}$$

$$\frac{\partial \gamma^{0,2}}{\partial t} - \frac{2}{h}\frac{\partial h}{\partial t}\gamma^{0,2} + u^0 \left(\frac{\partial \gamma^{0,2}}{\partial x} - \frac{2}{h}\frac{\partial h}{\partial x}\gamma^{0,2} \right) - 2\frac{\partial u^0}{\partial x}\gamma^{0,2} = 0 \tag{2.10.38}$$

Las ecuaciones (2.10.37) y (2.10.38) se pueden simplificar teniendo en cuenta (2.10.22), pues

$$-\frac{1}{h}\frac{\partial h}{\partial t}\gamma^{0,1} - u^0 \frac{1}{h}\frac{\partial h}{\partial x}\gamma^{0,1} - \frac{\partial u^0}{\partial x}\gamma^{0,1} = -\frac{1}{h}\gamma^{0,1}\left(\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} \right) = 0$$

$$-\frac{2}{h}\frac{\partial h}{\partial t}\gamma^{0,2} - u^0 \frac{2}{h}\frac{\partial h}{\partial x}\gamma^{0,2} - 2\frac{\partial u^0}{\partial x}\gamma^{0,2} = -\frac{2}{h}\gamma^{0,2}\left(\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} \right) = 0$$

de modo que (2.10.37) y (2.10.38) resultan:

$$\frac{\partial \gamma^{0,1}}{\partial t} + u^0 \frac{\partial \gamma^{0,1}}{\partial x} = 0 \tag{2.10.39}$$

$$\frac{\partial \gamma^{0,2}}{\partial t} + u^0 \frac{\partial \gamma^{0,2}}{\partial x} = 0 \tag{2.10.40}$$

Observación 2.5 *Una vez más, los términos* $\gamma^{0,k}$ $(k = 0, 1, 2)$ *se anularán o no dependiendo de las condiciones iniciales y de contorno.*

Una vez calculados $\gamma^{0,k}$ $(k = 0, 1, 2)$, teniendo en cuenta (2.10.35) se integra (2.10.34) respecto a z:

$$u^1 = u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2} \right) \tag{2.10.41}$$

donde $u_0^1(t,x) = u^1(t,x,0)$ está determinado por (2.10.28):

$$D_t u^1 + u^0 D_x u^1 + u^1 \frac{\partial u^0}{\partial x} + w^1 D_z u^1 = -g\frac{\partial s}{\partial x}$$

Comenzamos por sustituir u^1, según (2.10.41):

$$\begin{aligned}
D_t &\left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right] \\
&+ u^0 D_x \left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right] \\
&+ \left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right]\frac{\partial u^0}{\partial x} \\
&+ \left(u^0 H' - h\frac{\partial u^0}{\partial x}z\right) D_z \left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right] = -g\frac{\partial s}{\partial x}
\end{aligned}$$

Agrupando los coeficientes de cada potencia de z y simplificando (teniendo en cuenta (2.10.22), (2.10.36), (2.10.39) y (2.10.40)), obtenemos:

$$\frac{\partial u_0^1}{\partial t} + u^0\frac{\partial u_0^1}{\partial x} + u_0^1\frac{\partial u^0}{\partial x} = -g\frac{\partial s}{\partial x} \qquad (2.10.42)$$

Observación 2.6 *El término u_0^1, mediante la ecuación (2.10.42), está determinado de forma independiente a γ^0.*

A continuación, se calcula w^2 a partir de (2.10.29):

$$D_z w^2 = -D_x u^1$$

donde se sustituye u^1 por su expresión, vista en (2.10.41):

$$\begin{aligned}
\frac{\partial w^2}{\partial z} &= -h\left\{\frac{\partial u_0^1}{\partial x} + D_x\left[h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right]\right\} \\
&= -h\left\{\frac{\partial u_0^1}{\partial x} + h\frac{\partial}{\partial x}\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right) - \frac{\partial h}{\partial x}\left(\frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right. \\
&\quad \left. - H'\left(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2}\right)\right\}
\end{aligned}$$

y se integra respecto a z imponiendo la condición (2.10.31):

$$\begin{aligned}
w^2 &= u_0^1 H' - hz\left(\frac{\partial u_0^1}{\partial x} - H'\gamma^{0,0}\right) - \frac{1}{2}z^2 h\left(h\frac{\partial \gamma^{0,0}}{\partial x} - H'\gamma^{0,1}\right) \\
&- \frac{1}{3}z^3 h\left(\frac{1}{2}h\frac{\partial \gamma^{0,1}}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\gamma^{0,1} - H'\gamma^{0,2}\right) - \frac{1}{12}z^4 h\left(h\frac{\partial \gamma^{0,2}}{\partial x} - \frac{\partial h}{\partial x}\gamma^{0,2}\right) \qquad (2.10.43)
\end{aligned}$$

Ahora, usando (2.10.18) y (2.10.20) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 gh(1-z) \tag{2.10.44}$$

De igual modo por (2.10.17), (2.10.21) y (2.10.43) sabemos que:

$$\begin{aligned}
\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 = \varepsilon\left(u^0 H' - hz\frac{\partial u^0}{\partial x}\right) + \varepsilon^2\left[u_0^1 H' - hz\left(\frac{\partial u_0^1}{\partial x} - H'\gamma^{0,0}\right)\right.\\
-\frac{1}{2}z^2 h\left(h\frac{\partial \gamma^{0,0}}{\partial x} - H'\gamma^{0,1}\right) - \frac{1}{3}z^3 h\left(\frac{1}{2}h\frac{\partial \gamma^{0,1}}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\gamma^{0,1} - H'\gamma^{0,2}\right)\\
\left.-\frac{1}{12}z^4 h\left(h\frac{\partial \gamma^{0,2}}{\partial x} - \frac{\partial h}{\partial x}\gamma^{0,2}\right)\right]\\
= \left\{u^0 + \varepsilon\left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right]\right\}\varepsilon H'\\
-\varepsilon hz\left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1\right) + O(\varepsilon^2)
\end{aligned} \tag{2.10.45}$$

es decir,

$$\tilde{w}(\varepsilon) = \tilde{u}(\varepsilon)\varepsilon H' - \varepsilon hzD_x\tilde{u}(\varepsilon) + O(\varepsilon^2) \tag{2.10.46}$$

Se deshace, en este momento, el cambio de variable y se obtiene la siguiente aproximación de la solución en el dominio de partida

$$\begin{aligned}
\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{u}(\varepsilon)(t,x,z) = u^0(t,x) + \varepsilon u^1(t,x)\\
\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{w}(\varepsilon)(t,x,z) = \varepsilon w^1(t,x,z) + \varepsilon^2 w^2(t,x,z)\\
\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) &= \tilde{p}(\varepsilon)(t,x,z) = p^0(t,x) + \varepsilon p^1(t,x,z)
\end{aligned}$$

donde la aproximación de la presión en Ω^ε, si se realiza el cambio de variable en (2.10.44), es

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g(s^\varepsilon - z^\varepsilon) \tag{2.10.47}$$

Análogamente, deshaciendo el cambio de variable en (2.10.46) se logra la aproximación de la componente vertical de la velocidad

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2) \tag{2.10.48}$$

Para construir el modelo que vamos a proponer, se define $\breve{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \breve{u}(\varepsilon)(t, x) = u^0(t, x) + \varepsilon u_0^1(t, x)$, obsérvese que:

$$\begin{aligned}
\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} &= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u_0^1}{\partial t} + (u^0 + \varepsilon u_0^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x} \right) \\
&= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \varepsilon \left(\frac{\partial u_0^1}{\partial t} + u_0^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u_0^1}{\partial x} \right) + \varepsilon^2 u_0^1 \frac{\partial u_0^1}{\partial x} \\
&= -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + \varepsilon \left(-\frac{\partial s}{\partial x} g \right) + O(\varepsilon^2) = -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + O(\varepsilon^2) \quad (2.10.49)
\end{aligned}$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = \varepsilon \left[\frac{\partial h}{\partial t} + \frac{\partial (u^0 h)}{\partial x} + \varepsilon \frac{\partial (u_0^1 h)}{\partial x} \right] = O(\varepsilon^2) \quad (2.10.50)$$

Además, $\tilde{u}^\varepsilon$ se puede escribir en términos de $\breve{u}^\varepsilon$:

$$\begin{aligned}
\tilde{u}^\varepsilon &= u^0 + \varepsilon u^1 = u^0 + \varepsilon \left(u_0^1 + h \int_0^z \gamma^0 dz \right) = \breve{u}^\varepsilon + \int_{H^\varepsilon}^{z^\varepsilon} \gamma^{0,\varepsilon} \, dz^\varepsilon \\
&= \breve{u}^\varepsilon + \int_{H^\varepsilon}^{z^\varepsilon} \left(\gamma^{0,0,\varepsilon} + \frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \gamma^{0,1,\varepsilon} + \left(\frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \right)^2 \gamma^{0,2,\varepsilon} \right) dz^\varepsilon \\
&= \breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \gamma^{0,0,\varepsilon} + \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \gamma^{0,1,\varepsilon} + \frac{1}{3} \frac{(z^\varepsilon - H^\varepsilon)^3}{(h^\varepsilon)^2} \gamma^{0,2,\varepsilon} \quad (2.10.51)
\end{aligned}$$

(donde $\gamma^{0,\varepsilon}$, $\gamma^{0,k,\varepsilon}$ $\quad (k = 0, 1, 2)$ representan a γ^0, $\gamma^{0,k}$ $\quad (k = 0, 1, 2)$ tras el cambio de variable (2.3.3)).

Si en las ecuaciones (2.10.49) y (2.10.50) se desprecian los términos de orden $O(\varepsilon^2)$, así como en (2.10.48), se obtiene el siguiente modelo de aguas someras cuyo orden de precisión, al menos formalmente es $O(\varepsilon^2)$:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \quad (2.10.52)$$

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g \quad (2.10.53)$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon) \quad (2.10.54)$$

$$\frac{\partial \gamma^{0,k,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma^{0,k,\varepsilon}}{\partial x^\varepsilon} = 0 \quad k = 0, 1, 2 \quad (2.10.55)$$

$$\tilde{u}^\varepsilon = \breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \gamma^{0,0,\varepsilon} + \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \gamma^{0,1,\varepsilon} + \frac{1}{3} \frac{(z^\varepsilon - H^\varepsilon)^3}{(h^\varepsilon)^2} \gamma^{0,2,\varepsilon} \quad (2.10.56)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}(z^\varepsilon - H^\varepsilon) \tag{2.10.57}$$

Observación 2.7 *Las ecuaciones para el cálculo de* $\gamma^{0,k,\varepsilon}$ $(k = 0, 1, 2)$ *(*(2.10.55)*) se obtienen de* (2.10.36), (2.10.39) *y* (2.10.40) *tras deshacer el cambio de variable y sustituir* u^0 *por* $\breve{u}^\varepsilon$*.*

Buscamos la ecuación adecuada para el cálculo de la vorticidad:

$$\begin{aligned}
\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} &= D_t\tilde{\gamma}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{\gamma}(\varepsilon) + \tilde{w}(\varepsilon)D_z\tilde{\gamma}(\varepsilon) \\
&= D_t\gamma^0 + \varepsilon D_t\gamma^1 + (u^0 + \varepsilon u^1)\left[D_x\gamma^0 + \varepsilon D_x\gamma^1\right] \\
&+ (\varepsilon w^1 + \varepsilon^2 w^2)\frac{1}{\varepsilon}\left[D_z\gamma^0 + \varepsilon D_z\gamma^1\right] \\
&= D_t\gamma^0 + u^0 D_x\gamma^0 + w^1 D_z\gamma^0 \\
&+ \varepsilon\left(D_t\gamma^1 + u^0 D_x\gamma^1 + u^1 D_x\gamma^0 + w^1 D_z\gamma^1 + w^2 D_z\gamma^0\right) + O(\varepsilon^2) = O(\varepsilon^2)
\end{aligned}$$

Si se desprecian los términos en $O(\varepsilon^2)$ de esta ecuación, al igual que se hizo en las ecuaciones (2.10.49) y (2.10.50), la ecuación para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo es la siguiente:

$$\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{2.10.58}$$

Observación 2.8 *De la expresión (2.10.51) deducimos que* $\tilde{u}^\varepsilon$ *también se podría escribir*

$$\tilde{u}^\varepsilon = \breve{u}^\varepsilon + \int_{H^\varepsilon}^{z^\varepsilon} \tilde{\gamma}^\varepsilon \, dz^\varepsilon + O(\varepsilon^2) \tag{2.10.59}$$

y entonces se podría sustituir en el modelo que se acaba de proponer la igualdad (2.10.56) por esta otra pero, en ese caso sería necesario calcular $\tilde{\gamma}^\varepsilon$ *en lugar de* $\gamma^{0,\varepsilon}$ *para obtener* $\tilde{u}^\varepsilon$ *y en la ecuación propuesta para el cálculo de la vorticidad ((2.10.58)) sería necesario conocer* $\tilde{u}^\varepsilon$ *y* $\tilde{w}^\varepsilon$ *por lo que el modelo se acopla y se complica.*

El modelo (2.10.52)-(2.10.58) se puede escribir también en función de la velocidad media en la vertical. Para ello se tiene en cuenta que

$$\begin{aligned}
\bar{u}^\varepsilon &= \frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon} \tilde{u}^\varepsilon \, dz^\varepsilon \\
&= \frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left(\breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\,\gamma^{0,0,\varepsilon} + \frac{1}{2}\frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon}\gamma^{0,1,\varepsilon} + \frac{1}{3}\frac{(z^\varepsilon - H^\varepsilon)^3}{(h^\varepsilon)^2}\gamma^{0,2,\varepsilon}\right) dz^\varepsilon \\
&= \breve{u}^\varepsilon + h^\varepsilon\left(\frac{1}{2}\gamma^{0,0,\varepsilon} + \frac{1}{6}\gamma^{0,1,\varepsilon} + \frac{1}{12}\gamma^{0,2,\varepsilon}\right) \qquad (2.10.60)
\end{aligned}$$

se despeja $\breve{u}^\varepsilon$ y se sustituyen en las ecuaciones (2.10.52) y (2.10.53) de modo que, utilizando las ecuaciones (2.10.55), se obtiene:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{2.10.61}$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + O(\varepsilon^2) \tag{2.10.62}$$

Si en las ecuaciones (2.10.61) y (2.10.62) se desprecian los términos de orden $O(\varepsilon^2)$ como se hizo con (2.10.49) y (2.10.50) para obtener (2.10.52)-(2.10.57), se obtiene el siguiente modelo de aguas someras escrito en términos de la velocidad promediada en la vertical cuyo orden de precisión, al menos formalmente, también es $O(\varepsilon^2)$:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{2.10.63}$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g \tag{2.10.64}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon) \tag{2.10.65}$$

$$\frac{\partial \gamma^{0,k,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma^{0,k,\varepsilon}}{\partial x^\varepsilon} = 0 \quad k = 0, 1, 2 \tag{2.10.66}$$

$$\tilde{u}^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2})\,\gamma^{0,0,\varepsilon} + \left(\frac{(z^\varepsilon - H^\varepsilon)^2}{2h^\varepsilon} - \frac{h^\varepsilon}{6}\right)\gamma^{0,1,\varepsilon}$$
$$+ \left(\frac{(z^\varepsilon - H^\varepsilon)^3}{3(h^\varepsilon)^2} - \frac{h^\varepsilon}{12}\right)\gamma^{0,2,\varepsilon} \tag{2.10.67}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}(z^\varepsilon - H^\varepsilon) \tag{2.10.68}$$

Vamos a escribir, para acabar esta sección, cómo resulta el modelo en el caso particular de que γ^0 no dependa de z, es decir, $\gamma^{0,1} = \gamma^{0,2} = 0$. Bajo estas hipótesis, el modelo anterior resulta:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon)$$

$$\frac{\partial \gamma^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma^{0,\varepsilon}}{\partial x^\varepsilon} = 0$$

$$\tilde{u}^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2})\,\gamma^{0,\varepsilon}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}(z^\varepsilon - H^\varepsilon) \tag{2.10.69}$$

Observación 2.9 *Si en* (2.10.45) *se tiene en cuenta la hipótesis de que* $\gamma^{0,1} = \gamma^{0,2} = 0$*, resulta:*

$$\tilde{w}(\varepsilon) = \left(u^0 + \varepsilon u_0^1\right)\varepsilon H' - \varepsilon h z \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x} - \varepsilon H' \gamma^0\right) - \frac{1}{2}\varepsilon^2 z^2 h^2 \frac{\partial \gamma^0}{\partial x}$$

y realizando el cambio de variable en la expresión anterior tenemos:

$$\tilde{w}^\varepsilon = \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\gamma^{0,\varepsilon}\right) - \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \frac{\partial \gamma^{0,\varepsilon}}{\partial x^\varepsilon}$$

Con esta expresión para la velocidad vertical la condición de incompresibilidad se verifica de forma exacta mientras que la que hemos incluido en el modelo (2.10.69) *(y anteriores), obtenida de* (2.10.48) *despreciando los términos de orden* ε^2*, verifica dicha condición con un error de orden* $O(\varepsilon^2)$.

2.10.3. Aproximación de segundo orden

Se considera la aproximación de segundo orden:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{\gamma}(\varepsilon) = \varepsilon^{-1}\gamma^{-1} + \gamma^0 + \varepsilon\gamma^1 + \varepsilon^2\gamma^2$$

Los términos γ^{-1}, w^0, p^0, u^0, h, w^1, p^1, γ^0, u^1 y w^2 se calculan del mismo modo que en el apartado anterior para la aproximación de primer orden a partir de (2.10.12), (2.10.14), (2.10.17)-(2.10.22) y (2.10.36)-(2.10.42).

El término p^2 del desarrollo de la presión se puede obtener del mismo modo que en la sección 2.8, es decir, viene dado por la expresión (2.8.1)

$$p^2 = \rho_0 h(1-z)\left[\frac{\partial u^0}{\partial t}H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x}H'\right]$$

$$+ \frac{\rho_0}{2}h^2(1-z^2)\left[\left(\frac{\partial u^0}{\partial x}\right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2}\right] \tag{2.10.70}$$

No sucede lo mismo con u^2 pues su cálculo se realizaba a partir de una ecuación obtenida al suponer la vorticidad nula en el instante inicial. Ahora utilizaremos la relación (2.10.11.b) de donde se deduce:

$$D_z u^2 = D_x w^1 + \gamma^1 \tag{2.10.71}$$

Para poder integrar respecto a z la igualdad anterior es necesario conocer de qué modo depende γ^1 de z. Para ello se supone (igual que hicimos en (2.10.35)):

$$\gamma^1 = \gamma^{1,0} + z\gamma^{1,1} + z^2\gamma^{1,2} \tag{2.10.72}$$

De forma análoga a lo realizado en la sección anterior se sustituyen en (2.10.15) γ^0, γ^1, u^1, w^1 y w^2 (utilizando (2.10.35), (2.10.72), (2.10.41), (2.10.21) y (2.10.43) respectivamente) de modo que la dependencia de z sea explícita

$$\begin{aligned}
&D_t\left(\gamma^{1,0} + z\gamma^{1,1} + z^2\gamma^{1,2}\right) + u^0 D_x\left(\gamma^{1,0} + z\gamma^{1,1} + z^2\gamma^{1,2}\right)\\
&+\left[u_0^1 + h\left(z\gamma^{0,0} + \frac{1}{2}z^2\gamma^{0,1} + \frac{1}{3}z^3\gamma^{0,2}\right)\right] D_x\left(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2}\right)\\
&+\left(u^0 H' - h\frac{\partial u^0}{\partial x}z\right) D_z\left(\gamma^{1,0} + z\gamma^{1,1} + z^2\gamma^{1,2}\right)\\
&+\left[u_0^1 H' - hz\left(\frac{\partial u_0^1}{\partial x} - H'\gamma^{0,0}\right) - \frac{1}{2}z^2 h\left(h\frac{\partial \gamma^{0,0}}{\partial x} - H'\gamma^{0,1}\right)\right.\\
&-\frac{1}{3}z^3 h\left(\frac{1}{2}h\frac{\partial \gamma^{0,1}}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\gamma^{0,1} - H'\gamma^{0,2}\right)\\
&\left.-\frac{1}{12}z^4 h\left(h\frac{\partial \gamma^{0,2}}{\partial x} - \frac{\partial h}{\partial x}\gamma^{0,2}\right)\right] D_z\left(\gamma^{0,0} + z\gamma^{0,1} + z^2\gamma^{0,2}\right) = 0
\end{aligned}$$

agrupando según las potencias de z y simplificando:

$$\begin{aligned}
&\frac{\partial \gamma^{1,0}}{\partial t} + u^0\frac{\partial \gamma^{1,0}}{\partial x} + u_0^1\frac{\partial \gamma^{0,0}}{\partial x}\\
&\quad + z\left[\frac{\partial \gamma^{1,1}}{\partial t} - \frac{1}{h}\frac{\partial h}{\partial t}\gamma^{1,1} + u^0\left(\frac{\partial \gamma^{1,1}}{\partial x} - \frac{1}{h}\frac{\partial h}{\partial x}\gamma^{1,1}\right) + u_0^1\left(\frac{\partial \gamma^{0,1}}{\partial x} - \frac{1}{h}\frac{\partial h}{\partial x}\gamma^{0,1}\right)\right.\\
&\quad\left. + h\gamma^{0,0}\frac{\partial \gamma^{0,0}}{\partial x} - \frac{\partial u^0}{\partial x}\gamma^{1,1} - \frac{\partial u_0^1}{\partial x}\gamma^{0,1}\right]\\
&\quad + z^2\left[\frac{\partial \gamma^{1,2}}{\partial t} - \frac{2}{h}\frac{\partial h}{\partial t}\gamma^{1,2} + u^0\left(\frac{\partial \gamma^{1,2}}{\partial x} - \frac{2}{h}\frac{\partial h}{\partial x}\gamma^{1,2}\right) + u_0^1\left(\frac{\partial \gamma^{0,2}}{\partial x} - \frac{2}{h}\frac{\partial h}{\partial x}\gamma^{0,2}\right)\right.
\end{aligned}$$

$$+ h\gamma^{0,0}\frac{\partial \gamma^{0,1}}{\partial x} - \frac{\partial h}{\partial x}\gamma^{0,0}\gamma^{0,1} - 2\frac{\partial u^0}{\partial x}\gamma^{1,2} - 2\frac{\partial u_0^1}{\partial x}\gamma^{0,2}\Bigg]$$

$$+ z^3\left[h\left(-\frac{2}{3}\gamma^{0,2}\frac{\partial \gamma^{0,0}}{\partial x} + \frac{1}{3}\gamma^{0,1}\frac{\partial \gamma^{0,1}}{\partial x} + \gamma^{0,0}\frac{\partial \gamma^{0,2}}{\partial x}\right)\right.$$

$$\left. - \frac{\partial h}{\partial x}\left(2\gamma^{0,0}\gamma^{0,2} + \frac{1}{3}\left(\gamma^{0,1}\right)^2\right)\right]$$

$$+ z^4\left[\frac{5}{12}h\gamma^{0,1}\frac{\partial \gamma^{0,2}}{\partial x} - \frac{11}{12}\frac{\partial h}{\partial x}\gamma^{0,1}\gamma^{0,2}\right] + z^5\left[\frac{1}{6}h\gamma^{0,2}\frac{\partial \gamma^{0,2}}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\left(\gamma^{0,2}\right)^2\right] = 0$$

de donde deducimos que si se supone que γ^1 depende de z según (2.10.72) entonces

$$\gamma^{0,1} = \gamma^{0,2} = 0 \tag{2.10.73}$$

es decir, $\gamma^0 = \gamma^{0,0}$ es independiente de z. Bajo esta hipótesis y, utilizando (2.10.22) como se hizo en la sección anterior para obtener (2.10.39) y (2.10.40), tenemos las siguientes ecuaciones:

$$\frac{\partial \gamma^{1,0}}{\partial t} + u^0\frac{\partial \gamma^{1,0}}{\partial x} + u_0^1\frac{\partial \gamma^0}{\partial x} = 0 \tag{2.10.74}$$

$$\frac{\partial \gamma^{1,1}}{\partial t} + u^0\frac{\partial \gamma^{1,1}}{\partial x} + h\gamma^0\frac{\partial \gamma^0}{\partial x} = 0 \tag{2.10.75}$$

$$\frac{\partial \gamma^{1,2}}{\partial t} + u^0\frac{\partial \gamma^{1,2}}{\partial x} = 0 \tag{2.10.76}$$

Observación 2.10 *De nuevo, $\gamma^{1,2}$ se anulará o no dependiendo de las condiciones iniciales y de contorno, pero las ecuaciones (2.10.74) y (2.10.75) en ningún caso tienen solución cero si u_0^1 y γ^0 no lo son.*

Una vez calculados $\gamma^{1,k}$ $(k = 0, 1, 2)$, teniendo en cuenta (2.10.72) se integra (2.10.71) respecto a z:

$$u^2 = u_0^2 + hz\left(\gamma^{1,0} + u^0 H'' + 2H'\frac{\partial u^0}{\partial x}\right) + \frac{1}{2}hz^2\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right) + \frac{1}{3}hz^3\gamma^{1,2} \tag{2.10.77}$$

donde $u_0^2(t,x) = u^2(t,x,0)$ está determinado por (2.10.32):

$$D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2\frac{\partial u^0}{\partial x} + w^1 D_z u^2 + w^2 D_z u^1 + \frac{1}{\rho_0}D_x p^2 = 0$$

Si en esta ecuación sustituimos u^1, u^2, w^1, w^2 y p^2 por las expresiones (2.10.41), (2.10.77), (2.10.21), (2.10.43) y (2.10.70) respectivamente, de modo que la dependencia de z sea explícita, simplificamos y agrupamos en potencias de z, tenemos:

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u_0^1 \frac{\partial u_0^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} + zh \left(\frac{\partial \gamma^{1,0}}{\partial t} + u^0 \frac{\partial \gamma^{1,0}}{\partial x} + u_0^1 \frac{\partial \gamma^0}{\partial x} \right)$$
$$+ \frac{1}{2} z^2 h \left[\frac{\partial \gamma^{1,1}}{\partial t} - \frac{1}{h} \frac{\partial h}{\partial t} \gamma^{1,1} + u^0 \left(\frac{\partial \gamma^{1,1}}{\partial x} - \frac{1}{h} \frac{\partial h}{\partial x} \gamma^{1,1} \right) + h \gamma^0 \frac{\partial \gamma^0}{\partial x} - \gamma^{1,1} \frac{\partial u^0}{\partial x} \right]$$
$$+ \frac{1}{3} z^3 h \left[\frac{\partial \gamma^{1,2}}{\partial t} - \frac{2}{h} \frac{\partial h}{\partial t} \gamma^{1,2} + u^0 \left(\frac{\partial \gamma^{1,2}}{\partial x} - \frac{2}{h} \frac{\partial h}{\partial x} \gamma^{1,2} \right) - 2\gamma^{1,2} \frac{\partial u^0}{\partial x} \right]$$
$$= - \left\{ \frac{\partial}{\partial x} \left[h \left(\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 H' \frac{\partial u^0}{\partial x} \right) \right] \right.$$
$$+ \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right]$$
$$\left. + H' \left[\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 H' \frac{\partial u^0}{\partial x} \right] \right\}$$

teniendo en cuenta (2.10.74)-(2.10.76) los términos en z, z^2 y z^3 desaparecen y la ecuación para el cálculo de u_0^2 que obtenemos es la siguiente

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u_0^1 \frac{\partial u_0^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} = - \left\{ \frac{\partial}{\partial x} \left[h \left(\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right) \right] \right.$$
$$+ \frac{1}{2} \frac{\partial}{\partial x} \left[h^2 \left(\left(\frac{\partial u^0}{\partial x} \right)^2 - \frac{\partial^2 u^0}{\partial t \partial x} - u^0 \frac{\partial^2 u^0}{\partial x^2} \right) \right]$$
$$\left. + H' \left[\frac{\partial u^0}{\partial t} H' + \left(u^0\right)^2 H'' + u^0 \frac{\partial u^0}{\partial x} H' \right] \right\} = -\frac{1}{\rho_0} \, D_x p^2 \big|_{z=0} \tag{2.10.78}$$

que únicamente difiere de la ecuación análoga obtenida en el caso de vorticidad inicial nula en que en este caso en lugar de u^1 aparece u_0^1.

Una vez calculado u^2, w^3 se calcula integrando respecto a z (2.10.25)

$$D_x u^2 + D_z w^3 = 0$$

Para ello, en primer lugar se sustituye la expresión obtenida para u^2 en (2.10.77), se integra respecto a z, se despeja w^3 e imponiendo la condición de contorno ($w^3 =$

$u_0^2 H'$ en $z=0$) se obtiene la siguiente expresión de w^3 en función de u_0^2 y u^0

$$w^3 = u_0^2 H' + zh\left[H'\left(\gamma^{1,0} + u^0 H'' + 2\frac{\partial u^0}{\partial x}H'\right) - \frac{\partial u_0^2}{\partial x}\right]$$
$$- \frac{1}{2}z^2 h\left[h\frac{\partial}{\partial x}\left(\gamma^{1,0} + u^0 H'' + 2H'\frac{\partial u^0}{\partial x}\right) - H'\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right)\right]$$
$$- \frac{1}{6}z^3 h\left[h\frac{\partial}{\partial x}\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right) - \frac{\partial h}{\partial x}\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right) - 2H'\gamma^{1,2}\right]$$
$$- \frac{1}{12}z^4 h\left[h\frac{\partial \gamma^{1,2}}{\partial x} - 2\frac{\partial h}{\partial x}\gamma^{1,2}\right] \qquad (2.10.79)$$

Los términos p^0, p^1 y p^2 de la presión no han variado en absoluto al prescindir de la hipótesis sobre la vorticidad nula y por ello la aproximación de la presión en el dominio de referencia sigue siendo

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 h(1-z)\left[g + \varepsilon\left(\frac{\partial u^0}{\partial t}H' + \left(u^0\right)^2 H'' + u^0 H'\frac{\partial u^0}{\partial x}\right)\right]$$
$$+ \varepsilon^2\frac{\rho_0 h^2}{2}(1-z^2)\left[\left(\frac{\partial u^0}{\partial x}\right)^2 - \frac{\partial^2 u^0}{\partial t\partial x} - u^0\frac{\partial^2 u^0}{\partial x^2}\right] \qquad (2.10.80)$$

Usando las expresiones encontradas para w^0, w^1, w^2 y w^3, (2.10.17), (2.10.21), (2.10.43) y (2.10.79) respectivamente, tenemos las siguiente aproximación de la velocidad vertical en Ω

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$
$$= \varepsilon\left(u^0 H' - h\frac{\partial u^0}{\partial x}z\right) + \varepsilon^2\left[u_0^1 H' - hz\left(\frac{\partial u_0^1}{\partial x} - H'\gamma^0\right) - \frac{1}{2}z^2 h^2\frac{\partial \gamma^0}{\partial x}\right]$$
$$+ \varepsilon^3\left\{u_0^2 H' + zh\left[H'\left(\gamma^{1,0} + u^0 H'' + 2\frac{\partial u^0}{\partial x}H'\right) - \frac{\partial u_0^2}{\partial x}\right]\right.$$
$$- \frac{1}{2}z^2 h\left[h\frac{\partial}{\partial x}\left(\gamma^{1,0} + u^0 H'' + 2H'\frac{\partial u^0}{\partial x}\right) - H'\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right)\right]$$
$$- \frac{1}{6}z^3 h\left[h\frac{\partial}{\partial x}\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right) - \frac{\partial h}{\partial x}\left(\gamma^{1,1} - h\frac{\partial^2 u^0}{\partial x^2}\right) - 2H'\gamma^{1,2}\right]$$
$$\left.- \frac{1}{12}z^4 h\left[h\frac{\partial \gamma^{1,2}}{\partial x} - 2\frac{\partial h}{\partial x}\gamma^{1,2}\right]\right\}$$

$$= -\varepsilon \left[zh \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \frac{\partial u^2}{\partial x} \right) - H' \left(u^0 + \varepsilon u^1 + \varepsilon u^2 \right) \right.$$

$$\left. - z \left(H' + z \frac{\partial h}{\partial x} \right) \left(\varepsilon \frac{\partial u^1}{\partial z} + \varepsilon^2 \frac{\partial u^2}{\partial z} \right) \right]$$

$$+ \frac{1}{2} \varepsilon^2 zh \left[zh \left(\frac{\partial \gamma^0}{\partial x} + \varepsilon \frac{\partial \gamma^{1,0}}{\partial x} \right) - 2H' \left(\gamma^0 + \varepsilon \gamma^{1,0} \right) \right] + O(\varepsilon^3)$$

Si denotamos por

$$\check{\gamma}(\varepsilon)(t,x) = \gamma^0(t,x) + \varepsilon \gamma^{1,0}(t,x)$$

entonces la velocidad vertical se puede escribir como sigue

$$\tilde{w}(\varepsilon) = -\varepsilon \left[zh \frac{\partial \tilde{u}(\varepsilon)}{\partial x} - H' \tilde{u}(\varepsilon) - z \left(H' + z \frac{\partial h}{\partial x} \right) \frac{\partial \tilde{u}(\varepsilon)}{\partial z} \right]$$

$$+ \frac{1}{2} \varepsilon^2 zh \left(zh \frac{\partial \check{\gamma}(\varepsilon)}{\partial x} - 2H' \check{\gamma}(\varepsilon) \right) + O(\varepsilon^3)$$

$$= -\varepsilon D_x \left(zh \tilde{u}(\varepsilon) - \frac{1}{2} \varepsilon z^2 h^2 \check{\gamma}(\varepsilon) \right) + O(\varepsilon^3) \quad (2.10.81)$$

Si ahora se deshace el cambio de variable, volviendo al dominio original, obtenemos la siguiente aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,z) = u^0(t,x) + \varepsilon u^1(t,x) + \varepsilon^2 u^2(t,x,z)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,z) = \varepsilon w^1(t,x,z) + \varepsilon^2 w^2(t,x,z) + \varepsilon^3 w^3(t,x,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,z) = p^0(t,x) + \varepsilon p^1(t,x,z) + \varepsilon^2 p^2(t,x,z)$$

$$\tilde{\gamma}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{\gamma}(\varepsilon)(t,x,z) = \gamma^0(t,x) + \varepsilon \gamma^1(t,x,z) + \varepsilon^2 \gamma^2(t,x,z)$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (2.10.80):

$$\tilde{p}^\varepsilon = p_s + \rho_0 (s^\varepsilon - z^\varepsilon) \left[g + \frac{\partial u^0}{\partial t^\varepsilon} \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \left(u^0 \right)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + u^0 \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \frac{\partial u^0}{\partial x^\varepsilon} \right]$$

$$+ \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[\left(\frac{\partial u^0}{\partial x^\varepsilon} \right)^2 - \frac{\partial^2 u^0}{\partial t^\varepsilon \partial x^\varepsilon} - u^0 \frac{\partial^2 u^0}{\partial (x^\varepsilon)^2} \right] \quad (2.10.82)$$

Y de forma análoga, realizando el cambio de variable de Ω a Ω^ε en (2.10.81), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon = -\frac{\partial}{\partial x^\varepsilon} \left[\left(z^\varepsilon - H^\varepsilon \right) \tilde{u}^\varepsilon - \frac{1}{2} \left(z^\varepsilon - H^\varepsilon \right)^2 \check{\gamma}^\varepsilon \right] + O(\varepsilon^3) \quad (2.10.83)$$

donde $\check{\gamma}^\varepsilon(t^\varepsilon, x^\varepsilon) = \check{\gamma}(\varepsilon)(t,x)$.

Ahora, teniendo en cuenta (2.10.36) y (2.10.74), se verifica

$$\begin{aligned}\frac{\partial \check{\gamma}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{\gamma}^\varepsilon}{\partial x^\varepsilon} &= D_t \check{\gamma}(\varepsilon) + \check{u}(\varepsilon) D_x \check{\gamma}(\varepsilon) \\ &= \frac{\partial \gamma^0}{\partial t} + \varepsilon \frac{\partial \gamma^{1,0}}{\partial t} + (u^0 + \varepsilon u_0^1)\left(\frac{\partial \gamma^0}{\partial x} + \varepsilon \frac{\partial \gamma^{1,0}}{\partial x}\right) \\ &= \frac{\partial \gamma^0}{\partial t} + u^0 \frac{\partial \gamma^0}{\partial x} + \varepsilon \left(\frac{\partial \gamma^{1,0}}{\partial t} + u^0 \frac{\partial \gamma^{1,0}}{\partial x} + u_0^1 \frac{\partial \gamma^0}{\partial x}\right) \\ &+ \varepsilon^2 u_0^1 \frac{\partial \gamma^{1,0}}{\partial x} = O(\varepsilon^2) \end{aligned} \tag{2.10.84}$$

El modelo que vamos a proponer requiere que definamos también

$$\begin{aligned} &\check{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \check{u}(\varepsilon)(t,x) = u^0(t,x) + \varepsilon u_0^1(t,x) \\ &\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \hat{u}(\varepsilon)(t,x) = u^0(t,x) + \varepsilon u_0^1(t,x) + \varepsilon^2 u_0^2(t,x) \\ &\gamma^{1,k,\varepsilon}(t^\varepsilon, x^\varepsilon) = \varepsilon^k \gamma^{1,k}(t,x), \quad (k = 1,2) \end{aligned}$$

Recordamos que en la sección anterior deducimos que para h y $\check{u}^\varepsilon$ se tenían las ecuaciones (2.10.49) y (2.10.50):

$$\begin{aligned} &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \\ &\frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + O(\varepsilon^2) \end{aligned}$$

Expresamos $\tilde{p}^\varepsilon$ en función de $\check{u}^\varepsilon$ en lugar de $u^{0,\varepsilon}$:

$$\begin{aligned}\tilde{p}^\varepsilon &= p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (\check{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}\right] \\ &+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 \check{u}^\varepsilon}{\partial t^\varepsilon \partial x^\varepsilon} - \check{u}^\varepsilon \frac{\partial^2 \check{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right] + O(\varepsilon^3) \end{aligned} \tag{2.10.85}$$

Si se consideran las ecuaciones (2.10.19), (2.10.42), (2.10.78) y (2.10.82) se verifica

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u_0^1}{\partial t} + \varepsilon^2 \frac{\partial u_0^2}{\partial t} + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x} + \varepsilon^2 \frac{\partial u_0^2}{\partial x} \right)$$

$$= -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \varepsilon g \frac{\partial s}{\partial x} - \varepsilon^2 \frac{1}{\rho_0} \left. D_x p^2 \right|_{z=0} + O(\varepsilon^3)$$

$$= -\frac{1}{\rho_0} \left. \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} \right|_{z^\varepsilon = H^\varepsilon} + O(\varepsilon^3) \tag{2.10.86}$$

Las ecuaciones para el cálculo de $\gamma^{1,k,\varepsilon} \quad (k = 1, 2)$ se obtienen deshaciendo el cambio de variable en las ecuaciones (2.10.75) y (2.10.76) y sustituyendo $\gamma^{0,\varepsilon}$ y $u^{0,\varepsilon}$ por $\check{\gamma}^\varepsilon$ y $\check{u}^\varepsilon$ respectivamente

$$\frac{\partial \gamma^{1,1,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma^{1,1,\varepsilon}}{\partial x^\varepsilon} + h^\varepsilon \check{\gamma}^\varepsilon \frac{\partial \check{\gamma}^\varepsilon}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{2.10.87}$$

$$\frac{\partial \gamma^{1,2,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma^{1,2,\varepsilon}}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{2.10.88}$$

Veamos cómo se puede escribir $\tilde{u}^\varepsilon$ en función de $\hat{u}^\varepsilon$, $\check{\gamma}^\varepsilon$, $\gamma^{1,k,\varepsilon} \quad (k = 1, 2)$ y h^ε. Para ello usaremos (2.10.41) y (2.10.77)

$$\tilde{u}^\varepsilon = \tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 = u^0 + \varepsilon \left(u_0^1 + hz\gamma^0 \right)$$

$$+ \varepsilon^2 \left[u_0^2 + hz \left(\gamma^{1,0} + u^0 H'' + 2 \frac{\partial u^0}{\partial x} H' \right) + \frac{1}{2} hz^2 \left(\gamma^{1,1} - h \frac{\partial^2 u^0}{\partial x^2} \right) + \frac{1}{3} hz^3 \gamma^{1,2} \right]$$

$$= u^0 + \varepsilon u_0^1 + \varepsilon^2 u_0^2 + \varepsilon hz \left(\gamma^0 + \varepsilon \gamma^{1,0} + u^0 \varepsilon H'' + 2 \frac{\partial u^0}{\partial x} \varepsilon H' \right)$$

$$+ \frac{1}{2} \varepsilon^2 hz^2 \gamma^{1,1} + \varepsilon^2 \frac{1}{3} hz^3 \gamma^{1,2} - \frac{1}{2} \varepsilon^2 h^2 z^2 \frac{\partial^2 u^0}{\partial x^2}$$

$$= \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left(\check{\gamma}^\varepsilon + \check{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + 2 \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right) + \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \gamma^{1,1,\varepsilon}$$

$$+ \frac{1}{3} \frac{(z^\varepsilon - H^\varepsilon)^3}{(h^\varepsilon)^2} \gamma^{1,2,\varepsilon} - \frac{1}{2} (z^\varepsilon - H^\varepsilon)^2 \frac{\partial^2 \check{u}^\varepsilon}{\partial (x^\varepsilon)^2} + O(\varepsilon^3) \tag{2.10.89}$$

Proponemos el siguiente modelo resultado de despreciar los términos en $O(\varepsilon^2)$ de las ecuaciones (2.10.49), (2.10.50), (2.10.84), (2.10.87) y (2.10.88) y los términos en

$O(\varepsilon^3)$ de las expresiones (2.10.83), (2.10.85) y (2.10.89) y de la ecuación (2.10.86):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{2.10.90}$$

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g \tag{2.10.91}$$

$$\begin{aligned}\tilde{p}^\varepsilon &= p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (\breve{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}\right] \\ &+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 \breve{u}^\varepsilon}{\partial t^\varepsilon \partial x^\varepsilon} - \breve{u}^\varepsilon \frac{\partial^2 \breve{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right]\end{aligned} \tag{2.10.92}$$

$$\frac{\partial \breve{\gamma}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{\gamma}^\varepsilon}{\partial x^\varepsilon} = 0 \tag{2.10.93}$$

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \tag{2.10.94}$$

$$\frac{\partial \gamma^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma^{1,1,\varepsilon}}{\partial x^\varepsilon} + h^\varepsilon \breve{\gamma}^\varepsilon \frac{\partial \breve{\gamma}^\varepsilon}{\partial x^\varepsilon} = 0 \tag{2.10.95}$$

$$\frac{\partial \gamma^{1,2,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma^{1,2,\varepsilon}}{\partial x^\varepsilon} = 0 \tag{2.10.96}$$

$$\begin{aligned}\tilde{u}^\varepsilon &= \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left(\breve{\gamma}^\varepsilon + \breve{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + 2\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right) + \frac{1}{2}\frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon}\gamma^{1,1,\varepsilon} \\ &+ \frac{1}{3}\frac{(z^\varepsilon - H^\varepsilon)^3}{(h^\varepsilon)^2}\gamma^{1,2,\varepsilon} - \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \frac{\partial^2 \breve{u}^\varepsilon}{\partial (x^\varepsilon)^2}\end{aligned} \tag{2.10.97}$$

$$\tilde{w}^\varepsilon = -\frac{\partial}{\partial x^\varepsilon}\left[(z^\varepsilon - H^\varepsilon)\,\tilde{u}^\varepsilon - \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \breve{\gamma}^\varepsilon\right] \tag{2.10.98}$$

Buscamos la ecuación adecuada para el cálculo de la vorticidad:

$$\begin{aligned}\frac{\partial \tilde{\gamma}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}^\varepsilon}{\partial z^\varepsilon} &= D_t\tilde{\gamma}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{\gamma}(\varepsilon) + \tilde{w}(\varepsilon)D_z\tilde{\gamma}(\varepsilon) \\ &= D_t\gamma^0 + \varepsilon D_t\gamma^1 + \varepsilon^2 D_t\gamma^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left[D_x\gamma^0 + \varepsilon D_x\gamma^1 + \varepsilon^2 D_x\gamma^2\right] \\ &+ (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3)\frac{1}{\varepsilon}\left[D_z\gamma^0 + \varepsilon D_z\gamma^1 + \varepsilon^2 D_z\gamma^2\right] \\ &= D_t\gamma^0 + u^0 D_x\gamma^0 + w^1 D_z\gamma^0\end{aligned}$$

$$+\,\varepsilon\left(D_t\gamma^1+u^0D_x\gamma^1+u^1D_x\gamma^0+w^1D_z\gamma^1+w^2D_z\gamma^0\right)$$

$$+\,\varepsilon^2\left(D_t\gamma^2+u^0D_x\gamma^2+u^1D_x\gamma^1+u^2D_x\gamma^0+w^1D_z\gamma^2+w^2D_z\gamma^1+w^3D_z\gamma^0\right)$$

$$+\,O(\varepsilon^3)$$

Si se tiene en cuenta las ecuaciones (2.10.14)-(2.10.16) y se desprecian los términos en $O(\varepsilon^3)$ de esta ecuación, al igual que se hizo en las igualdades (2.10.83), (2.10.85) y (2.10.89) y en la ecuación (2.10.86), la ecuación para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo es la siguiente:

$$\frac{\partial\tilde{\gamma}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}^\varepsilon}{\partial x^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}^\varepsilon}{\partial z^\varepsilon}=0 \tag{2.10.99}$$

Del mismo modo que se hizo en la sección anterior para la aproximación de primer orden en ε, el modelo (2.10.90)-(2.10.98) se puede escribir en función de la velocidad media en la vertical. Para ello se tiene en cuenta que

$$\bar{u}^\varepsilon=\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\tilde{u}^\varepsilon\,dz^\varepsilon=\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left[\hat{u}^\varepsilon+(z^\varepsilon-H^\varepsilon)\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)\right.$$

$$\left.+\,\frac{1}{2}\frac{(z^\varepsilon-H^\varepsilon)^2}{h^\varepsilon}\gamma^{1,1,\varepsilon}+\frac{1}{3}\frac{(z^\varepsilon-H^\varepsilon)^3}{(h^\varepsilon)^2}\gamma^{1,2,\varepsilon}-\frac{1}{2}(z^\varepsilon-H^\varepsilon)^2\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right]dz^\varepsilon$$

$$=\hat{u}^\varepsilon+\frac{h^\varepsilon}{2}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)$$

$$+\,\frac{h^\varepsilon}{6}\gamma^{1,1,\varepsilon}+\frac{h^\varepsilon}{12}\gamma^{1,2,\varepsilon}-\frac{(h^\varepsilon)^2}{6}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2} \tag{2.10.100}$$

y comprobamos si $\bar{u}^\varepsilon$ verifica una ecuación similar a la que verifica $\hat{u}^\varepsilon$ ((2.10.94)):

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}=\frac{\partial}{\partial t^\varepsilon}\left[\hat{u}^\varepsilon+\frac{h^\varepsilon}{2}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right.\right.$$

$$\left.\left.+\,\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)\right]+\left[\hat{u}^\varepsilon+\frac{h^\varepsilon}{2}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right.\right.$$

$$\left.\left.+\,\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)\right]\frac{\partial}{\partial x^\varepsilon}\left[\hat{u}^\varepsilon+\frac{h^\varepsilon}{2}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}\right.\right.$$

$$\left.\left.+\,2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)\right]=\frac{\partial\hat{u}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon}$$

$$+\,\frac{1}{2}\left(\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial h^\varepsilon}{\partial x^\varepsilon}+h^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon}\right)\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right.$$

$$+\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\Big)$$

$$+\frac{h}{2}\left[\frac{\partial\check{\gamma}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial\check{\gamma}^\varepsilon}{\partial x^\varepsilon}+\frac{1}{3}\left(\frac{\partial\gamma^{1,1,\varepsilon}}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial\gamma^{1,1,\varepsilon}}{\partial x^\varepsilon}+h^\varepsilon\check{\gamma}^\varepsilon\frac{\partial\check{\gamma}^\varepsilon}{\partial x^\varepsilon}\right)-\frac{1}{3}h^\varepsilon\check{\gamma}^\varepsilon\frac{\partial\check{\gamma}^\varepsilon}{\partial x^\varepsilon}\right.$$

$$+\frac{1}{6}\left(\frac{\partial\gamma^{1,2,\varepsilon}}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial\gamma^{1,2,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{\partial}{\partial t^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)$$

$$+\hat{u}^\varepsilon\frac{\partial}{\partial x^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)$$

$$+\frac{1}{2}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)^2$$

$$\left.+\frac{h^\varepsilon}{4}\frac{\partial}{\partial x^\varepsilon}\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\frac{\gamma^{1,1,\varepsilon}}{3}+\frac{\gamma^{1,2,\varepsilon}}{6}-\frac{h^\varepsilon}{3}\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right)^2\right]$$

Se utilizan, ahora, las ecuaciones (2.10.84)-(2.10.88) y que $\hat{u}^\varepsilon=\check{u}^\varepsilon+O(\varepsilon^2)$, de modo que la expresión anterior se reduce a:

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\left.\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}+O(\varepsilon^2)=-\frac{1}{\rho_0h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}\,dz^\varepsilon+O(\varepsilon^2)$$

$$=-\frac{1}{\rho_0}\frac{\partial}{\partial x^\varepsilon}\left(\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\tilde{p}^\varepsilon\,dz^\varepsilon\right)+O(\varepsilon^2)$$

Si denotamos por $\bar{p}^\varepsilon=\dfrac{1}{h^\varepsilon}\displaystyle\int_{H^\varepsilon}^{s^\varepsilon}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}\,dz^\varepsilon$, finalmente la ecuación que permite el cálculo de $\bar{u}^\varepsilon$ se puede escribir como sigue:

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial\bar{p}^\varepsilon}{\partial x^\varepsilon}+O(\varepsilon^2)\qquad(2.10.101)$$

Si en el modelo (2.10.90)-(2.10.98) se sustituye $\hat{u}^\varepsilon$ por $\bar{u}^\varepsilon$ y por tanto, la ecuación (2.10.94) por la que resulta de despreciar en (2.10.101) los términos $O(\varepsilon^2)$ se pierde precisión, pues para obtener (2.10.101) los términos suprimidos eran de orden superior. El modelo sería

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial(\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon}=0\qquad(2.10.102)$$

$$\frac{\partial\check{u}^\varepsilon}{\partial t^\varepsilon}+\check{u}^\varepsilon\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}g\qquad(2.10.103)$$

$$\tilde{p}^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)\left[g+\frac{\partial\check{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+(\check{u}^\varepsilon)^2\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\right]$$

$$+\frac{\rho_0}{2}\left[(h^\varepsilon)^2-(z^\varepsilon-H^\varepsilon)^2\right]\left[\left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\right)^2-\frac{\partial^2\check{u}^\varepsilon}{\partial t^\varepsilon\partial x^\varepsilon}-\check{u}^\varepsilon\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\right] \quad (2.10.104)$$

$$\frac{\partial\check{\gamma}^\varepsilon}{\partial t^\varepsilon}+\check{u}^\varepsilon\frac{\partial\check{\gamma}^\varepsilon}{\partial x^\varepsilon}=0 \quad (2.10.105)$$

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial\bar{p}^\varepsilon}{\partial x^\varepsilon} \quad (2.10.106)$$

$$\frac{\partial\gamma^{1,1,\varepsilon}}{\partial t^\varepsilon}+\check{u}^\varepsilon\frac{\partial\gamma^{1,1,\varepsilon}}{\partial x^\varepsilon}+h^\varepsilon\check{\gamma}^\varepsilon\frac{\partial\check{\gamma}^\varepsilon}{\partial x^\varepsilon}=0 \quad (2.10.107)$$

$$\frac{\partial\gamma^{1,2,\varepsilon}}{\partial t^\varepsilon}+\check{u}^\varepsilon\frac{\partial\gamma^{1,2,\varepsilon}}{\partial x^\varepsilon}=0 \quad (2.10.108)$$

$$\begin{aligned}\tilde{u}^\varepsilon=\bar{u}^\varepsilon&+\left(z^\varepsilon-H^\varepsilon-\frac{h^\varepsilon}{2}\right)\left(\check{\gamma}^\varepsilon+\check{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}+2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)\\&+\left(\frac{1}{2}\frac{(z^\varepsilon-H^\varepsilon)^2}{h^\varepsilon}-\frac{h^\varepsilon}{6}\right)\gamma^{1,1,\varepsilon}+\left(\frac{1}{3}\frac{(z^\varepsilon-H^\varepsilon)^3}{(h^\varepsilon)^2}-\frac{h^\varepsilon}{12}\right)\gamma^{1,2,\varepsilon}\\&-\left(\frac{1}{2}(z^\varepsilon-H^\varepsilon)^2-\frac{(h^\varepsilon)^2}{6}\right)\frac{\partial^2\check{u}^\varepsilon}{\partial(x^\varepsilon)^2}\end{aligned} \quad (2.10.109)$$

$$\tilde{w}^\varepsilon=-\frac{\partial}{\partial x^\varepsilon}\left[\left(z^\varepsilon-H^\varepsilon\right)\tilde{u}^\varepsilon-\frac{1}{2}\left(z^\varepsilon-H^\varepsilon\right)^2\check{\gamma}^\varepsilon\right] \quad (2.10.110)$$

2.10.4. Modelo propuesto

El modelo (2.10.102)-(2.10.110) se descarta pues es más complicado de resolver que los modelos propuestos a partir de la aproximación de primer orden y no mejora el orden de aproximación. El modelo (2.10.90)-(2.10.98) sí que mejora, al menos formalmente, el orden de precisión respecto a los modelos propuestos en la sección 2.10.2 pero el esfuerzo necesario para resolverlo es mucho mayor, si se compara por ejemplo con el modelo (2.10.69) el número de ecuaciones a resolver es el doble. De entre los modelos deducidos en la sección 2.10.2 escogemos precisamente (2.10.69) que está escrito en términos de la velocidad promediada en altura (forma usual de escribir los modelos clásicos de aguas someras) y considera que el primer término de la vorticidad no depende de z. Se propone finalmente, suprimiendo ˜ para simplificar la notación:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon}=0$$

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}g$$

$$\frac{\partial \gamma^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma^{0,\varepsilon}}{\partial x^\varepsilon} = 0$$

$$\tilde{u}^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2})\, \gamma^{0,\varepsilon}$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0 g(s^\varepsilon - z^\varepsilon)$$

$$w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial u^\varepsilon}{\partial x^\varepsilon} \tag{2.10.111}$$

También en este caso, como en la sección 2.9, se podría introducir en el modelo la presión obtenida en la aproximación de orden 2:

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (\tilde{u}^\varepsilon)^2\,\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right]$$
$$+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 - \frac{\partial^2 \tilde{u}^\varepsilon}{\partial t^\varepsilon \partial x^\varepsilon} - \tilde{u}^\varepsilon \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right] \tag{2.10.112}$$

Observación 2.11 *La presión no se ve afectada por la hipótesis sobre la vorticidad inicial en ninguno de los modelos.*

2.11. Conclusiones

Hemos visto en este capítulo que el método de desarrollos asintóticos permite recuperar y generalizar el modelo clásico de aguas someras.

Comenzamos estudiando el caso en el que la vorticidad inicial es nula y no es hasta la aproximación de primer orden (véase el modelo (2.7.8)) que se obtiene un modelo comparable al modelo clásico de aguas poco profundas ((1.2.24)), con la ventaja de que el modelo obtenido no supone la velocidad vertical nula.

Si consideramos los distintos modelos de segundo orden propuestos (véase sección 2.8), observamos que todos ellos requieren un esfuerzo de cálculo mucho mayor que el modelo (2.7.8) (bien porque es necesario resolver un mayor número de ecuaciones del mismo tipo, o bien porque el acoplamiento es mucho mayor) para obtener una pequeña mejora en la precisión (teóricamente del orden de ε^2, aunque habría que comprobarlo numéricamente). Por lo que parece lógico proponer el modelo (2.9.1) para el caso de las ecuaciones de aguas someras sin vorticidad en dimensión uno. Además, parece factible mejorar la precisión del cálculo de la presión (véase (2.9.2)) sin que el coste aumente significativamente.

En el caso de que supongamos que la vorticidad inicial no es nula, no obtenemos diferencias significativas hasta la aproximación de primer orden en ε, cuando suponemos que $\gamma^0 \neq 0$. Bajo estas hipótesis, incluso en el caso más sencillo en el que suponemos que γ^0 sea independiente de z, obtenemos una modificación interesante del modelo (2.7.8), en la que la velocidad horizontal depende de forma explícita de la variable z^ε a través de la vorticidad. De nuevo, la aproximación de segundo orden, debido a que el número de ecuaciones a resolver aumenta mucho, se deshecha, por lo que finalmente el modelo propuesto es (2.10.111), al que la aproximación de orden dos podría aportarle la corrección a la presión (2.10.112).

Capítulo 3

Modelo unidimensional de aguas someras obtenido a partir de las ecuaciones de Navier-Stokes

3.1. Formulación del problema

En este capítulo pretendemos obtener un modelo unidimensional de aguas someras con viscosidad. Para ello actuaremos de forma similar a como lo hicimos en el capítulo 2, pero partiendo en esta ocasión de las ecuaciones de Navier-Stokes bidimensionales.

3.1.1. Ecuaciones de partida

Al igual que en el capítulo 2, el dominio sobre el que trabajaremos es un canal que representamos mediante el dominio Ω (Figura 2.1) definido por:

$$\Omega = \{(x, z)/x \in [0, L], z \in [H(x), H(x) + h(t, x)]\}$$

Ahora consideraremos que el flujo se rige por las ecuaciones bidimensionales de Navier-Stokes en Ω y que la única fuerza externa actuando sobre el fluido es la debida a la gravedad, esto es, se verifica que

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \Delta u \tag{3.1.1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \nu \Delta w \tag{3.1.2}$$

donde se ha empleado la misma notación que en el capítulo anterior.

El fluido es incompresible por lo que se cumple:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3.1.3}$$

En este caso también suponemos que la presión es la atmosférica en la superficie

$$p = p_s \quad \text{en } z = s(t,x) \tag{3.1.4}$$

y que el fluido no puede penetrar el fondo

$$(u,w)\cdot\vec{\mathbf{n}} = 0 \quad \text{en } z = H(x) \tag{3.1.5}$$

También se tiene en cuenta el efecto del viento en la superficie y del rozamiento en el fondo

$$(T\,\vec{\mathbf{n}})\cdot\vec{\tau} = \vec{\mathbf{f}}\cdot\vec{\tau} = f_W \quad \text{en } z = s(t,x) \tag{3.1.6}$$

$$(T\,\vec{\mathbf{n}})\cdot\vec{\tau} = -\vec{\mathbf{G}}(u,w)\cdot\vec{\tau} = -f_R \quad \text{en } z = H(x) \tag{3.1.7}$$

donde $\vec{\mathbf{f}}$ es la fuerza del viento, $-\vec{\mathbf{G}}(u,w)$ es la fuerza de rozamiento, T es el tensor de tensiones [1]:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ & \\ T_{21} & T_{22} \end{pmatrix} \quad \text{con } T_{12} = T_{21} \tag{3.1.8}$$

y $\vec{\mathbf{n}}$ y $\vec{\tau}$ son el vector normal unitario exterior y el vector unitario tangente a la frontera respectivamente, que en $z = H$ son los siguientes:

$$\vec{\mathbf{n}} = \left(\frac{H'}{\sqrt{(H')^2+1}}, \frac{-1}{\sqrt{(H')^2+1}} \right) \tag{3.1.9}$$

$$\vec{\tau} = \left(\frac{1}{\sqrt{(H')^2+1}}, \frac{H'}{\sqrt{(H')^2+1}} \right) \tag{3.1.10}$$

mientras que en $z = s$

$$\vec{\mathbf{n}} = \left(\frac{-\dfrac{\partial s}{\partial x}}{\sqrt{\left(\dfrac{\partial s}{\partial x}\right)^2+1}}, \frac{1}{\sqrt{\left(\dfrac{\partial s}{\partial x}\right)^2+1}} \right) \tag{3.1.11}$$

$$\vec{\tau} = \left(\frac{1}{\sqrt{\left(\dfrac{\partial s}{\partial x}\right)^2+1}}, \frac{\dfrac{\partial s}{\partial x}}{\sqrt{\left(\dfrac{\partial s}{\partial x}\right)^2+1}} \right) \tag{3.1.12}$$

[1] En realidad el tensor de tensiones es $\sigma = -pI + T$ y T representa las tensiones no debidas a la presión. Las expresiones (3.1.6)-(3.1.7) son sin embargo correctas, debido a que

$$\sigma\vec{\mathbf{n}}\cdot\vec{\tau} = (-p\vec{\mathbf{n}} + T\,\vec{\mathbf{n}})\cdot\vec{\tau} = (T\,\vec{\mathbf{n}})\cdot\vec{\tau}$$

Reescribimos, ahora, las condiciones (3.1.5)-(3.1.7) sustituyendo T, $\vec{\mathbf{n}}$ y $\vec{\tau}$ por las expresiones vistas en (3.1.8)-(3.1.12)

$$uH' - w = 0 \quad \text{en } z = H \tag{3.1.13}$$

$$\frac{T_{12} - \frac{\partial s}{\partial x}T_{11} + \frac{\partial s}{\partial x}T_{22} - \left(\frac{\partial s}{\partial x}\right)^2 T_{12}}{1 + \left(\frac{\partial s}{\partial x}\right)^2} = f_W \quad \text{en } z = s \tag{3.1.14}$$

$$\frac{H'T_{11} - T_{12} + (H')^2 T_{12} - H'T_{22}}{1 + (H')^2} = -f_R \quad \text{en } z = H \tag{3.1.15}$$

de donde se puede despejar T_{12} en la frontera superior e inferior:

$$T_{12} = \frac{f_W \left[1 + \left(\frac{\partial s}{\partial x}\right)^2\right] + \frac{\partial s}{\partial x}\left[T_{11} - T_{22}\right]}{1 - \left(\frac{\partial s}{\partial x}\right)^2} \quad \text{en } z = s \tag{3.1.16}$$

$$T_{12} = \frac{-f_R \left[1 + (H')^2\right] - H'\left[T_{11} - T_{22}\right]}{(H')^2 - 1} \quad \text{en } z = H \tag{3.1.17}$$

Suponemos, además, que el caudal de entrada (uh en $x = 0$) y el de salida (uh en $x = L$) son conocidos en cada instante.

Para cerrar el problema se deben fijar las condiciones iniciales:

$$u(0, x, z) = u_0(x, z) \tag{3.1.18}$$

$$w(0, x, z) = w_0(x, z) \tag{3.1.19}$$

Además se deduce de la conservación de la masa, como vimos en la sección 2.2 (ecuación (2.2.5)), que la altura del agua se puede calcular del modo siguiente:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_H^s u dz = 0 \tag{3.1.20}$$

El tensor de tensiones (véase nota al pie de la página 84) del fluido se define como:

$$T = \mu \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & 2\frac{\partial w}{\partial z} \end{pmatrix} \tag{3.1.21}$$

entonces, teniendo en cuenta la incompresibilidad del fluido se verifica:

$$\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial z} = \mu \left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \right) = \mu \Delta u$$

$$\frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial z} = \mu \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial z^2} \right) = \mu \Delta w$$

considerando además que $\nu = \dfrac{\mu}{\rho_0}$, se puede escribir:

$$\nu \Delta u = 2\nu \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial T_{12}}{\partial z} \tag{3.1.22}$$

$$\nu \Delta w = \frac{1}{\rho_0}\frac{\partial T_{12}}{\partial x} + 2\nu \frac{\partial^2 w}{\partial z^2} \tag{3.1.23}$$

Si se sustituyen las expresiones anteriores en las ecuaciones de Navier-Stokes obtenemos las siguientes ecuaciones, que serán nuestro punto de partida ya que facilitarán la incorporación de los efectos del viento y el rozamiento a la ecuación límite:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\nu \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial T_{12}}{\partial z} \tag{3.1.24}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \frac{1}{\rho_0}\frac{\partial T_{12}}{\partial x} + 2\nu \frac{\partial^2 w}{\partial z^2} \tag{3.1.25}$$

3.1.2. Cambio de notación

Como ya se explicó en la sección 2.1.2 para obtener el modelo de aguas someras que buscamos se introduce un pequeño parámetro adimensional, ε, del orden del cociente entre la profundidad media y la longitud del canal. Tanto el dominio como las variables y funciones mencionadas arriba dependen de este parámetro. Esta dependencia se indicará con el superíndice ε. Las ecuaciones se reescriben usando esta nueva notación, y así (3.1.24), (3.1.25) y (3.1.3) se escriben:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + w^\varepsilon\frac{\partial u^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial x^\varepsilon} + 2\nu \frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial T_{12}^\varepsilon}{\partial z^\varepsilon} \tag{3.1.26}$$

$$\frac{\partial w^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon\frac{\partial w^\varepsilon}{\partial x^\varepsilon} + w^\varepsilon\frac{\partial w^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial z^\varepsilon} - g + \frac{1}{\rho_0}\frac{\partial T_{12}^\varepsilon}{\partial x^\varepsilon} + 2\nu \frac{\partial^2 w^\varepsilon}{\partial (z^\varepsilon)^2} \tag{3.1.27}$$

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = 0 \tag{3.1.28}$$

en $[0, T] \times \Omega^\varepsilon$, donde

$$\Omega^\varepsilon = \{(x^\varepsilon, z^\varepsilon)/x^\varepsilon \in [0, L], z^\varepsilon \in [H^\varepsilon(x^\varepsilon), H^\varepsilon(x^\varepsilon) + h^\varepsilon(t^\varepsilon, x^\varepsilon)]\}$$

y ahora, $(u^\varepsilon, w^\varepsilon) = (u^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon), w^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon))$ es el vector velocidad y $p^\varepsilon = p^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$ es la presión.

Las condiciones de contorno (3.1.4), (3.1.13), (3.1.16) y (3.1.17) se escriben:

$$p^\varepsilon = p_s \quad \text{en } z^\varepsilon = s^\varepsilon \tag{3.1.29}$$

$$u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - w^\varepsilon = 0 \quad \text{en } z^\varepsilon = H^\varepsilon \tag{3.1.30}$$

$$T_{12}^\varepsilon = \frac{f_W^\varepsilon \left[1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right] + \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left[T_{11}^\varepsilon - T_{22}^\varepsilon\right]}{1 - \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2} \quad \text{en } z^\varepsilon = s^\varepsilon \tag{3.1.31}$$

$$T_{12}^\varepsilon = \frac{f_R^\varepsilon \left[1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right] + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left[T_{11}^\varepsilon - T_{22}^\varepsilon\right]}{1 - \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2} \quad \text{en } z^\varepsilon = H^\varepsilon \tag{3.1.32}$$

donde f_W^ε y f_R^ε son las fuerzas del viento y de rozamiento respectivamente. Las condiciones iniciales (3.1.18)-(3.1.19) resultan

$$u^\varepsilon(0, x^\varepsilon, z^\varepsilon) = u_0^\varepsilon(x^\varepsilon, z^\varepsilon) \tag{3.1.33}$$

$$w^\varepsilon(0, x^\varepsilon, z^\varepsilon) = w_0^\varepsilon(x^\varepsilon, z^\varepsilon) \tag{3.1.34}$$

la ecuación (3.1.20) para el cálculo de h^ε:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial}{\partial x^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} u^\varepsilon dz^\varepsilon = 0 \tag{3.1.35}$$

las componentes del tensor de tensiones, T^ε:

$$T_{11}^\varepsilon = 2\mu \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \tag{3.1.36}$$

$$T_{12}^\varepsilon = \mu \left(\frac{\partial u^\varepsilon}{\partial z^\varepsilon} + \frac{\partial w^\varepsilon}{\partial x^\varepsilon}\right) \tag{3.1.37}$$

$$T_{22}^\varepsilon = 2\mu \frac{\partial w^\varepsilon}{\partial z^\varepsilon} \tag{3.1.38}$$

y los laplacianos en términos de T_{12}^ε (3.1.22)-(3.1.23):

$$\nu\Delta^\varepsilon u^\varepsilon = 2\nu\frac{\partial^2 u^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial T_{12}^\varepsilon}{\partial z^\varepsilon} \tag{3.1.39}$$

$$\nu\Delta^\varepsilon w^\varepsilon = \frac{1}{\rho_0}\frac{\partial T_{12}^\varepsilon}{\partial x^\varepsilon} + 2\nu\frac{\partial^2 w^\varepsilon}{\partial(z^\varepsilon)^2} \tag{3.1.40}$$

Deberíamos añadir también que el caudal es conocido en $x^\varepsilon = 0$ y $x^\varepsilon = L$ en cada instante, pero, de nuevo, como estas condiciones son impuestas de varias formas en la literatura y no es necesario explicitarlas en lo que sigue, preferimos no incluirlas de momento, aunque éstas u otras condiciones similares serán necesarias en la resolución del modelo que obtengamos finalmente.

3.2. Ecuaciones en el dominio de referencia

En la sección 2.3 se justificaron los motivos que nos llevan a elegir trabajar en un dominio de referencia independiente de ε y del tiempo, en lugar de en el dominio original. En este caso esos mismos motivos siguen siendo válidos. Recordamos el cambio de variable que se proponía en la mencionada sección, que será el que apliquemos en este caso también.

Sea $\Omega = [0, L] \times [0, 1]$ el dominio de referencia. Se supone que:

$$h^\varepsilon(t^\varepsilon, x^\varepsilon) = \varepsilon h(t, x) \tag{3.2.1}$$

$$H^\varepsilon(x^\varepsilon) = \varepsilon H(x) \tag{3.2.2}$$

(por tanto $s^\varepsilon(t^\varepsilon, x^\varepsilon) = \varepsilon s(t, x)$) y se define el siguiente cambio de variable, de Ω a Ω^ε:

$$\begin{aligned} t^\varepsilon &= t \\ x^\varepsilon &= x \\ z^\varepsilon &= \varepsilon[H(x) + z h(t, x)] \end{aligned} \tag{3.2.3}$$

Dada una función F^ε cualquiera definida en $[0, T] \times \bar{\Omega}^\varepsilon$, se puede definir a partir de ella otra función $F(\varepsilon)$ definida en $[0, T] \times \bar{\Omega}$ utilizando para ello el cambio de variable del modo siguiente: $F(\varepsilon)(t, x, z) = F^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$. La relación entre las derivadas parciales de una y otra función era la siguiente:

$$\frac{\partial F^\varepsilon}{\partial t^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial F(\varepsilon)}{\partial z} = D_t F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial x^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial x} - \frac{H' + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_x F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial z^\varepsilon} = \frac{1}{\varepsilon h}\frac{\partial F(\varepsilon)}{\partial z} = \frac{1}{\varepsilon}D_z F(\varepsilon)$$

donde hemos introducido la siguiente notación

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial}{\partial z} \\ D_x &= \frac{\partial}{\partial x} - \frac{H' + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial}{\partial z} \\ D_z &= \frac{1}{h}\frac{\partial}{\partial z} \\ D_x^2 &= D_x(D_x), \quad D_z^2 = D_z(D_z) \end{aligned} \tag{3.2.4}$$

Luego si se definen,

$$u(\varepsilon)(t,x,z) = u^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$$

$$w(\varepsilon)(t,x,z) = w^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$$

$$p(\varepsilon)(t,x,z) = p^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon)$$

$$T_{ij}(\varepsilon)(t,x,z) = T_{ij}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) \quad i,j = 1,2$$

el problema (3.1.26)-(3.1.40) se puede escribir en el dominio de referencia Ω de la forma siguiente:

- las ecuaciones de Navier-Stokes:

$$\begin{aligned} D_t u(\varepsilon) + u(\varepsilon)D_x u(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z u(\varepsilon) &= -\frac{1}{\rho_0}D_x p(\varepsilon) \\ &+ 2\nu D_x^2 u(\varepsilon) + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z T_{12}(\varepsilon) \end{aligned} \tag{3.2.5}$$

$$\begin{aligned} D_t w(\varepsilon) + u(\varepsilon)D_x w(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z w(\varepsilon) &= -\frac{1}{\rho_0}\frac{1}{\varepsilon}D_z p(\varepsilon) - g \\ &+ 2\nu\frac{1}{\varepsilon^2}D_z^2 w(\varepsilon) + \frac{1}{\rho_0}D_x T_{12}(\varepsilon) \end{aligned} \tag{3.2.6}$$

- la condición de incompresibilidad

$$D_x u(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \tag{3.2.7}$$

- las condiciones de contorno

$$p(\varepsilon) = p_s \qquad \text{en } z = 1 \tag{3.2.8}$$

$$w(\varepsilon) = \varepsilon H' u(\varepsilon) \qquad \text{en } z = 0 \tag{3.2.9}$$

$$T_{12}(\varepsilon) = \frac{f_W(\varepsilon)\left[1 + \varepsilon^2 \left(\frac{\partial s}{\partial x}\right)^2\right] + \varepsilon \frac{\partial s}{\partial x}\left[T_{11}(\varepsilon) - T_{22}(\varepsilon)\right]}{1 - \varepsilon^2 \left(\frac{\partial s}{\partial x}\right)^2} \qquad \text{en } z = 1 \tag{3.2.10}$$

$$T_{12}(\varepsilon) = \frac{f_R(\varepsilon)\left[1 + \varepsilon^2 (H')^2\right] + \varepsilon H' \left[T_{11}(\varepsilon) - T_{22}(\varepsilon)\right]}{1 - \varepsilon^2 (H')^2} \qquad \text{en } z = 0 \tag{3.2.11}$$

(a las que habría que añadir las de caudal conocido en $x = 0$ y en $x = L$)

- las condiciones iniciales:

$$u(\varepsilon)(0, x, z) = u_0(\varepsilon)(x, z), \tag{3.2.12}$$

$$w(\varepsilon)(0, x, z) = w_0(\varepsilon)(x, z) \tag{3.2.13}$$

- la ecuación que determina el calado:

$$\frac{\partial h}{\partial t} + \int_0^1 \frac{\partial (u(\varepsilon) h)}{\partial x} dz = 0 \tag{3.2.14}$$

- las componentes del tensor de tensiones:

$$T_{11}(\varepsilon) = 2\mu D_x u(\varepsilon) \tag{3.2.15}$$

$$T_{12}(\varepsilon) = \mu \left(\frac{1}{\varepsilon} D_z u(\varepsilon) + D_x w(\varepsilon) \right) \tag{3.2.16}$$

$$T_{22}(\varepsilon) = 2\mu \frac{1}{\varepsilon} D_z w(\varepsilon) \tag{3.2.17}$$

- y los laplacianos en términos de las componentes del tensor de tensiones $T_{12}(\varepsilon)$:

$$\nu \left(\frac{1}{\varepsilon^2} D_z^2 u(\varepsilon) - D_x^2 u(\varepsilon) \right) = \frac{1}{\varepsilon \rho_0} D_z T_{12}(\varepsilon) \tag{3.2.18}$$

$$\nu \left(D_x^2 w(\varepsilon) - \frac{1}{\varepsilon^2} D_z^2 w(\varepsilon) \right) = \frac{1}{\rho_0} D_x T_{12}(\varepsilon) \tag{3.2.19}$$

3.3. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (3.2.5)-(3.2.19) admite un desarrollo en serie de potencias de ε en la forma:

$$\begin{aligned}
&u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots \\
&w(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots \\
&p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots \\
&T_{11}(\varepsilon) = T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2 + \cdots \\
&T_{i2}(\varepsilon) = \varepsilon^{-1} T_{i2}^{-1} + T_{i2}^0 + \varepsilon T_{i2}^1 + \varepsilon^2 T_{i2}^2 + \cdots \quad (i = 1, 2)
\end{aligned} \tag{3.3.1}$$

Observación 3.1 *Se supone que los términos principales de $T_{i2}(\varepsilon)$ son de orden ε^{-1} porque, si se sustituyen los desarrollos asintóticos de $u(\varepsilon)$ y $w(\varepsilon)$ en* (3.2.16) *y* (3.2.17)*, ésta es la hipótesis natural.*

También se supone que la fuerza de rozamiento y del viento admiten un desarrollo en serie de potencias de ε. Para construir estas series estudiamos en primer lugar las fórmulas empíricas a partir de las que se calculan la fuerza de rozamiento y del viento.

En [101] (páginas 32-34), por ejemplo, podemos encontrar la siguiente expresión para el cálculo de la fuerza de rozamiento en el caso de ríos

$$f_R = \frac{g\rho_0 h}{C^2 R} \bar{u}^2$$

donde C es el coeficiente de Chezy ($\sqrt{m}/s$), R es el radio hidraúlico (m), y $\bar{u}$ es la velocidad media de la sección transversal. C se determina experimentalmente y su valor para ríos naturales es entre 20 y 70. Por ello se puede considerar que $\dfrac{g}{C^2}$ es de orden ε.

La fórmula de Manning nos dice que ([101]):

$$C = \frac{R^{\frac{1}{6}}}{n}$$

donde n es el coeficiente de rozamiento de Manning (normalmente se considera constante).
Sustituyendo C por esta expresión se obtiene:

$$f_R = \frac{g\rho_0 h n^2}{R^{\frac{4}{3}}} \bar{u} \, |\bar{u}|$$

donde se ha escrito $\bar{u}^2$ como $\bar{u}\,|\bar{u}|$ para expresar la dirección de la fuerza de rozamiento (opuesta a la de la velocidad, recordemos que en (3.1.7) se escribe $-f_R$).

En estudios oceanográficos ([101]), se suele usar

$$f_R = \rho_0 \gamma \bar{u}\,|\bar{u}|$$

donde γ es un coeficiente de fricción del fondo marino (un valor posible es $\gamma = 2{,}6 \times 10^{-3}$) es decir, γ se puede pensar también de orden ε.

La fórmula propuesta en [21] (páginas 108-109) para los mismos cálculos es la siguiente

$$f_R = \frac{g\rho_0}{C^2}\bar{u}\,|\bar{u}| \tag{3.3.2}$$

Estos autores toman como valor del coeficiente de Chézy $C = 64$. De nuevo, $\dfrac{g}{C^2}$ se puede considerar de orden ε.

Del mismo modo, en [101] (páginas 28-29) se propone la siguiente fórmula para el cálculo de la fuerza debida al viento

$$f_W = \rho_{aire} C_D \omega\,|\omega|$$

donde ω es la velocidad del viento 10 ó 15 m por encima de la superficie del agua y C_D es un coeficiente adimensional que da idea de la resistencia a una altura determinada (10 ó 15 m). Se puede calcular empíricamente usando alguna de las siguientes fórmulas:

1. Fórmula de Wilson (1960), la altura que se considera es 10 m:

$$C_D = (0{,}9 + 0{,}08\omega) \times 10^{-3}$$

2. Fórmula del Instituto de Ciencias Oceanográficas (Gran Bretaña) :

$$C_D = (0{,}63 + 0{,}066\omega) \times 10^{-3}$$

3. Fórmula de Garrat (1971):

$$C_D = (0{,}75 + 0{,}067\omega) \times 10^{-3}$$

4. Fórmula de Heaps (1965), la altura que se considera es 15 m:

$$\begin{cases} C_D = 0{,}565 \times 10^{-3} & (\omega < 5m/s) \\ C_D = -0{,}12 + 0{,}137w & (5m/s \le \omega \le 19{,}22m/s) \\ C_D = 2{,}513 \times 10^{-3} & (\omega > 19{,}22m/s) \end{cases}$$

En [21] (pág. 108) se propone la misma expresión

$$f_W = \rho_{aire}\gamma_{10}\omega\,|\omega| \tag{3.3.3}$$

donde $\gamma_{10} = (0{,}75 + 0{,}067\,|\omega|) \times 10^{-3}$, es decir, $\gamma_{10} = C_D$ calculado según la fórmula de Garrat, tomando $\rho_{aire} = 1{,}28 kg/m^3$ y w es la velocidad del viento 10 m por encima de la superficie del agua.

Si se sustituye en (3.3.2) f_R y $\bar{u}$ por sus desarrollos en series de potencias de ε

$$\bar{u}^\varepsilon = \frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon} u^\varepsilon\, dz^\varepsilon = \frac{1}{\varepsilon h}\int_0^1 u(\varepsilon)\,\varepsilon\, h\, dz = \int_0^1 \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots\right)\, dz$$

$$= \int_0^1 u^0\, dz + \varepsilon\int_0^1 u^1\, dz + \varepsilon^2\int_0^1 u^2\, dz + \cdots = \bar{u}^0 + \varepsilon\bar{u}^1 + \varepsilon\bar{u}^2 + \cdots$$

donde $\bar{u}^k = \displaystyle\int_0^1 u^k\, dz$, y resulta:

$$f_R^0 + \varepsilon f_R^1 + \varepsilon^2 f_R^2 + \cdots = \frac{g\rho_0}{C^2}[\bar{u}^0 + \varepsilon\bar{u}^1 + \varepsilon^2\bar{u}^2 + \cdots]\left|\bar{u}^0 + \varepsilon\bar{u}^1 + \varepsilon^2\bar{u}^2 + \cdots\right|$$

Si se denota por $f(\varepsilon) = \bar{u}^0 + \varepsilon\bar{u}^1 + \varepsilon^2\bar{u}^2 + \cdots$, entonces, existe una función $g(\varepsilon) \geq 0$ tal que $|f(\varepsilon)| = g(\varepsilon)$, por tanto, $\left(f(\varepsilon)\right)^2 = \left(g(\varepsilon)\right)^2$, es decir,

$$(g^0)^2 + 2\varepsilon g^0 g^1 + \varepsilon^2[(g^1)^2 + 2g^0g^2] + \cdots = (\bar{u}^0)^2 + 2\varepsilon\bar{u}^0\bar{u}^1 + \varepsilon^2[(\bar{u}^1)^2 + 2\bar{u}^0\bar{u}^2] + \cdots$$

por lo que, $g^0 = |\bar{u}^0|$, $g^k = \text{signo}(\bar{u}^0)\bar{u}^k \quad k \geq 1$ y, si además se tiene en cuenta que $\dfrac{g}{C^2} = \varepsilon C_1$, se obtiene:

$$\begin{cases} f_R^0 = 0 \\ f_R^1 = \rho_0 C_1 \bar{u}^0\,|\bar{u}^0| \\ f_R^2 = 2\rho_0 C_1 \bar{u}^1\,|\bar{u}^0| \\ f_R^3 = \rho_0 C_1(\text{signo}(\bar{u}^0)(\bar{u}^1)^2 + 2\bar{u}^2\,|\bar{u}^0|) \end{cases} \tag{3.3.4}$$

Análogamente, partiendo de (3.3.3) y teniendo en cuenta que $\gamma_{10} = \varepsilon C_2$

$$\begin{cases} f_W^0 = 0 \\ f_W^1 = \rho_0 C_2\,\omega^0\,|\omega^0| \\ f_W^2 = 2\rho_0 C_2\,\omega^1\,|\omega^0| \\ f_W^3 = \rho_0 C_2((\text{signo}(\bar{\omega}^0)(\omega^1)^2 + 2\omega^2\,|\omega^0|) \end{cases} \tag{3.3.5}$$

Por tanto, podemos suponer que

$$
\begin{aligned}
f_R(\varepsilon) &= \varepsilon f_R^1 + \varepsilon^2 f_R^2 + \cdots \\
f_W(\varepsilon) &= \varepsilon f_W^1 + \varepsilon^2 f_W^2 + \cdots
\end{aligned}
\tag{3.3.6}
$$

Se sustituyen, ahora, estos desarrollos en serie de potencias en las ecuaciones (3.2.5)-(3.2.19). Realizando esta sustitución en la primera ecuación de Navier-Stokes (3.2.5) se obtiene:

$$
\begin{aligned}
&D_t u^0 + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + \cdots \\
&\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\left[D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots\right] \\
&\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\frac{1}{\varepsilon}\left[D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right] \\
&\quad = -\frac{1}{\rho_0}\left(D_x p^0 + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 + \cdots\right) + 2\nu\left(D_x^2 u^0 + \varepsilon D_x^2 u^1 + \varepsilon^2 D_x^2 u^2 + \cdots\right) \\
&\quad + \frac{1}{\rho_0}\frac{1}{\varepsilon}\left(\varepsilon^{-1} D_z T_{12}^{-1} + D_z T_{12}^0 + \varepsilon D_z T_{12}^1 + \varepsilon^2 D_z T_{12}^2 + \varepsilon^3 D_z T_{12}^3 + \cdots\right)
\end{aligned}
$$

El paso siguiente consiste en identificar los términos multiplicados por la misma potencia de ε. En este caso se tiene:

$$
\begin{aligned}
&-\varepsilon^{-2}\frac{1}{\rho_0}D_z T_{12}^{-1} + \varepsilon^{-1}\left(w^0 D_z u^0 - \frac{1}{\rho_0}D_z T_{12}^0\right) \\
&\quad + \varepsilon^0\left(D_t u^0 + u^0 D_x u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0}D_x p^0 - 2\nu D_x^2 u^0 - \frac{1}{\rho_0}D_z T_{12}^1\right) \\
&\quad + \varepsilon\Big(D_t u^1 + u^0 D_x u^1 + u^1 D_x u^0 + w^0 D_z u^2 + w^1 D_z u^1 + w^2 D_z u^0 + \frac{1}{\rho_0}D_x p^1 \\
&\quad - 2\nu D_x^2 u^1 - \frac{1}{\rho_0}D_z T_{12}^2\Big) \\
&\quad + \varepsilon^2\Big(D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 D_x u^0 + w^0 D_z u^3 + w^1 D_z u^2 \\
&\quad + w^2 D_z u^1 + w^3 D_z u^0 + \frac{1}{\rho_0}D_x p^2 - 2\nu D_x^2 u^2 - \frac{1}{\rho_0}D_z T_{12}^3\Big) + O(\varepsilon^3) = 0
\end{aligned}
\tag{3.3.7}
$$

Reemplazando $u(\varepsilon)$, $w(\varepsilon)$, $p(\varepsilon)$ y $T_{ij}(\varepsilon)$, $(i, j = 1, 2)$ por sus desarrollos en serie de potencias de ε, (3.3.1), en la segunda ecuación de Navier-Stokes ((3.2.6)), ésta resulta:

$$
\begin{aligned}
&D_t w^0 + \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + \varepsilon^3 D_t w^3 + \cdots \\
&\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots) \left[D_x w^0 + \varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \varepsilon^3 D_x w^3 + \cdots\right] \\
&\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\frac{1}{\varepsilon}\left[D_z w^0 + \varepsilon D_z w^1 + \varepsilon^2 D_z w^2\right. \\
&\quad \left. + \varepsilon^3 D_z w^3 + \cdots\right] = -\frac{1}{\rho_0}\frac{1}{\varepsilon}\left(D_z p^0 + \varepsilon D_z p^1 + \varepsilon^2 D_z p^2 + \varepsilon^3 D_z p^3 + \cdots\right) - g \\
&\quad + 2\nu\frac{1}{\varepsilon^2}\left(D_z^2 w^0 + \varepsilon D_z^2 w^1 + \varepsilon^2 D_z^2 w^2 + \varepsilon^3 D_z^2 w^3 + \cdots\right) \\
&\quad + \frac{1}{\rho_0}\left(\varepsilon^{-1} D_x T_{12}^{-1} + D_x T_{12}^0 + \varepsilon D_x T_{12}^1 + \varepsilon^2 D_x T_{12}^2 + \cdots\right)
\end{aligned}
$$

Agrupando en potencias de ε se obtiene:

$$
\begin{aligned}
&-\varepsilon^{-2} 2\nu D_z^2 w^0 + \varepsilon^{-1}\left(w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2\nu D_z^2 w^1 - \frac{1}{\rho_0} D_x T_{12}^{-1}\right) \\
&\quad + \varepsilon^0 \left(D_t w^0 + u^0 D_x w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g\right. \\
&\quad \left. - 2\nu D_z^2 w^2 - \frac{1}{\rho_0} D_x T_{12}^0\right) \\
&\quad + \varepsilon\left(D_t w^1 + u^0 D_x w^1 + u^1 D_x w^0 + w^0 D_z w^2 + w^1 D_z w^1 + w^2 D_z w^0\right. \\
&\quad \left. + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} D_x T_{12}^1\right) \\
&\quad + \varepsilon^2\left(D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + u^2 D_x w^0 + w^0 D_z w^3 + w^1 D_z w^2 + w^2 D_z w^1\right. \\
&\quad \left. + w^3 D_z w^0 + \frac{1}{\rho_0} D_z p^3 - 2\nu D_z^2 w^4 - \frac{1}{\rho_0} D_x T_{12}^2\right) + O(\varepsilon^3) = 0
\end{aligned}
\tag{3.3.8}
$$

Repetimos el proceso para la ecuación de la incompresibilidad ((3.2.7)). Se realiza la sustitución y después la identificación de los términos multiplicados por cada potencia de ε

$$
\begin{aligned}
&\varepsilon^{-1} D_z w^0 + D_x u^0 + D_z w^1 + \varepsilon\left(D_x u^1 + D_z w^2\right) \\
&\quad + \varepsilon^2\left(D_x u^2 + D_z w^3\right) + O(\varepsilon^3) = 0
\end{aligned}
\tag{3.3.9}
$$

De la condición de contorno (3.2.8), suponiendo $p_s^\varepsilon = p_s$ independiente de ε, se tiene

$$p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots = p_s \text{ en } z = 1 \tag{3.3.10}$$

Del mismo modo, a partir de la condición (3.2.9) se tiene

$$w^0 + \varepsilon(w^1 - u^0 H') + \varepsilon^2(w^2 - u^1 H') + \varepsilon^3(w^3 - u^2 H') + \cdots = 0 \text{ en } z = 0 \tag{3.3.11}$$

Si se sustituyen $T_{11}(\varepsilon)$, $T_{12}(\varepsilon)$, $T_{22}(\varepsilon)$, $f_R(\varepsilon)$ y $f_W(\varepsilon)$ por sus desarrollos en serie de potencias de ε en las condiciones de contorno (3.2.10)-(3.2.11) se obtiene:

$$\left(\varepsilon^{-1}T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \cdots\right)\left[1 - \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right]$$

$$= \left\{(\varepsilon f_W^1 + \varepsilon^2 f_W^2 + \cdots)\left[1 + \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right] + \varepsilon\frac{\partial s}{\partial x}\left[T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2 + \cdots\right.\right.$$

$$\left.\left. - (\varepsilon^{-1}T_{22}^{-1} + T_{22}^0 + \varepsilon T_{22}^1 + \varepsilon^2 T_{22}^2 + \cdots)\right]\right\} \qquad \text{en } z = 1$$

$$\left(\varepsilon^{-1}T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \cdots\right)\left[1 - \varepsilon^2 (H')^2\right]$$

$$= \left\{(\varepsilon f_R^1 + \varepsilon^2 f_R^2 + \cdots)\left[1 + \varepsilon^2 (H')^2\right] + \varepsilon H'\left[T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2 + \cdots\right.\right.$$

$$\left.\left. - (\varepsilon^{-1}T_{22}^{-1} + T_{22}^0 + \varepsilon T_{22}^1 + \varepsilon^2 T_{22}^2 + \cdots)\right]\right\} \qquad \text{en } z = 0$$

agrupando en potencias de ε,

$$\varepsilon^{-1}T_{12}^{-1} + \varepsilon^0\left(T_{12}^0 + \frac{\partial s}{\partial x}T_{22}^{-1}\right)$$

$$+ \varepsilon\left[T_{12}^1 - \left(\frac{\partial s}{\partial x}\right)^2 T_{12}^{-1} - f_W^1 - \frac{\partial s}{\partial x}\left(T_{11}^0 - T_{22}^0\right)\right]$$

$$+ \varepsilon^2\left[T_{12}^2 - \left(\frac{\partial s}{\partial x}\right)^2 T_{12}^0 - f_W^2 - \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right)\right]$$

$$+ \varepsilon^3\left[T_{12}^3 - \left(\frac{\partial s}{\partial x}\right)^2 T_{12}^1 - f_W^3 - \left(\frac{\partial s}{\partial x}\right)^2 f_W^1 - \frac{\partial s}{\partial x}\left(T_{11}^2 - T_{22}^2\right)\right] + O(\varepsilon^4) = 0$$

$$\text{en } z = 1 \tag{3.3.12}$$

$$\varepsilon^{-1}T_{12}^{-1} + \varepsilon^0\left(T_{12}^0 + H'T_{22}^{-1}\right) + \varepsilon\left[T_{12}^1 - (H')^2T_{12}^{-1} - f_R^1 - H'\left(T_{11}^0 - T_{22}^0\right)\right]$$
$$+ \varepsilon^2\left[T_{12}^2 - (H')^2T_{12}^0 - f_R^2 - H'\left(T_{11}^1 - T_{22}^1\right)\right]$$
$$+ \varepsilon^3\left[T_{12}^3 - (H')^2T_{12}^1 - f_R^3 - (H')^2f_R^1 - H'\left(T_{11}^2 - T_{22}^2\right)\right] + O(\varepsilon^4) = 0$$
$$\text{en } z = 0 \quad (3.3.13)$$

A partir de la ecuación (3.2.14) necesaria para la determinación del calado, sustituyendo $u(\varepsilon)$ por su desarrollo en serie de potencias de ε se obtiene:

$$\frac{\partial h}{\partial t} + \int_0^1 \left(\frac{\partial(hu^0)}{\partial x} + \varepsilon\frac{\partial(hu^1)}{\partial x} + \varepsilon^2\frac{\partial(hu^2)}{\partial x} + \cdots\right) dz = 0 \quad (3.3.14)$$

Se sustituyen, ahora, los desarrollos en serie de potencias en las expresiones de las componentes del tensor de tensiones en función de las derivadas de u y w y de p ((3.2.15)-(3.2.17))

$$T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2 + \cdots = 2\mu\left(D_xu^0 + \varepsilon D_xu^1 + \varepsilon^2 D_xu^2 + \cdots\right)$$

$$\varepsilon^{-1}T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \cdots = \mu\Bigg[\frac{1}{\varepsilon}\left(D_zu^0 + \varepsilon D_zu^1 + \varepsilon^2 D_zu^2 + \cdots\right)$$
$$+ D_xw^0 + \varepsilon D_xw^1 + \varepsilon^2 D_xw^2 + \cdots\Bigg]$$

$$\varepsilon^{-1}T_{22}^{-1} + T_{22}^0 + \varepsilon T_{22}^1 + \varepsilon^2 T_{22}^2 + \cdots = 2\mu\frac{1}{\varepsilon}\left(D_zw^0 + \varepsilon D_zw^1 + \varepsilon^2 D_zw^2\right)$$

Identificando los términos multiplicados por cada potencia de ε se tiene:

$$T_{11}^0 - 2\mu D_xu^0 + \varepsilon\left(T_{11}^1 - 2\mu D_xu^1\right) + \varepsilon^2\left(T_{11}^2 - 2\mu D_xu^2\right) = O(\varepsilon^3) \quad (3.3.15)$$

$$\varepsilon^{-1}\left(T_{12}^{-1} - \mu D_zu^0\right) + T_{12}^0 - \mu\left(D_zu^1 + D_xw^0\right)$$
$$+ \varepsilon\left[T_{12}^1 - \mu\left(D_zu^2 + D_xw^1\right)\right] = O(\varepsilon^2) \quad (3.3.16)$$

$$\varepsilon^{-1}\left(T_{22}^{-1} - 2\mu D_zw^0\right) + T_{22}^0 - 2\mu D_zw^1 + \varepsilon\left(T_{22}^1 - 2\mu D_zw^2\right)$$
$$+ \varepsilon^2\left(T_{22}^2 - 2\mu D_zw^3\right) = O(\varepsilon^3) \quad (3.3.17)$$

Reemplazando $u(\varepsilon)$, $w(\varepsilon)$ y $T_{12}(\varepsilon)$ por sus desarrollos en serie de potencias de ε, (3.3.1), en las ecuaciones (3.2.18) y (3.2.19) se obtiene

$$\nu\left[\frac{1}{\varepsilon^2}\left(D_z^2u^0+\varepsilon D_z^2u^1+\varepsilon^2 D_z^2u^2+\cdots\right)-\left(D_x^2u^0+\varepsilon D_x^2u^1+\varepsilon^2 D_x^2u^2+\cdots\right)\right]$$

$$=\frac{1}{\varepsilon\rho_0}\left(\varepsilon^{-1}D_zT_{12}^{-1}+D_zT_{12}^0+\varepsilon D_zT_{12}^1+\varepsilon^2 D_zT_{12}^2+\cdots\right)$$

$$\nu\left[D_x^2w^0+\varepsilon D_x^2w^1+\varepsilon^2 D_x^2w^2+\cdots-\frac{1}{\varepsilon^2}\left(D_z^2w^0+\varepsilon D_z^2w^1+\varepsilon^2 D_z^2w^2+\cdots\right)\right]$$

$$=\frac{1}{\rho_0}\left(\varepsilon^{-1}D_xT_{12}^{-1}+D_xT_{12}^0+\varepsilon D_xT_{12}^1+\varepsilon^2 D_xT_{12}^2+\cdots\right)$$

agrupando en potencias de ε y teniendo en cuenta que $\nu=\dfrac{\mu}{\rho_0}$ resulta:

$$\varepsilon^{-2}\left(\mu D_z^2u^0-D_zT_{12}^{-1}\right)+\varepsilon^{-1}\left(\mu D_z^2u^1-D_zT_{12}^0\right)$$
$$+\mu\left(D_z^2u^2-D_x^2u^0\right)-D_zT_{12}^1+O(\varepsilon)=0 \qquad (3.3.18)$$

$$\varepsilon^{-2}\mu D_z^2w^0+\varepsilon^{-1}\left(\mu D_z^2w^1+D_xT_{12}^{-1}\right)+\mu\left(D_z^2w^2-D_x^2w^0\right)+D_xT_{12}^0$$
$$+\varepsilon\left[\mu\left(D_z^2w^3-D_x^2w^1\right)+D_xT_{12}^1\right]+O(\varepsilon^2)=0 \qquad (3.3.19)$$

Razonando igual que en la sección 2.5, como u^0, w^0, p^0, T_{i2}^{-1} $(i=1,2)$, T_{ij}^0 $(i,j=1,2)$, u^1, w^1, etc. son independientes de ε, una vez agrupados los términos que multiplican a una misma potencia de ε, en las ecuaciones anteriores obtenemos un polinomio en ε igualado a cero, por lo que sus coeficientes han de ser nulos. De este modo se logra una serie de ecuaciones que nos permitirán determinar los términos u^0, w^0, p^0, T_{i2}^{-1} $\quad(i=1,2)$, T_{ij}^0 $\quad(i,j=1,2)$, u^1, w^1, etc.

Comenzamos por los coeficientes de ε^{-2} que aparecen en (3.3.7), (3.3.8), (3.3.18) y (3.3.19):

$$D_zT_{12}^{-1}=0 \qquad (3.3.20)$$

$$D_z^2w^0=0 \qquad (3.3.21)$$

$$\mu D_z^2u^0-D_zT_{12}^{-1}=0 \qquad (3.3.22)$$

Igualando a cero los coeficientes de ε^{-1} que aparecen en (3.3.7)-(3.3.9), (3.3.12), (3.3.13), (3.3.16)-(3.3.19) tenemos las siguientes igualdades:

$$w^0D_zu^0-\frac{1}{\rho_0}D_zT_{12}^0=0 \qquad (3.3.23)$$

$$w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2\nu D_z^2 w^1 - \frac{1}{\rho_0} D_x T_{12}^{-1} = 0 \tag{3.3.24}$$

$$D_z w^0 = 0 \tag{3.3.25}$$

$$T_{12}^{-1} = 0 \quad \text{en } z = 1 \tag{3.3.26}$$

$$T_{12}^{-1} = 0 \quad \text{en } z = 0 \tag{3.3.27}$$

$$T_{12}^{-1} - \mu D_z u^0 = 0 \tag{3.3.28}$$

$$T_{22}^{-1} - 2\mu D_z w^0 = 0 \tag{3.3.29}$$

$$\mu D_z^2 u^1 - D_z T_{12}^0 = 0 \tag{3.3.30}$$

$$\mu D_z^2 w^1 + D_x T_{12}^{-1} = 0 \tag{3.3.31}$$

Como consecuencia de (3.3.20) y (3.3.27) sabemos que

$$T_{12}^{-1} = 0 \tag{3.3.32}$$

lo que implica por la igualdad (3.3.28) que u^0 no depende de z

$$D_z u^0 = 0 \tag{3.3.33}$$

Sustituyendo $D_z w^0 = 0$ ((3.3.25)) en (3.3.29) obtenemos

$$T_{22}^{-1} = 0 \tag{3.3.34}$$

Continuamos igualando a cero los términos que multiplican a ε^0 en las ecuaciones (3.3.7)-(3.3.19), se tienen en cuenta a la hora de escribir estos términos que $D_z u^0 = 0$ y $T_{12}^{-1} = T_{22}^{-1} = 0$

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + w^0 D_z u^1 + \frac{1}{\rho_0} D_x p^0 - 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^1 = 0 \tag{3.3.35}$$

$$\begin{aligned} &D_t w^0 + u^0 D_x w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g \\ &\quad - 2\nu D_z^2 w^2 - \frac{1}{\rho_0} D_x T_{12}^0 = 0 \end{aligned} \tag{3.3.36}$$

$$\frac{\partial u^0}{\partial x} + D_z w^1 = 0 \tag{3.3.37}$$

$$p^0 = p_s \quad \text{en } z = 1 \tag{3.3.38}$$

$$w^0 = 0 \quad \text{en } z = 0 \tag{3.3.39}$$

$$T_{12}^0 = 0 \quad \text{en } z = 1 \tag{3.3.40}$$

$$T_{12}^0 = 0 \quad \text{en } z = 0 \tag{3.3.41}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} = 0 \tag{3.3.42}$$

$$T_{11}^0 - 2\mu \frac{\partial u^0}{\partial x} = 0 \tag{3.3.43}$$

$$T_{12}^0 - \mu \left(D_z u^1 + D_x w^0 \right) = 0 \tag{3.3.44}$$

$$T_{22}^0 - 2\mu D_z w^1 = 0 \tag{3.3.45}$$

$$\mu \left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} \right) - D_z T_{12}^1 = 0 \tag{3.3.46}$$

$$\mu \left(D_z^2 w^2 - D_x^2 w^0 \right) + D_x T_{12}^0 = 0 \tag{3.3.47}$$

Veamos las consecuencias que se pueden extraer de las igualdades anteriores. En primer lugar, usando (3.3.25) y (3.3.39) se deduce que

$$w^0 = 0 \tag{3.3.48}$$

Ahora, (3.3.23) se reduce a $D_z T_{12}^0 = 0$. Teniendo en cuenta (3.3.41), podemos escribir

$$T_{12}^0 = 0 \tag{3.3.49}$$

Como $w^0 = T_{12}^0 = 0$, la igualdad (3.3.44) resulta

$$D_z u^1 = 0 \tag{3.3.50}$$

es decir, el término u^1 tampoco depende de z.

La ecuación (3.3.31) por ser $T_{12}^{-1} = 0$ nos dice que

$$D_z^2 w^1 = 0 \tag{3.3.51}$$

y entonces, por (3.3.32), (3.3.48) y (3.3.51), de la expresión (3.3.24) se obtiene

$$D_z p^0 = 0 \tag{3.3.52}$$

que junto con la condición de contorno (3.3.38) nos permite obtener

$$p^0 = p_s \tag{3.3.53}$$

Teniendo en cuenta las igualdades (3.3.33), (3.3.48)-(3.3.50) y (3.3.53) se pueden reescribir las ecuaciones (3.3.35), (3.3.36), (3.3.42), (3.3.46) y (3.3.47) de la forma siguiente:

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^1 = 0 \qquad (3.3.54)$$

$$\frac{1}{\rho_0} D_z p^1 + g - 2\nu D_z^2 w^2 = 0 \qquad (3.3.55)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} = 0 \qquad (3.3.56)$$

$$\mu \left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} \right) - D_z T_{12}^1 = 0 \qquad (3.3.57)$$

$$D_z^2 w^2 = 0 \qquad (3.3.58)$$

Las ecuaciones (3.3.54) y (3.3.56) se utilizarán para calcular u^0 y h.

Una vez más, repetimos el proceso e igualamos, ahora, a cero los coeficientes de ε que aparecen en (3.3.7)-(3.3.17) y (3.3.19). Se tiene en cuenta que $T_{12}^{-1} = 0$ ((3.3.32)), $D_z u^0 = D_z u^1 = 0$ ((3.3.33),(3.3.50)) y $w^0 = 0$ (3.3.48) para simplificar las igualdades, y se obtiene:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} D_x p^1 - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^2 = 0 \qquad (3.3.59)$$

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} D_x T_{12}^1 = 0 \qquad (3.3.60)$$

$$\frac{\partial u^1}{\partial x} + D_z w^2 = 0 \qquad (3.3.61)$$

$$p^1 = 0 \quad \text{en } z = 1 \qquad (3.3.62)$$

$$w^1 - u^0 H' = 0 \quad \text{en } z = 0 \qquad (3.3.63)$$

$$T_{12}^1 = f_W^1 + \frac{\partial s}{\partial x} \left(T_{11}^0 - T_{22}^0 \right) \quad \text{en } z = 1 \qquad (3.3.64)$$

$$T_{12}^1 = f_R^1 + H' \left(T_{11}^0 - T_{22}^0 \right) \quad \text{en } z = 0 \qquad (3.3.65)$$

$$\frac{\partial (hu^1)}{\partial x} = 0 \qquad (3.3.66)$$

$$T_{11}^1 = 2\mu \frac{\partial u^1}{\partial x} \qquad (3.3.67)$$

$$T_{12}^1 = \mu \left(D_z u^2 + D_x w^1 \right) \qquad (3.3.68)$$

$$T_{22}^1 = 2\mu D_z w^2 \tag{3.3.69}$$

$$\mu\left(D_z^2 w^3 - D_x^2 w^1\right) + D_x T_{12}^1 = 0 \tag{3.3.70}$$

La ecuación (3.3.55), si tenemos en cuenta (3.3.58), se reduce a

$$\frac{1}{\rho_0} D_z p^1 + g = 0$$

Integrando respecto a z esta igualdad e imponiendo la condición (3.3.62), nos proporciona la siguiente expresión para el término de orden 1 de la presión:

$$p^1 = \rho_0 g h(1 - z) \tag{3.3.71}$$

Sustituyendo $D_x p^1$ en la ecuación (3.3.59) (que utilizaremos para calcular u^1) resulta:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g\frac{\partial s}{\partial x} - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^2 = 0 \tag{3.3.72}$$

Integramos también (3.3.37) respecto a z teniendo en cuenta que u^0 no depende de z, e imponiendo la condición (3.3.63), encontramos la expresión siguiente para w^1 en términos de u^0 y H:

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x} z \tag{3.3.73}$$

Igualando a cero los coeficientes de ε^2 que aparecen en (3.3.7)-(3.3.15) y (3.3.17) y usando (3.3.33), (3.3.48)-(3.3.50) tenemos las siguientes igualdades:

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + \frac{1}{\rho_0} D_x p^2$$
$$- 2\nu D_x^2 u^2 - \frac{1}{\rho_0} D_z T_{12}^3 = 0 \tag{3.3.74}$$

$$D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1 + \frac{1}{\rho_0} D_z p^3$$
$$- 2\nu D_z^2 w^4 - \frac{1}{\rho_0} D_x T_{12}^2 = 0 \tag{3.3.75}$$

$$D_x u^2 + D_z w^3 = 0 \tag{3.3.76}$$

$$p^2 = 0 \quad \text{en } z = 1 \tag{3.3.77}$$

$$w^2 - u^1 H' = 0 \quad \text{en } z = 0 \tag{3.3.78}$$

$$T_{12}^2 = f_W^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right) \qquad \text{en } z = 1 \tag{3.3.79}$$

$$T_{12}^2 = f_R^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right) \qquad \text{en } z = 0 \tag{3.3.80}$$

$$\int_0^1 \frac{\partial(hu^2)}{\partial x} dz = 0 \tag{3.3.81}$$

$$T_{11}^2 = 2\mu D_x u^2 \tag{3.3.82}$$

$$T_{22}^2 = 2\mu D_z w^3 \tag{3.3.83}$$

Procediendo del mismo modo que para obtener la expresión (3.3.73) obtenemos también w^2 en función de u^1 y H' (en este caso se integra (3.3.61) y se impone la condición (3.3.78)):

$$w^2 = u^1 H' - h\frac{\partial u^1}{\partial x} z \tag{3.3.84}$$

En resumen, tenemos las siguientes ecuaciones, igualdades y condiciones para el cálculo de h, T_{12}^{-1}, T_{22}^{-1}, u^k, w^k, p^k y $T_{i,j}^k$ $(k = 0, 1, 2, \cdots, i, j = 1, 2)$ que nos permitirán construir una aproximación de la solución del problema (3.2.5)-(3.2.14):

$$T_{12}^{-1} = 0 \tag{3.3.85}$$

$$D_z u^0 = 0 \tag{3.3.86}$$

$$T_{22}^{-1} = 0 \tag{3.3.87}$$

$$w^0 = 0 \tag{3.3.88}$$

$$T_{12}^0 = 0 \tag{3.3.89}$$

$$D_z u^1 = 0 \tag{3.3.90}$$

$$p^0 = p_s \tag{3.3.91}$$

$$\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\nu\frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^1 = 0 \tag{3.3.92}$$

$$p^1 = \rho_0 h g(1 - z) \tag{3.3.93}$$

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x} z \tag{3.3.94}$$

$$\frac{\partial h}{\partial t} + \frac{\partial(hu^0)}{\partial x} = 0 \tag{3.3.95}$$

$$T_{11}^0 = 2\mu \frac{\partial u^0}{\partial x} \tag{3.3.96}$$

$$T_{22}^0 = 2\mu D_z w^1 \tag{3.3.97}$$

$$\mu \left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} \right) = D_z T_{12}^1 \tag{3.3.98}$$

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g \frac{\partial s}{\partial x} - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^2 = 0 \tag{3.3.99}$$

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} D_x T_{12}^1 = 0 \tag{3.3.100}$$

$$w^2 = u^1 H' - h \frac{\partial u^1}{\partial x} z \tag{3.3.101}$$

$$T_{12}^1 = f_W^1 + \frac{\partial s}{\partial x} \left(T_{11}^0 - T_{22}^0 \right) \qquad \text{en } z = 1 \tag{3.3.102}$$

$$T_{12}^1 = f_R^1 + H' \left(T_{11}^0 - T_{22}^0 \right) \qquad \text{en } z = 0 \tag{3.3.103}$$

$$\frac{\partial (h u^1)}{\partial x} = 0 \tag{3.3.104}$$

$$T_{11}^1 = 2\mu \frac{\partial u^1}{\partial x} \tag{3.3.105}$$

$$T_{12}^1 = \mu \left(D_z u^2 + D_x w^1 \right) \tag{3.3.106}$$

$$T_{22}^1 = 2\mu D_z w^2 \tag{3.3.107}$$

$$\mu \left(D_z^2 w^3 - D_x^2 w^1 \right) + D_x T_{12}^1 = 0 \tag{3.3.108}$$

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + \frac{1}{\rho_0} D_x p^2$$
$$- 2\nu D_x^2 u^2 - \frac{1}{\rho_0} D_z T_{12}^3 = 0 \tag{3.3.109}$$

$$D_t w^2 + u^0 D_x w^2 + u^1 D_x w^1 + w^1 D_z w^2 + w^2 D_z w^1 + \frac{1}{\rho_0} D_z p^3$$
$$- 2\nu D_z^2 w^4 - \frac{1}{\rho_0} D_x T_{12}^2 = 0 \tag{3.3.110}$$

$$D_x u^2 + D_z w^3 = 0 \tag{3.3.111}$$

$$p^2 = 0 \quad \text{en } z = 1 \tag{3.3.112}$$

$$T_{12}^2 = f_W^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right) \quad \text{en } z = 1 \tag{3.3.113}$$

$$T_{12}^2 = f_R^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right) \quad \text{en } z = 0 \tag{3.3.114}$$

$$\int_0^1 \frac{\partial(hu^2)}{\partial x} dz = 0 \tag{3.3.115}$$

$$T_{11}^2 = 2\mu D_x u^2 \tag{3.3.116}$$

$$T_{22}^2 = 2\mu D_z w^3 \tag{3.3.117}$$

3.4. Aproximación de orden cero

Se considera la aproximación de orden cero:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

$$T_{11}(\varepsilon) = T_{11}^0$$

$$T_{12}(\varepsilon) = \varepsilon^{-1} T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1$$

$$T_{22}(\varepsilon) = \varepsilon^{-1} T_{22}^{-1} + T_{22}^0$$

$$f_R(\varepsilon) = \varepsilon f_R^1$$

$$f_W(\varepsilon) = \varepsilon f_W^1$$

donde w^0, p^0, $T_{12}^k \quad (k = -1, 0)$ y T_{22}^{-1} son conocidos ((3.3.88), (3.3.91), (3.3.85), (3.3.89), (3.3.87)).

Como ya se anunció más arriba, u^0 se calcula a partir de (3.3.92)

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^1 = 0 \tag{3.4.1}$$

donde deberíamos conocer previamente T_{12}^1. Para obtener este término se podría emplear la ecuación (3.3.106) pero u^2 y w^1 no son conocidos. El término T_{12}^1 sólo lo podemos obtener con esta aproximación en $z = 0$ y $z = 1$. Para solventar este inconveniente se integra la ecuación respecto de z entre 0 y 1. Se aprovecha el hecho de que u^0 y p^0 no dependen de z ((3.3.86), (3.3.91)). Y así,

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0}\left(T_{12}^1\big|_{z=1} - T_{12}^1\big|_{z=0} \right)$$

se utilizan ahora las ecuaciones (3.3.102) y (3.3.103) para obtener T_{12}^1 en $z = 1$ y $z = 0$:

$$T_{12}^1 = f_W^1 + \frac{\partial s}{\partial x}\left(T_{11}^0 - T_{22}^0 \right) \quad \text{en } z = 1 \tag{3.4.2}$$

$$T_{12}^1 = f_R^1 + H'\left(T_{11}^0 - T_{22}^0 \right) \quad \text{en } z = 0 \tag{3.4.3}$$

donde es necesario conocer T_{11}^0 y T_{22}^0 en $z = 0$ y $z = 1$. Esta información nos la proporcionan las igualdades (3.3.96) y (3.3.97) (donde hemos hecho la sustitución $\mu = \nu\rho_0$):

$$T_{11}^0 = 2\nu\rho_0 \frac{\partial u^0}{\partial x} \tag{3.4.4}$$

$$T_{22}^0 = 2\frac{\nu\rho_0}{h}\frac{\partial w^1}{\partial z} \tag{3.4.5}$$

y w^1 se puede sustituir por su expresión en función de u^0 ((3.3.94)) de modo que T_{22}^0 se exprese también en términos de u^0:

$$T_{22}^0 = -2\nu\rho_0 \frac{\partial u^0}{\partial x} \tag{3.4.6}$$

Entonces, como u^0 no depende de z:

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0}\left(f_W^1 - f_R^1 + 4\nu\rho_0 \frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right)$$

que se puede escribir:

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u^0}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right) = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{\rho_0 h}\left(f_W^1 - f_R^1 \right) \tag{3.4.7}$$

El calado se obtiene resolviendo la ecuación (3.3.95)

$$\frac{\partial h}{\partial t} + \frac{\partial (h u^0)}{\partial x} = 0$$

Para resolver el sistema de ecuaciones (3.3.95) y (3.4.7) suponemos conocidos $u^0(0, x)$, $h(0, x)$ y, por fijar ideas, $u^0(t, 0)$ y $h(t, 0)$, aunque otras elecciones son posibles, de aquí en adelante cuando condiciones de este tipo sean necesarias se supondrán conocidas.

Una vez conocidos u^0 y h, w^1 viene dado, como acabamos de utilizar, por (3.3.94)

$$w^1 = u^0 H' - h\frac{\partial u^0}{\partial x}z$$

Si, ahora, deshacemos el cambio de variable, volviendo al dominio original, la aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, z) = u^0(t, x)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, z) = \varepsilon w^1(t, x, z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, z) = p^0(t, x)$$

con

$$\tilde{f}_R^\varepsilon = \tilde{f}_R(\varepsilon) = \varepsilon f_R^1$$

$$\tilde{f}_W^\varepsilon = \tilde{f}_W(\varepsilon) = \varepsilon f_W^1$$

verifica,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{3.4.8}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{3.4.9}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{3.4.10}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{3.4.11}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu\left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right) = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right) \tag{3.4.12}$$

Veamos en qué medida verifica la aproximación de orden cero las ecuaciones de Navier-Stokes de partida.

Si sustituimos en la primera ecuación de Navier-Stokes obtenemos

$$\begin{aligned}
\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} &= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} \\
&= 2\nu\left(\frac{\partial^2 u^0}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x}\right) - \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{\rho_0 h}\left(f_W^1 - f_R^1\right) \\
&= -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\nu \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{4\nu}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right)
\end{aligned} \tag{3.4.13}$$

de modo que la primera ecuación de Navier-Stokes (3.1.26) se verificaría de forma exacta si

$$\frac{4\nu}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right) = \frac{1}{\rho_0}\frac{\partial \tilde{T}_{12}^\varepsilon}{\partial z^\varepsilon} \tag{3.4.14}$$

donde llamamos $\tilde{T}_{12}^\varepsilon = \varepsilon T_{12}^1$. Puesto que T_{12}^1 no es conocido para todo z, no podemos garantizar que se verifique (3.4.14), pero si lo integramos respecto a z^ε (teniendo en cuenta que $\tilde{u}^\varepsilon$, h^ε, $\tilde{f}_W^\varepsilon$, $\tilde{f}_R^\varepsilon$ no dependen de z^ε) obtenemos:

$$\frac{1}{\rho_0}\left(\tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=H^\varepsilon}\right) = \frac{4\nu}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right) \tag{3.4.15}$$

que se verifica sin más que tener en cuenta (3.4.2)-(3.4.6).

Por tanto, no podemos garantizar que nuestro modelo verifique de forma exacta la primera ecuación de Navier-Stokes, pero sí que lo hace con el promedio en altura de dicha ecuación, es decir, se verifica de forma exacta que

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\nu\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{h^\varepsilon \rho_0}\left(\tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=H^\varepsilon}\right)$$

La segunda ecuación de Navier-Stokes se verifica con un error de orden ε^0, es decir, no se verifica ni tan siquiera aproximadamente. Mientras que la condición de incompresibilidad se verifica exactamente, como se deduce de (3.4.9):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = 0$$

Es inmediato comprobar que las condiciones de contorno (3.1.4) y (3.1.5) se verifican exactamente (teniendo en cuenta (3.4.9) y (3.4.10)).

Las condiciones de contorno (3.1.31)-(3.1.32), que recogen el efecto del viento en la superficie y el rozamiento en el fondo, se verifican con un orden de precisión de ε^3, ya que si tenemos en cuenta (3.3.102) y (3.3.103), obtenemos (donde $\tilde{T}_{11}^\varepsilon = T_{11}^0$ y $\tilde{T}_{22}^\varepsilon = T_{22}^0$)

$$\left[\left(1 - \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}_{12}^\varepsilon - \left(1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}_W^\varepsilon - \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}_{11}^\varepsilon - \tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon=s^\varepsilon}$$

$$= \left[\left(1 - \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{T}_{12}(\varepsilon) - \left(1 + \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{f}_W(\varepsilon)\right.$$

$$\left. - \varepsilon\frac{\partial s}{\partial x}\left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon)\right)\right]_{z=1}$$

$$= \left[1 - \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right]\varepsilon\, T_{12}^1\big|_{z=1} - \left[1 + \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right]\varepsilon f_W^1 - \varepsilon\frac{\partial s}{\partial x}\left(T_{11}^0 - T_{22}^0\right)\big|_{z=1}$$

$$= \varepsilon \left[T_{12}^1 - f_W^1 - \frac{\partial s}{\partial x} \left(T_{11}^0 - T_{22}^0 \right) \right]_{z=1} + O(\varepsilon^3) = O(\varepsilon^3)$$

$$\left[\left(1 - \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right)^2 \right) \tilde{T}_{12}^\varepsilon - \left(1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right)^2 \right) \tilde{f}_R^\varepsilon - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\tilde{T}_{11}^\varepsilon - \tilde{T}_{22}^\varepsilon \right) \right]_{z^\varepsilon = H^\varepsilon}$$

$$= \left[\left(1 - \varepsilon^2 (H')^2 \right) T_{12}(\varepsilon) - \left(1 + \varepsilon^2 (H')^2 \right) \tilde{f}_R(\varepsilon) - \varepsilon H' \left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon) \right) \right]_{z=0}$$

$$= \left[1 - \varepsilon^2 (H')^2 \right] \varepsilon \, T_{12}^1 \big|_{z=0} - \left[1 + \varepsilon^2 (H')^2 \right] \varepsilon f_R^1 - \varepsilon H' \left(T_{11}^0 - T_{22}^0 \right) \big|_{z=0}$$

$$= \varepsilon \left[T_{12}^1 - f_R^1 - H' \left(T_{11}^0 - T_{22}^0 \right) \right]_{z=0} + O(\varepsilon^3) = O(\varepsilon^3)$$

Se propone un primer modelo que viene dado por las ecuaciones (3.4.8)-(3.4.12):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{3.4.16}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right) \tag{3.4.17}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{3.4.18}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{3.4.19}$$

donde $\tilde{u}^\varepsilon$ no depende de z^ε.

Observación 3.2 *El término*

$$2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right)$$

puede escribirse

$$\frac{2\nu}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right)$$

de modo que la ecuación (3.4.17) *resulta*

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \frac{2\nu}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right) \tag{3.4.20}$$

3.5. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{T}_{11}(\varepsilon) = T_{11}^0 + \varepsilon T_{11}^1$$

$$\tilde{T}_{12}(\varepsilon) = \varepsilon^{-1} T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2$$

$$\tilde{T}_{22}(\varepsilon) = \varepsilon^{-1} T_{22}^{-1} + T_{22}^0 + \varepsilon T_{22}^1$$

$$\tilde{f}_R(\varepsilon) = \varepsilon f_R^1 + \varepsilon^2 f_R^2$$

$$\tilde{f}_W(\varepsilon) = \varepsilon f_W^1 + \varepsilon^2 f_W^2$$

Recordemos que w^0, p^0, $T_{12}^k \quad (k=-1,0)$ y T_{22}^{-1} son conocidos ((3.3.88), (3.3.91), (3.3.85), (3.3.89), (3.3.87)), u^0 y h se calculan resolviendo (3.4.7) y (3.3.95) y w^1 está determinado por (3.3.94) en función de u^0 y h.

También tenemos una expresión para p^1 en la que sólo es necesario conocer la profundidad del agua ((3.3.93)):

$$p^1 = \rho_0 g h (1 - z)$$

Para obtener u^1 resolvemos (3.3.99):

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g \frac{\partial s}{\partial x} - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T_{12}^2 = 0$$

De nuevo, T_{12}^2 sólo es conocido en $z = 0$ y $z = 1$. Como u^0, u^1 y s no dependen de z, se integra la ecuación respecto de z entre 0 y 1:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} = -g \frac{\partial s}{\partial x} + 2\nu \frac{\partial^2 u^1}{\partial x^2} + \frac{1}{h\rho_0} \left(T_{12}^2\big|_{z=1} - T_{12}^2\big|_{z=0} \right)$$

Se usan ahora las ecuaciones (3.3.113) y (3.3.114) para obtener T_{12}^2 en 0 y 1:

$$T_{12}^2 = f_W^2 + \frac{\partial s}{\partial x} \left(T_{11}^1 - T_{22}^1 \right) \quad \text{en } z = 1 \tag{3.5.1}$$

$$T_{12}^2 = f_R^2 + H' \left(T_{11}^1 - T_{22}^1 \right) \quad \text{en } z = 0 \tag{3.5.2}$$

donde es necesario conocer T_{11}^1 y T_{22}^1 en $z = 0$ y $z = 1$. Para ello se usan las ecuaciones (3.3.105), (3.3.107) y (3.3.101):

$$T_{11}^1 = 2\nu\rho_0 \frac{\partial u^1}{\partial x} \tag{3.5.3}$$

$$T_{22}^1 = 2\frac{\nu\rho_0}{h}\frac{\partial w^2}{\partial z} = -2\nu\rho_0 \frac{\partial u^1}{\partial x} \tag{3.5.4}$$

Entonces, obtenemos

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} = -g\frac{\partial s}{\partial x} + 2\nu \frac{\partial^2 u^1}{\partial x^2} + \frac{1}{h\rho_0}\left(f_W^2 - f_R^2 + \frac{\partial h}{\partial x} 4\nu\rho_0 \frac{\partial u^1}{\partial x}\right)$$

que podemos escribir:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} - 2\nu\left(\frac{\partial^2 u^1}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x}\right) = -g\frac{\partial s}{\partial x} + \frac{1}{h\rho_0}\left(f_W^2 - f_R^2\right) \tag{3.5.5}$$

A continuación, w^2 viene dado por (3.3.101):

$$w^2 = u^1 H' - h\frac{\partial u^1}{\partial x} z$$

Ahora, usando (3.3.91) y (3.3.93) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 g h(1 - z) \tag{3.5.6}$$

De igual modo, por (3.3.88), (3.3.94) y (3.3.101), sabemos que:

$$\begin{aligned} \tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 &= \varepsilon\left(u^0 H' - h\frac{\partial u^0}{\partial x} z\right) + \varepsilon^2\left(u^1 H' - h\frac{\partial u^1}{\partial x} z\right) \\ &= (u^0 + \varepsilon u^1)\varepsilon H' - \frac{\partial (u^0 + \varepsilon u^1)}{\partial x}\varepsilon h z = \tilde{u}(\varepsilon)\varepsilon H' - \frac{\partial \tilde{u}(\varepsilon)}{\partial x}\varepsilon h z \end{aligned} \tag{3.5.7}$$

Se deshace el cambio de variable y, así, se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, z) = u^0(t, x) + \varepsilon u^1(t, x)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, z) = \varepsilon w^1(t, x, z) + \varepsilon^2 w^2(t, x, z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, z) = p^0(t, x) + \varepsilon p^1(t, x, z)$$

Definimos también

$$\tilde{f}_R^\varepsilon = \tilde{f}_R(\varepsilon) = \varepsilon f_R^1 + \varepsilon^2 f_R^2$$

$$\tilde{f}_W^\varepsilon = \tilde{f}_W(\varepsilon) = \varepsilon f_W^1 + \varepsilon^2 f_W^2$$

$$\tilde{T}_{11}^\varepsilon = \tilde{T}_{11}(\varepsilon) = T_{11}^0 + \varepsilon T_{11}^1$$

$$\tilde{T}_{12}^\varepsilon = \tilde{T}_{12}(\varepsilon) = \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2$$

$$\tilde{T}_{22}^\varepsilon = \tilde{T}_{22}(\varepsilon) = T_{22}^0 + \varepsilon T_{22}^1$$

Según lo visto en (3.3.86) y (3.3.90)

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{3.5.8}$$

Si se realiza el cambio de variable en (3.5.6), obtenemos la aproximación de la presión en Ω^ε:

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g(s^\varepsilon - z^\varepsilon) \tag{3.5.9}$$

Análogamente, deshaciendo el cambio de variable en (3.5.7), se logra la aproximación de la componente vertical de la velocidad:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{3.5.10}$$

Teniendo en cuenta (3.4.7), (3.5.5) y (3.3.95) obtenemos que las ecuaciones que verifican $\tilde{u}^\varepsilon$ y h^ε y que permiten su cálculo son las siguientes:

$$\begin{aligned}
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) \\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} \right) \\
&\quad - 2\nu \left(\frac{\partial^2 u^0}{\partial x^2} + \varepsilon \frac{\partial^2 u^1}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} + \varepsilon \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} \right) \\
&\quad = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u^0}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right) \\
&\quad + \varepsilon \left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} - 2\nu \left(\frac{\partial^2 u^1}{\partial x^2} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} \right) \right] + \varepsilon^2 u^1 \frac{\partial u^1}{\partial x} \\
&\quad = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0}\left(f_W^1 - f_R^1 \right) + \varepsilon \left[-g\frac{\partial s}{\partial x} + \frac{1}{h\rho_0}\left(f_W^2 - f_R^2 \right) \right] + \varepsilon^2 u^1 \frac{\partial u^1}{\partial x}
\end{aligned}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right) + O(\varepsilon^2)$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = \varepsilon\frac{\partial h}{\partial t} + \frac{\partial(u^0+\varepsilon u^1)}{\partial x}\varepsilon h + (u^0+\varepsilon u^1)\varepsilon\frac{\partial h}{\partial x}$$

$$= \varepsilon\left[\frac{\partial h}{\partial t} + \frac{\partial u^0}{\partial x}h + u^0\frac{\partial h}{\partial x} + \varepsilon\left(\frac{\partial u^1}{\partial x}h + u^1\frac{\partial h}{\partial x}\right)\right]$$

$$= \varepsilon\left(\frac{\partial h}{\partial t} + \frac{\partial(u^0 h)}{\partial x}\right) + \varepsilon^2\frac{\partial(u^1 h)}{\partial x} = O(\varepsilon^2)$$

(ya que no hemos exigido en ningún momento que se verifique (3.3.104)), lo que implica que,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu\left(\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)$$

$$= -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right) + O(\varepsilon^2) \tag{3.5.11}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{3.5.12}$$

Veamos en qué medida verifica la aproximación de primer orden las ecuaciones de Navier-Stokes de partida:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} - \frac{1}{\rho_0}\frac{\partial \tilde{T}_{12}^\varepsilon}{\partial z^\varepsilon} = 0$$

Al igual que pasaba en la sección anterior, no podemos evaluar directamente esta ecuación ya que no conocemos el término $\tilde{T}_{12}^\varepsilon$. Si promediamos la ecuación en altura, obtenemos

$$\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} - \frac{1}{\rho_0}\frac{\partial \tilde{T}_{12}^\varepsilon}{\partial z^\varepsilon}\right) dz^\varepsilon$$

$$= \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + g\frac{\partial s^\varepsilon}{\partial x^\varepsilon} - 2\nu\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} - \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}_{12}^\varepsilon\Big|_{z^\varepsilon=H^\varepsilon}\right)$$

$$= \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} - \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon + 4\nu\rho_0\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)$$

$$= \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu\left(\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right) - \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon\right)$$

$$= O(\varepsilon^2)$$

sin más que tener en cuenta (3.5.8), (3.5.9), (3.4.2)-(3.4.6), (3.5.1)-(3.5.4) y (3.5.11).

Por tanto, la primera ecuación de Navier-Stokes, promediada en altura se verifica con un error de orden ε^2.

Para la segunda ecuación de Navier-Stokes se tiene, usando (3.3.93), que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - \frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{12}}{\partial x^\varepsilon} - 2\nu \frac{\partial^2 \tilde{w}^\varepsilon}{\partial (z^\varepsilon)^2}$$

$$= D_t\tilde{w}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon) + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z\tilde{p}(\varepsilon) + g$$

$$- \frac{1}{\rho_0}D_x\tilde{T}_{12}(\varepsilon) - \frac{2\nu}{\varepsilon^2}D_z^2\tilde{w}(\varepsilon)$$

$$= \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + (u^0 + \varepsilon u^1)\left[\varepsilon D_x w^1 + \varepsilon^2 D_x w^2\right]$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2)\frac{1}{\varepsilon}\left[\varepsilon D_z w^1 + \varepsilon^2 D_z w^2\right] + \frac{1}{\rho_0}D_z p^1 + g$$

$$- \frac{1}{\rho_0}\left(\varepsilon D_x T^1_{12} + \varepsilon^2 D_x T^2_{12}\right) - \frac{2\nu}{\varepsilon^2}\left(\varepsilon D_z^2 w^1 + \varepsilon^2 D_z^2 w^2\right) = O(\varepsilon)$$

La aproximación de primer orden verifica la segunda ecuación de Navier-Stokes con un error $O(\varepsilon)$.

La ecuación de la incompresibilidad se verifica de forma exacta como se ve utilizando (3.3.94) y (3.3.101)

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = \frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon) = \frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + D_z w^1 + \varepsilon D_z w^2 = 0,$$

o directamente por (3.5.10).

Lo mismo sucede con las condiciones de contorno (3.1.29) y (3.1.30), teniendo en cuenta (3.5.9) y (3.5.10).

Veamos qué sucede con las condiciones de contorno que recogen el efecto del viento en la superficie y el rozamiento en el fondo. Teniendo en cuenta (3.3.102), (3.3.103), (3.3.113) y (3.3.114), se obtiene que:

$$\left[\left(1 - \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}^\varepsilon_{12} - \left(1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}^\varepsilon_W - \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}^\varepsilon_{11} - \tilde{T}^\varepsilon_{22}\right)\right]_{z^\varepsilon = s^\varepsilon}$$

$$= \left[\left(1 - \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{T}_{12}(\varepsilon) - \left(1 + \varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{f}_W(\varepsilon)\right.$$

$$\left. - \varepsilon\frac{\partial s}{\partial x}\left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon)\right)\right]_{z=1}$$

$$= \left(\varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 - \varepsilon f_W^1 - \varepsilon^2 f_W^2\right)\Big|_{z=1} - \varepsilon \frac{\partial s}{\partial x} \left(T_{11}^0 + \varepsilon T_{11}^1 - T_{22}^0 - \varepsilon T_{22}^1\right)\Big|_{z=1}$$

$$+ O(\varepsilon^3) = \varepsilon \left[T_{12}^1 - f_W^1 - \frac{\partial s}{\partial x} \left(T_{11}^0 - T_{22}^0\right)\right]_{z=1}$$

$$+ \varepsilon^2 \left[T_{12}^2 - f_W^2 - \frac{\partial s}{\partial x} \left(T_{11}^1 - T_{22}^1\right)\right]_{z=1} + O(\varepsilon^3) = O(\varepsilon^3)$$

$$\left[\left(1 - \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right) \tilde{T}_{12}^\varepsilon - \left(1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right) \tilde{f}_R^\varepsilon - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\tilde{T}_{11}^\varepsilon - \tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon = H^\varepsilon}$$

$$= \left[\left(1 - \varepsilon^2 (H')^2\right) \tilde{T}_{12}(\varepsilon) - \left(1 + \varepsilon^2 (H')^2\right) \tilde{f}_R(\varepsilon) - \varepsilon H' \left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon)\right)\right]_{z=0}$$

$$= \left(\varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 - \varepsilon f_R^1 - \varepsilon^2 f_R^2\right)\Big|_{z=0} - \varepsilon H' \left(T_{11}^0 + \varepsilon T_{11}^1 - T_{22}^0 - \varepsilon T_{22}^1\right)\Big|_{z=0}$$

$$+ O(\varepsilon^3) = \varepsilon \left[T_{12}^1 - f_R^1 - H' \left(T_{11}^0 - T_{22}^0\right)\right]_{z=0}$$

$$+ \varepsilon^2 \left[T_{12}^2 - f_R^2 - H' \left(T_{11}^1 - T_{22}^1\right)\right]_{z=0} + O(\varepsilon^3) = O(\varepsilon^3)$$

es decir, las condiciones de contorno (3.1.31) y (3.1.32) se verifican con una precisión de $O(\varepsilon^3)$.

Si en (3.5.11) y (3.5.12) se desprecian los términos de orden $O(\varepsilon^2)$ se obtiene el siguiente modelo de aguas someras:

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right) \\
&\qquad = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon\right) \\
&\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon) \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}
\end{aligned} \tag{3.5.13}$$

donde $\tilde{u}^\varepsilon$ no depende de z^ε.

Si se comparan otros modelos de aguas someras con éste, se puede apreciar que la diferencia está esencialmente en el término de viscosidad, como veremos en la sección 6.1.

3.6. Aproximación de segundo orden

Se considera la aproximación de segundo orden:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{T}_{11}(\varepsilon) = T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2$$

$$\tilde{T}_{12}(\varepsilon) = \varepsilon^{-1} T_{12}^{-1} + T_{12}^0 + \varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \varepsilon^3 T_{12}^3$$

$$\tilde{T}_{22}(\varepsilon) = \varepsilon^{-1} T_{22}^{-1} + T_{22}^0 + \varepsilon T_{22}^1 + \varepsilon^2 T_{22}^2$$

$$\tilde{f}_R(\varepsilon) = \varepsilon f_R^1 + \varepsilon^2 f_R^2 + \varepsilon^3 f_W^3$$

$$\tilde{f}_W(\varepsilon) = \varepsilon f_W^1 + \varepsilon^2 f_W^2 + \varepsilon^3 f_W^3$$

Los términos w^0, p^0, $T_{12}^k \quad (k = -1, 0)$ y T_{22}^{-1}, u^0, h, w^1, p^1, u^1 y w^2 se calculan del mismo modo que en la sección anterior para la aproximación de primer orden a partir de (3.3.85)-(3.3.95), (3.3.99) y (3.3.101).

Buscamos, ahora, p^2, para ello partimos de la ecuación (3.3.100):

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} D_x T_{12}^1 = 0 \tag{3.6.1}$$

Comenzamos por despejar $D_z^2 w^3$ de la expresión (3.3.108)

$$\nu D_z^2 w^3 = \nu D_x^2 w^1 - \frac{1}{\rho_0} D_x T_{12}^1$$

y lo sustituimos en (3.6.1)

$$D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_x^2 w^1 + \frac{1}{\rho_0} D_x T_{12}^1 = 0 \tag{3.6.2}$$

A continuación, como u^0 es conocido, calculamos T_{12}^1 a partir de (3.3.92):

$$\frac{1}{h\rho_0} \frac{\partial T_{12}^1}{\partial z} = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2}$$

Integrando respecto de z,

$$T_{12}^1 = \rho_0 h z \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} \right) + T_{12}^1\big|_{z=0}$$

Finalmente, si se sustituye $T_{12}^1|_{z=0}$ según lo visto en (3.4.3)-(3.4.6), T_{12}^1 resulta:

$$T_{12}^1 = \rho_0 \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} \right) hz + f_R^1 + 4\nu \rho_0 H' \frac{\partial u^0}{\partial x} \tag{3.6.3}$$

Sustituyendo ahora T_{12}^1 por la expresión anterior y w^1 por (3.3.94) en (3.6.2) resulta

$$\begin{aligned} &\frac{\partial u^0}{\partial t} H' - hz \frac{\partial^2 u^0}{\partial t \partial x} + u^0 \left(u^0 H'' + 2 \frac{\partial u^0}{\partial x} H' - h \frac{\partial^2 u^0}{\partial x^2} z \right) - \left(u^0 H' - h \frac{\partial u^0}{\partial x} z \right) \frac{\partial u^0}{\partial x} \\ &+ \frac{1}{\rho_0} D_z p^2 - 2\nu \left[3 \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} H' \right) + u^0 H''' - zh \frac{\partial^3 u^0}{\partial x^3} \right] \\ &+ \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} \right) hz + \frac{1}{\rho_0} \frac{\partial f_R^1}{\partial x} + 4\nu \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} H' \right) \\ &- H' \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} \right) = 0 \end{aligned}$$

de donde se puede despejar

$$\begin{aligned} \frac{1}{\rho_0 h} \frac{\partial p^2}{\partial z} = &-H''(u^0)^2 - \frac{1}{\rho_0} \frac{\partial f_R^1}{\partial x} + 2\nu \frac{\partial}{\partial x} (H'' u^0) + H' \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \\ &- \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right] hz \end{aligned} \tag{3.6.4}$$

Ahora se integra respecto a z aprovechando que ni H, ni u^0, ni p_s, ni f_R^1 dependen de z e imponiendo la condición de contorno $p^2(t, x, 1) = 0$ ((3.3.112)). Se obtiene la siguiente expresión para p^2:

$$\begin{aligned} p^2 = \rho_0 \Bigg\{ &\left[H''(u^0)^2 + \frac{1}{\rho_0} \frac{\partial f_R^1}{\partial x} - 2\nu \frac{\partial}{\partial x} (H'' u^0) - H' \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right] h(1 - z) + \\ &+ \frac{1}{2} \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right] h^2 (1 - z^2) \Bigg\} \end{aligned} \tag{3.6.5}$$

A continuación calculamos u^2 a partir de (3.3.106):

$$T_{12}^1 = \mu \left(D_z u^2 + D_x w^1 \right)$$

en donde sustituimos w^1 y T_{12}^1 por sus expresiones dadas en (3.3.94) y (3.6.3), y despejamos $D_z u^2$, resultando:

$$\begin{aligned} D_z u^2 = \frac{1}{\nu} &\left[\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} \right) hz + \frac{1}{\rho_0} f_R^1 + 4\nu H' \frac{\partial u^0}{\partial x} \right] \\ &- \frac{\partial}{\partial x} \left(H' u^0 - h \frac{\partial u^0}{\partial x} z \right) - \left(H' + z \frac{\partial h}{\partial x} \right) \frac{\partial u^0}{\partial x} \end{aligned}$$

Simplificando e integrando esta ecuación se obtiene la siguiente expresión para u^2:

$$u^2 = u_0^2 + \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0\right) hz$$
$$+ \frac{1}{2\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^2 z^2 \qquad (3.6.6)$$

donde $u_0^2(t,x) = u^2(t,x,0)$ está determinado por (3.3.109)

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 D_z u^2 + \frac{1}{\rho_0} D_x p^2$$
$$- 2\nu D_x^2 u^2 - \frac{1}{\rho_0} D_z T_{12}^3 = 0$$

donde T_{12}^3 sólo es conocido en $z = 0$ y $z = 1$. Por tanto, la ecuación se integrará respecto de z entre 0 y 1. Para ello es necesario conocer explícitamente la dependencia de z de los diferentes términos que intervienen en la ecuación. Es por eso que se sustituye el valor de u^2 obtenido en (3.6.6) y el de p^2 obtenido en (3.6.5), además de w^1 según (3.3.94), y la ecuación anterior se puede escribir:

$$\frac{\partial u_0^2}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0\right) hz$$
$$+ \frac{1}{2\nu}\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^2 z^2$$
$$+ u^0 \left\{\frac{\partial u_0^2}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0\right) hz\right.$$
$$+ \frac{1}{2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^2 z^2$$
$$\left. -H' \left[\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 + \frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) hz\right]\right\}$$
$$+ u^1 \frac{\partial u^1}{\partial x} + \left[u_0^2 + \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0\right) hz\right.$$
$$\left. + \frac{1}{2\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^2 z^2\right] \frac{\partial u^0}{\partial x}$$
$$+ \left(H' u^0 - h \frac{\partial u^0}{\partial x} z\right) \left[\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0\right.$$

$$+\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p^0}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)hz\Bigg]$$

$$+\frac{\partial}{\partial x}\left(H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)h(1-z)$$

$$+\left(H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)\frac{\partial h}{\partial x}$$

$$+\frac{1}{2}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]h^2(1-z^2)+\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]h\frac{\partial h}{\partial x}$$

$$+H'\Bigg\{H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}$$

$$+\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]hz\Bigg\}$$

$$-2\nu\Bigg\{\frac{\partial^2 u_0^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)hz$$

$$+\frac{1}{2\nu}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)h^2z^2$$

$$-H''\left[\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0+\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)hz\right]$$

$$-2H'\Bigg[\frac{\partial}{\partial x}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)$$

$$+\frac{1}{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)hz\Bigg]$$

$$+(H')^2\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\Bigg\}-\frac{1}{\rho_0}D_zT_{12}^3=0$$

Simplificamos, agrupamos en potencias de z e integramos respecto de z entre 0 y 1, obteniendo:

$$\frac{\partial u_0^2}{\partial t}+u^0\frac{\partial u_0^2}{\partial x}+u^1\frac{\partial u^1}{\partial x}+u_0^2\frac{\partial u^0}{\partial x}$$

$$+\frac{\partial}{\partial x}\left[\left(H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)h\right]$$

$$+\frac{1}{2}\frac{\partial}{\partial x}\left\{\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]h^2\right\}$$

$$+H'\left(H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)$$

$$-2\nu\left[\frac{\partial^2 u_0^2}{\partial x^2}-H''\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)\right.$$

$$-2H'\frac{\partial}{\partial x}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)$$

$$\left.+(H')^2\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\right]$$

$$+\frac{1}{2}h\left\{\frac{\partial}{\partial t}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)+u^0\frac{\partial}{\partial x}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)\right.$$

$$-\frac{\partial}{\partial x}\left(H''(u^0)^2+\frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x}-2\nu\frac{\partial}{\partial x}(H''u^0)-H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)$$

$$+H'\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]-2\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)\right.$$

$$-H''\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)$$

$$\left.\left.-2H'\frac{1}{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\right]\right\}$$

$$+\frac{1}{6}h^2\left\{\frac{1}{\nu}\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\right.$$

$$+u^0\frac{1}{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)$$

$$-\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\frac{\partial u^0}{\partial x}-\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right]$$

$$\left.-2\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)\right\}-\frac{1}{\rho_0 h}\left[T_{12}^3\right]_0^1=0$$

Calculamos ahora, $[T^3_{12}]^1_0 = T^3_{12}|_{z=1} - T^3_{12}|_{z=0}$, para lo que recurrimos a las expresiones (3.3.12) y (3.3.13). Igualando a 0 los términos multiplicados por ε^3 se obtiene:

$$T^3_{12} = f^3_W + \frac{\partial s}{\partial x}\left[T^2_{11} - T^2_{22}\right] + \left(\frac{\partial s}{\partial x}\right)^2 \left[T^1_{12} + f^1_W\right] \quad \text{en } z = 1 \tag{3.6.7}$$

$$T^3_{12} = f^3_R + H'\left[T^2_{11} - T^2_{22}\right] + (H')^2 \left[T^1_{12} + f^1_R\right] \quad \text{en } z = 0 \tag{3.6.8}$$

Usamos las expresiones (3.3.111), (3.3.116) y (3.3.117) para escribir T^2_{11} y T^2_{22} en función de u^2:

$$T^2_{11} = 2\mu D_x u^2 \tag{3.6.9}$$

$$T^2_{22} = -2\mu D_x u^2 \tag{3.6.10}$$

En cuanto a T^1_{12}, se procede igual que se hizo en (3.4.2)-(3.4.6) a partir de (3.3.102) y (3.3.103). Así, si en (3.6.7) y (3.6.8) tenemos en cuenta (3.4.2)-(3.4.6) y (3.6.9)-(3.6.10) resulta:

$$T^3_{12} = f^3_W + 4\nu\rho_0 \frac{\partial s}{\partial x} D_x u^2 + \left(\frac{\partial s}{\partial x}\right)^2 \left[2f^1_W + 4\nu\rho_0 \frac{\partial s}{\partial x}\frac{\partial u^0}{\partial x}\right] \quad \text{en } z = 1 \tag{3.6.11}$$

$$T^3_{12} = f^3_R + 4\nu\rho_0 H' D_x u^2 + (H')^2 \left[2f^1_R + 4\nu\rho_0 H' \frac{\partial u^0}{\partial x}\right] \quad \text{en } z = 0 \tag{3.6.12}$$

A continuación, u^2 se reemplaza por su expresión dada en (3.6.6) y se tiene que

$$\begin{aligned}
\left[T^3_{12}\right]^1_0 = f^3_W - f^3_R + 4\nu\rho_0 \Bigg\{ & \frac{\partial h}{\partial x}\left[\frac{\partial u^2_0}{\partial x} - H'\left(\frac{1}{\rho_0\nu} f^1_R + 2H'\frac{\partial u^0}{\partial x} - H'' u^0\right)\right] \\
&+ \frac{\partial s}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{1}{\rho_0\nu} f^1_R + 2H'\frac{\partial u^0}{\partial x} - H'' u^0\right) h\right. \\
&+ \frac{1}{2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu\frac{\partial^2 u^0}{\partial x^2}\right) h^2 \\
&\left. - H'\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu\frac{\partial^2 u^0}{\partial x^2}\right) h\right]\Bigg\} \\
&+ \left(\frac{\partial s}{\partial x}\right)^2\left[2f^1_W + 4\nu\rho_0\frac{\partial s}{\partial x}\frac{\partial u^0}{\partial x}\right] - (H')^2\left[2f^1_R + 4\nu\rho_0 H'\frac{\partial u^0}{\partial x}\right]
\end{aligned}$$

La ecuación para el cálculo de u_0^2 resulta

$$
\begin{aligned}
&\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u_0^2}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u_0^2}{\partial x} \right) \\
&\quad + \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right] \frac{\partial h}{\partial x} h \\
&\quad + \frac{\partial s}{\partial x} \left(H''(u^0)^2 + \frac{1}{\rho_0} \frac{\partial f_R^1}{\partial x} - 2\nu \frac{\partial}{\partial x}(H'' u^0) - H' \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right) \\
&\quad + 2\nu \left[H'' \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) - 2 \frac{\partial h}{\partial x} \frac{\partial}{\partial x} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) \right. \\
&\quad \left. - (H')^2 \frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right] \\
&\quad + \frac{1}{2} h \left\{ \frac{\partial}{\partial t} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) + u^0 \frac{\partial}{\partial x} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) \right. \\
&\quad + \frac{\partial}{\partial x} \left(H''(u^0)^2 + \frac{1}{\rho_0} \frac{\partial f_R^1}{\partial x} - 2\nu \frac{\partial}{\partial x}(H'' u^0) - H' \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right) \\
&\quad + H' \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right] - 2\nu \left[\frac{\partial^2}{\partial x^2} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) \right. \\
&\quad - H'' \frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \\
&\quad \left. \left. - 2 \frac{\partial h}{\partial x} \frac{1}{\nu} \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right] \right\} \\
&\quad + \frac{1}{6} h^2 \left\{ \frac{1}{\nu} \frac{\partial}{\partial t} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right. \\
&\quad + u^0 \frac{1}{\nu} \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \\
&\quad - \frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \frac{\partial u^0}{\partial x} + 2 \frac{\partial}{\partial x} \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right] \\
&\quad \left. - 2 \frac{\partial^2}{\partial x^2} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right\} - \frac{1}{h \rho_0} \left(f_W^3 - f_R^3 \right)
\end{aligned}
$$

$$+\frac{4\nu}{h}\frac{\partial h}{\partial x}H'\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)$$

$$+4\frac{\partial s}{\partial x}H'\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)$$

$$-\frac{2}{h\rho_0}\left[\left(\frac{\partial s}{\partial x}\right)^2 f_W^1-(H')^2 f_R^1\right]-\frac{4\nu}{h}\left[\left(\frac{\partial s}{\partial x}\right)^3-(H')^3\right]\frac{\partial u^0}{\partial x}=0$$

Se simplifica la ecuación anterior utilizando (3.4.7) y (3.3.95), y se obtiene:

$$\frac{\partial u_0^2}{\partial t}+u^0\frac{\partial u_0^2}{\partial x}+u^1\frac{\partial u^1}{\partial x}+u_0^2\frac{\partial u^0}{\partial x}-2\nu\left(\frac{\partial^2 u_0^2}{\partial x^2}+\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u_0^2}{\partial x}\right)$$

$$=\frac{h}{2}H''\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+2\nu\frac{\partial^2 u^0}{\partial x^2}-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\frac{2}{\rho_0 h}\left(f_R^1+f_W^1\right)\right)$$

$$-hH'\left(\frac{4\nu}{h}\frac{\partial^2 h}{\partial x^2}\frac{\partial u^0}{\partial x}-\frac{1}{h^2\rho_0}\frac{\partial}{\partial x}[h(f_W^1-f_R^1)]\right)+hH'''(u^0)^2$$

$$-\frac{2\nu}{h}\frac{\partial}{\partial x}(h^2u^0H''')+\frac{2\nu}{h^2}H''u^0\frac{\partial}{\partial x}\left(h^2H'\right)-\frac{2\nu}{h^2}\frac{\partial}{\partial x}\left(h^2(H')^2\frac{\partial u^0}{\partial x}\right)$$

$$+\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x}\right)^2\right]+\frac{1}{3\rho_0}\frac{\partial}{\partial x}\left(h^2\frac{\partial^2 p_s}{\partial x^2}\right)$$

$$-\frac{1}{6\nu\rho_0}h\left[u^0\left(\frac{\partial f_W^1}{\partial x}+2\frac{\partial f_R^1}{\partial x}\right)+\left(\frac{\partial f_W^1}{\partial t}+2\frac{\partial f_R^1}{\partial t}\right)\right]$$

$$+\frac{2\nu}{3}\left\{\frac{\partial^2}{\partial x^2}\left(h\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x}\right)+\frac{\partial h}{\partial x}\left[\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u^0}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x}\right)+\frac{\partial h}{\partial x}\frac{\partial^2 u^0}{\partial x^2}\right]\right\}$$

$$+\frac{1}{6}h\left[\frac{6}{h^2\rho_0}\left(\frac{\partial h}{\partial x}\right)^2(f_W^1+f_R^1)-\frac{1}{h\rho_0}\frac{\partial^2 h}{\partial x^2}(f_W^1-f_R^1)\right.$$

$$\left.+\frac{6}{h\rho_0}\frac{\partial h}{\partial x}\left(\frac{\partial f_W^1}{\partial x}+3\frac{\partial f_R^1}{\partial x}\right)+\frac{1}{\rho_0}\left(\frac{\partial^2 f_W^1}{\partial x^2}+8\frac{\partial^2 f_R^1}{\partial x^2}\right)\right]$$

$$-\frac{1}{\rho_0}\left.D_xp^2\right|_{z=0}+\frac{1}{\rho_0 h}\left(f_W^3-f_R^3\right) \tag{3.6.13}$$

Una vez calculado u^2, w^3 se calcula a partir de la ecuación (3.3.111):

$$D_xu^2+D_zw^3=0$$

donde se sustituye la expresión obtenida para u^2 en (3.6.6)

$$D_z w^3 = -\frac{\partial u_0^2}{\partial x} - \frac{\partial}{\partial x}\left(\frac{1}{\rho_0 \nu} f_R^1 + 2H'\frac{\partial u^0}{\partial x} - H'' u^0\right) hz$$
$$- \frac{1}{2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^2 z^2$$
$$+ H'\left[\frac{1}{\rho_0 \nu} f_R^1 + 2H'\frac{\partial u^0}{\partial x} - H'' u^0 + \frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) hz\right]$$

y se integra respecto de z imponiendo la condición de contorno (3.3.11), que teniendo en cuenta (3.6.6), resulta ser:

$$w^3 = u_0^2 H' \text{ en } z = 0$$

De este modo, w^3 se puede escribir en función de u_0^2 y u^0 como sigue:

$$w^3 = u_0^2 H' + \left[-\frac{\partial u_0^2}{\partial x} + H'\left(\frac{1}{\rho_0 \nu} f_R^1 + 2H'\frac{\partial u^0}{\partial x} - H'' u^0\right)\right] hz$$
$$+ \frac{1}{2}\left[-\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 \nu} f_R^1 + 2H'\frac{\partial u^0}{\partial x} - H'' u^0\right)\right.$$
$$\left. + H'\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right)\right] h^2 z^2$$
$$- \frac{1}{6}\frac{1}{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2}\right) h^3 z^3 \tag{3.6.14}$$

Usando las expresiones encontradas para p^0, p^1 y p^2, (3.3.91), (3.3.93) y (3.6.5) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$
$$= p_s + \varepsilon \rho_0 h(1-z)\left[g + \varepsilon\left(H''(u^0)^2 + \frac{1}{\rho_0}\frac{\partial f_R^1}{\partial x} - 2\nu\frac{\partial}{\partial x}(H'' u^0) - H'\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)\right]$$
$$+ \varepsilon^2 \frac{\rho_0 h^2}{2}(1-z^2)\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right] \tag{3.6.15}$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0, 1, 2, 3)$ ((3.3.88), (3.3.94), (3.3.101) y (3.6.14)) obtenemos una aproximación de la velocidad vertical en Ω

$$\begin{aligned}
\tilde{w}(\varepsilon) &= \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 \\
&= \varepsilon \left(u^0 H' - h \frac{\partial u^0}{\partial x} z \right) + \varepsilon^2 \left(u^1 H' - h \frac{\partial u^1}{\partial x} z \right) \\
&+ \varepsilon^3 \left\{ u_0^2 H' + zh \left[H' \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) - \frac{\partial u_0^2}{\partial x} \right] \right. \\
&- \frac{1}{2} z^2 h^2 \left[\frac{\partial}{\partial x} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) \right. \\
&\left. - H' \frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right] \\
&\left. - \frac{1}{6\nu} z^3 h^3 \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right\} \\
&= \left\{ u^0 + \varepsilon u^1 + \varepsilon^2 \left[u_0^2 + zh \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) + \right. \right. \\
&\left. \left. + \frac{1}{2} \frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) z^2 h^2 \right) \right\} \varepsilon H' \\
&- \varepsilon z h \left\{ \frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \left[\frac{\partial u_0^2}{\partial x} + \frac{1}{2} z h \frac{\partial}{\partial x} \left(\frac{1}{\rho_0 \nu} f_R^1 + 2H' \frac{\partial u^0}{\partial x} - H'' u^0 \right) \right. \right. \\
&\left. \left. - \frac{1}{6} \frac{1}{\nu} z^2 h^2 \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \nu \frac{\partial^2 u^0}{\partial x^2} \right) \right] \right\}
\end{aligned}$$

que se puede escribir

$$\tilde{w}(\varepsilon) = \varepsilon H' \tilde{u}(\varepsilon) - \varepsilon z h D_x \tilde{u}(\varepsilon) + O(\varepsilon^3) \tag{3.6.16}$$

Deshacemos ahora el cambio de variable, volviendo al dominio original, y obtenemos la siguiente aproximación de la solución en Ω^ε:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, z) = u^0(t, x) + \varepsilon u^1(t, x) + \varepsilon^2 u^2(t, x, z) \tag{3.6.17}$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, z) = \varepsilon w^1(t, x, z) + \varepsilon^2 w^2(t, x, z) + \varepsilon^3 w^3(t, x, z) \tag{3.6.18}$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, z) = p^0(t, x) + \varepsilon p^1(t, x, z) + \varepsilon^2 p^2(t, x, z) \tag{3.6.19}$$

$$\tilde{f}_R^\varepsilon = \tilde{f}_R(\varepsilon) = \varepsilon f_R^1 + \varepsilon^2 f_R^2 + \varepsilon^3 f_R^3 \tag{3.6.20}$$

$$\tilde{f}_W^\varepsilon = \tilde{f}_W(\varepsilon) = \varepsilon f_W^1 + \varepsilon^2 f_W^2 + \varepsilon^3 f_W^3 \tag{3.6.21}$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (3.6.15):

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \left(u^{0,\varepsilon}\right)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon} - 2\nu \frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} u^{0,\varepsilon}\right)\right.$$
$$\left. - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right] + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2}\right] \quad (3.6.22)$$

donde $u^{0,\varepsilon} = u^0$, $f_R^{1,\varepsilon} = \varepsilon f_R^1$. La presión también se puede escribir en función de $\tilde{u}^\varepsilon$ y $\tilde{f}_R^\varepsilon$ en lugar de $u^{0,\varepsilon}$ y $f_R^{1,\varepsilon}$:

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + (\tilde{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial \tilde{f}_R^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} \tilde{u}^\varepsilon\right)\right.$$
$$\left. - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right] + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2}\right] + O(\varepsilon^3)$$
$$(3.6.23)$$

De forma análoga, realizando el cambio de variable de Ω a Ω^ε en (3.6.16), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}(z^\varepsilon - H^\varepsilon) + O(\varepsilon^3) \quad (3.6.24)$$

Para el cálculo de la velocidad horizontal y el calado tenemos varias posibilidades. En primer lugar, se tiene el modelo que resulta de forma natural de aplicar el método de desarrollos asintóticos y que consiste en calcular u^0, h, u^1 y u_0^2 (en este orden pues, u^0 es necesario para el cálculo de los restantes, h para el cálculo de u^1 y u_0^2 y finalmente u^1 para el cálculo de u_0^2) resolviendo las ecuaciones (3.4.7), (3.3.95), (3.5.5) y (3.6.13). A continuación se construye u^2 según la expresión (3.6.6), y finalmente la aproximación de la velocidad horizontal en el dominio original resulta:

$$\tilde{u}^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

mientras que la profundidad del agua se obtiene deshaciendo el cambio de variable, como $h^\varepsilon = \varepsilon h$.

Tanto la aproximación de la presión como la de la velocidad vertical se podrían construir también así, es decir, obteniendo $p^k \quad (k = 0, 1, 2)$ y $w^k \quad (k = 0, 1, 2, 3)$ y después usando las expresiones (3.6.18) y (3.6.19).

Este esquema que acabamos de señalar es, sin embargo, demasiado complicado. Requiere resolver varias veces ecuaciones similares para aportar una pequeña mejora (al menos desde el punto de vista formal) al resultado final. Sería deseable obtener un modelo similar al conseguido para la aproximación de orden uno (ver (3.5.13)), pero añadiendo algún término que mejorase la precisión. Para ello obsérvese que, considerando las aproximaciones de segundo orden, tenemos ahora:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) + \frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0 h^\varepsilon} \left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon \right)$$

$$= D_t \tilde{u}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{u}(\varepsilon) - 2\nu \left(D_x^2 \tilde{u}(\varepsilon) + \frac{2}{h} \frac{\partial h}{\partial x} D_x \tilde{u}(\varepsilon) \right)$$

$$+ \frac{1}{\rho_0} D_x \tilde{p}(\varepsilon) - \frac{1}{\rho_0 \varepsilon h} \left(\tilde{f}(\varepsilon)_W - \tilde{f}(\varepsilon)_R \right)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 D_t u^2 + \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 \right) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2 \right)$$

$$- 2\nu \left[\frac{\partial^2 u^0}{\partial x^2} + \varepsilon \frac{\partial^2 u^1}{\partial x^2} + \varepsilon^2 D_x^2 u^2 + \frac{2}{h} \frac{\partial h}{\partial x} \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2 \right) \right]$$

$$+ \frac{1}{\rho_0} \left(\frac{\partial p_s}{\partial x} + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 \right) - \frac{1}{\rho_0 h} \left(f_W^1 + \varepsilon f_W^2 + \varepsilon^2 f_W^3 - f_R^1 - \varepsilon f_R^2 - \varepsilon f_R^3 \right)$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u^0}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^0}{\partial x} \right) + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \frac{1}{h \rho_0} \left(f_W^1 - f_R^1 \right)$$

$$+ \varepsilon \left[\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u^1}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^1}{\partial x} \right) + g \frac{\partial s}{\partial x} - \frac{1}{h \rho_0} \left(f_W^2 - f_R^2 \right) \right]$$

$$+ O(\varepsilon^2)$$

Teniendo en cuenta las ecuaciones (3.4.7) y (3.5.5), resulta

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) + \frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0 h^\varepsilon} \left(\tilde{f}_W^\varepsilon - \tilde{f}_R^\varepsilon \right) = O(\varepsilon^2) \tag{3.6.25}$$

Buscamos del mismo modo una ecuación para el cálculo del calado:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = \varepsilon \frac{\partial h}{\partial t} + \varepsilon D_x \left(\tilde{u}(\varepsilon) h \right)$$

$$= \varepsilon \frac{\partial h}{\partial t} + \varepsilon \left[\frac{\partial (u^0 h)}{\partial x} + \varepsilon \frac{\partial (u^1 h)}{\partial x} + \varepsilon^2 D_x (u^2 h) \right]$$

$$= \varepsilon \left(\frac{\partial h}{\partial t} + \frac{\partial (u^0 h)}{\partial x} \right) + \varepsilon^2 \frac{\partial (u^1 h)}{\partial x} + \varepsilon^3 D_x (u^2 h)$$

y por (3.3.95):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = O(\varepsilon^2) \tag{3.6.26}$$

A continuación se propone un modelo de aguas someras resultado de despreciar los términos en $O(\varepsilon^2)$ en la ecuaciones anteriores y los términos de orden ε^3 de (3.6.23) y (3.6.24) obteniéndose:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{3.6.27}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right) \tag{3.6.28}$$

$$\begin{aligned} \tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \Bigg[g + (\tilde{u}^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0} \frac{\partial f_R^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} \tilde{u}^\varepsilon \right) \\ - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} \Bigg] + \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} \right] \end{aligned} \tag{3.6.29}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} (z^\varepsilon - H^\varepsilon) \tag{3.6.30}$$

Observación 3.3 *En la ecuación (3.6.28) el segundo miembro no es conocido pues la expresión utilizada para la presión ((3.6.29)) depende de $\tilde{u}^\varepsilon$ y h^ε. Si se desea evitar esto se debe emplear para el cálculo de $\tilde{p}^\varepsilon$ la expresión (3.6.22) y calcular previamente u^0 y h^ε. En este caso el modelo sería:*

$$\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} + 2 \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon \rho_0} \left(f_W^{1,\varepsilon} - f_R^{1,\varepsilon} \right)$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(h^\varepsilon u^{0,\varepsilon})}{\partial x^\varepsilon} = 0$$

$$\begin{aligned} \tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \Bigg[g + \left(u^{0,\varepsilon}\right)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{\varepsilon}{\rho_0} \frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon} - 2\nu \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} u^{0,\varepsilon} \right) \\ - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} \Bigg] + \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} \right] \end{aligned}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} (z^\varepsilon - H^\varepsilon) \tag{3.6.31}$$

Este modelo (en sus versiones (3.6.27)-(3.6.30) o (3.6.31)) tiene formalmente el mismo orden de precisión que el de orden uno (se desprecian términos $O(\varepsilon^2)$ en las ecuaciones) y sin embargo es claramente más difícil de resolver. Esto debería ser suficiente para descartar la propuesta de este modelo. Además hay que añadir que $\tilde{p}^\varepsilon$ depende de z^ε, por lo que z^ε actúa como parámetro en (3.6.28), y habría que resolver esta ecuación tantas veces como en valores constantes de z^ε deseásemos conocer $\tilde{u}^\varepsilon$. Se podría considerar como algo positivo del modelo el hecho de que permita conocer la variación de $\tilde{u}^\varepsilon$ con la profundidad, aunque sea a costa de resolver varias veces el modelo.

Nuestro objetivo es buscar un modelo que mejore el orden de precisión del modelo de orden 1 y que no suponga la resolución de ecuaciones similares varias veces. Para ello denotamos por

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \hat{u}(\varepsilon)(t, x) = u^0(t, x) + \varepsilon u^1(t, x) + \varepsilon^2 u_0^2(t, x)$$

(es decir, $\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon) = \tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, 0)$) y observamos que:

$$\begin{aligned}
&\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \right) \\
&\quad = \frac{\partial \hat{u}(\varepsilon)}{\partial t} + \hat{u}(\varepsilon) \frac{\partial \hat{u}(\varepsilon)}{\partial x} - 2\nu \left(\frac{\partial^2 \hat{u}(\varepsilon)}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial \hat{u}(\varepsilon)}{\partial x} \right) \\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 \frac{\partial u_0^2}{\partial t} + (u^0 + \varepsilon u^1 + \varepsilon_0^2 u^2) \left[\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \frac{\partial u_0^2}{\partial x} \right] \\
&\quad - 2\nu \left[\frac{\partial^2 u^0}{\partial x^2} + \varepsilon \frac{\partial^2 u^1}{\partial x^2} + \varepsilon^2 \frac{\partial^2 u_0^2}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \frac{\partial u_0^2}{\partial x} \right) \right] \\
&\quad = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} - 2\nu \left(\frac{\partial^2 u^0}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^0}{\partial x} \right) \\
&\quad + \varepsilon \left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} - 2\nu \left(\frac{\partial^2 u^1}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^1}{\partial x} \right) \right] \\
&\quad + \varepsilon^2 \left[\frac{\partial u_0^2}{\partial t} + u_0^2 \frac{\partial u^0}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u^0 \frac{\partial u_0^2}{\partial x} - 2\nu \left(\frac{\partial^2 u_0^2}{\partial x^2} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u_0^2}{\partial x} \right) \right] + O(\varepsilon^3)
\end{aligned}$$

y teniendo en cuenta (3.4.7), (3.5.5) y (3.6.13) resulta:

$$\begin{aligned}
&\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \right) \\
&\quad = -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0} \left(f_W^1 - f_R^1 \right) + \varepsilon \left[-g \frac{\partial s}{\partial x} + \frac{1}{h\rho_0} \left(f_W^2 - f_R^2 \right) \right]
\end{aligned}$$

$$
\begin{aligned}
& -\varepsilon^2 \left\{ \frac{h}{2} H'' \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \frac{2}{\rho_0 h}\left(f_R^1 + f_W^1\right) \right) \right. \\
& - hH' \left(\frac{4\nu}{h}\frac{\partial^2 h}{\partial x^2}\frac{\partial u^0}{\partial x} - \frac{1}{h^2\rho_0}\frac{\partial}{\partial x}[h(f_W^1 - f_R^1)] \right) + hH'''(u^0)^2 \\
& - \frac{2\nu}{h}\frac{\partial}{\partial x}(h^2 u^0 H''') + \frac{2\nu}{h^2} H'' u^0 \frac{\partial}{\partial x}\left(h^2 H'\right) - \frac{2\nu}{h^2}\frac{\partial}{\partial x}\left(h^2 (H')^2 \frac{\partial u^0}{\partial x} \right) \\
& + \frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^0}{\partial x} \right)^2 \right] + \frac{1}{3\rho_0}\frac{\partial}{\partial x}\left(h^2 \frac{\partial^2 p_s}{\partial x^2} \right) \\
& - \frac{1}{6\nu\rho_0} h \left[u^0 \left(\frac{\partial f_W^1}{\partial x} + 2\frac{\partial f_R^1}{\partial x} \right) + \left(\frac{\partial f_W^1}{\partial t} + 2\frac{\partial f_R^1}{\partial t} \right) \right] \\
& + \frac{2\nu}{3}\left\{ \frac{\partial^2}{\partial x^2}\left(h\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right) + \frac{\partial h}{\partial x}\left[\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u^0}{\partial x} \right) + \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right) + \frac{\partial h}{\partial x}\frac{\partial^2 u^0}{\partial x^2} \right] \right\} \\
& + \frac{1}{6} h \left[\frac{6}{h^2\rho_0}\left(\frac{\partial h}{\partial x} \right)^2 (f_W^1 + f_R^1) - \frac{1}{h\rho_0}\frac{\partial^2 h}{\partial x^2}(f_W^1 - f_R^1) \right. \\
& \left. + \frac{6}{h\rho_0}\frac{\partial h}{\partial x}\left(\frac{\partial f_W^1}{\partial x} + 3\frac{\partial f_R^1}{\partial x} \right) + \frac{1}{\rho_0}\left(\frac{\partial^2 f_W^1}{\partial x^2} + 8\frac{\partial^2 f_R^1}{\partial x^2} \right) \right] \\
& \left. - \frac{1}{\rho_0} D_x p^2\Big|_{z=0} + \frac{1}{h\rho_0}\left(f_W^3 - f_R^3 \right) \right\} + O(\varepsilon^3) \\
= & -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon \right) \\
& + \frac{h^\varepsilon}{2}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + 2\nu\frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} - \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{2}{\rho_0 h^\varepsilon}\left(f_R^{1,\varepsilon} + f_W^{1,\varepsilon} \right) \right) \\
& - h^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{4\nu}{h^\varepsilon}\frac{\partial^2 h^\varepsilon}{\partial (x^\varepsilon)^2}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} - \frac{1}{(h^\varepsilon)^2\rho_0}\frac{\partial}{\partial x^\varepsilon}[h^\varepsilon(f_W^{1,\varepsilon} - f_R^{1,\varepsilon})] \right) + h^\varepsilon \frac{\partial^3 H^\varepsilon}{\partial (x^\varepsilon)^3}(u^{0,\varepsilon})^2 \\
& - \frac{2\nu}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2 u^{0,\varepsilon}\frac{\partial^3 H^\varepsilon}{\partial (x^\varepsilon)^3} \right) + \frac{2\nu}{(h^\varepsilon)^2}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2 \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right) \\
& - \frac{2\nu}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2 \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right)^2 \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) + \frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right)^2 \right]
\end{aligned}
$$

$$+\frac{1}{3\rho_0}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2}\right)-\frac{1}{6\nu\rho_0}h^\varepsilon\left[u^{0,\varepsilon}\left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon}+2\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon}\right)\right.$$

$$\left.+\left(\frac{\partial f_W^{1,\varepsilon}}{\partial t^\varepsilon}+2\frac{\partial f_R^{1,\varepsilon}}{\partial t^\varepsilon}\right)\right]+\frac{2\nu}{3}\left\{\frac{\partial^2}{(\partial x^\varepsilon)^2}\left(h^\varepsilon\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)\right.$$

$$\left.+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial^2}{\partial(x^\varepsilon)^2}\left(h^\varepsilon\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial^2 u^{0,\varepsilon}}{\partial(x^\varepsilon)^2}\right]\right\}$$

$$+\frac{1}{6}h^\varepsilon\left[\frac{6}{(h^\varepsilon)^2\rho_0}\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\right)^2(f_W^{1,\varepsilon}+f_R^{1,\varepsilon})-\frac{1}{h^\varepsilon\rho_0}\frac{\partial^2 h^\varepsilon}{\partial(x^\varepsilon)^2}(f_W^{1,\varepsilon}-f_R^{1,\varepsilon})\right.$$

$$\left.+\frac{6}{h^\varepsilon\rho_0}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon}+3\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{1}{\rho_0}\left(\frac{\partial^2 f_W^{1,\varepsilon}}{\partial(x^\varepsilon)^2}+8\frac{\partial^2 f_R^{1,\varepsilon}}{\partial(x^\varepsilon)^2}\right)\right]+O(\varepsilon^3) \qquad (3.6.32)$$

Una vez conocido $\hat{u}^\varepsilon$ veamos cómo se calcula $\tilde{u}^\varepsilon$ utilizando (3.6.17) y (3.6.6):

$$\tilde{u}^\varepsilon=\tilde{u}(\varepsilon)=u^0+\varepsilon u^1+\varepsilon^2u^2$$

$$=u^0+\varepsilon u^1+\varepsilon^2u_0^2+\varepsilon^2\left[\left(\frac{1}{\rho_0\nu}f_R^1+2H'\frac{\partial u^0}{\partial x}-H''u^0\right)hz\right.$$

$$\left.+\frac{1}{2\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-\nu\frac{\partial^2 u^0}{\partial x^2}\right)h^2z^2\right]$$

$$=\hat{u}^\varepsilon+(z^\varepsilon-H^\varepsilon)\left(\frac{1}{\rho_0\nu}f_R^{1,\varepsilon}+2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}-u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}\right)$$

$$+\frac{1}{2\nu}(z^\varepsilon-H^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}+u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}+\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-\nu\frac{\partial^2 u^{0,\varepsilon}}{\partial(x^\varepsilon)^2}\right) \qquad (3.6.33)$$

A continuación proponemos un modelo de aguas someras resultado de despreciar los términos en ε^3 de (3.6.24) y (3.6.32). Se calculan en primer lugar $u^{0,\varepsilon}$ y h^ε, y $\tilde{p}^\varepsilon$ se calcula a partir de ellos usando la expresión (3.6.22):

$$\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}+u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}-2\nu\left(\frac{\partial^2 u^{0,\varepsilon}}{\partial(x^\varepsilon)^2}+2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}+\frac{1}{h^\varepsilon\rho_0}\left(f_W^{1,\varepsilon}-f_R^{1,\varepsilon}\right)$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial(h^\varepsilon u^{0,\varepsilon})}{\partial x^\varepsilon}=0$$

$$\tilde{p}^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)\left[g+\left(u^{0,\varepsilon}\right)^2\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}+\frac{\varepsilon}{\rho_0}\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon}-2\nu\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}u^{0,\varepsilon}\right)\right.$$

$$- \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} \Bigg] + \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} \right]$$

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \right) = - \frac{1}{\rho_0} \left. \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} \right|_{z^\varepsilon = H^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right)$$

$$+ \frac{h^\varepsilon}{2} \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} \left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + 2\nu \frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} - \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{2}{\rho_0 h^\varepsilon} \left(f_R^{1,\varepsilon} + f_W^{1,\varepsilon} \right) \right)$$

$$- h^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{4\nu}{h^\varepsilon} \frac{\partial^2 h^\varepsilon}{\partial (x^\varepsilon)^2} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} - \frac{1}{(h^\varepsilon)^2 \rho_0} \frac{\partial}{\partial x^\varepsilon} [h^\varepsilon (f_W^{1,\varepsilon} - f_R^{1,\varepsilon})] \right) + h^\varepsilon \frac{\partial^3 H^\varepsilon}{\partial (x^\varepsilon)^3} (u^{0,\varepsilon})^2$$

$$- \frac{2\nu}{h^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 u^{0,\varepsilon} \frac{\partial^3 H^\varepsilon}{\partial (x^\varepsilon)^3} \right) + \frac{2\nu}{(h^\varepsilon)^2} \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} u^{0,\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right)$$

$$- \frac{2\nu}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right)^2 \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) + \frac{\partial}{\partial x^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right)^2 \right]$$

$$+ \frac{1}{3\rho_0} \frac{\partial}{\partial x^\varepsilon} \left((h^\varepsilon)^2 \frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} \right) - \frac{1}{6\nu\rho_0} h^\varepsilon \left[u^{0,\varepsilon} \left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon} + 2 \frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon} \right) \right.$$

$$\left. + \left(\frac{\partial f_W^{1,\varepsilon}}{\partial t^\varepsilon} + 2 \frac{\partial f_R^{1,\varepsilon}}{\partial t^\varepsilon} \right) \right] + \frac{2\nu}{3} \left\{ \frac{\partial^2}{(\partial x^\varepsilon)^2} \left(h^\varepsilon \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) \right.$$

$$\left. + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \left[\frac{\partial^2}{\partial (x^\varepsilon)^2} \left(h^\varepsilon \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) + \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right) + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} \right] \right\}$$

$$+ \frac{1}{6} h^\varepsilon \left[\frac{6}{(h^\varepsilon)^2 \rho_0} \left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon} \right)^2 (f_W^{1,\varepsilon} + f_R^{1,\varepsilon}) - \frac{1}{h^\varepsilon \rho_0} \frac{\partial^2 h^\varepsilon}{\partial (x^\varepsilon)^2} (f_W^{1,\varepsilon} - f_R^{1,\varepsilon}) \right.$$

$$\left. + \frac{6}{h^\varepsilon \rho_0} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon} + 3 \frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon} \right) + \frac{1}{\rho_0} \left(\frac{\partial^2 f_W^{1,\varepsilon}}{\partial (x^\varepsilon)^2} + 8 \frac{\partial^2 f_R^{1,\varepsilon}}{\partial (x^\varepsilon)^2} \right) \right]$$

$$\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left(\frac{1}{\rho_0 \nu} f_R^{1,\varepsilon} + 2 \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \frac{\partial H^\varepsilon}{\partial x^\varepsilon} - u^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} \right)$$

$$+ \frac{1}{2\nu} (z^\varepsilon - H^\varepsilon)^2 \left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \nu \frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} \right)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{3.6.34}$$

Un poco más adelante veremos como escribir el modelo (3.6.34) de una forma un poco más "compacta". Mientras veamos en qué medida verifica la aproximación de segundo orden las ecuaciones de Navier-Stokes. Comencemos por la primera ecuación, usaremos las igualdades (3.3.85), (3.3.86), (3.3.88)-(3.3.91), (3.3.93) y (3.6.17)-(3.6.19) para poder escribir:

$$
\begin{aligned}
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} - \frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{12}}{\partial z^\varepsilon} \\
&\quad = D_t\tilde{u}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{u}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon} D_z \tilde{u}(\varepsilon) + \frac{1}{\rho_0} D_x \tilde{p}(\varepsilon) - 2\nu D_x^2 \tilde{u}(\varepsilon) - \frac{1}{\varepsilon\rho_0} D_z \tilde{T}_{12}(\varepsilon) \\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 D_t u^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left[\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2\right] \\
&\quad + (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3)\varepsilon \frac{1}{h}\frac{\partial u^2}{\partial z} + \frac{1}{\rho_0}\left(\frac{\partial p^0}{\partial x} + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2\right) \\
&\quad - 2\nu\left(\frac{\partial^2 u^0}{\partial x^2} + \varepsilon \frac{\partial^2 u^1}{\partial x^2} + \varepsilon^2 D_x^2 u^2\right) - \frac{1}{\varepsilon\rho_0}\left(\varepsilon D_z T^1_{12} + \varepsilon^2 D_z T^2_{12} + \varepsilon^3 D_z T^3_{12}\right) \\
&\quad = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T^1_{12} \\
&\quad + \varepsilon\left(\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g\frac{\partial s}{\partial x} - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T^2_{12}\right) \\
&\quad + \varepsilon^2\left(D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 \frac{1}{h}\frac{\partial u^2}{\partial z} + \frac{1}{\rho_0} D_x p^2 - 2\nu D_x^2 u^2 \right. \\
&\quad \left. - \frac{1}{\rho_0} D_z T^3_{12}\right) + O(\varepsilon^3)
\end{aligned}
$$

Si ahora promediamos la ecuación anterior obtenemos que:

$$
\begin{aligned}
&\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} - \frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{12}}{\partial z^\varepsilon}\right) dz^\varepsilon \\
&\quad = \int_0^1 \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\nu \frac{\partial^2 u^0}{\partial x^2} - \frac{1}{\rho_0} D_z T^1_{12}\right) dz \\
&\quad + \varepsilon \int_0^1 \left(\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + g\frac{\partial s}{\partial x} - 2\nu \frac{\partial^2 u^1}{\partial x^2} - \frac{1}{\rho_0} D_z T^2_{12}\right) dz
\end{aligned}
$$

$$+\varepsilon^2 \int_0^1 \left(D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + w^1 \frac{1}{h}\frac{\partial u^2}{\partial z} + \frac{1}{\rho_0} D_x p^2 - 2\nu D_x^2 u^2 \right.$$

$$\left. - \frac{1}{\rho_0} D_z T_{12}^3 \right) dz + O(\varepsilon^3)$$

Las igualdades (3.4.7), (3.5.5) y (3.6.13) nos aseguran que las integrales a la derecha de la igualdad son cero. Por tanto, la primera ecuación de Navier-Stokes se verifican con un error de $O(\varepsilon^3)$

Para la segunda ecuación de Navier-Stokes, como $T_{12}^{-1} = T_{12}^0 = w^0 = \dfrac{\partial p^0}{\partial z} = 0$, se tiene que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - \frac{1}{\rho_0}\frac{\partial \tilde{T}_{12}^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial^2 \tilde{w}^\varepsilon}{\partial (z^\varepsilon)^2}$$

$$= D_t \tilde{w}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon) + \frac{1}{\rho_0}\frac{1}{\varepsilon} D_z \tilde{p}(\varepsilon) + g$$

$$- \frac{1}{\rho_0} D_x \tilde{T}_{12}(\varepsilon) - 2\nu \frac{1}{\varepsilon^2} D_z^2 \tilde{w}(\varepsilon) = \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + \varepsilon^3 D_t w^3$$

$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left(\varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \varepsilon^3 D_x w^3\right)$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3)\left(D_z w^1 + \varepsilon D_z w^2 + \varepsilon^2 D_z w^3\right) + \frac{1}{\rho_0}\left(D_z p^1 + \varepsilon D_z p^2\right)$$

$$+ g - \frac{1}{\rho_0}\left(\varepsilon D_x T_{12}^1 + \varepsilon^2 D_x T_{12}^2 + \varepsilon^3 D_x T_{12}^3\right) - 2\nu \frac{1}{\varepsilon}\left(D_z^2 w^1 + \varepsilon D_z^2 w^2 + \varepsilon^2 D_z^2 w^3\right)$$

Por las igualdades (3.3.94) y (3.3.101) sabemos que $D_z^2 w^1 = D_z^2 w^2 = 0$ y por (3.3.93) deducimos que $\dfrac{1}{\rho_0} D_z p^1 = -g$, así la expresión anterior, agrupando en potencias de ε y utilizando (3.3.100), resulta

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - \frac{1}{\rho_0}\frac{\partial \tilde{T}_{12}^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial^2 \tilde{w}^\varepsilon}{\partial (z^\varepsilon)^2}$$

$$= \varepsilon \left(D_t w^1 + u^0 D_x w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - \frac{1}{\rho_0} D_x T_{12}^1 - 2\nu D_z^2 w^3 \right) + O(\varepsilon^2)$$

$$= O(\varepsilon^2)$$

La aproximación de segundo orden verifica la segunda ecuación de Navier-Stokes con un error del orden de ε^2 .

Observación 3.4 *No se puede emplear la ecuación* (3.3.110) *para mejorar el orden de aproximación pues esa ecuación no se ha utilizado para construir el modelo.*

Para la ecuación de la incompresibilidad, se tiene por la ecuación (3.6.24):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = O(\varepsilon^3)$$

Por construcción de $\tilde{p}^\varepsilon$ y $\tilde{w}^\varepsilon$ (véase (3.6.23) y (3.6.24)) las condiciones de contorno (3.1.4) y (3.1.5) se verifican exactamente. Veamos qué sucede con las condiciones de contorno que recogen el efecto del viento en la superficie y el rozamiento en el fondo:

$$\left[\left(1-\left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}_{12}^\varepsilon - \left(1+\left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}_W^\varepsilon - \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}_{11}^\varepsilon - \tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon=s^\varepsilon}$$

$$= \left[\left(1-\varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{T}_{12}(\varepsilon) - \left(1+\varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2\right)\tilde{f}_W(\varepsilon)\right.$$

$$\left. - \varepsilon\frac{\partial s}{\partial x}\left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon)\right)\right]_{z=1} = \left[\varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \varepsilon^3 T_{12}^3 - \varepsilon^3\left(\frac{\partial s}{\partial x}\right)^2 T_{12}^1\right.$$

$$- \varepsilon f_W^1 - \varepsilon^2 f_W^2 - \varepsilon^3 f_W^3 - \varepsilon^3\left(\frac{\partial s}{\partial x}\right)^2 f_W^1 - \varepsilon\frac{\partial s}{\partial x}\left(T_{11}^0 + \varepsilon T_{11}^1 + \varepsilon^2 T_{11}^2\right.$$

$$\left.\left. - T_{22}^0 - \varepsilon T_{22}^1 - \varepsilon^2 T_{22}^2\right)\right]_{z=1} + O(\varepsilon^4) = \varepsilon\left[T_{12}^1 - f_W^1 - \frac{\partial s}{\partial x}\left(T_{11}^0 - T_{22}^0\right)\right]_{z=1}$$

$$+ \varepsilon^2\left[T_{12}^2 - f_W^2 - \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{22}^1\right)\right]_{z=1} + \varepsilon^3\left[T_{12}^3 - f_W^3 - \left(\frac{\partial s}{\partial x}\right)^2\left(T_{12}^1 + f_W^1\right)\right.$$

$$\left. - \frac{\partial s}{\partial x}\left(T_{11}^2 - T_{22}^2\right)\right]_{z=1} + O(\varepsilon^4) = O(\varepsilon^4)$$

$$\left[\left(1-\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}_{12}^\varepsilon - \left(1+\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}_R^\varepsilon - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}_{11}^\varepsilon - \tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon=H^\varepsilon}$$

$$= \left[\left(1-\varepsilon^2 (H')^2\right)\tilde{T}_{12}(\varepsilon) - \left(1+\varepsilon^2 (H')^2\right)\tilde{f}_R(\varepsilon) - \varepsilon H'\left(\tilde{T}_{11}(\varepsilon) - \tilde{T}_{22}(\varepsilon)\right)\right]_{z=0}$$

$$= \left[\left(1-\varepsilon^2 (H')^2\right)\left(\varepsilon T_{12}^1 + \varepsilon^2 T_{12}^2 + \varepsilon^3 T_{12}^3\right) - \left(1+\varepsilon^2 (H')^2\right)\left(\varepsilon f_R^1 + \varepsilon^2 f_R^2 + \varepsilon^3 f_R^3\right)\right.$$

$$\left. - \varepsilon H'\left(T_{11}^0 + \varepsilon T_{11}^1 - T_{22}^0 + \varepsilon^2 T_{11}^2 - \varepsilon T_{22}^1 - \varepsilon^2 T_{22}^2\right)\right]_{z=0} + O(\varepsilon^4)$$

$$= \varepsilon\left[T_{12}^1 - f_R^1 - H'\left(T_{11}^0 - T_{22}^0\right)\right]_{z=0} + \varepsilon^2\left[T_{12}^2 - f_R^2 - H'\left(T_{11}^1 - T_{22}^1\right)\right]_{z=0}$$

$$+ \varepsilon^3\left[T_{12}^3 - f_R^3 - (H')^2\left(T_{12}^1 + f_R^1\right) - H'\left(T_{11}^2 - T_{22}^2\right)\right]_{z=0} + O(\varepsilon^4)$$

Como en la construcción del modelo (3.6.34) se han tenido en cuenta las igualdades (3.3.102), (3.3.103), (3.3.113), (3.3.114), (3.6.7) y (3.6.8) las condiciones de contorno (3.1.16) y (3.1.17) se verifican con la precisión siguiente:

$$\left[\left(1-\left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}_{12}^\varepsilon-\left(1+\left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}_W^\varepsilon-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}_{11}^\varepsilon-\tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon=s^\varepsilon}=O(\varepsilon^4)$$

$$\left[\left(1-\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{T}_{12}^\varepsilon-\left(1+\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\right)\tilde{f}_R^\varepsilon-\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\tilde{T}_{11}^\varepsilon-\tilde{T}_{22}^\varepsilon\right)\right]_{z^\varepsilon=H^\varepsilon}=O(\varepsilon^4)$$

Nos interesaría ahora escribir el modelo (3.6.34) en una forma más manejable. Para ello definimos

$$\begin{aligned}
\Upsilon^\varepsilon(h^\varepsilon,u^{0,\varepsilon})=&\frac{h^\varepsilon}{2}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}+u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}+2\nu\frac{\partial^2 u^{0,\varepsilon}}{\partial(x^\varepsilon)^2}-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right.\\
&\left.-\frac{2}{\rho_0 h^\varepsilon}\left(f_R^{1,\varepsilon}+f_W^{1,\varepsilon}\right)\right)-h^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{4\nu}{h^\varepsilon}\frac{\partial^2 h^\varepsilon}{\partial(x^\varepsilon)^2}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}-\frac{1}{(h^\varepsilon)^2\rho_0}\frac{\partial}{\partial x^\varepsilon}[h^\varepsilon(f_W^{1,\varepsilon}-f_R^{1,\varepsilon})]\right)\\
&+h^\varepsilon\frac{\partial^3 H^\varepsilon}{\partial(x^\varepsilon)^3}(u^{0,\varepsilon})^2-\frac{2\nu}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2 u^{0,\varepsilon}\frac{\partial^3 H^\varepsilon}{\partial(x^\varepsilon)^3}\right)\\
&+\frac{2\nu}{(h^\varepsilon)^2}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)-\frac{2\nu}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)\\
&+\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2\right]+\frac{1}{3\rho_0}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2}\right)\\
&-\frac{1}{6\nu\rho_0}h^\varepsilon\left[u^{0,\varepsilon}\left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon}+2\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon}\right)+\left(\frac{\partial f_W^{1,\varepsilon}}{\partial t^\varepsilon}+2\frac{\partial f_R^{1,\varepsilon}}{\partial t^\varepsilon}\right)\right]\\
&+\frac{2\nu}{3}\left\{\frac{\partial^2}{(\partial x^\varepsilon)^2}\left(h^\varepsilon\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial^2}{\partial(x^\varepsilon)^2}\left(h^\varepsilon\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)\right.\right.\\
&\left.\left.+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial^2 u^{0,\varepsilon}}{\partial(x^\varepsilon)^2}\right]\right\}+\frac{1}{6}h^\varepsilon\left[\frac{6}{(h^\varepsilon)^2\rho_0}\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\right)^2(f_W^{1,\varepsilon}+f_R^{1,\varepsilon})-\frac{1}{h^\varepsilon\rho_0}\frac{\partial^2 h^\varepsilon}{\partial(x^\varepsilon)^2}(f_W^{1,\varepsilon}-f_R^{1,\varepsilon})\right.\\
&\left.+\frac{6}{h^\varepsilon\rho_0}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial f_W^{1,\varepsilon}}{\partial x^\varepsilon}+3\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon}\right)+\frac{1}{\rho_0}\left(\frac{\partial^2 f_W^{1,\varepsilon}}{\partial(x^\varepsilon)^2}+8\frac{\partial^2 f_R^{1,\varepsilon}}{\partial(x^\varepsilon)^2}\right)\right]
\end{aligned}\tag{3.6.35}$$

Claramente $\Upsilon^\varepsilon(h^\varepsilon, u^{0,\varepsilon}) = O(\varepsilon^2)$, por lo que el modelo (3.6.34) es equivalente al siguiente (en el sentido de que tan solo se diferencian en términos $O(\varepsilon^3)$:

$$\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} - 2\nu\left(\frac{\partial^2 u^{0,\varepsilon}}{\partial (x^\varepsilon)^2} + 2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right) = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon\rho_0}\left(f_W^{1,\varepsilon} - f_R^{1,\varepsilon}\right) \tag{3.6.36}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^{0,\varepsilon})}{\partial x^\varepsilon} = 0 \tag{3.6.37}$$

$$\begin{aligned}\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)&\left[g + \left(u^{0,\varepsilon}\right)^2\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{\varepsilon}{\rho_0}\frac{\partial f_R^{1,\varepsilon}}{\partial x^\varepsilon} - 2\nu\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}u^{0,\varepsilon}\right)\right.\\ &\left. - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right] + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2}\right]\end{aligned} \tag{3.6.38}$$

$$\begin{aligned}\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} - 2\nu\left(\frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right) &= -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon\right)\\ &+ \Upsilon^\varepsilon(h^\varepsilon, u^{0,\varepsilon})\end{aligned} \tag{3.6.39}$$

$$\begin{aligned}\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)&\left(\frac{1}{\rho_0\nu}f_R^{1,\varepsilon} + 2\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} - \hat{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right)\\ &+ \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \nu\frac{\partial^2 \hat{u}^\varepsilon}{\partial (x^\varepsilon)^2}\right)\end{aligned} \tag{3.6.40}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \tag{3.6.41}$$

3.7. Conclusiones

En este capítulo hemos obtenido distintos modelos de aguas someras con viscosidad. Los modelos de orden cero y orden uno son, aparentemente, muy parecidos, salvo en la expresión de la presión. Esta diferencia permite que la aproximación del modelo de primer orden sea, al menos formalmente, mejor. Este modelo (véase 3.5.13) es comparable a los modelos de aguas poco profundas que aparecen en la literatura (como veremos en 6.1) con un nuevo término de viscosidad y con una velocidad vertical que no se supone nula.

Al menos teóricamente, la aproximación de segundo orden es la que permite una mayor precisión. La ecuación de Navier-Stokes para la velocidad horizontal y la condición de incompresibilidad se verifican con un error de $O(\varepsilon^3)$, la segunda ecuación (para la velocidad vertical) se verifica con un error de $O(\varepsilon^2)$, las condiciones de contorno (3.1.4) y (3.1.5) se verifican de forma exacta, mientras que (3.1.16) y (3.1.17) se verifican con un error de $O(\varepsilon^4)$.

A partir de esta aproximación se han propuesto varios modelos. El primero de ellos, se obtiene "naturalmente"del método asintótico. Las ecuaciones a resolver son más sencillas y el modelo es más preciso, sin embargo la ecuación de conservación de la masa se verifica con un error de $O(\varepsilon^2)$ (obsérvese que es $u^{0,\varepsilon}$ quien verifica de forma exacta la ecuación y no la aproximación final $\tilde{u}^\varepsilon$, lo que representa, a nuestro entender, un grave inconveniente). Además, este modelo tiene el defecto de que, al calcular por separado u^0, u^1 y u_0^2, supone el triple de esfuerzo de cálculo que el modelo (3.5.13).

El modelo alternativo (3.6.27)-(3.6.30) proporciona la misma precisión que el modelo propuesto para la aproximación de primer orden con el inconveniente de que las ecuaciones son más complicadas. La modificación (3.6.31) logra simplificar las ecuaciones calculando en primer lugar $u^{0,\varepsilon}$ y h^ε, obteniendo a continuación $\tilde{p}^\varepsilon$ a partir de ellos de modo que la ecuación para el cálculo de $\tilde{u}^\varepsilon$ sea más sencilla al no estar acoplada con ninguna otra. Pero, el esfuerzo de cálculo ahora es el doble que para resolver (3.5.13) y la aproximación no mejora.

Finalmente, (3.6.34) o el modelo asintóticamente equivalente (3.6.36)-(3.6.41), sí que mejora el modelo propuesto para el primer orden, al menos formalmente, salvo para la ecuación necesaria para el cálculo de h^ε, en este caso el error que se comete sigue siendo de orden $O(\varepsilon^2)$ (de nuevo es $u^{0,\varepsilon}$ y no la aproximación $\tilde{u}^\varepsilon$ quien verifica la ecuación de continuidad). El esfuerzo de cálculo sigue siendo el doble del necesario para resolver el modelo (3.5.13).

Una vez realizado el análisis anterior, nos parece justificado que el modelo propuesto sea el modelo de orden uno (3.5.13)(hemos suprimido el ˜ para simplificar la notación):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{3.7.1}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right)$$

$$= -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon} \left(f_W^\varepsilon - f_R^\varepsilon \right) \tag{3.7.2}$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) g \tag{3.7.3}$$

$$w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon) \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \tag{3.7.4}$$

De forma análoga a lo que ya se indicó en las secciones 2.9 y 2.10.4, se podría cambiar la expresión obtenida para la presión ((3.5.9)) por la mejora obtenida en la aproximación de orden dos ((3.6.23)):

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + (u^\varepsilon)^2\,\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial f_R^\varepsilon}{\partial x^\varepsilon} - 2\nu\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}u^\varepsilon\right)\right.$$
$$\left. - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right] + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2}\right] \quad (3.7.5)$$

Capítulo 4

Modelo bidimensional de aguas someras obtenido a partir de las ecuaciones de Euler

4.1. Formulación del problema

En este capítulo pretendemos obtener un modelo bidimensional de aguas someras sin viscosidad. Para ello seguiremos los pasos dados en el capítulo 2 pero partiendo ahora de las ecuaciones de Euler tridimensionales, en un dominio cuyas variables x e y representarán las coordenadas horizontales y z la vertical, respectivamente. El dominio en el que vamos a trabajar se caracteriza porque la altura es pequeña comparada con sus otras dimensiones. Un río, una ría o una región del mar son ejemplos de este tipo de dominio.

4.1.1. Ecuaciones de partida

Comenzamos por considerar un dominio que representamos mediante el conjunto Ω (Figura 4.1) definido por:

$$\Omega = \{(x, y, z)/(x, y) \in D, z \in [H(x, y), s(t, x, y)]\} \tag{4.1.1}$$

donde x e y son las coordenadas horizontales, z la coordenada vertical, D es la proyección sobre el plano XY de Ω, $z = H(x, y)$ es la ecuación del fondo del río o del mar que suponemos conocido y $z = s(t, x, y) = H(x, y) + h(t, x, y)$ es la ecuación de la superficie (desconocida), siendo $h(t, x, y)$ la altura de agua sobre el fondo.

Consideramos que el flujo se rige por las ecuaciones tridimensionales de Euler ((1.1.2)) en Ω y que las únicas fuerzas externas actuando sobre el fluido son las debidas a la gravedad y a la aceleración de Coriolis. Utilizamos por tanto, (1.1.5) y

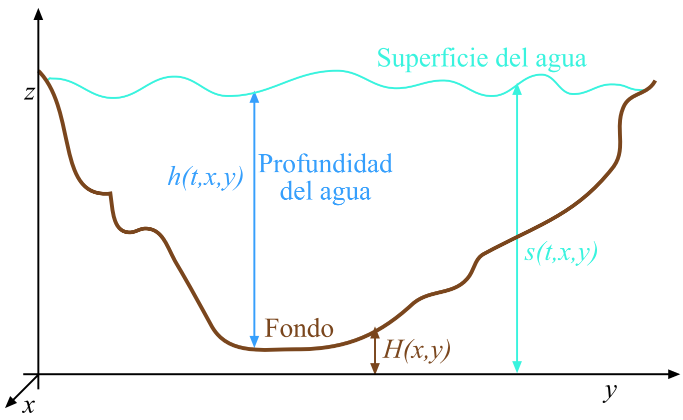


Figura 4.1: Dominio Ω

(1.1.6) para obtener F_x, F_y y F_z y sustituirlas en (1.1.2) de modo que se tiene:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+2\phi\left(\left(\operatorname{sen}\varphi\right)v-\left(\cos\varphi\right)w\right) \quad (4.1.2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)u \quad (4.1.3)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g+2\phi\left(\cos\varphi\right)u \quad (4.1.4)$$

El fluido se supone incompresible por lo que verifica:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (4.1.5)$$

Hemos de imponer las condiciones de contorno al sistema (4.1.2)-(4.1.5). Así, suponemos que la presión es la atmosférica en la superficie

$$p=p_s \quad \text{en } z=s(t,x,y) \quad (4.1.6)$$

($p_s=p_s(t,x,y)$ es la presión atmosférica en la superficie, que se supone conocida), y que el fluido no atraviesa el fondo

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}=0 \quad \text{en } z=H(x,y) \quad (4.1.7)$$

donde $\vec{\mathbf{n}}=(n_1,n_2,n_3)$ representa al vector normal exterior unitario en la frontera del dominio y $\vec{\mathbf{u}}=(u,v,w)$ es el vector velocidad.

Suponemos, además, que el caudal de entrada ($(u,v)h$ en $x=0$) y el de salida ($(u,v)h$ en $x=L$) son conocidos en cada instante.

Para cerrar el problema se deben fijar las condiciones iniciales:

$$\begin{aligned} u(0,x,z) &= u_0(x,z) \\ v(0,x,z) &= v_0(x,z) \\ w(0,x,z) &= w_0(x,z) \end{aligned} \tag{4.1.8}$$

Como veremos en lo que sigue, para aplicar el método de desarrollos asintóticos a las ecuaciones de Euler, necesitaremos de una ecuación más. Por ello se introduce la vorticidad:

$$\vec{\gamma} = \mathrm{rot}\vec{\mathbf{u}} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \tag{4.1.9}$$

que verifica la siguiente ecuación (véase [102]):

$$\frac{\partial \vec{\gamma}}{\partial t} + (\vec{\mathbf{u}} \cdot \nabla) \vec{\gamma} - (\vec{\gamma} \cdot \nabla) \vec{\mathbf{u}} = \mathrm{rot}\vec{\mathbf{F}} \tag{4.1.10}$$

donde $\vec{\mathbf{F}}$ es el vector de fuerzas exteriores volúmicas por unidad de masa. En nuestro caso,

$$\vec{\mathbf{F}} = (2\phi\left((\operatorname{sen}\varphi)\, v - (\cos\varphi)\, w\right), -2\phi\, (\operatorname{sen}\varphi)\, u, -g + 2\phi\, (\cos\varphi)\, u)$$

por tanto,

$$\begin{aligned} \mathrm{rot}\vec{\mathbf{F}} = 2\phi \Bigg(& \frac{\partial}{\partial y} \left((\cos\varphi)\, u\right) + (\operatorname{sen}\varphi) \frac{\partial u}{\partial z}, (\operatorname{sen}\varphi) \frac{\partial v}{\partial z} - (\cos\varphi) \frac{\partial w}{\partial z} - (\cos\varphi) \frac{\partial u}{\partial x}, \\ & - (\operatorname{sen}\varphi) \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} \left((\operatorname{sen}\varphi)\, v - (\cos\varphi)\, w\right) \Bigg) \end{aligned} \tag{4.1.11}$$

Sustituyendo en (4.1.10) la expresión (4.1.11) y utilizando (4.1.5) para reescribir la segunda componente de este rotacional, obtenemos:

$$\begin{aligned} & \frac{\partial \gamma_1}{\partial t} + u \frac{\partial \gamma_1}{\partial x} + v \frac{\partial \gamma_1}{\partial y} + w \frac{\partial \gamma_1}{\partial z} - \gamma_1 \frac{\partial u}{\partial x} - \gamma_2 \frac{\partial u}{\partial y} - \gamma_3 \frac{\partial u}{\partial z} \\ & = 2\phi \left[\frac{\partial}{\partial y} \left((\cos\varphi)\, u\right) + (\operatorname{sen}\varphi) \frac{\partial u}{\partial z} \right] \end{aligned} \tag{4.1.12}$$

$$\begin{aligned} & \frac{\partial \gamma_2}{\partial t} + u \frac{\partial \gamma_2}{\partial x} + v \frac{\partial \gamma_2}{\partial y} + w \frac{\partial \gamma_2}{\partial z} - \gamma_1 \frac{\partial v}{\partial x} - \gamma_2 \frac{\partial v}{\partial y} - \gamma_3 \frac{\partial v}{\partial z} \\ & = 2\phi \left((\operatorname{sen}\varphi) \frac{\partial v}{\partial z} + (\cos\varphi) \frac{\partial v}{\partial y} \right) \end{aligned} \tag{4.1.13}$$

$$\begin{aligned} & \frac{\partial \gamma_3}{\partial t} + u \frac{\partial \gamma_3}{\partial x} + v \frac{\partial \gamma_3}{\partial y} + w \frac{\partial \gamma_3}{\partial z} - \gamma_1 \frac{\partial w}{\partial x} - \gamma_2 \frac{\partial w}{\partial y} - \gamma_3 \frac{\partial w}{\partial z} \\ & = 2\phi \left[- (\operatorname{sen}\varphi) \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} \left((\operatorname{sen}\varphi)\, v - (\cos\varphi)\, w\right) \right] \end{aligned} \tag{4.1.14}$$

4.1.2. Cambio de notación

Como deseamos obtener un modelo de aguas someras, se considera que la profundidad es pequeña comparada con el diámetro del dominio, aunque la profundidad del agua no tiene porqué ser pequeña en términos absolutos. Introducimos de nuevo un pequeño parámetro adimensional, ε, del orden del cociente entre la profundidad media y el diámetro del dominio. Tanto el dominio como las variables y funciones mencionadas antes dependen de este parámetro. Indicaremos con el superíndice ε dicha dependencia. Las ecuaciones se reescriben de la siguiente forma:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial u^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial x^\varepsilon} + 2\phi\left(\left(\operatorname{sen}\varphi^\varepsilon\right) v^\varepsilon - \left(\cos\varphi^\varepsilon\right) w^\varepsilon\right) \quad (4.1.15)$$

$$\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial v^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial y^\varepsilon} - 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) u^\varepsilon \quad (4.1.16)$$

$$\frac{\partial w^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial w^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial w^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial z^\varepsilon} - g + 2\phi\left(\cos\varphi^\varepsilon\right) u^\varepsilon \quad (4.1.17)$$

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = 0 \quad (4.1.18)$$

en $[0,T] \times \Omega^\varepsilon$, siendo

$$\Omega^\varepsilon = \{(x^\varepsilon, y^\varepsilon, z^\varepsilon)/(x^\varepsilon, y^\varepsilon) \in D,\ z^\varepsilon \in [H^\varepsilon(x^\varepsilon, y^\varepsilon), s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)]\}$$

donde ahora, $\vec{\mathbf{u}}^\varepsilon = (u^\varepsilon, v^\varepsilon, w^\varepsilon) = (u^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), w^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon))$ es el vector velocidad y $p^\varepsilon = p^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$ es la presión.

Las condiciones de contorno se escriben

$$p^\varepsilon = p_s^\varepsilon \qquad \text{en } z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon), \quad (4.1.19)$$

$$\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{n}}^\varepsilon = 0 \qquad \text{en } z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon), \quad (4.1.20)$$

las condiciones iniciales

$$u^\varepsilon(0, x^\varepsilon, z^\varepsilon) = u_0^\varepsilon(x^\varepsilon, z^\varepsilon), \quad (4.1.21)$$

$$v^\varepsilon(0, x^\varepsilon, z^\varepsilon) = v_0^\varepsilon(x^\varepsilon, z^\varepsilon), \quad (4.1.22)$$

$$w^\varepsilon(0, x^\varepsilon, z^\varepsilon) = w_0^\varepsilon(x^\varepsilon, z^\varepsilon), \quad (4.1.23)$$

y las ecuaciones para la vorticidad resultan:

$$\frac{\partial \gamma_1^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial \gamma_1^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial \gamma_1^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial \gamma_1^\varepsilon}{\partial z^\varepsilon} - \gamma_1^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \gamma_2^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon} - \gamma_3^\varepsilon \frac{\partial u^\varepsilon}{\partial z^\varepsilon}$$

$$= 2\phi \left(\frac{\partial}{\partial y^\varepsilon} ((\cos \varphi^\varepsilon) \, u^\varepsilon) + (\operatorname{sen} \varphi^\varepsilon) \frac{\partial u^\varepsilon}{\partial z^\varepsilon} \right) \tag{4.1.24}$$

$$\frac{\partial \gamma_2^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial \gamma_2^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial \gamma_2^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial \gamma_2^\varepsilon}{\partial z^\varepsilon} - \gamma_1^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} - \gamma_2^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} - \gamma_3^\varepsilon \frac{\partial v^\varepsilon}{\partial z^\varepsilon}$$

$$= 2\phi \left((\operatorname{sen} \varphi^\varepsilon) \frac{\partial v^\varepsilon}{\partial z^\varepsilon} + (\cos \varphi^\varepsilon) \frac{\partial v^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.1.25}$$

$$\frac{\partial \gamma_3^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial \gamma_3^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial \gamma_3^\varepsilon}{\partial y^\varepsilon} + w^\varepsilon \frac{\partial \gamma_3^\varepsilon}{\partial z^\varepsilon} - \gamma_1^\varepsilon \frac{\partial w^\varepsilon}{\partial x^\varepsilon} - \gamma_2^\varepsilon \frac{\partial w^\varepsilon}{\partial y^\varepsilon} - \gamma_3^\varepsilon \frac{\partial w^\varepsilon}{\partial z^\varepsilon}$$

$$= 2\phi \left(-(\operatorname{sen} \varphi^\varepsilon) \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \frac{\partial}{\partial y^\varepsilon} ((\operatorname{sen} \varphi^\varepsilon) \, v^\varepsilon - (\cos \varphi^\varepsilon) \, w^\varepsilon) \right) \tag{4.1.26}$$

donde:

$$\begin{aligned} \gamma_1^\varepsilon &= \frac{\partial w^\varepsilon}{\partial y^\varepsilon} - \frac{\partial v^\varepsilon}{\partial z^\varepsilon} \\ \gamma_2^\varepsilon &= \frac{\partial u^\varepsilon}{\partial z^\varepsilon} - \frac{\partial w^\varepsilon}{\partial x^\varepsilon} \\ \gamma_3^\varepsilon &= \frac{\partial v^\varepsilon}{\partial x^\varepsilon} - \frac{\partial u^\varepsilon}{\partial y^\varepsilon} \end{aligned} \tag{4.1.27}$$

Deberíamos añadir también que el caudal de entrada y salida es conocido en cada instante, pero como estas condiciones son impuestas de varias formas en la literatura y no es necesario necesario explicitarlas en lo que sigue, preferimos no incluirlas de momento, aunque éstas u otras condiciones similares serán necesarias en la resolución del modelo finalmente obtenido.

4.2. Determinación de la altura de agua

Al igual que cuando trabajábamos en un dominio bidimensional, la función $H(x, y)$ que nos da el perfil del fondo puede considerarse conocida, pero $h(t, x, y)$ no, por lo que se debe introducir alguna ecuación para determinarla.

Para obtener dicha ecuación partimos de la ley de la conservación de la masa (del volumen en este caso, ya que la densidad es contante) como se hizo en 2.2. Utilizamos el hecho de que el fluido es incompresible ((4.1.18))

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + \frac{\partial w^\varepsilon}{\partial z^\varepsilon} = 0$$

por lo que toda variación de volumen del fluido entre dos puntos $(x_1^\varepsilon, y_1^\varepsilon)$ y $(x_2^\varepsilon, y_2^\varepsilon)$ ha de venir dada por la diferencia entre el fluido entrante y el saliente:

$$\int_{y_1^\varepsilon}^{y_2^\varepsilon} \left(\int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x_1^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x_1^\varepsilon, y^\varepsilon)} u^\varepsilon(t^\varepsilon, x_1^\varepsilon, y^\varepsilon, z^\varepsilon)\, dz^\varepsilon dt^\varepsilon \right.$$

$$\left. - \int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x_2^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x_2^\varepsilon, y^\varepsilon)} u^\varepsilon(t^\varepsilon, x_2^\varepsilon, y^\varepsilon, z^\varepsilon)\, dz^\varepsilon dt^\varepsilon \right) dy^\varepsilon +$$

$$+ \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left(\int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x^\varepsilon, y_1^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y_1^\varepsilon)} v^\varepsilon(t^\varepsilon, x^\varepsilon, y_1^\varepsilon, z^\varepsilon) dz^\varepsilon dt^\varepsilon \right.$$

$$\left. - \int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{H^\varepsilon(x^\varepsilon, y_2^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y_2^\varepsilon)} v^\varepsilon(t^\varepsilon, x^\varepsilon, y_2^\varepsilon, z^\varepsilon) dz^\varepsilon dt^\varepsilon \right) dx^\varepsilon =$$

$$= \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} h^\varepsilon(t_2^\varepsilon, x^\varepsilon, y^\varepsilon) dx^\varepsilon dy^\varepsilon - \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} h^\varepsilon(t_1^\varepsilon, x^\varepsilon, y^\varepsilon) dx^\varepsilon dy^\varepsilon$$

Usando que $F(b) - F(a) = \displaystyle\int_a^b \frac{\partial F}{\partial x}\, dx$, la igualdad anterior resulta:

$$\int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left[-\frac{\partial}{\partial x^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) dz^\varepsilon \right) \right] dx^\varepsilon dt^\varepsilon dy^\varepsilon$$

$$+ \int_{x_1^\varepsilon}^{x_2^\varepsilon} \int_{t_1^\varepsilon}^{t_2^\varepsilon} \int_{y_1^\varepsilon}^{y_2^\varepsilon} \left[-\frac{\partial}{\partial y^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)} v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) dz^\varepsilon \right) \right] dy^\varepsilon dt^\varepsilon dx^\varepsilon$$

$$= \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \left(h^\varepsilon(t_2^\varepsilon, x^\varepsilon, y^\varepsilon) - h^\varepsilon(t_1^\varepsilon, x^\varepsilon, y^\varepsilon) \right) dx^\varepsilon dy^\varepsilon$$

Ahora, dividiendo por $(t_2^\varepsilon - t_1^\varepsilon)$ y tomando el límite $t_2^\varepsilon \to t_1^\varepsilon = t^\varepsilon$ se obtiene:

$$- \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)} u^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) dz^\varepsilon \right) dx^\varepsilon dy^\varepsilon$$

$$- \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \frac{\partial}{\partial y^\varepsilon} \left(\int_{H^\varepsilon(x^\varepsilon, y^\varepsilon)}^{s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)} v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) dz^\varepsilon \right) dx^\varepsilon dy^\varepsilon$$

$$= \int_{y_1^\varepsilon}^{y_2^\varepsilon} \int_{x_1^\varepsilon}^{x_2^\varepsilon} \frac{\partial h^\varepsilon}{\partial t^\varepsilon}(t^\varepsilon, x^\varepsilon, y^\varepsilon) dx^\varepsilon dy^\varepsilon$$

que es equivalente, por el teorema del valor medio, a:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial}{\partial x^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon} u^\varepsilon dz^\varepsilon + \frac{\partial}{\partial y^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon} v^\varepsilon dz^\varepsilon = 0 \tag{4.2.1}$$

Observación 4.1 *Si, como en [108], se asume que* $\frac{\partial u^\varepsilon}{\partial z^\varepsilon} = \frac{\partial v^\varepsilon}{\partial z^\varepsilon} = 0$*, a partir de (4.2.1) se obtiene:*

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (u^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (v^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0$$

que es la ecuación que se propone en [108], (1.2.22).

4.3. Construcción del dominio de referencia

En el caso tridimensional el dominio de referencia independiente del parámetro ε y del tiempo que se considera es $\Omega = D \times [0,1]$. Se supone:

$$h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon h(t,x,y) \tag{4.3.1}$$

$$H^\varepsilon(x^\varepsilon, y^\varepsilon) = \varepsilon H(x,y) \tag{4.3.2}$$

(por tanto $s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon s(t,x,y)$) y se define el siguiente cambio de variable, de Ω a Ω^ε

$$\begin{aligned} t^\varepsilon &= t \\ x^\varepsilon &= x \\ y^\varepsilon &= y \\ z^\varepsilon &= \varepsilon[H(x,y) + zh(t,x,y)] \end{aligned} \tag{4.3.3}$$

Así el jacobiano del cambio de variable es:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial t}{\partial t^\varepsilon} & \frac{\partial x}{\partial t^\varepsilon} & \frac{\partial y}{\partial t^\varepsilon} & \frac{\partial z}{\partial t^\varepsilon} \\ \frac{\partial t}{\partial x^\varepsilon} & \frac{\partial x}{\partial x^\varepsilon} & \frac{\partial y}{\partial x^\varepsilon} & \frac{\partial z}{\partial x^\varepsilon} \\ \frac{\partial t}{\partial y^\varepsilon} & \frac{\partial x}{\partial y^\varepsilon} & \frac{\partial y}{\partial y^\varepsilon} & \frac{\partial z}{\partial y^\varepsilon} \\ \frac{\partial t}{\partial z^\varepsilon} & \frac{\partial x}{\partial z^\varepsilon} & \frac{\partial y}{\partial z^\varepsilon} & \frac{\partial z}{\partial z^\varepsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{z}{h}\frac{\partial h}{\partial t} \\ 0 & 1 & 0 & -\frac{\frac{\partial H}{\partial x} + z\frac{\partial h}{\partial x}}{h} \\ 0 & 0 & 1 & -\frac{\frac{\partial H}{\partial y} + z\frac{\partial h}{\partial y}}{h} \\ 0 & 0 & 0 & \frac{1}{\varepsilon h} \end{pmatrix}$$

y el jacobiano del cambio de variable inverso:

$$\mathbf{J^{-1}} = \begin{pmatrix} \frac{\partial t^\varepsilon}{\partial t} & \frac{\partial x^\varepsilon}{\partial t} & \frac{\partial y^\varepsilon}{\partial t} & \frac{\partial z^\varepsilon}{\partial t} \\ \frac{\partial t^\varepsilon}{\partial x} & \frac{\partial x^\varepsilon}{\partial x} & \frac{\partial y^\varepsilon}{\partial x} & \frac{\partial z^\varepsilon}{\partial x} \\ \frac{\partial t^\varepsilon}{\partial y} & \frac{\partial x^\varepsilon}{\partial y} & \frac{\partial y^\varepsilon}{\partial y} & \frac{\partial z^\varepsilon}{\partial y} \\ \frac{\partial t^\varepsilon}{\partial z} & \frac{\partial x^\varepsilon}{\partial z} & \frac{\partial y^\varepsilon}{\partial z} & \frac{\partial z^\varepsilon}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \varepsilon z \frac{\partial h}{\partial t} \\ 0 & 1 & 0 & \varepsilon \left[\frac{\partial H}{\partial x} + z \frac{\partial h}{\partial x} \right] \\ 0 & 0 & 1 & \varepsilon \left[\frac{\partial H}{\partial y} + z \frac{\partial h}{\partial y} \right] \\ 0 & 0 & 0 & \varepsilon h \end{pmatrix}$$

cuyos determinantes son: $|J| = \frac{1}{\varepsilon h}$ y $|J^{-1}| = \varepsilon h$.

Las hipótesis (4.3.1) y (4.3.2) únicamente explicitan que h^ε y H^ε son de orden ε, es decir, que son pequeñas comparadas con el diámetro del dominio.

Dada una función F^ε cualquiera definida en $[0,T] \times \bar{\Omega}^\varepsilon$, se puede construir a partir de ella otra función $F(\varepsilon)$ definida en $[0,T] \times \bar{\Omega}$ utilizando para ello el cambio de variable del modo natural: $F(\varepsilon)(t,x,y,z) = F^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$. La relación entre las derivadas parciales de una y otra función es:

$$\frac{\partial F^\varepsilon}{\partial t^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial t^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial t^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial y}\frac{\partial y}{\partial t^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial t^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial F(\varepsilon)}{\partial z}$$

$$= D_t F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial x^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial x^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial x^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial y}\frac{\partial y}{\partial x^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial x^\varepsilon}$$

$$= \frac{\partial F(\varepsilon)}{\partial x} - \frac{\frac{\partial H}{\partial x} + z\frac{\partial h}{\partial x}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_x F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial y^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial y^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial y^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial y}\frac{\partial y}{\partial y^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial y^\varepsilon}$$

$$= \frac{\partial F(\varepsilon)}{\partial y} - \frac{\frac{\partial H}{\partial y} + z\frac{\partial h}{\partial y}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_y F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial z^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t}\frac{\partial t}{\partial z^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial x}\frac{\partial x}{\partial z^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial y}\frac{\partial y}{\partial z^\varepsilon} + \frac{\partial F(\varepsilon)}{\partial z}\frac{\partial z}{\partial z^\varepsilon} = \frac{1}{\varepsilon h}\frac{\partial F(\varepsilon)}{\partial z}$$

$$= \frac{1}{\varepsilon} D_z F(\varepsilon)$$

donde hemos introducido la siguiente notación:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial}{\partial z} \\
D_x &= \frac{\partial}{\partial x} - \frac{\dfrac{\partial H}{\partial x} + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial}{\partial z} \\
D_y &= \frac{\partial}{\partial y} - \frac{\dfrac{\partial H}{\partial y} + z\dfrac{\partial h}{\partial y}}{h}\frac{\partial}{\partial z} \\
D_z &= \frac{1}{h}\frac{\partial}{\partial z}
\end{aligned} \tag{4.3.4}$$

Si ahora definimos,

$$\begin{aligned}
u(\varepsilon)(t,x,y,z) &= u^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
v(\varepsilon)(t,x,y,z) &= v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
w(\varepsilon)(t,x,y,z) &= w^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
p(\varepsilon)(t,x,y,z) &= p^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
\gamma_i(\varepsilon)(t,x,y,z) &= \gamma_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \quad (i = 1, 2, 3)
\end{aligned}$$

el problema (4.1.15)-(4.1.18) se puede escribir en el dominio de referencia Ω de la forma siguiente:

$$D_t u(\varepsilon) + u(\varepsilon) D_x u(\varepsilon) + v(\varepsilon) D_y u(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon} D_z u(\varepsilon) = -\frac{1}{\rho_0} D_x p(\varepsilon)$$

$$+ 2\phi\left((\operatorname{sen}\varphi)\, v(\varepsilon) - (\cos\varphi)\, w(\varepsilon)\right) \tag{4.3.5}$$

$$D_t v(\varepsilon) + u(\varepsilon) D_x v(\varepsilon) + v(\varepsilon) D_y v(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon} D_z v(\varepsilon) = -\frac{1}{\rho_0} D_y p(\varepsilon)$$

$$- 2\phi\,(\operatorname{sen}\varphi)\, u(\varepsilon) \tag{4.3.6}$$

$$D_t w(\varepsilon) + u(\varepsilon) D_x w(\varepsilon) + v(\varepsilon) D_y w(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon} D_z w(\varepsilon) = -\frac{1}{\rho_0}\frac{1}{\varepsilon} D_z p(\varepsilon)$$

$$- g + 2\phi\,(\cos\varphi)\, u(\varepsilon) \tag{4.3.7}$$

$$D_x u(\varepsilon) + D_y v(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \tag{4.3.8}$$

Observación 4.2 *Podemos suponer que la función latitud Norte no depende del parámetro ε (por lo que denotaremos $\varphi^\varepsilon = \varphi$), y se puede considerar o bien constante o bien dependiente sólo de y^ε.*

La condición de contorno (4.1.19) tras el cambio de variable se escribe:

$$p(\varepsilon) = p_s \text{ en } z = 1 \tag{4.3.9}$$

(donde implícitamente hemos supuesto que $p_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = p_s(t, x, y)$, es decir, la presión atmosférica superficial es independiente de ε, lo que nos parece una hipótesis razonable dado que no depende ni de z^ε ni de la profundidad).

Para aplicar el cambio de variable a la condición (4.1.20) tendremos en cuenta que $\vec{\mathbf{n}}^\varepsilon$ es la normal exterior unitaria en $z^\varepsilon = H^\varepsilon$, es decir, $\vec{\mathbf{n}}^\varepsilon$ debe ser paralelo al vector $\left(\varepsilon\dfrac{\partial H}{\partial x}, \varepsilon\dfrac{\partial H}{\partial y}, -1\right)$, y por ser unitario:

$$\vec{\mathbf{n}}^\varepsilon = \frac{1}{\sqrt{\varepsilon^2\left[\left(\dfrac{\partial H}{\partial x}\right)^2 + \left(\dfrac{\partial H}{\partial y}\right)^2\right] + 1}}\left(\varepsilon\frac{\partial H}{\partial x}, \varepsilon\frac{\partial H}{\partial y}, -1\right)$$

Ahora, si aplicamos el cambio de variable a (4.1.20), resulta:

$$w(\varepsilon) = u(\varepsilon)\varepsilon\frac{\partial H}{\partial x} + v(\varepsilon)\varepsilon\frac{\partial H}{\partial y} \text{ en } z = 0 \tag{4.3.10}$$

Las condiciones iniciales (4.1.21)-(4.1.22) se escriben en el dominio de referencia como sigue

$$u(\varepsilon)(0, x, y, z) = u_0(\varepsilon)(x, y, z) \tag{4.3.11}$$

$$v(\varepsilon)(0, x, y, z) = v_0(\varepsilon)(x, y, z) \tag{4.3.12}$$

$$w(\varepsilon)(0, x, y, z) = w_0(\varepsilon)(x, y, z) \tag{4.3.13}$$

Las ecuaciones de vorticidad (4.1.24)-(4.1.26) tras el cambio de variable resultan:

$$\begin{aligned} &D_t\gamma_1(\varepsilon) + u(\varepsilon)D_x\gamma_1(\varepsilon) + v(\varepsilon)D_y\gamma_1(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma_1(\varepsilon) \\ &\quad - \gamma_1(\varepsilon)D_xu(\varepsilon) - \gamma_2(\varepsilon)D_yu(\varepsilon) - \gamma_3(\varepsilon)\frac{1}{\varepsilon}D_zu(\varepsilon) \\ &\quad = 2\phi\left(D_y((\cos\varphi)\, u(\varepsilon)) + (\operatorname{sen}\varphi)\,\frac{1}{\varepsilon}D_zu(\varepsilon)\right) \end{aligned} \tag{4.3.14}$$

$$D_t\gamma_2(\varepsilon) + u(\varepsilon)D_x\gamma_2(\varepsilon) + v(\varepsilon)D_y\gamma_2(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma_2(\varepsilon)$$

$$
\begin{aligned}
&- \gamma_1(\varepsilon) D_x v(\varepsilon) - \gamma_2(\varepsilon) D_y v(\varepsilon) - \gamma_3(\varepsilon) \frac{1}{\varepsilon} D_z v(\varepsilon) \\
&= 2\phi \left((\operatorname{sen} \varphi) \frac{1}{\varepsilon} D_z v(\varepsilon) + (\cos \varphi) D_y v(\varepsilon) \right) \qquad (4.3.15)
\end{aligned}
$$

$$
\begin{aligned}
D_t \gamma_3(\varepsilon) &+ u(\varepsilon) D_x \gamma_3(\varepsilon) + v(\varepsilon) D_y \gamma_3(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z \gamma_3(\varepsilon) \\
&- \gamma_1(\varepsilon) D_x w(\varepsilon) - \gamma_2(\varepsilon) D_y w(\varepsilon) - \gamma_3(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon) \\
&= -2\phi \left[(\operatorname{sen} \varphi) D_x u(\varepsilon) + D_y ((\operatorname{sen} \varphi) v(\varepsilon) - (\cos \varphi) w(\varepsilon)) \right] \qquad (4.3.16)
\end{aligned}
$$

y las expresiones de las componentes de la vorticidad se escriben en Ω de la forma siguiente:

$$
\begin{aligned}
\gamma_1(\varepsilon) &= D_y w(\varepsilon) - \frac{1}{\varepsilon} D_z v(\varepsilon) \\
\gamma_2(\varepsilon) &= \frac{1}{\varepsilon} D_z u(\varepsilon) - D_x w(\varepsilon) \\
\gamma_3(\varepsilon) &= D_x v(\varepsilon) - D_y u(\varepsilon)
\end{aligned} \qquad (4.3.17)
$$

Finalmente, si se aplica el cambio de variable a la ecuación obtenida para el cálculo del calado ((4.2.1)) teniendo en cuenta que h^ε no depende de z^ε, obtenemos:

$$
\frac{\partial h}{\partial t} + \int_0^1 \left[\frac{\partial (u(\varepsilon) h)}{\partial x} + \frac{\partial (v(\varepsilon) h)}{\partial y} \right] dz = 0 \qquad (4.3.18)
$$

4.4. Ecuaciones en el dominio de referencia

En resumen, las ecuaciones que determinan el problema a resolver en el dominio de referencia Ω son las siguientes:

- las ecuaciones de Euler:

$$
\begin{aligned}
D_t u(\varepsilon) &+ u(\varepsilon) D_x u(\varepsilon) + v(\varepsilon) D_y u(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z u(\varepsilon) = -\frac{1}{\rho_0} D_x p(\varepsilon) \\
&+ 2\phi \left((\operatorname{sen} \varphi) v(\varepsilon) - (\cos \varphi) w(\varepsilon) \right) \qquad (4.4.1)
\end{aligned}
$$

$$
\begin{aligned}
D_t v(\varepsilon) &+ u(\varepsilon) D_x v(\varepsilon) + v(\varepsilon) D_y v(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z v(\varepsilon) = -\frac{1}{\rho_0} D_y p(\varepsilon) \\
&- 2\phi (\operatorname{sen} \varphi) u(\varepsilon) \qquad (4.4.2)
\end{aligned}
$$

$$
D_t w(\varepsilon) + u(\varepsilon) D_x w(\varepsilon) + v(\varepsilon) D_y w(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon) = -\frac{1}{\rho_0} \frac{1}{\varepsilon} D_z p(\varepsilon)
$$

$$- g + 2\phi\,(\cos\varphi)\,u(\varepsilon) \tag{4.4.3}$$

- la condición de incompresibilidad:

$$D_x u(\varepsilon) + D_y v(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \tag{4.4.4}$$

- las condiciones de contorno:

$$p(\varepsilon) = p_s \qquad \text{en } z = 1, \tag{4.4.5}$$

$$w(\varepsilon) = u(\varepsilon)\varepsilon\frac{\partial H}{\partial x} + v(\varepsilon)\varepsilon\frac{\partial H}{\partial y} \qquad \text{en } z = 0 \tag{4.4.6}$$

(a las que habría que añadir las de caudal de entrada y salida conocido en cada instante)

- las condiciones iniciales:

$$u(\varepsilon)(0, x, y, z) = u_0(\varepsilon)(x, y, z) \tag{4.4.7}$$

$$v(\varepsilon)(0, x, y, z) = v_0(\varepsilon)(x, y, z) \tag{4.4.8}$$

$$w(\varepsilon)(0, x, y, z) = w_0(\varepsilon)(x, y, z) \tag{4.4.9}$$

- las ecuaciones de vorticidad:

$$\begin{aligned} & D_t\gamma_1(\varepsilon) + u(\varepsilon)D_x\gamma_1(\varepsilon) + v(\varepsilon)D_y\gamma_1(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma_1(\varepsilon) \\ & \quad - \gamma_1(\varepsilon)D_x u(\varepsilon) - \gamma_2(\varepsilon)D_y u(\varepsilon) - \gamma_3(\varepsilon)\frac{1}{\varepsilon}D_z u(\varepsilon) \\ & \quad = 2\phi\left(D_y((\cos\varphi)\,u(\varepsilon)) + (\operatorname{sen}\varphi)\,\frac{1}{\varepsilon}D_z u(\varepsilon)\right) \end{aligned} \tag{4.4.10}$$

$$\begin{aligned} & D_t\gamma_2(\varepsilon) + u(\varepsilon)D_x\gamma_2(\varepsilon) + v(\varepsilon)D_y\gamma_2(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma_2(\varepsilon) \\ & \quad - \gamma_1(\varepsilon)D_x v(\varepsilon) - \gamma_2(\varepsilon)D_y v(\varepsilon) - \gamma_3(\varepsilon)\frac{1}{\varepsilon}D_z v(\varepsilon) \\ & \quad = 2\phi\left((\operatorname{sen}\varphi)\,\frac{1}{\varepsilon}D_z v(\varepsilon) + (\cos\varphi)\,D_y v(\varepsilon)\right) \end{aligned} \tag{4.4.11}$$

$$\begin{aligned} & D_t\gamma_3(\varepsilon) + u(\varepsilon)D_x\gamma_3(\varepsilon) + v(\varepsilon)D_y\gamma_3(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z\gamma_3(\varepsilon) \\ & \quad - \gamma_1(\varepsilon)D_x w(\varepsilon) - \gamma_2(\varepsilon)D_y w(\varepsilon) - \gamma_3(\varepsilon)\frac{1}{\varepsilon}D_z w(\varepsilon) \\ & \quad = -2\phi\,[(\operatorname{sen}\varphi)\,D_x u(\varepsilon) + D_y((\operatorname{sen}\varphi)\,v(\varepsilon) - (\cos\varphi)\,w(\varepsilon))] \end{aligned} \tag{4.4.12}$$

- las componentes de la vorticidad en función de las componentes de la velocidad:

$$\begin{aligned} \gamma_1(\varepsilon) &= D_y w(\varepsilon) - \frac{1}{\varepsilon} D_z v(\varepsilon) \\ \gamma_2(\varepsilon) &= \frac{1}{\varepsilon} D_z u(\varepsilon) - D_x w(\varepsilon) \\ \gamma_3(\varepsilon) &= D_x v(\varepsilon) - D_y u(\varepsilon) \end{aligned} \tag{4.4.13}$$

- la ecuación que determina la función h:

$$\frac{\partial h}{\partial t} + \int_0^1 \left[\frac{\partial(u(\varepsilon)h)}{\partial x} + \frac{\partial(v(\varepsilon)h)}{\partial y} \right] dz = 0 \tag{4.4.14}$$

4.5. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (4.4.1)-(4.4.14) admite un desarrollo en serie de potencias de ε en la forma:

$$\begin{aligned} & u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots \\ & v(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots \\ & w(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots \\ & p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots \\ & \gamma_i(\varepsilon) = \varepsilon^{-1} \gamma_i^{-1} + \gamma_i^0 + \varepsilon \gamma_i^1 + \varepsilon^2 \gamma_i^2 + \cdots \quad (i = 1, 2) \\ & \gamma_3(\varepsilon) = \gamma_3^0 + \varepsilon \gamma_3^1 + \varepsilon^2 \gamma_3^2 + \cdots \end{aligned} \tag{4.5.1}$$

En (4.5.1) suponemos que el desarrollo en serie de potencias de $\gamma_i(\varepsilon)$ $(i = 1, 2)$ comienza en término de orden -1 en ε. Esta hipótesis resulta ser la natural si sustituimos (4.5.1) en (4.4.13) (véase (4.5.12)-(4.5.13)).

Se sustituyen estos desarrollos en serie de potencias en las ecuaciones (4.4.1)-(4.4.14). Realizando esta sustitución en la primera ecuación de Euler ((4.4.1)) se obtiene:

$$
\begin{aligned}
&D_t u^0 + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + \cdots \\
&\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\left[D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots\right] \\
&\quad + (v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots)\left[D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots\right] \\
&\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\frac{1}{\varepsilon}\left[D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right] \\
&\quad = -\frac{1}{\rho_0}\left(D_x p^0 + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 + \cdots\right) + 2\phi\left[(\operatorname{sen}\varphi)(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots)\right. \\
&\quad \left. - (\cos\varphi)(w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\right]
\end{aligned}
$$

El paso siguiente consiste en identificar los términos multiplicados por la misma potencia de ε. En este caso se tiene:

$$
\begin{aligned}
&\varepsilon^{-1} w^0 D_z u^0 + \varepsilon^0 \Bigg[D_t u^0 + u^0 D_x u^0 + v^0 D_y u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0 \\
&\quad - 2\phi\left((\operatorname{sen}\varphi)\, v^0 - (\cos\varphi)\, w^0\right)\Bigg] + \varepsilon\left[D_t u^1 + u^0 D_x u^1 + u^1 D_x u^0 + v^0 D_y u^1\right. \\
&\quad + v^1 D_y u^0 + w^0 D_z u^2 + w^1 D_z u^1 + w^2 D_z u^0 + \frac{1}{\rho_0} D_x p^1 \\
&\quad \left. - 2\phi\left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1\right)\right] + \varepsilon^2 \Bigg[D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 D_x u^0 \\
&\quad + v^0 D_y u^2 + v^1 D_y u^1 + v^2 D_y u^0 + w^0 D_z u^3 + w^1 D_z u^2 + w^2 D_z u^1 + w^3 D_z u^0 \\
&\quad + \frac{1}{\rho_0} D_x p^2 - 2\phi\left((\operatorname{sen}\varphi)\, v^2 - (\cos\varphi)\, w^2\right)\Bigg] + O(\varepsilon^3) = 0
\end{aligned}
\tag{4.5.2}
$$

Reemplazando $u(\varepsilon)$, $v(\varepsilon)$, $w(\varepsilon)$ y $p(\varepsilon)$ por sus desarrollos en serie de potencias de ε, (4.5.1), en la segunda ecuación de Euler ((4.4.2)), ésta resulta:

$$
\begin{aligned}
&D_t v^0 + \varepsilon D_t v^1 + \varepsilon^2 D_t v^2 + \cdots \\
&\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\left[D_x v^0 + \varepsilon D_x v^1 + \varepsilon^2 D_x v^2 + \cdots\right] \\
&\quad + (v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots)\left[D_y v^0 + \varepsilon D_y v^1 + \varepsilon^2 D_y v^2 + \cdots\right] \\
&\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\frac{1}{\varepsilon}\left[D_z v^0 + \varepsilon D_z v^1 + \varepsilon^2 D_z v^2 + \cdots\right] \\
&\quad = -\frac{1}{\rho_0}\left(D_y p^0 + \varepsilon D_y p^1 + \varepsilon^2 D_y p^2 + \cdots\right) - 2\phi\,(\operatorname{sen}\varphi)\,(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)
\end{aligned}
$$

Agrupando en potencias de ε se obtiene:

$$\varepsilon^{-1}w^0D_zv^0 + \varepsilon^0\left(D_tv^0 + u^0D_xv^0 + v^0D_yv^0 + w^0D_zv^1 + w^1D_zv^0 + \frac{1}{\rho_0}D_yp^0\right.$$
$$\left. + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) + \varepsilon\left(D_tv^1 + u^0D_xv^1 + u^1D_xv^0 + v^0D_yv^1 + v^1D_yv^0\right.$$
$$\left. + w^0D_zv^2 + w^1D_zv^1 + w^2D_zv^0 + \frac{1}{\rho_0}D_yp^1 + 2\phi\,(\operatorname{sen}\varphi)\,u^1\right)$$
$$+ \varepsilon^2\left(D_tv^2 + u^0D_xv^2 + u^1D_xv^1 + u^2D_xv^0 + v^0D_yv^2 + v^1D_yv^1 + v^2D_yv^0\right.$$
$$\left. + w^0D_zv^3 + w^1D_zv^2 + w^2D_zv^1 + w^3D_zv^0 + \frac{1}{\rho_0}D_yp^2 + 2\phi\,(\operatorname{sen}\varphi)\,u^2\right)$$
$$+ O(\varepsilon^3) = 0 \tag{4.5.3}$$

Sustituyendo en la ecuación de Euler para $w(\varepsilon)$ ((4.4.3)), $u(\varepsilon)$, $v(\varepsilon)$, $w(\varepsilon)$ y $p(\varepsilon)$ por sus expresiones en serie de potencias de ε podemos escribir:

$$D_tw^0 + \varepsilon D_tw^1 + \varepsilon^2D_tw^2 + \varepsilon^3D_tw^3 + \cdots$$
$$+ (u^0 + \varepsilon u^1 + \varepsilon^2u^2 + \cdots)\left[D_xw^0 + \varepsilon D_xw^1 + \varepsilon^2D_xw^2 + \varepsilon^3D_xw^3 + \cdots\right]$$
$$+ (v^0 + \varepsilon v^1 + \varepsilon^2v^2 + \cdots)\left[D_yw^0 + \varepsilon D_yw^1 + \varepsilon^2D_yw^2 + \varepsilon^3D_yw^3 + \cdots\right]$$
$$+ (w^0 + \varepsilon w^1 + \varepsilon^2w^2 + \varepsilon^3w^3 + \cdots)\frac{1}{\varepsilon}\left[D_zw^0 + \varepsilon D_zw^1 + \varepsilon^2D_zw^2\right.$$
$$\left. + \varepsilon^3D_zw^3 + \cdots\right] = -\frac{1}{\rho_0}\frac{1}{\varepsilon}\left(D_zp^0 + \varepsilon D_zp^1 + \varepsilon^2D_zp^2 + \varepsilon^3D_zp^3 + \cdots\right) - g$$
$$+ 2\phi\,(\cos\varphi)\,(u^0 + \varepsilon u^1 + \varepsilon^2u^2 + \cdots)$$

Agrupando en potencias de ε se obtiene:

$$\varepsilon^{-1}\left(w^0D_zw^0 + \frac{1}{\rho_0}D_zp^0\right) + \varepsilon^0\left(D_tw^0 + u^0D_xw^0 + v^0D_yw^0 + w^0D_zw^1\right.$$
$$\left. + w^1D_zw^0 + \frac{1}{\rho_0}D_zp^1 + g - 2\phi\,(\cos\varphi)\,u^0\right) + \varepsilon\left(D_tw^1 + u^0D_xw^1 + u^1D_xw^0\right.$$
$$+ v^0D_yw^1 + v^1D_yw^0 + w^0D_zw^2 + w^1D_zw^1 + w^2D_zw^0 + \frac{1}{\rho_0}D_zp^2$$
$$\left. - 2\phi\,(\cos\varphi)\,u^1\right) + \varepsilon^2\left(D_tw^2 + u^0D_xw^2 + u^1D_xw^1 + u^2D_xw^0 + v^0D_yw^2\right.$$

$$+ v^1 D_y w^1 + v^2 D_y w^0 + w^0 D_z w^3 + w^1 D_z w^2 + w^2 D_z w^1 + w^3 D_z w^0$$

$$\left. + \frac{1}{\rho_0} D_z p^3 - 2\phi \left(\cos \varphi\right) u^1 \right) + O(\varepsilon^3) = 0 \tag{4.5.4}$$

Repetimos el proceso para la ecuación de la incompresibilidad ((4.4.4)). En primer lugar se realiza la sustitución:

$$D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots + D_y v^0 + \varepsilon D_y v^1 + \varepsilon^2 D_y v^2 + \cdots$$

$$+ \frac{1}{\varepsilon}\left(D_z w^0 + \varepsilon D_z w^1 + \varepsilon^2 D_z w^2 + \varepsilon^3 D_z w^3 + \cdots\right) = 0$$

y después la identificación de los términos multiplicados por cada potencia de ε

$$\varepsilon^{-1} D_z w^0 + D_x u^0 + D_y v^0 + D_z w^1 + \varepsilon\left(D_x u^1 + D_y v^1 + D_z w^2\right)$$

$$+ \varepsilon^2\left(D_x u^2 + D_y v^2 + D_z w^3\right) + O(\varepsilon^3) = 0 \tag{4.5.5}$$

De la condición de contorno (4.4.5) se tiene

$$p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots = p_s \text{ en } z = 1 \tag{4.5.6}$$

Al sustituir los desarrollos (4.5.1) en la condición (4.4.6) resulta

$$w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots = \varepsilon(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\frac{\partial H}{\partial x}$$

$$+ \varepsilon(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots)\frac{\partial H}{\partial y} \text{ en } z = 0$$

y agrupando

$$w^0 + \varepsilon\left(w^1 - u^0\frac{\partial H}{\partial x} - v^0\frac{\partial H}{\partial y}\right) + \varepsilon^2\left(w^2 - u^1\frac{\partial H}{\partial x} - v^1\frac{\partial H}{\partial y}\right)$$

$$+ \varepsilon^3\left(w^3 - u^2\frac{\partial H}{\partial x} - v^2\frac{\partial H}{\partial y}\right) + \cdots = 0 \text{ en } z = 0 \tag{4.5.7}$$

A partir de la ecuación (4.4.14) necesaria para la determinación del calado, sustituyendo $u(\varepsilon)$ y $v(\varepsilon)$ por sus desarrollos en serie de potencias de ε se obtiene:

$$\frac{\partial h}{\partial t} + \int_0^1 \left(\frac{\partial(hu^0)}{\partial x} + \varepsilon\frac{\partial(hu^1)}{\partial x} + \varepsilon^2\frac{\partial(hu^2)}{\partial x} + \cdots\right.$$

$$\left. + \frac{\partial(hv^0)}{\partial y} + \varepsilon\frac{\partial(hv^1)}{\partial y} + \varepsilon^2\frac{\partial(hv^2)}{\partial y} + \cdots\right) dz = 0 \tag{4.5.8}$$

Sustituyendo en la primera ecuación de la vorticidad (4.4.10) $u(\varepsilon)$, $v(\varepsilon)$, $w(\varepsilon)$ y $\gamma_i(\varepsilon)\quad (i = 1, 2, 3)$ por sus expresiones en serie de potencias de ε se obtiene:

$$
\begin{aligned}
&\varepsilon^{-1} D_t \gamma_1^{-1} + D_t \gamma_1^0 + \varepsilon D_t \gamma_1^1 + \varepsilon^2 D_t \gamma_1^2 + \cdots \\
&\quad + \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots\right) \left[\varepsilon^{-1} D_x \gamma_1^{-1} + D_x \gamma_1^0 + \varepsilon D_x \gamma_1^1 + \varepsilon^2 D_x \gamma_1^2 + \cdots\right] \\
&\quad + \left(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots\right) \left[\varepsilon^{-1} D_y \gamma_1^{-1} + D_y \gamma_1^0 + \varepsilon D_y \gamma_1^1 + \varepsilon^2 D_y \gamma_1^2 + \cdots\right] \\
&\quad + \left(w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots\right) \frac{1}{\varepsilon} \left[\varepsilon^{-1} D_z \gamma_1^{-1} + D_z \gamma_1^0\right. \\
&\quad \left. + \varepsilon D_z \gamma_1^1 + \varepsilon^2 D_z \gamma_1^2 + \cdots\right] \\
&\quad - \left(\varepsilon^{-1} \gamma_1^{-1} + \gamma_1^0 + \varepsilon \gamma_1^1 + \varepsilon^2 \gamma_1^2 + \cdots\right) \left[D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots\right] \\
&\quad - \left(\varepsilon^{-1} \gamma_2^{-1} + \gamma_2^0 + \varepsilon \gamma_2^1 + \varepsilon^2 \gamma_2^2 + \cdots\right) \left[D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots\right] \\
&\quad - \left(\gamma_3^0 + \varepsilon \gamma_3^1 + \varepsilon^2 \gamma_3^2 + \cdots\right) \frac{1}{\varepsilon} \left[D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right] \\
&\quad = 2\phi \left\{ D_y \left[(\cos\varphi) \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots\right)\right] + (\operatorname{sen}\varphi) \frac{1}{\varepsilon} \left(D_z u^0 + \varepsilon D_z u^1 \varepsilon^2 D_z u^2\right) \right\}
\end{aligned}
$$

Agrupando en potencias de ε,

$$
\begin{aligned}
&\varepsilon^{-2} w^0 D_z \gamma_1^{-1} + \varepsilon^{-1} \left(D_t \gamma_1^{-1} + u^0 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^{-1} + w^0 D_z \gamma_1^0 + w^1 D_z \gamma_1^{-1}\right. \\
&\quad \left. - \gamma_1^{-1} D_x u^0 - \gamma_2^{-1} D_y u^0 - \gamma_3^0 D_z u^0 - 2\phi \left(\operatorname{sen}\varphi\right) D_z u^0\right) \\
&\quad + \varepsilon^0 \left[D_t \gamma_1^0 + u^0 D_x \gamma_1^0 + u^1 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^0 + v^1 D_y \gamma_1^{-1}\right. \\
&\quad + w^0 D_z \gamma_1^1 + w^1 D_z \gamma_1^0 + w^2 D_z \gamma_1^{-1} - \gamma_1^{-1} D_x u^1 - \gamma_1^0 D_x u^0 - \gamma_2^{-1} D_y u^1 - \gamma_2^0 D_y u^0 \\
&\quad \left. - \gamma_3^0 D_z u^1 - \gamma_3^1 D_z u^0 - 2\phi \left(D_y((\cos\varphi)\, u^0) + (\operatorname{sen}\varphi)\, D_z u^1\right)\right] \\
&\quad + \varepsilon \left[D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + u^1 D_x \gamma_1^0 + u^2 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^1 + v^1 D_y \gamma_1^0 + v^2 D_y \gamma_1^{-1}\right. \\
&\quad + w^0 D_z \gamma_1^2 + w^1 D_z \gamma_1^1 + w^2 D_z \gamma_1^0 + w^3 D_z \gamma_1^{-1} - \gamma_1^{-1} D_x u^2 - \gamma_1^0 D_x u^1 \\
&\quad - \gamma_1^1 D_x u^0 - \gamma_2^{-1} D_y u^2 - \gamma_2^0 D_y u^1 - \gamma_2^1 D_y u^0 - \gamma_3^0 D_z u^2 - \gamma_3^1 D_z u^1 - \gamma_3^2 D_z u^0 \\
&\quad \left. - 2\phi \left(D_y((\cos\varphi)\, u^1) + (\operatorname{sen}\varphi)\, D_z u^2\right)\right] \\
&\quad + \varepsilon^2 \left[D_t \gamma_1^2 + u^0 D_x \gamma_1^2 + u^1 D_x \gamma_1^1 + u^2 D_x \gamma_1^0 + u^3 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^2 + v^1 D_y \gamma_1^1\right. \\
&\quad + v^2 D_y \gamma_1^0 + v^3 D_y \gamma_1^{-1} + w^0 D_z \gamma_1^3 + w^1 D_z \gamma_1^2 + w^2 D_z \gamma_1^1 + w^3 D_z \gamma_1^0 + w^4 D_z \gamma_1^{-1}
\end{aligned}
$$

$$- \gamma_1^{-1} D_x u^3 - \gamma_1^0 D_x u^2 - \gamma_1^1 D_x u^1 - \gamma_1^2 D_x u^0 - \gamma_2^{-1} D_y u^3 - \gamma_2^0 D_y u^2$$
$$- \gamma_2^1 D_y u^1 - \gamma_2^2 D_y u^0 - \gamma_3^0 D_z u^3 - \gamma_3^1 D_z u^2 - \gamma_3^2 D_z u^1 - \gamma_3^3 D_z u^0$$
$$- 2\phi \left(D_y((\cos\varphi)\, u^2) + (\operatorname{sen}\varphi)\, D_z u^3 \right) \Big] + O(\varepsilon^3) = 0 \qquad (4.5.9)$$

Repetimos el proceso para la segunda ecuación de la vorticidad ((4.4.11)) obteniendo en este caso al agrupar:

$$\varepsilon^{-2} w^0 D_z \gamma_2^{-1} + \varepsilon^{-1} \left(D_t \gamma_2^{-1} + u^0 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^{-1} + w^0 D_z \gamma_2^0 + w^1 D_z \gamma_2^{-1} \right.$$
$$\left. - \gamma_1^{-1} D_x v^0 - \gamma_2^{-1} D_y v^0 - \gamma_3^0 D_z v^0 - 2\phi\, (\operatorname{sen}\varphi)\, D_z v^0 \right)$$
$$+ \varepsilon^0 \left[D_t \gamma_2^0 + u^0 D_x \gamma_2^0 + u^1 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^0 + v^1 D_y \gamma_2^{-1} \right.$$
$$+ w^0 D_z \gamma_2^1 + w^1 D_z \gamma_2^0 + w^2 D_z \gamma_2^{-1} - \gamma_1^{-1} D_x v^1 - \gamma_1^0 D_x v^0 - \gamma_2^{-1} D_y v^1 - \gamma_2^0 D_y v^0$$
$$\left. - \gamma_3^0 D_z v^1 - \gamma_3^1 D_z v^0 - 2\phi \left((\operatorname{sen}\varphi)\, D_z v^1 + (\cos\varphi)\, D_y v^0 \right) \right]$$
$$+ \varepsilon \left[D_t \gamma_2^1 + u^0 D_x \gamma_2^1 + u^1 D_x \gamma_2^0 + u^2 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^1 + v^1 D_y \gamma_2^0 + v^2 D_y \gamma_2^{-1} \right.$$
$$+ w^0 D_z \gamma_2^2 + w^1 D_z \gamma_2^1 + w^2 D_z \gamma_2^0 + w^3 D_z \gamma_2^{-1} - \gamma_1^{-1} D_x v^2 - \gamma_1^0 D_x v^1$$
$$- \gamma_1^1 D_x v^0 - \gamma_2^{-1} D_y v^2 - \gamma_2^0 D_y v^1 - \gamma_2^1 D_y v^0 - \gamma_3^0 D_z v^2 - \gamma_3^1 D_z v^1 - \gamma_3^2 D_z v^0$$
$$\left. - 2\phi \left((\operatorname{sen}\varphi)\, D_z v^2 + (\cos\varphi)\, D_y v^1 \right) \right]$$
$$+ \varepsilon^2 \left[D_t \gamma_2^2 + u^0 D_x \gamma_2^2 + u^1 D_x \gamma_2^1 + u^2 D_x \gamma_2^0 + u^3 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^2 + v^1 D_y \gamma_2^1 \right.$$
$$+ v^2 D_y \gamma_2^0 + v^3 D_y \gamma_2^{-1} + w^0 D_z \gamma_2^3 + w^1 D_z \gamma_2^2 + w^2 D_z \gamma_2^1 + w^3 D_z \gamma_2^0 + w^4 D_z \gamma_2^{-1}$$
$$- \gamma_1^{-1} D_x v^3 - \gamma_1^0 D_x v^2 - \gamma_1^1 D_x v^1 - \gamma_1^2 D_x v^0 - \gamma_2^{-1} D_y v^3 - \gamma_2^0 D_y v^2$$
$$- \gamma_2^1 D_y v^1 - \gamma_2^2 D_y v^0 - \gamma_3^0 D_z v^3 - \gamma_3^1 D_z v^2 - \gamma_3^2 D_z v^1 - \gamma_3^3 D_z v^0$$
$$\left. - 2\phi \left((\operatorname{sen}\varphi)\, D_z v^3 + (\cos\varphi)\, D_y v^2 \right) \right] + O(\varepsilon^3) = 0 \qquad (4.5.10)$$

La tercera ecuación de vorticidad ((4.4.12)) al sustituir los desarrollos (4.5.1) resulta:

$$D_t \gamma_3^0 + \varepsilon D_t \gamma_3^1 + \varepsilon^2 D_t \gamma_3^2 + \cdots$$
$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots) \left[D_x \gamma_3^0 + \varepsilon D_x \gamma_3^1 + \varepsilon^2 D_x \gamma_3^2 + \cdots \right]$$
$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots) \left[D_y \gamma_3^0 + \varepsilon D_y \gamma_3^1 + \varepsilon^2 D_y \gamma_3^2 + \cdots \right]$$
$$+ (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots) \frac{1}{\varepsilon} \left[D_z \gamma_3^0 + \varepsilon D_z \gamma_3^1 + \varepsilon^2 D_z \gamma_3^2 + \cdots \right]$$

$$
\begin{aligned}
&- \left(\varepsilon^{-1}\gamma_1^{-1} + \gamma_1^0 + \varepsilon\gamma_1^1 + \varepsilon^2\gamma_1^2 + \cdots\right)\left[D_x w^0 + \varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \cdots\right] \\
&- \left(\varepsilon^{-1}\gamma_2^{-1} + \gamma_2^0 + \varepsilon\gamma_2^1 + \varepsilon^2\gamma_2^2 + \cdots\right)\left[D_y w^0 + \varepsilon D_y w^1 + \varepsilon^2 D_y w^2 + \cdots\right] \\
&- \left(\gamma_3^0 + \varepsilon\gamma_3^1 + \varepsilon^2\gamma_3^2 + \cdots\right)\frac{1}{\varepsilon}\left[D_z w^0 + \varepsilon D_z w^1 + \varepsilon^2 D_z w^2 + \cdots\right] \\
&= -2\phi\left\{(\operatorname{sen}\varphi)\left(D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots\right)\right. \\
&\left. + D_y\left[(\operatorname{sen}\varphi)\left(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots\right) - (\cos\varphi)\left(w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \cdots\right)\right]\right\}
\end{aligned}
$$

Agrupando los coeficientes de cada potencia de ε se tiene:

$$
\begin{aligned}
&\varepsilon^{-1}\left(w^0 D_z\gamma_3^0 - \gamma_1^{-1}D_x w^0 - \gamma_2^{-1}D_y w^0 - \gamma_3^0 D_z w^0\right) \\
&\quad + \varepsilon^0\left\{D_t\gamma_3^0 + u^0 D_x\gamma_3^0 + v^0 D_y\gamma_3^0 + w^0 D_z\gamma_3^1 + w^1 D_z\gamma_3^0 - \gamma_1^{-1}D_x w^1\right. \\
&\quad - \gamma_1^0 D_x w^0 - \gamma_2^{-1}D_y w^1 - \gamma_2^0 D_y w^0 - \gamma_3^0 D_z w^1 - \gamma_3^1 D_z w^0 \\
&\quad \left. + 2\phi\left[(\operatorname{sen}\varphi)D_x u^0 + D_y\left((\operatorname{sen}\varphi)v^0 - (\cos\varphi)w^0\right)\right]\right\} \\
&\quad + \varepsilon\left\{D_t\gamma_3^1 + u^0 D_x\gamma_3^1 + u^1 D_x\gamma_3^0 + v^0 D_y\gamma_3^1 + v^1 D_y\gamma_3^0 + w^0 D_z\gamma_3^2 + w^1 D_z\gamma_3^1\right. \\
&\quad + w^2 D_z\gamma_3^0 - \gamma_1^{-1}D_x w^2 - \gamma_1^0 D_x w^1 - \gamma_1^1 D_x w^0 - \gamma_2^{-1}D_y w^2 - \gamma_2^0 D_y w^1 - \gamma_2^1 D_y w^0 \\
&\quad - \gamma_3^0 D_z w^2 - \gamma_3^1 D_z w^1 - \gamma_3^2 D_z w^0 \\
&\quad \left. + 2\phi\left[(\operatorname{sen}\varphi)D_x u^1 + D_y\left((\operatorname{sen}\varphi)v^1 - (\cos\varphi)w^1\right)\right]\right\} + O(\varepsilon^2) = 0 \qquad (4.5.11)
\end{aligned}
$$

Si se sustituyen, ahora, en las relaciones entre las componentes de la vorticidad y la velocidad (4.4.13) $u(\varepsilon)$, $v(\varepsilon)$, $w(\varepsilon)$, $\gamma_1(\varepsilon)$, $\gamma_2(\varepsilon)$ y $\gamma_3(\varepsilon)$ por sus expresiones en serie de potencias de ε se obtiene :

$$
\begin{aligned}
&\varepsilon^{-1}\gamma_1^{-1} + \gamma_1^0 + \varepsilon\gamma_1^1 + \varepsilon^2\gamma_1^2 + \cdots = D_y w^0 + \varepsilon D_y w^1 + \varepsilon^2 D_y w^2 + \cdots \\
&\qquad - \frac{1}{\varepsilon}\left(D_z v^0 + \varepsilon D_z v^1 + \varepsilon^2 D_z v^2 + \varepsilon^3 D_z v^3 + \cdots\right) \\
&\varepsilon^{-1}\gamma_2^{-1} + \gamma_2^0 + \varepsilon\gamma_2^1 + \varepsilon^2\gamma_2^2 + \cdots = \frac{1}{\varepsilon}\left(D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \varepsilon^3 D_z u^3 + \cdots\right) \\
&\qquad - \left(D_x w^0 + \varepsilon D_x w^1 + \varepsilon^2 D_x w^2 + \cdots\right) \\
&\gamma_3^0 + \varepsilon\gamma_3^1 + \varepsilon^2\gamma_3^2 + \cdots = D_x v^0 + \varepsilon D_x v^1 + \varepsilon^2 D_x v^2 + \cdots \\
&\qquad - \left(D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots\right)
\end{aligned}
$$

Agrupando en potencias de ε se obtiene:

$$\varepsilon^{-1}\left(\gamma_1^{-1} + D_z v^0\right) + \gamma_1^0 - D_y w^0 + D_z v^1 + \varepsilon\left(\gamma_1^1 - D_y w^1 + D_z v^2\right)$$
$$+ \varepsilon^2\left(\gamma_1^2 - D_y w^2 + D_z v^3\right) + O(\varepsilon^3) = 0 \tag{4.5.12}$$

$$\varepsilon^{-1}\left(\gamma_2^{-1} - D_z u^0\right) + \gamma_2^0 - D_z u^1 + D_x w^0 + \varepsilon\left(\gamma_2^1 - D_z u^2 + D_x w^1\right)$$
$$+ \varepsilon^2\left(\gamma_2^2 - D_z u^3 + D_x w^2\right) + O(\varepsilon^3) = 0 \tag{4.5.13}$$

$$\gamma_3^0 - D_x v^0 + D_y u^0 + \varepsilon\left(\gamma_3^1 - D_x v^1 + D_y u^1\right) + \varepsilon^2\left(\gamma_3^2 - D_x v^2 + D_y u^2\right)$$
$$+ O(\varepsilon^3) = 0 \tag{4.5.14}$$

Puesto que u^0, v^0, w^0, p^0, γ_i^{-1}, u^1, v^1, w^1, etc. son independientes de ε, una vez agrupados los términos que multiplican a una misma potencia de ε, en las ecuaciones anteriores obtenemos un polinomio en ε igualado a cero, por lo que sus coeficientes han de ser nulos. De este modo se logra una serie de ecuaciones que nos permitirán determinar u^0, v^0, w^0, p^0, γ_i^{-1}, u^1, v^1, w^1, etc.

Comenzamos por los coeficientes de ε^{-2} que aparecen en (4.5.9)-(4.5.10):

$$w^0 D_z \gamma_1^{-1} = 0 \tag{4.5.15}$$

$$w^0 D_z \gamma_2^{-1} = 0 \tag{4.5.16}$$

Lo mismo hacemos con los coeficiente de ε^{-1} de las ecuaciones (4.5.2)-(4.5.5) y (4.5.9)-(4.5.13):

$$w^0 D_z u^0 = 0 \tag{4.5.17}$$

$$w^0 D_z v^0 = 0 \tag{4.5.18}$$

$$w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 = 0 \tag{4.5.19}$$

$$D_z w^0 = 0 \tag{4.5.20}$$

$$D_t \gamma_1^{-1} + u^0 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^{-1} + w^0 D_z \gamma_1^0 + w^1 D_z \gamma_1^{-1}$$
$$- \gamma_1^{-1} D_x u^0 - \gamma_2^{-1} D_y u^0 - \gamma_3^0 D_z u^0 - 2\phi\,(\operatorname{sen}\varphi)\, D_z u^0 = 0 \tag{4.5.21}$$

$$D_t \gamma_2^{-1} + u^0 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^{-1} + w^0 D_z \gamma_2^0 + w^1 D_z \gamma_2^{-1}$$
$$- \gamma_1^{-1} D_x v^0 - \gamma_2^{-1} D_y v^0 - \gamma_3^0 D_z v^0 - 2\phi\,(\operatorname{sen}\varphi)\, D_z v^0 = 0 \tag{4.5.22}$$

$$w^0 D_z \gamma_3^0 - \gamma_1^{-1} D_x w^0 - \gamma_2^{-1} D_y w^0 - \gamma_3^0 D_z w^0 = 0 \tag{4.5.23}$$

$$\gamma_1^{-1} + D_z v^0 = 0 \tag{4.5.24}$$

$$\gamma_2^{-1} - D_z u^0 = 0 \tag{4.5.25}$$

Igualando a cero los coeficientes de ε^0 que aparecen en (4.5.2)-(4.5.14) tenemos las siguientes igualdades:

$$D_t u^0 + u^0 D_x u^0 + v^0 D_y u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0$$
$$- 2\phi \left((\operatorname{sen}\varphi)\, v^0 - (\cos\varphi)\, w^0 \right) = 0 \tag{4.5.26}$$

$$D_t v^0 + u^0 D_x v^0 + v^0 D_y v^0 + w^0 D_z v^1 + w^1 D_z v^0 + \frac{1}{\rho_0} D_y p^0$$
$$+ 2\phi\, (\operatorname{sen}\varphi)\, u^0 = 0 \tag{4.5.27}$$

$$D_t w^0 + u^0 D_x w^0 + v^0 D_y w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1$$
$$+ g - 2\phi\, (\cos\varphi)\, u^0 = 0 \tag{4.5.28}$$

$$D_x u^0 + D_y v^0 + D_z w^1 = 0 \tag{4.5.29}$$

$$p^0 = p_s \text{ en } z = 1 \tag{4.5.30}$$

$$w^0 = 0 \text{ en } z = 0 \tag{4.5.31}$$

$$\frac{\partial h}{\partial t} + \int_0^1 \left(\frac{\partial (h u^0)}{\partial x} + \frac{\partial (h v^0)}{\partial y} \right) dz = 0 \tag{4.5.32}$$

$$D_t \gamma_1^0 + u^0 D_x \gamma_1^0 + u^1 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^0 + v^1 D_y \gamma_1^{-1}$$
$$+ w^0 D_z \gamma_1^1 + w^1 D_z \gamma_1^0 + w^2 D_z \gamma_1^{-1} - \gamma_1^{-1} D_x u^1 - \gamma_1^0 D_x u^0 - \gamma_2^{-1} D_y u^1 - \gamma_2^0 D_y u^0$$
$$- \gamma_3^0 D_z u^1 - \gamma_3^1 D_z u^0 - 2\phi \left(D_y((\cos\varphi)\, u^0) + (\operatorname{sen}\varphi)\, D_z u^1 \right) = 0 \tag{4.5.33}$$

$$D_t \gamma_2^0 + u^0 D_x \gamma_2^0 + u^1 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^0 + v^1 D_y \gamma_2^{-1}$$
$$+ w^0 D_z \gamma_2^1 + w^1 D_z \gamma_2^0 + w^2 D_z \gamma_2^{-1} - \gamma_1^{-1} D_x v^1 - \gamma_1^0 D_x v^0 - \gamma_2^{-1} D_y v^1$$
$$- \gamma_2^0 D_y v^0 - \gamma_3^0 D_z v^1 - \gamma_3^1 D_z v^0 - 2\phi \left((\operatorname{sen}\varphi)\, D_z v^1 + (\cos\varphi)\, D_y v^0 \right) = 0 \tag{4.5.34}$$

$$D_t \gamma_3^0 + u^0 D_x \gamma_3^0 + v^0 D_y \gamma_3^0 + w^0 D_z \gamma_3^1 + w^1 D_z \gamma_3^0 - \gamma_1^{-1} D_x w^1$$
$$- \gamma_1^0 D_x w^0 - \gamma_2^{-1} D_y w^1 - \gamma_2^0 D_y w^0 - \gamma_3^0 D_z w^1 - \gamma_3^1 D_z w^0$$

$$+ 2\phi \left[(\operatorname{sen}\varphi)\, D_x u^0 + D_y \left((\operatorname{sen}\varphi)\, v^0 - (\cos\varphi)\, w^0 \right) \right] = 0 \tag{4.5.35}$$

$$\gamma_1^0 - D_y w^0 + D_z v^1 = 0 \tag{4.5.36}$$

$$\gamma_2^0 - D_z u^1 + D_x w^0 = 0 \tag{4.5.37}$$

$$\gamma_3^0 - D_x v^0 + D_y u^0 = 0 \tag{4.5.38}$$

Como consecuencia de las igualdades (4.5.20) y (4.5.31):

$$w^0 = 0, \tag{4.5.39}$$

con lo que:

$$w(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots .$$

Usando (4.5.39), la igualdad (4.5.19) se reduce a $D_z p^0 = 0$ que, junto con (4.5.30), nos permite obtener el término de orden 0 de la presión:

$$p^0 = p_s(t, x, y) \tag{4.5.40}$$

Las ecuaciones (4.5.21) y (4.5.22), teniendo en cuenta que $w^0 = 0$ ((4.5.39)), $D_z v^0 = -\gamma_1^{-1}$ ((4.5.24)) y $D_z u^0 = \gamma_2^{-1}$ ((4.5.25)), se pueden escribir como sigue:

$$D_t \gamma_1^{-1} + u^0 D_x \gamma_1^{-1} + v^0 D_y \gamma_1^{-1} + w^1 D_z \gamma_1^{-1}$$
$$- \gamma_1^{-1} D_x u^0 - \gamma_2^{-1} D_y u^0 - \gamma_3^0 \gamma_2^{-1} - 2\phi\, (\operatorname{sen}\varphi)\, \gamma_2^{-1} = 0 \tag{4.5.41}$$

$$D_t \gamma_2^{-1} + u^0 D_x \gamma_2^{-1} + v^0 D_y \gamma_2^{-1} + w^1 D_z \gamma_2^{-1}$$
$$- \gamma_1^{-1} D_x v^0 - \gamma_2^{-1} D_y v^0 + \gamma_3^0 \gamma_1^{-1} + 2\phi\, (\operatorname{sen}\varphi)\, \gamma_1^{-1} = 0 \tag{4.5.42}$$

Si las condiciones iniciales y de contorno son las adecuadas (homogéneas), este sistema tiene como solución única $\gamma_1^{-1} = \gamma_2^{-1} = 0$ (véase página 129 de [20]). El caso contrario es lo mismo que suponer que la vorticidad inicial o en la frontera es muy grande (de orden ε^{-1}), lo que nos obligaría probablemente a emplear las ecuaciones de Navier-Stokes por estar en un régimen turbulento. Es por ello que supondremos que γ_1^{-1} y γ_2^{-1} se anulan en el instante inicial y en la frontera (es decir, $\gamma_1(\varepsilon)$ y $\gamma_2(\varepsilon)$ son $O(1)$ en $t = 0$ y en la frontera) y como consecuencia obtenemos que

$$\gamma_1^{-1} = \gamma_2^{-1} = 0 \tag{4.5.43}$$

Se deduce ahora de (4.5.24) y (4.5.25) que

$$\frac{\partial u^0}{\partial z} = 0 \tag{4.5.44}$$

$$\frac{\partial v^0}{\partial z} = 0 \tag{4.5.45}$$

Teniendo en cuenta las igualdades (4.5.39), (4.5.40), (4.5.43)-(4.5.45), así como (4.3.4), se pueden reescribir las ecuaciones (4.5.26)-(4.5.29) y (4.5.32)-(4.5.38) de la forma siguiente:

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\,v^0 \tag{4.5.46}$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,u^0 \tag{4.5.47}$$

$$\frac{1}{\rho_0} D_z p^1 = -g + 2\phi\,(\cos\varphi)\,u^0 \tag{4.5.48}$$

$$\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} + D_z w^1 = 0 \tag{4.5.49}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (h u^0)}{\partial x} + \frac{\partial (h v^0)}{\partial y} = 0 \tag{4.5.50}$$

$$\begin{aligned} &D_t\gamma_1^0 + u^0 D_x\gamma_1^0 + v^0 D_y\gamma_1^0 + w^1 D_z\gamma_1^0 - \gamma_1^0\frac{\partial u^0}{\partial x} \\ &\quad - \gamma_2^0\frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^1 - 2\phi\left(\frac{\partial}{\partial y}((\cos\varphi)\,u^0) + (\operatorname{sen}\varphi)\,D_z u^1\right) = 0 \end{aligned} \tag{4.5.51}$$

$$\begin{aligned} &D_t\gamma_2^0 + u^0 D_x\gamma_2^0 + v^0 D_y\gamma_2^0 + w^1 D_z\gamma_2^0 - \gamma_1^0\frac{\partial v^0}{\partial x} \\ &\quad - \gamma_2^0\frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^1 - 2\phi\left((\operatorname{sen}\varphi)\,D_z v^1 + (\cos\varphi)\,\frac{\partial v^0}{\partial y}\right) = 0 \end{aligned} \tag{4.5.52}$$

$$\begin{aligned} &D_t\gamma_3^0 + u^0 D_x\gamma_3^0 + v^0 D_y\gamma_3^0 + w^1 D_z\gamma_3^0 - \gamma_3^0 D_z w^1 \\ &\quad + 2\phi\left[(\operatorname{sen}\varphi)\,\frac{\partial u^0}{\partial x} + \frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,v^0)\right] = 0 \end{aligned} \tag{4.5.53}$$

$$D_z v^1 = -\gamma_1^0 \tag{4.5.54}$$

$$D_z u^1 = \gamma_2^0 \tag{4.5.55}$$

$$\gamma_3^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \tag{4.5.56}$$

Continuamos igualando a cero los términos que multiplican a ε en las ecuaciones (4.5.2)-(4.5.7) y (4.5.9)-(4.5.14). Tenemos en cuenta a la hora de reescribir estos términos que $w^0 = \gamma_i^{-1} = \dfrac{\partial u^0}{\partial z} = \dfrac{\partial v^0}{\partial z} = 0 \quad (i = 1, 2)$ ((4.5.39), (4.5.43)-(4.5.45)), y obtenemos:

$$D_t u^1 + u^0 D_x u^1 + u^1 \frac{\partial u^0}{\partial x} + v^0 D_y u^1 + v^1 \frac{\partial u^0}{\partial y}$$

$$+ w^1 D_z u^1 + \frac{1}{\rho_0} D_x p^1 - 2\phi \left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1 \right) = 0 \tag{4.5.57}$$

$$D_t v^1 + u^0 D_x v^1 + u^1 \frac{\partial v^0}{\partial x} + v^0 D_y v^1 + v^1 \frac{\partial v^0}{\partial y}$$

$$+ w^1 D_z v^1 + \frac{1}{\rho_0} D_y p^1 + 2\phi\, (\operatorname{sen}\varphi)\, u^1 = 0 \tag{4.5.58}$$

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\phi\, (\cos\varphi)\, u^1 = 0 \tag{4.5.59}$$

$$D_x u^1 + D_y v^1 + D_z w^2 = 0 \tag{4.5.60}$$

$$p^1 = 0 \text{ en } z = 1 \tag{4.5.61}$$

$$w^1 = u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \text{ en } z = 0 \tag{4.5.62}$$

$$D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + u^1 D_x \gamma_1^0 + v^0 D_y \gamma_1^1 + v^1 D_y \gamma_1^0 + w^1 D_z \gamma_1^1 + w^2 D_z \gamma_1^0$$

$$- \gamma_1^0 D_x u^1 - \gamma_1^1 \frac{\partial u^0}{\partial x} - \gamma_2^0 D_y u^1 - \gamma_2^1 \frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^2 - \gamma_3^1 D_z u^1$$

$$- 2\phi \left(D_y((\cos\varphi)\, u^1) + (\operatorname{sen}\varphi)\, D_z u^2 \right) = 0 \tag{4.5.63}$$

$$D_t \gamma_2^1 + u^0 D_x \gamma_2^1 + u^1 D_x \gamma_2^0 + v^0 D_y \gamma_2^1 + v^1 D_y \gamma_2^0 + w^1 D_z \gamma_2^1 + w^2 D_z \gamma_2^0$$

$$- \gamma_1^0 D_x v^1 - \gamma_1^1 \frac{\partial v^0}{\partial x} - \gamma_2^0 D_y v^1 - \gamma_2^1 \frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^2 - \gamma_3^1 D_z v^1$$

$$- 2\phi \left((\operatorname{sen}\varphi)\, D_z v^2 + (\cos\varphi)\, D_y v^1 \right) = 0 \tag{4.5.64}$$

$$D_t \gamma_3^1 + u^0 D_x \gamma_3^1 + u^1 D_x \gamma_3^0 + v^0 D_y \gamma_3^1 + v^1 D_y \gamma_3^0 + w^1 D_z \gamma_3^1 + w^2 D_z \gamma_3^0$$

$$- \gamma_1^0 D_x w^1 - \gamma_2^0 D_y w^1 - \gamma_3^0 D_z w^2 - \gamma_3^1 D_z w^1$$

$$+ 2\phi \left[(\operatorname{sen}\varphi)\, D_x u^1 + D_y \left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1 \right) \right] = 0 \tag{4.5.65}$$

$$\gamma_1^1 = D_y w^1 - D_z v^2 \tag{4.5.66}$$

$$\gamma_2^1 = D_z u^2 - D_x w^1 \tag{4.5.67}$$

$$\gamma_3^1 = D_x v^1 - D_y u^1 \tag{4.5.68}$$

La integración de la ecuación (4.5.48) respecto a z, imponiendo la condición (4.5.61), nos proporciona la siguiente expresión para el término de orden 1 de la presión:

$$p^1 = \rho_0 h \left(2\phi (\cos\varphi)\, u^0 - g\right)(z-1) \tag{4.5.69}$$

lo que nos permite calcular

$$D_x p^1 = \rho_0 \left[2\phi (\cos\varphi) \frac{\partial u^0}{\partial x} h(z-1) - \frac{\partial s}{\partial x}\left(2\phi(\cos\varphi)\, u^0 - g\right)\right] \tag{4.5.70}$$

$$D_y p^1 = \rho_0 \left[2\phi \frac{\partial}{\partial y}\left((\cos\varphi)\, u^0\right) h(z-1) - \frac{\partial s}{\partial y}\left(2\phi(\cos\varphi)\, u^0 - g\right)\right] \tag{4.5.71}$$

Integramos también (4.5.49) respecto a z teniendo en cuenta que u^0 y v^0 no dependen de z e imponiendo la condición (4.5.62). Encontramos la siguiente expresión para w^1 en términos de u^0, v^0, H y h:

$$w^1 = u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \tag{4.5.72}$$

En resumen, tenemos las ecuaciones, igualdades y condiciones (4.5.39), (4.5.40), (4.5.43)-(4.5.47), (4.5.50)-(4.5.60), (4.5.63)-(4.5.72) y las que se obtienen al igualar a cero los coeficientes de ε^2 que aparecen en (4.5.2), (4.5.3), (4.5.5)-(4.5.7), (4.5.9)-(4.5.10) y (4.5.12)-(4.5.14), para el cálculo de h, u^k, v^k, w^k, p^k, γ_3^k $(k = 0, 1, 2, \ldots)$ y γ_i^j $(j = -1, 0, 1, \ldots, i = 1, 2)$ que nos permitirán construir una aproximación de la solución del problema (4.4.1)-(4.4.14):

$$w^0 = 0 \tag{4.5.73}$$

$$p^0 = p_s(t, x, y) \tag{4.5.74}$$

$$\gamma_1^{-1} = \gamma_2^{-1} = 0 \tag{4.5.75}$$

$$\frac{\partial u^0}{\partial z} = 0 \tag{4.5.76}$$

$$\frac{\partial v^0}{\partial z} = 0 \tag{4.5.77}$$

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi(\operatorname{sen}\varphi)\, v^0 \tag{4.5.78}$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} - 2\phi(\operatorname{sen}\varphi)\, u^0 \tag{4.5.79}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0 \tag{4.5.80}$$

$$p^1 = \rho_0 h \left(2\phi\,(\cos\varphi)\,u^0 - g\right)(z-1) \tag{4.5.81}$$

$$w^1 = u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \tag{4.5.82}$$

$$\begin{aligned} &D_t\gamma_1^0 + u^0 D_x\gamma_1^0 + v^0 D_y\gamma_1^0 + w^1 D_z\gamma_1^0 - \gamma_1^0\frac{\partial u^0}{\partial x} \\ &\quad - \gamma_2^0\frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^1 - 2\phi\left(\frac{\partial}{\partial y}((\cos\varphi)\,u^0) + (\operatorname{sen}\varphi)\,D_z u^1\right) = 0 \end{aligned} \tag{4.5.83}$$

$$\begin{aligned} &D_t\gamma_2^0 + u^0 D_x\gamma_2^0 + v^0 D_y\gamma_2^0 + w^1 D_z\gamma_2^0 - \gamma_1^0\frac{\partial v^0}{\partial x} \\ &\quad - \gamma_2^0\frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^1 - 2\phi\left((\operatorname{sen}\varphi)\,D_z v^1 + (\cos\varphi)\,\frac{\partial v^0}{\partial y}\right) = 0 \end{aligned} \tag{4.5.84}$$

$$\begin{aligned} &D_t\gamma_3^0 + u^0 D_x\gamma_3^0 + v^0 D_y\gamma_3^0 + w^1 D_z\gamma_3^0 - \gamma_3^0 D_z w^1 \\ &\quad + 2\phi\left[(\operatorname{sen}\varphi)\,\frac{\partial u^0}{\partial x} + \frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,v^0)\right] = 0 \end{aligned} \tag{4.5.85}$$

$$D_z v^1 = -\gamma_1^0 \tag{4.5.86}$$

$$D_z u^1 = \gamma_2^0 \tag{4.5.87}$$

$$\gamma_3^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \tag{4.5.88}$$

$$\begin{aligned} &D_t u^1 + u^0 D_x u^1 + u^1\frac{\partial u^0}{\partial x} + v^0 D_y u^1 + v^1\frac{\partial u^0}{\partial y} + w^1 D_z u^1 + 2\phi\,(\cos\varphi)\,\frac{\partial u^0}{\partial x}h(z-1) \\ &\quad - \frac{\partial s}{\partial x}\left(2\phi\,(\cos\varphi)\,u^0 - g\right) - 2\phi\left((\operatorname{sen}\varphi)\,v^1 - (\cos\varphi)\,w^1\right) = 0 \end{aligned} \tag{4.5.89}$$

$$\begin{aligned} &D_t v^1 + u^0 D_x v^1 + u^1\frac{\partial v^0}{\partial x} + v^0 D_y v^1 + v^1\frac{\partial v^0}{\partial y} + w^1 D_z v^1 + 2\phi\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h(z-1) \\ &\quad - \frac{\partial s}{\partial y}\left(2\phi\,(\cos\varphi)\,u^0 - g\right) + 2\phi\,(\operatorname{sen}\varphi)\,u^1 = 0 \end{aligned} \tag{4.5.90}$$

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0}D_z p^2 - 2\phi\,(\cos\varphi)\,u^1 = 0 \tag{4.5.91}$$

$$D_x u^1 + D_y v^1 + D_z w^2 = 0 \tag{4.5.92}$$

$$\begin{aligned} &D_t\gamma_1^1 + u^0 D_x\gamma_1^1 + u^1 D_x\gamma_1^0 + v^0 D_y\gamma_1^1 + v^1 D_y\gamma_1^0 + w^1 D_z\gamma_1^1 + w^2 D_z\gamma_1^0 \\ &\quad - \gamma_1^0 D_x u^1 - \gamma_1^1 \frac{\partial u^0}{\partial x} - \gamma_2^0 D_y u^1 - \gamma_2^1 \frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^2 - \gamma_3^1 D_z u^1 \\ &\quad - 2\phi\left(D_y((\cos\varphi)\, u^1) + (\operatorname{sen}\varphi)\, D_z u^2\right) = 0 \end{aligned} \tag{4.5.93}$$

$$\begin{aligned} &D_t\gamma_2^1 + u^0 D_x\gamma_2^1 + u^1 D_x\gamma_2^0 + v^0 D_y\gamma_2^1 + v^1 D_y\gamma_2^0 + w^1 D_z\gamma_2^1 + w^2 D_z\gamma_2^0 \\ &\quad - \gamma_1^0 D_x v^1 - \gamma_1^1 \frac{\partial v^0}{\partial x} - \gamma_2^0 D_y v^1 - \gamma_2^1 \frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^2 - \gamma_3^1 D_z v^1 \\ &\quad - 2\phi\left((\operatorname{sen}\varphi)\, D_z v^2 + (\cos\varphi)\, D_y v^1\right) = 0 \end{aligned} \tag{4.5.94}$$

$$\begin{aligned} &D_t\gamma_3^1 + u^0 D_x\gamma_3^1 + u^1 D_x\gamma_3^0 + v^0 D_y\gamma_3^1 + v^1 D_y\gamma_3^0 + w^1 D_z\gamma_3^1 + w^2 D_z\gamma_3^0 \\ &\quad - \gamma_1^0 D_x w^1 - \gamma_2^0 D_y w^1 - \gamma_3^0 D_z w^2 - \gamma_3^1 D_z w^1 \\ &\quad + 2\phi\left[(\operatorname{sen}\varphi)\, D_x u^1 + D_y\left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1\right)\right] = 0 \end{aligned} \tag{4.5.95}$$

$$\gamma_1^1 = D_y w^1 - D_z v^2 \tag{4.5.96}$$

$$\gamma_2^1 = D_z u^2 - D_x w^1 \tag{4.5.97}$$

$$\gamma_3^1 = D_x v^1 - D_y u^1 \tag{4.5.98}$$

$$\begin{aligned} &D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 \frac{\partial u^0}{\partial x} + v^0 D_y u^2 + v^1 D_y u^1 + v^2 \frac{\partial u^0}{\partial y} \\ &\quad + w^1 D_z u^2 + w^2 D_z u^1 + \frac{1}{\rho_0} D_x p^2 - 2\phi\left((\operatorname{sen}\varphi)\, v^2 - (\cos\varphi)\, w^2\right) = 0 \end{aligned} \tag{4.5.99}$$

$$\begin{aligned} &D_t v^2 + u^0 D_x v^2 + u^1 D_x v^1 + u^2 \frac{\partial v^0}{\partial x} + v^0 D_y v^2 + v^1 D_y v^1 + v^2 \frac{\partial v^0}{\partial y} \\ &\quad + w^1 D_z v^2 + w^2 D_z v^1 + \frac{1}{\rho_0} D_y p^2 + 2\phi\,(\operatorname{sen}\varphi)\, u^2 = 0 \end{aligned} \tag{4.5.100}$$

$$D_x u^2 + D_y v^2 + D_z w^3 = 0 \tag{4.5.101}$$

$$p^2 = 0 \text{ en } z = 1 \tag{4.5.102}$$

$$w^2 = u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} = 0 \text{ en } z = 0 \tag{4.5.103}$$

$$D_t\gamma_1^2 + u^0 D_x\gamma_1^2 + u^1 D_x\gamma_1^1 + u^2 D_x\gamma_1^0 + v^0 D_y\gamma_1^2 + v^1 D_y\gamma_1^1 + v^2 D_y\gamma_1^0$$
$$+ w^1 D_z\gamma_1^2 + w^2 D_z\gamma_1^1 + w^3 D_z\gamma_1^0 - \gamma_1^0 D_x u^2 - \gamma_1^1 D_x u^1 - \gamma_1^2 \frac{\partial u^0}{\partial x}$$
$$- \gamma_2^0 D_y u^2 - \gamma_2^1 D_y u^1 - \gamma_2^2 \frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^3 - \gamma_3^1 D_z u^2 - \gamma_3^2 D_z u^1$$
$$- 2\phi \left(D_y((\cos\varphi)\, u^2) + (\operatorname{sen}\varphi)\, D_z u^3\right) = 0 \tag{4.5.104}$$

$$D_t\gamma_2^2 + u^0 D_x\gamma_2^2 + u^1 D_x\gamma_2^1 + u^2 D_x\gamma_2^0 + v^0 D_y\gamma_2^2 + v^1 D_y\gamma_2^1 + v^2 D_y\gamma_2^0$$
$$+ w^1 D_z\gamma_2^2 + w^2 D_z\gamma_2^1 + w^3 D_z\gamma_2^0 - \gamma_1^0 D_x v^2 - \gamma_1^1 D_x v^1 - \gamma_1^2 \frac{\partial v^0}{\partial x}$$
$$- \gamma_2^0 D_y v^2 - \gamma_2^1 D_y v^1 - \gamma_2^2 \frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^3 - \gamma_3^1 D_z v^2 - \gamma_3^2 D_z v^1$$
$$- 2\phi \left((\operatorname{sen}\varphi)\, D_z v^3 + (\cos\varphi)\, D_y v^2\right) = 0 \tag{4.5.105}$$

$$\gamma_1^2 = D_y w^2 - D_z v^3 \tag{4.5.106}$$
$$\gamma_2^2 = D_z u^3 - D_x w^2 \tag{4.5.107}$$
$$\gamma_3^2 = D_x v^2 - D_y u^2 \tag{4.5.108}$$

4.6. Aproximación de orden cero

Se considera la aproximación de orden cero en ε:

$$\tilde{u}(\varepsilon) = u^0$$
$$\tilde{v}(\varepsilon) = v^0$$
$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$
$$\tilde{p}(\varepsilon) = p^0$$
$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1}\gamma_i^{-1} + \gamma_i^0 \quad (i = 1, 2)$$
$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0$$

donde w^0, p^0 y γ_i^{-1} $(i = 1, 2)$ son conocidos ((4.5.73)-(4.5.75)).

Las ecuaciones a partir de las que se calculan u^0 y v^0 son (4.5.78) y (4.5.79)

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\,v^0$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,u^0$$

y el calado, resolviendo la ecuación (4.5.80)

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0$$

(suponemos conocidos $u^0(0,x,y)$, $v^0(0,x,y)$, $h(0,x,y)$ y el flujo de entrada y de salida, de aquí en adelante cuando condiciones de este tipo sean necesarias se supondrán conocidas).

Una vez calculados u^0, v^0 y h, w^1 viene dado por (4.5.82)

$$w^1 = u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)$$

La tercera componente de la vorticidad se puede obtener a partir de (4.5.88):

$$\gamma_3^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}$$

Para calcular las otras dos componentes se resuelven las ecuaciones (4.5.83) y (4.5.84) donde se sustituye según las expresiones (4.5.86)-(4.5.88): $D_z v^1 = -\gamma_1^0$, $D_z u^1 = \gamma_2^0$ y $\gamma_3^0 = \dfrac{\partial v^0}{\partial x} - \dfrac{\partial u^0}{\partial y}$

$$D_t\gamma_1^0 + u^0 D_x\gamma_1^0 + v^0 D_y\gamma_1^0 + w^1 D_z\gamma_1^0 - \gamma_1^0\frac{\partial u^0}{\partial x} - \gamma_2^0\frac{\partial v^0}{\partial x} - 2\phi\left(\frac{\partial}{\partial y}((\cos\varphi)\,u^0) + (\operatorname{sen}\varphi)\,\gamma_2^0\right) = 0 \qquad (4.6.1)$$

$$D_t\gamma_2^0 + u^0 D_x\gamma_2^0 + v^0 D_y\gamma_2^0 + w^1 D_z\gamma_2^0 - \gamma_1^0\frac{\partial u^0}{\partial y} - \gamma_2^0\frac{\partial v^0}{\partial y} - 2\phi\left(-(\operatorname{sen}\varphi)\,\gamma_1^0 + (\cos\varphi)\frac{\partial v^0}{\partial y}\right) = 0 \qquad (4.6.2)$$

(es necesario conocer las condiciones iniciales y de contorno).

Si deshacemos el cambio de variable, volviendo al dominio original, la aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, y, z) = u^0(t, x, y)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t, x, y, z) = v^0(t, x, y)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, y, z) = \varepsilon w^1(t, x, y, z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, y, z) = p^0(t, x, y)$$

$$\tilde{\gamma}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_i(\varepsilon)(t, x, y, z) = \gamma_i^0(t, x, y, z) \quad (i = 1, 2)$$

$$\tilde{\gamma}_3^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_3(\varepsilon)(t, x, y, z) = \gamma_3^0(t, x, y)$$

verifica,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{4.6.3}$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial z^\varepsilon} = 0 \tag{4.6.4}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{v}^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{4.6.5}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \tilde{v}^\varepsilon \tag{4.6.6}$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \tilde{u}^\varepsilon \tag{4.6.7}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{4.6.8}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.6.9}$$

$$\begin{aligned} &\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \\ &\quad - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - 2\phi \left(\frac{\partial}{\partial y^\varepsilon} ((\cos \varphi^\varepsilon)\, \tilde{u}^\varepsilon) + (\operatorname{sen} \varphi^\varepsilon)\, \tilde{\gamma}_2^\varepsilon \right) = 0 \end{aligned} \tag{4.6.10}$$

$$\begin{aligned} &\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \\ &\quad - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(-(\operatorname{sen} \varphi^\varepsilon)\, \tilde{\gamma}_1^\varepsilon + (\cos \varphi^\varepsilon) \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) = 0 \end{aligned} \tag{4.6.11}$$

$$\tilde{\gamma}_3^{\varepsilon} = \frac{\partial \tilde{v}^{\varepsilon}}{\partial x^{\varepsilon}} - \frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} \tag{4.6.12}$$

Nos interesa ahora estudiar el error que cometemos al sustituir la solución exacta de las ecuaciones de Euler por la aproximación que acabamos de construir. Para ello sustituimos la solución aproximada en las ecuaciones de Euler y observamos con qué orden de ε se verifican.

Si sustituimos la aproximación de orden cero en la primera de las ecuaciones de Euler, obtenemos:

$$\frac{\partial \tilde{u}^{\varepsilon}}{\partial t^{\varepsilon}} + \tilde{u}^{\varepsilon}\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon}\frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} + \tilde{w}^{\varepsilon}\frac{\partial \tilde{u}^{\varepsilon}}{\partial z^{\varepsilon}} = \frac{\partial \tilde{u}(\varepsilon)}{\partial t} + \tilde{u}(\varepsilon)\frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \tilde{v}(\varepsilon)\frac{\partial \tilde{u}(\varepsilon)}{\partial y}$$

$$= \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\,v^0$$

$$= -\frac{1}{\rho_0}\frac{\partial \tilde{p}^{\varepsilon}}{\partial x^{\varepsilon}} + 2\phi\,(\operatorname{sen}\varphi^{\varepsilon})\,\tilde{v}^{\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^{\varepsilon}}{\partial x^{\varepsilon}} + 2\phi\left((\operatorname{sen}\varphi^{\varepsilon})\,\tilde{v}^{\varepsilon} - (\cos\varphi^{\varepsilon})\,\tilde{w}^{\varepsilon}\right) + O(\varepsilon)$$

Por lo tanto, la primera ecuación se verifica con un error de orden ε.

Para la segunda ecuación de Euler se tiene que:

$$\frac{\partial \tilde{v}^{\varepsilon}}{\partial t^{\varepsilon}} + \tilde{u}^{\varepsilon}\frac{\partial \tilde{v}^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon}\frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}} + \tilde{w}^{\varepsilon}\frac{\partial \tilde{v}^{\varepsilon}}{\partial z^{\varepsilon}} = \frac{\partial \tilde{u}(\varepsilon)}{\partial t} + \tilde{u}(\varepsilon)\frac{\partial \tilde{v}(\varepsilon)}{\partial x} + \tilde{v}(\varepsilon)\frac{\partial \tilde{v}(\varepsilon)}{\partial y}$$

$$= \frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,u^0$$

$$= -\frac{1}{\rho_0}\frac{\partial \tilde{p}^{\varepsilon}}{\partial y^{\varepsilon}} - 2\phi\,(\operatorname{sen}\varphi^{\varepsilon})\,\tilde{u}^{\varepsilon}$$

Es decir, la segunda ecuación se verifica exactamente.

Veamos lo que sucede con la ecuación de Euler para la velocidad vertical:

$$\frac{\partial \tilde{w}^{\varepsilon}}{\partial t^{\varepsilon}} + \tilde{u}^{\varepsilon}\frac{\partial \tilde{w}^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon}\frac{\partial \tilde{w}^{\varepsilon}}{\partial y^{\varepsilon}} + \tilde{w}^{\varepsilon}\frac{\partial \tilde{w}^{\varepsilon}}{\partial z^{\varepsilon}} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^{\varepsilon}}{\partial z^{\varepsilon}} + g - 2\phi\,(\cos\varphi^{\varepsilon})\,u^{\varepsilon}$$

$$= D_t\tilde{w}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{w}(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon) + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z\tilde{p}(\varepsilon)$$

$$+ g - 2\phi\,(\cos\varphi)\,u(\varepsilon)$$

$$= \varepsilon D_t w^1 + \varepsilon u^0 D_x w^1 + \varepsilon v^0 D_y w^1 + \varepsilon w^1 D_z w^1 + g - 2\phi\,(\cos\varphi)\,u^0 = O(1)$$

Esta ecuación se cumple con un error de orden ε^0, o lo que es lo mismo, no se verifica ni tan siquiera aproximadamente.

La ecuación de la incompresibilidad se cumple de forma exacta, como se deduce de (4.6.9):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = 0$$

Es inmediato comprobar que las condiciones de contorno (4.1.6) y (4.1.7) se verifican exactamente (teniendo en cuenta (4.6.9) y (4.6.8)).

Se puede proponer un modelo de orden 0 en ε (al menos formalmente) que viene dado por las ecuaciones (4.6.3)-(4.6.9):

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{v}^\varepsilon)}{\partial y^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi)\,\tilde{v}^\varepsilon \\
&\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi)\,\tilde{u}^\varepsilon \\
&\tilde{p}^\varepsilon = p_s^\varepsilon \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)
\end{aligned} \tag{4.6.13}$$

donde $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ no dependen de z^ε, y es necesario conocer las condiciones iniciales y de contorno.

Además, una vez conocidos $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y $\tilde{w}^\varepsilon$, se pueden calcular las componentes de la vorticidad, las dos primeras resolviendo el sistema de ecuaciones (4.6.10)-(4.6.11)

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \\
&\qquad - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - 2\phi\left(\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon) + (\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_2^\varepsilon\right) = 0
\end{aligned} \tag{4.6.14}$$

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \\
&\qquad - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - 2\phi\left(-(\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_1^\varepsilon + (\cos\varphi^\varepsilon)\,\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) = 0
\end{aligned} \tag{4.6.15}$$

y la tercera componente mediante la expresión (4.6.12):

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \tag{4.6.16}$$

Observación 4.3 *La aproximación de* $\tilde{\gamma}_1^\varepsilon$ *y* $\tilde{\gamma}_2^\varepsilon$ *mediante las expresiones*

$$\tilde{\gamma}_1^\varepsilon = \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} - \frac{\partial \tilde{v}^\varepsilon}{\partial z^\varepsilon}, \quad \tilde{\gamma}_2^\varepsilon = \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} - \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon}$$

es peor que la que dan (4.6.14) *y* (4.6.15), *ya que de estas dos expresiones obtenemos*

$$\tilde{\gamma}_1^\varepsilon = \tilde{\gamma}_1(\varepsilon) = \varepsilon D_y w^1 - \frac{1}{\varepsilon} D_z v^0 = \varepsilon D_y w^1$$

$$\tilde{\gamma}_2^\varepsilon = \tilde{\gamma}_2(\varepsilon) = \frac{1}{\varepsilon} D_z u^0 - \varepsilon D_x w^1 = -\varepsilon D_x w^1$$

cuando de (4.5.86)-(4.5.87) *obtenemos*

$$\tilde{\gamma}_1^\varepsilon = \gamma_1^0 = -D_z v^1, \quad \tilde{\gamma}_2^\varepsilon = \gamma_2^0 = D_z u^1$$

es decir, en el primer caso obtenemos valores de orden $O(\varepsilon)$ *y en el segundo del orden* $O(1)$.

4.7. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación de orden 1 en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1}\gamma_i^{-1} + \gamma_i^0 + \varepsilon \gamma_i^1 \quad (i = 1, 2)$$

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0 + \varepsilon \gamma_3^1$$

Recordemos que w^0, p^0 y γ_i^{-1} $(i = 1, 2)$ son conocidos ((4.5.73)-(4.5.75)), u^0, v^0 y h se calculan resolviendo (4.5.78)-(4.5.80), y w^1 está determinado por (4.5.82) en función de u^0, v^0 y h.

También tenemos una expresión para p^1 en la que sólo es necesario conocer la profundidad del agua y u^0 ((4.5.81)):

$$p^1 = \rho_0 h \left(2\phi \left(\cos\varphi\right) u^0 - g\right)(z - 1)$$

Además, γ_1^0 y γ_2^0 se calculan resolviendo (4.6.1) y (4.6.2) mientras que γ_3^0 viene dado por la expresión (4.5.88).

La dependencia de z de u^1 y v^1 viene dada por (4.5.86)-(4.5.87):

$$D_z v^1 = -\gamma_1^0$$

$$D_z u^1 = \gamma_2^0$$

Es claro ahora que

$$u^1 = u^1\big|_{z=0} + \int_0^z \gamma_2^0 dz$$

$$v^1 = v^1\big|_{z=0} - \int_0^z \gamma_1^0 dz$$

Para obtener u^1 y v^1 a partir de estas expresiones debemos conocer explícitamente cómo dependen γ_1^0 y γ_2^0 de z, pero sólo sabemos que son solución de (4.6.1)-(4.6.2). Sin embargo, de (4.5.76), (4.5.77), (4.5.82) y (4.6.1)-(4.6.2) se deduce que, si las condiciones iniciales y de contorno sobre γ_1^0 y γ_2^0 son polinómicas en z, entonces γ_1^0 y γ_2^0 son también polinómicos en z.

Teniendo en cuenta que buscamos un modelo de aguas someras, no sería excesivamente restrictivo suponer que las condiciones iniciales y de contorno sobre γ_1^0 y γ_2^0 son independientes de z. Supongamos inicialmente que la dependencia en z es cuadrática, es decir,

$$\gamma_i^0 = \gamma_i^{0,0} + z\gamma_i^{0,1} + z^2\gamma_i^{0,2} \quad (i = 1, 2) \tag{4.7.1}$$

Sustituyendo en (4.6.1)-(4.6.2) tanto $\gamma_i^0 \quad (i = 1, 2)$ como w^1 (utilizando (4.7.1) y (4.5.82), respectivamente), de modo que la dependencia de z sea explícita, obtenemos:

$$\begin{aligned}
D_t(\gamma_1^{0,0} &+ z\gamma_1^{0,1} + z^2\gamma_1^{0,2}) + u^0 D_x(\gamma_1^{0,0} + z\gamma_1^{0,1} + z^2\gamma_1^{0,2}) \\
&+ v^0 D_y(\gamma_1^{0,0} + z\gamma_1^{0,1} + z^2\gamma_1^{0,2}) \\
&+ \left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] D_z(\gamma_1^{0,0} + z\gamma_1^{0,1} + z^2\gamma_1^{0,2}) \\
&- (\gamma_1^{0,0} + z\gamma_1^{0,1} + z^2\gamma_1^{0,2})\frac{\partial u^0}{\partial x} - (\gamma_2^{0,0} + z\gamma_2^{0,1} + z^2\gamma_2^{0,2})\frac{\partial v^0}{\partial x} \\
&- 2\phi\left[\frac{\partial}{\partial y}((\cos\varphi)\, u^0) + (\operatorname{sen}\varphi)\,(\gamma_2^{0,0} + z\gamma_2^{0,1} + z^2\gamma_2^{0,2})\right] = 0
\end{aligned}$$

$$\begin{aligned}
D_t(\gamma_2^{0,0} &+ z\gamma_2^{0,1} + z^2\gamma_2^{0,2}) + u^0 D_x(\gamma_2^{0,0} + z\gamma_2^{0,1} + z^2\gamma_2^{0,2}) \\
&+ v^0 D_y(\gamma_2^{0,0} + z\gamma_2^{0,1} + z^2\gamma_2^{0,2})
\end{aligned}$$

$$+\left[u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}-hz\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]D_z(\gamma_2^{0,0}+z\gamma_2^{0,1}+z^2\gamma_2^{0,2})$$

$$-(\gamma_1^{0,0}+z\gamma_1^{0,1}+z^2\gamma_1^{0,2})\frac{\partial u^0}{\partial y}-(\gamma_2^{0,0}+z\gamma_2^{0,1}+z^2\gamma_2^{0,2})\frac{\partial v^0}{\partial y}$$

$$-2\phi\left(-(\operatorname{sen}\varphi)(\gamma_1^{0,0}+z\gamma_1^{0,1}+z^2\gamma_1^{0,2})+(\cos\varphi)\frac{\partial v^0}{\partial y}\right)=0$$

Agrupando según las potencias de z y utilizando (4.5.80) obtenemos las seis ecuaciones siguientes:

$$\frac{\partial\gamma_1^{0,0}}{\partial t}+u^0\frac{\partial\gamma_1^{0,0}}{\partial x}+v^0\frac{\partial\gamma_1^{0,0}}{\partial y}-\gamma_1^{0,0}\frac{\partial u^0}{\partial x}-\gamma_2^{0,0}\frac{\partial v^0}{\partial x}$$

$$=2\phi\left(\frac{\partial}{\partial y}((\cos\varphi)\,u^0)+(\operatorname{sen}\varphi)\,\gamma_2^{0,0}\right)\tag{4.7.2}$$

$$\frac{\partial\gamma_1^{0,1}}{\partial t}+u^0\frac{\partial\gamma_1^{0,1}}{\partial x}+v^0\frac{\partial\gamma_1^{0,1}}{\partial y}-\gamma_1^{0,1}\frac{\partial u^0}{\partial x}-\gamma_2^{0,1}\frac{\partial v^0}{\partial x}=2\phi\,(\operatorname{sen}\varphi)\,\gamma_2^{0,1}\tag{4.7.3}$$

$$\frac{\partial\gamma_1^{0,2}}{\partial t}+u^0\frac{\partial\gamma_1^{0,2}}{\partial x}+v^0\frac{\partial\gamma_1^{0,2}}{\partial y}-\gamma_1^{0,2}\frac{\partial u^0}{\partial x}-\gamma_2^{0,2}\frac{\partial v^0}{\partial x}=2\phi\,(\operatorname{sen}\varphi)\,\gamma_2^{0,2}\tag{4.7.4}$$

$$\frac{\partial\gamma_2^{0,0}}{\partial t}+u^0\frac{\partial\gamma_2^{0,0}}{\partial x}+v^0\frac{\partial\gamma_2^{0,0}}{\partial y}-\gamma_1^{0,0}\frac{\partial u^0}{\partial y}-\gamma_2^{0,0}\frac{\partial v^0}{\partial y}$$

$$=2\phi\left(-\gamma_1^{0,0}\,(\operatorname{sen}\varphi)+(\cos\varphi)\,\frac{\partial v^0}{\partial y}\right)\tag{4.7.5}$$

$$\frac{\partial\gamma_2^{0,1}}{\partial t}+u^0\frac{\partial\gamma_2^{0,1}}{\partial x}+v^0\frac{\partial\gamma_2^{0,1}}{\partial y}-\gamma_1^{0,1}\frac{\partial u^0}{\partial y}-\gamma_2^{0,1}\frac{\partial v^0}{\partial y}=-2\phi\,(\operatorname{sen}\varphi)\,\gamma_1^{0,1}\tag{4.7.6}$$

$$\frac{\partial\gamma_2^{0,2}}{\partial t}+u^0\frac{\partial\gamma_2^{0,2}}{\partial x}+v^0\frac{\partial\gamma_2^{0,2}}{\partial y}-\gamma_1^{0,2}\frac{\partial u^0}{\partial y}-\gamma_2^{0,2}\frac{\partial v^0}{\partial y}=-2\phi\,(\operatorname{sen}\varphi)\,\gamma_1^{0,2}\tag{4.7.7}$$

En el caso más sencillo en el que las condiciones iniciales y de contorno de γ_1^0 y γ_2^0 no dependen de z obtenemos que el sistema formado por las ecuaciones (4.7.3) y (4.7.6) tiene por solución $\gamma_1^{0,1}=\gamma_2^{0,1}=0$, que se puede probar que es única (ver página 129 de [20]). Lo mismo sucede con el sistema (4.7.4) y (4.7.7). Por tanto, $\gamma_1^0=\gamma_1^{0,0}$ y $\gamma_2^0=\gamma_2^{0,0}$ no dependen de z y se pueden calcular resolviendo el sistema (4.7.2), (4.7.5). Hemos obtenido así que suponer que las condiciones iniciales de γ_1^0 y γ_2^0 no dependen de z, implica que γ_1^0 y γ_2^0 no dependen de z en todo el dominio. Esto justifica, en cierto modo, que trabajemos con γ_1^0 y γ_2^0 independientes de z, aunque otras elecciones son posibles (y llevan a modelos mucho más complicados). Ahora, al integrar (4.5.86)-(4.5.87) respecto a z, obtenemos:

$$u^1 = u_0^1 + h\gamma_2^0 z \tag{4.7.8}$$

$$v^1 = v_0^1 - h\gamma_1^0 z \tag{4.7.9}$$

donde $u_0^1(t,x,y) = u^1(t,x,y,0)$, $v_0^1(t,x,y) = v^1(t,x,y,0)$ están determinados por (4.5.89)-(4.5.90):

$$D_t u^1 + u^0 D_x u^1 + u^1 \frac{\partial u^0}{\partial x} + v^0 D_y u^1 + v^1 \frac{\partial u^0}{\partial y} + w^1 D_z u^1 + 2\phi\left(\cos\varphi\right)\frac{\partial u^0}{\partial x} h(z-1)$$
$$- \frac{\partial s}{\partial x}\big(2\phi\left(\cos\varphi\right)u^0 - g\big) - 2\phi\big(\left(\operatorname{sen}\varphi\right)v^1 - \left(\cos\varphi\right)w^1\big) = 0$$

$$D_t v^1 + u^0 D_x v^1 + u^1 \frac{\partial v^0}{\partial x} + v^0 D_y v^1 + v^1 \frac{\partial v^0}{\partial y} + w^1 D_z v^1$$
$$+ 2\phi\frac{\partial}{\partial y}\big(\left(\cos\varphi\right)u^0\big)h(z-1) - \frac{\partial s}{\partial y}\big(2\phi\left(\cos\varphi\right)u^0 - g\big) + 2\phi\left(\operatorname{sen}\varphi\right)u^1 = 0$$

Comenzamos por sustituir u^1 y v^1 (según (4.7.8) y (4.7.9)), y w^1 (según (4.5.82)) de modo que la dependencia de z sea explícita:

$$D_t\left(u_0^1 + h\gamma_2^0 z\right) + u^0 D_x\left(u_0^1 + h\gamma_2^0 z\right) + \left(u_0^1 + h\gamma_2^0 z\right)\frac{\partial u^0}{\partial x} + v^0 D_y\left(u_0^1 + h\gamma_2^0 z\right)$$
$$+ \left(v_0^1 - h\gamma_1^0 z\right)\frac{\partial u^0}{\partial y} + \left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] D_z\left(u_0^1 + h\gamma_2^0 z\right)$$
$$+ 2\phi\left(\cos\varphi\right)\frac{\partial u^0}{\partial x}h(z-1) - \frac{\partial s}{\partial x}\left(2\phi\left(\cos\varphi\right)u^0 - g\right)$$
$$- 2\phi\left\{\left(\operatorname{sen}\varphi\right)\left(v_0^1 - h\gamma_1^0 z\right) - \left(\cos\varphi\right)\left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\right\} = 0$$

$$D_t\left(v_0^1 - h\gamma_1^0 z\right) + u^0 D_x\left(v_0^1 - h\gamma_1^0 z\right) + \left(u_0^1 + h\gamma_2^0 z\right)\frac{\partial v^0}{\partial x} + v^0 D_y\left(v_0^1 - h\gamma_1^0 z\right)$$
$$+ \left(v_0^1 - h\gamma_1^0 z\right)\frac{\partial v^0}{\partial y} + \left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] D_z\left(v_0^1 - h\gamma_1^0 z\right)$$
$$+ 2\phi\frac{\partial}{\partial y}\left(\left(\cos\varphi\right)u^0\right)h(z-1) - \frac{\partial s}{\partial y}\left(2\phi\left(\cos\varphi\right)u^0 - g\right) + 2\phi\left(\operatorname{sen}\varphi\right)\left(u_0^1 + h\gamma_2^0 z\right) = 0$$

Agrupando los coeficientes de cada potencia de z y simplificando (teniendo en cuenta (4.7.2) y (4.7.5)), obtenemos:

$$\frac{\partial u_0^1}{\partial t} + u^0 \frac{\partial u_0^1}{\partial x} + u_0^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u_0^1}{\partial y} + v_0^1 \frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v_0^1$$

$$= 2\phi\,(\cos\varphi)\left(\frac{\partial(u^0 h)}{\partial x} - v^0 \frac{\partial H}{\partial y}\right) - g\frac{\partial s}{\partial x} \tag{4.7.10}$$

$$\frac{\partial v_0^1}{\partial t} + u^0 \frac{\partial v_0^1}{\partial x} + u_0^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v_0^1}{\partial y} + v_0^1 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u_0^1$$

$$= 2\phi\left(\frac{\partial}{\partial y}\big((\cos\varphi)\, u^0 h\big) + (\cos\varphi)\, u^0 \frac{\partial H}{\partial y}\right) - g\frac{\partial s}{\partial y} \tag{4.7.11}$$

(es necesario conocer las condiciones iniciales y de contorno).

Observación 4.4 *Los términos u_0^1 y v_0^1 están determinados de forma independiente a γ_i^0 $(i = 1, 2)$ mediante las ecuaciones* (4.7.10)-(4.7.11).

Además, sucede que los segundos miembros de estas ecuaciones se pueden escribir:

$$2\phi\,(\cos\varphi)\left(\frac{\partial(u^0 h)}{\partial x} - v^0 \frac{\partial H}{\partial y}\right) - g\frac{\partial s}{\partial x} = \left[-\frac{1}{\rho_0} D_x p^1 - 2\phi\,(\cos\varphi)\, w^1\right]_{z=0} \tag{4.7.12}$$

$$2\phi\left(\frac{\partial}{\partial y}\big((\cos\varphi)\, u^0 h\big) + (\cos\varphi)\, u^0 \frac{\partial H}{\partial y}\right) - g\frac{\partial s}{\partial y} = -\frac{1}{\rho_0}\, D_y p^1\big|_{z=0} \tag{4.7.13}$$

Observación 4.5 *Si al integrar para obtener u^1 y v^1 a partir de γ_2^0 y γ_1^0 respectivamente se hace entre z y 1 en lugar de entre 0 y z se obtiene en lugar de* (4.7.8)-(4.7.9) *las siguientes expresiones*

$$u^1 = u_1^1 + h\gamma_2^0(z-1) \tag{4.7.14}$$

$$v^1 = v_1^1 + h\gamma_1^0(1-z) \tag{4.7.15}$$

donde $u_1^1(t,x,y) = u^1(t,x,y,1)$, $v_1^1(t,x,y) = v^1(t,x,y,1)$. En este caso, las ecuaciones para obtener u_1^1 y v_1^1 son:

$$\frac{\partial u_1^1}{\partial t} + u^0 \frac{\partial u_1^1}{\partial x} + u_1^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u_1^1}{\partial y} + v_1^1 \frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v_1^1 = \frac{\partial s}{\partial x}(2\phi\,(\cos\varphi)\, u^0 - g)$$

$$- 2\phi\,(\cos\varphi)\left[u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - h\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$

$$= \left[-\frac{1}{\rho_0} D_x p^1 - 2\phi\,(\cos\varphi)\, w^1\right]_{z=1}$$

$$\frac{\partial v_1^1}{\partial t} + u^0 \frac{\partial v_1^1}{\partial x} + u_1^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v_1^1}{\partial y} + v_1^1 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u_1^1 = \frac{\partial s}{\partial y}((\cos\varphi)\,u^0 - g)$$

$$= -\frac{1}{\rho_0}\, D_y p^1\big|_{z=1}$$

A continuación, se calcula w^2 partiendo de (4.5.92),

$$D_z w^2 = -D_x u^1 - D_y v^1$$

sustituimos u^1 y v^1 utilizando (4.7.8)-(4.7.9) y agrupamos según las potencias de z,

$$D_z w^2 = -\frac{\partial u_0^1}{\partial x} + \frac{\partial H}{\partial x}\gamma_2^0 - \frac{\partial v_0^1}{\partial y} - \frac{\partial H}{\partial y}\gamma_1^0 - zh\left(\frac{\partial \gamma_2^0}{\partial x} - \frac{\partial \gamma_1^0}{\partial y}\right)$$

y finalmente, se integra respecto de z imponiendo la condición de contorno (4.5.103):

$$w^2 = u_0^1 \frac{\partial H}{\partial x} + v_0^1 \frac{\partial H}{\partial y} - hz\left(\frac{\partial u_0^1}{\partial x} + \frac{\partial v_0^1}{\partial y} - \frac{\partial H}{\partial x}\gamma_2^0 + \frac{\partial H}{\partial y}\gamma_1^0\right)$$
$$- \frac{1}{2} z^2 h^2 \left(\frac{\partial \gamma_2^0}{\partial x} - \frac{\partial \gamma_1^0}{\partial y}\right) \tag{4.7.16}$$

La tercera componente de la vorticidad viene dada por (4.5.98):

$$\gamma_3^1 = D_x v^1 - D_y u^1$$

Para calcular las otras dos componentes se resuelven las ecuaciones (4.5.93) y (4.5.94) donde se tiene en cuenta que γ_1^0 y γ_2^0 no dependen de z y se sustituye $D_z u^1$ y $D_z v^1$ por las expresiones (4.5.86)-(4.5.87), $D_z u^2$ y $D_z v^2$ se despejan de (4.5.96)-(4.5.97), y, γ_3^0, γ_3^1 se reemplazan por (4.5.88) y (4.5.98):

$$D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + u^1 \frac{\partial \gamma_1^0}{\partial x} + v^0 D_y \gamma_1^1 + v^1 \frac{\partial \gamma_1^0}{\partial y} + w^1 D_z \gamma_1^1 - \gamma_1^0 D_x u^1 - \gamma_1^1 \frac{\partial u^0}{\partial x}$$
$$- \gamma_2^1 \frac{\partial v^0}{\partial x} - \gamma_2^0 D_x v^1 - 2\phi\,(\operatorname{sen}\varphi)\,\gamma_2^1 = \left(\frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) D_x w^1$$
$$+ 2\phi\left(D_y((\cos\varphi)\,u^1) + (\operatorname{sen}\varphi)\,D_x w^1\right) \tag{4.7.17}$$

$$D_t \gamma_2^1 + u^0 D_x \gamma_2^1 + u^1 \frac{\partial \gamma_2^0}{\partial x} + v^0 D_y \gamma_2^1 + v^1 \frac{\partial \gamma_2^0}{\partial y} + w^1 D_z \gamma_2^1 - \gamma_1^0 D_y u^1 - \gamma_1^1 \frac{\partial u^0}{\partial y}$$
$$- \gamma_2^1 \frac{\partial v^0}{\partial y} - \gamma_2^0 D_y v^1 + 2\phi\,(\operatorname{sen}\varphi)\,\gamma_1^1 = \left(\frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) D_y w^1$$
$$+ 2\phi\left[(\cos\varphi)\,D_y v^1 + (\operatorname{sen}\varphi)\,D_y w^1\right] \tag{4.7.18}$$

Ahora, usando (4.5.74) y (4.5.81) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 h \left(2\phi\,(\cos\varphi)\,u^0 - g\right)(z-1) \qquad (4.7.19)$$

De igual modo, por (4.5.73), (4.5.82) y (4.7.16), sabemos que:

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 = \varepsilon\left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$

$$+\,\varepsilon^2\left[u_0^1\frac{\partial H}{\partial x} + v_0^1\frac{\partial H}{\partial y} - hz\left(\frac{\partial u_0^1}{\partial x} + \frac{\partial v_0^1}{\partial y} - \frac{\partial H}{\partial x}\gamma_2^0 + \frac{\partial H}{\partial y}\gamma_1^0\right)\right.$$

$$\left. -\,\frac{1}{2}z^2h^2\left(\frac{\partial \gamma_2^0}{\partial x} - \frac{\partial \gamma_1^0}{\partial y}\right)\right]$$

es decir,

$$\tilde{w}(\varepsilon) = \varepsilon\left(u^0 + \varepsilon u_0^1\right)\frac{\partial H}{\partial x} + \varepsilon\left(v^0 + \varepsilon v_0^1\right)\frac{\partial H}{\partial y} - \varepsilon hz\left[\frac{\partial(u^0 + \varepsilon u_0^1)}{\partial x} + \frac{\partial(v^0 + \varepsilon v_0^1)}{\partial y}\right.$$

$$\left. +\,\varepsilon\left(\gamma_1^0\frac{\partial H}{\partial y} - \gamma_2^0\frac{\partial H}{\partial x}\right)\right] + \frac{1}{2}\varepsilon^2 z^2 h^2\left(\frac{\partial \gamma_1^0}{\partial y} - \frac{\partial \gamma_2^0}{\partial x}\right) \qquad (4.7.20)$$

Observación 4.6 *También podemos escribir $\tilde{w}(\varepsilon)$ en términos de $\tilde{u}(\varepsilon)$ y $\tilde{v}(\varepsilon)$, y es fácil comprobar que, en ese caso*

$$\tilde{w}(\varepsilon) = \tilde{u}(\varepsilon)\varepsilon\frac{\partial H}{\partial x} + \tilde{v}(\varepsilon)\varepsilon\frac{\partial H}{\partial y} - \varepsilon hz\left(D_x\tilde{u}(\varepsilon) + D_y\tilde{v}(\varepsilon)\right) + O(\varepsilon^2) \qquad (4.7.21)$$

Se deshace el cambio de variable y se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y,z)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y,z)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z)$$

$$\tilde{\gamma}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_i(\varepsilon)(t,x,y,z) = \gamma_i^0(t,x,y) + \varepsilon\gamma_i^1(t,x,y,z) \quad (i=1,2)$$

$$\tilde{\gamma}_3^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_3(\varepsilon)(t,x,y,z) = \gamma_3^0(t,x,y) + \varepsilon\gamma_3^1(t,x,y,z)$$

y definimos también

$$\check{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u_0^1(t,x,y)$$

$$\check{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v_0^1(t,x,y)$$

La aproximación de la presión en Ω^ε, si se realiza el cambio de variable en (4.7.19), es

$$\tilde{p}^\varepsilon = p_s + \rho_0(s^\varepsilon - z^\varepsilon)\left(g - 2\phi\,(\cos\varphi^\varepsilon)\,u^{0,\varepsilon}\right) \tag{4.7.22}$$

(donde $u^{0,\varepsilon}(t^\varepsilon, x^\varepsilon, y^\varepsilon) = u^0(t,x,y)$) o en términos de $\check{u}^\varepsilon$ en lugar de $u^{0,\varepsilon}$:

$$\tilde{p}^\varepsilon = p_s + \rho_0(s^\varepsilon - z^\varepsilon)\,(g - 2\phi\,(\cos\varphi^\varepsilon)\,\check{u}^\varepsilon) + O(\varepsilon^2) \tag{4.7.23}$$

Análogamente, deshaciendo el cambio de variable en (4.7.20), se tiene que la aproximación de la componente vertical de la velocidad se puede obtener:

$$\tilde{w}^\varepsilon = \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left[\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} + \gamma_1^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right]$$
$$+ \frac{1}{2}(H^\varepsilon - z^\varepsilon)^2\left(\frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon}\right) \tag{4.7.24}$$

(donde $\gamma_i^{0,\varepsilon}(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \gamma_i^0(t,x,y)$ $(i = 1,2)$)

Observación 4.7 *Si utilizásemos (4.7.21) obtendríamos:*

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial v^\varepsilon}\right) + O(\varepsilon^2) \tag{4.7.25}$$

Podemos ahora proponer varios modelos (asintóticamente equivalentes) a partir de lo que hemos visto en esta sección. En primer lugar, se tiene el modelo que resulta de forma natural de aplicar el método de desarrollos asintóticos y que consiste en calcular u^0, v^0, h, u_0^1, v_0^1, γ_1^0 y γ_2^0 (u^0 y v^0 son necesarios para el cálculo de los restantes y h para el cálculo de u_0^1 y v_0^1) resolviendo las ecuaciones (4.5.78), (4.5.79), (4.5.80), (4.7.10), (4.7.11), (4.7.2) y (4.7.5):

$$\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\,v^0$$

$$\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,u^0$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0$$

$$\frac{\partial u_0^1}{\partial t} + u^0\frac{\partial u_0^1}{\partial x} + u_0^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u_0^1}{\partial y} + v_0^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v_0^1$$
$$= 2\phi\,(\cos\varphi)\left(\frac{\partial (u^0 h)}{\partial x} - v^0\frac{\partial H}{\partial y}\right) - g\frac{\partial s}{\partial x}$$

$$\frac{\partial v_0^1}{\partial t} + u^0 \frac{\partial v_0^1}{\partial x} + u_0^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v_0^1}{\partial y} + v_0^1 \frac{\partial v^0}{\partial y} + 2\phi \left(\operatorname{sen} \varphi\right) u_0^1$$

$$= 2\phi \left(\frac{\partial}{\partial y} \big((\cos \varphi)\, u^0 h \big) + (\cos \varphi)\, u^0 \frac{\partial H}{\partial y} \right) - g \frac{\partial s}{\partial y}$$

$$\frac{\partial \gamma_1^0}{\partial t} + u^0 \frac{\partial \gamma_1^0}{\partial x} + v^0 \frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0 \frac{\partial u^0}{\partial x} - \gamma_2^0 \frac{\partial v^0}{\partial x}$$

$$= 2\phi \left(\frac{\partial}{\partial y} ((\cos \varphi)\, u^0) + (\operatorname{sen} \varphi)\, \gamma_2^0 \right)$$

$$\frac{\partial \gamma_2^0}{\partial t} + u^0 \frac{\partial \gamma_2^0}{\partial x} + v^0 \frac{\partial \gamma_2^0}{\partial y} - \gamma_1^0 \frac{\partial u^0}{\partial y} - \gamma_2^0 \frac{\partial v^0}{\partial y}$$

$$= 2\phi \left(-\gamma_1^0 \left(\operatorname{sen} \varphi\right) + (\cos \varphi) \frac{\partial v^0}{\partial y} \right)$$

A continuación se construyen u^1 y v^1 según las expresiones (4.7.9)-(4.7.8):

$$u^1 = u_0^1 + h\gamma_2^0 z$$

$$v^1 = v_0^1 - h\gamma_1^0 z$$

y finalmente las aproximaciones de las velocidades horizontales en el dominio original resultan:

$$\tilde{u}^\varepsilon = u^0 + \varepsilon u^1$$

$$\tilde{v}^\varepsilon = v^0 + \varepsilon v^1$$

La profundidad del agua se obtiene deshaciendo el cambio de variable como $h^\varepsilon = \varepsilon h$.

Si pudiéramos obtener u^0 y u_0^1 (y v^0 y v_0^1) resolviendo una única ecuación supondría un ahorro de esfuerzo de cálculo. Para ello utilizamos $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ definidos anteriormente y observamos que:

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u_0^1}{\partial t} + (u^0 + \varepsilon u_0^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x} \right)$$

$$+ (v^0 + \varepsilon v_0^1) \left(\frac{\partial u^0}{\partial y} + \varepsilon \frac{\partial u_0^1}{\partial y} - 2\phi \left(\operatorname{sen} \varphi\right) \right)$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \left(\frac{\partial u^0}{\partial y} - 2\phi \left(\operatorname{sen} \varphi\right) \right)$$

$$+ \varepsilon \left[\frac{\partial u_0^1}{\partial t} + u_0^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u_0^1}{\partial x} + v^0 \frac{\partial u_0^1}{\partial y} + v_0^1 \left(\frac{\partial u^0}{\partial y} - 2\phi \left(\operatorname{sen} \varphi\right) \right) \right]$$

$$+ \varepsilon^2 \left(u_0^1 \frac{\partial u_0^1}{\partial x} + v_0^1 \frac{\partial u_0^1}{\partial y} \right)$$

Utilizando las ecuaciones (4.5.78) y (4.7.10) obtenemos:

$$\frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right)$$

$$= -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + \varepsilon \left[2\phi \left(\cos \varphi\right) \left(\frac{\partial (u^0 h)}{\partial x} - v^0 \frac{\partial H}{\partial y} \right) - g \frac{\partial s}{\partial x} \right] + O(\varepsilon^2)$$

$$= -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\cos \varphi^\varepsilon\right) \left[\frac{\partial (\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} - \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right] - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2) \tag{4.7.26}$$

Análogamente

$$\frac{\partial \check{v}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) + \check{v}^\varepsilon \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon}$$

$$+ 2\phi \left(\frac{\partial}{\partial y^\varepsilon} \left(\left(\cos \varphi^\varepsilon\right) \check{u}^\varepsilon h^\varepsilon \right) + \left(\cos \varphi^\varepsilon\right) \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} + O(\varepsilon^2) \tag{4.7.27}$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\check{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \tag{4.7.28}$$

Si en las ecuaciones (4.7.2) y (4.7.5) se realiza el cambio de variable y se sustituye $u^{0,\varepsilon}$ por $\check{u}^\varepsilon$ (y, del mismo modo, $v^{0,\varepsilon} = v^0(t,x,y)$ por $\check{v}^\varepsilon$), resultan:

$$\frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon}$$

$$= 2\phi \left(\frac{\partial}{\partial y^\varepsilon} \left(\left(\cos \varphi^\varepsilon\right) \check{u}^\varepsilon \right) + \left(\operatorname{sen} \varphi^\varepsilon\right) \gamma_2^{0,\varepsilon} \right) + O(\varepsilon) \tag{4.7.29}$$

$$\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -2\phi \left(\gamma_1^{0,\varepsilon} \left(\operatorname{sen} \varphi^\varepsilon\right) - \left(\cos \varphi^\varepsilon\right) \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.7.30}$$

Además, $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ se pueden escribir en términos de $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$:

$$\tilde{u}^\varepsilon = u^0 + \varepsilon u^1 = u^0 + \varepsilon \left(u_0^1 + h\gamma_2^0 z\right) = \breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\,\gamma_2^{0,\varepsilon} \tag{4.7.31}$$

$$\tilde{v}^\varepsilon = v^0 + \varepsilon v^1 = u^0 + \varepsilon \left(v_0^1 - h\gamma_1^0 z\right) = \breve{v}^\varepsilon - (z^\varepsilon - H^\varepsilon)\,\gamma_1^{0,\varepsilon} \tag{4.7.32}$$

Despreciando en las ecuaciones (4.7.26)-(4.7.28) los términos de orden $O(\varepsilon^2)$, así como en (4.7.23) y (4.7.24) y en las ecuaciones (4.7.29)-(4.7.30) los términos en ε (aunque en estas ecuaciones se desprecien términos de orden $O(\varepsilon)$, como $\gamma_1^{0,\varepsilon}$ y $\gamma_2^{0,\varepsilon}$ aparecen multiplicados por términos de orden ε en (4.7.31)-(4.7.32), finalmente el error que se comete es de orden ε^2), se obtiene el siguiente modelo de aguas someras cuyo orden de precisión, al menos formalmente es $O(\varepsilon^2)$:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\breve{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{4.7.33}$$

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{v}^\varepsilon$$

$$+ 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial(\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} - \breve{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.7.34}$$

$$\frac{\partial \breve{v}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{u}^\varepsilon$$

$$+ 2\phi\left(\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\breve{u}^\varepsilon h^\varepsilon) + (\cos\varphi^\varepsilon)\,\breve{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.7.35}$$

$$\frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon}\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon}$$

$$= 2\phi\left(\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\breve{u}^\varepsilon) + (\operatorname{sen}\varphi^\varepsilon)\,\gamma_2^{0,\varepsilon}\right) \tag{4.7.36}$$

$$\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon}\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -2\phi\left(\gamma_1^{0,\varepsilon}\,(\operatorname{sen}\varphi^\varepsilon) - (\cos\varphi^\varepsilon)\,\frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.7.37}$$

$$\tilde{u}^\varepsilon = \breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\,\gamma_2^{0,\varepsilon} \tag{4.7.38}$$

$$\tilde{v}^\varepsilon = \breve{v}^\varepsilon - (z^\varepsilon - H^\varepsilon)\,\gamma_1^{0,\varepsilon} \tag{4.7.39}$$

$$\tilde{w}^\varepsilon = \breve{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left[\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} + \gamma_1^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right]$$

$$+\frac{1}{2}(H^\varepsilon - z^\varepsilon)^2 \left(\frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon}\right) \tag{4.7.40}$$

$$\tilde{p}^\varepsilon = p_s + \rho_0 (s^\varepsilon - z^\varepsilon)\left(g - 2\phi\,(\cos\varphi^\varepsilon)\,\check{u}^\varepsilon\right) \tag{4.7.41}$$

Una vez conocidos $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y $\tilde{w}^\varepsilon$, buscamos las ecuaciones o expresiones adecuadas para el cálculo de la vorticidad. Comenzamos por las dos primeras componentes:

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \\
&\quad - 2\phi \left(\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon) + (\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_2^\varepsilon\right) - \left(2\phi\,(\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} \\
&\quad = D_t\tilde{\gamma}_1(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{\gamma}_1(\varepsilon) + \tilde{v}(\varepsilon) D_y \tilde{\gamma}_1(\varepsilon) + \tilde{w}(\varepsilon) D_z \tilde{\gamma}_1(\varepsilon) - \tilde{\gamma}_1(\varepsilon) D_x \tilde{u}(\varepsilon) \\
&\quad - \tilde{\gamma}_2(\varepsilon) D_x v(\varepsilon) - 2\phi\left(D_y((\cos\varphi)\ \tilde{u}(\varepsilon)) + (\operatorname{sen}\varphi)\ \tilde{\gamma}_2(\varepsilon)\right) \\
&\quad - \left(2\phi\,(\operatorname{sen}\varphi) + D_x\tilde{v}(\varepsilon) - D_y\tilde{u}(\varepsilon)\right) D_x\tilde{w}(\varepsilon) \\
&\quad = \frac{\partial \gamma_1^0}{\partial t} + \varepsilon D_t \gamma_1^1 + (u^0 + \varepsilon u^1)\left(\frac{\partial \gamma_1^0}{\partial x} + \varepsilon D_x \gamma_1^1\right) + (v^0 + \varepsilon v^1)\left(\frac{\partial \gamma_1^0}{\partial y} + \varepsilon D_y \gamma_1^1\right) \\
&\quad + \varepsilon (w^1 + \varepsilon w^2) D_z \gamma_1^1 - (\gamma_1^0 + \varepsilon \gamma_1^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1\right) \\
&\quad - (\gamma_2^0 + \varepsilon \gamma_2^1)\left(\frac{\partial v^0}{\partial x} + \varepsilon D_x v^1\right) - 2\phi\left(D_y((\cos\varphi)\ (u^0 + \varepsilon u^1)) + (\operatorname{sen}\varphi)\ (\gamma_2^0 + \varepsilon\gamma_2^1)\right) \\
&\quad - \left[2\phi\,(\operatorname{sen}\varphi) + D_x(v^0 + \varepsilon v^1) - D_y(u^0 + \varepsilon u^1)\right] \varepsilon D_x (w^1 + \varepsilon w^2) \\
&\quad = \frac{\partial \gamma_1^0}{\partial t} + u^0 \frac{\partial \gamma_1^0}{\partial x} + v^0 \frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0 \frac{\partial u^0}{\partial x} - \gamma_2^0 \frac{\partial v^0}{\partial x} - 2\phi\left(D_y((\cos\varphi)\ u^0) + (\operatorname{sen}\varphi)\ \gamma_2^0\right) \\
&\quad + \varepsilon \Bigg[D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + u^1 \frac{\partial \gamma_1^0}{\partial x} + v^0 D_y \gamma_1^1 + v^1 \frac{\partial \gamma_1^0}{\partial y} + w^1 D_z \gamma_1^1 - \gamma_1^0 D_x u^1 \\
&\quad - \gamma_1^1 \frac{\partial u^0}{\partial x} - \gamma_2^0 D_x v^1 - \gamma_2^1 \frac{\partial v^0}{\partial x} - 2\phi\left(D_y((\cos\varphi)\ u^1) + (\operatorname{sen}\varphi)\ \gamma_2^1\right) \\
&\quad - \left(2\phi \operatorname{sen}\varphi + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) D_x w^1\Bigg] + O(\varepsilon^2) = O(\varepsilon^2)
\end{aligned}$$

De manera análoga,

$$\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}$$

$$- 2\phi \left(- (\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_1^\varepsilon + (\cos\varphi^\varepsilon)\,\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) - \left(2\phi \operatorname{sen}\varphi^\varepsilon + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} = O(\varepsilon^2)$$

Si se desprecian los términos en $O(\varepsilon^2)$ de estas ecuaciones, al igual que se hizo en las ecuaciones (4.7.26)-(4.7.28), las ecuaciones para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo son las siguientes:

$$\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - 2\phi \frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon)$$

$$- 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_2^\varepsilon = \left(2\phi \operatorname{sen}\varphi^\varepsilon + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} \tag{4.7.42}$$

$$\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\cos\varphi^\varepsilon)\,\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}$$

$$+ 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{\gamma}_1^\varepsilon = \left(2\phi \operatorname{sen}\varphi^\varepsilon + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} \tag{4.7.43}$$

Para la tercera componente se tiene:

$$\tilde{\gamma}_3^\varepsilon = \gamma_3^0 + \varepsilon\gamma_3^1 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} + \varepsilon\left(D_x v^1 - D_y u^1\right) = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \tag{4.7.44}$$

Veamos en qué medida verifica la aproximación de primer orden las ecuaciones de Euler de partida:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon\right)$$

$$= D_t\tilde{u}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{u}(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{u}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{u}(\varepsilon) + \frac{1}{\rho_0}D_x\tilde{p}(\varepsilon)$$

$$- 2\phi\left((\operatorname{sen}\varphi)\ \tilde{v}(\varepsilon) - (\cos\varphi)\ \tilde{w}(\varepsilon)\right)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon D_t u^1 + (u^0 + \varepsilon u^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1\right) + (v^0 + \varepsilon v^1)\left(\frac{\partial u^0}{\partial y} + \varepsilon D_y u^1\right)$$

$$+ (w^1 + \varepsilon w^2)\varepsilon D_z u^1 + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \varepsilon\frac{1}{\rho_0}D_x p^1$$

$$- 2\phi\left((\operatorname{sen}\varphi)\ (v^0 + \varepsilon v^1) - (\cos\varphi)\ \varepsilon(w^1 + \varepsilon w^2)\right)$$

$$= \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\ v^0$$

$$+ \varepsilon\left[D_t u^1 + u^1\frac{\partial u^0}{\partial x} + u^0 D_x u^1 + v^1\frac{\partial u^0}{\partial y} + v^0 D_y u^1 + w^1 D_z u^1\right.$$

$$+ 2\phi\,(\cos\varphi)\,\frac{\partial u^0}{\partial x}h(z-1) - \frac{\partial s}{\partial x}\big(2\phi\,(\cos\varphi)\,u^0 - g\big)$$

$$- 2\phi\,\big((\operatorname{sen}\varphi)\ v^1 - (\cos\varphi)\ w^1\big)\big] + O(\varepsilon^2)$$

Usando (4.5.78) y (4.5.89) se puede escribir:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon) + O(\varepsilon^2)$$

Así, la primera ecuación de Euler se verifica con un error de orden ε^2. La segunda ecuación de Euler se comporta del mismo modo.

Para la tercera ecuación de Euler se tiene, usando (4.5.81), que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - 2\phi\,(\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon$$

$$= D_t\tilde{w}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{w}(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon)$$

$$+ \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z\tilde{p}(\varepsilon) + g - 2\phi\,(\cos\varphi)\,u(\varepsilon) = \varepsilon D_t w^1 + \varepsilon^2 D_t w^2$$

$$+ (u^0 + \varepsilon u^1)\varepsilon\left[D_x w^1 + \varepsilon D_x w^2\right] + (v^0 + \varepsilon v^1)\varepsilon\left[D_y w^1 + \varepsilon D_y w^2\right]$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2)\left[D_z w^1 + \varepsilon D_z w^2\right] + \frac{1}{\rho_0}D_z p^1 + g - 2\phi\,(\cos\varphi)\,(u^0 + \varepsilon u^1) = O(\varepsilon)$$

La aproximación de primer orden verifica la tercera ecuación de Euler con un error $O(\varepsilon)$.

La ecuación de la incompresibilidad se verifica de forma exacta como se ve utilizando (4.5.49) y (4.5.92)

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = 0,$$

o directamente por (4.7.24), (4.7.31) y (4.7.32).

Lo mismo sucede con las condiciones de contorno ((4.1.6) y (4.1.7)), teniendo en cuenta (4.7.24) y (4.7.23).

Veamos ahora lo que sucede con las ecuaciones de la vorticidad (4.1.12)-(4.1.14),

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} \\
&\quad - 2\phi \left(\frac{\partial}{\partial y^\varepsilon} ((\cos\varphi^\varepsilon)\, \tilde{u}^\varepsilon) + (\operatorname{sen}\varphi^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} \right) \\
&\quad = \frac{\partial \gamma_1^0}{\partial t} + \varepsilon D_t \gamma_1^1 + (u^0 + \varepsilon u^1) \left(\frac{\partial \gamma_1^0}{\partial x} + \varepsilon D_x \gamma_1^1 \right) + (v^0 + \varepsilon v^1) \left(\frac{\partial \gamma_1^0}{\partial y} + \varepsilon D_y \gamma_1^1 \right) \\
&\quad + \varepsilon (w^1 + \varepsilon w^2) D_z \gamma_1^1 - (\gamma_1^0 + \varepsilon \gamma_1^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1 \right) \\
&\quad - (\gamma_2^0 + \varepsilon \gamma_2^1) \left(\frac{\partial u^0}{\partial y} + \varepsilon D_y u^1 \right) - (\gamma_3^0 + \varepsilon \gamma_3^1) D_z u^1 \\
&\quad - 2\phi \left[D_y \left((\cos\varphi)\, (u^0 + \varepsilon u^1) \right) + (\operatorname{sen}\varphi)\, D_z u^1 \right] \\
&\quad = \frac{\partial \gamma_1^0}{\partial t} + u^0 \frac{\partial \gamma_1^0}{\partial x} + v^0 \frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0 \frac{\partial u^0}{\partial x} - \gamma_2^0 \frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^1 \\
&\quad - 2\phi \frac{\partial}{\partial y} \left((\cos\varphi)\, u^0 \right) - 2\phi\, (\operatorname{sen}\varphi)\, D_z u^1 \\
&\quad + \varepsilon \Bigg[D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + u^1 \frac{\partial \gamma_1^0}{\partial x} + v^0 D_y \gamma_1^1 + v^1 \frac{\partial \gamma_1^0}{\partial y} + w^1 D_z \gamma_1^1 - \gamma_1^0 D_x u^1 \\
&\quad - \gamma_1^1 \frac{\partial u^0}{\partial x} - \gamma_2^0 D_y u^1 - \gamma_2^1 \frac{\partial u^0}{\partial y} - \gamma_3^1 D_z u^1 - 2\phi D_y ((\cos\varphi)\, u^1) \Bigg] + O(\varepsilon^2)
\end{aligned}$$

usando (4.5.87), (4.5.88), (4.5.97), (4.5.98), (4.7.2) y (4.7.17) resulta:

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} \\
&\quad - 2\phi \left(\frac{\partial}{\partial y^\varepsilon} ((\cos\varphi^\varepsilon)\, \tilde{u}^\varepsilon) + (\operatorname{sen}\varphi^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} \right) = O(\varepsilon)
\end{aligned}$$

es decir, la primera ecucación de la vorticidad se verifica con un error del orden de ε. Lo mismo sucede para la segunda ecuación.

Para la tercera ecuación de vorticidad se tiene que:

$$\frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon}$$

$$+ 2\phi \left((\operatorname{sen}\varphi^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\, \tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\, \tilde{w}^\varepsilon) \right)$$

$$= \frac{\partial \gamma_3^0}{\partial t} + \varepsilon D_t \gamma_3^1 + (u^0 + \varepsilon u^1) \left(\frac{\partial \gamma_3^0}{\partial x} + \varepsilon D_x \gamma_3^1 \right) + (v^0 + \varepsilon v^1) \left(\frac{\partial \gamma_3^0}{\partial y} + \varepsilon D_y \gamma_3^1 \right)$$

$$+ \varepsilon (w^1 + \varepsilon w^2) D_z \gamma_3^1 - \left(\gamma_1^0 + \varepsilon \gamma_1^1\right) \varepsilon \left(D_x w^1 + \varepsilon D_x w^2\right)$$

$$- \left(\gamma_2^0 + \varepsilon \gamma_2^1\right) \varepsilon \left(D_y w^1 + \varepsilon D_y w^2\right) - \left(\gamma_3^0 + \varepsilon \gamma_3^1\right) \left(D_z w^1 + \varepsilon D_z w^2\right)$$

$$+ 2\phi \left[(\operatorname{sen}\varphi) \left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1 \right) + D_y((\operatorname{sen}\varphi)\,(v^0 + \varepsilon v^1) - (\cos\varphi)\, \varepsilon (w^1 + \varepsilon w^2)) \right]$$

$$= \frac{\partial \gamma_3^0}{\partial t} + u^0 \frac{\partial \gamma_3^0}{\partial x} + v^0 \frac{\partial \gamma_3^0}{\partial y} - \gamma_3^0 D_z w^1 + 2\phi \left((\operatorname{sen}\varphi) \frac{\partial u^0}{\partial x} + D_y((\operatorname{sen}\varphi)\, v^0) \right)$$

$$+ \varepsilon \left[D_t \gamma_3^1 + u^0 D_x \gamma_3^1 + u^1 \frac{\partial \gamma_3^0}{\partial x} + v^0 D_y \gamma_3^1 + v^1 \frac{\partial \gamma_3^0}{\partial y} + w^1 D_z \gamma_3^1 - \gamma_1^0 D_x w^1 \right.$$

$$- \gamma_2^0 D_y w^1 - \gamma_3^0 D_z w^2 - \gamma_3^1 D_z w^1$$

$$\left. + 2\phi \left((\operatorname{sen}\varphi)\, D_x u^1 + D_y((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1) \right) \right] + O(\varepsilon^2)$$

Se sustituye γ_i^0 $(i = 1, 2)$, según (4.5.86)-(4.5.87), γ_3^k $(k = 0, 1)$, según (4.5.88) y (4.5.98) y $D_z w^k$ $(k = 1, 2)$, según (4.5.82) y (4.7.16):

$$\frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon}$$

$$+ 2\phi \left((\operatorname{sen}\varphi^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\, \tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\, \tilde{w}^\varepsilon) \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + 2\phi\, (\operatorname{sen}\varphi)\, u^0 \right)$$

$$- \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} - 2\phi\, (\operatorname{sen}\varphi)\, v^0 \right)$$

$$+ \varepsilon \left\{ D_x \left[D_t v^1 + u^0 D_x v^1 + u^1 \frac{\partial v^0}{\partial x} + v^0 D_y v^1 + v^1 \frac{\partial v^0}{\partial y} + w^1 D_z v^1 + 2\phi\, (\operatorname{sen}\varphi)\, u^1 \right] \right.$$

$$- D_y \left[D_t u^1 + u^0 D_x u^1 + u^1 \frac{\partial u^0}{\partial x} + v^0 D_y u^1 + v^1 \frac{\partial u^0}{\partial y} + w^1 D_z u^1 \right.$$

$$\left. \left. - 2\phi \left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1 \right) \right] \right\} + O(\varepsilon^2)$$

por las ecuaciones (4.5.78), (4.5.79), (4.5.89) y (4.5.90) se sabe que:

$$\frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_3^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon}$$

$$+ 2\phi \left((\operatorname{sen} \varphi^\varepsilon) \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial}{\partial y^\varepsilon} ((\operatorname{sen} \varphi^\varepsilon) \tilde{v}^\varepsilon - (\cos \varphi^\varepsilon) \tilde{w}^\varepsilon) \right)$$

$$= \frac{\partial}{\partial x} \left(-\frac{1}{\rho_0} \frac{\partial p_s}{\partial y} \right) - \frac{\partial}{\partial y} \left(-\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right)$$

$$+\varepsilon \left\{ D_x \left[-2\phi \frac{\partial}{\partial y} \left((\cos \varphi) u^0 \right) h(z-1) + \frac{\partial s}{\partial y} \left(2\phi (\cos \varphi) u^0 - g \right) \right] \right.$$

$$\left. - D_y \left[-2\phi (\cos \varphi) \frac{\partial u^0}{\partial x} h(z-1) + \frac{\partial s}{\partial x} \left(2\phi (\cos \varphi) u^0 - g \right) \right] \right\} + O(\varepsilon^2) = O(\varepsilon^2)$$

es decir, la tercera ecucación de la vorticidad se verifica con un error del orden de ε^2.

4.7.1. Modelo en función de las velocidades medias

El modelo (4.7.33)-(4.7.41) se puede escribir también en función de las velocidades medias en la vertical. Para ello se tiene en cuenta que

$$\bar{u}^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \tilde{u}^\varepsilon \, dz^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \left(\breve{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \gamma_2^{0,\varepsilon} \right) dz^\varepsilon = \breve{u}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \quad (4.7.45)$$

$$\bar{v}^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \tilde{v}^\varepsilon \, dz^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \left(\breve{v}^\varepsilon - (z^\varepsilon - H^\varepsilon) \gamma_1^{0,\varepsilon} \right) dz^\varepsilon = \breve{v}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \quad (4.7.46)$$

se despejan $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ y se sustituyen en las ecuaciones (4.7.33)-(4.7.39). Para el cálculo del calado se tiene:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\bar{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \quad (4.7.47)$$

Realizando las sustitución en la ecuación (4.7.36), se llega a:

$$\frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right)$$

$$- \gamma_2^{0,\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) - 2\phi \left[\frac{\partial}{\partial y^\varepsilon} \left((\cos \varphi^\varepsilon) \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) \right) + (\operatorname{sen} \varphi^\varepsilon) \gamma_2^{0,\varepsilon} \right]$$

$$= \frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \bar{v}^\varepsilon}{\partial x^\varepsilon}$$

$$- 2\phi \left[\frac{\partial}{\partial y^\varepsilon} \left((\cos \varphi^\varepsilon) \bar{u}^\varepsilon \right) + (\operatorname{sen} \varphi^\varepsilon) \gamma_2^{0,\varepsilon} \right] + O(\varepsilon) = 0 \tag{4.7.48}$$

Análogamente, a partir de (4.7.37) se tiene:

$$\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -2\phi \left(\gamma_1^{0,\varepsilon} (\operatorname{sen} \varphi^\varepsilon) - (\cos \varphi^\varepsilon) \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.7.49}$$

Vamos a repetir el proceso para las ecuaciones a partir de las que se calculaban $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ ((4.7.34)-(4.7.35)):

$$\frac{\partial}{\partial t^\varepsilon} \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) + \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) \frac{\partial}{\partial x^\varepsilon} \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right)$$

$$+ \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) \left[\frac{\partial}{\partial y^\varepsilon} \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) - 2\phi (\operatorname{sen} \varphi^\varepsilon) \right] = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon}$$

$$+ 2\phi (\cos \varphi^\varepsilon) \left[\frac{\partial}{\partial x^\varepsilon} \left(\bar{u}^\varepsilon h^\varepsilon - \frac{1}{2} (h^\varepsilon)^2 \gamma_2^{0,\varepsilon} \right) - \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right] \tag{4.7.50}$$

$$\frac{\partial}{\partial t^\varepsilon} \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) + \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) \left[\frac{\partial}{\partial x^\varepsilon} \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) + 2\phi (\operatorname{sen} \varphi^\varepsilon) \right]$$

$$+ \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) \frac{\partial}{\partial y^\varepsilon} \left(\bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon \gamma_1^{0,\varepsilon} \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon}$$

$$+ 2\phi \left[\frac{\partial}{\partial y^\varepsilon} \left((\cos \varphi^\varepsilon) \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) h^\varepsilon \right) \right.$$

$$\left. + (\cos \varphi^\varepsilon) \left(\bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon \gamma_2^{0,\varepsilon} \right) \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right] \tag{4.7.51}$$

Desarrollamos la ecuación (4.7.50) y agrupamos:

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \left(\frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi (\operatorname{sen} \varphi^\varepsilon) \right)$$

$$- \frac{1}{2} \gamma_2^{0,\varepsilon} \left(\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + h^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \right)$$

$$- \frac{1}{2} h^\varepsilon \left(\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} + 2\phi (\operatorname{sen} \varphi^\varepsilon) \gamma_1^{0,\varepsilon} \right)$$

$$+\frac{1}{4}h^\varepsilon\gamma_2^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon\gamma_2^{0,\varepsilon}\right)-\frac{1}{4}h^\varepsilon\gamma_1^{0,\varepsilon}\frac{\partial}{\partial y^\varepsilon}\left(h^\varepsilon\gamma_2^{0,\varepsilon}\right)$$

$$-2\phi\left(\cos\varphi^\varepsilon\right)\left[\frac{\partial}{\partial x^\varepsilon}\left(-\frac{1}{2}(h^\varepsilon)^2\gamma_2^{0,\varepsilon}\right)-\frac{1}{2}h^\varepsilon\gamma_1^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right]$$

$$=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}+2\phi\left(\cos\varphi^\varepsilon\right)\left(\frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon}-\bar{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)-g\frac{\partial s^\varepsilon}{\partial x^\varepsilon}$$

utilizando (4.7.47) y (4.7.49) se puede reescribir la expresión anterior:

$$\frac{\partial\bar{u}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial x^\varepsilon}+\bar{v}^\varepsilon\frac{\partial\bar{u}^\varepsilon}{\partial y^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-g\frac{\partial s^\varepsilon}{\partial x^\varepsilon}$$
$$+2\phi\left[\left(\operatorname{sen}\varphi^\varepsilon\right)\bar{v}^\varepsilon+\left(\cos\varphi^\varepsilon\right)\left(\frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon}+\frac{1}{2}h^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y^\varepsilon}-\bar{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right]+O(\varepsilon^2) \quad (4.7.52)$$

De manera análoga, a partir de (4.7.51) utilizando (4.7.47) y (4.7.48) se tiene:

$$\frac{\partial\bar{v}^\varepsilon}{\partial t^\varepsilon}+\bar{u}^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial x^\varepsilon}+\bar{v}^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon}-g\frac{\partial s^\varepsilon}{\partial y^\varepsilon}+2\phi\left[-\left(\operatorname{sen}\varphi^\varepsilon\right)\bar{u}^\varepsilon+\frac{\partial}{\partial y^\varepsilon}\left(\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon h^\varepsilon\right)\right.$$
$$\left.-\frac{1}{2}h^\varepsilon\frac{\partial}{\partial y^\varepsilon}\left(\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon\right)+\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right]+O(\varepsilon^2) \quad (4.7.53)$$

También $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y $\tilde{p}^\varepsilon$ ((4.7.23)) se pueden escribir en función de $\bar{u}^\varepsilon$ y $\bar{v}^\varepsilon$:

$$\tilde{u}^\varepsilon=\bar{u}^\varepsilon+\left(z^\varepsilon-H^\varepsilon-\frac{1}{2}h^\varepsilon\right)\gamma_2^{0,\varepsilon} \quad (4.7.54)$$

$$\tilde{v}^\varepsilon=\bar{v}^\varepsilon-\left(z^\varepsilon-H^\varepsilon-\frac{1}{2}h^\varepsilon\right)\gamma_1^{0,\varepsilon} \quad (4.7.55)$$

$$\tilde{p}^\varepsilon=p_s+\rho_0(s^\varepsilon-z^\varepsilon)\left[g-2\phi\left(\cos\varphi^\varepsilon\right)\left(\bar{u}^\varepsilon-\frac{1}{2}h^\varepsilon\gamma_2^{0,\varepsilon}\right)\right]+O(\varepsilon^2)$$
$$=p_s+\rho_0(s^\varepsilon-z^\varepsilon)\left[g-2\phi\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon\right]+O(\varepsilon^2) \quad (4.7.56)$$

Si en las ecuaciones e igualdades (4.7.24), (4.7.47), (4.7.52)-(4.7.56) se desprecian los términos de orden $O(\varepsilon^2)$ y en (4.7.48)-(4.7.49) los términos de orden $O(\varepsilon)$, al igual que se hizo para obtener (4.7.33)-(4.7.41), se obtiene el siguiente modelo de aguas someras escrito en términos de la velocidad promediada en la vertical cuyo orden de precisión, al menos formalmente, también es $O(\varepsilon^2)$:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\bar{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{4.7.57}$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\bar{v}^\varepsilon$$

$$+ 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{1}{2}h^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} - \bar{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.7.58}$$

$$\frac{\partial \bar{v}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\bar{u}^\varepsilon$$

$$+ 2\phi\left[\frac{\partial}{\partial y^\varepsilon}\left((\cos\varphi^\varepsilon)\,\bar{u}^\varepsilon h^\varepsilon\right) - \frac{1}{2}h^\varepsilon \frac{\partial}{\partial y^\varepsilon}\left((\cos\varphi^\varepsilon)\,\bar{u}^\varepsilon\right) + (\cos\varphi^\varepsilon)\,\bar{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right] \tag{4.7.59}$$

$$\frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon}\frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial \bar{v}^\varepsilon}{\partial x^\varepsilon}$$

$$= 2\phi\left[\frac{\partial}{\partial y^\varepsilon}\left((\cos\varphi^\varepsilon)\,\bar{u}^\varepsilon\right) + (\operatorname{sen}\varphi^\varepsilon)\,\gamma_2^{0,\varepsilon}\right] \tag{4.7.60}$$

$$\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon}\frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -2\phi\left(\gamma_1^{0,\varepsilon}\,(\operatorname{sen}\varphi^\varepsilon) - (\cos\varphi^\varepsilon)\,\frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.7.61}$$

$$\tilde{u}^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{1}{2}h^\varepsilon)\,\gamma_2^{0,\varepsilon}, \quad \check{u}^\varepsilon = \tilde{u}^\varepsilon|_{z^\varepsilon = H^\varepsilon} = \bar{u}^\varepsilon - \frac{1}{2}h^\varepsilon\,\gamma_2^{0,\varepsilon} \tag{4.7.62}$$

$$\tilde{v}^\varepsilon = \bar{v}^\varepsilon - (z^\varepsilon - H^\varepsilon - \frac{1}{2}h^\varepsilon)\,\gamma_1^{0,\varepsilon}, \quad \check{v}^\varepsilon = \tilde{v}^\varepsilon|_{z^\varepsilon = H^\varepsilon} = \bar{v}^\varepsilon + \frac{1}{2}h^\varepsilon\,\gamma_1^{0,\varepsilon} \tag{4.7.63}$$

$$\tilde{w}^\varepsilon = \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left[\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} + \gamma_1^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right]$$

$$+ \frac{1}{2}(H^\varepsilon - z^\varepsilon)^2\left(\frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon}\right) \tag{4.7.64}$$

$$\tilde{p}^\varepsilon = p_s + \rho_0(s^\varepsilon - z^\varepsilon)\,(g - 2\phi\,(\cos\varphi^\varepsilon)\,\bar{u}^\varepsilon) \tag{4.7.65}$$

4.8. Aproximación de segundo orden

Se considera la aproximación de segundo orden en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1}\gamma_i^{-1} + \gamma_i^0 + \varepsilon\gamma_i^1 + \varepsilon^2\gamma_i^2 \quad (i = 1, 2)$$

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0 + \varepsilon\gamma_3^1 + \varepsilon^2\gamma_3^2$$

Los términos w^0, p^0, γ_i^{-1} $(i = 1, 2)$, u^0, v^0, h, w^1, p^1, γ_i^0 $(i = 1, 2, 3)$, u^1, v^1, w^2 y γ_i^1 $(i = 1, 2, 3)$, se calculan del mismo modo que en la sección anterior para la aproximación de primer orden a partir de (4.5.73)-(4.5.75), (4.5.78)-(4.5.82), (4.5.88), (4.5.98), (4.7.2), (4.7.5), (4.7.9)-(4.7.11) y (4.7.16)-(4.7.18).

Buscamos ahora p^2, para ello partimos de la ecuación (4.5.91):

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 - 2\phi\left(\cos\varphi\right) u^1 = -\frac{1}{\rho_0} D_z p^2$$

Utilizamos las expresiones de w^1 y u^1 ((4.5.82), (4.7.8)) donde la dependencia de z es explícita y reescribimos (4.5.91) simplificando:

$$\begin{aligned}
-\frac{1}{\rho_0} D_z p^2 = {} & \frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - hz\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \\
& + u^0\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - hz\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& + v^0\left[\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - hz\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& + hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2 - 2\phi\left(\cos\varphi\right)\left(u_0^1 + hz\gamma_2^0\right)
\end{aligned}$$

Integrando respecto a z e imponiendo la condición (4.5.102) ($p^2 = 0$ en $z = 1$) se obtiene:

$$p^2 = \rho_0 h(1-z)\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.$$

$$+ v^0 \frac{\partial}{\partial y}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right) - 2\phi\,(\cos\varphi)\, u_0^1\Bigg]$$

$$- \frac{1}{2}\rho_0 h^2 (1-z^2)\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$

$$\left. + v^0 \frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2 + 2\phi\,(\cos\varphi)\,\gamma_2^0\right] \tag{4.8.1}$$

A continuación calculamos u^2 y v^2 a partir de (4.5.96)-(4.5.97):

$$D_z u^2 = D_x w^1 + \gamma_2^1$$

$$D_z v^2 = D_y w^1 - \gamma_1^1$$

Para poder integrar respecto a z las igualdades anteriores (y obtener un resultado explícito en z) es necesario conocer de qué modo dependen γ_i^1 $(i = 1, 2)$ de z. Para ello se supone (igual que hicimos en (4.7.1)):

$$\gamma_1^1 = \gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2} \tag{4.8.2}$$

$$\gamma_2^1 = \gamma_2^{1,0} + z\gamma_2^{1,1} + z^2\gamma_2^{1,2} \tag{4.8.3}$$

y, de manera análoga a la sección anterior, se sustituyen en (4.7.17)-(4.7.18) γ_i^1 $(i = 1, 2)$, u^1, v^1, w^1 y w^2 (utilizando (4.8.2)-(4.8.3), (4.7.8), (4.7.9), (4.5.82) y (4.7.16) respectivamente) de modo que la dependencia de z sea explícita. Veámoslo para (4.7.17):

$$D_t\left(\gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2}\right) + u^0 D_x\left(\gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2}\right) + \left(u_0^1 + h\gamma_2^0 z\right)\frac{\partial \gamma_1^0}{\partial x}$$

$$+ v^0 D_y\left(\gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2}\right) + \left(v_0^1 - h\gamma_1^0 z\right)\frac{\partial \gamma_1^0}{\partial y}$$

$$+ \left[u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] D_z\left(\gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2}\right)$$

$$- \gamma_1^0 D_x\left(u_0^1 + h\gamma_2^0 z\right) - \left(\gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2}\right)\frac{\partial u^0}{\partial x}$$

$$- \left(\gamma_2^{1,0} + z\gamma_2^{1,1} + z^2\gamma_2^{1,2}\right)\frac{\partial v^0}{\partial x} - \gamma_2^0 D_x\left(v_0^1 - h\gamma_1^0 z\right)$$

$$- 2\phi\,(\operatorname{sen}\varphi)\left(\gamma_2^{1,0} + z\gamma_2^{1,1} + z^2\gamma_2^{1,2}\right) = 2\phi D_y((\cos\varphi)\left(u_0^1 + h\gamma_2^0 z\right))$$

$$+ \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) D_x\left[u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$

De cada ecuación se obtienen otras tres al agrupar según las potencias de z (utilizamos (4.5.80) para simplificar):

$$\frac{\partial\gamma_1^{1,0}}{\partial t}+u^0\frac{\partial\gamma_1^{1,0}}{\partial x}+u_0^1\frac{\partial\gamma_1^0}{\partial x}+v^0\frac{\partial\gamma_1^{1,0}}{\partial y}+v_0^1\frac{\partial\gamma_1^0}{\partial y}-\gamma_1^0\frac{\partial u_0^1}{\partial x}-\gamma_1^{1,0}\frac{\partial u^0}{\partial x}-\gamma_2^0\frac{\partial v_0^1}{\partial x}$$

$$-\gamma_2^{1,0}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)=2\phi\left[\frac{\partial}{\partial y}\left(\left(\cos\varphi\right)u_0^1\right)-\frac{\partial H}{\partial y}\left(\cos\varphi\right)\gamma_2^0\right]$$

$$+\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$

$$\left.+\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right] \tag{4.8.4}$$

$$\frac{\partial\gamma_1^{1,1}}{\partial t}+u^0\frac{\partial\gamma_1^{1,1}}{\partial x}+2h\gamma_2^0\frac{\partial\gamma_1^0}{\partial x}+v^0\frac{\partial\gamma_1^{1,1}}{\partial y}-h\gamma_1^0\frac{\partial\gamma_1^0}{\partial y}-h\gamma_1^0\frac{\partial\gamma_2^0}{\partial x}-\gamma_1^{1,1}\frac{\partial u^0}{\partial x}$$

$$-\gamma_2^{1,1}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)=2\phi h\frac{\partial}{\partial y}\left(\left(\cos\varphi\right)\gamma_2^0\right)$$

$$-h\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right) \tag{4.8.5}$$

$$\frac{\partial\gamma_1^{1,2}}{\partial t}+u^0\frac{\partial\gamma_1^{1,2}}{\partial x}+v^0\frac{\partial\gamma_1^{1,2}}{\partial y}-\gamma_1^{1,2}\frac{\partial u^0}{\partial x}-\gamma_2^{1,2}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)=0 \tag{4.8.6}$$

Análogamente a partir de (4.7.18) obtenemos otras tres ecuaciones:

$$\frac{\partial\gamma_2^{1,0}}{\partial t}+u^0\frac{\partial\gamma_2^{1,0}}{\partial x}+u_0^1\frac{\partial\gamma_2^0}{\partial x}+v^0\frac{\partial\gamma_2^{1,0}}{\partial y}+v_0^1\frac{\partial\gamma_2^0}{\partial y}-\gamma_1^0\frac{\partial u_0^1}{\partial y}-\gamma_1^{1,0}\left(\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)\right)$$

$$-\gamma_2^0\frac{\partial v_0^1}{\partial y}-\gamma_2^{1,0}\frac{\partial v^0}{\partial y}=2\phi\left(\cos\varphi\right)\left(\frac{\partial v_0^1}{\partial y}+\frac{\partial H}{\partial y}\gamma_1^0\right)$$

$$+\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\left[\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$

$$\left.+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right] \tag{4.8.7}$$

$$\frac{\partial\gamma_2^{1,1}}{\partial t}+u^0\frac{\partial\gamma_2^{1,1}}{\partial x}+h\gamma_2^0\frac{\partial\gamma_2^0}{\partial x}+v^0\frac{\partial\gamma_2^{1,1}}{\partial y}-2h\gamma_1^0\frac{\partial\gamma_2^0}{\partial y}-\gamma_1^{1,1}\left(\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)\right)$$

$$+h\gamma_2^0\frac{\partial\gamma_1^0}{\partial y}-\gamma_2^{1,1}\frac{\partial v^0}{\partial y}=-2\phi\left(\cos\varphi\right)h\frac{\partial\gamma_1^0}{\partial y}$$

$$- h\left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right)\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \tag{4.8.8}$$

$$\frac{\partial \gamma_2^{1,2}}{\partial t} + u^0\frac{\partial \gamma_2^{1,2}}{\partial x} + v^0\frac{\partial \gamma_2^{1,2}}{\partial y} - \gamma_1^{1,2}\left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\right) - \gamma_2^{1,2}\frac{\partial v^0}{\partial y} = 0 \tag{4.8.9}$$

El sistema formado por las ecuaciones (4.8.6) y (4.8.9) tiene por solución $\gamma_1^{1,2} = \gamma_2^{1,2} = 0$ (si las condiciones iniciales y de contorno para $\gamma_1^{1,2}$ y $\gamma_2^{2,2,}$ son también nulas) que se puede probar que es única del mismo modo que se hizo en la sección anterior. Por tanto, γ_1^1 y γ_2^1 dependen de linealmente de z:

$$\gamma_1^1 = \gamma_1^{1,0} + z\gamma_1^{1,1} \tag{4.8.10}$$

$$\gamma_2^1 = \gamma_2^{1,0} + z\gamma_2^{1,1} \tag{4.8.11}$$

donde $\gamma_i^{1,k} \quad (i = 1,2, k = 0,1)$ se obtienen resolviendo (4.8.4), (4.8.5), (4.8.7) y (4.8.8).

Sustituimos w^1 por su expresión dada en (4.5.82) y $\gamma_i^1 \quad (i = 1,2)$ por las expresiones anteriores e integrando (4.5.96)-(4.5.97) obtenemos u^2 y v^2:

$$u^2 = u_0^2 + hz\left[\gamma_2^{1,0} + \frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$
$$+ \frac{1}{2}z^2 h\left[\gamma_2^{1,1} - h\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \tag{4.8.12}$$

$$v^2 = v_0^2 + hz\left[-\gamma_1^{1,0} + \frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$
$$- \frac{1}{2}z^2 h\left[\gamma_1^{1,1} + h\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \tag{4.8.13}$$

donde $u_0^2(t,x,y) = u^2(t,x,y,0)$, $v_0^2(t,x,y) = v^2(t,x,y,0)$ están determinados por (4.5.99)-(4.5.100). En estas ecuaciones se sustituyen u^1, v^1, u^2, v^2, p^2, w^1 y w^2 de modo que la dependencia de z sea explícita, se agrupan los términos multiplicados por las distintas potencias de z y se simplifica utilizando para ello (4.8.4), (4.8.5), (4.8.7) y (4.8.8). Obtenemos finalmente las siguientes ecuaciones:

$$\frac{\partial u_0^2}{\partial t} + u^0\frac{\partial u_0^2}{\partial x} + u_0^1\frac{\partial u_0^1}{\partial x} + u_0^2\frac{\partial u^0}{\partial x} + v^0\frac{\partial u_0^2}{\partial y} + v_0^1\frac{\partial u_0^1}{\partial y} + v_0^2\frac{\partial u^0}{\partial y}$$
$$= -\frac{\partial}{\partial x}\left\{h\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\right.$$
$$\left.\left. + v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - 2\phi\,(\cos\varphi)\,u_0^1\right]\right\}$$

$$+\frac{1}{2}\frac{\partial}{\partial x}\left\{h^2\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+u^0\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right.\right.$$

$$\left.\left.+v^0\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)-\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)^2+2\phi\left(\cos\varphi\right)\gamma_2^0\right]\right\}$$

$$-\frac{\partial H}{\partial x}\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)+u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$

$$\left.+v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)-2\phi\left(\cos\varphi\right)u_0^1\right]$$

$$+2\phi\left[\left(\operatorname{sen}\varphi\right)v_0^2-\left(\cos\varphi\right)\left(u_0^1\frac{\partial H}{\partial x}+v_0^1\frac{\partial H}{\partial y}\right)\right]$$

$$=-\frac{1}{\rho_0}\left.D_xp^2\right|_{z=0}+2\phi\left(\left(\operatorname{sen}\varphi\right)v_0^2-\left(\cos\varphi\right)\left.w^2\right|_{z=0}\right) \tag{4.8.14}$$

$$\frac{\partial v_0^2}{\partial t}+u^0\frac{\partial v_0^2}{\partial x}+u_0^1\frac{\partial v_0^1}{\partial x}+u_0^2\frac{\partial v^0}{\partial x}+v^0\frac{\partial v_0^2}{\partial y}+v_0^1\frac{\partial v_0^1}{\partial y}+v_0^2\frac{\partial v^0}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u_0^2$$

$$=-\frac{\partial}{\partial y}\left\{h\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)+u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.\right.$$

$$\left.\left.+v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)-2\phi\left(\cos\varphi\right)u_0^1\right]\right\}$$

$$-\frac{1}{2}\frac{\partial}{\partial y}\left\{h^2\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+u^0\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right.\right.$$

$$\left.\left.+v^0\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)-\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)^2-2\phi\left(\cos\varphi\right)\gamma_1^0\right]\right\}$$

$$-\frac{\partial H}{\partial y}\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)+u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$

$$\left.+v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)-2\phi\left(\cos\varphi\right)u_0^1\right]=-\frac{1}{\rho_0}\left.D_yp^2\right|_{z=0} \tag{4.8.15}$$

Una vez calculados u^2 y v^2, w^3 se calcula integrando respecto a z (4.5.101)

$$D_zw^3=-D_xu^2-D_xv^2$$

Para ello, en primer lugar se sustituyen las expresiones obtenidas para u^2 y v^2 en (4.8.12)-(4.8.13):

$$
\begin{aligned}
-D_z w^3 = {} & \frac{\partial u_0^2}{\partial x} + zh\left[\frac{\partial \gamma_2^{1,0}}{\partial x} + \frac{\partial^2}{\partial x^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right. \\
& \left. + \frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] - \frac{1}{2}z^2\frac{\partial h}{\partial x}\left[\gamma_2^{1,1} - h\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& + \frac{1}{2}z^2 h\left[\frac{\partial \gamma_2^{1,1}}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - h\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& - \frac{\partial H}{\partial x}\left[\gamma_2^{1,0} + \frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right. \\
& \left. + z\left(\gamma_2^{1,1} - h\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right)\right] \\
& + \frac{\partial v_0^2}{\partial y} + zh\left[-\frac{\partial \gamma_1^{1,0}}{\partial y} + \frac{\partial^2}{\partial y^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial^2 H}{\partial y^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right. \\
& \left. + \frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \frac{1}{2}z^2\frac{\partial h}{\partial y}\left[\gamma_1^{1,1} + h\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& - \frac{1}{2}z^2 h\left[\frac{\partial \gamma_1^{1,1}}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + h\frac{\partial^2}{\partial y^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] \\
& - \frac{\partial H}{\partial y}\left\{-\gamma_1^{1,0} + \frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right. \\
& \left. - z\left[\gamma_1^{1,1} + h\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\right\}
\end{aligned}
$$

A continuación simplificamos, integramos respecto a z e imponemos la condición (derivada de (4.5.7)), $w^3 = u_0^2\frac{\partial H}{\partial x} + v_0^2\frac{\partial H}{\partial y}$ en $z = 0$, para llegar a la expresión siguiente de w^3 en función de u_0^2, v_0^2, u^0 y v^0

$$
\begin{aligned}
w^3 = {} & u_0^2\frac{\partial H}{\partial x} + v_0^2\frac{\partial H}{\partial y} - hz\left\{\frac{\partial u_0^2}{\partial x} - \frac{\partial H}{\partial x}\left[\gamma_2^{1,0} + \frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\right. \\
& \left. + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \frac{\partial v_0^2}{\partial y} - \frac{\partial H}{\partial y}\left[-\gamma_1^{1,0} + \frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.
\end{aligned}
$$

$$+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\Big]\Big\}-\frac{1}{2}z^2h\Big\{h\Big[\frac{\partial\gamma_2^{1,0}}{\partial x}+\frac{\partial^2}{\partial x^2}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)$$

$$+\frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\Big]$$

$$-\frac{\partial H}{\partial x}\left[\gamma_2^{1,1}-h\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]+h\Big[-\frac{\partial\gamma_1^{1,0}}{\partial y}+\frac{\partial^2}{\partial y^2}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)$$

$$+\frac{\partial^2 H}{\partial y^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\Big]$$

$$+\frac{\partial H}{\partial y}\left[\gamma_1^{1,1}+h\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\Big\}-\frac{1}{6}z^3h^2\Big\{\frac{\partial\gamma_2^{1,1}}{\partial x}-\frac{1}{h}\frac{\partial h}{\partial x}\gamma_2^{1,1}$$

$$-\frac{\partial\gamma_1^{1,1}}{\partial y}+\frac{1}{h}\frac{\partial h}{\partial y}\gamma_1^{1,1}-h\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\Big\} \quad (4.8.16)$$

La tercera componente de la vorticidad viene dada por (4.5.108):

$$\gamma_3^2 = D_x v^2 - D_y u^2$$

Para calcular las otras dos componentes se resuelven las ecuaciones (4.5.104)-(4.5.105) donde se sustituye γ_3^k $(k = 0,1,2)$ por sus expresiones en función de u^k, v^k $(k = 0,1,2)$, además de $D_z u^k$ y $D_z v^k$ $(k = 1,2,3)$, por las expresiones (4.5.86)-(4.5.87), (4.5.96)-(4.5.97) y (4.5.106)-(4.5.107):

$$D_t\gamma_1^2+u^0D_x\gamma_1^2+u^1D_x\gamma_1^1+u^2\frac{\partial\gamma_1^0}{\partial x}+v^0D_y\gamma_1^2+v^1D_y\gamma_1^1+v^2\frac{\partial\gamma_1^0}{\partial y}$$

$$+w^1D_z\gamma_1^2+w^2D_z\gamma_1^1-\gamma_1^0D_xu^2-\gamma_1^1D_xu^1-\gamma_1^2\frac{\partial u^0}{\partial x}-\gamma_2^0D_xv^2$$

$$-\gamma_2^1D_xv^1-\gamma_2^2\left(\frac{\partial v^0}{\partial x}+2\phi\,(\operatorname{sen}\varphi)\right)=\left(\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\right)D_xw^2$$

$$+\left(D_xv^1-D_yu^1\right)D_xw^1+2\phi D_y((\cos\varphi)\,u^2) \quad (4.8.17)$$

$$D_t\gamma_2^2+u^0D_x\gamma_2^2+u^1D_x\gamma_2^1+u^2\frac{\partial\gamma_2^0}{\partial x}+v^0D_y\gamma_2^2+v^1D_y\gamma_2^1+v^2\frac{\partial\gamma_2^0}{\partial y}$$

$$+w^1D_z\gamma_2^2+w^2D_z\gamma_2^1-\gamma_1^0D_yu^2-\gamma_1^1D_yu^1-\gamma_1^2\left(\frac{\partial u^0}{\partial y}-2\phi\,(\operatorname{sen}\varphi)\right)$$

$$-\gamma_2^0D_yv^2-\gamma_2^1D_yv^1-\gamma_2^2\frac{\partial v^0}{\partial y}=\left(\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\right)D_yw^2$$

$$+\left(D_x v^1 - D_y u^1\right) D_y w^1 + 2\phi\left(\cos\varphi\right) D_y v^2 \tag{4.8.18}$$

Usando las expresiones encontradas para p^0, p^1 y p^2, (4.5.74), (4.5.81) y (4.8.1) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 = p_s + \varepsilon\rho_0 h\left(1 - z\right)(g - 2\phi\left(\cos\varphi\right) u^0\right)$$
$$+ \varepsilon^2 \rho_0 h(1-z)\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.$$
$$\left. + v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - 2\phi\left(\cos\varphi\right)u_0^1\right] - \frac{1}{2}\varepsilon^2\rho_0 h^2(1-z^2)\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$
$$\left. + u^0\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + v^0\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2 + 2\phi\left(\cos\varphi\right)\gamma_2^0\right] \tag{4.8.19}$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0, 1, 2, 3)$ ((4.5.73), (4.5.82), (4.7.16) y (4.8.16)), obtenemos una aproximación de la velocidad vertical en Ω:

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$
$$= \varepsilon\left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \varepsilon^2\left[u_0^1\frac{\partial H}{\partial x} + v_0^1\frac{\partial H}{\partial y}\right.$$
$$\left. - hz\left(\frac{\partial u_0^1}{\partial x} + \frac{\partial v_0^1}{\partial y} - \frac{\partial H}{\partial x}\gamma_2^0 + \frac{\partial H}{\partial y}\gamma_1^0\right) - \frac{1}{2}z^2h^2\left(\frac{\partial \gamma_2^0}{\partial x} - \frac{\partial \gamma_1^0}{\partial y}\right)\right]$$
$$+ \varepsilon^3\left(u_0^2\frac{\partial H}{\partial x} + v_0^2\frac{\partial H}{\partial y} - hz\left\{\frac{\partial u_0^2}{\partial x} - \frac{\partial H}{\partial x}\left[\gamma_2^{1,0} + \frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\right.\right.$$
$$\left. + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \frac{\partial v_0^2}{\partial y} - \frac{\partial H}{\partial y}\left[-\gamma_1^{1,0} + \frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.$$
$$\left.\left. + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\right\} - \frac{1}{2}z^2h\left\{h\left[\frac{\partial \gamma_2^{1,0}}{\partial x} + \frac{\partial^2}{\partial x^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\right.$$
$$\left. + \frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$
$$- \frac{\partial H}{\partial x}\left[\gamma_2^{1,1} - h\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + h\left[-\frac{\partial \gamma_1^{1,0}}{\partial y} + \frac{\partial^2}{\partial y^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.$$

$$+\frac{\partial^2 H}{\partial y^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\bigg]$$

$$+\frac{\partial H}{\partial y}\left[\gamma_1^{1,1}+h\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\bigg\}-\frac{1}{6}z^3h^2\bigg\{\frac{\partial \gamma_2^{1,1}}{\partial x}-\frac{1}{h}\frac{\partial h}{\partial x}\gamma_2^{1,1}$$

$$-\frac{\partial \gamma_1^{1,1}}{\partial y}+\frac{1}{h}\frac{\partial h}{\partial y}\gamma_1^{1,1}-h\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\bigg\}\bigg)$$

Agrupamos de forma conveniente para obtener:

$$\tilde{w}(\varepsilon)=\varepsilon\frac{\partial H}{\partial x}\left(u^0+\varepsilon u^1+\varepsilon^2u^2\right)+\varepsilon\frac{\partial H}{\partial y}\left(v^0+\varepsilon v^1+\varepsilon^2v^2\right)$$

$$-\varepsilon zh\left(\frac{\partial u^0}{\partial x}+\varepsilon D_xu^1+\varepsilon^2D_xu^2+\frac{\partial v^0}{\partial y}+\varepsilon D_yv^1+\varepsilon^2D_yv^2\right)$$

$$+\varepsilon zh\left[\varepsilon\frac{\partial H}{\partial x}\left(\gamma_2^0+\varepsilon\gamma_2^1\right)-\varepsilon\frac{\partial H}{\partial y}\left(\gamma_1^0+\varepsilon\gamma_1^1\right)\right]$$

$$-\frac{1}{2}\varepsilon^2z^2h^2\left(\frac{\partial \gamma_2^0}{\partial x}+\varepsilon D_x\gamma_2^1-\frac{\partial \gamma_1^0}{\partial y}-\varepsilon D_y\gamma_1^1\right)+O(\varepsilon^3)$$

y por tanto,

$$\tilde{w}(\varepsilon)=\varepsilon\frac{\partial H}{\partial x}\tilde{u}(\varepsilon)+\varepsilon\frac{\partial H}{\partial y}\tilde{v}(\varepsilon)-\varepsilon hz\left(D_x\tilde{u}(\varepsilon)+D_y\tilde{v}(\varepsilon)\right)$$

$$+\varepsilon^2zh\left(\frac{\partial H}{\partial x}\tilde{\gamma}_2(\varepsilon)-\frac{\partial H}{\partial y}\tilde{\gamma}_1(\varepsilon)\right)-\frac{1}{2}\varepsilon^2z^2h^2\left(D_x\tilde{\gamma}_2(\varepsilon)-D_y\tilde{\gamma}_1(\varepsilon)\right)+O(\varepsilon^3) \quad (4.8.20)$$

Si denotamos por

$$\check{\gamma}_i(\varepsilon)(t,x)=\gamma_i^0(t,x)+\varepsilon\gamma_i^{1,0}(t,x)\quad(i=1,2)$$

también se puede escribir la velocidad vertical de la forma siguiente:

$$\tilde{w}(\varepsilon)=\varepsilon\frac{\partial H}{\partial x}\tilde{u}(\varepsilon)+\varepsilon\frac{\partial H}{\partial y}\tilde{v}(\varepsilon)-\varepsilon hz\left(D_x\tilde{u}(\varepsilon)+D_y\tilde{v}(\varepsilon)\right)$$

$$+\varepsilon^2zh\left(\frac{\partial H}{\partial x}\check{\gamma}_2(\varepsilon)-\frac{\partial H}{\partial y}\check{\gamma}_1(\varepsilon)\right)-\frac{1}{2}\varepsilon^2z^2h^2\left(D_x\check{\gamma}_2(\varepsilon)-D_y\check{\gamma}_1(\varepsilon)\right)+O(\varepsilon^3)$$

$$=-\varepsilon\left[D_x\left(zh\tilde{u}(\varepsilon)+\frac{1}{2}\varepsilon z^2h^2\check{\gamma}_2(\varepsilon)\right)+D_y\left(zh\tilde{v}(\varepsilon)-\frac{1}{2}\varepsilon z^2h^2\check{\gamma}_1(\varepsilon)\right)\right]$$

$$+O(\varepsilon^3) \quad (4.8.21)$$

Además, para la tercera componente de la vorticidad tenemos:

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0 + \varepsilon\gamma_3^1 + \varepsilon^2\gamma_3^2 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} + \varepsilon\left(D_x v^1 - D_y u^1\right)$$
$$+ \varepsilon^2\left(D_x v^2 - D_y u^2\right) = D_x\tilde{v}(\varepsilon) - D_y\tilde{u}(\varepsilon) \quad (4.8.22)$$

Si ahora se deshace el cambio de variable, volviendo al dominio original, obtenemos la siguiente aproximación de la solución en Ω^ε:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y,z) + \varepsilon^2 u^2(t,x,y,z)$$
$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y,z) + \varepsilon^2 v^2(t,x,y,z)$$
$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z) + \varepsilon^3 w^3(t,x,y,z)$$
$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z) + \varepsilon^2 p^2(t,x,y,z)$$
$$\tilde{\gamma}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_i(\varepsilon)(t,x,y,z) = \gamma_i^0(t,x,y) + \varepsilon\gamma_i^1(t,x,y,z) + \varepsilon^2\gamma_i^2(t,x,y,z) \quad (i=1,2)$$
$$\tilde{\gamma}_3^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_3(\varepsilon)(t,x,y,z) = \gamma_3^0(t,x,y) + \varepsilon\gamma_3^1(t,x,y,z) + \varepsilon^2\gamma_3^2(t,x,y,z)$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (4.8.19):

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)\left(u^{0,\varepsilon} + \varepsilon u_0^{1,\varepsilon}\right) + \frac{\partial}{\partial t^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right.$$
$$\left. + u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + v^{0,\varepsilon}\frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right]$$
$$- \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\phi\left(\cos\varphi^\varepsilon\right)\gamma_2^{0,\varepsilon} + \frac{\partial}{\partial t^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)\right.$$
$$+ u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + v^{0,\varepsilon}\frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)$$
$$\left. - \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)^2\right] \quad (4.8.23)$$

donde $u^{0,\varepsilon} = u^0$, $v^{0,\varepsilon} = v^0$, $u_0^{1,\varepsilon} = \varepsilon u_0^1$ y $\gamma_2^{0,\varepsilon} = \gamma_2^0$, tras el cambio de variable de Ω a Ω^ε.

Y de forma análoga, realizando el cambio de variable de Ω a Ω^ε en (4.8.20), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \tilde{\gamma}_2^\varepsilon + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \tilde{\gamma}_1^\varepsilon \right)$$

$$- \frac{1}{2}(z^\varepsilon - H^\varepsilon)^2 \left(\frac{\partial \tilde{\gamma_2}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{\gamma_1}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon^3) \tag{4.8.24}$$

Y al cambiar de variable en la expresión vista para la tercera componente de la vorticidad:

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}$$

El modelo que vamos a proponer requiere que definamos también

$$\check{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u_0^1(t,x,y)$$

$$\check{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v_0^1(t,x,y)$$

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u_0^1(t,x,y) + \varepsilon^2 u_0^2(t,x,y)$$

$$\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v_0^1(t,x,y) + \varepsilon^2 v_0^2(t,x,y)$$

$$\gamma^{1,k,\varepsilon}(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon^k \gamma^{1,k}(t,x,y), \quad (k = 1,2)$$

Recordamos que en la sección anterior deducimos que para h, $\check{u}^\varepsilon$ y $\check{v}^\varepsilon$ se tenían las ecuaciones (4.7.26)-(4.7.28).

Ahora, teniendo en cuenta (4.7.2) y (4.8.4), se verifica

$$\frac{\partial \check{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \check{\gamma}_1^\varepsilon}{\partial y^\varepsilon} - \check{\gamma}_1^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} - \check{\gamma}_2^\varepsilon \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right)$$

$$- 2\phi \left[\frac{\partial}{\partial y^\varepsilon} \left((\cos\varphi^\varepsilon)\, \check{u}^\varepsilon \right) - \frac{\partial H^\varepsilon}{\partial y^\varepsilon} (\cos\varphi^\varepsilon)\, \check{\gamma}_2^\varepsilon \right]$$

$$- \left(2\phi\,(\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$= \frac{\partial \check{\gamma}_1(\varepsilon)}{\partial t} + \check{u}(\varepsilon) \frac{\partial \check{\gamma}_1(\varepsilon)}{\partial x} + \check{v}(\varepsilon) \frac{\partial \check{\gamma}_1(\varepsilon)}{\partial y} - \check{\gamma}_1(\varepsilon) \frac{\partial \check{u}(\varepsilon)}{\partial x}$$

$$- \check{\gamma}_2(\varepsilon) \left(\frac{\partial \check{v}(\varepsilon)}{\partial x} + 2\phi\,(\operatorname{sen}\varphi) \right) - 2\phi \left[\frac{\partial}{\partial y} \left((\cos\varphi)\ \check{u}(\varepsilon) \right) - \varepsilon \frac{\partial H}{\partial y} (\cos\varphi)\, \check{\gamma}_2(\varepsilon) \right]$$

$$- \varepsilon \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial \check{v}(\varepsilon)}{\partial x} - \frac{\partial \check{u}(\varepsilon)}{\partial y} \right) \left[\frac{\partial}{\partial x} \left(\check{u}(\varepsilon) \frac{\partial H}{\partial x} + \check{v}(\varepsilon) \frac{\partial H}{\partial y} \right) \right.$$

$$
\begin{aligned}
&+ \frac{\partial H}{\partial x}\left(\frac{\partial \check{u}(\varepsilon)}{\partial x} + \frac{\partial \check{v}(\varepsilon)}{\partial y}\right)\Bigg]\\
&= \frac{\partial \gamma_1^0}{\partial t} + \varepsilon \frac{\partial \gamma_1^{1,0}}{\partial t} + (u^0 + \varepsilon u_0^1)\left(\frac{\partial \gamma_1^0}{\partial x} + \varepsilon \frac{\partial \gamma_1^{1,0}}{\partial x}\right) + (v^0 + \varepsilon v_0^1)\left(\frac{\partial \gamma_1^0}{\partial y} + \varepsilon \frac{\partial \gamma_1^{1,0}}{\partial y}\right)\\
&- \left(\gamma_1^0 + \varepsilon \gamma_1^{1,0}\right)\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x}\right) - \left(\gamma_2^0 + \varepsilon \gamma_2^{1,0}\right)\left(\frac{\partial v^0}{\partial x} + \varepsilon \frac{\partial v_0^1}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\right)\\
&- 2\phi\left\{\frac{\partial}{\partial y}\left[(\cos\varphi)\,(u^0 + \varepsilon u_0^1)\right] - \varepsilon \frac{\partial H}{\partial y}(\cos\varphi)\left(\gamma_2^0 + \varepsilon \gamma_2^{1,0}\right)\right\}\\
&- \varepsilon\left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} + \varepsilon + \frac{\partial v_0^1}{\partial x} - \frac{\partial u^0}{\partial y} - \varepsilon \frac{\partial u_0^1}{\partial y}\right)\left[\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}(u^0 + \varepsilon u_0^1)\right.\right.\\
&\left.\left. + \frac{\partial H}{\partial y}(v^0 + \varepsilon v_0^1)\right) + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u_0^1}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon \frac{\partial v_0^1}{\partial y}\right)\right]\\
&= \frac{\partial \gamma_1^0}{\partial t} + u^0 \frac{\partial \gamma_1^0}{\partial x} + v^0 \frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0 \frac{\partial u^0}{\partial x} - \gamma_2^0\left(\frac{\partial v^0}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\right) - 2\phi \frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)\\
&+ \varepsilon\left\{\frac{\partial \gamma_1^{1,0}}{\partial t} + u^0 \frac{\partial \gamma_1^{1,0}}{\partial x} + u_0^1 \frac{\partial \gamma_1^0}{\partial x} + v^0 \frac{\partial \gamma_1^{1,0}}{\partial y} + v_0^1 \frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0 \frac{\partial u_0^1}{\partial x} - \gamma_1^{1,0} \frac{\partial u^0}{\partial x}\right.\\
&- \gamma_2^0 \frac{\partial v_0^1}{\partial x} - \gamma_2^{1,0}\left(\frac{\partial v^0}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\right) - 2\phi\left(\frac{\partial}{\partial y}\left((\cos\varphi)\,u_0^1\right) - \frac{\partial H}{\partial y}(\cos\varphi)\,\gamma_2^0\right)\\
&\left. - \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right)\left[\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}u^0 + \frac{\partial H}{\partial y}v^0\right) + \frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\right\}\\
&+ O(\varepsilon^2) = O(\varepsilon^2) \qquad (4.8.25)
\end{aligned}
$$

Y análogamente, a partir (4.7.5) y (4.8.7), tenemos

$$
\begin{aligned}
&\frac{\partial \check{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \check{\gamma}_2^\varepsilon}{\partial y^\varepsilon} - \check{\gamma}_1^\varepsilon\left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right) - \check{\gamma}_2^\varepsilon \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\\
&\quad - 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\check{\gamma}_1^\varepsilon\right) - \left(2\phi\,(\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon}\right)\left[\frac{\partial}{\partial y^\varepsilon}\left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right.\right.\\
&\quad\left.\left. + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] = O(\varepsilon^2) \qquad (4.8.26)
\end{aligned}
$$

Expresamos $\tilde{p}^\varepsilon$ en función de $\check{u}^\varepsilon$, $\check{v}^\varepsilon$ y $\check{\gamma}_2^\varepsilon$ en lugar de $u^{0,\varepsilon}$, $v^{0,\varepsilon}$, $u_0^{1,\varepsilon}$ y $\gamma_2^{0,\varepsilon}$:

$$\begin{aligned}\tilde{p}^\varepsilon &= p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\,(\cos\varphi^\varepsilon)\,\check{u}^\varepsilon + \frac{\partial}{\partial t^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right.\\
&\quad\left. + \check{u}^\varepsilon\frac{\partial}{\partial x^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + \check{v}^\varepsilon\frac{\partial}{\partial y^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right]\\
&\quad - \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\phi\,(\cos\varphi^\varepsilon)\,\check{\gamma}_2^\varepsilon + \frac{\partial}{\partial t^\varepsilon}\left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right.\\
&\quad\left. + \check{u}^\varepsilon\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon}\right) + \check{v}^\varepsilon\frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon}\right) - \left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon}\right)^2\right]\\
&\quad + O(\varepsilon^3)\end{aligned}\tag{4.8.27}$$

Si se consideran las ecuaciones (4.5.78), (4.7.10) y (4.8.14) se verifica

$$\begin{aligned}\frac{\partial\hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} &= \frac{\partial u^0}{\partial t} + \varepsilon\frac{\partial u_0^1}{\partial t} + \varepsilon^2\frac{\partial u_0^2}{\partial t}\\
&\quad + (u^0 + \varepsilon u_0^1 + \varepsilon^2 u_0^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u_0^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x}\right)\\
&\quad + (v^0 + \varepsilon v_0^1 + \varepsilon^2 v_0^2)\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u_0^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y}\right)\\
&= \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \varepsilon\left(\frac{\partial u_0^1}{\partial t} + u_0^1\frac{\partial u^0}{\partial x} + u^0\frac{\partial u_0^1}{\partial x} + v_0^1\frac{\partial u^0}{\partial y} + v^0\frac{\partial u_0^1}{\partial y}\right)\\
&\quad + \varepsilon^2\left[\frac{\partial u_0^2}{\partial t} + u_0^2\frac{\partial u^0}{\partial x} + u_0^1\frac{\partial u_0^1}{\partial x} + u^0\frac{\partial u_0^2}{\partial x} + v_0^2\frac{\partial u^0}{\partial y} + v_0^1\frac{\partial u_0^1}{\partial y} + v^0\frac{\partial u_0^2}{\partial y}\right] + O(\varepsilon^3)\\
&= -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\,v^0\\
&\quad + \varepsilon\left[-g\frac{\partial s}{\partial x} + 2\phi\,(\cos\varphi)\left(\frac{\partial(u^0 h)}{\partial x} - v^0\frac{\partial H}{\partial y}\right) + 2\phi\,(\operatorname{sen}\varphi)\,v_0^1\right]\\
&\quad + \varepsilon^2\left[-\frac{1}{\rho_0}\left.D_x p^2\right|_{z=0} + 2\phi\left((\operatorname{sen}\varphi)\,v_0^2 - (\cos\varphi)\left.w^2\right|_{z=0}\right)\right] + O(\varepsilon^3)\\
&= -\frac{1}{\rho_0}\left(\frac{\partial p^0}{\partial x} + \varepsilon\left.D_x p^1\right|_{z=0} + \varepsilon^2\left.D_x p^2\right|_{z=0}\right) - \varepsilon 2\phi\,(\cos\varphi)\left(\left.w^1\right|_{z=0} + \varepsilon\left.w^2\right|_{z=0}\right)\\
&\quad + 2\phi\,(\operatorname{sen}\varphi)\,(v^0 + \varepsilon v_0^1 + \varepsilon^2 v_0^2) + O(\varepsilon^3)\end{aligned}$$

por tanto,

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \hat{v}^\varepsilon$$

$$- 2\phi \left(\cos \varphi^\varepsilon\right) \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon^3) \tag{4.8.28}$$

De manera similar utilizando las ecuaciones (4.5.79), (4.7.11) y (4.8.15) se obtiene

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \left(\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + O(\varepsilon^3) \tag{4.8.29}$$

Las ecuaciones para el cálculo de $\gamma_i^{1,1,\varepsilon} \quad (i = 1, 2)$, se obtienen deshaciendo el cambio de variable en las ecuaciones (4.8.5) y (4.8.8) y sustituyendo $\gamma_i^0 \quad (i = 1, 2)$, u^0 y v^0 por $\breve{\gamma}_i^\varepsilon \quad (i = 1, 2)$, $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ respectivamente

$$\frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial x^\varepsilon} + 2h^\varepsilon \breve{\gamma}_2^\varepsilon \frac{\partial \breve{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial y^\varepsilon} - h^\varepsilon \breve{\gamma}_1^\varepsilon \left(\frac{\partial \breve{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \breve{\gamma}_2^\varepsilon}{\partial x^\varepsilon} \right) - \gamma_1^{1,1,\varepsilon} \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}$$

$$- \gamma_2^{1,1,\varepsilon} \left(\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) = 2\phi h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\left(\cos \varphi^\varepsilon\right) \breve{\gamma}_2^\varepsilon \right)$$

$$-h^\varepsilon \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.8.30}$$

$$\frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial x^\varepsilon} + h^\varepsilon \breve{\gamma}_2^\varepsilon \left(\frac{\partial \breve{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{\gamma}_1^\varepsilon}{\partial y^\varepsilon} \right) + \breve{v}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial y^\varepsilon} - 2h^\varepsilon \breve{\gamma}_1^\varepsilon \frac{\partial \breve{\gamma}_2^\varepsilon}{\partial y^\varepsilon}$$

$$- \gamma_1^{1,1,\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) - \gamma_2^{1,1,\varepsilon} \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} = 2\phi \left(\cos \varphi^\varepsilon\right) h^\varepsilon \frac{\partial \breve{\gamma}_1^\varepsilon}{\partial y^\varepsilon}$$

$$- h^\varepsilon \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.8.31}$$

Veamos cómo se pueden escribir $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ en función de $\hat{u}^\varepsilon$, $\hat{v}^\varepsilon$, $\breve{\gamma}_i^\varepsilon$, $\gamma_i^{1,1,\varepsilon} \quad (i = 1, 2)$ y h^ε. Comenzamos por $\tilde{u}^\varepsilon$ utilizando (4.7.8) y (4.8.12)

$$\tilde{u}^\varepsilon = \tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 = u^0 + \varepsilon \left(u_0^1 + hz\gamma_2^0 \right)$$

$$+ \varepsilon^2 \left\{ u_0^2 + hz \left[\gamma_2^{1,0} + \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) + \frac{\partial H}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right.$$

$$\left. + \frac{1}{2} z^2 h \left[\gamma_2^{1,1} - h \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right\}$$

$$= u^0 + \varepsilon u_0^1 + \varepsilon^2 u_0^2 + \varepsilon hz \left\{ \gamma_2^0 + \varepsilon \left[\gamma_2^{1,0} + \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right.\right.$$

$$\left.\left. + \frac{\partial H}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right\} + \frac{1}{2} \varepsilon^2 h z^2 \left[\gamma_2^{1,1} - h \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right]$$

$$= \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[\breve{\gamma}_2^\varepsilon + \frac{\partial}{\partial x^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$+ \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_2^{1,1,\varepsilon} - h^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon^3) \tag{4.8.32}$$

Para obtener una expresión análoga para $\tilde{v}^\varepsilon$ se tienen en cuenta las ecuaciones (4.7.9) y (4.8.13)

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[-\breve{\gamma}_1^\varepsilon + \frac{\partial}{\partial y^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$- \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_1^{1,1,\varepsilon} + h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon^3) \tag{4.8.33}$$

Proponemos el siguiente modelo resultado de despreciar los términos en $O(\varepsilon)$ de las ecuaciones (4.8.30)-(4.8.31), los términos en $O(\varepsilon^2)$ de las ecuaciones (4.7.26)-(4.7.28), (4.8.25) y (4.8.26) y los términos en $O(\varepsilon^3)$ de las expresiones (4.8.24), (4.8.27), (4.8.32) y (4.8.33) y de las ecuaciones (4.8.28)-(4.8.29):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\breve{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{4.8.34}$$

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) \breve{v}^\varepsilon$$

$$+ 2\phi \left(\cos \varphi^\varepsilon \right) \left[\frac{\partial (\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} - \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right] \tag{4.8.35}$$

$$\frac{\partial \breve{v}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) \breve{u}^\varepsilon$$

$$+ 2\phi \left(\frac{\partial}{\partial y^\varepsilon} ((\cos \varphi^\varepsilon) \breve{u}^\varepsilon h^\varepsilon) + (\cos \varphi^\varepsilon) \breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.8.36}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \left[g - 2\phi \left(\cos \varphi^\varepsilon \right) \breve{u}^\varepsilon + \frac{\partial}{\partial t^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \breve{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \breve{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$- \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[2\phi \left(\cos\varphi^\varepsilon\right) \check{\gamma}_2^\varepsilon + \frac{\partial}{\partial t^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \check{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) + \check{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) - \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right)^2 \right] \tag{4.8.37}$$

$$\frac{\partial \check{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \check{\gamma}_1^\varepsilon}{\partial y^\varepsilon} - \check{\gamma}_1^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} - \check{\gamma}_2^\varepsilon \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) - 2\phi \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\cos\varphi^\varepsilon\right) \right)$$

$$= 2\phi \frac{\partial}{\partial y^\varepsilon} \left(\left(\cos\varphi^\varepsilon\right) \check{u}^\varepsilon \right) + \left(2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.8.38}$$

$$\frac{\partial \check{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \check{\gamma}_2^\varepsilon}{\partial y^\varepsilon} - \check{\gamma}_1^\varepsilon \left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) + 2\phi \left(\cos\varphi^\varepsilon\right) \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) - \check{\gamma}_2^\varepsilon \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= 2\phi \left(\cos\varphi^\varepsilon\right) \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} + \left(2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial y^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.8.39}$$

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} = - \frac{1}{\rho_0} \left. \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} \right|_{z^\varepsilon = H^\varepsilon} + 2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) \hat{v}^\varepsilon$$

$$- 2\phi \left(\cos\varphi^\varepsilon\right) \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.8.40}$$

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} = - \frac{1}{\rho_0} \left. \frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon} \right|_{z^\varepsilon = H^\varepsilon} - 2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) \hat{u}^\varepsilon \tag{4.8.41}$$

$$\frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial x^\varepsilon} + 2h^\varepsilon \check{\gamma}_2^\varepsilon \frac{\partial \check{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial y^\varepsilon} - h^\varepsilon \check{\gamma}_1^\varepsilon \left(\frac{\partial \check{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \check{\gamma}_2^\varepsilon}{\partial x^\varepsilon} \right) - \gamma_1^{1,1,\varepsilon} \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}$$

$$- \gamma_2^{1,1,\varepsilon} \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) \right) = 2\phi h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\left(\cos\varphi^\varepsilon\right) \check{\gamma}_2^\varepsilon \right)$$

$$-h^\varepsilon \left(2\phi \left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.8.42}$$

$$\frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial x^\varepsilon} + h^\varepsilon \check{\gamma}_2^\varepsilon \left(\frac{\partial \check{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{\gamma}_1^\varepsilon}{\partial y^\varepsilon} \right) + \check{v}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial y^\varepsilon} - 2h^\varepsilon \check{\gamma}_1^\varepsilon \frac{\partial \check{\gamma}_2^\varepsilon}{\partial y^\varepsilon}$$

$$-\gamma_1^{1,1,\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\right)-\gamma_2^{1,1,\varepsilon}\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}=2\phi\left(\cos\varphi^\varepsilon\right)h^\varepsilon\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon}$$

$$-h^\varepsilon\left(2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)+\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.8.43}$$

$$\tilde{u}^\varepsilon=\hat{u}^\varepsilon+\left(z^\varepsilon-H^\varepsilon\right)\left[\tilde{\gamma}_2^\varepsilon+\frac{\partial}{\partial x^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\check{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]$$

$$+\frac{1}{2}\frac{\left(z^\varepsilon-H^\varepsilon\right)^2}{h^\varepsilon}\left[\gamma_2^{1,1,\varepsilon}-h^\varepsilon\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] \tag{4.8.44}$$

$$\tilde{v}^\varepsilon=\hat{v}^\varepsilon+\left(z^\varepsilon-H^\varepsilon\right)\left[-\tilde{\gamma}_1^\varepsilon+\frac{\partial}{\partial y^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\check{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]$$

$$-\frac{1}{2}\frac{\left(z^\varepsilon-H^\varepsilon\right)^2}{h^\varepsilon}\left[\gamma_1^{1,1,\varepsilon}+h^\varepsilon\frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] \tag{4.8.45}$$

$$\tilde{w}^\varepsilon=\tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}-\left(z^\varepsilon-H^\varepsilon\right)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}-\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\tilde{\gamma}_2^\varepsilon+\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\tilde{\gamma}_1^\varepsilon\right)$$

$$-\frac{1}{2}(z^\varepsilon-H^\varepsilon)^2\left(\frac{\partial \tilde{\gamma}_2{}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial \tilde{\gamma}_1{}^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.8.46}$$

donde es necesario conocer los datos iniciales y de contorno.

El modelo (4.8.34)-(4.8.46) es más preciso (al menos formalmente) que el modelo (4.7.33)-(4.7.41), pues los términos despreciados son del orden de $O(\varepsilon^3)$ en lugar de $O(\varepsilon^2)$, salvo en el caso de la ecuación para el cálculo de h^ε, que sigue siendo de orden $O(\varepsilon^2)$. Sin embargo, el esfuerzo para resolver el sistema (4.8.34)-(4.8.46) es el doble del necesario para resolver el modelo (4.7.33)-(4.7.41), y como h^ε se calcula con un error del orden $O(\varepsilon^2)$ el error global del modelo será también de orden $O(\varepsilon^2)$, por lo que no vale la pena el mayor esfuerzo de cálculo que hay que realizar.

Nuestra opinión es que la supuesta mejora en el orden de precisión que introduce este modelo no justifica la complejidad que presenta su resolución. En todo caso, podría utilizarse la expresión de $\tilde{p}^\varepsilon$ escrita en términos de $\tilde{u}^\varepsilon$ y de $\tilde{v}^\varepsilon$ como una mejora de la obtenida en la sección anterior (véase (4.7.22)).

Si de todos modos, estuviésemos interesados en utilizar el modelo (4.8.34)-(4.8.46), sería útil obtener las expresiones adecuadas para el cálculo de la vorticidad a partir de $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y $\tilde{w}^\varepsilon$. Para el cálculo de la primera componente, será necesario tener en cuenta (4.7.2), (4.7.17) y (4.8.17):

$$\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}-\tilde{\gamma}_2^\varepsilon\left(\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\right)$$

$$- 2\phi \frac{\partial}{\partial y^{\varepsilon}}((\cos\varphi^{\varepsilon})\,\tilde{u}^{\varepsilon}) - \left(2\phi\,(\operatorname{sen}\varphi^{\varepsilon}) + \frac{\partial \tilde{v}^{\varepsilon}}{\partial x^{\varepsilon}} - \frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}}\right)\frac{\partial \tilde{w}^{\varepsilon}}{\partial x^{\varepsilon}}$$

$$= D_t\tilde{\gamma}_1(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{\gamma}_1(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{\gamma}_1(\varepsilon) + \tilde{w}(\varepsilon)D_z\tilde{\gamma}_1(\varepsilon) - \tilde{\gamma}_1(\varepsilon)D_x\tilde{u}(\varepsilon)$$

$$- \tilde{\gamma}_2(\varepsilon)\,(D_x v(\varepsilon) + 2\phi\,(\operatorname{sen}\varphi)) - 2\phi D_y((\cos\varphi)\ \tilde{u}(\varepsilon))$$

$$- (2\phi\,(\operatorname{sen}\varphi) + D_x\tilde{v}(\varepsilon) - D_y\tilde{u}(\varepsilon))\,D_x\tilde{w}(\varepsilon)$$

$$= \frac{\partial \gamma_1^0}{\partial t} + \varepsilon D_t\gamma_1^1 + \varepsilon^2 D_t\gamma_1^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left(\frac{\partial \gamma_1^0}{\partial x} + \varepsilon D_x\gamma_1^1 + \varepsilon^2 D_x\gamma_1^2\right)$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\left(\frac{\partial \gamma_1^0}{\partial y} + \varepsilon D_y\gamma_1^1 + \varepsilon^2 D_y\gamma_1^2\right)$$

$$+ \varepsilon(w^1 + \varepsilon w^2 + \varepsilon^2 w^3)\left(D_z\gamma_1^1 + \varepsilon D_z\gamma_1^2\right)$$

$$- (\gamma_1^0 + \varepsilon\gamma_1^1 + \varepsilon^2\gamma_1^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2\right)$$

$$- (\gamma_2^0 + \varepsilon\gamma_2^1 + \varepsilon^2\gamma_2^2)\left(\frac{\partial v^0}{\partial x} + \varepsilon D_x v^1 + \varepsilon^2 D_x v^2 + 2\phi\,(\operatorname{sen}\varphi)\right)$$

$$- 2\phi D_y((\cos\varphi)\ (u^0 + \varepsilon u^1)) - \big[2\phi\,(\operatorname{sen}\varphi) + D_x(v^0 + \varepsilon v^1 + \varepsilon^2 v^2)$$

$$- D_y(u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\big]\,\varepsilon D_x(w^1 + \varepsilon w^2 + \varepsilon^2 w^3)$$

$$= \frac{\partial \gamma_1^0}{\partial t} + u^0\frac{\partial \gamma_1^0}{\partial x} + v^0\frac{\partial \gamma_1^0}{\partial y} - \gamma_1^0\frac{\partial u^0}{\partial x} - \gamma_2^0\frac{\partial v^0}{\partial x} - 2\phi\left(\frac{\partial}{\partial y}((\cos\varphi)\ u^0) + (\operatorname{sen}\varphi)\ \gamma_2^0\right)$$

$$+ \varepsilon\left[D_t\gamma_1^1 + u^0 D_x\gamma_1^1 + u^1\frac{\partial \gamma_1^0}{\partial x} + v^0 D_y\gamma_1^1 + v^1\frac{\partial \gamma_1^0}{\partial y} + w^1 D_z\gamma_1^1 - \gamma_1^0 D_x u^1\right.$$

$$- \gamma_1^1\frac{\partial u^0}{\partial x} - \gamma_2^0 D_x v^1 - \gamma_2^1\frac{\partial v^0}{\partial x} - 2\phi\left(D_y((\cos\varphi)\ u^1) + (\operatorname{sen}\varphi)\ \gamma_2^1\right)$$

$$\left. - \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) D_x w^1\right]$$

$$+ \varepsilon^2\left[D_t\gamma_1^2 + u^0 D_x\gamma_1^2 + u^1 D_x\gamma_1^1 + u^2\frac{\partial \gamma_1^0}{\partial x} + v^0 D_y\gamma_1^2 + v^1 D_y\gamma_1^1 + v^2\frac{\partial \gamma_1^0}{\partial y}\right.$$

$$+ w^1 D_z\gamma_1^2 + w^2 D_z\gamma_1^1 - \gamma_1^0 D_x u^2 - \gamma_1^1 D_x u^1 - \gamma_1^2\frac{\partial u^0}{\partial x} - \gamma_2^0 D_x v^2 - \gamma_2^1 D_x v^1$$

$$- \gamma_2^2 \left(\frac{\partial v^0}{\partial x} + 2\phi\, (\operatorname{sen} \varphi) \right) - 2\phi D_y((\cos\varphi)\ u^2) - \left(2\phi\, (\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \right) D_x w^2$$

$$- \left(D_x v^1 - D_y u^1 \right) D_x w^1 \Big] + O(\varepsilon^3) = O(\varepsilon^3)$$

De manera análoga,

$$\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}$$

$$- 2\phi \left(- (\operatorname{sen}\varphi^\varepsilon)\, \tilde{\gamma}_1^\varepsilon + (\cos\varphi^\varepsilon) \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) - \left(2\phi\, (\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} = O(\varepsilon^3)$$

Si se desprecian los términos en $O(\varepsilon^3)$, las ecuaciones para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo son las siguientes:

$$\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \left(\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\, (\operatorname{sen}\varphi^\varepsilon) \right)$$

$$= 2\phi \frac{\partial}{\partial y^\varepsilon} ((\cos\varphi^\varepsilon)\, \tilde{u}^\varepsilon) + \left(2\phi\, (\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} \quad (4.8.47)$$

$$\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\, (\operatorname{sen}\varphi^\varepsilon) \right) - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= 2\phi\, (\cos\varphi^\varepsilon) \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \left(2\phi\, (\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} \quad (4.8.48)$$

Observación 4.8 *En las expresiones anteriores la variable z^ε actúa como parámetro. Sería posible obtener ecuaciones para $\tilde{\gamma}_1^\varepsilon$ y $\tilde{\gamma}_2^\varepsilon$ con dependencia de z^ε explícita. Para ello sería suficiente con suponer que γ_1^2 y γ_2^2 son polinomios de z y sustituir dichas expresiones en las ecuaciones (4.8.17)-(4.8.18) identificando los términos que multiplican a las distintas potencias de z. Las ecuaciones resultantes son mucho más largas que (4.8.17)-(4.8.18), pero la dependencia de z^ε es explícita.*

Para la tercera componente de la vorticidad se tiene:

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \quad (4.8.49)$$

Por último, veamos en qué medida verifica la aproximación de segundo orden las ecuaciones de Euler. Comencemos por la primera de estas ecuaciones:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon\right)$$

$$= D_t\tilde{u}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{u}(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{u}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{u}(\varepsilon) + \frac{1}{\rho_0}D_x\tilde{p}(\varepsilon)$$

$$- 2\phi\left((\operatorname{sen}\varphi)\;\tilde{v}(\varepsilon) - (\cos\varphi)\;\tilde{w}(\varepsilon)\right)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2\right)$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon v^2)\left(\frac{\partial u^0}{\partial y} + \varepsilon D_y u^1 + \varepsilon D_y u^2\right)$$

$$+ (w^1 + \varepsilon w^2 + \varepsilon^2 w^3)\left(\varepsilon D_z u^1 + \varepsilon^2 D_z u^2\right) + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \varepsilon\frac{1}{\rho_0}D_x p^1 + \varepsilon^2\frac{1}{\rho_0}D_x p^2$$

$$- 2\phi\left((\operatorname{sen}\varphi)\;(v^0 + \varepsilon v^1 + \varepsilon^2 v^2) - (\cos\varphi)\;\varepsilon(w^1 + \varepsilon w^2 + \varepsilon^2 w^3)\right)$$

$$= \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\;v^0$$

$$+ \varepsilon\left[D_t u^1 + u^1\frac{\partial u^0}{\partial x} + u^0 D_x u^1 + v^1\frac{\partial u^0}{\partial y} + v^0 D_y u^1 + w^1 D_z u^1\right.$$

$$\left. + 2\phi\,(\cos\varphi)\,\frac{\partial u^0}{\partial x}h(z-1) - \frac{\partial s}{\partial x}\left(2\phi\,(\cos\varphi)\,u^0 - g\right) - 2\phi\left((\operatorname{sen}\varphi)\;v^1 - (\cos\varphi)\;w^1\right)\right]$$

$$+ \varepsilon^2\left[D_t u^2 + u^2\frac{\partial u^0}{\partial x} + u^1 D_x u^1 + u^0 D_x u^2 + v^2\frac{\partial u^0}{\partial y} + v^1 D_y u^1 + v^0 D_y u^2\right.$$

$$\left. + w^2 D_z u^1 + w^1 D_z u^2 + \frac{1}{\rho_0}D_x p^2 - 2\phi\left((\operatorname{sen}\varphi)\;v^2 - (\cos\varphi)\;w^2\right)\right] + O(\varepsilon^3)$$

Usando (4.5.78), (4.5.89) y (4.5.99) se puede escribir:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon\right) + O(\varepsilon^3)$$

Así, la primera ecuación de Euler se verifica con un error de orden ε^3. La segunda ecuación de Euler se comporta del mismo modo.

Para la tercera ecuación de Euler se tiene, usando (4.5.74), (4.5.81) y (4.5.91), que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - 2\phi\left(\cos\varphi^\varepsilon\right)u^\varepsilon$$

$$= D_t\tilde{w}(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{w}(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{w}(\varepsilon) + \tilde{w}(\varepsilon)\frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon)$$

$$+ \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z\tilde{p}(\varepsilon) + g - 2\phi\left(\cos\varphi\right)u(\varepsilon) = \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + \varepsilon^3 D_t w^3$$

$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\varepsilon\left[D_x w^1 + \varepsilon D_x w^2 + \varepsilon^2 D_x w^3\right]$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\varepsilon\left[D_y w^1 + \varepsilon D_y w^2 + \varepsilon^2 D_y w^3\right]$$

$$+ (\varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3)\left[D_z w^1 + \varepsilon D_z w^2 + \varepsilon^2 D_z w^3\right]$$

$$+ \frac{1}{\rho_0}D_z p^1 + \varepsilon\frac{1}{\rho_0}D_z p^2 + g - 2\phi\left(\cos\varphi\right)(u^0 + \varepsilon u^1 + \varepsilon^2 u^2)$$

$$= \frac{1}{\rho_0}D_z p^1 + g - 2\phi\left(\cos\varphi\right)u^0 + \varepsilon\Big(D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1$$

$$+ w^1 D_z w^1 + \frac{1}{\rho_0}D_z p^2 - 2\phi\left(\cos\varphi\right)u^1\Big) + O(\varepsilon^2)$$

La aproximación de segundo orden verifica la tercera ecuación de Euler con un error del orden de ε^2.

De las ecuaciones (4.8.44)-(4.8.46) se deduce que la ecuación de incompresibilidad se verifica con un error de orden $O(\varepsilon^2)$.

Por construcción de $\tilde{p}^\varepsilon$ y $\tilde{w}^\varepsilon$ (véase (4.8.37) y (4.8.46)) las condiciones de contorno (4.1.6) y (4.1.7) se verifican exactamente.

Veamos ahora lo que sucede con las ecuaciones de la vorticidad (4.1.12)-(4.1.14),

$$\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \tilde{\gamma}_3^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon}$$

$$- 2\phi\left(\frac{\partial}{\partial y^\varepsilon}\left(\left(\cos\varphi^\varepsilon\right)\tilde{u}^\varepsilon\right) + \left(\operatorname{sen}\varphi^\varepsilon\right)\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon}\right)$$

$$= \frac{\partial \gamma_1^0}{\partial t} + \varepsilon D_t\gamma_1^1 + \varepsilon^2 D_t\gamma_2^1 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left(\frac{\partial \gamma_1^0}{\partial x} + \varepsilon D_x\gamma_1^1 + \varepsilon^2 D_x\gamma_1^2\right)$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon v^2)\left(\frac{\partial \gamma_1^0}{\partial y} + \varepsilon D_y\gamma_1^1 + \varepsilon^2 D_y\gamma_1^2\right)$$

$$
\begin{aligned}
&+\varepsilon(w^1+\varepsilon w^2+\varepsilon^2 w^3)\left(D_z\gamma_1^1+\varepsilon D_z\gamma_1^2\right)\\
&-(\gamma_1^0+\varepsilon\gamma_1^1+\varepsilon^2\gamma_1^2)\left(\frac{\partial u^0}{\partial x}+\varepsilon D_x u^1+\varepsilon^2 D_x u^2\right)\\
&-(\gamma_2^0+\varepsilon\gamma_2^1+\varepsilon^2\gamma_2^2)\left(\frac{\partial u^0}{\partial y}+\varepsilon D_y u^1+\varepsilon^2 D_y u^2\right)\\
&-(\gamma_3^0+\varepsilon\gamma_3^1+\varepsilon^2\gamma_3^2)\left(D_z u^1+\varepsilon D_z u^2\right)\\
&-2\phi\left[D_y\left((\cos\varphi)\,(u^0+\varepsilon u^1+\varepsilon^2 u^2)\right)+(\operatorname{sen}\varphi)\left(D_z u^1+\varepsilon D_z u^2\right)\right]\\
&=\frac{\partial\gamma_1^0}{\partial t}+u^0\frac{\partial\gamma_1^0}{\partial x}+v^0\frac{\partial\gamma_1^0}{\partial y}-\gamma_1^0\frac{\partial u^0}{\partial x}-\gamma_2^0\frac{\partial u^0}{\partial y}-\gamma_3^0 D_z u^1\\
&-2\phi\left(D_y((\cos\varphi)\,u^0)+(\operatorname{sen}\varphi)\,D_z u^1\right)+\varepsilon\left[D_t\gamma_1^1+u^0 D_x\gamma_1^1+u^1\frac{\partial\gamma_1^0}{\partial x}+v^0 D_y\gamma_1^1\right.\\
&+v^1\frac{\partial\gamma_1^0}{\partial y}+w^1 D_z\gamma_1^1-\gamma_1^0 D_x u^1-\gamma_1^1\frac{\partial u^0}{\partial x}-\gamma_2^0 D_y u^1-\gamma_2^1\frac{\partial u^0}{\partial y}-\gamma_3^0 D_z u^2\\
&\left.-\gamma_3^1 D_z u^1-2\phi\left(D_y((\cos\varphi)\,u^1)+(\operatorname{sen}\varphi)\,D_z u^2\right)\right]+O(\varepsilon^2)
\end{aligned}
$$

usando (4.5.83) y (4.5.93) resulta:

$$
\begin{aligned}
&\frac{\partial\tilde\gamma_1^\varepsilon}{\partial t^\varepsilon}+\tilde u^\varepsilon\frac{\partial\tilde\gamma_1^\varepsilon}{\partial x^\varepsilon}+\tilde v^\varepsilon\frac{\partial\tilde\gamma_1^\varepsilon}{\partial y^\varepsilon}+\tilde w^\varepsilon\frac{\partial\tilde\gamma_1^\varepsilon}{\partial z^\varepsilon}-\tilde\gamma_1^\varepsilon\frac{\partial\tilde u^\varepsilon}{\partial x^\varepsilon}-\tilde\gamma_2^\varepsilon\frac{\partial\tilde u^\varepsilon}{\partial y^\varepsilon}-\tilde\gamma_3^\varepsilon\frac{\partial\tilde u^\varepsilon}{\partial z^\varepsilon}\\
&-2\phi\left(\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\tilde u^\varepsilon)+(\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial\tilde u^\varepsilon}{\partial z^\varepsilon}\right)=O(\varepsilon^2)
\end{aligned}
$$

es decir, la primera ecucación de la vorticidad se verifica con un error del orden de ε^2. Lo mismo sucede para la segunda ecuación.

Para la tercera ecuación de vorticidad se tiene que:

$$
\begin{aligned}
&\frac{\partial\tilde\gamma_3^\varepsilon}{\partial t^\varepsilon}+\tilde u^\varepsilon\frac{\partial\tilde\gamma_3^\varepsilon}{\partial x^\varepsilon}+\tilde v^\varepsilon\frac{\partial\tilde\gamma_3^\varepsilon}{\partial y^\varepsilon}+\tilde w^\varepsilon\frac{\partial\tilde\gamma_3^\varepsilon}{\partial z^\varepsilon}-\tilde\gamma_1^\varepsilon\frac{\partial\tilde w^\varepsilon}{\partial x^\varepsilon}-\tilde\gamma_2^\varepsilon\frac{\partial\tilde w^\varepsilon}{\partial y^\varepsilon}+\tilde\gamma_3^\varepsilon\frac{\partial\tilde w^\varepsilon}{\partial z^\varepsilon}\\
&+2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial\tilde u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,\tilde v^\varepsilon-(\cos\varphi^\varepsilon)\,\tilde w^\varepsilon)\right)\\
&=\frac{\partial\gamma_3^0}{\partial t}+\varepsilon D_t\gamma_3^1+\varepsilon^2 D_t\gamma_3^2+(u^0+\varepsilon u^1+\varepsilon^2 u^2)\left(\frac{\partial\gamma_3^0}{\partial x}+\varepsilon D_x\gamma_3^1+\varepsilon^2 D_x\gamma_3^2\right)\\
&+(v^0+\varepsilon v^1+\varepsilon^2 v^2)\left(\frac{\partial\gamma_3^0}{\partial y}+\varepsilon D_y\gamma_3^1+\varepsilon^2 D_y\gamma_3^2\right)
\end{aligned}
$$

$$+\varepsilon(w^1+\varepsilon w^2+\varepsilon^2 w^3)\left(D_z\gamma_3^1+\varepsilon^2 D_z\gamma_3^2\right)$$

$$-\left(\gamma_1^0+\varepsilon\gamma_1^1+\varepsilon^2\gamma_1^2\right)\varepsilon\left(D_x w^1+\varepsilon D_x w^2+\varepsilon^2 D_x w^3\right)$$

$$-\left(\gamma_2^0+\varepsilon\gamma_2^1+\varepsilon^2\gamma_2^2\right)\varepsilon\left(D_y w^1+\varepsilon D_y w^2+\varepsilon^2 D_y w^3\right)$$

$$-\left(\gamma_3^0+\varepsilon\gamma_3^1+\varepsilon^2\gamma_3^2\right)\left(D_z w^1+\varepsilon D_z w^2+\varepsilon^2 D_z w^3\right)$$

$$+2\phi\left[(\operatorname{sen}\varphi)\left(\frac{\partial u^0}{\partial x}+\varepsilon D_x u^1+\varepsilon^2 D_x u^2\right)\right.$$

$$\left.+D_y((\operatorname{sen}\varphi)(v^0+\varepsilon v^1+\varepsilon^2 v^2)-\varepsilon(\cos\varphi)(w^1+\varepsilon w^2+\varepsilon^2 w^3))\right]$$

$$=\frac{\partial\gamma_3^0}{\partial t}+u^0\frac{\partial\gamma_3^0}{\partial x}+v^0\frac{\partial\gamma_3^0}{\partial y}-\gamma_3^0 D_z w^1+2\phi\left((\operatorname{sen}\varphi)\frac{\partial u^0}{\partial x}+D_y((\operatorname{sen}\varphi)v^0)\right)$$

$$+\varepsilon^2\left[D_t\gamma_3^1+u^0 D_x\gamma_3^1+u^1\frac{\partial\gamma_3^0}{\partial x}+v^0 D_y\gamma_3^1+v^1\frac{\partial\gamma_3^0}{\partial y}+w^1 D_z\gamma_3^1-\gamma_1^0 D_x w^1\right.$$

$$\left.-\gamma_2^0 D_y w^1-\gamma_3^0 D_z w^2-\gamma_3^1 D_z w^1+2\phi\left((\operatorname{sen}\varphi)D_x u^1+D_y((\operatorname{sen}\varphi)v^1-(\cos\varphi)w^1)\right)\right]$$

$$+\varepsilon\left[D_t\gamma_3^2+u^0 D_x\gamma_3^2+u^1 D_x\gamma_3^1+u^2\frac{\partial\gamma_3^0}{\partial x}+v^0 D_y\gamma_3^2+v_y^1\gamma_3^1+v^2\frac{\partial\gamma_3^0}{\partial y}\right.$$

$$+w^1 D_z\gamma_3^2+w^2 D_z\gamma_3^1-\gamma_1^0 D_x w^2-\gamma_1^1 D_x w^1-\gamma_2^0 D_y w^2-\gamma_2^1 D_y w^1$$

$$-\gamma_3^0 D_z w^3-\gamma_3^1 D_z w^2-\gamma_3^2 D_z w^1$$

$$\left.+2\phi\left((\operatorname{sen}\varphi)D_x u^2+D_y((\operatorname{sen}\varphi)v^2-(\cos\varphi)w^2)\right)\right]+O(\varepsilon^3)$$

Se sustituye γ_i^0 $(i=1,2)$, según (4.5.86)-(4.5.87), γ_3^k $(k=0,1,3)$, según (4.5.88), (4.5.98) y (4.5.108) y $D_z w^k$ $(k=1,2,3)$, según (4.5.82), (4.7.16) y (4.8.16):

$$\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial x^\varepsilon}-\tilde{\gamma}_2^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial y^\varepsilon}-\tilde{\gamma}_3^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial z^\varepsilon}$$

$$+2\phi\left((\operatorname{sen}\varphi^\varepsilon)\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\tilde{v}^\varepsilon-(\cos\varphi^\varepsilon)\tilde{w}^\varepsilon)\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+2\phi(\operatorname{sen}\varphi)u^0\right)$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}-2\phi(\operatorname{sen}\varphi)v^0\right)$$

$$+\varepsilon\left\{D_x\left[D_t v^1+u^0D_xv^1+u^1\frac{\partial v^0}{\partial x}+v^0D_yv^1+v^1\frac{\partial v^0}{\partial y}+w^1D_zv^1+2\phi\,(\operatorname{sen}\varphi)\,u^1\right]\right.$$

$$-D_y\left[D_tu^1+u^0D_xu^1+u^1\frac{\partial u^0}{\partial x}+v^0D_yu^1+v^1\frac{\partial u^0}{\partial y}+w^1D_zu^1\right.$$

$$\left.\left.-2\phi\left((\operatorname{sen}\varphi)\,v^1-(\cos\varphi)\,w^1\right)\right]\right\}$$

$$+\varepsilon^2\left\{D_x\left[D_tv^2+u^0D_xv^2+u^1D_xv^1+u^2\frac{\partial v^0}{\partial x}+v^0D_yv^2+v^1D_yv^1+v^2\frac{\partial v^0}{\partial y}\right.\right.$$

$$\left.+w^1D_zv^2+w^2D_zv^1+2\phi\,(\operatorname{sen}\varphi)\,u^2\right]-D_y\left[D_tu^2+u^0D_xu^2+u^1D_xu^1+u^2\frac{\partial u^0}{\partial x}\right.$$

$$+v^0D_yu^2+v^1D_yu^1+v^2\frac{\partial u^0}{\partial y}+w^1D_zu^2+w^2D_zu^1$$

$$\left.\left.-2\phi\left((\operatorname{sen}\varphi)\,v^2-(\cos\varphi)\,w^2\right)\right]\right\}+O(\varepsilon^3)$$

Por las ecuaciones (4.5.78), (4.5.79), (4.5.89), (4.5.90), (4.5.99) y (4.5.100) se sabe que:

$$\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}_3^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial x^\varepsilon}-\tilde{\gamma}_2^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial y^\varepsilon}-\tilde{\gamma}_3^\varepsilon\frac{\partial\tilde{w}^\varepsilon}{\partial z^\varepsilon}$$

$$+2\phi\left((\operatorname{sen}\varphi^\varepsilon)\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon-(\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon)\right)$$

$$=\frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right)-\frac{\partial}{\partial y}\left(-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right)$$

$$+\varepsilon\left\{D_x\left[-2\phi\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h(z-1)+\frac{\partial s}{\partial y}\left(2\phi\,(\cos\varphi)\,u^0-g\right)\right]\right.$$

$$\left.-D_y\left[-2\phi\,(\cos\varphi)\frac{\partial u^0}{\partial x}h(z-1)+\frac{\partial s}{\partial x}\left(2\phi\,(\cos\varphi)\,u^0-g\right)\right]\right\}$$

$$+\varepsilon^2\left[D_x\left(-\frac{1}{\rho_0}D_yp^2\right)-D_y\left(-\frac{1}{\rho_0}D_xp^2\right)\right]+O(\varepsilon^3)=O(\varepsilon^3)$$

es decir, la tercera ecucación de la vorticidad se verifica con un error del orden de ε^3.

El modelo (4.8.34)-(4.8.46) se podría escribir también en función de las velocidades medias en la vertical, teniendo en cuenta que

$$\bar{u}^\varepsilon=\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\tilde{u}^\varepsilon\,dz^\varepsilon$$

$$= \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \left\{ \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[\check{\gamma}_2^\varepsilon + \frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.\right.$$

$$\left.\left. + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_2^{1,1,\varepsilon} - h^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \right\} dz^\varepsilon$$

$$= \hat{u}^\varepsilon + \frac{1}{2} h^\varepsilon \left[\check{\gamma}_2^\varepsilon + \frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$+ \frac{1}{6} h^\varepsilon \left[\gamma_2^{1,1,\varepsilon} - h^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.8.50}$$

$$\bar{v}^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \tilde{v}^\varepsilon \, dz^\varepsilon$$

$$= \frac{1}{h^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} \left\{ \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[-\check{\gamma}_1^\varepsilon + \frac{\partial}{\partial y^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.\right.$$

$$\left.\left. + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] - \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_1^{1,1,\varepsilon} + h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \right\} dz^\varepsilon$$

$$= \hat{v}^\varepsilon + \frac{1}{2} h^\varepsilon \left[-\check{\gamma}_1^\varepsilon + \frac{\partial}{\partial y^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$- \frac{1}{6} h^\varepsilon \left[\gamma_1^{1,1,\varepsilon} + h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.8.51}$$

pero si se despejan $\hat{u}^\varepsilon$ y $\hat{v}^\varepsilon$ y se sustituyen en las ecuaciones (4.8.40)-(4.8.41), se pierde precisión.

4.9. Modelo propuesto

A continuación propondremos el modelo que pensamos es el mejor de los obtenidos en este capítulo. El modelo (4.8.34)-(4.8.46) es, formalmente, el de mayor precisión, pero también el que requiere mayor esfuerzo de cálculo. Como ya comentamos, el error cometido al calcular h^ε en este modelo es el mismo que en el de orden uno, por lo que parece probable que con este último consigamos una precisión similar con mucho menos esfuerzo. Puesto que el modelo de orden cero, aunque sencillo, es menos preciso que el de orden uno, escogemos el modelo de primer orden (en su formulación en velocidad media (4.7.57)-(4.7.65)) como modelo a proponer (suprimiendo la ˜ para simplificar la notación):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\bar{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial x^\varepsilon}$$

$$+ 2\phi \left[(\operatorname{sen}\varphi^\varepsilon)\, \bar{v}^\varepsilon + (\cos\varphi^\varepsilon) \left(\frac{\partial (\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} - \bar{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$\frac{\partial \bar{v}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial y^\varepsilon} + 2\phi \left[- (\operatorname{sen}\varphi^\varepsilon)\, \bar{u}^\varepsilon \right.$$

$$\left. + \frac{\partial}{\partial y^\varepsilon} \left((\cos\varphi^\varepsilon)\, \bar{u}^\varepsilon h^\varepsilon \right) - \frac{1}{2} h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left((\cos\varphi^\varepsilon)\, \bar{u}^\varepsilon \right) + (\cos\varphi^\varepsilon)\, \bar{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right]$$

$$\frac{\partial \gamma_1^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \bar{v}^\varepsilon}{\partial x^\varepsilon}$$

$$= 2\phi \left[\frac{\partial}{\partial y^\varepsilon} \left((\cos\varphi^\varepsilon)\, \bar{u}^\varepsilon \right) + (\operatorname{sen}\varphi^\varepsilon)\, \gamma_2^{0,\varepsilon} \right]$$

$$\frac{\partial \gamma_2^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial \gamma_2^{0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{0,\varepsilon} \frac{\partial \bar{u}^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -2\phi \left(\gamma_1^{0,\varepsilon} (\operatorname{sen}\varphi^\varepsilon) - (\cos\varphi^\varepsilon) \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

$$u^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{1}{2} h^\varepsilon)\, \gamma_2^{0,\varepsilon}, \quad \check{u}^\varepsilon = u^\varepsilon|_{z^\varepsilon = H^\varepsilon} = \bar{u}^\varepsilon - \frac{1}{2} h^\varepsilon\, \gamma_2^{0,\varepsilon}$$

$$v^\varepsilon = \bar{v}^\varepsilon - (z^\varepsilon - H^\varepsilon - \frac{1}{2} h^\varepsilon)\, \gamma_1^{0,\varepsilon}, \quad \check{v}^\varepsilon = v^\varepsilon|_{z^\varepsilon = H^\varepsilon} = \bar{v}^\varepsilon + \frac{1}{2} h^\varepsilon\, \gamma_1^{0,\varepsilon}$$

$$w^\varepsilon = \check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon) \left[\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} + \gamma_1^{0,\varepsilon} \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \gamma_2^{0,\varepsilon} \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \right]$$

$$+ \frac{1}{2} (H^\varepsilon - z^\varepsilon)^2 \left(\frac{\partial \gamma_1^{0,\varepsilon}}{\partial y^\varepsilon} - \frac{\partial \gamma_2^{0,\varepsilon}}{\partial x^\varepsilon} \right)$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \left[g - 2\phi\, (\cos\varphi^\varepsilon)\, \bar{u}^\varepsilon \right] \tag{4.9.1}$$

Este modelo se puede escribir en forma vectorial como sigue:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = \mathbf{0}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g \nabla h^\varepsilon = -\frac{1}{\rho_0} \nabla p_s^\varepsilon - g \nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon$$

$$\frac{\partial \vec{\gamma}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\gamma}^\varepsilon \cdot \vec{\bar{\mathbf{u}}}^\varepsilon - (\nabla \vec{\bar{\mathbf{u}}}^\varepsilon)^T \cdot \vec{\gamma}^\varepsilon = 2\phi \vec{\mathbf{F}}_V^\varepsilon$$

$$u^\varepsilon = \bar{u}^\varepsilon + \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_2^\varepsilon, \quad v^\varepsilon = \bar{v}^\varepsilon - \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_1^\varepsilon$$

$$\vec{\tilde{\mathbf{u}}}^\varepsilon = \vec{\mathbf{u}}^\varepsilon\big|_{z^\varepsilon = H^\varepsilon}$$

$$w^\varepsilon = \vec{\tilde{\mathbf{u}}}^\varepsilon \cdot \nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon)(\operatorname{div} \vec{\tilde{\mathbf{u}}}^\varepsilon - \vec{\gamma}^\varepsilon \cdot \operatorname{rot} H^\varepsilon) + \frac{1}{2}(H^\varepsilon - z^\varepsilon)^2 \operatorname{rot} \vec{\gamma}^\varepsilon$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon\right] \tag{4.9.2}$$

donde $\operatorname{rot} \vec{\alpha}^\varepsilon = \dfrac{\partial \alpha_2^\varepsilon}{\partial x^\varepsilon} - \dfrac{\partial \alpha_1^\varepsilon}{\partial y^\varepsilon}$, $\operatorname{rot} \alpha = \left(\dfrac{\partial \alpha^\varepsilon}{\partial y^\varepsilon}, -\dfrac{\partial \alpha^\varepsilon}{\partial x^\varepsilon}\right)$,

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon)\bar{v}^\varepsilon + \cos\varphi^\varepsilon\left(\dfrac{\partial(h^\varepsilon \bar{u}^\varepsilon)}{\partial x^\varepsilon} + \dfrac{h^\varepsilon}{2}\dfrac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} - \bar{v}^\varepsilon \dfrac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\operatorname{sen}\varphi^\varepsilon)\bar{u}^\varepsilon + \dfrac{h^\varepsilon}{2}\dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] + \dfrac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] \end{pmatrix}$$

y

$$\vec{\mathbf{F}}_V^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon)\gamma_2^\varepsilon + \dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] \\ -(\operatorname{sen}\varphi^\varepsilon)\gamma_1^\varepsilon + (\cos\varphi^\varepsilon)\dfrac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} \end{pmatrix}$$

Podría aprovecharse del modelo de orden dos la mejora en la aproximación de la presión (que no requiere resolver nuevas ecuaciones) escrita en función de la velocidad (para lo cual recordamos que (4.8.37) se obtiene de (4.8.23) despreciando términos de $O(\varepsilon^3)$).

$$\begin{aligned} p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\Bigg[g - 2\phi\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon + \frac{\partial}{\partial t^\varepsilon}\left(\bar{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ + \bar{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon}\left(\bar{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + \bar{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon}\left(\bar{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \bar{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\Bigg] \\ - \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\Bigg[2\phi\left(\cos\varphi^\varepsilon\right)\gamma_2^{0,\varepsilon} + \frac{\partial}{\partial t^\varepsilon}\left(\frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}\right) \\ + \bar{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}\right) + \bar{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}\right) - \left(\frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon}\right)^2\Bigg] \end{aligned} \tag{4.9.3}$$

4.10. Simplificaciones de la oceanografía dinámica

Como ya mencionamos en la observación 1.1, es usual en oceanografía dinámica considerar que la aceleración de Coriolis viene dada por (1.1.9) en lugar de (1.1.6). En esta sección queremos considerar qué modelo se obtiene si se aplica el método de desarrollos asintóticos bajo esta hipótesis.

Siguiendo los pasos dados con anterioridad en este capítulo, obtendremos modelos de orden cero, uno y dos, deduciremos que (en lo que al modelo de orden uno se refiere y bajo ciertas hipótesis razonables) la hipótesis oceanográfica (1.1.9) obliga a que las dos primeras componentes de la vorticidad (γ_1 y γ_2 en (4.10.4)-(4.10.6)) sean nulas, es decir, que si admitimos la hipótesis de la oceanografía dinámica (1.1.9), la vorticidad sólo puede tener componente vertical no nula.

4.10.1. Ecuaciones de partida

Si se realizan las simplificaciones expuestas en la Observación 1.1 sobre el término debido a la aceleración de Coriolis, tomándola según (1.1.9):

$$-2\vec{\phi} \times \vec{\mathbf{u}} = -2\phi \left(-\left(\operatorname{sen}\varphi\right) v, \left(\operatorname{sen}\varphi\right) u, 0 \right)$$

las ecuaciones de Euler de partida se escriben ahora:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi\left(\operatorname{sen}\varphi\right) v \tag{4.10.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi\left(\operatorname{sen}\varphi\right) u \tag{4.10.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g \tag{4.10.3}$$

y las ecuaciones de la vorticidad:

$$\frac{\partial \gamma_1}{\partial t} + u\frac{\partial \gamma_1}{\partial x} + v\frac{\partial \gamma_1}{\partial y} + w\frac{\partial \gamma_1}{\partial z} - \gamma_1\frac{\partial u}{\partial x} - \gamma_2\frac{\partial u}{\partial y} - \gamma_3\frac{\partial u}{\partial z} = 2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial u}{\partial z} \tag{4.10.4}$$

$$\frac{\partial \gamma_2}{\partial t} + u\frac{\partial \gamma_2}{\partial x} + v\frac{\partial \gamma_2}{\partial y} + w\frac{\partial \gamma_2}{\partial z} - \gamma_1\frac{\partial v}{\partial x} - \gamma_2\frac{\partial v}{\partial y} - \gamma_3\frac{\partial v}{\partial z} = 2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v}{\partial z} \tag{4.10.5}$$

$$\frac{\partial \gamma_3}{\partial t} + u\frac{\partial \gamma_3}{\partial x} + v\frac{\partial \gamma_3}{\partial y} + w\frac{\partial \gamma_3}{\partial z} - \gamma_1\frac{\partial w}{\partial x} - \gamma_2\frac{\partial w}{\partial y} - \gamma_3\frac{\partial w}{\partial z}$$

$$= -2\phi\left[\left(\operatorname{sen}\varphi\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left(\left(\operatorname{sen}\varphi\right) v\right)\right] \tag{4.10.6}$$

4.10.2. Ecuaciones en el dominio de referencia

Bajo las hipótesis realizadas, las ecuaciones que determinan el problema a resolver en el dominio de referencia Ω son las siguientes:

- las ecuaciones de Euler:

$$D_t u(\varepsilon) + u(\varepsilon) D_x u(\varepsilon) + v(\varepsilon) D_y u(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z u(\varepsilon)$$

$$= -\frac{1}{\rho_0} D_x p(\varepsilon) + 2\phi \, (\operatorname{sen} \varphi) \, v(\varepsilon) \tag{4.10.7}$$

$$D_t v(\varepsilon) + u(\varepsilon) D_x v(\varepsilon) + v(\varepsilon) D_y v(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z v(\varepsilon)$$

$$= -\frac{1}{\rho_0} D_y p(\varepsilon) - 2\phi \, (\operatorname{sen} \varphi) \, u(\varepsilon) \tag{4.10.8}$$

$$D_t w(\varepsilon) + u(\varepsilon) D_x w(\varepsilon) + v(\varepsilon) D_y w(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon)$$

$$= -\frac{1}{\rho_0} \frac{1}{\varepsilon} D_z p(\varepsilon) - g \tag{4.10.9}$$

- las ecuaciones de vorticidad:

$$D_t \gamma_1(\varepsilon) + u(\varepsilon) D_x \gamma_1(\varepsilon) + v(\varepsilon) D_y \gamma_1(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z \gamma_1(\varepsilon) - \gamma_1(\varepsilon) D_x u(\varepsilon)$$

$$- \gamma_2(\varepsilon) D_y u(\varepsilon) - \gamma_3(\varepsilon) \frac{1}{\varepsilon} D_z u(\varepsilon) = 2\phi \, (\operatorname{sen} \varphi) \, \frac{1}{\varepsilon} D_z u(\varepsilon) \tag{4.10.10}$$

$$D_t \gamma_2(\varepsilon) + u(\varepsilon) D_x \gamma_2(\varepsilon) + v(\varepsilon) D_y \gamma_2(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z \gamma_2(\varepsilon) - \gamma_1(\varepsilon) D_x v(\varepsilon)$$

$$- \gamma_2(\varepsilon) D_y v(\varepsilon) - \gamma_3(\varepsilon) \frac{1}{\varepsilon} D_z v(\varepsilon) = 2\phi \, (\operatorname{sen} \varphi) \, \frac{1}{\varepsilon} D_z v(\varepsilon) \tag{4.10.11}$$

$$D_t \gamma_3(\varepsilon) + u(\varepsilon) D_x \gamma_3(\varepsilon) + v(\varepsilon) D_y \gamma_3(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z \gamma_3(\varepsilon)$$

$$- \gamma_1(\varepsilon) D_x w(\varepsilon) - \gamma_2(\varepsilon) D_y w(\varepsilon) - \gamma_3(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon)$$

$$= -2\phi \left[(\operatorname{sen} \varphi) \, D_x u(\varepsilon) + D_y ((\operatorname{sen} \varphi) \, v(\varepsilon)) \right] \tag{4.10.12}$$

además de la condición de incompresibilidad ((4.4.4)), las condiciones de contorno e iniciales ((4.4.5)-(4.4.9)), las relaciones de las componentes de la vorticidad y de la velocidad ((4.4.13)) y la ecuación para el cálculo del calado ((4.4.14)) que no varían bajo estas nuevas hipótesis.

4.10.3. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (4.4.4)-(4.4.9), (4.4.13)-(4.4.14), (4.10.7)-(4.10.12) admite un desarrollo en serie de potencias de ε en la forma (4.5.1). Se sustituye este desarrollo en las ecuaciones que han variado respecto a las secciones anteriores. Agrupamos los términos multiplicados por la misma potencia de ε e igualando a cero los coeficientes, obtenemos las siguientes ecuaciones:

$$p^1 = \rho_0 g h(1-z) \tag{4.10.13}$$

$$\begin{aligned} &D_t\gamma_1^0 + u^0 D_x\gamma_1^0 + v^0 D_y\gamma_1^0 + w^1 D_z\gamma_1^0 - \gamma_1^0\frac{\partial u^0}{\partial x} - \gamma_2^0\frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^1 \\ &\quad - 2\phi\,(\operatorname{sen}\varphi)\, D_z u^1 = 0 \end{aligned} \tag{4.10.14}$$

$$\begin{aligned} &D_t\gamma_2^0 + u^0 D_x\gamma_2^0 + v^0 D_y\gamma_2^0 + w^1 D_z\gamma_2^0 - \gamma_1^0\frac{\partial v^0}{\partial x} - \gamma_2^0\frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^1 \\ &\quad - 2\phi\,(\operatorname{sen}\varphi)\, D_z v^1 = 0 \end{aligned} \tag{4.10.15}$$

$$\begin{aligned} &D_t\gamma_3^0 + u^0 D_x\gamma_3^0 + v^0 D_y\gamma_3^0 + w^1 D_z\gamma_3^0 - \gamma_3^0 D_z w^1 \\ &\quad + 2\phi\left[(\operatorname{sen}\varphi)\frac{\partial u^0}{\partial x} + \frac{\partial}{\partial y}((\operatorname{sen}\varphi)\, v^0)\right] = 0 \end{aligned} \tag{4.10.16}$$

$$\begin{aligned} &D_t u^1 + u^0 D_x u^1 + u^1\frac{\partial u^0}{\partial x} + v^0 D_y u^1 + v^1\frac{\partial u^0}{\partial y} + w^1 D_z u^1 \\ &\quad = -g\frac{\partial s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\, v^1 \end{aligned} \tag{4.10.17}$$

$$\begin{aligned} &D_t v^1 + u^0 D_x v^1 + u^1\frac{\partial v^0}{\partial x} + v^0 D_y v^1 + v^1\frac{\partial v^0}{\partial y} + w^1 D_z v^1 \\ &\quad = -g\frac{\partial s}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, u^1 \end{aligned} \tag{4.10.18}$$

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 = 0 \tag{4.10.19}$$

$$\begin{aligned} &D_t\gamma_1^1 + u^0 D_x\gamma_1^1 + u^1 D_x\gamma_1^0 + v^0 D_y\gamma_1^1 + v^1 D_y\gamma_1^0 + w^1 D_z\gamma_1^1 + w^2 D_z\gamma_1^0 \\ &\quad - \gamma_1^0 D_x u^1 - \gamma_1^1\frac{\partial u^0}{\partial x} - \gamma_2^0 D_y u^1 - \gamma_2^1\frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^2 - \gamma_3^1 D_z u^1 \\ &\quad - 2\phi\,(\operatorname{sen}\varphi)\, D_z u^2 = 0 \end{aligned} \tag{4.10.20}$$

$$
\begin{aligned}
&D_t\gamma_2^1 + u^0 D_x\gamma_2^1 + u^1 D_x\gamma_2^0 + v^0 D_y\gamma_2^1 + v^1 D_y\gamma_2^0 + w^1 D_z\gamma_2^1 + w^2 D_z\gamma_2^0 \\
&\quad - \gamma_1^0 D_x v^1 - \gamma_1^1 \frac{\partial v^0}{\partial x} - \gamma_2^0 D_y v^1 - \gamma_2^1 \frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^2 - \gamma_3^1 D_z v^1 \\
&\quad - 2\phi\,(\operatorname{sen}\varphi)\, D_z v^2 = 0 \qquad (4.10.21)
\end{aligned}
$$

$$
\begin{aligned}
&D_t\gamma_3^1 + u^0 D_x\gamma_3^1 + u^1 D_x\gamma_3^0 + v^0 D_y\gamma_3^1 + v^1 D_y\gamma_3^0 + w^1 D_z\gamma_3^1 + w^2 D_z\gamma_3^0 \\
&\quad - \gamma_1^0 D_x w^1 - \gamma_2^0 D_y w^1 - \gamma_3^0 D_z w^2 - \gamma_3^1 D_z w^1 \\
&\quad + 2\phi \left[(\operatorname{sen}\varphi)\, D_x u^1 + D_y\left((\operatorname{sen}\varphi)\, v^1\right)\right] = 0 \qquad (4.10.22)
\end{aligned}
$$

$$
\begin{aligned}
&D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 \frac{\partial u^0}{\partial x} + v^0 D_y u^2 + v^1 D_y u^1 + v^2 \frac{\partial u^0}{\partial y} \\
&\quad + w^1 D_z u^2 + w^2 D_z u^1 + \frac{1}{\rho_0} D_x p^2 - 2\phi\,(\operatorname{sen}\varphi)\, v^2 = 0 \qquad (4.10.23)
\end{aligned}
$$

$$
\begin{aligned}
&D_t v^2 + u^0 D_x v^2 + u^1 D_x v^1 + u^2 \frac{\partial v^0}{\partial x} + v^0 D_y v^2 + v^1 D_y v^1 + v^2 \frac{\partial v^0}{\partial y} \\
&\quad + w^1 D_z v^2 + w^2 D_z v^1 + \frac{1}{\rho_0} D_y p^2 + 2\phi\,(\operatorname{sen}\varphi)\, u^2 = 0 \qquad (4.10.24)
\end{aligned}
$$

$$
\begin{aligned}
&D_t\gamma_1^2 + u^0 D_x\gamma_1^2 + u^1 D_x\gamma_1^1 + u^2 D_x\gamma_1^0 + v^0 D_y\gamma_1^2 + v^1 D_y\gamma_1^1 + v^2 D_y\gamma_1^0 + w^1 D_z\gamma_1^2 \\
&\quad + w^2 D_z\gamma_1^1 + w^3 D_z\gamma_1^0 - \gamma_1^0 D_x u^2 - \gamma_1^1 D_x u^1 - \gamma_1^2 \frac{\partial u^0}{\partial x} - \gamma_2^0 D_y u^2 - \gamma_2^1 D_y u^1 \\
&\quad - \gamma_2^2 \frac{\partial u^0}{\partial y} - \gamma_3^0 D_z u^3 - \gamma_3^1 D_z u^2 - \gamma_3^2 D_z u^1 - 2\phi\,(\operatorname{sen}\varphi)\, D_z u^3 = 0 \qquad (4.10.25)
\end{aligned}
$$

$$
\begin{aligned}
&D_t\gamma_2^2 + u^0 D_x\gamma_2^2 + u^1 D_x\gamma_2^1 + u^2 D_x\gamma_2^0 + v^0 D_y\gamma_2^2 + v^1 D_y\gamma_2^1 + v^2 D_y\gamma_2^0 + w^1 D_z\gamma_2^2 \\
&\quad + w^2 D_z\gamma_2^1 + w^3 D_z\gamma_2^0 - \gamma_1^0 D_x v^2 - \gamma_1^1 D_x v^1 - \gamma_1^2 \frac{\partial v^0}{\partial x} - \gamma_2^0 D_y v^2 - \gamma_2^1 D_y v^1 \\
&\quad - \gamma_2^2 \frac{\partial v^0}{\partial y} - \gamma_3^0 D_z v^3 - \gamma_3^1 D_z v^2 - \gamma_3^2 D_z v^1 - 2\phi\,(\operatorname{sen}\varphi)\, D_z v^3 = 0 \qquad (4.10.26)
\end{aligned}
$$

que junto con (4.5.73)-(4.5.80), (4.5.82), (4.5.86)-(4.5.88), (4.5.92), (4.5.96)-(4.5.98), (4.5.101)-(4.5.103) y (4.5.106)-(4.5.108) nos permitirán el cálculo de h, u^k, v^k, w^k, p^k, γ_3^k $(k = 0, 1, 2, \ldots)$ y γ_i^j $(j = -1, 0, 1, \ldots, i = 1, 2)$ y a partir de ellos construiremos una aproximación de la solución del problema (4.4.4)-(4.4.9), (4.4.13)-(4.4.14), (4.10.7)-(4.10.12).

4.10.4. Aproximación de orden cero

Si se considera la aproximación de orden cero en ε:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1}\gamma_i^{-1} + \gamma_i^0 \quad (i = 1, 2)$$

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0$$

se obtiene el mismo modelo ((4.6.13)) que en la sección 4.6 salvo para las dos primeras componentes de la vorticidad:

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(h^\varepsilon \tilde{v}^\varepsilon)}{\partial y^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi)\,\tilde{v}^\varepsilon \\
&\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi)\,\tilde{u}^\varepsilon \\
&\tilde{p}^\varepsilon = p_s^\varepsilon \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)
\end{aligned} \tag{4.10.27}$$

Sabemos que $\gamma_i^{-1} = 0$ $(i = 1, 2)$ ((4.5.75)), y en el caso de no suponer las simplificaciones que se proponen en oceanografía dinámica, γ_i^0 $(i = 1, 2)$ se calculan resolviendo (4.5.83)-(4.5.84). Pero, bajo las hipótesis realizadas, estas ecuaciones resultan (4.10.14)-(4.10.15), donde se puede sustituir $D_z u^1$ y $D_z v^1$ en función de γ_i^0 $(i = 1, 2)$, por las relaciones (4.5.86)-(4.5.87):

$$\begin{aligned}
&D_t\gamma_1^0 + u^0 D_x\gamma_1^0 + v^0 D_y\gamma_1^0 + w^1 D_z\gamma_1^0 - \gamma_1^0\frac{\partial u^0}{\partial x} - \gamma_2^0\frac{\partial u^0}{\partial y} \\
&\qquad - \gamma_2^0\left(\gamma_3^0 + 2\phi\,(\operatorname{sen}\varphi)\right) = 0
\end{aligned} \tag{4.10.28}$$

$$\begin{aligned}
&D_t\gamma_2^0 + u^0 D_x\gamma_2^0 + v^0 D_y\gamma_2^0 + w^1 D_z\gamma_2^0 - \gamma_1^0\frac{\partial v^0}{\partial x} - \gamma_2^0\frac{\partial v^0}{\partial y} \\
&\qquad + \gamma_1^0\left(\gamma_3^0 + 2\phi\,(\operatorname{sen}\varphi)\right) = 0
\end{aligned} \tag{4.10.29}$$

Este sistema de ecuaciones tiene por única solución

$$\gamma_1^0 = \gamma_2^0 = 0 \tag{4.10.30}$$

si las condiciones de contorno e iniciales son también nulas. En este caso las dos primeras componentes de la vorticidad son nulas:

$$\tilde{\gamma}_i^\varepsilon = 0 \quad (i = 1, 2) \tag{4.10.31}$$

mientras que la tercera sigue siendo:

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \tag{4.10.32}$$

Observación 4.9 *Si las condiciones iniciales de contorno e iniciales en* (4.10.28)-(4.10.29) *no fuesen nulas no obtendríamos* (4.10.30)*, pero queremos resaltar que sin la hipótesis oceanográfica, ni siquiera en el caso de que las condiciones iniciales y de contorno fuesen nulas tendríamos* (4.10.30)*, ya que en* (4.5.83)-(4.5.84) *bastaría con que u^0 o v^0 fuesen no nulos (que es lo habitual) para no tener* (4.10.30).

4.10.5. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación de orden 1 en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1}\gamma_i^{-1} + \gamma_i^0 + \varepsilon\gamma_i^1 \quad (i = 1, 2)$$

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0 + \varepsilon\gamma_3^1$$

Recordemos que w^0, p^0 y γ_i^{-1} $(i = 1, 2)$ son conocidos ((4.5.73)-(4.5.75)), u^0, v^0 y h se calculan resolviendo (4.5.78)-(4.5.80), y w^1 está determinado por (4.5.82) en función de u^0, v^0 y h.

También tenemos una expresión para p^1 en la que sólo es necesario conocer la profundidad del agua ((4.10.13)):

$$p^1 = \rho_0 g h(1 - z)$$

Además, $\gamma_1^0 = \gamma_2^0 = 0$ ((4.10.30)) mientras que γ_3^0 viene dado por la expresión (4.5.88).

La dependencia de z de u^1 y v^1 nos la proporcionan (4.5.86)-(4.5.87):

$$D_z v^1 = -\gamma_1^0 = 0$$

$$D_z u^1 = \gamma_2^0 = 0$$

es decir, u^1 y v^1 no dependen de z. Para calcularlos se resuelven (4.10.17)-(4.10.18) teniendo en cuenta que $D_z u^1 = D_z v^1 = 0$:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} = -\frac{\partial s}{\partial x} g + 2\phi\,(\operatorname{sen}\varphi)\, v^1 \tag{4.10.33}$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} = -\frac{\partial s}{\partial y} g - 2\phi\,(\operatorname{sen}\varphi)\, u^1 \tag{4.10.34}$$

A continuación, se calcula w^2 partiendo de (4.5.92):

$$D_z w^2 = -\frac{\partial u^1}{\partial x} - \frac{\partial v^1}{\partial y}$$

Se integra respecto de z imponiendo la condición de contorno (4.5.103) y se obtiene:

$$w^2 = u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - hz \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \tag{4.10.35}$$

La tercera componente de la vorticidad viene dada por (4.5.98):

$$\gamma_3^1 = \frac{\partial v^1}{\partial x} - \frac{\partial u^1}{\partial y}$$

y para calcular las otras dos componentes se resuelven las ecuaciones (4.10.20) y (4.10.21) donde se tiene en cuenta que $\gamma_1^0 = \gamma_2^0 = D_z u^1 = D_z u^1 = 0$, $D_z u^2$ y $D_z v^2$ se despejan de (4.5.96)-(4.5.97), y γ_3^0 se reemplaza por (4.5.88):

$$D_t \gamma_1^1 + u^0 D_x \gamma_1^1 + v^0 D_y \gamma_1^1 + w^1 D_z \gamma_1^1 - \gamma_1^1 \frac{\partial u^0}{\partial x} - \gamma_2^1 \left(\frac{\partial v^0}{\partial x} + 2\phi \operatorname{sen}\varphi \right)$$

$$= \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \right) D_x w^1 \tag{4.10.36}$$

$$D_t \gamma_2^1 + u^0 D_x \gamma_2^1 + v^0 D_y \gamma_2^1 + w^1 D_z \gamma_2^1 - \gamma_1^1 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) - \gamma_2^1 \frac{\partial v^0}{\partial y}$$

$$= \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \right) D_y w^1 \tag{4.10.37}$$

Estamos ahora en condiciones de escribir quienes son $\tilde{p}(\varepsilon)$ y $\tilde{w}(\varepsilon)$.

Usando (4.5.74) y (4.10.13) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 gh(1-z) \tag{4.10.38}$$

De igual modo, por (4.5.73), (4.5.82) y (4.10.35), sabemos que:

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 = \varepsilon\left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$

$$+\,\varepsilon^2\left[u^1\frac{\partial H}{\partial x} + v^1\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right]$$

$$= \left(u^0 + \varepsilon u^1\right)\varepsilon\frac{\partial H}{\partial x} + \left(v^0 + \varepsilon v^1\right)\varepsilon\frac{\partial H}{\partial y} - \varepsilon hz\left[\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right]$$

es decir,

$$\tilde{w}(\varepsilon) = \tilde{u}(\varepsilon)\varepsilon\frac{\partial H}{\partial x} + \tilde{v}(\varepsilon)\varepsilon\frac{\partial H}{\partial y} - \varepsilon hz\left(D_x\tilde{u}(\varepsilon) + D_y\tilde{v}(\varepsilon)\right) \tag{4.10.39}$$

Se deshace el cambio de variable y se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y,z)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y,z)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z)$$

$$\tilde{\gamma}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_i(\varepsilon)(t,x,y,z) = \gamma_i^0(t,x,y) + \varepsilon\gamma_i^1(t,x,y,z) \quad (i=1,2)$$

$$\tilde{\gamma}_3^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_3(\varepsilon)(t,x,y,z) = \gamma_3^0(t,x,y) + \varepsilon\gamma_3^1(t,x,y,z)$$

La aproximación de la presión en Ω^ε, si se realiza el cambio de variable en (4.10.38), es

$$\tilde{p}^\varepsilon = p_s + \rho_0 g(s^\varepsilon - z^\varepsilon) \tag{4.10.40}$$

Análogamente, deshaciendo el cambio de variable en (4.10.39), se tiene que la aproximación de la componente vertical de la velocidad viene dada por:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\tilde{v}^\varepsilon}{\partial v^\varepsilon}\right) \tag{4.10.41}$$

Obsérvese que se tiene:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right)$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} \right)$$

$$+ (v^0 + \varepsilon v^1) \left(\frac{\partial u^0}{\partial y} + \varepsilon \frac{\partial u^1}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right)$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right)$$

$$+ \varepsilon \left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + v^1 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) + v^0 \frac{\partial u^1}{\partial y} \right] + O(\varepsilon^2)$$

$$= -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \varepsilon g \frac{\partial s}{\partial x} + O(\varepsilon^2) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2)$$

y procediendo de manera análoga, se obtiene una ecuación similar para $\tilde{v}^\varepsilon$. Se tiene:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2) \quad (4.10.42)$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \left(\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} + O(\varepsilon^2) \quad (4.10.43)$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \quad (4.10.44)$$

A continuación se propone un modelo resultado de despreciar los términos en $O(\varepsilon^2)$ en la ecuaciones anteriores, obteniéndose:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, \tilde{v}^\varepsilon$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, \tilde{u}^\varepsilon$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.10.45}$$

donde es necesario conocer $\tilde{u}^\varepsilon(0, x^\varepsilon, y^\varepsilon)$, $\tilde{v}^\varepsilon(0, x^\varepsilon, y^\varepsilon)$ y $h^\varepsilon(0, x^\varepsilon, y^\varepsilon)$, y las condiciones de contorno.

Si se desea calcular la vorticidad debe resolverse un sistema de ecuaciones para las dos primeras componentes mientras que la tercera se obtiene directamente de las componentes horizontales de la velocidad (véase (4.10.36)-(4.10.37), (4.5.88) y (4.5.98)):

$$\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \left(\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) \right)$$

$$= \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} \tag{4.10.46}$$

$$\frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) \right) - \tilde{\gamma}_2^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} \tag{4.10.47}$$

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} \tag{4.10.48}$$

donde es necesario conocer las condiciones iniciales y de contorno para (4.10.46)-(4.10.47).

El orden de precisión con el que se verifican las ecuaciones de partida es exactamente el mismo que el comprobado para la aproximación de primer orden sin imponer que la aceleración de Coriolis sea (1.1.9).

4.10.6. Aproximación de segundo orden

Se considera la aproximación de segundo orden en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{\gamma}_i(\varepsilon) = \varepsilon^{-1} \gamma_i^{-1} + \gamma_i^0 + \varepsilon \gamma_i^1 + \varepsilon^2 \gamma_i^2 \quad (i = 1, 2)$$

$$\tilde{\gamma}_3(\varepsilon) = \gamma_3^0 + \varepsilon \gamma_3^1 + \varepsilon^2 \gamma_3^2$$

Los términos w^0, p^0, γ_i^{-1} $(i = 1, 2)$, u^0, v^0, h, w^1, p^1, γ_i^0 $(i = 1, 2, 3)$, u^1, v^1, w^2 y γ_i^1 $(i = 1, 2, 3)$, se calculan del mismo modo que en la sección anterior para la aproximación de primer orden a partir de (4.5.73)-(4.5.75), (4.5.78)-(4.5.80), (4.5.82), (4.10.13), (4.10.30), (4.5.88) y (4.10.33)-(4.10.37).

Buscamos ahora p^2, y para ello partimos de la ecuación (4.10.19):

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 = -\frac{1}{\rho_0} D_z p^2$$

Utilizamos la expresión de w^1((4.5.82)) donde la dependencia de z es explícita, integrando respecto a z e imponiendo la condición (4.5.102) ($p^2 = 0$ en $z = 1$), se obtiene:

$$\begin{aligned} p^2 = {} & \rho_0 h(1-z)\left[\frac{\partial}{\partial t}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) + u^0\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right. \\ & \left. + v^0\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right] \\ & - \frac{1}{2}\rho_0 h^2(1-z^2)\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + u^0\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right. \\ & \left. + v^0\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2\right] \end{aligned} \tag{4.10.49}$$

A continuación calculamos u^2 y v^2 a partir de (4.5.96)-(4.5.97):

$$D_z u^2 = D_x w^1 + \gamma_2^1$$

$$D_z v^2 = D_y w^1 - \gamma_1^1$$

Para poder integrar respecto a z las igualdades anteriores es necesario conocer de qué modo dependen γ_i^1 $(i = 1, 2)$ de z. Para ello se supone:

$$\gamma_1^1 = \gamma_1^{1,0} + z\gamma_1^{1,1} + z^2\gamma_1^{1,2} \tag{4.10.50}$$

$$\gamma_2^1 = \gamma_2^{1,0} + z\gamma_2^{1,1} + z^2\gamma_2^{1,2} \tag{4.10.51}$$

se sustituyen en (4.10.36)-(4.10.37) γ_i^1 $(i = 1, 2)$, y w^1 (utilizando (4.10.50)-(4.10.51) y (4.5.82), respectivamente) de modo que la dependencia de z sea explícita. Veámoslo para (4.10.36):

$$D_t\left(\gamma_1^{1,0}+z\gamma_1^{1,1}+z^2\gamma_1^{1,2}\right)+u^0D_x\left(\gamma_1^{1,0}+z\gamma_1^{1,1}+z^2\gamma_1^{1,2}\right)$$
$$+v^0D_y\left(\gamma_1^{1,0}+z\gamma_1^{1,1}+z^2\gamma_1^{1,2}\right)$$
$$+\left[u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}-hz\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]D_z\left(\gamma_1^{1,0}+z\gamma_1^{1,1}+z^2\gamma_1^{1,2}\right)$$
$$-\left(\gamma_1^{1,0}+z\gamma_1^{1,1}+z^2\gamma_1^{1,2}\right)\frac{\partial u^0}{\partial x}-\left(\gamma_2^{1,0}+z\gamma_2^{1,1}+z^2\gamma_2^{1,2}\right)\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)$$
$$=\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)D_x\left[u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}-hz\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]$$

De cada ecuación se obtienen otras tres al agrupar según las potencias de z (utilizamos (4.5.80) para simplificar):

$$\frac{\partial\gamma_1^{1,0}}{\partial t}+u^0\frac{\partial\gamma_1^{1,0}}{\partial x}+v^0\frac{\partial\gamma_1^{1,0}}{\partial y}-\gamma_1^{1,0}\frac{\partial u^0}{\partial x}-\gamma_2^{1,0}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)$$
$$=\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$
$$\left.+\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right] \tag{4.10.52}$$

$$\frac{\partial\gamma_1^{1,1}}{\partial t}+u^0\frac{\partial\gamma_1^{1,1}}{\partial x}+v^0\frac{\partial\gamma_1^{1,1}}{\partial y}-\gamma_1^{1,1}\frac{\partial u^0}{\partial x}-\gamma_2^{1,1}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)$$
$$=-h\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right) \tag{4.10.53}$$

$$\frac{\partial\gamma_1^{1,2}}{\partial t}+u^0\frac{\partial\gamma_1^{1,2}}{\partial x}+v^0\frac{\partial\gamma_1^{1,2}}{\partial y}-\gamma_1^{1,2}\frac{\partial u^0}{\partial x}-\gamma_2^{1,2}\left(\frac{\partial v^0}{\partial x}+2\phi\left(\operatorname{sen}\varphi\right)\right)=0 \tag{4.10.54}$$

Análogamente a partir de (4.10.37) obtenemos otras tres ecuaciones:

$$\frac{\partial\gamma_2^{1,0}}{\partial t}+u^0\frac{\partial\gamma_2^{1,0}}{\partial x}+v^0\frac{\partial\gamma_2^{1,0}}{\partial y}-\gamma_1^{1,0}\left(\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)\right)-\gamma_2^{1,0}\frac{\partial v^0}{\partial y}$$
$$=\left(2\phi\left(\operatorname{sen}\varphi\right)+\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)\left[\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x}+v^0\frac{\partial H}{\partial y}\right)\right.$$
$$\left.+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right] \tag{4.10.55}$$

$$\frac{\partial \gamma_2^{1,1}}{\partial t} + u^0 \frac{\partial \gamma_2^{1,1}}{\partial x} + v^0 \frac{\partial \gamma_2^{1,1}}{\partial y} - \gamma_1^{1,1} \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) - \gamma_2^{1,1} \frac{\partial v^0}{\partial y}$$

$$= -h \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \right) \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \quad (4.10.56)$$

$$\frac{\partial \gamma_2^{1,2}}{\partial t} + u^0 \frac{\partial \gamma_2^{1,2}}{\partial x} + v^0 \frac{\partial \gamma_2^{1,2}}{\partial y} - \gamma_1^{1,2} \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) - \gamma_2^{1,2} \frac{\partial v^0}{\partial y} = 0 \quad (4.10.57)$$

El sistema formado por las ecuaciones (4.10.54) y (4.10.57) tiene por solución $\gamma_1^{1,2} = \gamma_2^{1,2} = 0$ que se puede probar que es única (véase [20]) si las condiciones iniciales y de contorno son nulas. Por tanto, γ_1^1 y γ_2^1 dependen linealmente de z:

$$\gamma_1^1 = \gamma_1^{1,0} + z\gamma_1^{1,1} \quad (4.10.58)$$

$$\gamma_2^1 = \gamma_2^{1,0} + z\gamma_2^{1,1} \quad (4.10.59)$$

donde $\gamma_i^{1,k} \quad (i = 1, 2, k = 0, 1)$ se obtienen resolviendo (4.10.52), (4.10.53), (4.10.55) y (4.10.56).

Sustituimos w^1 por su expresión dada en (4.5.82) y $\gamma_i^1 \quad (i = 1, 2)$ por las expresiones anteriores e integrando (4.5.96)-(4.5.97) obtenemos u^2 y v^2:

$$u^2 = u_0^2 + hz \left[\gamma_2^{1,0} + \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) + \frac{\partial H}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right]$$

$$+ \frac{1}{2} z^2 h \left[\gamma_2^{1,1} - h \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \quad (4.10.60)$$

$$v^2 = v_0^2 + hz \left[-\gamma_1^{1,0} + \frac{\partial}{\partial y} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right]$$

$$- \frac{1}{2} z^2 h \left[\gamma_1^{1,1} + h \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \quad (4.10.61)$$

donde $u_0^2(t,x,y) = u^2(t,x,y,0)$, $v_0^2(t,x,y) = v^2(t,x,y,0)$ están determinados por (4.10.23)-(4.10.24). En estas ecuaciones se sustituyen u^2, v^2, p^2, w^1 y w^2 de modo que la dependencia de z sea explícita, se agrupan los términos multiplicados por las distintas potencias de z y se simplifica utilizando para ello (4.10.52), (4.10.53), (4.10.55) y (4.10.56). Obtenemos finalmente las siguientes ecuaciones:

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u_0^2}{\partial y} + v^1 \frac{\partial u^1}{\partial y} + v_0^2 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right)$$

$$= -\frac{\partial}{\partial x} \left\{ h \left[\frac{\partial}{\partial t} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) + u^0 \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right.\right.$$

$$+ v^0 \frac{\partial}{\partial y}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\Bigg]\Bigg\}$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\left\{h^2\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\right.$$

$$\left.\left. + v^0 \frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2\right]\right\}$$

$$- \frac{\partial H}{\partial x}\left[\frac{\partial}{\partial t}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right.$$

$$\left. + v^0 \frac{\partial}{\partial y}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right] = -\frac{1}{\rho_0}\, D_x p^2\Big|_{z=0} \tag{4.10.62}$$

$$\frac{\partial v_0^2}{\partial t} + u^0 \frac{\partial v_0^2}{\partial x} + u^1 \frac{\partial v^1}{\partial x} + u_0^2 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v_0^2}{\partial y} + v^1 \frac{\partial v^1}{\partial y} + v_0^2 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u_0^2$$

$$= -\frac{\partial}{\partial y}\left\{h\left[\frac{\partial}{\partial t}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right.\right.$$

$$\left.\left. + v^0 \frac{\partial}{\partial y}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right]\right\}$$

$$- \frac{1}{2}\frac{\partial}{\partial y}\left\{h^2\left[\frac{\partial}{\partial t}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\right.$$

$$\left.\left. + v^0 \frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)^2\right]\right\}$$

$$- \frac{\partial H}{\partial y}\left[\frac{\partial}{\partial t}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right) + u^0 \frac{\partial}{\partial x}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right.$$

$$\left. + v^0 \frac{\partial}{\partial y}\left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y}\right)\right] = -\frac{1}{\rho_0}\, D_y p^2\Big|_{z=0} \tag{4.10.63}$$

Una vez calculados u^2 y v^2, w^3 se calcula integrando respecto a z (4.5.101)

$$D_z w^3 = -D_x u^2 - D_x v^2$$

Para ello, en primer lugar se sustituyen las expresiones obtenidas para u^2 y v^2 en (4.10.60)-(4.10.61), integramos respecto a z e imponemos la condición (derivada de (4.5.7)), $w^3 = u_0^2 \dfrac{\partial H}{\partial x} + v_0^2 \dfrac{\partial H}{\partial y}$ en $z = 0$, para llegar a la misma expresión de w^3 en función de u_0^2, v_0^2, u^0 y v^0 que en la sección 4.8, ((4.8.16)).

La tercera componente de la vorticidad viene dada por (4.5.108):

$$\gamma_3^2 = D_x v^2 - D_y u^2$$

Para calcular las otras dos componentes de la vorticidad se resuelven las ecuaciones (4.10.25)-(4.10.26) donde se sustituye γ_3^k $(k = 0, 1)$, por sus expresiones en función de u^k, v^k $(k = 0, 1)$, además de $D_z u^k$ y $D_z v^k$ $(k = 2, 3)$, por las expresiones (4.5.96)-(4.5.97) y (4.5.106)-(4.5.107):

$$\begin{aligned} D_t\gamma_1^2 &+ u^0 D_x\gamma_1^2 + u^1 D_x\gamma_1^1 + v^0 D_y\gamma_1^2 + v^1 D_y\gamma_1^1 + w^1 D_z\gamma_1^2 + w^2 D_z\gamma_1^1 \\ &- \gamma_1^1 \frac{\partial u^1}{\partial x} - \gamma_1^2 \frac{\partial u^0}{\partial x} - \gamma_2^1 \frac{\partial v^1}{\partial x} - \gamma_2^2 \left(\frac{\partial v^0}{\partial x} + 2\phi\,(\operatorname{sen}\varphi) \right) \\ &= \left(\frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi) \right) D_x w^2 + \left(\frac{\partial v^1}{\partial x} - \frac{\partial u^1}{\partial y} \right) D_x w^1 \qquad (4.10.64) \end{aligned}$$

$$\begin{aligned} D_t\gamma_2^2 &+ u^0 D_x\gamma_2^2 + u^1 D_x\gamma_2^1 + v^0 D_y\gamma_2^2 + v^1 D_y\gamma_2^1 + w^1 D_z\gamma_2^2 + w^2 D_z\gamma_2^1 \\ &- \gamma_1^1 \frac{\partial u^1}{\partial y} - \gamma_1^2 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) - \gamma_2^1 \frac{\partial v^1}{\partial y} - \gamma_2^2 \frac{\partial v^0}{\partial y} \\ &= \left(\frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi) \right) D_y w^2 + \left(\frac{\partial v^1}{\partial x} - \frac{\partial u^1}{\partial y} \right) D_y w^1 \qquad (4.10.65) \end{aligned}$$

Usando las expresiones encontradas para p^0, p^1 y p^2, (4.5.74), (4.10.13) y (4.10.49) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia

$$\begin{aligned} \tilde{p}(\varepsilon) &= p^0 + \varepsilon p^1 + \varepsilon^2 p^2 = p_s + \varepsilon \rho_0 g h\,(1 - z) \\ &+ \varepsilon^2 \rho_0 h (1 - z) \left[\frac{\partial}{\partial t} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) + u^0 \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right. \\ &\left. + v^0 \frac{\partial}{\partial y} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right] - \frac{1}{2} \varepsilon^2 \rho_0 h^2 (1 - z^2) \left[\frac{\partial}{\partial t} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right. \\ &\left. + u^0 \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + v^0 \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) - \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right)^2 \right] \qquad (4.10.66) \end{aligned}$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0, 1, 2, 3)$ ((4.5.73), (4.5.82), (4.10.35) y (4.8.16)), obtenemos una aproximación de la velocidad vertical en Ω:

$$
\begin{aligned}
\tilde{w}(\varepsilon) &= \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 \\
&= \varepsilon \left[u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \\
&+ \varepsilon^2 \left[u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - hz \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] \\
&+ \varepsilon^3 \left(u_0^2 \frac{\partial H}{\partial x} + v_0^2 \frac{\partial H}{\partial y} - hz \left\{ \frac{\partial u_0^2}{\partial x} - \frac{\partial H}{\partial x} \left[\gamma_2^{1,0} + \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right.\right.\right. \\
&+ \left. \frac{\partial H}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] + \frac{\partial v_0^2}{\partial y} - \frac{\partial H}{\partial y} \left[-\gamma_1^{1,0} + \frac{\partial}{\partial y} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right. \\
&+ \left.\left. \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right\} - \frac{1}{2} z^2 h \left\{ h \left[\frac{\partial \gamma_2^{1,0}}{\partial x} + \frac{\partial^2}{\partial x^2} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right.\right. \\
&+ \left. \frac{\partial^2 H}{\partial x^2} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \\
&- \frac{\partial H}{\partial x} \left[\gamma_2^{1,1} - h \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] + h \left[- \frac{\partial \gamma_1^{1,0}}{\partial y} + \frac{\partial^2}{\partial y^2} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right. \\
&+ \left. \frac{\partial^2 H}{\partial y^2} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \\
&+ \left. \frac{\partial H}{\partial y} \left[\gamma_1^{1,1} + h \frac{\partial}{\partial y} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right\} - \frac{1}{6} z^3 h^2 \left\{ \frac{\partial \gamma_2^{1,1}}{\partial x} - \frac{1}{h} \frac{\partial h}{\partial x} \gamma_2^{1,1} \right. \\
&- \left.\left. \frac{\partial \gamma_1^{1,1}}{\partial y} + \frac{1}{h} \frac{\partial h}{\partial y} \gamma_1^{1,1} - h \left[\frac{\partial^2}{\partial x^2} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right\} \right) \\
&= \varepsilon \frac{\partial H}{\partial x} \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 \right) + \varepsilon \frac{\partial H}{\partial y} \left(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 \right) \\
&- \varepsilon z h \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2 + \frac{\partial v^0}{\partial y} + \varepsilon \frac{\partial v^1}{\partial y} + \varepsilon^2 D_y v^2 \right) + O(\varepsilon^3)
\end{aligned}
$$

y por tanto,

$$
\tilde{w}(\varepsilon) = \varepsilon \frac{\partial H}{\partial x} \tilde{u}(\varepsilon) + \varepsilon \frac{\partial H}{\partial y} \tilde{v}(\varepsilon) - \varepsilon h z \left(D_x \tilde{u}(\varepsilon) + D_y \tilde{v}(\varepsilon) \right) + O(\varepsilon^3) \tag{4.10.67}
$$

Además, para la tercera componente de la vorticidad tenemos:

$$\tilde{\gamma}_3(\varepsilon) = D_x\tilde{v}(\varepsilon) - D_y\tilde{u}(\varepsilon) \tag{4.10.68}$$

Si ahora se deshace el cambio de variable, volviendo al dominio original, obtenemos la siguiente aproximación de la solución en Ω^ε:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y,z) + \varepsilon^2 u^2(t,x,y,z)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y,z) + \varepsilon^2 v^2(t,x,y,z)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z) + \varepsilon^3 w^3(t,x,y,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z) + \varepsilon^2 p^2(t,x,y,z)$$

$$\tilde{\gamma}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_i(\varepsilon)(t,x,y,z) = \gamma_i^0(t,x,y) + \varepsilon \gamma_i^1(t,x,y,z) + \varepsilon^2 \gamma_i^2(t,x,y,z) \quad (i=1,2)$$

$$\tilde{\gamma}_3^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{\gamma}_3(\varepsilon)(t,x,y,z) = \gamma_3^0(t,x,y) + \varepsilon \gamma_3^1(t,x,y,z) + \varepsilon^2 \gamma_3^2(t,x,y,z)$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (4.10.66):

$$\begin{aligned}
\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon) \Bigg[g + \frac{\partial}{\partial t^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\
+ u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + v^{0,\varepsilon}\frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\Bigg] \\
- \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\Bigg[\frac{\partial}{\partial t^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + u^{0,\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) \\
+ v^{0,\varepsilon}\frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) - \left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)^2\Bigg]
\end{aligned} \tag{4.10.69}$$

Y de forma análoga, realizando el cambio de variable de Ω a Ω^ε en (4.10.67), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) + O(\varepsilon^3) \tag{4.10.70}$$

Y al cambiar de variable en la expresión vista para la tercera componente de la vorticidad:

$$\tilde{\gamma}_3^\varepsilon = \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}$$

El modelo que vamos a proponer requiere que definamos

$$\check{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u^1(t,x,y)$$

$$\check{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \check{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v^1(t,x,y)$$

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u^1(t,x,y) + \varepsilon^2 u_0^2(t,x,y)$$

$$\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v^1(t,x,y) + \varepsilon^2 v_0^2(t,x,y)$$

y observamos que:

$$\begin{aligned}
&\frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) \\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} \right) \\
&\quad + (v^0 + \varepsilon v^1) \left(\frac{\partial u^0}{\partial y} + \varepsilon \frac{\partial u^1}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) \\
&\quad = \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) \\
&\quad + \varepsilon \left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi) \right) \right] + O(\varepsilon^2)
\end{aligned}$$

Utilizando las ecuaciones (4.5.78) y (4.10.33) obtenemos:

$$\begin{aligned}
\frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) &= -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - \varepsilon g \frac{\partial s}{\partial x} + O(\varepsilon^2) \\
&= -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + -g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + O(\varepsilon^2) \qquad (4.10.71)
\end{aligned}$$

análogamente

$$\frac{\partial \check{v}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) + \check{v}^\varepsilon \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} + O(\varepsilon^2) \qquad (4.10.72)$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\check{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \qquad (4.10.73)$$

Expresamos $\tilde{p}^\varepsilon$ en función de $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ en lugar de $u^{0,\varepsilon}$ y $v^{0,\varepsilon}$:

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + \frac{\partial}{\partial t^\varepsilon}\left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) + \breve{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon}\left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right.$$
$$\left. + \breve{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon}\left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)\right] - \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[\frac{\partial}{\partial t^\varepsilon}\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right)\right.$$
$$\left. + \breve{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right) + \breve{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon}\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right) - \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right)^2\right]$$
$$+ O(\varepsilon^3) \tag{4.10.74}$$

Si se consideran las ecuaciones (4.5.78), (4.10.33) y (4.10.62) se verifica

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right) = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + \varepsilon^2 \frac{\partial u_0^2}{\partial t}$$
$$+ (u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 \frac{\partial u_0^2}{\partial x}\right)$$
$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)\left(\frac{\partial u^0}{\partial y} + \varepsilon \frac{\partial u^1}{\partial y} + \varepsilon^2 \frac{\partial u_0^2}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\right)$$
$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0\left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\right)$$
$$+ \varepsilon\left[\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + v^1\left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\right) + v^0 \frac{\partial u^1}{\partial y}\right]$$
$$+ \varepsilon^2\left[\frac{\partial u_0^2}{\partial t} + u_0^2 \frac{\partial u^0}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u^0 \frac{\partial u_0^2}{\partial x} + v_0^2\left(\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\right)\right.$$
$$\left. + v^1 \frac{\partial u^1}{\partial y} + v^0 \frac{\partial u_0^2}{\partial y}\right] + O(\varepsilon^3)$$
$$= -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - \varepsilon g \frac{\partial s}{\partial x} - \varepsilon^2 \frac{1}{\rho_0}\left. D_x p^2\right|_{z=0} + O(\varepsilon^3)$$

por tanto,

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right) = -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + O(\varepsilon^3) \tag{4.10.75}$$

De manera similar utilizando las ecuaciones (4.5.79), (4.10.34) y (4.10.63) se obtiene

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \left(\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + O(\varepsilon^3) \tag{4.10.76}$$

Las ecuaciones para el cálculo de $\gamma_i^{1,k,\varepsilon}$ ($i = 1, 2$, $k = 0, 1$), se obtienen deshaciendo el cambio de variable en las ecuaciones (4.10.52), (4.10.53), (4.10.55) y (4.10.56) y sustituyendo u^0 y v^0 por $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$ respectivamente

$$\frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,0,\varepsilon} \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{1,0,\varepsilon} \left(\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right)$$

$$= \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial x^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon) \tag{4.10.77}$$

$$\frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,1,\varepsilon} \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{1,1,\varepsilon} \left(\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right)$$

$$= -h^\varepsilon \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.10.78}$$

$$\frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,0,\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) - \gamma_2^{1,0,\varepsilon} \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial y^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon) \tag{4.10.79}$$

$$\frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,1,\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) - \gamma_2^{1,1,\varepsilon} \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}$$

$$= -h^\varepsilon \left(2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon) \tag{4.10.80}$$

Veamos cómo se pueden escribir $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ en función de $\hat{u}^\varepsilon$, $\hat{v}^\varepsilon$, $\gamma_i^{1,k,\varepsilon}$ ($i = 1, 2$, $k = 0, 1$) y h^ε. Comenzamos por $\tilde{u}^\varepsilon$ utilizando (4.10.60):

$$\tilde{u}^\varepsilon = \tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$= u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2 + \varepsilon^2 hz \left[\gamma_2^{1,0} + \frac{\partial}{\partial x} \left(u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} \right) \right.$$

$$\left. + \frac{\partial H}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] + \frac{1}{2} \varepsilon^2 hz^2 \left[\gamma_2^{1,1} - h \frac{\partial}{\partial x} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right]$$

$$= \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[\gamma_2^{1,0,\varepsilon} + \frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$+ \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_2^{1,1,\varepsilon} - h^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon^3) \tag{4.10.81}$$

Para obtener una expresión análoga para $\tilde{v}^\varepsilon$ se tiene en cuenta la ecuación (4.10.61)

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon) \left[-\gamma_1^{1,0,\varepsilon} + \frac{\partial}{\partial y^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$

$$- \frac{1}{2} \frac{(z^\varepsilon - H^\varepsilon)^2}{h^\varepsilon} \left[\gamma_1^{1,1,\varepsilon} + h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + O(\varepsilon^3) \tag{4.10.82}$$

Proponemos el siguiente modelo resultado de despreciar los términos en $O(\varepsilon)$ de las ecuaciones (4.10.77)-(4.10.80), los términos en $O(\varepsilon^2)$ de las ecuaciones (4.10.71)-(4.10.73) y los términos en $O(\varepsilon^3)$ de las expresiones (4.10.70), (4.10.74), (4.10.81) y (4.10.82) y de las ecuaciones (4.10.75)-(4.10.76):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\check{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\check{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{4.10.83}$$

$$\frac{\partial \check{u}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \left(\frac{\partial \check{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) = - \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} \tag{4.10.84}$$

$$\frac{\partial \check{v}^\varepsilon}{\partial t^\varepsilon} + \check{u}^\varepsilon \left(\frac{\partial \check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) + \check{v}^\varepsilon \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} = - \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g \tag{4.10.85}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \left[g + \frac{\partial}{\partial t^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \check{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \check{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\check{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \check{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right] - \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[\frac{\partial}{\partial t^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) \right.$$

$$\left. + \check{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) + \check{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right) - \left(\frac{\partial \check{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \check{v}^\varepsilon}{\partial y^\varepsilon} \right)^2 \right] \tag{4.10.86}$$

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \right) = - \frac{1}{\rho_0} \left. \frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon} \right|_{z^\varepsilon = H^\varepsilon} \tag{4.10.87}$$

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \left(\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) \right) + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \tag{4.10.88}$$

$$\frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{1,0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,0,\varepsilon} \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{1,0,\varepsilon} \left(\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) \right)$$
$$= \left(2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial x^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$
$$\left. + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.10.89}$$

$$\frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_1^{1,1,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,1,\varepsilon} \frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} - \gamma_2^{1,1,\varepsilon} \left(\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) \right)$$
$$= -h^\varepsilon \left(2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.10.90}$$

$$\frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_2^{1,0,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,0,\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) \right) - \gamma_2^{1,0,\varepsilon} \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}$$
$$= \left(2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \left[\frac{\partial}{\partial y^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \right.$$
$$\left. + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.10.91}$$

$$\frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial t^\varepsilon} + \breve{u}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial \gamma_2^{1,1,\varepsilon}}{\partial y^\varepsilon} - \gamma_1^{1,1,\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) \right) - \gamma_2^{1,1,\varepsilon} \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}$$
$$= -h^\varepsilon \left(2\phi\left(\operatorname{sen}\varphi^\varepsilon\right) + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} \right) \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \tag{4.10.92}$$

$$\tilde{u}^\varepsilon = \hat{u}^\varepsilon + \left(z^\varepsilon - H^\varepsilon\right) \left[\gamma_2^{1,0,\varepsilon} + \frac{\partial}{\partial x^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$
$$+ \frac{1}{2} \frac{\left(z^\varepsilon - H^\varepsilon\right)^2}{h^\varepsilon} \left[\gamma_2^{1,1,\varepsilon} - h^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.10.93}$$

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + \left(z^\varepsilon - H^\varepsilon\right) \left[-\gamma_1^{1,0,\varepsilon} + \frac{\partial}{\partial y^\varepsilon} \left(\breve{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right]$$
$$- \frac{1}{2} \frac{\left(z^\varepsilon - H^\varepsilon\right)^2}{h^\varepsilon} \left[\gamma_1^{1,1,\varepsilon} + h^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \tag{4.10.94}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{4.10.95}$$

donde es necesario conocer las condiciones iniciales y de contorno.

Una vez conocidos $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y $\tilde{w}^\varepsilon$, buscamos las ecuaciones o expresiones adecuadas para el cálculo de la vorticidad. Comenzamos por la primera componente; será necesario tener en cuenta (4.10.30), (4.10.36) y (4.10.64):

$$\begin{aligned}
&\frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon} - \tilde{\gamma}_1^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} - \tilde{\gamma}_2^\varepsilon \left(\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right)\\
&\quad - \left(2\phi\,(\operatorname{sen}\varphi^\varepsilon) + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon}\\
&\quad = D_t\tilde{\gamma}_1(\varepsilon) + \tilde{u}(\varepsilon)D_x\tilde{\gamma}_1(\varepsilon) + \tilde{v}(\varepsilon)D_y\tilde{\gamma}_1(\varepsilon) + \tilde{w}(\varepsilon)D_z\tilde{\gamma}_1(\varepsilon) - \tilde{\gamma}_1(\varepsilon)D_x\tilde{u}(\varepsilon)\\
&\quad - \tilde{\gamma}_2(\varepsilon)\left(D_x v(\varepsilon) + 2\phi\,(\operatorname{sen}\varphi)\right) - \left(2\phi\,(\operatorname{sen}\varphi) + D_x\tilde{v}(\varepsilon) - D_y\tilde{u}(\varepsilon)\right)D_x\tilde{w}(\varepsilon)\\
&\quad = \varepsilon D_t\gamma_1^1 + \varepsilon^2 D_t\gamma_1^2 + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\left(\varepsilon D_x\gamma_1^1 + \varepsilon^2 D_x\gamma_1^2\right)\\
&\quad + (v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\left(\varepsilon D_y\gamma_1^1 + \varepsilon^2 D_y\gamma_1^2\right) + \varepsilon(w^1 + \varepsilon w^2 + \varepsilon^2 w^3)\left(D_z\gamma_1^1 + \varepsilon D_z\gamma_1^2\right)\\
&\quad - (\varepsilon\gamma_1^1 + \varepsilon^2\gamma_1^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2\right)\\
&\quad - (\varepsilon\gamma_2^1 + \varepsilon^2\gamma_2^2)\left(\frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x} + \varepsilon^2 D_x v^2 + 2\phi\,(\operatorname{sen}\varphi)\right)\\
&\quad - \big[2\phi\,(\operatorname{sen}\varphi) + D_x(v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\\
&\quad - D_y(u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\big]\,\varepsilon D_x(w^1 + \varepsilon w^2 + \varepsilon^2 w^3)\\
&\quad = \varepsilon\left[D_t\gamma_1^1 + u^0 D_x\gamma_1^1 + v^0 D_y\gamma_1^1 + w^1 D_z\gamma_1^1 - \gamma_1^1\frac{\partial u^0}{\partial x} - \gamma_2^1\left(\frac{\partial v^0}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\right)\right.\\
&\quad \left. - \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right)D_x w^1\right]\\
&\quad + \varepsilon^2\left[D_t\gamma_1^2 + u^0 D_x\gamma_1^2 + u^1 D_x\gamma_1^1 + v^0 D_y\gamma_1^2 + v^1 D_y\gamma_1^1 + w^1 D_z\gamma_1^2 + w^2 D_z\gamma_1^1\right.\\
&\quad - \gamma_1^1 D_x u^1 - \gamma_1^2\frac{\partial u^0}{\partial x} - \gamma_2^1 D_x v^1 - \gamma_2^2\left(\frac{\partial v^0}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\right)\\
&\quad \left. - \left(2\phi\,(\operatorname{sen}\varphi) + \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right)D_x w^2 - \left(D_x v^1 - D_y u^1\right)D_x w^1\right] + O(\varepsilon^3) = O(\varepsilon^3)
\end{aligned}$$

De manera análoga,

$$\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\left(\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right)-\tilde{\gamma}_2^\varepsilon\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}$$
$$-\left(2\phi\,(\operatorname{sen}\varphi^\varepsilon)+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial\tilde{w}^\varepsilon}{\partial y^\varepsilon}=O(\varepsilon^3)$$

Si se desprecian los términos en $O(\varepsilon^3)$, las ecuaciones para el cálculo de la vorticidad con el mismo orden de precisión que el resto de ecuaciones del modelo son las siguientes:

$$\frac{\partial\tilde{\gamma}_1^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}_1^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{\gamma}_1^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}_1^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}-\tilde{\gamma}_2^\varepsilon\left(\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right)$$
$$=\left(2\phi\,(\operatorname{sen}\varphi^\varepsilon)+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial\tilde{w}^\varepsilon}{\partial x^\varepsilon} \tag{4.10.96}$$

$$\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{\gamma}_2^\varepsilon}{\partial z^\varepsilon}-\tilde{\gamma}_1^\varepsilon\left(\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\right)-\tilde{\gamma}_2^\varepsilon\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}$$
$$=\left(2\phi\,(\operatorname{sen}\varphi^\varepsilon)+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}\right)\frac{\partial\tilde{w}^\varepsilon}{\partial y^\varepsilon} \tag{4.10.97}$$

Para la tercera componente se tiene:

$$\tilde{\gamma}_3^\varepsilon=\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}-\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon} \tag{4.10.98}$$

El orden de precisión con el que se verifican las ecuaciones de partida es exactamente el mismo que en la sección 4.8.

4.10.7. Modelo propuesto

De forma similar a lo que comentábamos en la sección 4.9 al analizar los modelos de la sección 4.8, nos encontramos de nuevo con que el modelo (4.10.83)-(4.10.95) es más preciso (al menos formalmente) que el modelo (4.10.45), pero el esfuerzo para resolver el sistema (4.10.83)-(4.10.95) es mucho mayor que el necesario para resolver el modelo (4.10.45). Por ello concluimos que la supuesta mejora en el orden de precisión que introduce este modelo no justifica la complejidad que presenta su resolución. Se propone finalmente, suprimiendo ˜ para simplificar la notación, el siguiente modelo:

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (u^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (v^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \\
&\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} \\
&\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \left(\frac{\partial v^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \right) + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g \frac{\partial s^\varepsilon}{\partial y^\varepsilon} \\
&w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon} \right) \\
&p^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon)
\end{aligned}
\tag{4.10.99}
$$

Este modelo se puede escribir en forma vectorial como sigue:

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0 \\
&\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g \nabla h^\varepsilon = -\frac{1}{\rho_0} \nabla p_s^\varepsilon - g \nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon \\
&w^\varepsilon = \vec{\mathbf{u}}^\varepsilon \cdot \nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon) \operatorname{div} \vec{\mathbf{u}}^\varepsilon \\
&p^\varepsilon = p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon)
\end{aligned}
\tag{4.10.100}
$$

donde

$$
\vec{\mathbf{F}}_C^\varepsilon = (\operatorname{sen}\varphi) \begin{pmatrix} v^\varepsilon \\ -u^\varepsilon \end{pmatrix}
$$

Si comparamos este modelo con (4.9.2) observamos que al aceptar la hipótesis oceanográfica la velocidad horizontal no depende de z^ε y por tanto no es necesario escribir el modelo en términos de la velocidad media, los términos debidos a la aceleración de Coriolis ($\vec{\mathbf{F}}_C^\varepsilon$) se simplifican notablemente y el cálculo de la vorticidad deja de ser necesario para conocer la velocidad horizontal y vertical. Es decir, reobtenemos el modelo clásico de aguas someras ((1.2.23)).

Se podría emplear la expresión de $\tilde{p}^\varepsilon$ (escrita en términos de $\tilde{u}^\varepsilon$ y de $\tilde{v}^\varepsilon$) en la aproximación de orden dos como una mejora de (4.10.40):

$$
\begin{aligned}
\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \Bigg[g &+ \frac{\partial}{\partial t^\varepsilon} \left(\tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) + \tilde{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{y^\varepsilon} \right) \\
&+ \tilde{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \Bigg] - \frac{\rho_0}{2} \left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[\frac{\partial}{\partial t^\varepsilon} \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right.
\end{aligned}
$$

$$+ \tilde{u}^\varepsilon \frac{\partial}{\partial x^\varepsilon} \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) + \tilde{v}^\varepsilon \frac{\partial}{\partial y^\varepsilon} \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) - \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)^2 \Bigg] \quad (4.10.101)$$

4.11. Conclusiones

Hemos visto en este capítulo que el método de desarrollos asintóticos permite obtener distintos modelos de aguas someras bidimensionales sin viscosidad. Hemos trabajado, en un principio, a partir de la expresión completa de la aceleración de Coriolis, sin aplicar ninguna de las hipótesis que hemos llamado de la oceanografía dinámica y a continuación realizando estas simplificaciones.

En ambos casos los modelos de segundo orden propuestos ((4.8.34)-(4.8.46) y (4.10.83)-(4.10.95)), requieren un esfuerzo de cálculo mucho mayor que los modelos (4.7.33)-(4.7.41) y (4.10.45) para obtener una pequeña mejora en la precisión (teóricamente del orden de ε^2). Por ello proponemos el modelo (4.9.1) (o (4.9.2)) en el caso general (expresado en velocidades medias, como se suele hacer en la literatura) y el modelo (4.10.99) (o (4.10.100)) en el caso de que aceptemos las hipótesis de la oceanografía dinámica. Además, parece factible mejorar la precisión del cálculo de la presión (usando (4.8.23), (4.10.69), respectivamente) sin que el coste aumente significativamente.

Queremos señalar que con el modelo (4.9.1) (o (4.9.2)) las componentes de la velocidad horizontal dependen de la profundidad si la vorticidad es no nula, lo que supone una interesante novedad respecto a los modelos que se encuentran en la literatura. Observamos también que la hipótesis simplificadora sobre la aceleración de Coriolis de la oceanografía dinámica implica que las dos primeras componentes de la vorticidad se anulan, por lo que las velocidades horizontales no dependen de la profundidad (y son, por tanto, iguales a las velocidades medias) y recuperamos el modelo clásico de aguas poco profundas.

Capítulo 5

Modelo bidimensional de aguas someras obtenido a partir de las ecuaciones de Navier-Stokes

5.1. Formulación del problema

El objetivo de este capítulo es obtener un modelo bidimensional de aguas someras con viscosidad. Para ello actuaremos de forma similar a como lo hicimos en los capítulos 3 y 4, pero partiendo en esta ocasión de las ecuaciones de Navier-Stokes tridimensionales.

5.1.1. Ecuaciones de partida

Al igual que en el capítulo 4, el dominio Ω (Figura 4.1) sobre el que trabajaremos representa, por ejemplo, un río, una ría o una región del mar y está definido por:

$$\Omega = \{(x, y, z)/(x, y) \in D, z \in [H(x, y), s(t, x, y)]\} \tag{5.1.1}$$

Ahora consideraremos que el flujo se rige por las ecuaciones tridimensionales de Navier-Stokes ((1.1.3)) en Ω y que las únicas fuerzas externas actuando sobre el fluido son las debidas a la gravedad y a la aceleración de Coriolis. Utilizamos por tanto, (1.1.5) y (1.1.6) para obtener F_x, F_y y F_z y sustituirlas en (1.1.3) de modo que se tiene:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi\left((\operatorname{sen}\varphi)\, v - (\cos\varphi)\, w\right) + \nu\Delta u \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, u + \nu\Delta v \\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + 2\phi\,(\cos\varphi)\, u + \nu\Delta w
\end{aligned} \tag{5.1.2}$$

donde se ha empleado la misma notación que en los capítulos anteriores.

El fluido es incompresible por lo que se cumple:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.1.3}$$

En este caso también suponemos que la presión es la atmosférica en la superficie

$$p = p_s \quad \text{en } z = s(t,x,y) \tag{5.1.4}$$

y que el fluido no puede penetrar el fondo

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{n}} = 0 \quad \text{en } z = H(x,y) \tag{5.1.5}$$

donde $\vec{\mathbf{u}} = (u,v,w)$ es el vector de velocidades del fluido y $\vec{\mathbf{n}}$ el vector normal unitario exterior a la frontera de Ω.

Para introducir el efecto del viento en la superficie y del rozamiento en el fondo es necesario introducir el tensor de tensiones[1]:

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \quad \text{con } T_{ij} = T_{ji} \quad (i,j = 1,2,3) \tag{5.1.6}$$

y entonces, ambos efectos se pueden escribir como sigue:

$$(T\vec{\mathbf{n}})_{\vec{\tau}} = \left(\vec{\mathbf{f}}_W\right)_{\vec{\tau}} \quad \text{en } z = s(t,x,y) \tag{5.1.7}$$

$$(T\vec{\mathbf{n}})_{\vec{\tau}} = -\vec{\mathbf{f}}_R \quad \text{en } z = H(x,y) \tag{5.1.8}$$

donde $()_{\vec{\tau}}$ denota la proyección sobre el plano tangente a la frontera, $\vec{\mathbf{f}}_W = (f_{W_1}, f_{W_2}, 0)$ es la fuerza del viento y $\vec{\mathbf{f}}_R = (f_{R_1}, f_{R_2})$ es la fuerza de rozamiento.

Observación 5.1 *Se supone la fuerza del viento es "horizontal" ($f_{W_3} = 0$) y que la fuerza de rozamiento se expresa en la base $\{\vec{\tau_1}, \vec{\tau_2}\}$ (es decir, $\vec{f}_R = f_{R_1}\vec{\tau_1} + f_{R_2}\vec{\tau_2}$).*

[1] En realidad el tensor de tensiones es $\sigma = -pI + T$ y T representa las tensiones no debidas a la presión. Las expresiones (5.1.7)-(5.1.8) son sin embargo correctas, debido a que

$$\sigma\vec{\mathbf{n}} \cdot \vec{\tau_i} = (-p\vec{\mathbf{n}} + T\,\vec{\mathbf{n}}) \cdot \vec{\tau_i} = (T\,\vec{\mathbf{n}}) \cdot \vec{\tau_i} \quad (i = 1,2)$$

donde $\vec{\tau_1}$ y $\vec{\tau_2}$ generan el plano tangente

En $z = H$ el vector normal y los vectores tangentes unitarios son los siguientes:

$$\vec{\mathbf{n}} = \frac{1}{\sqrt{1 + \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2}} \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, -1\right) \tag{5.1.9}$$

$$\vec{\tau}_1 = \frac{1}{\sqrt{1 + \left(\frac{\partial H}{\partial x}\right)^2}} \left(1, 0, \frac{\partial H}{\partial x}\right) \tag{5.1.10}$$

$$\vec{\tau}_2 = \frac{1}{\sqrt{1 + \left(\frac{\partial H}{\partial x}\right)^2}\sqrt{1 + \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2}} \left(-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}, 1 + \left(\frac{\partial H}{\partial x}\right)^2, \frac{\partial H}{\partial y}\right) \tag{5.1.11}$$

mientras que en $z = s$

$$\vec{\mathbf{n}} = \frac{1}{\sqrt{1 + \left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2}} \left(-\frac{\partial s}{\partial x}, -\frac{\partial s}{\partial y}, 1\right) \tag{5.1.12}$$

$$\vec{\tau}_1 = \frac{1}{\sqrt{1 + \left(\frac{\partial s}{\partial x}\right)^2}} \left(1, 0, \frac{\partial s}{\partial x}\right) \tag{5.1.13}$$

$$\vec{\tau}_2 = \frac{1}{\sqrt{1 + \left(\frac{\partial s}{\partial x}\right)^2}\sqrt{1 + \left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2}} \left(-\frac{\partial s}{\partial x}\frac{\partial s}{\partial y}, 1 + \left(\frac{\partial s}{\partial x}\right)^2, \frac{\partial s}{\partial y}\right) \tag{5.1.14}$$

Reescribimos, ahora, las condiciones (5.1.5), (5.1.7) y (5.1.8) sustituyendo T y $\vec{\mathbf{n}}$ por las expresiones vistas en (5.1.6) y (5.1.9)-(5.1.14):

$$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} - w = 0 \quad \text{en } z = H \tag{5.1.15}$$

$$\frac{-\frac{\partial s}{\partial x}T_{11} - \frac{\partial s}{\partial y}T_{12} + T_{13} + \frac{\partial s}{\partial x}\left(-\frac{\partial s}{\partial x}T_{13} - \frac{\partial s}{\partial y}T_{23} + T_{33}\right)}{\sqrt{1 + \left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2}} = f_{W_1} \quad \text{en } z = s \tag{5.1.16}$$

$$\frac{-\frac{\partial s}{\partial x}T_{12}-\frac{\partial s}{\partial y}T_{22}+T_{23}+\frac{\partial s}{\partial y}\left(-\frac{\partial s}{\partial x}T_{13}-\frac{\partial s}{\partial y}T_{23}+T_{33}\right)}{\sqrt{1+\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2}}=f_{W_2}\quad \text{en } z=s \qquad (5.1.17)$$

$$\frac{\frac{\partial H}{\partial x}T_{11}+\frac{\partial H}{\partial y}T_{12}-T_{13}+\frac{\partial H}{\partial x}\left(\frac{\partial H}{\partial x}T_{13}+\frac{\partial H}{\partial y}T_{23}-T_{33}\right)}{\sqrt{1+\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2}\sqrt{1+\left(\frac{\partial H}{\partial x}\right)^2}}=-f_{R_1}\quad \text{en } z=H \qquad (5.1.18)$$

$$\frac{1}{\left[1+\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right]\sqrt{1+\left(\frac{\partial H}{\partial x}\right)^2}}\left[-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial H}{\partial x}T_{11}+\frac{\partial H}{\partial y}T_{12}-T_{13}\right)\right.$$
$$\left.+\left[1+\left(\frac{\partial H}{\partial x}\right)^2\right]\left(\frac{\partial H}{\partial x}T_{12}+\frac{\partial H}{\partial y}T_{22}-T_{23}\right)+\frac{\partial H}{\partial y}\left(\frac{\partial H}{\partial x}T_{13}+\frac{\partial H}{\partial y}T_{23}-T_{33}\right)\right]$$
$$=-f_{R_2}\quad \text{en } z=H \qquad (5.1.19)$$

Suponemos además que en $\partial D\times[H(x,y),s(t,x,y)]$ es conocido el comportamiento del fluido, de modo que o se verifica que $\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}=0$ o bien es conocido el caudal (de entrada o salida) en cada instante.

Para cerrar el problema se deben fijar las condiciones iniciales:

$$\begin{aligned} u(0,x,y,z) &= u_0(x,y,z)\\ v(0,x,y,z) &= v_0(x,y,z)\\ w(0,x,y,z) &= w_0(x,y,z) \end{aligned} \qquad (5.1.20)$$

Además se deduce de la conservación de la masa como vimos en la sección 4.2 (ecuación (4.2.1)) que la altura del agua se puede calcular del modo siguiente:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\int_H^s u dz+\frac{\partial}{\partial y}\int_H^s v dz=0 \qquad (5.1.21)$$

El tensor de tensiones del fluido se define como:

$$T = \mu \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} & \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} & 2\dfrac{\partial w}{\partial z} \end{pmatrix} \tag{5.1.22}$$

Teniendo en cuenta la incompresibilidad del fluido se verifica:

$$\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + \frac{\partial T_{13}}{\partial z} = \mu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right) = \mu \Delta u$$

$$\frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{23}}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z}\right) = \mu \Delta v$$

$$\frac{\partial T_{31}}{\partial x} + \frac{\partial T_{32}}{\partial y} + \frac{\partial T_{33}}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2}\right) = \mu \Delta w$$

y considerando además que $\nu = \dfrac{\mu}{\rho_0}$, se puede escribir:

$$\nu \Delta u = \nu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \frac{1}{\rho_0}\frac{\partial T_{13}}{\partial z} \tag{5.1.23}$$

$$\nu \Delta v = \nu\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{\rho_0}\frac{\partial T_{23}}{\partial z} \tag{5.1.24}$$

$$\nu \Delta w = \frac{1}{\rho_0}\frac{\partial T_{31}}{\partial x} + \frac{1}{\rho_0}\frac{\partial T_{32}}{\partial y} + 2\nu\frac{\partial^2 w}{\partial z^2} \tag{5.1.25}$$

Si se sustituyen las expresiones anteriores en las ecuaciones de Navier-Stokes obtenemos las siguientes ecuaciones que serán nuestro punto de partida ya que facilitarán la incorporación de los efectos del viento y el rozamiento a la ecuación límite:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi\left((\operatorname{sen}\varphi)\, v - (\cos\varphi)\, w\right) \\ &+ \nu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \frac{1}{\rho_0}\frac{\partial T_{13}}{\partial z} \end{aligned} \tag{5.1.26}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, u$$

$$+\nu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right)+\frac{1}{\rho_0}\frac{\partial T_{23}}{\partial z} \tag{5.1.27}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g+2\phi\left(\cos\varphi\right)u$$

$$+\frac{1}{\rho_0}\frac{\partial T_{31}}{\partial x}+\frac{1}{\rho_0}\frac{\partial T_{32}}{\partial y}+2\nu\frac{\partial^2 w}{\partial z^2} \tag{5.1.28}$$

5.1.2. Cambio de notación

Como ya se explicó en los capítulos anteriores para obtener el modelo de aguas someras que buscamos se introduce un pequeño parámetro adimensional ε, del orden del cociente entre la profundidad media y el diámetro de Ω. Tanto el dominio como las variables y funciones mencionadas arriba dependen de este parámetro. Esta dependencia se indicará con el superíndice ε. Las ecuaciones se reescriben usando esta nueva notación, y así (5.1.26)- (5.1.28) y (5.1.3) se escriben:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon}+v^\varepsilon\frac{\partial u^\varepsilon}{\partial y^\varepsilon}+w^\varepsilon\frac{\partial u^\varepsilon}{\partial z^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial x^\varepsilon}+2\phi\left(\left(\operatorname{sen}\varphi^\varepsilon\right)v^\varepsilon-\left(\cos\varphi^\varepsilon\right)w^\varepsilon\right)$$

$$+\nu\left(2\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2}+\frac{\partial^2 u^\varepsilon}{\partial (y^\varepsilon)^2}+\frac{\partial^2 v^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right)+\frac{1}{\rho_0}\frac{\partial T_{13}^\varepsilon}{\partial z^\varepsilon} \tag{5.1.29}$$

$$\frac{\partial v^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial v^\varepsilon}{\partial x^\varepsilon}+v^\varepsilon\frac{\partial v^\varepsilon}{\partial y^\varepsilon}+w^\varepsilon\frac{\partial v^\varepsilon}{\partial z^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial y^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)u^\varepsilon$$

$$+\nu\left(\frac{\partial^2 u^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}+\frac{\partial^2 v^\varepsilon}{\partial (x^\varepsilon)^2}+2\frac{\partial^2 v^\varepsilon}{\partial (y^\varepsilon)^2}\right)+\frac{1}{\rho_0}\frac{\partial T_{23}^\varepsilon}{\partial z^\varepsilon} \tag{5.1.30}$$

$$\frac{\partial w^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial w^\varepsilon}{\partial x^\varepsilon}+v^\varepsilon\frac{\partial w^\varepsilon}{\partial y^\varepsilon}+w^\varepsilon\frac{\partial w^\varepsilon}{\partial z^\varepsilon}=-\frac{1}{\rho_0}\frac{\partial p^\varepsilon}{\partial z^\varepsilon}-g+2\phi\left(\cos\varphi^\varepsilon\right)u^\varepsilon$$

$$+\frac{1}{\rho_0}\frac{\partial T_{13}^\varepsilon}{\partial x^\varepsilon}+\frac{1}{\rho_0}\frac{\partial T_{23}^\varepsilon}{\partial y^\varepsilon}+2\nu\frac{\partial^2 w^\varepsilon}{\partial (z^\varepsilon)^2} \tag{5.1.31}$$

$$\frac{\partial u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial v^\varepsilon}{\partial y^\varepsilon}+\frac{\partial w^\varepsilon}{\partial z^\varepsilon}=0 \tag{5.1.32}$$

en $[0,T]\times\Omega^\varepsilon$, donde

$$\Omega^\varepsilon=\{(x^\varepsilon,y^\varepsilon,z^\varepsilon)/(x^\varepsilon,y^\varepsilon)\in D,\ z^\varepsilon\in[H^\varepsilon(x^\varepsilon,y^\varepsilon),s^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon)]\}$$

Las condiciones de contorno (5.1.4) y (5.1.15)-(5.1.19) se escriben:

$$p^\varepsilon = p_s \quad \text{en } z^\varepsilon = s^\varepsilon \tag{5.1.33}$$

$$u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - w^\varepsilon = 0 \quad \text{en } z^\varepsilon = H^\varepsilon \tag{5.1.34}$$

$$\begin{aligned} -\frac{\partial s^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{11} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{12} + T^\varepsilon_{13} + \frac{\partial s^\varepsilon}{\partial x^\varepsilon} \left(-\frac{\partial s^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{13} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{23} + T^\varepsilon_{33} \right) \\ = f^\varepsilon_{W_1} \sqrt{1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\right)^2} \qquad \text{en } z^\varepsilon = s^\varepsilon \end{aligned} \tag{5.1.35}$$

$$\begin{aligned} -\frac{\partial s^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{12} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{22} + T^\varepsilon_{23} + \frac{\partial s^\varepsilon}{\partial y^\varepsilon} \left(-\frac{\partial s^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{13} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{23} + T^\varepsilon_{33} \right) \\ = f^\varepsilon_{W_2} \sqrt{1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\right)^2} \qquad \text{en } z^\varepsilon = s^\varepsilon \end{aligned} \tag{5.1.36}$$

$$\begin{aligned} \frac{\partial H^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{11} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{12} - T^\varepsilon_{13} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{13} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{23} - T^\varepsilon_{33} \right) \\ = -f^\varepsilon_{R_1} \left(\sqrt{1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)^2} \sqrt{1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2} \right) \\ \text{en } z^\varepsilon = H^\varepsilon \end{aligned} \tag{5.1.37}$$

$$\begin{aligned} -\frac{\partial H^\varepsilon}{\partial x^\varepsilon} \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{11} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{12} - T^\varepsilon_{13} \right) + \left[1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2 \right] \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{12} \right. \\ \left. + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} T^\varepsilon_{22} - T^\varepsilon_{23} \right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon} T^\varepsilon_{13} + \frac{\partial H^\varepsilon}{\partial y} T^\varepsilon_{23} - T^\varepsilon_{33} \right) \\ = -f^\varepsilon_{R_2} \left(\left[1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right)^2 \right] \sqrt{1 + \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)^2} \right) \\ \text{en } z^\varepsilon = H^\varepsilon \end{aligned} \tag{5.1.38}$$

donde $f^\varepsilon_{W_i}$ y $f^\varepsilon_{R_i}$ $(i = 1, 2)$ son las componentes de fuerzas del viento y de rozamiento respectivamente. Las condiciones iniciales (5.1.20) resultan

$$u^\varepsilon(0, x^\varepsilon, y^\varepsilon, z^\varepsilon) \;=\; u^\varepsilon_0(x^\varepsilon, y^\varepsilon, z^\varepsilon), \tag{5.1.39}$$

$$v^{\varepsilon}(0, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}) = v_0^{\varepsilon}(x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}), \tag{5.1.40}$$

$$w^{\varepsilon}(0, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}) = w_0^{\varepsilon}(x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}), \tag{5.1.41}$$

La ecuación (5.1.21) para el cálculo de h^{ε}:

$$\frac{\partial h^{\varepsilon}}{\partial t^{\varepsilon}} + \frac{\partial}{\partial x^{\varepsilon}} \int_{H^{\varepsilon}}^{s^{\varepsilon}} u^{\varepsilon} dz^{\varepsilon} + \frac{\partial}{\partial y^{\varepsilon}} \int_{H^{\varepsilon}}^{s^{\varepsilon}} v^{\varepsilon} dz^{\varepsilon} = 0 \tag{5.1.42}$$

Las componentes del tensor de tensiones T^{ε}:

$$T_{11}^{\varepsilon} = 2\mu \frac{\partial u^{\varepsilon}}{\partial x^{\varepsilon}} \tag{5.1.43}$$

$$T_{12}^{\varepsilon} = \mu \left(\frac{\partial u^{\varepsilon}}{\partial y^{\varepsilon}} + \frac{\partial v^{\varepsilon}}{\partial x^{\varepsilon}} \right) \tag{5.1.44}$$

$$T_{13}^{\varepsilon} = \mu \left(\frac{\partial u^{\varepsilon}}{\partial z^{\varepsilon}} + \frac{\partial w^{\varepsilon}}{\partial x^{\varepsilon}} \right) \tag{5.1.45}$$

$$T_{22}^{\varepsilon} = 2\mu \frac{\partial v^{\varepsilon}}{\partial y^{\varepsilon}} \tag{5.1.46}$$

$$T_{23}^{\varepsilon} = \mu \left(\frac{\partial v^{\varepsilon}}{\partial z^{\varepsilon}} + \frac{\partial w^{\varepsilon}}{\partial y^{\varepsilon}} \right) \tag{5.1.47}$$

$$T_{33}^{\varepsilon} = 2\mu \frac{\partial w^{\varepsilon}}{\partial z^{\varepsilon}} \tag{5.1.48}$$

y los laplacianos en términos de T_{i3}^{ε} $(i = 1, 2)$, (5.1.23)-(5.1.25):

$$\nu \Delta^{\varepsilon} u^{\varepsilon} = \nu \left(2 \frac{\partial^2 u^{\varepsilon}}{\partial (x^{\varepsilon})^2} + \frac{\partial^2 u^{\varepsilon}}{\partial (y^{\varepsilon})^2} + \frac{\partial^2 v^{\varepsilon}}{\partial x^{\varepsilon} \partial y^{\varepsilon}} \right) + \frac{1}{\rho_0} \frac{\partial T_{13}^{\varepsilon}}{\partial z^{\varepsilon}} \tag{5.1.49}$$

$$\nu \Delta^{\varepsilon} v^{\varepsilon} = \nu \left(\frac{\partial^2 u^{\varepsilon}}{\partial x^{\varepsilon} \partial y^{\varepsilon}} + \frac{\partial^2 v^{\varepsilon}}{\partial (x^{\varepsilon})^2} + 2 \frac{\partial^2 v^{\varepsilon}}{\partial (y^{\varepsilon})^2} \right) + \frac{1}{\rho_0} \frac{\partial T_{23}^{\varepsilon}}{\partial z^{\varepsilon}} \tag{5.1.50}$$

$$\nu \Delta^{\varepsilon} w^{\varepsilon} = \frac{1}{\rho_0} \frac{\partial T_{13}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{1}{\rho_0} \frac{\partial T_{23}^{\varepsilon}}{\partial y^{\varepsilon}} + 2\nu \frac{\partial^2 w^{\varepsilon}}{\partial (z^{\varepsilon})^2} \tag{5.1.51}$$

Deberíamos añadir también que el caudal de entrada y salida es conocido en cada instante, pero como estas condiciones son impuestas de varias formas en la literatura y no es necesario explicitarlas en lo que sigue, preferimos no incluirlas de momento, aunque éstas u otras condiciones similares serán necesarias en la resolución del modelo que obtengamos finalmente.

5.2. Ecuaciones en el dominio de referencia

Como en los capítulos anteriores trabajaremos en un dominio de referencia independiente de ε y del tiempo en lugar de en el dominio original. El cambio de variable que se va a aplicar será el mismo que vimos en el capítulo anterior (sección 4.3).

Sea $\Omega = D \times [0,1]$ el dominio de referencia. Se supone que:

$$h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon h(t,x,y) \tag{5.2.1}$$

$$H^\varepsilon(x^\varepsilon, y^\varepsilon) = \varepsilon H(x,y) \tag{5.2.2}$$

(y por tanto $s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon s(t,x,y)) = \varepsilon\left(H(x,y) + h(t,x,y)\right)$ y se define el siguiente cambio de variable, de Ω a Ω^ε:

$$\begin{aligned} t^\varepsilon &= t \\ x^\varepsilon &= x \\ y^\varepsilon &= y \\ z^\varepsilon &= \varepsilon[H(x,y) + zh(t,x,y)] \end{aligned} \tag{5.2.3}$$

Dada una función F^ε cualquiera definida en $[0,T] \times \bar{\Omega}^\varepsilon$, se puede construir a partir de ella otra función $F(\varepsilon)$ definida en $[0,T] \times \bar{\Omega}$ utilizando para ello el cambio de variable del modo natural: $F(\varepsilon)(t,x,y,z) = F^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$. La relación entre las derivadas parciales de una y otra función es:

$$\frac{\partial F^\varepsilon}{\partial t^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial F(\varepsilon)}{\partial z} = D_t F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial x^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial x} - \frac{\dfrac{\partial H}{\partial x} + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_x F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial y^\varepsilon} = \frac{\partial F(\varepsilon)}{\partial y} - \frac{\dfrac{\partial H}{\partial y} + z\dfrac{\partial h}{\partial y}}{h}\frac{\partial F(\varepsilon)}{\partial z} = D_y F(\varepsilon)$$

$$\frac{\partial F^\varepsilon}{\partial z^\varepsilon} = \frac{1}{\varepsilon h}\frac{\partial F(\varepsilon)}{\partial z} = \frac{1}{\varepsilon} D_z F(\varepsilon)$$

donde hemos introducido la siguiente notación:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} - \frac{z}{h}\frac{\partial h}{\partial t}\frac{\partial}{\partial z} \\
D_x &= \frac{\partial}{\partial x} - \frac{\dfrac{\partial H}{\partial x} + z\dfrac{\partial h}{\partial x}}{h}\frac{\partial}{\partial z} \\
D_y &= \frac{\partial}{\partial y} - \frac{\dfrac{\partial H}{\partial y} + z\dfrac{\partial h}{\partial y}}{h}\frac{\partial}{\partial z} \\
D_z &= \frac{1}{h}\frac{\partial}{\partial z}
\end{aligned} \tag{5.2.4}$$

Aprovechamos para definir también:

$$\begin{aligned}
&D_x^2 = D_x(D_x), \quad D_y^2 = D_y(D_y), \quad D_{xy}^2 = D_x(D_y) = D_y(D_x) \\
&D_z^2 = D_z(D_z)
\end{aligned} \tag{5.2.5}$$

Si ahora definimos,

$$\begin{aligned}
u(\varepsilon)(t,x,y,z) &= u^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
v(\varepsilon)(t,x,y,z) &= v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
w(\varepsilon)(t,x,y,z) &= w^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
p(\varepsilon)(t,x,y,z) &= p^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \\
T_{ij}(\varepsilon)(t,x,y,z) &= T_{ij}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \quad (i,j = 1,2,3)
\end{aligned}$$

el problema (5.1.29)-(5.1.51) se puede escribir en el dominio de referencia Ω de la forma siguiente:

- las ecuaciones de Navier-Stokes:

$$\begin{aligned}
&D_t u(\varepsilon) + u(\varepsilon)D_x u(\varepsilon) + v(\varepsilon)D_y u(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z u(\varepsilon) = -\frac{1}{\rho_0}D_x p(\varepsilon) \\
&\quad + 2\phi\left((\operatorname{sen}\varphi)\, v(\varepsilon) - (\cos\varphi)\, w(\varepsilon)\right) + \nu\left(2D_x^2 u(\varepsilon) + D_y^2 u(\varepsilon) + D_{xy}^2 v(\varepsilon)\right) \\
&\quad + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z T_{13}(\varepsilon)
\end{aligned} \tag{5.2.6}$$

$$D_t v(\varepsilon) + u(\varepsilon)D_x v(\varepsilon) + v(\varepsilon)D_y v(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z v(\varepsilon) = -\frac{1}{\rho_0}D_y p(\varepsilon)$$

$$- 2\phi\,(\operatorname{sen}\varphi)\,u(\varepsilon) + \nu\left(D^2_{xy}u(\varepsilon) + D^2_x v(\varepsilon) + 2D^2_y v(\varepsilon)\right)$$

$$+ \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z T_{23}(\varepsilon) \tag{5.2.7}$$

$$D_t w(\varepsilon) + u(\varepsilon)D_x w(\varepsilon) + v(\varepsilon)D_y w(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z w(\varepsilon) = -\frac{1}{\rho_0}\frac{1}{\varepsilon}D_z p(\varepsilon)$$

$$- g + 2\phi\,(\cos\varphi)\,u(\varepsilon) + 2\nu\frac{1}{\varepsilon^2}D^2_z w(\varepsilon)$$

$$+ \frac{1}{\rho_0}D_x T_{13}(\varepsilon) + \frac{1}{\rho_0}D_y T_{23}(\varepsilon) \tag{5.2.8}$$

- la condición de incompresibilidad

$$D_x u(\varepsilon) + D_y v(\varepsilon) + \frac{1}{\varepsilon}D_z w(\varepsilon) = 0 \tag{5.2.9}$$

- las condiciones de contorno

$$p(\varepsilon) = p_s \qquad \text{en } z = 1 \tag{5.2.10}$$

$$w(\varepsilon) = u(\varepsilon)\varepsilon\frac{\partial H}{\partial x} + v(\varepsilon)\varepsilon\frac{\partial H}{\partial y} \text{ en } z = 0 \tag{5.2.11}$$

$$-\varepsilon\frac{\partial s}{\partial x}T_{11}(\varepsilon) - \varepsilon\frac{\partial s}{\partial y}T_{12}(\varepsilon) + T_{13}(\varepsilon) + \varepsilon\frac{\partial s}{\partial x}\left(-\varepsilon\frac{\partial s}{\partial x}T_{13}(\varepsilon) - \varepsilon\frac{\partial s}{\partial y}T_{23}(\varepsilon) + T_{33}(\varepsilon)\right)$$

$$= f_{W_1}(\varepsilon)\sqrt{1 + \left(\varepsilon\frac{\partial s}{\partial x}\right)^2 + \left(\varepsilon\frac{\partial s}{\partial y}\right)^2} \quad \text{en } z = 1 \tag{5.2.12}$$

$$-\varepsilon\frac{\partial s}{\partial x}T_{12}(\varepsilon) - \varepsilon\frac{\partial s}{\partial y}T_{22}(\varepsilon) + T_{23}(\varepsilon) + \varepsilon\frac{\partial s}{\partial y}\left(-\varepsilon\frac{\partial s}{\partial x}T_{13}(\varepsilon) - \varepsilon\frac{\partial s}{\partial y}T_{23}(\varepsilon) + T_{33}(\varepsilon)\right)$$

$$= f_{W_2}(\varepsilon)\sqrt{1 + \left(\varepsilon\frac{\partial s}{\partial x}\right)^2 + \left(\varepsilon\frac{\partial s}{\partial y}\right)^2} \qquad \text{en } z = 1 \tag{5.2.13}$$

$$\varepsilon\frac{\partial H}{\partial x}T_{11}(\varepsilon) + \varepsilon\frac{\partial H}{\partial y}T_{12}(\varepsilon) - T_{13}(\varepsilon) + \varepsilon\frac{\partial H}{\partial x}\left(\varepsilon\frac{\partial H}{\partial x}T_{13}(\varepsilon) + \varepsilon\frac{\partial H}{\partial y}T_{23}(\varepsilon) - T_{33}(\varepsilon)\right)$$

$$= -f_{R_1}(\varepsilon)\left(\sqrt{1 + \left(\varepsilon\frac{\partial H}{\partial x}\right)^2 + \left(\varepsilon\frac{\partial H}{\partial y}\right)^2}\sqrt{1 + \left(\varepsilon\frac{\partial H}{\partial x}\right)^2}\right) \quad \text{en } z = 0 \tag{5.2.14}$$

$$-\varepsilon^2 \frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\varepsilon\frac{\partial H}{\partial x}T_{11}(\varepsilon)+\varepsilon\frac{\partial H}{\partial y}T_{12}(\varepsilon)-T_{13}(\varepsilon)\right)$$

$$+\left[1+\left(\varepsilon\frac{\partial H}{\partial x}\right)^2\right]\left(\varepsilon\frac{\partial H}{\partial x}T_{12}(\varepsilon)+\varepsilon\frac{\partial H}{\partial y}T_{22}(\varepsilon)-T_{23}(\varepsilon)\right)$$

$$+\varepsilon\frac{\partial H}{\partial y}\left(\varepsilon\frac{\partial H}{\partial x}T_{13}(\varepsilon)+\varepsilon\frac{\partial H}{\partial y}T_{23}(\varepsilon)-T_{33}(\varepsilon)\right)$$

$$=-f_{R_2}(\varepsilon)\left(\left[1+\left(\varepsilon\frac{\partial H}{\partial x}\right)^2+\left(\varepsilon\frac{\partial H}{\partial y}\right)^2\right]\sqrt{1+\left(\varepsilon\frac{\partial H}{\partial x}\right)^2}\right)\quad \text{en } z=0 \quad (5.2.15)$$

- las condiciones iniciales:

$$u(\varepsilon)(0,x,y,z) = u_0(\varepsilon)(x,y,z) \quad (5.2.16)$$

$$v(\varepsilon)(0,x,y,z) = v_0(\varepsilon)(x,y,z) \quad (5.2.17)$$

$$w(\varepsilon)(0,x,y,z) = w_0(\varepsilon)(x,y,z) \quad (5.2.18)$$

- la ecuación que determina el calado:

$$\frac{\partial h}{\partial t}+\int_0^1\left[\frac{\partial(u(\varepsilon)h)}{\partial x}+\frac{\partial(v(\varepsilon)h)}{\partial y}\right]dz=0 \quad (5.2.19)$$

- las componentes del tensor de tensiones:

$$T_{11}(\varepsilon) = 2\mu D_x u(\varepsilon) \quad (5.2.20)$$

$$T_{12}(\varepsilon) = \mu\left(D_y u(\varepsilon)+D_x v(\varepsilon)\right) \quad (5.2.21)$$

$$T_{13}(\varepsilon) = \mu\left(\frac{1}{\varepsilon}D_z u(\varepsilon)+D_x w(\varepsilon)\right) \quad (5.2.22)$$

$$T_{22}(\varepsilon) = 2\mu D_y v(\varepsilon) \quad (5.2.23)$$

$$T_{23}(\varepsilon) = \mu\left(\frac{1}{\varepsilon}D_z v(\varepsilon)+D_y w(\varepsilon)\right) \quad (5.2.24)$$

$$T_{33}(\varepsilon) = 2\mu\frac{1}{\varepsilon}D_z w(\varepsilon) \quad (5.2.25)$$

- y los laplacianos en términos de las componentes del tensor de tensiones $T_{i3}(\varepsilon)$ $(i = 1, 2)$:

$$\nu\left(\frac{1}{\varepsilon^2}D_z^2u(\varepsilon) - D_x^2u(\varepsilon) - D_{xy}^2v(\varepsilon)\right) = \frac{1}{\varepsilon\rho_0}D_zT_{13}(\varepsilon) \tag{5.2.26}$$

$$\nu\left(\frac{1}{\varepsilon^2}D_z^2v(\varepsilon) - D_{xy}^2u(\varepsilon) - D_y^2v(\varepsilon)\right) = \frac{1}{\varepsilon\rho_0}D_zT_{23}(\varepsilon) \tag{5.2.27}$$

$$\nu\left(D_x^2w(\varepsilon) + D_y^2w(\varepsilon) - \frac{1}{\varepsilon^2}D_z^2w(\varepsilon)\right) = \frac{1}{\rho_0}\left(D_xT_{13}(\varepsilon) + D_yT_{23}(\varepsilon)\right) \tag{5.2.28}$$

5.3. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (5.2.6)-(5.2.28) admite un desarrollo en serie de potencias de ε en la forma:

$$\begin{aligned}
&u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots \\
&v(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots \\
&w(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots \\
&p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots \\
&T_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 + \varepsilon^2 T_{ij}^2 + \cdots \quad (i, j = 1, 2) \\
&T_{i3}(\varepsilon) = \varepsilon^{-1}T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 + \varepsilon^2 T_{i3}^2 + \cdots \quad (i = 1, 2, 3)
\end{aligned} \tag{5.3.1}$$

Observación 5.2 *Se supone que los términos principales de $T_{i3}(\varepsilon)$ son de orden ε^{-1} porque, si se sustituyen los desarrollos asintóticos de $u(\varepsilon)$, $v(\varepsilon)$ y $w(\varepsilon)$ en* (5.2.22), (5.2.24) *y* (5.2.25), *ésta es la hipótesis natural.*

También se supone que la fuerza de rozamiento y del viento admiten un desarrollo en serie de potencias de ε. Para construir estas series estudiamos, como se hizo en la sección 3.3, las fórmulas empíricas a partir de las que se calculan la fuerza de rozamiento y del viento, y este estudio nos permite suponer que ambas son de orden ε:

$$\begin{aligned}
&f_{R_1}(\varepsilon) = \varepsilon f_{R_1}^1 + \varepsilon^2 f_{R_1}^2 + \cdots \\
&f_{R_2}(\varepsilon) = \varepsilon f_{R_2}^1 + \varepsilon^2 f_{R_2}^2 + \cdots \\
&f_{W_1}(\varepsilon) = \varepsilon f_{W_1}^1 + \varepsilon^2 f_{W_1}^2 + \cdots \\
&f_{W_2}(\varepsilon) = \varepsilon f_{W_2}^1 + \varepsilon^2 f_{W_2}^2 + \cdots
\end{aligned} \tag{5.3.2}$$

Se sustituyen, ahora, estos desarrollos en serie de potencias en las ecuaciones (5.2.6)-(5.2.28). Realizando esta sustitución en la primera ecuación de Navier-Stokes ((5.2.6)) se obtiene:

$$
\begin{aligned}
&D_t u^0 + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + \cdots \\
&\quad + (u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots)\left[D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots\right] \\
&\quad + (v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots)\left[D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots\right] \\
&\quad + (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\frac{1}{\varepsilon}\left[D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots\right] \\
&\quad = -\frac{1}{\rho_0}\left(D_x p^0 + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 + \cdots\right) \\
&\quad + 2\phi\left[(\operatorname{sen}\varphi)(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots) - (\cos\varphi)(w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots)\right] \\
&\quad + \nu\left[2\left(D_x^2 u^0 + \varepsilon D_x^2 u^1 + \varepsilon^2 D_x^2 u^2 + \cdots\right) + D_y^2 u^0 + \varepsilon D_y^2 u^1 + \varepsilon^2 D_y^2 u^2 + \cdots\right. \\
&\quad \left. + D_{xy}^2 v^0 + \varepsilon D_{xy}^2 v^1 + \varepsilon^2 D_{xy}^2 v^2 + \cdots\right] \\
&\quad + \frac{1}{\rho_0}\frac{1}{\varepsilon}\left(\varepsilon^{-1} D_z T_{13}^{-1} + D_z T_{13}^0 + \varepsilon D_z T_{13}^1 + \varepsilon^2 D_z T_{13}^2 + \varepsilon^3 D_z T_{13}^3 + \cdots\right)
\end{aligned}
$$

El paso siguiente consiste en identificar los términos multiplicados por la misma potencia de ε. En este caso se tiene:

$$
\begin{aligned}
&-\varepsilon^{-2}\frac{1}{\rho_0} D_z T_{13}^{-1} + \varepsilon^{-1}\left(w^0 D_z u^0 - \frac{1}{\rho_0} D_z T_{13}^0\right) \\
&\quad + \varepsilon^0\left[D_t u^0 + u^0 D_x u^0 + v^0 D_y u^0 + w^0 D_z u^1 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0\right. \\
&\quad \left. - 2\phi\left((\operatorname{sen}\varphi) v^0 - (\cos\varphi) w^0\right) - \nu\left(2D_x^2 u^0 + D_y^2 u^0 + D_{xy}^2 v^0\right) - \frac{1}{\rho_0} D_z T_{13}^1\right] \\
&\quad + \varepsilon\left[D_t u^1 + u^0 D_x u^1 + u^1 D_x u^0 + v^0 D_y u^1 + v^1 D_y u^0 + w^0 D_z u^2 + w^1 D_z u^1\right. \\
&\quad + w^2 D_z u^0 + \frac{1}{\rho_0} D_x p^1 - 2\phi\left((\operatorname{sen}\varphi) v^1 - (\cos\varphi) w^1\right) - \nu\left(2D_x^2 u^1 + D_y^2 u^1 + D_{xy}^2 v^1\right) \\
&\quad \left. - \frac{1}{\rho_0} D_z T_{13}^2\right] + \varepsilon^2\left[D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 D_x u^0 + v^0 D_y u^2 + v^1 D_y u^1\right. \\
&\quad + v^2 D_y u^0 + w^0 D_z u^3 + w^1 D_z u^2 + w^2 D_z u^1 + w^3 D_z u^0 + \frac{1}{\rho_0} D_x p^2
\end{aligned}
$$

$$- 2\phi \left((\operatorname{sen}\varphi)\, v^2 - (\cos\varphi)\, w^2 \right) - \nu \left(2D_x^2 u^2 + D_y^2 u^2 + D_{xy}^2 v^2 \right) - \frac{1}{\rho_0} D_z T_{13}^3 \Big]$$

$$+ O(\varepsilon^3) = 0 \tag{5.3.3}$$

Repetimos el proceso a partir de la segunda ecuación de Navier-Stokes ((5.2.7)) y llegamos a:

$$-\varepsilon^{-2} \frac{1}{\rho_0} D_z T_{23}^{-1} + \varepsilon^{-1} \left(w^0 D_z v^0 - \frac{1}{\rho_0} D_z T_{23}^0 \right)$$

$$+ \varepsilon^0 \Big[D_t v^0 + u^0 D_x v^0 + v^0 D_y v^0 + w^0 D_z v^1 + w^1 D_z v^0 + \frac{1}{\rho_0} D_y p^0$$

$$+ 2\phi\, (\operatorname{sen}\varphi)\, u^0 - \nu \left(D_{xy}^2 u^0 + D_x^2 v^0 + 2D_y^2 v^0 \right) - \frac{1}{\rho_0} D_z T_{23}^1 \Big]$$

$$+ \varepsilon \big[D_t v^1 + u^0 D_x v^1 + u^1 D_x v^0 + v^0 D_y v^1 + v^1 D_y v^0 + w^0 D_z v^2 + w^1 D_z v^1$$

$$+ w^2 D_z v^0 + \frac{1}{\rho_0} D_y p^1 + 2\phi\, (\operatorname{sen}\varphi)\, u^1 - \nu \left(D_{xy}^2 u^1 + D_x^2 v^1 + 2D_y^2 v^1 \right) - \frac{1}{\rho_0} D_z T_{23}^2 \Big]$$

$$+ \varepsilon^2 \big[D_t v^2 + u^0 D_x v^2 + u^1 D_x v^1 + u^2 D_x v^0 + v^0 D_y v^2 + v^1 D_y v^1 + v^2 D_y v^0$$

$$+ w^0 D_z v^3 + w^1 D_z v^2 + w^2 D_z v^1 + w^3 D_z v^0 + \frac{1}{\rho_0} D_y p^2 + 2\phi\, (\operatorname{sen}\varphi)\, u^2$$

$$- \nu \left(D_{xy}^2 u^2 + D_x^2 v^2 + 2D_y^2 v^2 \right) - \frac{1}{\rho_0} D_z T_{23}^3 \Big] + O(\varepsilon^3) = 0 \tag{5.3.4}$$

Reemplazando $u(\varepsilon)$, $w(\varepsilon)$, $p(\varepsilon)$ y $T_{ij}(\varepsilon)$ $(i, j = 1, 2)$, por sus desarrollos en serie de potencias de ε ((5.3.1)) en la tercera ecuación de Navier-Stokes ((5.2.8)), y agrupando en potencias de ε se obtiene:

$$-\varepsilon^{-2} 2\nu D_z^2 w^0 + \varepsilon^{-1} \left[w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2\nu D_z^2 w^1 - \frac{1}{\rho_0} \left(D_x T_{13}^{-1} + D_y T_{23}^{-1} \right) \right]$$

$$+ \varepsilon^0 \Big[D_t w^0 + u^0 D_x w^0 + v^0 D_y w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g$$

$$- 2\phi\, (\cos\varphi)\, u^0 - 2\nu D_z^2 w^2 - \frac{1}{\rho_0} \left(D_x T_{13}^0 - D_y T_{23}^0 \right) \Big]$$

$$+ \varepsilon \big[D_t w^1 + u^0 D_x w^1 + u^1 D_x w^0 + v^0 D_y w^1 + v^1 D_y w^0 + w^0 D_z w^2 + w^1 D_z w^1$$

$$+ w^2 D_z w^0 + \frac{1}{\rho_0} D_z p^2 - 2\phi\, (\cos\varphi)\, u^1 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1 \right) \Big]$$

$$+ O(\varepsilon^2) = 0 \tag{5.3.5}$$

Una vez más, realizamos la sustitución y después la identificación de los términos multiplicados por cada potencia de ε a partir de la ecuación de la incompresibilidad ((5.2.9))

$$\varepsilon^{-1} D_z w^0 + D_x u^0 + D_y v^0 + D_z w^1 + \varepsilon \left(D_x u^1 + D_y v^1 + D_z w^2 \right)$$
$$+ \varepsilon^2 \left(D_x u^2 + D_y v^2 + D_z w^3 \right) + O(\varepsilon^3) = 0 \tag{5.3.6}$$

De la condición de contorno (5.2.10), suponiendo $p_s^\varepsilon = p_s$ independiente de ε, se tiene

$$p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots = p_s \text{ en } z = 1 \tag{5.3.7}$$

Del mismo modo, a partir de la condición (5.2.11) se tiene

$$w^0 + \varepsilon \left(w^1 - u^0 \frac{\partial H}{\partial x} - v^0 \frac{\partial H}{\partial y} \right) + \varepsilon^2 \left(w^2 - u^1 \frac{\partial H}{\partial x} - v^1 \frac{\partial H}{\partial y} \right)$$
$$+ \varepsilon^3 \left(w^3 - u^2 \frac{\partial H}{\partial x} - v^2 \frac{\partial H}{\partial y} \right) + \cdots = 0 \text{ en } z = 0 \tag{5.3.8}$$

Para realizar la sustitución en las condiciones de contorno (5.2.12)-(5.2.15) es necesario conocer el desarrollo en serie de potencias de ε de los términos:

$$\sqrt{1 + \left(\varepsilon \frac{\partial s}{\partial x} \right)^2 + \left(\varepsilon \frac{\partial s}{\partial y} \right)^2}, \quad \sqrt{1 + \left(\varepsilon \frac{\partial H}{\partial x} \right)^2 + \left(\varepsilon \frac{\partial H}{\partial y} \right)^2}, \quad \sqrt{1 + \left(\varepsilon \frac{\partial H}{\partial x} \right)^2}$$

Los tres términos son de la forma

$$\alpha(\varepsilon) = \sqrt{1 + \varepsilon^2 A}$$

por lo que si denotamos $\beta(\varepsilon) = 1 + \varepsilon^2 A$ y, ya que sabemos que $[\alpha(\varepsilon)]^2 = \beta(\varepsilon)$, si suponemos que $\alpha(\varepsilon) = \alpha^0 + \varepsilon \alpha^1 + \varepsilon^2 \alpha^2 + \cdots$, entonces

$$(\alpha^0)^2 + \varepsilon 2\alpha^0 \alpha^1 + \varepsilon^2 [2\alpha^0 \alpha^2 + (\alpha^1)^2] + \varepsilon^3 [2\alpha^0 \alpha^3 + 2\alpha^1 \alpha^2]$$
$$+ \varepsilon^4 [2\alpha^0 \alpha^4 + 2\alpha^1 \alpha^3 + (\alpha^2)^2] + \cdots = 1 + \varepsilon^2 A$$

de donde se deduce que:

$$\alpha^0 = 1, \quad \alpha^1 = 0, \quad \alpha^2 = \frac{1}{2} A, \quad \alpha^3 = 0, \quad \alpha^4 = -\frac{1}{8} A^2, \quad \cdots$$

y por tanto,

$$\sqrt{1+\varepsilon^2\left(\frac{\partial H}{\partial x}\right)^2} = 1+\varepsilon^2\frac{1}{2}\left(\frac{\partial H}{\partial x}\right)^2 - \varepsilon^4\frac{1}{8}\left(\frac{\partial H}{\partial x}\right)^4 + \cdots$$

$$\sqrt{1+\varepsilon^2\left[\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right]} = 1+\varepsilon^2\frac{1}{2}\left[\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right]$$
$$-\varepsilon^4\frac{1}{8}\left[\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right]^2 + \cdots$$

$$\sqrt{1+\varepsilon^2\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]} = 1+\varepsilon^2\frac{1}{2}\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]$$
$$-\varepsilon^4\frac{1}{8}\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]^2 + \cdots$$

Ahora sustituimos estos desarrollos en serie de potencias de ε junto con los de $T_{11}(\varepsilon)$, $T_{12}(\varepsilon)$, $T_{22}(\varepsilon)$, $f_{R_i}(\varepsilon)$ y $f_{W_i}(\varepsilon)$ $(i=1,2)$, en (5.2.12)-(5.2.15). Comenzamos por (5.2.12):

$$-\varepsilon\frac{\partial s}{\partial x}\left(T_{11}^0+\varepsilon T_{11}^1+\varepsilon^2T_{11}^2+\cdots\right)-\varepsilon\frac{\partial s}{\partial y}\left(T_{12}^0+\varepsilon T_{12}^1+\varepsilon^2T_{12}^2+\cdots\right)$$
$$+\varepsilon^{-1}T_{13}^{-1}+T_{13}^0+\varepsilon T_{13}^1+\varepsilon^2T_{13}^2+\cdots$$
$$+\varepsilon\frac{\partial s}{\partial x}\left[-\varepsilon\frac{\partial s}{\partial x}\left(\varepsilon^{-1}T_{13}^{-1}+T_{13}^0+\varepsilon T_{13}^1+\varepsilon^2T_{13}^2+\cdots\right)\right.$$
$$-\varepsilon\frac{\partial s}{\partial y}\left(\varepsilon^{-1}T_{23}^{-1}+T_{23}^0+\varepsilon T_{23}^1+\varepsilon^2T_{23}^2+\cdots\right)$$
$$\left.+\varepsilon^{-1}T_{33}^{-1}+T_{33}^0+\varepsilon T_{33}^1+\varepsilon^2T_{33}^2+\cdots\right]$$
$$=\left(\varepsilon f_{W_1}^1+\varepsilon^2f_{W_1}^2+\cdots\right)\left\{1+\varepsilon^2\frac{1}{2}\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]\right.$$
$$\left.-\varepsilon^4\frac{1}{4}\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]^2+\cdots\right\} \quad \text{en } z=1$$

Agrupando en potencias de ε:

$$\begin{aligned}
&\varepsilon^{-1}T_{13}^{-1} + T_{13}^{0} + \frac{\partial s}{\partial x}T_{33}^{-1} \\
&\quad + \varepsilon\left[-\frac{\partial s}{\partial x}T_{11}^{0} - \frac{\partial s}{\partial y}T_{12}^{0} + T_{13}^{1} + \frac{\partial s}{\partial x}\left(-\frac{\partial s}{\partial x}T_{13}^{-1} - \frac{\partial s}{\partial y}T_{23}^{-1} + T_{33}^{0}\right) - f_{W_1}^{1}\right] \\
&\quad + \varepsilon^2\left[-\frac{\partial s}{\partial x}T_{11}^{1} - \frac{\partial s}{\partial y}T_{12}^{1} + T_{13}^{2} + \frac{\partial s}{\partial x}\left(-\frac{\partial s}{\partial x}T_{13}^{0} - \frac{\partial s}{\partial y}T_{23}^{0} + T_{33}^{1}\right) - f_{W_1}^{2}\right] \\
&\quad + \varepsilon^3\left[-\frac{\partial s}{\partial x}T_{11}^{2} - \frac{\partial s}{\partial y}T_{12}^{2} + T_{13}^{3} + \frac{\partial s}{\partial x}\left(-\frac{\partial s}{\partial x}T_{13}^{1} - \frac{\partial s}{\partial y}T_{23}^{1} + T_{33}^{2}\right) - f_{W_1}^{3}\right. \\
&\quad \left. - f_{W_1}^{1}\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right)\right] + O(\varepsilon^4) = 0 \quad \text{en } z = 1
\end{aligned} \tag{5.3.9}$$

De forma análoga, a partir de (5.2.13)-(5.2.15) se obtienen:

$$\begin{aligned}
&\varepsilon^{-1}T_{23}^{-1} + T_{23}^{0} + \frac{\partial s}{\partial y}T_{33}^{-1} \\
&\quad + \varepsilon\left[-\frac{\partial s}{\partial x}T_{12}^{0} - \frac{\partial s}{\partial y}T_{22}^{0} + T_{23}^{1} + \frac{\partial s}{\partial y}\left(-\frac{\partial s}{\partial x}T_{13}^{-1} - \frac{\partial s}{\partial y}T_{23}^{-1} + T_{33}^{0}\right) - f_{W_2}^{1}\right] \\
&\quad + \varepsilon^2\left[-\frac{\partial s}{\partial x}T_{12}^{1} - \frac{\partial s}{\partial y}T_{22}^{1} + T_{23}^{2} + \frac{\partial s}{\partial y}\left(-\frac{\partial s}{\partial x}T_{13}^{0} - \frac{\partial s}{\partial y}T_{23}^{0} + T_{33}^{1}\right) - f_{W_2}^{2}\right] \\
&\quad + \varepsilon^3\left[-\frac{\partial s}{\partial x}T_{12}^{2} - \frac{\partial s}{\partial y}T_{22}^{2} + T_{23}^{3} + \frac{\partial s}{\partial y}\left(-\frac{\partial s}{\partial x}T_{13}^{1} - \frac{\partial s}{\partial y}T_{23}^{1} + T_{33}^{2}\right) - f_{W_2}^{3}\right. \\
&\quad \left. - f_{W_2}^{1}\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right)\right] + O(\varepsilon^4) = 0 \quad \text{en } z = 1
\end{aligned} \tag{5.3.10}$$

$$\begin{aligned}
&\varepsilon^{-1}T_{13}^{-1} + T_{13}^{0} + \frac{\partial H}{\partial x}T_{33}^{-1} \\
&\quad + \varepsilon\left[-\frac{\partial H}{\partial x}T_{11}^{0} - \frac{\partial H}{\partial y}T_{12}^{0} + T_{13}^{1} + \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial x}T_{13}^{-1} - \frac{\partial H}{\partial y}T_{23}^{-1} + T_{33}^{0}\right) - f_{R_1}^{1}\right] \\
&\quad + \varepsilon^2\left[-\frac{\partial H}{\partial x}T_{11}^{1} - \frac{\partial H}{\partial y}T_{12}^{1} + T_{13}^{2} + \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial x}T_{13}^{0} - \frac{\partial H}{\partial y}T_{23}^{0} + T_{33}^{1}\right) - f_{R_1}^{2}\right] \\
&\quad + \varepsilon^3\left[-\frac{\partial H}{\partial x}T_{11}^{2} - \frac{\partial H}{\partial y}T_{12}^{2} + T_{13}^{3} + \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial x}T_{13}^{1} - \frac{\partial H}{\partial y}T_{23}^{1} + T_{33}^{2}\right) - f_{R_1}^{3}\right.
\end{aligned}$$

$$- f_{R_1}^1 \left(\left(\frac{\partial H}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial H}{\partial y} \right)^2 \right) \Bigg] + O(\varepsilon^4) = 0 \quad \text{en } z = 0 \tag{5.3.11}$$

$$\varepsilon^{-1} T_{23}^{-1} + T_{23}^0 + \frac{\partial H}{\partial y} T_{33}^{-1} + \varepsilon \left[-\frac{\partial H}{\partial x} T_{12}^0 - \frac{\partial H}{\partial y} T_{22}^0 + T_{23}^1 \right.$$

$$\left. + \frac{\partial H}{\partial y} \left(-2 \frac{\partial H}{\partial x} T_{13}^{-1} - \frac{\partial H}{\partial y} T_{23}^{-1} + T_{33}^0 \right) + \left(\frac{\partial H}{\partial x} \right)^2 T_{23}^{-1} - f_{R_2}^1 \right]$$

$$+ \varepsilon^2 \left[-\frac{\partial H}{\partial x} T_{12}^1 - \frac{\partial H}{\partial y} T_{22}^1 + T_{23}^2 + \frac{\partial H}{\partial y} \left(-2 \frac{\partial H}{\partial x} T_{13}^0 - \frac{\partial H}{\partial y} T_{23}^0 + T_{33}^1 \right) \right.$$

$$\left. + \left(\frac{\partial H}{\partial x} \right)^2 T_{23}^0 - f_{R_2}^2 \right] + \varepsilon^3 \left[\frac{\partial H}{\partial x} \frac{\partial H}{\partial y} \left(\frac{\partial H}{\partial x} T_{11}^0 + \frac{\partial H}{\partial y} T_{12}^0 - T_{13}^1 \right) \right.$$

$$- \frac{\partial H}{\partial x} T_{12}^2 - \frac{\partial H}{\partial y} T_{22}^2 + T_{23}^3 - \left(\frac{\partial H}{\partial x} \right)^2 \left(\frac{\partial H}{\partial x} T_{12}^0 + \frac{\partial H}{\partial y} T_{22}^0 - T_{23}^1 \right)$$

$$+ \frac{\partial H}{\partial y} \left(-\frac{\partial H}{\partial x} T_{13}^1 - \frac{\partial H}{\partial y} T_{23}^1 + T_{33}^2 \right) - f_{R_2}^3$$

$$- f_{R_2}^1 \left(\left(\frac{3}{2} \frac{\partial H}{\partial x} \right)^2 + \left(\frac{\partial H}{\partial y} \right)^2 \right) \Bigg] + O(\varepsilon^4) = 0 \quad \text{en } z = 0 \tag{5.3.12}$$

A partir de la ecuación (5.2.19) necesaria para la determinación del calado, sustituyendo $u(\varepsilon)$ y $v(\varepsilon)$ por su desarrollo en serie de potencias de ε, se obtiene:

$$\frac{\partial h}{\partial t} + \int_0^1 \left(\frac{\partial (hu^0)}{\partial x} + \varepsilon \frac{\partial (hu^1)}{\partial x} + \varepsilon^2 \frac{\partial (hu^2)}{\partial x} + \cdots \right.$$

$$\left. + \frac{\partial (hv^0)}{\partial y} + \varepsilon \frac{\partial (hv^1)}{\partial y} + \varepsilon^2 \frac{\partial (hv^2)}{\partial y} + \cdots \right) dz = 0 \tag{5.3.13}$$

Sustituimos, ahora, los desarrollos en serie de potencias en las expresiones de las componentes del tensor de tensiones en función de las derivadas de u, v y w y de p ((5.2.20)-(5.2.25)) e identificando los términos multiplicados por cada potencia de ε tenemos:

$$T_{11}^0 - 2\mu D_x u^0 + \varepsilon \left(T_{11}^1 - 2\mu D_x u^1 \right) + \varepsilon^2 \left(T_{11}^2 - 2\mu D_x u^2 \right) = O(\varepsilon^3) \tag{5.3.14}$$

$$T_{12}^0 - \mu D_y u^0 - \mu D_x v^0 + \varepsilon \left(T_{12}^1 - \mu D_y u^1 - \mu D_x v^1 \right)$$

$$+ \varepsilon^2 \left(T_{12}^2 - \mu D_y u^2 - \mu D_x v^2 \right) = O(\varepsilon^3) \tag{5.3.15}$$

$$\varepsilon^{-1}\left(T_{13}^{-1}-\mu D_z u^0\right)+T_{13}^0-\mu\left(D_z u^1+D_x w^0\right)$$
$$+\varepsilon\left[T_{13}^1-\mu\left(D_z u^2+D_x w^1\right)\right]=O(\varepsilon^2) \tag{5.3.16}$$
$$T_{22}^0-2\mu D_y v^0+\varepsilon\left(T_{22}^1-2\mu D_y v^1\right)+\varepsilon^2\left(T_{22}^2-2\mu D_y v^2\right)=O(\varepsilon^3) \tag{5.3.17}$$
$$\varepsilon^{-1}\left(T_{23}^{-1}-\mu D_z v^0\right)+T_{23}^0-\mu\left(D_z v^1+D_y w^0\right)$$
$$+\varepsilon\left[T_{23}^1-\mu\left(D_z v^2+D_y w^1\right)\right]=O(\varepsilon^2) \tag{5.3.18}$$
$$\varepsilon^{-1}\left(T_{33}^{-1}-2\mu D_z w^0\right)+T_{33}^0-2\mu D_z w^1$$
$$+\varepsilon\left(T_{33}^1-2\mu D_z w^2\right)+\varepsilon^2\left(T_{33}^2-2\mu D_z w^3\right)=O(\varepsilon^3) \tag{5.3.19}$$

Reemplazando $u(\varepsilon)$, $v(\varepsilon)$, $w(\varepsilon)$, $T_{13}(\varepsilon)$ y $T_{23}(\varepsilon)$ por sus desarrollos en serie de potencias de ε, ((5.3.1)), en las ecuaciones (5.2.26)-(5.2.28), agrupando en potencias de ε y teniendo en cuenta que $\nu=\dfrac{\mu}{\rho_0}$ resulta:

$$\varepsilon^{-2}\left(\mu D_z^2 u^0-D_z T_{13}^{-1}\right)+\varepsilon^{-1}\left(\mu D_z^2 u^1-D_z T_{13}^0\right)$$
$$+\mu\left(D_z^2 u^2-D_x^2 u^0-D_{xy}^2 v^0\right)-D_z T_{13}^1+O(\varepsilon)=0 \tag{5.3.20}$$
$$\varepsilon^{-2}\left(\mu D_z^2 v^0-D_z T_{23}^{-1}\right)+\varepsilon^{-1}\left(\mu D_z^2 v^1-D_z T_{23}^0\right)$$
$$+\mu\left(D_z^2 v^2-D_y^2 v^0-D_{xy}^2 u^0\right)-D_z T_{23}^1+O(\varepsilon)=0 \tag{5.3.21}$$
$$\varepsilon^{-2}D_z^2 w^0+\varepsilon^{-1}\left(D_z^2 w^1+D_x T_{13}^{-1}+D_y T_{23}^{-1}\right)+\mu\left(D_z^2 w^2-D_x^2 w^0-D_y^2 w^0\right)$$
$$+D_x T_{13}^0+D_y T_{23}^0+\varepsilon\left[\mu\left(D_z^2 w^3-D_x^2 w^1-D_y^2 w^1\right)+D_x T_{13}^1+D_y T_{23}^1\right]$$
$$+O(\varepsilon^2)=0 \tag{5.3.22}$$

Razonando igual que en los capítulos anteriores, como u^0, v^0, w^0, p^0, $T_{i3}^{-1}\quad(i=1,2,3)$, $T_{ij}^0\quad(i,j=1,2,3)$, u^1, v^1, w^1, etc. son independientes de ε, una vez agrupados los términos que multiplican a una misma potencia de ε, en las ecuaciones anteriores obtenemos un polinomio en ε igualado a cero, por lo que sus coeficientes han de ser nulos. De este modo se logra una serie de ecuaciones que nos permitirán determinar los términos u^0, v^0, w^0, p^0, $T_{i3}^{-1}\quad(i=1,2,3)$, $T_{ij}^0\quad(i,j=1,2,3)$, u^1, v^1, w^1, etc.

Comenzamos por los coeficientes de ε^{-2} que aparecen en (5.3.3)-(5.3.5) y (5.3.20)-(5.3.22):

$$D_z T_{13}^{-1}=0 \tag{5.3.23}$$
$$D_z T_{23}^{-1}=0 \tag{5.3.24}$$

$$D_z^2 w^0 = 0 \tag{5.3.25}$$

$$\mu D_z^2 u^0 - D_z T_{13}^{-1} = 0 \tag{5.3.26}$$

$$\mu D_z^2 v^0 - D_z T_{23}^{-1} = 0 \tag{5.3.27}$$

Igualando a cero los coeficientes de ε^{-1} que aparecen en (5.3.3)-(5.3.6), (5.3.9)-(5.3.12), (5.3.16), (5.3.18)-(5.3.22) tenemos las siguientes igualdades:

$$w^0 D_z u^0 - \frac{1}{\rho_0} D_z T_{13}^0 = 0 \tag{5.3.28}$$

$$w^0 D_z v^0 - \frac{1}{\rho_0} D_z T_{23}^0 = 0 \tag{5.3.29}$$

$$w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2\nu D_z^2 w^1 - \frac{1}{\rho_0}\left(D_x T_{13}^{-1} + D_y T_{23}^{-1}\right) = 0 \tag{5.3.30}$$

$$D_z w^0 = 0 \tag{5.3.31}$$

$$T_{13}^{-1} = 0 \quad \text{en } z = 1 \tag{5.3.32}$$

$$T_{23}^{-1} = 0 \quad \text{en } z = 1 \tag{5.3.33}$$

$$T_{13}^{-1} = 0 \quad \text{en } z = 0 \tag{5.3.34}$$

$$T_{23}^{-1} = 0 \quad \text{en } z = 0 \tag{5.3.35}$$

$$T_{13}^{-1} - \mu D_z u^0 = 0 \tag{5.3.36}$$

$$T_{23}^{-1} - \mu D_z v^0 = 0 \tag{5.3.37}$$

$$T_{33}^{-1} - 2\mu D_z w^0 = 0 \tag{5.3.38}$$

$$\mu D_z^2 u^1 - D_z T_{13}^0 = 0 \tag{5.3.39}$$

$$\mu D_z^2 v^1 - D_z T_{23}^0 = 0 \tag{5.3.40}$$

$$D_z^2 w^1 + D_x T_{13}^{-1} + D_y T_{23}^{-1} = 0 \tag{5.3.41}$$

Como consecuencia de (5.3.23) y (5.3.34), sabemos que

$$T_{13}^{-1} = 0 \tag{5.3.42}$$

y análogamente, por (5.3.24) y (5.3.35), se tiene

$$T_{23}^{-1} = 0 \tag{5.3.43}$$

lo que implica por las igualdades (5.3.36) y (5.3.37) que u^0 y v^0 no dependen de z

$$D_z u^0 = D_z v^0 = 0 \tag{5.3.44}$$

Sustituyendo $D_z w^0 = 0$ ((5.3.31)) en (5.3.38) obtenemos

$$T_{33}^{-1} = 0 \tag{5.3.45}$$

Continuamos igualando a cero los términos que multiplican a ε^0 en las ecuaciones (5.3.3)-(5.3.22), y se tiene en cuenta a la hora de escribir estos términos que $D_z u^0 = D_z v^0 = 0$ y $T_{13}^{-1} = T_{23}^{-1} = T_{33}^{-1} = 0$

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + w^0 D_z u^1 + \frac{1}{\rho_0} D_x p^0 - 2\phi \left((\operatorname{sen}\varphi)\, v^0 - (\cos\varphi)\, w^0 \right)$$

$$- \nu \left(2 \frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y} \right) - \frac{1}{\rho_0} D_z T_{13}^1 = 0 \tag{5.3.46}$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + w^0 D_z v^1 + \frac{1}{\rho_0} D_y p^0 + 2\phi \,(\operatorname{sen}\varphi)\, u^0$$

$$- \nu \left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2 \frac{\partial^2 v^0}{\partial y^2} \right) - \frac{1}{\rho_0} D_z T_{23}^1 = 0 \tag{5.3.47}$$

$$D_t w^0 + u^0 D_x w^0 + v^0 D_y w^0 + w^0 D_z w^1 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 + g$$

$$- 2\phi \,(\cos\varphi)\, u^0 - 2\nu D_z^2 w^2 - \frac{1}{\rho_0} \left(D_x T_{13}^0 - D_y T_{23}^0 \right) = 0 \tag{5.3.48}$$

$$\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} + D_z w^1 = 0 \tag{5.3.49}$$

$$p^0 = p_s \quad \text{en } z = 1 \tag{5.3.50}$$

$$w^0 = 0 \quad \text{en } z = 0 \tag{5.3.51}$$

$$T_{13}^0 = 0 \quad \text{en } z = 1 \tag{5.3.52}$$

$$T_{23}^0 = 0 \quad \text{en } z = 1 \tag{5.3.53}$$

$$T_{13}^0 = 0 \quad \text{en } z = 0 \tag{5.3.54}$$

$$T_{23}^0 = 0 \quad \text{en } z = 0 \tag{5.3.55}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (h u^0)}{\partial x} + \frac{\partial (h v^0)}{\partial y} = 0 \tag{5.3.56}$$

$$T_{11}^0 - 2\mu\frac{\partial u^0}{\partial x} = 0 \tag{5.3.57}$$

$$T_{12}^0 - \mu\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) = 0 \tag{5.3.58}$$

$$T_{13}^0 - \mu\left(D_z u^1 + D_x w^0\right) = 0 \tag{5.3.59}$$

$$T_{22}^0 - 2\mu\frac{\partial v^0}{\partial y} = 0 \tag{5.3.60}$$

$$T_{23}^0 - \mu\left(D_z v^1 + D_y w^0\right) = 0 \tag{5.3.61}$$

$$T_{33}^0 - 2\mu D_z w^1 = 0 \tag{5.3.62}$$

$$\mu\left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} - \frac{\partial^2 v^0}{\partial x \partial y}\right) - D_z T_{13}^1 = 0 \tag{5.3.63}$$

$$\mu\left(D_z^2 v^2 - \frac{\partial^2 v^0}{\partial y^2} - \frac{\partial^2 u^0}{\partial x \partial y}\right) - D_z T_{23}^1 = 0 \tag{5.3.64}$$

$$\mu\left(D_z^2 w^2 - D_x^2 w^0 - D_y^2 w^0\right) + D_x T_{13}^0 + D_y T_{23}^0 = 0 \tag{5.3.65}$$

Veamos las consecuencias que se pueden extraer de las igualdades anteriores. En primer lugar, usando (5.3.31) y (5.3.51), se deduce que

$$w^0 = 0 \tag{5.3.66}$$

Ahora, (5.3.28) y (5.3.29) se reducen a $D_z T_{13}^0 = D_z T_{23}^0 = 0$. Teniendo en cuenta (5.3.54) y (5.3.55), podemos escribir

$$T_{13}^0 = 0 \tag{5.3.67}$$

$$T_{23}^0 = 0 \tag{5.3.68}$$

Como $w^0 = T_{13}^0 = T_{23}^0 = 0$, las igualdades (5.3.59) y (5.3.61) resultan

$$D_z u^1 = 0 \tag{5.3.69}$$

$$D_z v^1 = 0 \tag{5.3.70}$$

es decir, los términos u^1 y v^1 tampoco dependen de z.

La ecuación (5.3.41), por ser $T_{13}^{-1} = T_{23}^{-1} = 0$, nos dice que

$$D_z^2 w^1 = 0 \tag{5.3.71}$$

y entonces, por (5.3.42), (5.3.43), (5.3.66) y (5.3.71), de la expresión (5.3.30) se obtiene

$$D_z p^0 = 0 \tag{5.3.72}$$

que junto con la condición de contorno (5.3.50) nos permite obtener

$$p^0 = p_s \tag{5.3.73}$$

Teniendo en cuenta las igualdades (5.3.44), (5.3.66)-(5.3.70) y (5.3.73) se pueden reescribir las ecuaciones (5.3.46)-(5.3.49), (5.3.56)-(5.3.58), (5.3.60), (5.3.62)-(5.3.65) de la forma siguiente:

$$\begin{aligned}\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 \\ - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right) - \frac{1}{\rho_0}D_z T^1_{13} = 0\end{aligned} \tag{5.3.74}$$

$$\begin{aligned}\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 \\ - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right) - \frac{1}{\rho_0}D_z T^1_{23} = 0\end{aligned} \tag{5.3.75}$$

$$\frac{1}{\rho_0}D_z p^1 + g - 2\phi\,(\cos\varphi)\,u^0 - 2\nu D_z^2 w^2 = 0 \tag{5.3.76}$$

$$\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} + D_z w^1 = 0 \tag{5.3.77}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0 \tag{5.3.78}$$

$$T^0_{11} - 2\mu\frac{\partial u^0}{\partial x} = 0 \tag{5.3.79}$$

$$T^0_{12} - \mu\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) = 0 \tag{5.3.80}$$

$$T^0_{22} - 2\mu\frac{\partial v^0}{\partial y} = 0 \tag{5.3.81}$$

$$T^0_{33} - 2\mu D_z w^1 = 0 \tag{5.3.82}$$

$$\mu\left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} - \frac{\partial^2 v^0}{\partial x \partial y}\right) - D_z T^1_{13} = 0 \tag{5.3.83}$$

$$\mu\left(D_z^2 v^2 - \frac{\partial^2 v^0}{\partial y^2} - \frac{\partial^2 u^0}{\partial x \partial y}\right) - D_z T_{23}^1 = 0 \tag{5.3.84}$$

$$D_z^2 w^2 = 0 \tag{5.3.85}$$

Una vez más, repetimos el proceso e igualamos, ahora, a cero los coeficientes de ε que aparecen en (5.3.3)-(5.3.19) y (5.3.22). Se tiene en cuenta que $T_{13}^{-1} = T_{23}^{-1} = 0$ ((5.3.42)-(5.3.43)), $D_z u^0 = D_z u^1 = D_z v^0 = D_z v^1 = 0$ ((5.3.44),(5.3.69), (5.3.70)) y $w^0 = 0$ (5.3.66) para simplificar las igualdades, y se obtiene:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} D_x p^1 - 2\phi\left((\operatorname{sen}\varphi)\, v^1 - (\cos\varphi)\, w^1\right)$$
$$- \nu\left(2\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial x \partial y}\right) - \frac{1}{\rho_0} D_z T_{13}^2 = 0 \tag{5.3.86}$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} D_y p^1 + 2\phi\,(\operatorname{sen}\varphi)\, u^1$$
$$- \nu\left(\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2}\right) - \frac{1}{\rho_0} D_z T_{23}^2 = 0 \tag{5.3.87}$$

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\phi\,(\cos\varphi)\, u^1 - 2\nu D_z^2 w^3$$
$$- \frac{1}{\rho_0}\left(D_x T_{13}^1 + D_y T_{23}^1\right) = 0 \tag{5.3.88}$$

$$\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} + D_z w^2 = 0 \tag{5.3.89}$$

$$p^1 = 0 \quad \text{en } z = 1 \tag{5.3.90}$$

$$w^1 - u^0 \frac{\partial H}{\partial x} - v^0 \frac{\partial H}{\partial y} = 0 \quad \text{en } z = 0 \tag{5.3.91}$$

$$T_{13}^1 = f_{W_1}^1 + \frac{\partial s}{\partial x}\left(T_{11}^0 - T_{33}^0\right) + \frac{\partial s}{\partial y} T_{12}^0 \quad \text{en } z = 1 \tag{5.3.92}$$

$$T_{23}^1 = f_{W_2}^1 + \frac{\partial s}{\partial x} T_{12}^0 + \frac{\partial s}{\partial y}\left(T_{22}^0 - T_{33}^0\right) \quad \text{en } z = 1 \tag{5.3.93}$$

$$T_{13}^1 = f_{R_1}^1 + \frac{\partial H}{\partial x}\left(T_{11}^0 - T_{33}^0\right) + \frac{\partial H}{\partial y} T_{12}^0 \quad \text{en } z = 0 \tag{5.3.94}$$

$$T_{23}^1 = f_{R_2}^1 + \frac{\partial H}{\partial x} T_{12}^0 + \frac{\partial H}{\partial y}\left(T_{22}^0 - T_{33}^0\right) \quad \text{en } z = 0 \tag{5.3.95}$$

$$\frac{\partial(hu^1)}{\partial x} + \frac{\partial(hv^1)}{\partial y} = 0 \tag{5.3.96}$$

$$T_{11}^1 = 2\mu\frac{\partial u^1}{\partial x} \tag{5.3.97}$$

$$T_{12}^1 = \mu\left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right) \tag{5.3.98}$$

$$T_{13}^1 = \mu\left(D_z u^2 + D_x w^1\right) \tag{5.3.99}$$

$$T_{22}^1 = 2\mu\frac{\partial v^1}{\partial y} \tag{5.3.100}$$

$$T_{23}^1 = \mu\left(D_z v^2 + D_y w^1\right) \tag{5.3.101}$$

$$T_{33}^1 = 2\mu D_z w^2 \tag{5.3.102}$$

$$\mu\left(D_z^2 w^3 - D_x^2 w^1 - D_y^2 w^1\right) + D_x T_{13}^1 + D_y T_{23}^1 = 0 \tag{5.3.103}$$

La ecuación (5.3.76), si tenemos en cuenta (5.3.85), se reduce a

$$\frac{1}{\rho_0} D_z p^1 + g - 2\phi\left(\cos\varphi\right) u^0 = 0$$

Integrando respecto a z esta igualdad e imponiendo la condición (5.3.90), nos proporciona la siguiente expresión para el término de orden 1 de la presión:

$$p^1 = \rho_0 h\left(g - 2\phi\left(\cos\varphi\right) u^0\right)(1 - z) \tag{5.3.104}$$

Integramos también (5.3.77) respecto a z teniendo en cuenta que u^0 y v^0 no dependen de z, e imponiendo la condición (5.3.91), encontramos la expresión siguiente para w^1 en términos de u^0, v^0 y H:

$$w^1 = u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - zh\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \tag{5.3.105}$$

Igualando a cero los coeficientes de ε^2 que aparecen en (5.3.3)-(5.3.4), (5.3.6)-(5.3.15), (5.3.17) y (5.3.19) y usando (5.3.44), (5.3.66)-(5.3.70) tenemos las siguientes igualdades:

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + v^0 D_y u^2 + v^1 \frac{\partial u^1}{\partial y} + v^2 \frac{\partial u^0}{\partial y} + w^1 D_z u^2$$

$$+ \frac{1}{\rho_0} D_x p^2 - 2\phi \left((\operatorname{sen} \varphi)\, v^2 - (\cos \varphi)\, w^2 \right) - \nu \left(2D_x^2 u^2 + D_y^2 u^2 + D_{xy}^2 v^2 \right)$$

$$- \frac{1}{\rho_0} D_z T_{13}^3 = 0 \tag{5.3.106}$$

$$D_t v^2 + u^0 D_x v^2 + u^1 \frac{\partial v^1}{\partial x} + u^2 \frac{\partial v^0}{\partial x} + v^0 D_y v^2 + v^1 \frac{\partial v^1}{\partial y} + v^2 \frac{\partial v^0}{\partial y} + w^1 D_z v^2 + \frac{1}{\rho_0} D_y p^2$$

$$+ 2\phi \,(\operatorname{sen} \varphi)\, u^2 - \nu \left(D_{xy}^2 u^2 + D_x^2 v^2 + 2D_y^2 v^2 \right) - \frac{1}{\rho_0} D_z T_{23}^3 = 0 \tag{5.3.107}$$

$$D_x u^2 + D_y v^2 + D_z w^3 = 0 \tag{5.3.108}$$

$$p^2 = 0 \quad \text{en } z = 1 \tag{5.3.109}$$

$$w^2 - u^1 \frac{\partial H}{\partial x} - v^1 \frac{\partial H}{\partial y} = 0 \quad \text{en } z = 0 \tag{5.3.110}$$

$$T_{13}^2 = f_{W_1}^2 + \frac{\partial s}{\partial x} \left(T_{11}^1 - T_{33}^1 \right) + \frac{\partial s}{\partial y} T_{12}^1 \quad \text{en } z = 1 \tag{5.3.111}$$

$$T_{23}^2 = f_{W_2}^2 + \frac{\partial s}{\partial x} T_{12}^1 + \frac{\partial s}{\partial y} \left(T_{22}^1 - T_{33}^1 \right) \quad \text{en } z = 1 \tag{5.3.112}$$

$$T_{13}^2 = f_{R_1}^2 + \frac{\partial H}{\partial x} \left(T_{11}^1 - T_{33}^1 \right) + \frac{\partial H}{\partial y} T_{12}^1 \quad \text{en } z = 0 \tag{5.3.113}$$

$$T_{23}^2 = f_{R_2}^2 + \frac{\partial H}{\partial x} T_{12}^1 + \frac{\partial H}{\partial y} \left(T_{22}^1 - T_{33}^1 \right) \quad \text{en } z = 0 \tag{5.3.114}$$

$$\int_0^1 \left(\frac{\partial (h u^2)}{\partial x} + \frac{\partial (h v^2)}{\partial y} \right) dz = 0 \tag{5.3.115}$$

$$T_{11}^2 = 2\mu D_x u^2 \tag{5.3.116}$$

$$T_{12}^2 = \mu \left(D_y u^2 + D_x v^2 \right) \tag{5.3.117}$$

$$T_{22}^2 = 2\mu D_y v^2 \tag{5.3.118}$$

$$T_{33}^2 = 2\mu D_z w^3 \tag{5.3.119}$$

Procediendo del mismo modo que para obtener la expresión (5.3.105) obtenemos también w^2 en función de u^1, v^1 y H (en este caso se integra (5.3.89) y se impone la condición (5.3.110)):

$$w^2 = u^1\frac{\partial H}{\partial x} + v^1\frac{\partial H}{\partial y} - zh\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right) \tag{5.3.120}$$

En resumen, tenemos las siguientes ecuaciones, igualdades y condiciones para el cálculo de h, $T_{i3}^{-1}(i = 1, 2, 3)$, u^k, v^k, w^k, p^k y $T_{i,j}^k$ $(k = 0, 1, 2, \cdots, i, j = 1, 2)$ que nos permitirán construir una aproximación de la solución del problema (5.2.6)-(5.2.19):

$$T_{13}^{-1} = 0 \tag{5.3.121}$$

$$T_{23}^{-1} = 0 \tag{5.3.122}$$

$$D_z u^0 = 0 \tag{5.3.123}$$

$$D_z v^0 = 0 \tag{5.3.124}$$

$$T_{33}^{-1} = 0 \tag{5.3.125}$$

$$w^0 = 0 \tag{5.3.126}$$

$$T_{13}^{0} = 0 \tag{5.3.127}$$

$$T_{23}^{0} = 0 \tag{5.3.128}$$

$$D_z u^1 = 0 \tag{5.3.129}$$

$$D_z v^1 = 0 \tag{5.3.130}$$

$$p^0 = p_s \tag{5.3.131}$$

$$\begin{aligned}&\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 \\ &\qquad - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right) - \frac{1}{\rho_0} D_z T_{13}^1 = 0\end{aligned} \tag{5.3.132}$$

$$\begin{aligned}&\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 \\ &\qquad - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right) - \frac{1}{\rho_0} D_z T_{23}^1 = 0\end{aligned} \tag{5.3.133}$$

$$p^1 = \rho_0 h\left(g - 2\phi\,(\cos\varphi)\,u^0\right)(1 - z) \tag{5.3.134}$$

$$w^1 = u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - zh \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \tag{5.3.135}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0 \tag{5.3.136}$$

$$T_{11}^0 = 2\mu \frac{\partial u^0}{\partial x} \tag{5.3.137}$$

$$T_{12}^0 = \mu \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \tag{5.3.138}$$

$$T_{22}^0 = 2\mu \frac{\partial v^0}{\partial y} \tag{5.3.139}$$

$$T_{33}^0 = 2\mu D_z w^1 \tag{5.3.140}$$

$$\mu \left(D_z^2 u^2 - \frac{\partial^2 u^0}{\partial x^2} - \frac{\partial^2 v^0}{\partial x \partial y} \right) = D_z T_{13}^1 \tag{5.3.141}$$

$$\mu \left(D_z^2 v^2 - \frac{\partial^2 v^0}{\partial y^2} - \frac{\partial^2 u^0}{\partial x \partial y} \right) = D_z T_{23}^1 \tag{5.3.142}$$

$$\begin{aligned} &\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} D_x p^1 - 2\phi \left((\operatorname{sen} \varphi)\, v^1 - (\cos \varphi)\, w^1 \right) \\ &\quad - \nu \left(2 \frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial x \partial y} \right) - \frac{1}{\rho_0} D_z T_{13}^2 = 0 \end{aligned} \tag{5.3.143}$$

$$\begin{aligned} &\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} D_y p^1 + 2\phi (\operatorname{sen} \varphi)\, u^1 \\ &\quad - \nu \left(\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 v^1}{\partial x^2} + 2 \frac{\partial^2 v^1}{\partial y^2} \right) - \frac{1}{\rho_0} D_z T_{23}^2 = 0 \end{aligned} \tag{5.3.144}$$

$$\begin{aligned} &D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\phi (\cos \varphi)\, u^1 - 2\nu D_z^2 w^3 \\ &\quad - \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1 \right) = 0 \end{aligned} \tag{5.3.145}$$

$$w^2 = u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - zh \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \tag{5.3.146}$$

$$T_{13}^1 = f_{W_1}^1 + \frac{\partial s}{\partial x}\left(T_{11}^0 - T_{33}^0\right) + \frac{\partial s}{\partial y}T_{12}^0 \quad \text{en } z = 1 \tag{5.3.147}$$

$$T_{23}^1 = f_{W_2}^1 + \frac{\partial s}{\partial x}T_{12}^0 + \frac{\partial s}{\partial y}\left(T_{22}^0 - T_{33}^0\right) \quad \text{en } z = 1 \tag{5.3.148}$$

$$T_{13}^1 = f_{R_1}^1 + \frac{\partial H}{\partial x}\left(T_{11}^0 - T_{33}^0\right) + \frac{\partial H}{\partial y}T_{12}^0 \quad \text{en } z = 0 \tag{5.3.149}$$

$$T_{23}^1 = f_{R_2}^1 + \frac{\partial H}{\partial x}T_{12}^0 + \frac{\partial H}{\partial y}\left(T_{22}^0 - T_{33}^0\right) \quad \text{en } z = 0 \tag{5.3.150}$$

$$\frac{\partial(hu^1)}{\partial x} + \frac{\partial(hv^1)}{\partial y} = 0 \tag{5.3.151}$$

$$T_{11}^1 = 2\mu\frac{\partial u^1}{\partial x} \tag{5.3.152}$$

$$T_{12}^1 = \mu\left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right) \tag{5.3.153}$$

$$T_{13}^1 = \mu\left(D_z u^2 + D_x w^1\right) \tag{5.3.154}$$

$$T_{22}^1 = 2\mu\frac{\partial v^1}{\partial y} \tag{5.3.155}$$

$$T_{23}^1 = \mu\left(D_z v^2 + D_y w^1\right) \tag{5.3.156}$$

$$T_{33}^1 = 2\mu D_z w^2 \tag{5.3.157}$$

$$\mu\left(D_z^2 w^3 - D_x^2 w^1 - D_y^2 w^1\right) + D_x T_{13}^1 + D_y T_{23}^1 = 0 \tag{5.3.158}$$

$$\begin{aligned} &D_t u^2 + u^0 D_x u^2 + u^1\frac{\partial u^1}{\partial x} + u^2\frac{\partial u^0}{\partial x} + v^0 D_y u^2 + v^1\frac{\partial u^1}{\partial y} + v^2\frac{\partial u^0}{\partial y} + w^1 D_z u^2 \\ &\quad + \frac{1}{\rho_0}D_x p^2 - 2\phi\left((\operatorname{sen}\varphi)\, v^2 - (\cos\varphi)\, w^2\right) - \nu\left(2D_x^2 u^2 + D_y^2 u^2 + D_{xy}^2 v^2\right) \\ &\quad - \frac{1}{\rho_0}D_z T_{13}^3 = 0 \end{aligned} \tag{5.3.159}$$

$$\begin{aligned} &D_t v^2 + u^0 D_x v^2 + u^1\frac{\partial v^1}{\partial x} + u^2\frac{\partial v^0}{\partial x} + v^0 D_y v^2 + v^1\frac{\partial v^1}{\partial y} + v^2\frac{\partial v^0}{\partial y} + w^1 D_z v^2 + \frac{1}{\rho_0}D_y p^2 \\ &\quad + 2\phi\,(\operatorname{sen}\varphi)\, u^2 - \nu\left(D_{xy}^2 u^2 + D_x^2 v^2 + 2D_y^2 v^2\right) - \frac{1}{\rho_0}D_z T_{23}^3 = 0 \end{aligned} \tag{5.3.160}$$

$$D_x u^2 + D_y v^2 + D_z w^3 = 0 \tag{5.3.161}$$

$$p^2 = 0 \quad \text{en } z = 1 \tag{5.3.162}$$

$$T_{13}^2 = f_{W_1}^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{33}^1\right) + \frac{\partial s}{\partial y} T_{12}^1 \quad \text{en } z = 1 \tag{5.3.163}$$

$$T_{23}^2 = f_{W_2}^2 + \frac{\partial s}{\partial x} T_{12}^1 + \frac{\partial s}{\partial y}\left(T_{22}^1 - T_{33}^1\right) \quad \text{en } z = 1 \tag{5.3.164}$$

$$T_{13}^2 = f_{R_1}^2 + \frac{\partial H}{\partial x}\left(T_{11}^1 - T_{33}^1\right) + \frac{\partial H}{\partial y} T_{12}^1 \quad \text{en } z = 0 \tag{5.3.165}$$

$$T_{23}^2 = f_{R_2}^2 + \frac{\partial H}{\partial x} T_{12}^1 + \frac{\partial H}{\partial y}\left(T_{22}^1 - T_{33}^1\right) \quad \text{en } z = 0 \tag{5.3.166}$$

$$\int_0^1 \left(\frac{\partial(hu^2)}{\partial x} + \frac{\partial(hv^2)}{\partial y}\right) dz = 0 \tag{5.3.167}$$

$$T_{11}^2 = 2\mu D_x u^2 \tag{5.3.168}$$

$$T_{12}^2 = \mu\left(D_y u^2 + D_x v^2\right) \tag{5.3.169}$$

$$T_{22}^2 = 2\mu D_y v^2 \tag{5.3.170}$$

$$T_{33}^2 = 2\mu D_z w^3 \tag{5.3.171}$$

5.4. Aproximación de orden cero

Se considera la aproximación de orden cero:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{v}(\varepsilon) = v^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

$$\tilde{T}_{ij}(\varepsilon) = T_{ij}^0 \quad (i, j = 1, 2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1} T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 \quad (i = 1, 2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1} T_{33}^{-1} + T_{33}^0$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^1 \quad (i = 1, 2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} \quad (i = 1, 2)$$

donde w^0, p^0, $T^k_{i3} \quad (k = -1, 0, i = 1, 2)$ y T^{-1}_{33} son conocidos ((5.3.121), (5.3.122), (5.3.125)-(5.3.128), (5.3.131).

Calculamos u^0 y v^0 a partir de (5.3.132)-(5.3.133)

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\, v^0$$

$$- \nu \left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y} \right) - \frac{1}{\rho_0} D_z T^1_{13} = 0 \tag{5.4.1}$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^0$$

$$- \nu \left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2} \right) - \frac{1}{\rho_0} D_z T^1_{23} = 0 \tag{5.4.2}$$

donde deberíamos conocer previamente T^1_{13} y T^1_{23}. Para obtener estos términos se podrían emplear las ecuaciones (5.3.154) y (5.3.156), pero u^2, v^2 y w^1 no son conocidos. Los términos $T^1_{i3} \quad (i = 1, 2)$ sólo los podemos obtener con esta aproximación en $z = 0$ y $z = 1$. Para solventar este inconveniente se integran las ecuaciones respecto de z entre 0 y 1. Se aprovecha el hecho de que u^0, v^0 y p_s no dependen de z ((5.3.123), (5.3.124), (5.3.131)). Y así,

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\, v^0$$

$$- \nu \left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y} \right) - \frac{1}{\rho_0 h} \left(T^1_{13}\big|_{z=1} - T^1_{13}\big|_{z=0} \right) = 0$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^0$$

$$- \nu \left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2} \right) - \frac{1}{\rho_0 h} \left(T^1_{23}\big|_{z=1} - T^1_{23}\big|_{z=0} \right) = 0$$

Se utilizan ahora las ecuaciones (5.3.147)-(5.3.150) para obtener $T^1_{i3} \quad (i = 1, 2)$ en $z = 1$ y $z = 0$:

$$T^1_{13} = f^1_{W_1} + \frac{\partial s}{\partial x} \left(T^0_{11} - T^0_{33} \right) + \frac{\partial s}{\partial y} T^0_{12} \quad \text{en } z = 1 \tag{5.4.3}$$

$$T^1_{23} = f^1_{W_2} + \frac{\partial s}{\partial x} T^0_{12} + \frac{\partial s}{\partial y} \left(T^0_{22} - T^0_{33} \right) \quad \text{en } z = 1 \tag{5.4.4}$$

$$T_{13}^1 = f_{R_1}^1 + \frac{\partial H}{\partial x}\left(T_{11}^0 - T_{33}^0\right) + \frac{\partial H}{\partial y} T_{12}^0 \quad \text{en } z = 0 \tag{5.4.5}$$

$$T_{23}^1 = f_{R_2}^1 + \frac{\partial H}{\partial x} T_{12}^0 + \frac{\partial H}{\partial y}\left(T_{22}^0 - T_{33}^0\right) \quad \text{en } z = 0 \tag{5.4.6}$$

donde es necesario conocer T_{11}^0, T_{12}^0, T_{22}^0 y T_{33}^0 en $z = 0$ y $z = 1$. Esta información nos la proporcionan las igualdades (5.3.137)-(5.3.140) (donde hemos hecho la sustitución $\mu = \nu\rho_0$):

$$T_{11}^0 = 2\nu\rho_0 \frac{\partial u^0}{\partial x} \tag{5.4.7}$$

$$T_{12}^0 = \nu\rho_0 \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) \tag{5.4.8}$$

$$T_{22}^0 = 2\nu\rho_0 \frac{\partial v^0}{\partial y} \tag{5.4.9}$$

$$T_{33}^0 = 2\nu\rho_0 D_z w^1 \tag{5.4.10}$$

y donde w^1 se puede sustituir por su expresión (5.3.135) de modo que T_{33}^0 se exprese también en términos de u^0 y de v^0:

$$T_{33}^0 = -2\nu\rho_0 \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) \tag{5.4.11}$$

Entonces, como u^0 y v^0 no dependen de z:

$$\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\, v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)$$
$$- \frac{1}{h\rho_0}(f_{W_1}^1 - f_{R_1}^1) - \frac{\nu}{h}\left[2\frac{\partial h}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right] = 0$$

$$\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)$$
$$- \frac{1}{h\rho_0}(f_{W_2}^1 - f_{R_2}^1) - \frac{\nu}{h}\left[2\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right] = 0$$

y que también se puede escribir:

$$\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} - \nu\left\{\Delta_{xy} u^0 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\right.$$
$$\left. + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right\} = -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + 2\phi\,(\operatorname{sen}\varphi)\, v^0 + \frac{1}{h\rho_0}(f_{W_1}^1 - f_{R_1}^1) \tag{5.4.12}$$

$$\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} - \nu \left\{ \Delta_{xy} v^0 + \frac{2}{h} \frac{\partial h}{\partial y} \frac{\partial v^0}{\partial y} + \frac{1}{h^2} \frac{\partial}{\partial y} \left[h^2 \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] \right.$$
$$\left. + \frac{1}{h} \frac{\partial h}{\partial x} \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right\} = -\frac{1}{\rho_0} \frac{\partial p_s}{\partial y} - 2\phi \left(\operatorname{sen} \varphi \right) u^0 + \frac{1}{h\rho_0} (f^1_{W_2} - f^1_{R_2}) \quad (5.4.13)$$

donde $\Delta_{xy} F = \dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2}$.

Acoplada a este sistema de ecuaciones debemos resolver también la ecuación (5.3.136)

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^0)}{\partial x} + \frac{\partial (hv^0)}{\partial y} = 0$$

Una vez conocidos u^0, v^0 y h, w^1 viene dado por (5.3.135)

$$w^1 = u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - zh \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right)$$

Si ahora deshacemos el cambio de variable, volviendo al dominio original, la aproximación de la solución en Ω^ε

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, y, z) = u^0(t, x, y)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t, x, y, z) = v^0(t, x, y)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, y, z) = \varepsilon w^1(t, x, y, z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, y, z) = p^0(t, x, y)$$

con

$$\tilde{f}^\varepsilon_{R_i} = \tilde{f}_{R_i}(\varepsilon) = \varepsilon f^1_{R_i} \quad (i = 1, 2)$$

$$\tilde{f}^\varepsilon_{W_i} = \tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} \quad (i = 1, 2)$$

verifica,

$$\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon} = 0 \quad (5.4.14)$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial z^\varepsilon} = 0 \quad (5.4.15)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \quad (5.4.16)$$

$$\tilde{p}^\varepsilon = p_s \quad (5.4.17)$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial(h^\varepsilon\tilde{u}^\varepsilon)}{\partial x^\varepsilon}+\frac{\partial(h^\varepsilon\tilde{v}^\varepsilon)}{\partial y^\varepsilon}=0 \tag{5.4.18}$$

$$\frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{u}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$

$$\left.+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}=-\frac{1}{\rho_0}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1}-\tilde{f}^\varepsilon_{R_1}) \tag{5.4.19}$$

$$\frac{\partial\tilde{v}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{v}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$

$$\left.+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}=-\frac{1}{\rho_0}\frac{\partial\tilde{p}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2}-\tilde{f}^\varepsilon_{R_2}) \tag{5.4.20}$$

donde $\Delta_{x^\varepsilon y^\varepsilon}F^\varepsilon=\dfrac{\partial^2F^\varepsilon}{\partial(x^\varepsilon)^2}+\dfrac{\partial^2F^\varepsilon}{\partial(y^\varepsilon)^2}$.

Veamos en qué medida verifica la aproximación de orden cero las ecuaciones de Navier-Stokes de partida.

Si sustituimos en la primera ecuación de Navier-Stokes obtenemos

$$\frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial z^\varepsilon}=\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}$$

$$=\nu\left\{\Delta_{xy}u^0+\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x}+\frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\right.$$

$$\left.+\frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right]\right\}-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^0+\frac{1}{h\rho_0}(f^1_{W_1}-f^1_{R_1})$$

$$=\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{u}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$

$$\left.+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}-\frac{1}{\rho_0}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1}-\tilde{f}^\varepsilon_{R_1}) \tag{5.4.21}$$

de modo que la primera ecuación de Navier-Stokes (5.1.29) se verificaría de forma exacta si

$$\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1}-\tilde{f}^\varepsilon_{R_1})+\frac{\nu}{h^\varepsilon}\left[2\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}\right)\right]$$

$$=\frac{1}{\rho_0}\frac{\partial\tilde{T}^\varepsilon_{13}}{\partial z^\varepsilon}-2\phi\,(\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon \tag{5.4.22}$$

donde llamamos $\tilde{T}^\varepsilon_{13}=\varepsilon T^1_{13}$. Puesto que T^1_{13} no es conocido para todo z, no podemos garantizar que se verifique (5.4.22), pero si lo integramos respecto a z^ε (teniendo en cuenta que $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$, h^ε, $\tilde{f}^\varepsilon_{W_1}$, $\tilde{f}^\varepsilon_{R_1}$ no dependen de z^ε) obtenemos:

$$\frac{1}{\rho_0}\left(\tilde{T}^\varepsilon_{13}\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}^\varepsilon_{13}\Big|_{z^\varepsilon=H^\varepsilon}\right) = \frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1})$$
$$+\frac{\nu}{h^\varepsilon}\left[2\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right)\right]+O(\varepsilon) \qquad (5.4.23)$$

que se verifica sin más que tener en cuenta (5.3.137), (5.3.138), (5.3.140), (5.3.147), (5.3.149) y $2\phi\,(\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon = 2\phi\,(\cos\varphi)\,\varepsilon w^1 = O(\varepsilon)$.

Por tanto, no podemos saber con qué precisión nuestro modelo verifica la primera ecuación de Navier-Stokes, pero sí que el promedio en altura de dicha ecuación se verifica con un error de orden ε:

$$\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial z^\varepsilon}+\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^\varepsilon\right.$$
$$\left.+\,2\phi\,(\cos\varphi^\varepsilon)\,w^\varepsilon-\nu\left(2\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2}+\frac{\partial^2 u^\varepsilon}{\partial (y^\varepsilon)^2}+\frac{\partial^2 v^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right)-\frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{13}}{\partial z^\varepsilon}\right]dz^\varepsilon = O(\varepsilon)$$

Veamos qué sucede con la segunda ecuación de Navier-Stokes:

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon}+\tilde{u}^\varepsilon\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}+\tilde{w}^\varepsilon\frac{\partial \tilde{v}^\varepsilon}{\partial z^\varepsilon}=\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}$$
$$=\nu\left\{\Delta_{xy}v^0+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^0}{\partial y}+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]\right.$$
$$\left.+\frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right\}-\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}-2\phi\,(\operatorname{sen}\varphi)\,u^0+\frac{1}{h\rho_0}(f^1_{W_2}-f^1_{R_2})$$
$$=\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{v}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$
$$\left.+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\}-\frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi)\,\tilde{u}^\varepsilon+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2}-\tilde{f}^\varepsilon_{R_2}) \qquad (5.4.24)$$

La ecuación (5.1.30) se verificaría de forma exacta si

$$\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2}-\tilde{f}^\varepsilon_{R_2})+\frac{\nu}{h^\varepsilon}\left[2\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon}+2\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right)\right]=\frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{23}}{\partial z^\varepsilon} \qquad (5.4.25)$$

donde llamamos $\tilde{T}^\varepsilon_{23} = \varepsilon T^1_{23}$. Como T^1_{23} no es conocido para todo z, no podemos garantizar que se verifique (5.4.25), pero si lo integramos respecto a z^ε (teniendo en cuenta que $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$, h^ε, $\tilde{f}^\varepsilon_{W_2}$, $\tilde{f}^\varepsilon_{R_2}$ no dependen de z^ε) obtenemos:

$$\frac{1}{\rho_0}\left(\tilde{T}^\varepsilon_{23}\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}^\varepsilon_{23}\Big|_{z^\varepsilon=H^\varepsilon}\right) = \frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2})$$

$$+ \frac{\nu}{h^\varepsilon}\left[2\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + 2\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right)\right] \tag{5.4.26}$$

que se verifica sin más que tener en cuenta (5.3.138), (5.3.139), (5.3.140), (5.3.148) y (5.3.150).

Desconocemos, por tanto, si nuestro modelo verifica de forma exacta la segunda ecuación de Navier-Stokes pero podemos afirmar que sí verifica dicha ecuación promediada en altura.

La tercera ecuación se verifica con un error de orden ε^0, es decir, no se verifica ni tan siquiera aproximadamente, mientras que la condición de incompresibilidad se verifica exactamente, como se deduce de (5.4.16):

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = 0$$

Es inmediato comprobar que las condiciones de contorno (5.1.4) y (5.1.5) se verifican exactamente (teniendo en cuenta (5.4.16) y (5.4.17)).

Las condiciones de contorno (5.1.35)-(5.1.38), que recogen el efecto del viento en la superficie y el rozamiento en el fondo, se verifican con un orden de precisión de ε^3, ya que si tenemos en cuenta (5.3.147), para la condición (5.1.35) obtenemos (donde $\tilde{T}^\varepsilon_{ij} = T^0_{ij} \quad (i,j=1,2)$)

$$\left[-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\tilde{T}^\varepsilon_{11} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon}\tilde{T}^\varepsilon_{12} + \tilde{T}^\varepsilon_{13} + \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\tilde{T}^\varepsilon_{13} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon}\tilde{T}^\varepsilon_{23} + \tilde{T}^\varepsilon_{33}\right)\right]_{z^\varepsilon=s^\varepsilon}$$

$$- \tilde{f}^\varepsilon_{W_1}\sqrt{1 + \left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\right)^2}$$

$$= \left[-\varepsilon\frac{\partial s}{\partial x}T^0_{11} - \varepsilon\frac{\partial s}{\partial y}T^0_{12} + \varepsilon T^1_{13} + \varepsilon\frac{\partial s}{\partial x}\left(-\varepsilon^2\frac{\partial s}{\partial x}T^1_{13} - \varepsilon^2\frac{\partial s}{\partial y}T^1_{23} + T^0_{33}\right)\right]_{z=1}$$

$$- \varepsilon f^1_{W_1}\sqrt{1 + \varepsilon^2\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right]}$$

$$= \left[\varepsilon\left(-\frac{\partial s}{\partial x}T^0_{11} - \frac{\partial s}{\partial y}T^0_{12} + T^1_{13} + \frac{\partial s}{\partial x}T^0_{33}\right) - \varepsilon^3\frac{\partial s}{\partial x}\left(\frac{\partial s}{\partial x}T^1_{13} + \frac{\partial s}{\partial y}T^1_{23}\right)\right]_{z=1}$$

$$- \varepsilon f^1_{W_1}\left\{1 + \varepsilon^2\frac{1}{2}\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right]\right\} + O(\varepsilon^5)$$

$$= \varepsilon\left[-\frac{\partial s}{\partial x}T_{11}^0 - \frac{\partial s}{\partial y}T_{12}^0 + T_{13}^1 + \frac{\partial s}{\partial x}T_{33}^0 - f_{W_1}^1\right]_{z=1} + O(\varepsilon^3) = O(\varepsilon^3)$$

Se puede razonar de modo análogo para las condiciones (5.1.36)-(5.1.38) teniendo en cuenta (5.3.147)-(5.3.150).

Se propone un primer modelo que viene dado por las ecuaciones (5.4.14)-(5.4.20):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(h^\varepsilon \tilde{v}^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{5.4.27}$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$

$$\left. + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\tilde{v}^\varepsilon + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon) \tag{5.4.28}$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\tilde{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$

$$\left. + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\tilde{u}^\varepsilon + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_2}^\varepsilon - \tilde{f}_{R_2}^\varepsilon) \tag{5.4.29}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.4.30}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{5.4.31}$$

(donde h^ε, u^ε y v^ε son independientes de z^ε).

5.5. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{T}_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 \quad (i, j = 1, 2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1}T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 + \varepsilon^2 T_{i3}^2 \quad (i = 1, 2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1}T_{33}^{-1} + T_{33}^0 + \varepsilon T_{33}^1$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f^1_{R_i} + \varepsilon^2 f^2_{R_i} \quad (i = 1, 2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} + \varepsilon^2 f^2_{W_i} \quad (i = 1, 2)$$

Recordemos que $w^0, p^0, T^k_{i3} \quad (k = -1, 0, i = 1, 2)$ y T^{-1}_{33} son conocidos ((5.3.121), (5.3.122), (5.3.125)-(5.3.128), (5.3.131)), u^0, v^0 y h se calculan resolviendo (5.4.12), (5.4.13) y (5.3.136) y w^1 está determinado por (5.3.135) en función de u^0, v^0 y h.

También tenemos una expresión para p^1 en términos de la profundidad del agua y u^0 ((5.3.134)):

$$p^1 = \rho_0 h \left(g - 2\phi \left(\cos\varphi\right) u^0\right)(1 - z)$$

Para obtener u^1 y v^1 resolvemos (5.3.143)-(5.3.144):

$$\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\left(\left(\operatorname{sen}\varphi\right) v^1 - \left(\cos\varphi\right) w^1\right)$$

$$- \nu\left(2\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial x \partial y}\right) = -\frac{1}{\rho_0} D_x p^1 + \frac{1}{\rho_0} D_z T^2_{13}$$

$$\frac{\partial v^1}{\partial t} + u^0\frac{\partial v^1}{\partial x} + u^1\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^1}{\partial y} + v^1\frac{\partial v^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right) u^1$$

$$- \nu\left(\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2}\right) = -\frac{1}{\rho_0} D_y p^1 + \frac{1}{\rho_0} D_z T^2_{23}$$

Se sustituyen w^1 y p^1 por sus expresiones de modo que la dependencia de z sea explícita:

$$\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\left(\operatorname{sen}\varphi\right) v^1$$

$$- \nu\left(2\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial y \partial x}\right) = -\frac{\partial s}{\partial x} g$$

$$+ 2\phi\left(\cos\varphi\right)\left(\frac{\partial u^0}{\partial x} h + \frac{\partial h}{\partial x} u^0 - v^0\frac{\partial H}{\partial y} + zh\frac{\partial v^0}{\partial y}\right) + \frac{1}{\rho_0} D_z T^2_{13}$$

$$\frac{\partial v^1}{\partial t} + u^0\frac{\partial v^1}{\partial x} + u^1\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^1}{\partial y} + v^1\frac{\partial v^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right) u^1$$

$$- \nu\left(\frac{\partial^2 u^1}{\partial y \partial x} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2}\right) = -\frac{\partial s}{\partial y} g$$

$$+ 2\phi\left(\frac{\partial}{\partial y}\left(\left(\cos\varphi\right) u^0\right) h(1 - z) + \frac{\partial s}{\partial y}\left(\cos\varphi\right) u^0\right) + \frac{1}{\rho_0} D_z T^2_{23}$$

De nuevo, T_{13}^2 y T_{23}^2 sólo son conocidos en $z = 0$ y $z = 1$. Como u^0, v^0, u^1, v^1 y s no dependen de z, se integra la ecuación respecto de z entre 0 y 1:

$$\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^1$$

$$-\,\nu\left(2\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial y\partial x}\right) = -\frac{\partial s}{\partial x}g$$

$$+\,2\phi\,(\cos\varphi)\left(\frac{\partial u^0}{\partial x}h + \frac{\partial h}{\partial x}u^0 - v^0\frac{\partial H}{\partial y} + \frac{1}{2}h\frac{\partial v^0}{\partial y}\right) + \frac{1}{h\rho_0}\left(T_{13}^2\big|_{z=1} - T_{13}^2\big|_{z=0}\right)$$

$$\frac{\partial v^1}{\partial t} + u^0\frac{\partial v^1}{\partial x} + u^1\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^1}{\partial y} + v^1\frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^1$$

$$-\,\nu\left(\frac{\partial^2 u^1}{\partial y\partial x} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2}\right) = -\frac{\partial s}{\partial y}g$$

$$+\,2\phi\left(\frac{1}{2}\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h + \frac{\partial s}{\partial y}(\cos\varphi)\,u^0\right) + \frac{1}{h\rho_0}\left(T_{23}^2\big|_{z=1} - T_{23}^2\big|_{z=0}\right)$$

Se usan ahora las ecuaciones (5.3.163)-(5.3.166) para obtener $T_{i3}^2 \quad (i = 1, 2)$ en 1 y 0:

$$T_{13}^2 = f_{W_1}^2 + \frac{\partial s}{\partial x}\left(T_{11}^1 - T_{33}^1\right) + \frac{\partial s}{\partial y}T_{12}^1 \qquad \text{en } z = 1 \tag{5.5.1}$$

$$T_{23}^2 = f_{W_2}^2 + \frac{\partial s}{\partial x}T_{12}^1 + \frac{\partial s}{\partial y}\left(T_{22}^1 - T_{33}^1\right) \qquad \text{en } z = 1 \tag{5.5.2}$$

$$T_{13}^2 = f_{R_1}^2 + \frac{\partial H}{\partial x}\left(T_{11}^1 - T_{33}^1\right) + \frac{\partial H}{\partial y}T_{12}^1 \qquad \text{en } z = 0 \tag{5.5.3}$$

$$T_{23}^2 = f_{R_2}^2 + \frac{\partial H}{\partial x}T_{12}^1 + \frac{\partial H}{\partial y}\left(T_{22}^1 - T_{33}^1\right) \qquad \text{en } z = 0 \tag{5.5.4}$$

donde es necesario conocer $T_{ii}^1 \quad (i = 1, 2, 3)$ y T_{12}^1 en $z = 0$ y $z = 1$. Esta información nos la proporcionan las igualdades (5.3.152), (5.3.153), (5.3.155) y (5.3.157) (donde hemos hecho la sustitución $\mu = \nu\rho_0$):

$$T_{11}^1 = 2\nu\rho_0\frac{\partial u^1}{\partial x} \tag{5.5.5}$$

$$T_{12}^1 = \nu\rho_0\left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right) \tag{5.5.6}$$

$$T_{22}^1 = 2\nu\rho_0\frac{\partial v^1}{\partial y} \tag{5.5.7}$$

$$T_{33}^1 = 2\nu\rho_0 D_z w^2 \tag{5.5.8}$$

y donde w^2 se puede sustituir por su expresión (5.3.146), de modo que T_{33}^1 se exprese también en términos de u^1 y de v^1:

$$T_{33}^1 = -2\nu\rho_0 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right) \tag{5.5.9}$$

Entonces, obtenemos:

$$\begin{aligned}
&\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^1 \\
&\quad - \nu\left(2\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial y \partial x}\right) = -\frac{\partial s}{\partial x}g \\
&\quad + 2\phi\,(\cos\varphi)\left(\frac{\partial u^0}{\partial x}h + \frac{\partial h}{\partial x}u^0 - v^0\frac{\partial H}{\partial y} + \frac{1}{2}h\frac{\partial v^0}{\partial y}\right) + \frac{1}{h\rho_0}(f_{W_1}^2 - f_{R_1}^2) \\
&\quad + \frac{\nu}{h}\left[2\frac{\partial h}{\partial x}\left(2\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right)\right] \\
&\frac{\partial v^1}{\partial t} + u^0\frac{\partial v^1}{\partial x} + u^1\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^1}{\partial y} + v^1\frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^1 \\
&\quad - \nu\left(\frac{\partial^2 u^1}{\partial y \partial x} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2}\right) = -\frac{\partial s}{\partial y}g \\
&\quad + 2\phi\left(\frac{1}{2}\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h + \frac{\partial s}{\partial y}(\cos\varphi)\,u^0\right) + \frac{1}{h\rho_0}(f_{W_2}^2 - f_{R_2}^2) \\
&\quad + \frac{\nu}{h}\left[2\frac{\partial h}{\partial y}\left(\frac{\partial u^1}{\partial x} + 2\frac{\partial v^1}{\partial y}\right) + \frac{\partial h}{\partial x}\left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right)\right]
\end{aligned}$$

y que también se puede escribir:

$$\begin{aligned}
&\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^1 \\
&\quad - \nu\left\{\Delta_{xy}u^1 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right]\right\} \\
&= -\frac{\partial s}{\partial x}g + 2\phi\,(\cos\varphi)\left(\frac{\partial u^0}{\partial x}h + \frac{\partial h}{\partial x}u^0 - v^0\frac{\partial H}{\partial y} + \frac{1}{2}h\frac{\partial v^0}{\partial y}\right) \\
&+ \frac{1}{h\rho_0}(f_{W_1}^2 - f_{R_1}^2)
\end{aligned} \tag{5.5.10}$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^1$$
$$- \nu \left\{ \Delta_{xy} v^1 + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^1}{\partial y} + \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right]\right\}$$
$$= -\frac{\partial s}{\partial y} g + 2\phi\left(\frac{1}{2}\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h + \frac{\partial s}{\partial y}(\cos\varphi)\,u^0\right) + \frac{1}{h\rho_0}(f^2_{W_2} - f^2_{R_2}) \quad (5.5.11)$$

Como ya mencionamos, w^2 viene dado por (5.3.146):

$$w^2 = u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - zh\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)$$

Ahora, usando (5.3.131) y (5.3.134) se llega a:

$$\tilde{p}(\varepsilon) = p_s + \varepsilon\rho_0 h\left(2\phi\,(\cos\varphi)\,u^0 - g\right)(z-1) \quad (5.5.12)$$

De igual modo, por (5.3.126), (5.3.135) y (5.3.146), sabemos que:

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 = \varepsilon\left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$
$$+ \varepsilon^2\left[u^1\frac{\partial H}{\partial x} + v^1\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right]$$
$$= \varepsilon\left[(u^0 + \varepsilon u^1)\frac{\partial H}{\partial x} + (v^0 + \varepsilon v^1)\frac{\partial H}{\partial y} - hz\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y}\right)\right]$$
$$= \varepsilon\left[\tilde{u}(\varepsilon)\frac{\partial H}{\partial x} + \tilde{v}(\varepsilon)\frac{\partial H}{\partial y} - hz\left(\frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{\partial \tilde{v}(\varepsilon)}{\partial y}\right)\right] \quad (5.5.13)$$

Se deshace el cambio de variable y, así, se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y)$$
$$v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y)$$
$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z)$$
$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z)$$

Definimos también

$$\tilde{T}_{ij}^{\varepsilon} = \tilde{T}_{ij}(\varepsilon) = T_{ij}^{0} + \varepsilon T_{ij}^{1} \quad (i,j=1,2)$$

$$\tilde{T}_{i3}^{\varepsilon} = \tilde{T}_{i3}(\varepsilon) = \varepsilon T_{i3}^{1} + \varepsilon^2 T_{i3}^{2} \quad (i=1,2)$$

$$\tilde{T}_{33}^{\varepsilon} = \tilde{T}_{33}(\varepsilon) = T_{33}^{0} + \varepsilon T_{33}^{1}$$

$$\tilde{f}_{R_i}^{\varepsilon} = \tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^{1} + \varepsilon^2 f_{R_i}^{2},$$

$$\tilde{f}_{W_i}^{\varepsilon} = \tilde{f}_{W_i}(\varepsilon) = \varepsilon f_{W_i}^{1} + \varepsilon^2 f_{W_i}^{2} \quad (i=1,2)$$

Según lo visto en (5.3.123), (5.3.124), (5.3.129) y (5.3.130):

$$\frac{\partial \tilde{u}^{\varepsilon}}{\partial z^{\varepsilon}} = \frac{\partial \tilde{v}^{\varepsilon}}{\partial z^{\varepsilon}} = 0 \tag{5.5.14}$$

Si se realiza el cambio de variable en (5.5.12), obtenemos la aproximación de la presión en Ω^{ε}:

$$\tilde{p}^{\varepsilon} = p_s^{\varepsilon} + \rho_0 (z^{\varepsilon} - s^{\varepsilon}) \left(2\phi \left(\cos\varphi^{\varepsilon}\right) \tilde{u}^{\varepsilon} - g\right) + O(\varepsilon^2) \tag{5.5.15}$$

Análogamente, deshaciendo el cambio de variable en (5.5.13), se logra la aproximación de la componente vertical de la velocidad:

$$\tilde{w}^{\varepsilon} = \tilde{u}^{\varepsilon} \frac{\partial H^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon} \frac{\partial H^{\varepsilon}}{\partial y^{\varepsilon}} - (z^{\varepsilon} - H^{\varepsilon}) \left(\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}} \right) \tag{5.5.16}$$

Teniendo en cuenta (5.4.12) y (5.5.10) obtenemos la siguiente ecuación para el cálculo de $\tilde{u}^{\varepsilon}$:

$$\begin{aligned}
&\frac{\partial \tilde{u}^{\varepsilon}}{\partial t^{\varepsilon}} + \tilde{u}^{\varepsilon} \frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon} \frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} - 2\phi \left(\operatorname{sen}\varphi^{\varepsilon}\right) \tilde{v}^{\varepsilon} - \nu \left\{ \Delta_{x^{\varepsilon} y^{\varepsilon}} \tilde{u}^{\varepsilon} + \frac{2}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial x^{\varepsilon}} \frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} \right.\\
&\quad \left. + \frac{1}{(h^{\varepsilon})^2} \frac{\partial}{\partial x^{\varepsilon}} \left[(h^{\varepsilon})^2 \left(\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}} \right) \right] + \frac{1}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial y^{\varepsilon}} \left[\frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial x^{\varepsilon}} \right] \right\}\\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon \frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} \right) + (v^0 + \varepsilon v^1) \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial y} \right)\\
&\quad - 2\phi \left(\operatorname{sen}\varphi\right) (v^0 + \varepsilon v^1) - \nu \left\{ \Delta_{xy} u^0 + \varepsilon \Delta_{xy} u^1 + \frac{2}{h} \frac{\partial h}{\partial x} \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} \right) \right.\\
&\quad + \frac{1}{h^2} \frac{\partial}{\partial x} \left[h^2 \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon \frac{\partial v^1}{\partial y} \right) \right]\\
&\quad \left. + \frac{1}{h} \frac{\partial h}{\partial y} \left(\frac{\partial u^0}{\partial y} + \varepsilon \frac{\partial u^1}{\partial y} + \frac{\partial v^0}{\partial x} + \varepsilon \frac{\partial v^1}{\partial x} \right) \right\}
\end{aligned}$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} - 2\phi (\operatorname{sen} \varphi) v^0 - \nu \left\{ \Delta_{xy} u^0 + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^0}{\partial x} \right.$$

$$\left. + \frac{1}{h^2} \frac{\partial}{\partial x} \left[h^2 \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] + \frac{1}{h} \frac{\partial h}{\partial y} \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right\}$$

$$+ \varepsilon \left(\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + v^1 \frac{\partial u^0}{\partial y} + v^0 \frac{\partial u^1}{\partial y} - 2\phi (\operatorname{sen} \varphi) v^1 \right.$$

$$\left. - \nu \left\{ \Delta_{xy} u^1 + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial u^1}{\partial x} + \frac{1}{h^2} \frac{\partial}{\partial x} \left[h^2 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] + \frac{1}{h} \frac{\partial h}{\partial y} \left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right] \right\} \right)$$

$$+ \varepsilon^2 \left(u^1 \frac{\partial u^1}{\partial x} + v^1 \frac{\partial u^1}{\partial y} \right) = -\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0} (f^1_{W_1} - f^1_{R_1}) + \varepsilon \left[-\frac{\partial s}{\partial x} g \right.$$

$$\left. + \frac{1}{h\rho_0} (f^2_{W_1} - f^2_{R_1}) + 2\phi (\cos \varphi) \left(\frac{\partial u^0}{\partial x} h + \frac{\partial h}{\partial x} u^0 - v^0 \frac{\partial H}{\partial y} + \frac{1}{2} h \frac{\partial v^0}{\partial y} \right) \right] + O(\varepsilon^2)$$

$$= -\frac{1}{\rho_0} \frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + 2\phi (\cos \varphi^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tilde{u}^\varepsilon - \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

$$+ \frac{1}{h^\varepsilon \rho_0} (\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1}) + O(\varepsilon^2)$$

De manera análoga, empleando (5.4.13) y (5.5.11) se obtiene una ecuación para el cálculo de $\tilde{v}^\varepsilon$. Se tiene por tanto:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi (\operatorname{sen} \varphi^\varepsilon) \tilde{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}$$

$$= -\frac{1}{\rho_0} \frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + 2\phi (\cos \varphi^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tilde{u}^\varepsilon - \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

$$+ \frac{1}{h^\varepsilon \rho_0} (\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1}) + O(\varepsilon^2) \tag{5.5.17}$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi (\operatorname{sen} \varphi^\varepsilon) \tilde{u}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{v}^\varepsilon + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2} \frac{\partial}{\partial y^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon}g + 2\phi\left(\frac{1}{2}h^\varepsilon\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon) + (\cos\varphi^\varepsilon)\,\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\tilde{u}^\varepsilon\right)$$

$$+ \frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2}) + O(\varepsilon^2) \tag{5.5.18}$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = \varepsilon\frac{\partial h}{\partial t} + \varepsilon\frac{\partial}{\partial x}\left[h\left(u^0 + \varepsilon u^1\right)\right] + \varepsilon\frac{\partial}{\partial y}\left[\left(v^0 + \varepsilon v^1\right)h\right]$$

$$= \varepsilon\left[\frac{\partial h}{\partial t} + \frac{\partial(u^0 h)}{\partial x} + \frac{\partial(v^0 h)}{\partial x}\right] + O(\varepsilon^2)$$

por lo que

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \tag{5.5.19}$$

Veamos en qué medida verifica la aproximación de primer orden las ecuaciones de Navier-Stokes de partida:

$$\frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon\right)$$

$$-\nu\left(2\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{\partial^2\tilde{u}^\varepsilon}{\partial(y^\varepsilon)^2} + \frac{\partial^2\tilde{v}^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right) - \frac{1}{\rho_0}\frac{\partial\tilde{T}^\varepsilon_{13}}{\partial z^\varepsilon} = 0$$

Al igual que pasaba en la sección anterior, no podemos evaluar directamente esta ecuación ya que no conocemos el término en $\tilde{T}^\varepsilon_{13}$. Si promediamos la ecuación en altura, obtenemos

$$\frac{1}{h^\varepsilon}\int_{H^\varepsilon}^{s^\varepsilon}\left[\frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon} - 2\phi\left((\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - (\cos\varphi^\varepsilon)\,\tilde{w}^\varepsilon\right)\right.$$

$$\left.-\nu\left(2\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{\partial^2\tilde{u}^\varepsilon}{\partial(y^\varepsilon)^2} + \frac{\partial^2\tilde{v}^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right) - \frac{1}{\rho_0}\frac{\partial\tilde{T}^\varepsilon_{13}}{\partial z^\varepsilon}\right]dz^\varepsilon$$

$$= \frac{\partial\tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(2\phi\left(\cos\varphi^\varepsilon\right)\tilde{u}^\varepsilon - g\right) - h^\varepsilon\phi\left(\cos\varphi^\varepsilon\right)\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}$$

$$-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\tilde{v}^\varepsilon + 2\phi\left(\cos\varphi^\varepsilon\right)\left[\tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \frac{h^\varepsilon}{2}\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]$$

$$-\nu\left(2\frac{\partial^2\tilde{u}^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{\partial^2\tilde{u}^\varepsilon}{\partial(y^\varepsilon)^2} + \frac{\partial^2\tilde{v}^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right) - \frac{1}{\rho_0 h^\varepsilon}\left(\tilde{T}^\varepsilon_{13}\Big|_{z^\varepsilon=s^\varepsilon} - \tilde{T}^\varepsilon_{13}\Big|_{z^\varepsilon=H^\varepsilon}\right)$$

$$= \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \tilde{v}^\varepsilon$$

$$- 2\phi \left(\cos \varphi^\varepsilon\right) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tilde{u}^\varepsilon - \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

$$- \nu \left(2 \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{\partial^2 \tilde{u}^\varepsilon}{\partial (y^\varepsilon)^2} + \frac{\partial^2 \tilde{v}^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon} \right) - \frac{1}{\rho_0 h^\varepsilon} \left(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon \right)$$

$$- \frac{\nu}{h^\varepsilon} \left[2 \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \left(2 \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) + \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \left(\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right) \right]$$

$$= \frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon\right) \tilde{v}^\varepsilon$$

$$- \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] \right.$$

$$\left. + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\} - 2\phi \left(\cos \varphi^\varepsilon\right) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tilde{u}^\varepsilon - \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

$$- \frac{1}{\rho_0 h^\varepsilon} \left(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon \right) = O(\varepsilon^2)$$

sin más que tener en cuenta (5.5.14), (5.5.15), (5.4.3), (5.4.5), (5.4.7), (5.4.8), (5.4.11), (5.5.1), (5.5.3), (5.5.5), (5.5.6), (5.5.9) y (5.5.17).

Por tanto, la primera ecuación de Navier-Stokes, promediada en altura se verifica con un error de orden ε^2. Lo mismo sucede con la segunda ecuación de Navier-Stokes, y en este caso es necesario emplear las igualdades y ecuaciones (5.5.14), (5.5.15), (5.4.4), (5.4.6), (5.4.8), (5.4.9), (5.4.11), (5.5.2), (5.5.4), (5.5.6), (5.5.7), (5.5.9) y (5.5.18).

Para la tercera ecuación de Navier-Stokes se tiene, usando (5.3.134), que:

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0} \frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - 2\phi \left(\cos \varphi^\varepsilon\right) \tilde{u}^\varepsilon - \frac{1}{\rho_0} \frac{\partial \tilde{T}_{13}^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0} \frac{\partial \tilde{T}_{23}^\varepsilon}{\partial y^\varepsilon}$$

$$- 2\nu \frac{\partial^2 w^\varepsilon}{\partial (z^\varepsilon)^2} = D_t \tilde{w}(\varepsilon) + \tilde{u}(\varepsilon) D_x \tilde{w}(\varepsilon) + \tilde{v}(\varepsilon) D_y \tilde{w}(\varepsilon) + \tilde{w}(\varepsilon) \frac{1}{\varepsilon} D_z \tilde{w}(\varepsilon)$$

$$+ \frac{1}{\rho_0} \frac{1}{\varepsilon} D_z \tilde{p}(\varepsilon) + g - 2\phi \left(\cos \varphi\right) \tilde{u}(\varepsilon) - \frac{1}{\rho_0} D_x \tilde{T}_{13}(\varepsilon) - \frac{1}{\rho_0} D_y \tilde{T}_{23}(\varepsilon) - \frac{2\nu}{\varepsilon^2} D_z^2 \tilde{w}(\varepsilon)$$

$$= \varepsilon D_t w^1 + \varepsilon^2 D_t w^2 + (u^0 + \varepsilon u^1) \left[\varepsilon D_x w^1 + \varepsilon^2 D_x w^2 \right]$$

$$+ (v^0 + \varepsilon v^1) \left[\varepsilon D_y w^1 + \varepsilon^2 D_y w^2 \right] + (\varepsilon w^1 + \varepsilon^2 w^2) \frac{1}{\varepsilon} \left[\varepsilon D_z w^1 + \varepsilon^2 D_z w^2 \right]$$

$$+\frac{1}{\rho_0}D_z p^1 + g - 2\phi\left(\cos\varphi\right)\left(u^0+\varepsilon u^1\right) - \frac{1}{\rho_0}\left(\varepsilon D_x T^1_{13} + \varepsilon^2 D_x T^2_{13}\right)$$

$$-\frac{1}{\rho_0}\left(\varepsilon D_y T^1_{23} + \varepsilon^2 D_y T^2_{23}\right) - \frac{2\nu}{\varepsilon^2}\left(\varepsilon D_z^2 w^1 + \varepsilon^2 D_z^2 w^2\right) = O(\varepsilon)$$

La aproximación de primer orden verifica la tercera ecuación de Navier-Stokes con un error $O(\varepsilon)$.

La ecuación de la incompresibilidad se verifica de forma exacta como se ve utilizando (5.3.135) y (5.3.146)

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = \frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{\partial \tilde{v}(\varepsilon)}{\partial y} + \frac{1}{\varepsilon}D_z\tilde{w}(\varepsilon)$$

$$= \frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + D_z w^1 + \varepsilon D_z w^2 = 0,$$

o directamente por (5.5.16).

Lo mismo sucede con las condiciones de contorno (5.1.33) y (5.1.34), teniendo en cuenta (5.5.15) y (5.5.16).

Veamos qué sucede con las condiciones de contorno que recogen el efecto del viento en la superficie y el rozamiento en el fondo. Teniendo en cuenta (5.3.147) y (5.3.163), para la condición (5.1.35) obtenemos

$$\left[-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\tilde{T}^\varepsilon_{11} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon}\tilde{T}^\varepsilon_{12} + \tilde{T}^\varepsilon_{13} + \frac{\partial s^\varepsilon}{\partial x^\varepsilon}\left(-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\tilde{T}^\varepsilon_{13} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon}\tilde{T}^\varepsilon_{23} + \tilde{T}^\varepsilon_{33}\right)\right]_{z^\varepsilon = s^\varepsilon}$$

$$-\tilde{f}^\varepsilon_{W_1}\sqrt{1+\left(\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right)^2 + \left(\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\right)^2}$$

$$= \left[-\varepsilon\frac{\partial s}{\partial x}\left(T^0_{11}+\varepsilon T^1_{11}\right) - \varepsilon\frac{\partial s}{\partial y}\left(T^0_{12}+\varepsilon T^1_{12}\right) + \varepsilon T^1_{13} + \varepsilon^2 T^2_{13}\right.$$

$$\left.+\varepsilon\frac{\partial s}{\partial x}\left(-\varepsilon^2\frac{\partial s}{\partial x}\left(T^1_{13}+\varepsilon T^2_{13}\right) - \varepsilon^2\frac{\partial s}{\partial y}\left(T^1_{23}+\varepsilon T^2_{23}\right) + T^0_{33} + \varepsilon T^1_{33}\right)\right]_{z=1}$$

$$-\varepsilon\left(f^1_{W_1}+\varepsilon f^2_{W_1}\right)\sqrt{1+\varepsilon^2\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right]}$$

$$= \left[\varepsilon\left(-\frac{\partial s}{\partial x}T^0_{11} - \frac{\partial s}{\partial y}T^0_{12} + T^1_{13} + \frac{\partial s}{\partial x}T^0_{33}\right) + \varepsilon^2\left(-\frac{\partial s}{\partial x}T^1_{11} - \frac{\partial s}{\partial y}T^1_{12} + T^2_{13}\right.\right.$$

$$\left.\left.+\frac{\partial s}{\partial x}T^1_{33}\right)\right]_{z=1} - \varepsilon f^1_{W_1} - \varepsilon^2 f^2_{W_1} + O(\varepsilon^3) = O(\varepsilon^3)$$

Se puede razonar de modo análogo para las condiciones (5.1.36)-(5.1.38) teniendo en cuenta (5.3.148)-(5.3.150) y (5.3.164)-(5.3.166).

Si en (5.5.15), (5.5.17)-(5.5.19) se desprecian los términos de orden $O(\varepsilon^2)$ se obtiene el siguiente modelo de aguas someras:

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right. \\
&\qquad \left. + \frac{1}{(h^\varepsilon)^2} \frac{\partial}{\partial x^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\} \\
&\qquad = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + 2\phi\,(\cos\varphi^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tilde{u}^\varepsilon - \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \\
&\qquad + \frac{1}{h^\varepsilon \rho_0} (\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon) \\
&\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{v}^\varepsilon + \frac{2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y^\varepsilon} \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right. \\
&\qquad \left. + \frac{1}{(h^\varepsilon)^2} \frac{\partial}{\partial y^\varepsilon} \left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial x^\varepsilon} \left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\} \\
&\qquad = -\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g + 2\phi \left(\frac{1}{2} h^\varepsilon \frac{\partial}{\partial y^\varepsilon} ((\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon) + (\cos\varphi^\varepsilon) \frac{\partial s^\varepsilon}{\partial y^\varepsilon} \tilde{u}^\varepsilon \right) \\
&\qquad + \frac{1}{h^\varepsilon \rho_0} (\tilde{f}_{W_2}^\varepsilon - \tilde{f}_{R_2}^\varepsilon) \\
&\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (z^\varepsilon - s^\varepsilon)\,(2\phi\,(\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon - g) \\
&\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon) \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)
\end{aligned}
\tag{5.5.20}
$$

donde $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y h^ε no dependen de z^ε.

Si se compara el modelo clásico de “shallow waters”((1.2.23), ver por ejemplo [5] (pág. 3) o en [101] (pág. 38)) con éste, se puede apreciar que la diferencia está esencialmente en el término de viscosidad y los que proceden de la aceleración de Coriolis.

5.6. Aproximación de segundo orden

Se considera la aproximación de segundo orden:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{T}_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 + \varepsilon^2 T_{ij}^2 \quad (i, j = 1, 2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1} T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 + \varepsilon^2 T_{i3}^2 + \varepsilon^3 T_{i3}^3 \quad (i = 1, 2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1} T_{33}^{-1} + T_{33}^0 + \varepsilon T_{33}^1 + \varepsilon^2 T_{33}^2$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^1 + \varepsilon^2 f_{R_i}^2 + \varepsilon^3 f_{R_i}^3 \quad (i = 1, 2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f_{W_i}^1 + \varepsilon^2 f_{W_i}^2 + \varepsilon^3 f_{W_i}^3 \quad (i = 1, 2)$$

Los términos w^0, p^0, $T_{i3}^k \quad (k = -1, 0, i = 1, 2)$, T_{33}^{-1}, u^0, v^0, h, w^1, p^1, u^1, v^1 y w^2 se calculan del mismo modo que en la sección anterior, para la aproximación de primer orden, a partir de (5.3.121)-(5.3.136), (5.3.143), (5.3.144) y (5.3.146).

Buscamos ahora p^2. Para ello partimos de la ecuación (5.3.145):

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\phi \left(\cos\varphi\right) u^1 - 2\nu D_z^2 w^3$$
$$- \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1\right) = 0 \qquad (5.6.1)$$

Comenzamos por despejar $D_z^2 w^3$ de la expresión (5.3.158)

$$\nu D_z^2 w^3 = \nu \left(D_x^2 w^1 + D_y^2 w^1\right) - \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1\right)$$

y lo sustituimos en (5.6.1)

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\phi \left(\cos\varphi\right) u^1$$
$$- 2\nu \left(D_x^2 w^1 + D_y^2 w^1\right) + \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1\right) = 0 \qquad (5.6.2)$$

A continuación, como u^0 y v^0 son conocidos, calculamos $T^1_{i3} \quad (i=1,2)$ a partir de (5.3.132)-(5.3.133):

$$\frac{1}{\rho_0 h}\frac{\partial T^1_{13}}{\partial z} = \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)$$

$$\frac{1}{\rho_0 h}\frac{\partial T^1_{23}}{\partial z} = \frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)$$

Integrando respecto de z,

$$T^1_{13} = \rho_0 h z\left[\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)\right] + T^1_{13}\big|_{z=0}$$

$$T^1_{23} = \rho_0 h z\left[\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)\right] + T^1_{23}\big|_{z=0}$$

Finalmente, si se sustituyen $T^1_{i3}|_{z=0}$ $(i=1,2)$ según lo visto en (5.4.5)-(5.4.11), T^1_{i3} $(i=1,2)$ resultan:

$$T^1_{13} = \rho_0\left[\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)\right] hz + f^1_{R_1} + 2\nu\rho_0\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \nu\rho_0\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) \tag{5.6.3}$$

$$T^1_{23} = \rho_0\left[\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)\right] hz + f^1_{R_2} + 2\nu\rho_0\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \nu\rho_0\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) \tag{5.6.4}$$

Sustituyendo ahora T_{i3}^1 $(i = 1, 2)$ por las expresiones anteriores y w^1 por (5.3.135) en (5.6.2) resulta

$$
\begin{aligned}
&\frac{\partial u^0}{\partial t}\frac{\partial H}{\partial x} + \frac{\partial v^0}{\partial t}\frac{\partial H}{\partial y} - hz\left(\frac{\partial^2 u^0}{\partial x \partial t} + \frac{\partial^2 v^0}{\partial y \partial t}\right) + u^0\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\\
&\quad - hz\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \left.\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + v^0\left[\frac{\partial}{\partial y}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right)\right.\\
&\quad - hz\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \left.\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] - \left[u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right.\\
&\quad - \left. zh\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{1}{\rho_0}D_z p^2 - 2\phi\,(\cos\varphi)\,u^1\\
&\quad - 2\nu\left[\frac{\partial^2}{\partial x^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - zh\frac{\partial^2}{\partial x^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\\
&\quad + 2\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(u^0\frac{\partial H}{\partial x} + v^0\frac{\partial H}{\partial y}\right) - zh\frac{\partial^2}{\partial y^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\\
&\quad + \left.\frac{\partial^2 H}{\partial y^2}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + 2\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + hz\frac{\partial}{\partial x}\left[\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x}\right.\\
&\quad + \left. v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)\right] + \frac{1}{\rho_0}\frac{\partial f_{R_1}^1}{\partial x}\\
&\quad + 2\nu\frac{\partial^2 H}{\partial x^2}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + 2\nu\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \nu\frac{\partial^2 H}{\partial y \partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\\
&\quad + \nu\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - \frac{\partial H}{\partial x}\left[\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right.\\
&\quad - \left. 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left(2\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)\right] + hz\frac{\partial}{\partial y}\left[\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y}\right.\\
&\quad + \left.\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)\right] + \frac{1}{\rho_0}\frac{\partial f_{R_2}^1}{\partial y}\\
&\quad + 2\nu\frac{\partial^2 H}{\partial y^2}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + 2\nu\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \nu\frac{\partial^2 H}{\partial y \partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\\
&\quad + \nu\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - \frac{\partial H}{\partial y}\left[\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.
\end{aligned}
$$

$$+ 2\phi\,(\operatorname{sen}\varphi)\,u^0 - \nu\left(\frac{\partial^2 u^0}{\partial x\partial y} + \frac{\partial^2 v^0}{\partial x^2} + 2\frac{\partial^2 v^0}{\partial y^2}\right)\Bigg] = 0$$

de donde, simplificando, se puede despejar

$$-\frac{1}{\rho_0 h}\frac{\partial p^2}{\partial z} = (u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x} + (v^0)^2\frac{\partial^2 H}{\partial y^2} - 2\phi\,(\cos\varphi)\,u^1$$

$$+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right)$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}\right) + \frac{\partial}{\partial y}\left(v^0\frac{\partial^2 H}{\partial y^2}\right)\right] + zh\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

Ahora se integra respecto a z teniendo en cuenta que ninguna de las funciones que aparecen en la expresión anterior dependen de z e imponiendo la condición de contorno $p^2(t,x,y,1) = 0$ ((5.3.162)). Se obtiene la siguiente expresión para p^2:

$$p^2 = \rho_0 h(1-z)\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x} + (v^0)^2\frac{\partial^2 H}{\partial y^2} - 2\phi\,(\cos\varphi)\,u^1\right.$$

$$+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right)$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x} + v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\}$$

$$+\frac{1}{2}\rho_0 h^2(1-z^2)\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right.$$

$$\left.+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right] \qquad (5.6.5)$$

A continuación calculamos u^2 y v^2 a partir de (5.3.154) y (5.3.156):

$$T_{13}^1 = \mu\left(D_z u^2 + D_x w^1\right)$$

$$T_{23}^1 = \mu\left(D_z v^2 + D_y w^1\right)$$

en donde sustituimos w^1, T_{13}^1 y T_{23}^1 por sus expresiones dadas en (5.3.135), (5.6.3) y (5.6.4), y despejamos $D_z u^2$ y $D_z v^2$, resultando:

$$D_z u^2 = hz\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \frac{\partial^2 u^0}{\partial x^2} - \frac{\partial^2 u^0}{\partial y^2}\right]$$
$$+ \frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}$$

$$D_z v^2 = zh\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \frac{\partial^2 v^0}{\partial x^2} - \frac{\partial^2 v^0}{\partial y^2}\right]$$
$$+ \frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}$$

Integrando estas igualdades se obtienen las siguientes expresiones para u^2 y v^2:

$$u^2 = u_0^2 + hz\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}\right]$$
$$+ \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right] \quad (5.6.6)$$

$$v^2 = v_0^2 + hz\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]$$
$$+ \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right] \quad (5.6.7)$$

donde $u_0^2(t,x,y) = u^2(t,x,y,0)$, $v_0^2(t,x,y) = v^2(t,x,y,0)$ están determinados por (5.3.159) y (5.3.160). En estas ecuaciones aparecen T_{13}^3 y T_{23}^3 que sólo son conocidos en $z=0$ y $z=1$. Es por ello que se integran las ecuaciones respecto de z entre 0 y 1. Para realizar la integración es necesario conocer explícitamente la dependencia de z de los diferentes términos que intervienen en las ecuaciones. Por eso se sustituyen u^2, v^2, p^2, w^1 y w^2 por las expresiones vistas en (5.6.5)-(5.6.7), (5.3.135) y (5.3.146), y la ecuación (5.3.159) se puede escribir:

$$
\begin{aligned}
&\frac{\partial u_0^2}{\partial t} + hz\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] \\
&\quad + \frac{1}{2}z^2h^2\frac{\partial}{\partial t}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right] \\
&\quad + u^0\left\{\frac{\partial u_0^2}{\partial x} + hz\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2}\right.\right. \\
&\quad \left. - v^0\frac{\partial^2 H}{\partial y\partial x}\right] + \frac{1}{2}z^2h^2\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right. \\
&\quad \left.\left. - \Delta_{xy}u^0\right]\right\} + u^1\frac{\partial u^1}{\partial x} + \left\{u_0^2 + \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right.\right.\right. \\
&\quad \left.\left.\left. - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right]\right\}\frac{\partial u^0}{\partial x} + v^0\left\{\frac{\partial u_0^2}{\partial y} + hz\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\right. \\
&\quad \left. + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial y}\right] + \frac{1}{2}z^2h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y}\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right]\right\} + v^1\frac{\partial u^1}{\partial y} + \left\{v_0^2 + hz\left[\frac{1}{\nu\rho_0}f_{R_2}^1\right.\right. \\
&\quad \left. + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right] \\
&\quad \left. + \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right]\right\}\frac{\partial u^0}{\partial y} \\
&\quad - hz\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]\frac{\partial v^0}{\partial y} \\
&\quad -z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right. \\
&\quad \left. - \Delta_{xy}u^0\right]\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial h}{\partial x}\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x} + (v^0)^2\frac{\partial^2 H}{\partial y^2}\right. \\
&\quad - 2\phi\left(\cos\varphi\right)u^1 + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^1}{\partial x} + \frac{\partial f_{R_2}^1}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)
\end{aligned}
$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\}+h(1-z)\frac{\partial}{\partial x}\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2}+2u^0v^0\frac{\partial^2 H}{\partial y\partial x}\right.$$

$$+(v^0)^2\frac{\partial^2 H}{\partial y^2}-2\phi\left(\cos\varphi\right)u^1+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x}+\frac{\partial f^1_{R_2}}{\partial y}\right)-\frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\}+h\frac{\partial h}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+\frac{1}{2}h^2(1-z^2)\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}+2\left(\frac{\partial v^0}{\partial y}\right)^2\right.$$

$$\left.+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]+\frac{\partial H}{\partial x}\left(\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2}\right.\right.$$

$$+2u^0v^0\frac{\partial^2 H}{\partial y\partial x}+(v^0)^2\frac{\partial^2 H}{\partial y^2}-2\phi\left(\cos\varphi\right)u^1+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x}+\frac{\partial f^1_{R_2}}{\partial y}\right)-\frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right.$$

$$\left.-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\}+hz\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]\right)$$

$$-2\phi\left((\operatorname{sen}\varphi)\left\{v_0^2+hz\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.\right.\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]+\frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$- \Delta_{xy} v^0 \big] \Big\} - (\cos\varphi) \left[u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - zh \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] \Big)$$

$$- \nu \left(2 \left\{ \frac{\partial^2 u_0^2}{\partial x^2} + hz \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right. \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + \frac{1}{2} z^2 h^2 \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] - \frac{\partial^2 H}{\partial x^2} \left(\left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + hz \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. \left. - \Delta_{xy} u^0 \right] \right) - \frac{\partial H}{\partial x} \left(2 \frac{\partial}{\partial x} \left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + 2hz \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] - \frac{\partial H}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. \left. - \Delta_{xy} u^0 \right] \right) \Big\} + \frac{\partial^2 u_0^2}{\partial y^2} + hz \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} \right.$$

$$\left. - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + \frac{1}{2} z^2 h^2 \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right. \right.$$

$$\left. \left. - 2\phi (\operatorname{sen}\varphi) v^0 \right) - \Delta_{xy} u^0 \right] - \frac{\partial^2 H}{\partial y^2} \left(\left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} \right. \right.$$

$$\left. - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + hz \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. \left. - \Delta_{xy} u^0 \right] \right) - \frac{\partial H}{\partial y} \left(2 \frac{\partial}{\partial y} \left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] + 2hz \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] - \frac{\partial H}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi (\operatorname{sen}\varphi) v^0 \right) \right.$$

$$- \Delta_{xy} u^0]) + \frac{\partial^2 v_0^2}{\partial x \partial y} + hz \frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu \rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} \right.$$

$$\left. - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right] + \frac{1}{2} z^2 h^2 \frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} \right. \right.$$

$$\left. \left. + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] - \frac{\partial^2 H}{\partial x \partial y} \left(\left[\frac{1}{\nu \rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} \right. \right.$$

$$\left. - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right] + hz \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) \right.$$

$$\left. \left. - \Delta_{xy} v^0 \right] \right) - \frac{\partial H}{\partial x} \left\{ \frac{\partial}{\partial y} \left[\frac{1}{\nu \rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] + hz \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) \right.$$

$$\left. \left. - \Delta_{xy} v^0 \right] \right\} - \frac{\partial H}{\partial y} \left\{ \frac{\partial}{\partial x} \left[\frac{1}{\nu \rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] + hz \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) \right.$$

$$\left. - \Delta_{xy} v^0 \right] - \frac{\partial H}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) \right.$$

$$\left. \left. \left. - \Delta_{xy} v^0 \right] \right\} \right) - \frac{1}{\rho_0} D_z T_{13}^3 = 0$$

Simplificamos, agrupamos en potencias de z e integramos respecto de z entre 0 y 1, obteniendo:

$$\frac{\partial u_0^2}{\partial t} + u^0 \frac{\partial u_0^2}{\partial x} + u^1 \frac{\partial u^1}{\partial x} + u_0^2 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u_0^2}{\partial y} + v^1 \frac{\partial u^1}{\partial y} + v_0^2 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \left. D_x p^2 \right|_{z=0}$$

$$- 2\phi \left(\operatorname{sen} \varphi \right) v_0^2 + 2\phi \left(\cos \varphi \right) \left(u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} \right) - 2\nu \frac{\partial^2 u_0^2}{\partial x^2} - \nu \frac{\partial^2 u_0^2}{\partial y^2} - \nu \frac{\partial^2 v_0^2}{\partial x \partial y}$$

$$+ \nu \left(2 \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) \left[\frac{1}{\nu \rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] - \nu \left[2 \left(\frac{\partial H}{\partial x} \right)^2 + \left(\frac{\partial H}{\partial y} \right)^2 \right] \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} \right. \right.$$

$$+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\Big)-\Delta_{xy}u^0\Bigg]$$

$$+4\nu\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+2\nu\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+\nu\frac{\partial^2 H}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+\nu\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+\nu\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$-\nu\frac{\partial H}{\partial y}\frac{\partial H}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$+\frac{h}{2}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]\right.$$

$$+u^0\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+v^0\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial y}\right]$$

$$+\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]\frac{\partial u^0}{\partial y}$$

$$-\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]\frac{\partial v^0}{\partial y}$$

$$-\frac{\partial}{\partial x}\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2}+2u^0v^0\frac{\partial^2 H}{\partial y\partial x}+(v^0)^2\frac{\partial^2 H}{\partial y^2}-2\phi\left(\cos\varphi\right)u^1\right.$$

$$+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x}+\frac{\partial f^1_{R_2}}{\partial y}\right)-\frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\Bigg]\Bigg\}+\frac{\partial H}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$-2\phi\left(\operatorname{sen}\varphi\right)\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$-2\phi\left(\cos\varphi\right)\left(\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)-2\nu\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left.+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+2\nu\frac{\partial^2 H}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]$$

$$+4\nu\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]$$

$$-\nu\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+\nu\frac{\partial^2 H}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]$$

$$+2\nu\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]$$

$$-\nu\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+\nu\frac{\partial^2 H}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$+\nu\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$+\nu\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\Bigg\}$$

$$+\frac{h^2}{6}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]\right.$$

$$+u^0\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]$$

$$+v^0\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]$$

$$+\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^0\right)-\Delta_{xy}v^0\right]\frac{\partial u^0}{\partial y}$$

$$-\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)-\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$-2\phi\,(\operatorname{sen}\varphi)\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^0\right)-\Delta_{xy}v^0\right]$$

$$-2\nu\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]$$

$$-\nu\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]$$

$$\left.-\nu\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^0\right)-\Delta_{xy}v^0\right]\right\}$$

$$-\frac{1}{\rho_0 h}\left(T^3_{13}\big|_{z=1}-T^3_{13}\big|_{z=0}\right)=0$$

Calculamos ahora, $T^3_{13}\big|_{z=1}-T^3_{13}\big|_{z=0}$, para lo que recurrimos a las expresiones (5.3.9) y (5.3.11). Igualando a 0 los términos multiplicados por ε^3 se obtiene:

$$T^3_{13}=\frac{\partial s}{\partial x}T^2_{11}+\frac{\partial s}{\partial y}T^2_{12}-\frac{\partial s}{\partial x}\left(-\frac{\partial s}{\partial x}T^1_{13}-\frac{\partial s}{\partial y}T^1_{23}+T^2_{33}\right)+f^3_{W_1}$$

$$+f^1_{W_1}\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)\qquad\text{en } z=1 \tag{5.6.8}$$

$$T_{13}^3 = \frac{\partial H}{\partial x} T_{11}^2 + \frac{\partial H}{\partial y} T_{12}^2 - \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial x} T_{13}^1 - \frac{\partial H}{\partial y} T_{23}^1 + T_{33}^2\right) + f_{R_1}^3$$
$$+ f_{R_1}^1 \left(\left(\frac{\partial H}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial H}{\partial y}\right)^2\right) \quad \text{en } z = 0 \tag{5.6.9}$$

Usamos las expresiones (5.3.161), (5.3.168)-(5.3.171) para escribir $T_{ij}^2 \quad (i, j = 1, 2)$ y T_{33}^2 en función de u^2 y v^2:

$$T_{11}^2 = 2\mu D_x u^2 \tag{5.6.10}$$

$$T_{12}^2 = \mu\left(D_y u^2 + D_x v^2\right) \tag{5.6.11}$$

$$T_{22}^2 = 2\mu D_y v^2 \tag{5.6.12}$$

$$T_{33}^2 = -2\mu\left(D_x u^2 + D_y v^2\right) \tag{5.6.13}$$

En cuanto a T_{13}^1 y T_{23}^1, se procede igual que se hizo en (5.4.3)-(5.4.11) a partir de (5.3.147)- (5.3.150). Así, si en (5.6.8) y (5.6.9) tenemos en cuenta (5.4.3)-(5.4.11) y (5.6.10)-(5.6.13), y u^2, v^2 se reemplazan por sus expresiones dadas en (5.6.6)-(5.6.7), resultan:

$$T_{13}^3 = 2\mu \frac{\partial s}{\partial x}\left\{2\frac{\partial u_0^2}{\partial x} + 2h\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2}\right.\right.$$
$$\left. - v^0\frac{\partial^2 H}{\partial y \partial x}\right] + h^2\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right) v^0\right)\right.$$
$$\left. - \Delta_{xy} u^0\right] - 2\frac{\partial H}{\partial x}\left(\frac{1}{\nu\rho_0} f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2}\right.$$
$$\left. - v^0\frac{\partial^2 H}{\partial y \partial x} + h\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right) v^0\right) - \Delta_{xy} u^0\right]\right)$$
$$+ \frac{\partial v_0^2}{\partial y} + h\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]$$
$$+ \frac{1}{2}h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right) u^0\right) - \Delta_{xy} v^0\right]$$
$$- \frac{\partial H}{\partial y}\left(\frac{1}{\nu\rho_0} f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right.$$
$$\left.\left. + h\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right) u^0\right) - \Delta_{xy} v^0\right]\right)\right\}$$

$$
\begin{aligned}
&+\mu\frac{\partial s}{\partial y}\left\{\frac{\partial u_0^2}{\partial y}+h\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\frac{1}{2}h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.\\
&\left.-\Delta_{xy}u^0\right]-\frac{\partial H}{\partial y}\left(\frac{1}{\nu\rho_0}f_{R_1}^1+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y\partial x}+h\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]\right)\\
&+\frac{\partial v_0^2}{\partial x}+h\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_2}^1+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]\\
&+\frac{1}{2}h^2\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\\
&-\frac{\partial H}{\partial x}\left(\frac{1}{\nu\rho_0}f_{R_2}^1+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right.\\
&\left.\left.+h\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\right)\right\}\\
&+\left(\frac{\partial s}{\partial x}\right)^2\left[f_{W_1}^1+2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\\
&+\frac{\partial s}{\partial x}\frac{\partial s}{\partial y}\left[f_{W_2}^1+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]+f_{W_1}^3\\
&+f_{W_1}^1\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)\qquad\text{en } z=1
\end{aligned}
\tag{5.6.14}
$$

$$
\begin{aligned}
T_{13}^3=&\,2\mu\frac{\partial H}{\partial x}\left\{2\frac{\partial u_0^2}{\partial x}-2\frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\frac{\partial v_0^2}{\partial y}-\frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_2}^1+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.\\
&\left.\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]\right\}+\mu\frac{\partial H}{\partial y}\left\{\frac{\partial u_0^2}{\partial y}-\frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}\right.\right.\\
&\left.-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\frac{\partial v_0^2}{\partial x}-\frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_2}^1+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}\right.
\end{aligned}
$$

$$- u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\bigg]\bigg\} + \left(\frac{\partial H}{\partial x}\right)^2\left[f^1_{R_1} + 2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$

$$+ \mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\bigg] + \frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left[f^1_{R_2} + 2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left. + \mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right] + f^3_{R_1} + f^1_{R_1}\left(\left(\frac{\partial H}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial H}{\partial y}\right)^2\right) \qquad \text{en } z = 0 \tag{5.6.15}$$

Obtenemos por tanto, la siguiente ecuación para el cálculo de u_0^2:

$$\frac{\partial u_0^2}{\partial t} + u^0\frac{\partial u_0^2}{\partial x} + u^1\frac{\partial u^1}{\partial x} + u_0^2\frac{\partial u^0}{\partial x} + v^0\frac{\partial u_0^2}{\partial y} + v^1\frac{\partial u^1}{\partial y} + v_0^2\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v_0^2$$

$$- \nu\left[2\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \frac{\partial^2 v_0^2}{\partial x\partial y} + \frac{2}{h}\frac{\partial h}{\partial x}\left(2\frac{\partial u_0^2}{\partial x} + \frac{\partial v_0^2}{\partial y}\right) + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u_0^2}{\partial y} + \frac{\partial v_0^2}{\partial x}\right)\right]$$

$$= -\frac{1}{\rho_0}\, D_x p^2\big|_{z=0} - 2\phi\,(\cos\varphi)\left(u^1\frac{\partial H}{\partial x} + v^1\frac{\partial H}{\partial y}\right) - \left\{\nu\left(2\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right.\right.$$

$$\left.\left. + \frac{4}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + \frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial y}\right) - \frac{h}{2}\frac{\partial v^0}{\partial y}\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}\right.$$

$$\left. - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] - \left(-4\nu\frac{\partial h}{\partial x} + \frac{h}{2}u^0\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$

$$\left. + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] - \left(\nu\frac{\partial H}{\partial y} - \nu\frac{\partial h}{\partial y} + \frac{h}{2}v^0\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}\right.$$

$$\left. + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$- \frac{h}{2}\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+ \nu h\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+ \nu\frac{h}{2}\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$- \left\{\nu\left(\frac{\partial^2 H}{\partial x\partial y} + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial y} + \frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right) + \frac{h}{2}\frac{\partial u^0}{\partial y} - \phi\,(\operatorname{sen}\varphi)\,h\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_2}\right.$$

$$+ \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\Bigg]$$

$$+ \nu\frac{\partial h}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+ \nu\left(\frac{\partial H}{\partial x} + 2\frac{\partial h}{\partial x}\right)\frac{\partial}{\partial y}\Bigg[\frac{1}{\nu\rho_0}f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y}$$

$$- v^0\frac{\partial^2 H}{\partial y^2}\Bigg] + \nu\frac{h}{2}\frac{\partial^2}{\partial x \partial y}\Bigg[\frac{1}{\nu\rho_0}f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y}$$

$$- v^0\frac{\partial^2 H}{\partial y^2}\Bigg] - \Bigg\{\nu\left[2\left(\frac{\partial H}{\partial x}\right)^2 + h\frac{\partial^2 H}{\partial x^2} + \frac{h}{2}\frac{\partial^2 H}{\partial y^2} + 4\frac{\partial H}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial h}{\partial y}\right]$$

$$- \frac{h^2}{6}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right)\Bigg\}\Bigg[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)$$

$$- \Delta_{xy}u^0\Bigg] - \left(-2\nu h\frac{\partial h}{\partial x} + \frac{h^2}{6}u^0\right)\frac{\partial}{\partial x}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x}$$

$$- 2\phi\left(\operatorname{sen}\varphi\right)v^0\Bigg) - \Delta_{xy}u^0\Bigg] - \left\{\nu\frac{h}{2}\left(\frac{\partial H}{\partial y} - \frac{\partial h}{\partial y}\right) + \frac{h^2}{6}v^0\right\}\frac{\partial}{\partial y}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x}$$

$$+ v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\Bigg) - \Delta_{xy}u^0\Bigg] - \frac{h^2}{6}\frac{\partial}{\partial t}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y}$$

$$+ \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\Bigg) - \Delta_{xy}u^0\Bigg] + \nu\frac{h^2}{3}\frac{\partial^2}{\partial x^2}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y}$$

$$+ \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\Bigg) - \Delta_{xy}u^0\Bigg] + \nu\frac{h^2}{6}\frac{\partial^2}{\partial y^2}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y}$$

$$+ \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\Bigg) - \Delta_{xy}u^0\Bigg] - \Bigg\{\nu\left(\frac{h}{2}\frac{\partial^2 H}{\partial x \partial y} + 2\frac{\partial s}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right)$$

$$+ \frac{h^2}{6}\frac{\partial u^0}{\partial y} - \frac{h^2}{3}\phi\left(\operatorname{sen}\varphi\right)\Bigg\}\Bigg[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right)$$

$$- \Delta_{xy}v^0\Bigg] + \nu\frac{h}{2}\frac{\partial h}{\partial y}\frac{\partial}{\partial x}\Bigg[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right)$$

$$- \Delta_{xy}v^0\Bigg] - \nu\left(\frac{h}{2}\frac{\partial H}{\partial x} - h\frac{\partial s}{\partial x}\right)\frac{\partial}{\partial y}\Bigg[\frac{1}{\nu}\Bigg(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y}$$

$$
\begin{aligned}
&+2\phi\,(\operatorname{sen}\varphi)\,u^0\big)-\Delta_{xy}v^0\big]+\nu\frac{h^2}{6}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.\right.\\
&+2\phi\,(\operatorname{sen}\varphi)\,u^0\big)-\Delta_{xy}v^0\big]+\frac{h}{2}\left\{\frac{\partial}{\partial x}\left((u^0)^2\frac{\partial^2 H}{\partial x^2}+2u^0v^0\frac{\partial^2 H}{\partial y\partial x}+(v^0)^2\frac{\partial^2 H}{\partial y^2}\right.\right.\\
&-2\phi\,(\cos\varphi)\,u^1+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x}+\frac{\partial f^1_{R_2}}{\partial y}\right)-\frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)\\
&-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.\\
&\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right)-\frac{\partial H}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.\\
&\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]\\
&\left.+2\phi\,(\cos\varphi)\left(\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)\right\}+\frac{h^2}{6}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.\\
&\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]\\
&+\frac{1}{\rho_0 h}\left\{\left(\frac{\partial s}{\partial x}\right)^2\left[2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\right.\\
&-\left(\frac{\partial H}{\partial x}\right)^2\left[2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\\
&+\frac{\partial s}{\partial x}\frac{\partial s}{\partial y}\left[f^1_{W_2}+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\\
&-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left[f^1_{R_2}+2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\\
&\left.+f^3_{W_1}-f^3_{R_1}+f^1_{W_1}\left(\frac{3}{2}\left(\frac{\partial s}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial s}{\partial y}\right)^2\right)-f^1_{R_1}\left(2\left(\frac{\partial H}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial H}{\partial y}\right)^2\right)\right\}
\end{aligned}
\tag{5.6.16}
$$

Repetimos el proceso a partir de (5.3.160) comenzando por sustituir u^2, v^2, p^2, w^1 y w^2 por las expresiones vistas en (5.6.5)-(5.6.7), (5.3.135) y (5.3.146).

A continuación, agrupamos en potencias de z e integramos respecto a esta variable entre $z=0$ y $z=1$, obteniendo:

$$
\begin{aligned}
&\frac{\partial v_0^2}{\partial t} + u^0\frac{\partial v_0^2}{\partial x} + u^1\frac{\partial v^1}{\partial x} + u_0^2\frac{\partial v^0}{\partial x} + v^0\frac{\partial v_0^2}{\partial y} + v^1\frac{\partial v^1}{\partial y} + v_0^2\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\left. D_y p^2\right|_{z=0} \\
&\quad + 2\phi\,(\operatorname{sen}\varphi)\,u_0^2 - \nu\frac{\partial^2 u_0^2}{\partial x\partial y} - \nu\frac{\partial^2 v_0^2}{\partial x^2} - 2\nu\frac{\partial^2 v_0^2}{\partial y^2} \\
&\quad + \nu\left\{\frac{\partial^2 H}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]\right. \\
&\quad + \frac{\partial H}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] \\
&\quad + \frac{\partial H}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] \\
&\quad - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right) - \Delta_{xy}u^0\right] \\
&\quad + \frac{\partial^2 H}{\partial x^2}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right] \\
&\quad + 2\frac{\partial H}{\partial x}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right] \\
&\quad - \left(\frac{\partial H}{\partial x}\right)^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy}v^0\right] \\
&\quad + 2\frac{\partial^2 H}{\partial y^2}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right] \\
&\quad + 4\frac{\partial H}{\partial y}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right] \\
&\quad \left. - 2\left(\frac{\partial H}{\partial y}\right)^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy}v^0\right]\right\} \\
&\quad + \frac{h}{2}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]\right.
\end{aligned}
$$

$$+ u^0 \frac{\partial}{\partial x} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$+ \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] \frac{\partial v^0}{\partial x}$$

$$+ v^0 \frac{\partial}{\partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$- \frac{\partial u^0}{\partial x} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$- \frac{\partial}{\partial y} \left((u^0)^2 \frac{\partial^2 H}{\partial x^2} + 2 u^0 v^0 \frac{\partial^2 H}{\partial y \partial x} + (v^0)^2 \frac{\partial^2 H}{\partial y^2} - 2\phi \left(\cos \varphi \right) u^1 + \frac{1}{\rho_0} \left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y} \right) \right.$$

$$- \frac{\partial H}{\partial x} \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \frac{\partial H}{\partial y} \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right)$$

$$\left. - 2\nu \left[\frac{\partial}{\partial x} \left(u^0 \frac{\partial^2 H}{\partial x^2} + v^0 \frac{\partial^2 H}{\partial y \partial x} \right) + \frac{\partial}{\partial y} \left(u^0 \frac{\partial^2 H}{\partial y \partial x} + v^0 \frac{\partial^2 H}{\partial y^2} \right) \right] \right)$$

$$+ \frac{\partial H}{\partial y} \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + 2 \frac{\partial v^0}{\partial x} \frac{\partial u^0}{\partial y} + 2 \frac{\partial u^0}{\partial x} \frac{\partial v^0}{\partial y} + 2 \left(\frac{\partial v^0}{\partial y} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} \right.$$

$$\left. + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial y^2} - 2\phi \left(\operatorname{sen} \varphi \right) \frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y} \left(\left(\operatorname{sen} \varphi \right) u^0 \right) \right]$$

$$+ 2\phi \operatorname{sen} \varphi \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right]$$

$$- \nu \left(\frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] \right.$$

$$- \frac{\partial^2 H}{\partial x \partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right]$$

$$- \frac{\partial H}{\partial x} \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right]$$

$$- \frac{\partial H}{\partial y} \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right]$$

$$+ \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$- \frac{\partial^2 H}{\partial x^2} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$- 2 \frac{\partial H}{\partial x} \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$+ 2 \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$- 2 \frac{\partial^2 H}{\partial y^2} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$- 4 \frac{\partial H}{\partial y} \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] \Bigg) \Bigg\}$$

$$+ \frac{h^2}{6} \Bigg\{ \frac{\partial}{\partial t} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$+ u^0 \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$+ \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] \frac{\partial v^0}{\partial x}$$

$$+ v^0 \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$- \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] \frac{\partial v^0}{\partial y}$$

$$- 2 \frac{\partial u^0}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right]$$

$$- \frac{\partial}{\partial y} \Bigg[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + 2 \frac{\partial v^0}{\partial x} \frac{\partial u^0}{\partial y} + 2 \frac{\partial u^0}{\partial x} \frac{\partial v^0}{\partial y} + 2 \left(\frac{\partial v^0}{\partial y} \right)^2$$

$$+ \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial y^2} - 2\phi \left(\operatorname{sen} \varphi \right) \frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y} \left(\left(\operatorname{sen} \varphi \right) u^0 \right) \Bigg]$$

$$+ 2\phi \left(\operatorname{sen} \varphi \right) \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right]$$

$$- \nu \left(\frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] \right.$$

$$+\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$+2\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\Bigg)\Bigg\}$$

$$-\frac{1}{\rho_0 h}\left(T^3_{23}\big|_{z=1}-T^3_{23}\big|_{z=0}\right)=0$$

Calculamos ahora, $T^3_{23}|_{z=1}-T^3_{23}|_{z=0}$, para lo que recurrimos a las expresiones (5.3.10) y (5.3.12). Igualando a 0 los términos multiplicados por ε^3 se obtiene:

$$T^3_{23}=\frac{\partial s}{\partial x}T^2_{12}+\frac{\partial s}{\partial y}T^2_{22}-\frac{\partial s}{\partial y}\left(-\frac{\partial s}{\partial x}T^1_{13}-\frac{\partial s}{\partial y}T^1_{23}+T^2_{33}\right)+f^3_{W_2}$$

$$+f^1_{W_2}\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)\qquad \text{en } z=1 \tag{5.6.17}$$

$$T^3_{23}=-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial H}{\partial x}T^0_{11}+\frac{\partial H}{\partial y}T^0_{12}-T^1_{13}\right)+\frac{\partial H}{\partial x}T^2_{12}+\frac{\partial H}{\partial y}T^2_{22}$$

$$+\left(\frac{\partial H}{\partial x}\right)^2\left(\frac{\partial H}{\partial x}T^0_{12}+\frac{\partial H}{\partial y}T^0_{22}-T^1_{23}\right)-\frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}T^1_{13}-\frac{\partial H}{\partial y}T^1_{23}+T^2_{33}\right)$$

$$+f^3_{R_2}+f^1_{R_2}\left(\left(\frac{3}{2}\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right)\qquad \text{en } z=0 \tag{5.6.18}$$

Teniendo en cuenta (5.4.3)-(5.4.11) y (5.6.11)-(5.6.13), y reemplazando u^2 y v^2 por sus expresiones dadas en (5.6.6)-(5.6.7), (5.6.17)-(5.6.18) resultan:

$$T^3_{23}=\mu\frac{\partial s}{\partial x}\Bigg\{\frac{\partial u^2_0}{\partial y}+h\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\frac{1}{2}h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]-\frac{\partial H}{\partial y}\left(\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}+h\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]\right)$$

$$+\frac{\partial v^2_0}{\partial x}+h\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+\frac{1}{2}h^2\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$-\frac{\partial H}{\partial x}\left(\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right.$$

$$\left.\left.+h\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\right)\right\}$$

$$+2\mu\frac{\partial s}{\partial y}\left\{\frac{\partial u_0^2}{\partial x}+h\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\frac{h^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]-\frac{\partial H}{\partial x}\left(\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}+h\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]\right)$$

$$+2\frac{\partial v_0^2}{\partial y}+2h\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$-2\frac{\partial H}{\partial y}\left(\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right.$$

$$\left.\left.+h\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\right)\right\}$$

$$+\frac{\partial s}{\partial y}\frac{\partial s}{\partial x}\left[f^1_{W_1}+2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+\left(\frac{\partial s}{\partial y}\right)^2\left[f^1_{W_2}+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]+f^3_{W_2}$$

$$+f^1_{W_2}\frac{1}{2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)\qquad \text{en } z=1 \tag{5.6.19}$$

$$T_{23}^3 = \mu\frac{\partial H}{\partial x}\left\{2\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial v^0}{\partial y} - \frac{\partial u^0}{\partial x}\right) + \left[\left(\frac{\partial H}{\partial x}\right)^2 - \left(\frac{\partial H}{\partial y}\right)^2\right]\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right\}$$

$$+ \mu\frac{\partial H}{\partial x}\left\{\frac{\partial u_0^2}{\partial y} - \frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}\right.\right.$$

$$\left. - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] + \frac{\partial v_0^2}{\partial x} - \frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}\right.$$

$$\left.\left. - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]\right\} + 2\mu\frac{\partial H}{\partial y}\left\{\frac{\partial u_0^2}{\partial x} - \frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\right.$$

$$\left. + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right] + 2\frac{\partial v_0^2}{\partial y} - 2\frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left.\left. + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]\right\} + 2\frac{\partial H}{\partial y}\frac{\partial H}{\partial x}\left[f_{R_1}^1 + 2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$

$$\left. + \mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right] + \left[\left(\frac{\partial H}{\partial y}\right)^2 - \left(\frac{\partial H}{\partial x}\right)^2\right]\left[f_{R_2}^1 + 2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left. + \mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right] + f_{R_2}^3 + f_{R_2}^1\left(\left(\frac{3}{2}\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2\right) \quad \text{en } z = 0 \qquad (5.6.20)$$

Para el cálculo de v_0^2 obtenemos la siguiente ecuación:

$$\frac{\partial v_0^2}{\partial t} + u^0\frac{\partial v_0^2}{\partial x} + u^1\frac{\partial v^1}{\partial x} + u_0^2\frac{\partial v^0}{\partial x} + v^0\frac{\partial v_0^2}{\partial y} + v^1\frac{\partial v^1}{\partial y} + v_0^2\frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u_0^2$$

$$- \nu\left[\frac{\partial^2 u_0^2}{\partial x\partial y} + \frac{\partial^2 v_0^2}{\partial x^2} + 2\frac{\partial^2 v_0^2}{\partial y^2} + \frac{2}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u_0^2}{\partial x} + 2\frac{\partial v_0^2}{\partial y}\right) + \frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u_0^2}{\partial y} + \frac{\partial v_0^2}{\partial x}\right)\right]$$

$$= -\frac{1}{\rho_0}\left.D_y p^2\right|_{z=0} - \left\{\nu\left(\frac{\partial^2 H}{\partial x\partial y} + \frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial y} + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right) + \frac{h}{2}\frac{\partial v^0}{\partial x}\right.$$

$$\left. + \phi\,(\operatorname{sen}\varphi)\,h\right\}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+ \nu\frac{\partial h}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+ \nu\left(\frac{\partial s}{\partial y} + \frac{\partial h}{\partial y}\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$- v^0 \frac{\partial^2 H}{\partial y \partial x}\Bigg] + \nu \frac{h}{2} \frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y \partial x} \right] - \left\{ \nu \left(\frac{\partial^2 H}{\partial x^2} + 2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{h} \frac{\partial h}{\partial x} \frac{\partial H}{\partial x} + \frac{4}{h} \frac{\partial h}{\partial y} \frac{\partial H}{\partial y} \right) - \frac{h}{2} \frac{\partial u^0}{\partial x} \right\} \left[\frac{1}{\nu \rho_0} f^1_{R_2} \right.$$

$$\left. + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right] - \left\{ \nu \left(\frac{\partial H}{\partial x} - \frac{\partial h}{\partial x} \right) \right.$$

$$\left. + \frac{h}{2} u^0 \right\} \frac{\partial}{\partial x} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$- \left(\frac{h}{2} v^0 - 4\nu \frac{\partial h}{\partial y} \right) \frac{\partial}{\partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] - \frac{h}{2} \left\{ \frac{\partial}{\partial t} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right. \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] - \nu \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] - 2\nu \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right.$$

$$\left. \left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] \right\} - \left\{ \nu \left(\frac{\partial H}{\partial y} \frac{\partial h}{\partial x} + 2 \frac{\partial s}{\partial y} \frac{\partial H}{\partial x} + \frac{h}{2} \frac{\partial^2 H}{\partial x \partial y} \right) + \frac{h^2}{6} \frac{\partial v^0}{\partial x} \right.$$

$$\left. + \frac{h^2}{3} \phi \left(\operatorname{sen} \varphi \right) \right\} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right]$$

$$- \nu \left(\frac{h}{2} \frac{\partial H}{\partial y} - h \frac{\partial s}{\partial y} \right) \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] + \nu \frac{h}{2} \frac{\partial h}{\partial x} \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] + \nu \frac{h^2}{6} \frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] - \left\{ \nu \left[\frac{\partial h}{\partial x} \frac{\partial H}{\partial x} + 2 \left(\frac{\partial H}{\partial y} \right)^2 + 4 \frac{\partial h}{\partial y} \frac{\partial H}{\partial y} + \frac{h}{2} \left(\frac{\partial^2 H}{\partial x^2} + 2 \frac{\partial^2 H}{\partial y^2} \right) \right] \right.$$

$$\left. - \frac{h^2}{6} \left(\frac{\partial v^0}{\partial y} + 2 \frac{\partial u^0}{\partial x} \right) \right\} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) \right.$$

$$
\begin{aligned}
&- \Delta_{xy} v^0 \Big] - \left\{ \nu \frac{h}{2} \left(\frac{\partial H}{\partial x} - \frac{\partial h}{\partial x} \right) + \frac{h^2}{6} u^0 \right\} \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} \right.\right. \\
&\left.\left. + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] + \left(2\nu h \frac{\partial h}{\partial y} - \frac{h^2}{6} v^0 \right) \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} \right.\right. \\
&\left.\left. + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] - \frac{h^2}{6} \left\{ \frac{\partial}{\partial t} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} \right.\right.\right. \\
&\left.\left. + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] - \nu \left(\frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} \right.\right.\right. \\
&\left.\left. + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] + 2 \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} \right.\right. \\
&\left.\left.\left.\left. + \frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] \right) \right\} + \frac{h}{2} \frac{\partial}{\partial y} \left\{ (u^0)^2 \frac{\partial^2 H}{\partial x^2} + 2 u^0 v^0 \frac{\partial^2 H}{\partial y \partial x} \right. \\
&+ (v^0)^2 \frac{\partial^2 H}{\partial y^2} - 2\phi \left(\cos \varphi \right) u^1 + \frac{1}{\rho_0} \left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y} \right) - \frac{\partial H}{\partial x} \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) \\
&- \frac{\partial H}{\partial y} \left(\frac{1}{\rho_0} \frac{\partial p_s}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - 2\nu \left[\frac{\partial}{\partial x} \left(u^0 \frac{\partial^2 H}{\partial x^2} + v^0 \frac{\partial^2 H}{\partial y \partial x} \right) \right. \\
&\left.\left. + \frac{\partial}{\partial y} \left(u^0 \frac{\partial^2 H}{\partial y \partial x} + v^0 \frac{\partial^2 H}{\partial y^2} \right) \right] \right\} - \frac{h}{2} \frac{\partial H}{\partial y} \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + 2 \frac{\partial v^0}{\partial x} \frac{\partial u^0}{\partial y} + 2 \frac{\partial u^0}{\partial x} \frac{\partial v^0}{\partial y} \right. \\
&\left. + 2 \left(\frac{\partial v^0}{\partial y} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial y^2} - 2\phi \left(\operatorname{sen} \varphi \right) \frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y} ((\operatorname{sen} \varphi) u^0) \right] \\
&+ \frac{h^2}{6} \frac{\partial}{\partial y} \left[2 \left(\frac{\partial u^0}{\partial x} \right)^2 + 2 \frac{\partial v^0}{\partial x} \frac{\partial u^0}{\partial y} + 2 \frac{\partial u^0}{\partial x} \frac{\partial v^0}{\partial y} + 2 \left(\frac{\partial v^0}{\partial y} \right)^2 \right. \\
&\left. + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0} \frac{\partial^2 p_s}{\partial y^2} - 2\phi \left(\operatorname{sen} \varphi \right) \frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y} ((\operatorname{sen} \varphi) u^0) \right] \\
&+ \frac{1}{\rho_0 h} \left\{ 2 \frac{\partial s}{\partial y} \frac{\partial s}{\partial x} \left[f^1_{W_1} + 2\mu \frac{\partial s}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \mu \frac{\partial s}{\partial y} \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right] \right. \\
&- 2 \frac{\partial H}{\partial y} \frac{\partial H}{\partial x} \left[f^1_{R_1} + 2\mu \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \mu \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right]
\end{aligned}
$$

$$+\left(\frac{\partial s}{\partial y}\right)^2\left[f^1_{W_2}+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$-\left[\left(\frac{\partial H}{\partial y}\right)^2-\left(\frac{\partial H}{\partial x}\right)^2\right]\left[f^1_{R_2}+2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+f^3_{W_2}-f^3_{R_2}+\frac{1}{2}f^1_{W_2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)-f^1_{R_2}\left(\left(\frac{3}{2}\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right)$$

$$-\mu\frac{\partial H}{\partial x}\left(2\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial v^0}{\partial y}-\frac{\partial u^0}{\partial x}\right)+\left[\left(\frac{\partial H}{\partial x}\right)^2-\left(\frac{\partial H}{\partial y}\right)^2\right]\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right)\Bigg\}$$

(5.6.21)

Una vez calculados u^2 y v^2, w^3 se calcula a partir de la ecuación (5.3.161):

$$D_x u^2+D_y v^2+D_z w^3=0$$

donde se sustituyen la expresiones obtenidas para u^2 y v^2 en (5.6.6) y (5.6.7):

$$D_z w^3=-\frac{\partial u^2_0}{\partial x}-zh\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left.+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$-\frac{1}{2}z^2h^2\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p^0}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]$$

$$+\frac{\partial H}{\partial x}\left\{\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right.$$

$$\left.+zh\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p^0}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]\right\}$$

$$-\frac{\partial v^2_0}{\partial y}-zh\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$-\frac{1}{2}z^2h^2\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p^0}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]$$

$$+\frac{\partial H}{\partial y}\left\{\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right.$$

$$+ zh\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right]\right\}$$

y se integra respecto de z imponiendo la condición de contorno (5.3.8), que teniendo en cuenta (5.6.6) y (5.6.7), resulta ser:

$$w^3 = u_0^2\frac{\partial H}{\partial x} + v_0^2\frac{\partial H}{\partial y} \text{ en } z = 0$$

De este modo, w^3 se puede escribir en función de u_0^2, v_0^2 y u^0, v^0 como sigue:

$$w^3 = u_0^2\frac{\partial H}{\partial x} + v_0^2\frac{\partial H}{\partial y} - hz\left\{\frac{\partial u_0^2}{\partial x} - \frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.\right.$$
$$\left. + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$
$$\left. - \frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]\right\}$$
$$- \frac{1}{2}z^2h^2\left\{\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]\right.$$
$$- \frac{\partial H}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right]$$
$$+ \frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]$$
$$\left. - \frac{\partial H}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right]\right\}$$
$$- \frac{1}{6}z^3h^3\left\{\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left[\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right] - \Delta_{xy}u^0\right]\right.$$
$$\left. + \frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right]\right\} \quad (5.6.22)$$

Usando las expresiones encontradas para p^0, p^1 y p^2, (5.3.131), (5.3.134) y (5.6.5) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia:

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 = p_s + \varepsilon \rho_0 h \left(1 - z\right)(g - 2\phi\,(\cos\varphi)\,u^0)$$
$$+ \varepsilon^2 \rho_0 h (1-z) \left\{ (u^0)^2 \frac{\partial^2 H}{\partial x^2} + 2u^0 v^0 \frac{\partial^2 H}{\partial y \partial x} + (v^0)^2 \frac{\partial^2 H}{\partial y^2} - 2\phi\,(\cos\varphi)\,u^1 \right.$$
$$+ \frac{1}{\rho_0}\left[\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right] - \frac{\partial H}{\partial x}\left[\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right] - \frac{\partial H}{\partial y}\left[\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right]$$
$$\left. - 2\nu \left[\frac{\partial}{\partial x}\left(u^0 \frac{\partial^2 H}{\partial x^2} + v^0 \frac{\partial^2 H}{\partial y \partial x}\right) + \frac{\partial}{\partial y}\left(u^0 \frac{\partial^2 H}{\partial y \partial x} + v^0 \frac{\partial^2 H}{\partial y^2}\right)\right]\right\}$$
$$+ \varepsilon^2 \frac{1}{2} \rho_0 h^2 (1 - z^2) \left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} \right.$$
$$\left. + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right] \quad (5.6.23)$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0, 1, 2, 3)$ ((5.3.126), (5.3.135), (5.3.146) y (5.6.22)) obtenemos una aproximación de la velocidad vertical en Ω:

$$\tilde{w}(\varepsilon) = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 = \varepsilon \left[u^0 \frac{\partial H}{\partial x} + v^0 \frac{\partial H}{\partial y} - hz \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right]$$
$$+ \varepsilon^2 \left[u^1 \frac{\partial H}{\partial x} + v^1 \frac{\partial H}{\partial y} - hz \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right] + \varepsilon^3 \left(u^2_0 \frac{\partial H}{\partial x} + v^2_0 \frac{\partial H}{\partial y} - hz \left\{ \frac{\partial u^2_0}{\partial x} \right.\right.$$
$$- \frac{\partial H}{\partial x}\left[\frac{1}{\nu\rho_0} f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x}\right]$$
$$\left. + \frac{\partial v^2_0}{\partial y} - \frac{\partial H}{\partial y}\left[\frac{1}{\nu\rho_0} f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2}\right]\right\}$$
$$- \frac{1}{2} z^2 h^2 \left\{ \frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0} f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x}\right] \right.$$
$$- \frac{\partial H}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p^0}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right) - \Delta_{xy} u^0\right]$$
$$+ \frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0} f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2}\right]$$

$$- \frac{\partial H}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p^0}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] \Bigg\}$$

$$- \frac{1}{6} z^3 h^3 \left\{ \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p^0}{\partial x} - 2\phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] \right.$$

$$\left. + \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p^0}{\partial y} + 2\phi \left(\operatorname{sen} \varphi \right) u^0 \right) - \Delta_{xy} v^0 \right] \right\} \Bigg)$$

$$= \varepsilon \left[\frac{\partial H}{\partial x} \left(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 \right) + \frac{\partial H}{\partial y} \left(v^0 + \varepsilon v^1 + \varepsilon^2 v^2 \right) \right.$$

$$\left. - zh \left(\frac{\partial u^0}{\partial x} + \varepsilon \frac{\partial u^1}{\partial x} + \varepsilon^2 D_x u^2 + \frac{\partial v^0}{\partial y} + \varepsilon \frac{\partial v^1}{\partial y} + \varepsilon^2 D_y v^2 \right) \right] + O(\varepsilon^3)$$

que se puede escribir

$$\tilde{w}(\varepsilon) = \varepsilon \frac{\partial H}{\partial x} \tilde{u}(\varepsilon) + \varepsilon \frac{\partial H}{\partial y} \tilde{v}(\varepsilon) - \varepsilon hz \left(D_x \tilde{u}(\varepsilon) + D_y \tilde{v}(\varepsilon) \right) + O(\varepsilon^3) \qquad (5.6.24)$$

Deshacemos ahora el cambio de variable, volviendo al dominio original, y obtenemos la siguiente aproximación de la solución en Ω^ε:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, y, z) = u^0(t, x, y) + \varepsilon u^1(t, x, y) + \varepsilon^2 u^2(t, x, y, z) \quad (5.6.25)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t, x, y, z) = v^0(t, x, y) + \varepsilon v^1(t, x, y) + \varepsilon^2 v^2(t, x, y, z) \quad (5.6.26)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, y, z) = \varepsilon w^1(t, x, y, z) + \varepsilon^2 w^2(t, x, y, z) + \varepsilon^3 w^3(t, x, y, z) \quad (5.6.27)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, y, z) = p^0(t, x, y) + \varepsilon p^1(t, x, y, z) + \varepsilon^2 p^2(t, x, y, z) \quad (5.6.28)$$

$$\tilde{f}^\varepsilon_{R_i} = \tilde{f}_{R_i}(\varepsilon) = \varepsilon f^1_{R_i} + \varepsilon^2 f^2_{R_i} + \varepsilon^3 f^3_{R_i} \quad (i = 1, 2) \qquad (5.6.29)$$

$$\tilde{f}^\varepsilon_{W_i} = \tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} + \varepsilon^2 f^2_{W_i} + \varepsilon^3 f^3_{W_i} \quad (i = 1, 2) \qquad (5.6.30)$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (5.6.23):

$$\tilde{p}^\varepsilon = p^\varepsilon_s - \rho_0 2\phi \left(\cos \varphi^\varepsilon \right) (s^\varepsilon - z^\varepsilon)(u^{0,\varepsilon} + u^{1,\varepsilon})$$

$$+ \rho_0 (s^\varepsilon - z^\varepsilon) \left\{ g + (u^{0,\varepsilon})^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + 2u^{0,\varepsilon} v^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + (v^{0,\varepsilon})^2 \frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2} \right.$$

$$+ \frac{1}{\rho_0} \left(\frac{\partial f^{1,\varepsilon}_{R_1}}{\partial x^\varepsilon} + \frac{\partial f^{1,\varepsilon}_{R_2}}{\partial y^\varepsilon} \right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon} \left(\frac{1}{\rho_0} \frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - 2\phi \left(\operatorname{sen} \varphi^\varepsilon \right) v^{0,\varepsilon} \right)$$

$$
\begin{aligned}
&- \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)u^{0,\varepsilon}\right) - 2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon}\right)\right.\\
&\left.\left. + \frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2}\right)\right]\right\}\\
&+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2 + 2\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + 2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right.\\
&\left. + 2\left(\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (y^\varepsilon)^2} - 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + 2\phi\frac{\partial}{\partial y^\varepsilon}\left(\left(\operatorname{sen}\varphi^\varepsilon\right)u^{0,\varepsilon}\right)\right]
\end{aligned}
\tag{5.6.31}
$$

donde $u^{0,\varepsilon} = u^0$, $v^{0,\varepsilon} = v^0$, $u^{1,\varepsilon} = \varepsilon u^1$, $f_{R_i}^{1,\varepsilon} = \varepsilon f_{R_i}^1$ $(i = 1, 2)$, tras el cambio de variable de Ω a Ω^ε.

De forma análoga, realizando el cambio de variable de Ω a Ω^ε en (5.6.24), llegamos a la siguiente expresión de la velocidad vertical:

$$
\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) + O(\varepsilon^3) \tag{5.6.32}
$$

Para el cálculo de la velocidad horizontal y el calado tenemos varias posibilidades. En primer lugar, se tiene el modelo que resulta de forma natural de aplicar el método de desarrollos asintóticos y que consiste en calcular u^0, v^0 y h resolviendo el sistema de ecuaciones (5.3.136), (5.4.12) y (5.4.13), una vez conocidos estos términos, se obtienen u^1 y v^1 a partir de (5.5.10)-(5.5.11), finalmente es necesario determinar u_0^2 y v_0^2 resolviendo (5.6.16) y (5.6.21). A continuación se construyen u^2 y v^2 según las expresiones (5.6.6) y (5.6.7), y la aproximación de las componentes de la velocidad horizontal en el dominio original resultan:

$$
\tilde{u}^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2
$$

$$
\tilde{v}^\varepsilon = v^0 + \varepsilon v^1 + \varepsilon^2 v^2
$$

mientras que la profundidad del agua se obtiene deshaciendo el cambio de variable, como $h^\varepsilon = \varepsilon h$.

Tanto la aproximación de la presión como la de la velocidad vertical se podrían construir también así, es decir, obteniendo p^k $\quad (k = 0, 1, 2)$ y w^k $\quad (k = 0, 1, 2, 3)$ y después usando las expresiones (5.6.27) y (5.6.28).

Este esquema que acabamos de señalar es, sin embargo, demasiado complicado. Requiere resolver varias veces ecuaciones similares para aportar una pequeña mejora (al menos desde el punto de vista formal) al resultado final. Sería deseable obtener un modelo similar al conseguido para la aproximación de orden uno (véase (5.5.20)), pero añadiendo algún término que mejorase la precisión. Para ello introducimos la notación

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{u}(\varepsilon)(t,x,y) = u^0(t,x,y) + \varepsilon u^1(t,x,y) + \varepsilon^2 u_0^2(t,x,y)$$

$$\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{v}(\varepsilon)(t,x,y) = v^0(t,x,y) + \varepsilon v^1(t,x,y) + \varepsilon^2 v_0^2(t,x,y)$$

(es decir, $\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, 0)$, $\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, 0)$) y observamos que:

$$\begin{aligned}
&\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \hat{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \right.\\
&\quad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\}\\
&\quad = \frac{\partial \hat{u}(\varepsilon)}{\partial t} + \hat{u}(\varepsilon)\frac{\partial \hat{u}(\varepsilon)}{\partial x} + \hat{v}(\varepsilon)\frac{\partial \hat{u}(\varepsilon)}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,\hat{v}(\varepsilon) - \nu \left\{ \Delta_{xy}\hat{u}(\varepsilon) + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial \hat{u}(\varepsilon)}{\partial x} \right.\\
&\quad \left. + \frac{1}{h^2}\frac{\partial}{\partial x^\varepsilon}\left[h^2\left(\frac{\partial \hat{u}(\varepsilon)}{\partial x} + \frac{\partial \hat{v}(\varepsilon)}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial \hat{u}(\varepsilon)}{\partial y} + \frac{\partial \hat{v}(\varepsilon)}{\partial x}\right)\right\}\\
&\quad = \frac{\partial u^0}{\partial t} + \varepsilon\frac{\partial u^1}{\partial t} + \varepsilon^2\frac{\partial u_0^2}{\partial t} + (u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x}\right)\\
&\quad + (v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y}\right) - 2\phi\,(\operatorname{sen}\varphi)\,(v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)\\
&\quad - \nu\left\{\Delta_{xy}u^0 + \varepsilon\Delta_{xy}u^1 + \varepsilon^2\Delta_{xy}u_0^2 + \frac{2}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x}\right)\right.\\
&\quad + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + \varepsilon^2\frac{\partial v_0^2}{\partial y}\right)\right]\\
&\quad \left. + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y} + \frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x} + \varepsilon^2\frac{\partial v_0^2}{\partial x}\right)\right\}\\
&\quad = \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu\left\{\Delta_{xy}u^0 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x}\right.\\
&\quad \left. + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right\}\\
&\quad + \varepsilon\left(\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^1\right.
\end{aligned}$$

$$-\nu\left\{\Delta_{xy}u^1+\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x}+\frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right]\right\}\Bigg)$$

$$+\varepsilon^2\left(\frac{\partial u_0^2}{\partial t}+u^0\frac{\partial u_0^2}{\partial x}+u^1\frac{\partial u^1}{\partial x}+u_0^2\frac{\partial u^0}{\partial x}+v^0\frac{\partial u_0^2}{\partial y}+v^1\frac{\partial u^1}{\partial y}+v_0^2\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)v_0^2\right.$$

$$\left.-\nu\left\{\Delta_{xy}u_0^2+\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u_0^2}{\partial x}+\frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u_0^2}{\partial x}+\frac{\partial v_0^2}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u_0^2}{\partial y}+\frac{\partial v_0^2}{\partial x}\right]\right\}\right)$$

$$+O(\varepsilon^3)$$

$$\frac{\partial\hat v^\varepsilon}{\partial t^\varepsilon}+\hat u^\varepsilon\frac{\partial\hat v^\varepsilon}{\partial x^\varepsilon}+\hat v^\varepsilon\frac{\partial\hat v^\varepsilon}{\partial y^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\hat u^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat v^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial\hat v^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\hat u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\hat v^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial\hat u^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\hat v^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$=\frac{\partial\hat v(\varepsilon)}{\partial t^\varepsilon}+\hat u(\varepsilon)\frac{\partial\hat v(\varepsilon)}{\partial x^\varepsilon}+\hat v(\varepsilon)\frac{\partial\hat v(\varepsilon)}{\partial y^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\hat u(\varepsilon)-\nu\left\{\Delta_{xy}\hat v(\varepsilon)+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial\hat v(\varepsilon)}{\partial y}\right.$$

$$\left.+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial\hat u(\varepsilon)}{\partial x}+\frac{\partial\hat v(\varepsilon)}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial\hat u(\varepsilon)}{\partial y}+\frac{\partial\hat v(\varepsilon)}{\partial x}\right]\right\}$$

$$=\frac{\partial v^0}{\partial t}+\varepsilon\frac{\partial v^1}{\partial t}+\varepsilon^2\frac{\partial v_0^2}{\partial t}+(u^0+\varepsilon u^1+\varepsilon^2u_0^2)\left(\frac{\partial v^0}{\partial x}+\varepsilon\frac{\partial v^1}{\partial x}+\varepsilon^2\frac{\partial v_0^2}{\partial x}\right)$$

$$+(v^0+\varepsilon v^1+\varepsilon^2v_0^2)\left(\frac{\partial v^0}{\partial y}+\varepsilon\frac{\partial v^1}{\partial y}+\varepsilon^2\frac{\partial v_0^2}{\partial y}\right)+2\phi\left(\operatorname{sen}\varphi\right)(u^0+\varepsilon u^1+\varepsilon^2u_0^2)$$

$$-\nu\left\{\Delta_{xy}v^0+\varepsilon\Delta_{xy}v^1+\varepsilon^2\Delta_{xy}v_0^2+\frac{2}{h}\frac{\partial h}{\partial y}\left(\frac{\partial v^0}{\partial y}+\varepsilon\frac{\partial v^1}{\partial y}+\varepsilon^2\frac{\partial v_0^2}{\partial y}\right)\right.$$

$$+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^0}{\partial x}+\varepsilon\frac{\partial u^1}{\partial x}+\varepsilon^2\frac{\partial u_0^2}{\partial x}+\frac{\partial v^0}{\partial y}+\varepsilon\frac{\partial v^1}{\partial y}+\varepsilon^2\frac{\partial v_0^2}{\partial y}\right)\right]$$

$$\left.+\frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y}+\varepsilon\frac{\partial u^1}{\partial y}+\varepsilon^2\frac{\partial u_0^2}{\partial y}+\frac{\partial v^0}{\partial x}+\varepsilon\frac{\partial v^1}{\partial x}+\varepsilon^2\frac{\partial v_0^2}{\partial x}\right)\right\}$$

$$=\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0-\nu\left\{\Delta_{xy}v^0+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^0}{\partial y}\right.$$

$$\left.+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right\}$$

$$+\varepsilon\left(\frac{\partial v^1}{\partial t}+u^0\frac{\partial v^1}{\partial x}+u^1\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^1}{\partial y}+v^1\frac{\partial v^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^1\right.$$

$$\left.-\nu\left\{\Delta_{xy}v^1+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^1}{\partial y}+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right]\right\}\right)$$

$$+\varepsilon^2\left(\frac{\partial v_0^2}{\partial t}+u^0\frac{\partial v_0^2}{\partial x}+u^1\frac{\partial v^1}{\partial x}+u_0^2\frac{\partial v^0}{\partial x}+v^0\frac{\partial v_0^2}{\partial y}+v^1\frac{\partial v^1}{\partial y}+v_0^2\frac{\partial v^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u_0^2\right.$$

$$\left.-\nu\left\{\Delta_{xy}v_0^2+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v_0^2}{\partial y}+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u_0^2}{\partial x}+\frac{\partial v_0^2}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u_0^2}{\partial y}+\frac{\partial v_0^2}{\partial x}\right]\right\}\right)$$

$$+O(\varepsilon^3)$$

y teniendo en cuenta (5.4.12), (5.4.13), (5.5.10), (5.5.11), (5.6.16) y (5.6.21) resulta:

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\hat{v}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\}$$

$$=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}+\frac{1}{h\rho_0}(f_{W_1}^1-f_{R_1}^1)$$

$$+\varepsilon\left[-\frac{\partial s}{\partial x}g+2\phi\,(\cos\varphi)\left(\frac{\partial u^0}{\partial x}h+\frac{\partial h}{\partial x}u^0-v^0\frac{\partial H}{\partial y}+\frac{h}{2}\frac{\partial v^0}{\partial y}\right)+\frac{1}{h\rho_0}(f_{W_1}^2-f_{R_1}^2)\right]$$

$$+\varepsilon^2\left[-\frac{1}{\rho_0}\left.D_xp^2\right|_{z=0}-2\phi\,(\cos\varphi)\left(u^1\frac{\partial H}{\partial x}+v^1\frac{\partial H}{\partial y}-\frac{h}{2}\frac{\partial v^1}{\partial y}\right)+\frac{1}{h\rho_0}(f_{W_1}^3-f_{R_1}^3)\right.$$

$$\left.+\Upsilon_1(h,u^0,v^0)\right]+O(\varepsilon^3)$$

$$=-\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}-2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\hat{u}^\varepsilon+\hat{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}-\frac{1}{2}h^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon-\tilde{f}_{R_1}^\varepsilon)$$

$$+\Upsilon_1^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})+O(\varepsilon^3) \tag{5.6.33}$$

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}+\hat{v}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+\frac{1}{h\rho_0}(f_{W_2}^1-f_{R_2}^1)$$

$$+\varepsilon\left[-\frac{\partial s}{\partial y}g+2\phi\left(\frac{1}{2}\frac{\partial}{\partial y}\left((\cos\varphi)\,u^0\right)h+\frac{\partial s}{\partial y}(\cos\varphi)\,u^0\right)+\frac{1}{h\rho_0}(f^2_{W_2}-f^2_{R_2})\right]$$

$$+\varepsilon^2\left[-\frac{1}{\rho_0}\,D_yp^2\Big|_{z=0}-h\phi\frac{\partial}{\partial y}\left((\cos\varphi)\,u^1\right)+\frac{1}{h\rho_0}(f^3_{W_2}-f^3_{R_2})+\Upsilon_2(h,u^0,v^0)\right]$$

$$+O(\varepsilon^3)=-\frac{1}{\rho_0}\left.\frac{\partial\tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}-h^\varepsilon\phi\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\hat{u}^\varepsilon)+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2}-\tilde{f}^\varepsilon_{R_2})$$

$$+\Upsilon^\varepsilon_2(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})+O(\varepsilon^3)\qquad(5.6.34)$$

donde

$$\Upsilon^\varepsilon_i(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})=\varepsilon^2\Upsilon_i(h,u^0,v^0)\quad(i=1,2)\qquad(5.6.35)$$

y,

$$\Upsilon_1(h,u^0,v^0)=-\left\{\nu\left(2\frac{\partial^2H}{\partial x^2}+\frac{\partial^2H}{\partial y^2}+\frac{4}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial x}+\frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial y}\right)\right.$$

$$\left.-\frac{h}{2}\frac{\partial v^0}{\partial y}\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}-v^0\frac{\partial^2H}{\partial y\partial x}\right]$$

$$-\left(-4\nu\frac{\partial h}{\partial x}+\frac{h}{2}u^0\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2H}{\partial y\partial x}\right]-\left(\nu\frac{\partial H}{\partial y}-\nu\frac{\partial h}{\partial y}+\frac{h}{2}v^0\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)\right.$$

$$\left.+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}-v^0\frac{\partial^2H}{\partial y\partial x}\right]$$

$$-\frac{h}{2}\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}-v^0\frac{\partial^2H}{\partial y\partial x}\right]$$

$$+\nu h\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}-v^0\frac{\partial^2H}{\partial y\partial x}\right]$$

$$+\nu\frac{h}{2}\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2H}{\partial x^2}-v^0\frac{\partial^2H}{\partial y\partial x}\right]$$

$$-\left\{\nu\left(\frac{\partial^2H}{\partial x\partial y}+\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial y}+\frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right)+\frac{h}{2}\frac{\partial u^0}{\partial y}-\phi\,(\operatorname{sen}\varphi)\,h\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_2}\right.$$

$$\left.+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2H}{\partial x\partial y}-v^0\frac{\partial^2H}{\partial y^2}\right]$$

$$+\nu\frac{\partial h}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+\nu\left(\frac{\partial H}{\partial x}+2\frac{\partial h}{\partial x}\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]+\nu\frac{h}{2}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]-\left\{\nu\left[2\left(\frac{\partial H}{\partial x}\right)^2+h\frac{\partial^2 H}{\partial x^2}+\frac{h}{2}\frac{\partial^2 H}{\partial y^2}+4\frac{\partial H}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial H}{\partial y}\frac{\partial h}{\partial y}\right]\right.$$

$$\left.-\frac{h^2}{6}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)\right\}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]-\left(-2\nu h\frac{\partial h}{\partial x}+\frac{h^2}{6}u^0\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}\right.\right.$$

$$\left.\left.-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]-\left\{\nu\frac{h}{2}\left(\frac{\partial H}{\partial y}-\frac{\partial h}{\partial y}\right)+\frac{h^2}{6}v^0\right\}\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}\right.\right.$$

$$\left.\left.+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]-\frac{h^2}{6}\frac{\partial}{\partial t}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}\right.\right.$$

$$\left.\left.+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]+\nu\frac{h^2}{3}\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}\right.\right.$$

$$\left.\left.+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]+\nu\frac{h^2}{6}\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}\right.\right.$$

$$\left.\left.+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]-\left\{\nu\left(\frac{h}{2}\frac{\partial^2 H}{\partial x\partial y}+2\frac{\partial s}{\partial x}\frac{\partial H}{\partial y}+\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right)\right.$$

$$\left.+\frac{h^2}{6}\frac{\partial u^0}{\partial y}-\frac{h^2}{3}\phi\left(\operatorname{sen}\varphi\right)\right\}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$\left.-\Delta_{xy}v^0\right]+\nu\frac{h}{2}\frac{\partial h}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$\left.-\Delta_{xy}v^0\right]-\nu\left(\frac{h}{2}\frac{\partial H}{\partial x}-h\frac{\partial s}{\partial x}\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.\right.$$

$$\left.\left.+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]+\nu\frac{h^2}{6}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.\right.$$

$$+ 2\phi\,(\operatorname{sen}\varphi)\,u^0\Big) - \Delta_{xy}v^0\Big] + \frac{h}{2}\left\{\frac{\partial}{\partial x}\left((u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x} + (v^0)^2\frac{\partial^2 H}{\partial y^2}\right.\right.$$

$$+ \frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right)$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left. + \frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x} + v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right) - \frac{\partial H}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.\left. + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]\right\}$$

$$+ \frac{h^2}{6}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left. + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+ \frac{1}{\rho_0 h}\left\{\left(\frac{\partial s}{\partial x}\right)^2\left[2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]\right.$$

$$- \left(\frac{\partial H}{\partial x}\right)^2\left[2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$+ \frac{\partial s}{\partial x}\frac{\partial s}{\partial y}\left[f^1_{W_2} + 2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$- \frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left[f^1_{R_2} + 2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$\left. + f^1_{W_1}\left(\frac{3}{2}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial s}{\partial y}\right)^2\right) - f^1_{R_1}\left(2\left(\frac{\partial H}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial H}{\partial y}\right)^2\right)\right\} \tag{5.6.36}$$

$$\Upsilon_2(h, u^0, v^0) = -\left\{\nu\left(\frac{\partial^2 H}{\partial x\partial y} + \frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial y} + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right) + \frac{h}{2}\frac{\partial v^0}{\partial x}\right.$$

$$\left. + \phi\,(\operatorname{sen}\varphi)\,h\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_1} + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+\nu\frac{\partial h}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]$$

$$+\nu\left(\frac{\partial s}{\partial y}+\frac{\partial h}{\partial y}\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\nu\frac{h}{2}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]-\left\{\nu\left(\frac{\partial^2 H}{\partial x^2}+2\frac{\partial^2 H}{\partial y^2}+\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial x}+\frac{4}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial y}\right)-\frac{h}{2}\frac{\partial u^0}{\partial x}\right\}\left[\frac{1}{\nu\rho_0}f^1_{R_2}\right.$$

$$\left.+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]-\left\{\nu\left(\frac{\partial H}{\partial x}-\frac{\partial h}{\partial x}\right)\right.$$

$$\left.+\frac{h}{2}u^0\right\}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$-\left(\frac{h}{2}v^0-4\nu\frac{\partial h}{\partial y}\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]-\frac{h}{2}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]-\nu\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]-2\nu\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left.\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]\right\}-\left\{\nu\left(\frac{\partial H}{\partial y}\frac{\partial h}{\partial x}+2\frac{\partial s}{\partial y}\frac{\partial H}{\partial x}+\frac{h}{2}\frac{\partial^2 H}{\partial x\partial y}\right)+\frac{h^2}{6}\frac{\partial v^0}{\partial x}\right.$$

$$\left.+\frac{h^2}{3}\phi\,(\operatorname{sen}\varphi)\right\}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)-\Delta_{xy}u^0\right]$$

$$-\nu\left(\frac{h}{2}\frac{\partial H}{\partial y}-h\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]+\nu\frac{h}{2}\frac{\partial h}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)\right.$$

$$\left.-\Delta_{xy}u^0\right]+\nu\frac{h^2}{6}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\,(\operatorname{sen}\varphi)\,v^0\right)\right.$$

$$- \Delta_{xy} u^0 \Big] - \Bigg\{ \nu \left[\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + 2\left(\frac{\partial H}{\partial y}\right)^2 + 4\frac{\partial h}{\partial y}\frac{\partial H}{\partial y} + \frac{h}{2}\left(\frac{\partial^2 H}{\partial x^2} + 2\frac{\partial^2 H}{\partial y^2}\right)\right]$$

$$- \frac{h^2}{6}\left(\frac{\partial v^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\right)\Bigg\} \left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right)\right.$$

$$\left. - \Delta_{xy} v^0\right] - \left\{\nu\frac{h}{2}\left(\frac{\partial H}{\partial x} - \frac{\partial h}{\partial x}\right) + \frac{h^2}{6}u^0\right\}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y}\right.\right.$$

$$\left.\left. + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy} v^0\right] + \left(2\nu h\frac{\partial h}{\partial y} - \frac{h^2}{6}v^0\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy} v^0\right] - \frac{h^2}{6}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy} v^0\right] - \nu\left(\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy} v^0\right] + 2\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y}\right.\right.$$

$$\left.\left.\left.\left. + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy} v^0\right]\right)\right\} + \frac{h}{2}\frac{\partial}{\partial y}\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x}\right.$$

$$+ (v^0)^2\frac{\partial^2 H}{\partial y^2} + \frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right)$$

$$- \frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left. + \frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x} + v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\} - \frac{h}{2}\frac{\partial H}{\partial y}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left. + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+ \frac{h^2}{6}\frac{\partial}{\partial y}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2\right.$$

$$\left. + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+\frac{1}{\rho_0 h}\left\{2\frac{\partial s}{\partial y}\frac{\partial s}{\partial x}\left[f^1_{W_1}+2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\right.$$

$$-2\frac{\partial H}{\partial y}\frac{\partial H}{\partial x}\left[f^1_{R_1}+2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+\left(\frac{\partial s}{\partial y}\right)^2\left[f^1_{W_2}+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$-\left[\left(\frac{\partial H}{\partial y}\right)^2-\left(\frac{\partial H}{\partial x}\right)^2\right]\left[f^1_{R_2}+2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+\frac{1}{2}f^1_{W_2}\left(\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right)-f^1_{R_2}\left(\left(\frac{3}{2}\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2\right)$$

$$\left.-\mu\frac{\partial H}{\partial x}\left(2\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial v^0}{\partial y}-\frac{\partial u^0}{\partial x}\right)+\left[\left(\frac{\partial H}{\partial x}\right)^2-\left(\frac{\partial H}{\partial y}\right)^2\right]\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right)\right\} \tag{5.6.37}$$

Una vez conocidos $\hat{u}^\varepsilon$ y $\hat{v}^\varepsilon$ veamos cómo se calculan $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ utilizando (5.6.6), (5.6.7), (5.6.25) y (5.6.26):

$$\tilde{u}^\varepsilon=\tilde{u}(\varepsilon)=u^0+\varepsilon u^1+\varepsilon^2 u^2=u^0+\varepsilon u^1+\varepsilon^2 u^2_0$$

$$+\varepsilon^2\left\{hz\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]\right.$$

$$\left.+\frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)-\Delta_{xy}u^0\right]\right\} \tag{5.6.38}$$

$$=\hat{u}^\varepsilon+(z^\varepsilon-H^\varepsilon)\left[\frac{1}{\rho_0\nu}f^{1,\varepsilon}_{R_1}+\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}+\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)+\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon}\right.$$

$$\left.-u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}-v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right]+\frac{1}{2\nu}(z^\varepsilon-H^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon}+u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}+v^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon}\right.$$

$$\left.+\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)v^{0,\varepsilon}-\nu\Delta_{x^\varepsilon y^\varepsilon}u^{0,\varepsilon}\right) \tag{5.6.39}$$

$$\tilde{v}^\varepsilon=\tilde{v}(\varepsilon)=v^0+\varepsilon v^1+\varepsilon^2 v^2=v^0+\varepsilon v^1+\varepsilon^2 v^2_0$$

$$+\varepsilon^2\left\{hz\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]\right.$$
$$\left.+\frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]\right\}$$
(5.6.40)

$$=\hat{v}^\varepsilon+(z^\varepsilon-H^\varepsilon)\left[\frac{1}{\rho_0\nu}f^{1,\varepsilon}_{R_2}+\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}+2\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)+\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\right.$$
$$\left.-u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial\partial x^\varepsilon\partial y^\varepsilon}-v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{(y^\varepsilon)^2}\right]+\frac{1}{2\nu}(z^\varepsilon-H^\varepsilon)^2\left(\frac{\partial v^{0,\varepsilon}}{\partial t^\varepsilon}+u^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}+v^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right.$$
$$\left.+\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)u^{0,\varepsilon}-\nu\Delta_{x^\varepsilon y^\varepsilon}v^{0,\varepsilon}\right)\tag{5.6.41}$$

A continuación proponemos un modelo de aguas someras resultado de despreciar los términos en ε^3 de (5.6.32)-(5.6.34). Se calcula en primer lugar una aproximación $\breve{u}^\varepsilon=u^{0,\varepsilon}+u^{1,\varepsilon}$, $\breve{v}^\varepsilon=v^{0,\varepsilon}+v^{1,\varepsilon}$ y h^ε utilizando (5.5.20), de modo que

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial(\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon}+\frac{\partial(\breve{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon}=0$$

$$\frac{\partial\breve{u}^\varepsilon}{\partial t^\varepsilon}+\breve{u}^\varepsilon\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}+\breve{v}^\varepsilon\frac{\partial\breve{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\breve{v}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\breve{u}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}\right.$$
$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial\breve{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\breve{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$
$$=-\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon}-\frac{\partial s^\varepsilon}{\partial x^\varepsilon}g+2\phi\left(\cos\varphi^\varepsilon\right)\left(\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}h^\varepsilon+\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\breve{u}^\varepsilon-\breve{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}+\frac{1}{2}h^\varepsilon\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right)$$
$$+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1}-\tilde{f}^\varepsilon_{R_1})$$

$$\frac{\partial\breve{v}^\varepsilon}{\partial t^\varepsilon}+\breve{u}^\varepsilon\frac{\partial\breve{v}^\varepsilon}{\partial x^\varepsilon}+\breve{v}^\varepsilon\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\breve{u}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\breve{v}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right.$$
$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial\breve{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial\breve{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$
$$=-\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial y^\varepsilon}-\frac{\partial s^\varepsilon}{\partial y^\varepsilon}g+2\phi\left(\frac{1}{2}h^\varepsilon\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\breve{u}^\varepsilon)+(\cos\varphi^\varepsilon)\,\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\breve{u}^\varepsilon\right)$$

$$+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2}-\tilde{f}^\varepsilon_{R_2})$$

A continuación se calcula $\tilde{p}^\varepsilon$ usando la expresión (5.6.31), donde los términos en $u^{0,\varepsilon}$ y $v^{0,\varepsilon}$ son sustituidos por $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$, respectivamente, introduciendo un error $O(\varepsilon^3)$:

$$\begin{aligned}
\tilde{p}^\varepsilon &= p^\varepsilon_s - \rho_0 2\phi\,(\cos\varphi^\varepsilon)\,[(s^\varepsilon - z^\varepsilon)\breve{u}^\varepsilon] \\
&+ \rho_0(s^\varepsilon - z^\varepsilon)\left\{g + (\breve{u}^\varepsilon)^2\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + 2\breve{u}^\varepsilon\breve{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon} + (\breve{v}^\varepsilon)^2\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2}\right. \\
&+ \frac{1}{\rho_0}\left(\frac{\partial f^{1,\varepsilon}_{R_1}}{\partial x^\varepsilon} + \frac{\partial f^{1,\varepsilon}_{R_2}}{\partial y^\varepsilon}\right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{v}^\varepsilon\right) \\
&- \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{u}^\varepsilon\right) - 2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(\breve{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + \breve{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}\right)\right. \\
&\left.\left. + \frac{\partial}{\partial y^\varepsilon}\left(\breve{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon} + \breve{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2}\right)\right]\right\} \\
&+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 + 2\frac{\partial\breve{v}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\breve{u}^\varepsilon}{\partial y^\varepsilon} + 2\frac{\partial\breve{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right. \\
&\left. + 2\left(\frac{\partial\breve{v}^\varepsilon}{\partial y^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p^\varepsilon_s}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial^2 p^\varepsilon_s}{\partial(y^\varepsilon)^2} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial\breve{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,\breve{u}^\varepsilon)\right]
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon}\right. \\
&\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\} = -\frac{1}{\rho_0}\left.\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \\
&- 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\hat{u}^\varepsilon + \hat{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \frac{1}{2}h^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1}) + \Upsilon^\varepsilon_1(h^\varepsilon, \breve{u}^\varepsilon, \breve{v}^\varepsilon)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right. \\
&\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\} \\
&= -\frac{1}{\rho_0}\left.\frac{\partial\tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} - h^\varepsilon\phi\frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\hat{u}^\varepsilon) + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2}) + \Upsilon^\varepsilon_2(h^\varepsilon, \breve{u}^\varepsilon, \breve{v}^\varepsilon)
\end{aligned}$$

$$\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu} f_{R_1}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} - u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right.$$

$$\left. - v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left. - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, v^{0,\varepsilon} - \nu\Delta_{x^\varepsilon y^\varepsilon} u^{0,\varepsilon}\right)$$

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu} f_{R_2}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + 2\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} - u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial\partial x^\varepsilon \partial y^\varepsilon}\right.$$

$$\left. - v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{(y^\varepsilon)^2}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial v^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left. + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, u^{0,\varepsilon} - \nu\Delta_{x^\varepsilon y^\varepsilon} v^{0,\varepsilon}\right)$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.6.42}$$

donde $\Upsilon_i^\varepsilon(h^\varepsilon, \breve{u}^\varepsilon, \breve{v}^\varepsilon)$ $(i = 1, 2)$, vienen dados por (5.6.35)-(5.6.37) (donde los términos en $u^{0,\varepsilon}$ y $v^{0,\varepsilon}$ son sustituidos por $\breve{u}^\varepsilon$ y $\breve{v}^\varepsilon$, respectivamente, introduciendo un error $O(\varepsilon^3)$), expresiones en las que está claro que $\Upsilon_i^\varepsilon(h^\varepsilon, \breve{u}^\varepsilon, \breve{v}^\varepsilon) = O(\varepsilon^2)$ $(i = 1, 2)$, por lo que el modelo (5.6.42) es equivalente al siguiente (en el sentido de que tan solo se diferencian en términos $O(\varepsilon^3)$):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\breve{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial(\breve{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{5.6.43}$$

$$\frac{\partial \breve{u}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{v}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\breve{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon}g + 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon}h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon}\breve{u}^\varepsilon - \breve{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2}h^\varepsilon\frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right)$$

$$+ \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon) \tag{5.6.44}$$

$$\frac{\partial \breve{v}^\varepsilon}{\partial t^\varepsilon} + \breve{u}^\varepsilon\frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon} + \breve{v}^\varepsilon\frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\breve{u}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\breve{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \breve{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \breve{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \breve{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g + 2\phi\left(\frac{1}{2}h^\varepsilon \frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\check{u}^\varepsilon) + (\cos\varphi^\varepsilon)\,\frac{\partial s^\varepsilon}{\partial y^\varepsilon}\check{u}^\varepsilon\right)$$

$$+ \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_2}^\varepsilon - \tilde{f}_{R_2}^\varepsilon) \tag{5.6.45}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon - \rho_0 2\phi\,(\cos\varphi^\varepsilon)\,[(s^\varepsilon - z^\varepsilon)\check{u}^\varepsilon]$$

$$+ \rho_0(s^\varepsilon - z^\varepsilon)\left\{g + (\check{u}^\varepsilon)^2\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + 2\check{u}^\varepsilon\check{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon} + (\check{v}^\varepsilon)^2\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2}\right.$$

$$+ \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon}\right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\check{v}^\varepsilon\right)$$

$$- \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\check{u}^\varepsilon\right) - 2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + \check{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}\right)\right.$$

$$\left.\left. + \frac{\partial}{\partial y^\varepsilon}\left(\check{u}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon} + \check{v}^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2}\right)\right]\right\}$$

$$+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\right)^2 + 2\frac{\partial\check{v}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\check{u}^\varepsilon}{\partial y^\varepsilon} + 2\frac{\partial\check{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon} + 2\left(\frac{\partial\check{v}^\varepsilon}{\partial y^\varepsilon}\right)^2\right.$$

$$\left. + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(y^\varepsilon)^2} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial\check{v}^\varepsilon}{\partial x^\varepsilon} + 2\phi\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,\check{u}^\varepsilon)\right] \tag{5.6.46}$$

$$\frac{\partial\hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\} = -\frac{1}{\rho_0}\left.\frac{\partial\tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon}$$

$$- 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\hat{u}^\varepsilon + \hat{v}^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon} - \frac{1}{2}h^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon)$$

$$+ \Upsilon_1^\varepsilon(h^\varepsilon, \check{u}^\varepsilon, \check{v}^\varepsilon) \tag{5.6.47}$$

$$\frac{\partial\hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial\hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial\hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial\hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$= -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon} - h^\varepsilon \phi \frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\,\hat{u}^\varepsilon) + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2}) + \Upsilon^\varepsilon_2(h^\varepsilon, \check{u}^\varepsilon, \check{v}^\varepsilon) \tag{5.6.48}$$

$$\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0 \nu} f^{1,\varepsilon}_{R_1} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - \hat{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right.$$
$$\left. - \hat{v}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon}\right.$$
$$\left. - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu \Delta_{x^\varepsilon y^\varepsilon} \hat{u}^\varepsilon\right) \tag{5.6.49}$$

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0 \nu} f^{1,\varepsilon}_{R_2} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + 2\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} - \hat{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial \partial x^\varepsilon \partial y^\varepsilon}\right.$$
$$\left. - \hat{v}^\varepsilon \frac{\partial^2 H^\varepsilon}{(y^\varepsilon)^2}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon}\right.$$
$$\left. + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon - \nu \Delta_{x^\varepsilon y^\varepsilon} \hat{v}^\varepsilon\right) \tag{5.6.50}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.6.51}$$

Veamos en qué medida verifica la aproximación de segundo orden las ecuaciones de Navier-Stokes.

Si promediamos en altura la primera y segunda ecuaciones de Navier-Stokes, y usamos las igualdades (5.4.12), (5.4.13), (5.5.10), (5.5.11), (5.6.16), (5.6.21), (5.6.38) y (5.6.40), obtenemos que se verifican con un error $O(\varepsilon^3)$. Los cálculos a realizar son semejantes a los que hicimos en las secciones 5.4 y 5.5, pero debido a su extensión, preferimos omitirlos.

Siguiendo también los pasos de las secciones 5.4 y 5.5 obtenemos que

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial y^\varepsilon} + \tilde{w}^\varepsilon \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} + \frac{1}{\rho_0}\frac{\partial \tilde{p}^\varepsilon}{\partial z^\varepsilon} + g - 2\phi\,(\cos\varphi^\varepsilon)\,\tilde{u}^\varepsilon - \frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{13}}{\partial x^\varepsilon} - \frac{1}{\rho_0}\frac{\partial \tilde{T}^\varepsilon_{23}}{\partial y^\varepsilon}$$
$$- 2\nu \frac{\partial^2 w^\varepsilon}{\partial (z^\varepsilon)^2} = O(\varepsilon^2)$$

y que la ecuación de incompresibilidad no se verifica de forma exacta:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{w}^\varepsilon}{\partial z^\varepsilon} = O(\varepsilon^2)$$

Por construcción de $\tilde{p}^\varepsilon$ y $\tilde{w}^\varepsilon$ (véase (5.6.46) y (5.6.51)) las condiciones de contorno (5.1.4) y (5.1.5) se verifican exactamente.

Finalmente, también podemos comprobar las condiciones de contorno que recogen los efectos del viento en la superficie y del rozamiento en el fondo, (5.1.35)-(5.1.38), se verifican con un error de orden $O(\varepsilon^4)$.

5.7. Modelo propuesto

De los modelos obtenidos en este capítulo vamos a tratar de justificar cuál es el que consideramos "mejor" y por tanto proponemos. El modelo de segundo orden parece demasiado complicado, sin que resulte evidente que su precisión sea mayor, y presenta además algunos inconvenientes, como que la ecuación de incompresibilidad no se verifica exactamente. Como el modelo de orden cero, aunque sencillo, es claramente menos preciso que el de orden uno, proponemos el modelo (5.5.20) (suprimiendo la ˜ para simplificar la notación):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (u^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (v^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{5.7.1}$$

$$\begin{aligned}
&\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, v^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} u^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right.\\
&\qquad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial u^\varepsilon}{\partial y^\varepsilon} + \frac{\partial v^\varepsilon}{\partial x^\varepsilon}\right]\right\}\\
&\qquad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + 2\phi\,(\cos\varphi^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} h^\varepsilon + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} u^\varepsilon - v^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{2} h^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\\
&\qquad + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1})
\end{aligned} \tag{5.7.2}$$

$$\begin{aligned}
&\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, u^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} v^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial v^\varepsilon}{\partial y^\varepsilon} \right.\\
&\qquad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial u^\varepsilon}{\partial y^\varepsilon} + \frac{\partial v^\varepsilon}{\partial x^\varepsilon}\right]\right\}\\
&\qquad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g + 2\phi\left(\frac{1}{2} h^\varepsilon \frac{\partial}{\partial y^\varepsilon}((\cos\varphi^\varepsilon)\, u^\varepsilon) + (\cos\varphi^\varepsilon)\frac{\partial s^\varepsilon}{\partial y^\varepsilon} u^\varepsilon\right)\\
&\qquad + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2})
\end{aligned} \tag{5.7.3}$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\,[g - 2\phi\,(\cos\varphi^\varepsilon)\, u^\varepsilon] \tag{5.7.4}$$

$$w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.7.5}$$

que se puede escribir en forma vectorial como sigue:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - \nu \left\{ \Delta \vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla \vec{\mathbf{u}}^\varepsilon)^T + \nabla \vec{\mathbf{u}}^\varepsilon\right] \nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2} \nabla \left[(h^\varepsilon)^2 (\operatorname{div} \vec{\mathbf{u}}^\varepsilon)\right] \right\}$$

$$+ g \nabla h^\varepsilon = -\frac{1}{\rho_0} \nabla p_s^\varepsilon - g \nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon\right)$$

$$w^\varepsilon = \vec{\mathbf{u}}^\varepsilon \cdot \nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon) \operatorname{div} \vec{\mathbf{u}}^\varepsilon$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon)\left[g - 2\phi (\cos\varphi^\varepsilon) u^\varepsilon\right] \tag{5.7.6}$$

donde

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon) v^\varepsilon + \cos\varphi^\varepsilon \left(\dfrac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \dfrac{h^\varepsilon}{2}\dfrac{\partial v^\varepsilon}{\partial y^\varepsilon} - v^\varepsilon \dfrac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\operatorname{sen}\varphi^\varepsilon) u^\varepsilon + \dfrac{h^\varepsilon}{2}\dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon) u^\varepsilon] + \dfrac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon) u^\varepsilon] \end{pmatrix}$$

y h^ε y $\vec{\mathbf{u}}^\varepsilon$ son independientes de z^ε.

Sin realizar un mayor esfuerzo podríamos sustituir la expresión de la presión en (5.5.20) por la aproximación de segundo orden (5.6.31) en la que, para no calcular de nuevo $u^{0,\varepsilon}$ y $v^{0,\varepsilon}$ los sustituimos por $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ (lo que no afecta al orden de la aproximación).

$$p^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) \left\{ g - 2\phi (\cos\varphi^\varepsilon) u^\varepsilon + (u^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + 2 u^\varepsilon v^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} \right.$$

$$+ (v^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2} + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon}\right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - 2\phi (\operatorname{sen}\varphi^\varepsilon) v^\varepsilon\right)$$

$$- \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} + 2\phi (\operatorname{sen}\varphi^\varepsilon) u^\varepsilon\right) - 2\nu \left[\frac{\partial}{\partial x^\varepsilon}\left(u^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + v^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon}\right)\right.$$

$$\left.\left. + \frac{\partial}{\partial y^\varepsilon}\left(u^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + v^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2}\right)\right]\right\}$$

$$+ \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right)^2 + 2\frac{\partial v^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial y^\varepsilon} + 2\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\frac{\partial v^\varepsilon}{\partial y^\varepsilon} + 2\left(\frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)^2\right.$$

$$\left. + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (y^\varepsilon)^2} - 2\phi (\operatorname{sen}\varphi^\varepsilon)\frac{\partial v^\varepsilon}{\partial x^\varepsilon} + 2\phi \frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon) u^\varepsilon)\right] \tag{5.7.7}$$

5.8. Simplificaciones de la oceanografía dinámica

En la sección 4.10 estudiamos lo que sucedía si tomábamos como ecuaciones de partida las ecuaciones de Euler y suponíamos que la ecuación de Coriolis se puede aproximar por (1.1.9) (que es la hipótesis habitual en oceanografía dinámica). En esta sección realizaremos el mismo estudio, pero partiendo ahora de las ecuaciones de Navier-Stokes.

5.8.1. Ecuaciones de partida

Si se realizan las simplificaciones expuestas en la Observación 1.1 sobre el término debido a la aceleración de Coriolis, tomándola según (1.1.9):

$$-2\vec{\phi} \times \vec{\mathbf{u}} = -2\phi\left(-\left(\operatorname{sen}\varphi\right)v, \left(\operatorname{sen}\varphi\right)u, 0\right)$$

las ecuaciones de Navier-Stokes se escriben ahora:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\phi\left(\operatorname{sen}\varphi\right)v + \nu\Delta u \tag{5.8.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\phi\left(\operatorname{sen}\varphi\right)u + \nu\Delta v \tag{5.8.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \nu\Delta w \tag{5.8.3}$$

5.8.2. Ecuaciones en el dominio de referencia

Bajo las hipótesis realizadas, las ecuaciones de Navier-Stokes en el dominio de referencia Ω son las siguientes:

$$\begin{aligned} D_t u(\varepsilon) + u(\varepsilon)D_x u(\varepsilon) + v(\varepsilon)D_y u(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z u(\varepsilon) &= -\frac{1}{\rho_0}D_x p(\varepsilon) \\ &+ 2\phi\left(\operatorname{sen}\varphi\right)v(\varepsilon) + \nu\left(2D_x^2 u(\varepsilon) + D_y^2 u(\varepsilon) + D_{xy}^2 v(\varepsilon)\right) + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z T_{13}(\varepsilon) \end{aligned} \tag{5.8.4}$$

$$\begin{aligned} D_t v(\varepsilon) + u(\varepsilon)D_x v(\varepsilon) + v(\varepsilon)D_y v(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z v(\varepsilon) &= -\frac{1}{\rho_0}D_y p(\varepsilon) \\ &- 2\phi\left(\operatorname{sen}\varphi\right)u(\varepsilon) + \nu\left(D_{xy}^2 u(\varepsilon) + D_x^2 v(\varepsilon) + 2D_y^2 v(\varepsilon)\right) + \frac{1}{\rho_0}\frac{1}{\varepsilon}D_z T_{23}(\varepsilon) \end{aligned} \tag{5.8.5}$$

$$D_t w(\varepsilon) + u(\varepsilon)D_x w(\varepsilon) + v(\varepsilon)D_y w(\varepsilon) + w(\varepsilon)\frac{1}{\varepsilon}D_z w(\varepsilon) = -\frac{1}{\rho_0}\frac{1}{\varepsilon}D_z p(\varepsilon)$$

$$- g + 2\nu \frac{1}{\varepsilon^2} D_z^2 w(\varepsilon) + \frac{1}{\rho_0} D_x T_{13}(\varepsilon) + \frac{1}{\rho_0} D_y T_{23}(\varepsilon) \tag{5.8.6}$$

El problema además está determinado por la condición de incompresibilidad ((5.2.9)), las condiciones de contorno e iniciales ((5.2.10)-(5.2.18)), la ecuación para el cálculo del calado ((5.2.19)), las componentes del tensor de tensiones ((5.2.20)-(5.2.25)) y y los laplacianos en términos de las componentes del tensor de tensiones ((5.2.26)-(5.2.28)) que no varían bajo estas nuevas hipótesis.

5.8.3. Desarrollo asintótico en ε

Suponemos ahora que la solución del problema (5.2.9)-(5.2.28), (5.8.4)-(5.8.6) admite un desarrollo en serie de potencias de ε en la forma (5.3.1). Se sustituye este desarrollo en las ecuaciones que han variado respecto a las secciones anteriores. Agrupamos los términos multiplicados por la misma potencia de ε e igualando a cero los coeficientes, obtenemos las siguientes ecuaciones:

$$p^1 = \rho_0 h g (1 - z) \tag{5.8.7}$$

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} D_x p^1 - 2\phi (\operatorname{sen} \varphi) v^1$$
$$- \nu \left(2 \frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial x \partial y} \right) - \frac{1}{\rho_0} D_z T_{13}^2 = 0 \tag{5.8.8}$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + \frac{1}{\rho_0} D_y p^1 + 2\phi (\operatorname{sen} \varphi) u^1$$
$$- \nu \left(\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 v^1}{\partial x^2} + 2 \frac{\partial^2 v^1}{\partial y^2} \right) - \frac{1}{\rho_0} D_z T_{23}^2 = 0 \tag{5.8.9}$$

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3$$
$$- \frac{1}{\rho_0} \left(D_x T_{13}^1 + D_y T_{23}^1 \right) = 0 \tag{5.8.10}$$

$$D_t u^2 + u^0 D_x u^2 + u^1 \frac{\partial u^1}{\partial x} + u^2 \frac{\partial u^0}{\partial x} + v^0 D_y u^2 + v^1 \frac{\partial u^1}{\partial y} + v^2 \frac{\partial u^0}{\partial y} + w^1 D_z u^2$$
$$+ \frac{1}{\rho_0} D_x p^2 - 2\phi (\operatorname{sen} \varphi) v^2 - \nu \left(2 D_x^2 u^2 + D_y^2 u^2 + D_{xy}^2 v^2 \right) - \frac{1}{\rho_0} D_z T_{13}^3 = 0 \tag{5.8.11}$$

$$D_t v^2 + u^0 D_x v^2 + u^1 \frac{\partial v^1}{\partial x} + u^2 \frac{\partial v^0}{\partial x} + v^0 D_y v^2 + v^1 \frac{\partial v^1}{\partial y} + v^2 \frac{\partial v^0}{\partial y} + w^1 D_z v^2 + \frac{1}{\rho_0} D_y p^2$$

$$+ 2\phi\,(\operatorname{sen}\varphi)\,u^2 - \nu\left(D^2_{xy}u^2 + D^2_x v^2 + 2D^2_y v^2\right) - \frac{1}{\rho_0}D_z T^3_{23} = 0 \qquad (5.8.12)$$

que junto con (5.3.121)-(5.3.133), (5.3.135)-(5.3.142), (5.3.146)-(5.3.158) y (5.3.161)-(5.3.171) nos permitirán el cálculo de h, u^k, v^k, w^k y p^k y a partir de ellos construiremos una aproximación de la solución del problema (5.2.9)-(5.2.28), (5.8.4)-(5.8.6).

5.8.4. Aproximación de orden cero

Si se considera la aproximación de orden cero en ε:

$$\tilde{u}(\varepsilon) = u^0$$

$$\tilde{v}(\varepsilon) = v^0$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1$$

$$\tilde{p}(\varepsilon) = p^0$$

$$\tilde{T}_{ij}(\varepsilon) = T^0_{ij} \quad (i,j = 1,2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1}T^{-1}_{i3} + T^0_{i3} + \varepsilon T^1_{i3} \quad (i = 1,2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1}T^{-1}_{33} + T^0_{33}$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f^1_{R_i} \quad (i = 1,2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} \quad (i = 1,2)$$

y se obtiene el mismo modelo ((5.4.27)-(5.4.31)) que en la sección 5.4:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon \tilde{v}^\varepsilon)}{\partial y^\varepsilon} = 0$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$
$$\left. + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1})$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]\right.$$
$$\left. + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\} = -\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2})$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon \tag{5.8.13}$$

5.8.5. Aproximación de primer orden

Se considera, ahora, la siguiente aproximación de orden 1 en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1$$

$$\tilde{T}_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 \quad (i, j = 1, 2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1} T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 + \varepsilon^2 T_{i3}^2 \quad (i = 1, 2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1} T_{33}^{-1} + T_{33}^0 + \varepsilon T_{33}^1$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^1 + \varepsilon^2 f_{R_i}^2 \quad (i = 1, 2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f_{W_i}^1 + \varepsilon^2 f_{W_i}^2 \quad (i = 1, 2)$$

Recordemos que w^0, p^0, $T_{i3}^k \quad (k = -1, 0, i = 1, 2)$ y T_{33}^{-1} son conocidos ((5.3.121), (5.3.122), (5.3.125)-(5.3.128), (5.3.131)), u^0, v^0 y h se calculan resolviendo (5.4.12), (5.4.13) y (5.3.136) y w^1 está determinado por (5.3.135) en función de u^0, v^0 y h.

También tenemos una expresión para p^1 en la que sólo es necesario conocer la profundidad del agua ((5.8.7)):

$$p^1 = \rho_0 h g (1 - z)$$

Para obtener u^1 y v^1 es necesario resolver (5.8.8)-(5.8.9). Para ello, se sustituye p^1 por la expresión anterior de modo que:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v^1$$

$$- \nu \left(2 \frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial^2 v^1}{\partial y \partial x}\right) = -\frac{\partial s}{\partial x} g + \frac{1}{\rho_0} D_z T_{13}^2$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^1$$

$$-\nu\left(\frac{\partial^2 u^1}{\partial y\partial x}+\frac{\partial^2 v^1}{\partial x^2}+2\frac{\partial^2 v^1}{\partial y^2}\right)=-\frac{\partial s}{\partial y}g+\frac{1}{\rho_0}D_z T_{23}^2$$

Como T_{13}^2 y T_{23}^2 sólo son conocidos en $z=0$ y $z=1$ y u^0, v^0, u^1, v^1 y s no dependen de z, se integra la ecuación respecto de z entre 0 y 1, obteniendo:

$$\frac{\partial u^1}{\partial t}+u^0\frac{\partial u^1}{\partial x}+u^1\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^1}{\partial y}+v^1\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)v^1$$

$$-\nu\left(2\frac{\partial^2 u^1}{\partial x^2}+\frac{\partial^2 u^1}{\partial y^2}+\frac{\partial^2 v^1}{\partial y\partial x}\right)=-\frac{\partial s}{\partial x}g+\frac{1}{h\rho_0}\left(T_{13}^2\big|_{z=1}-T_{13}^2\big|_{z=0}\right)$$

$$\frac{\partial v^1}{\partial t}+u^0\frac{\partial v^1}{\partial x}+u^1\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^1}{\partial y}+v^1\frac{\partial v^0}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^1$$

$$-\nu\left(\frac{\partial^2 u^1}{\partial y\partial x}+\frac{\partial^2 v^1}{\partial x^2}+2\frac{\partial^2 v^1}{\partial y^2}\right)=-\frac{\partial s}{\partial y}g+\frac{1}{h\rho_0}\left(T_{23}^2\big|_{z=1}-T_{23}^2\big|_{z=0}\right)$$

Se usan ahora las ecuaciones (5.3.152), (5.3.153), (5.3.155), (5.3.157) y (5.3.163)-(5.3.166), como se hizo en la sección 5.5 para obtener $T_{i3}^2 \quad (i=1,2)$ en 1 y 0:

$$T_{13}^2=f_{W_1}^2+2\nu\rho_0\frac{\partial s}{\partial x}\left(2\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)+\nu\rho_0\frac{\partial s}{\partial y}\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)\qquad \text{en } z=1$$

$$T_{23}^2=f_{W_2}^2+\nu\rho_0\frac{\partial s}{\partial x}\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)+2\nu\rho_0\frac{\partial s}{\partial y}\left(2\frac{\partial v^1}{\partial y}+\frac{\partial u^1}{\partial x}\right)\qquad \text{en } z=1$$

$$T_{13}^2=f_{R_1}^2+2\nu\rho_0\frac{\partial H}{\partial x}\left(2\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)+\nu\rho_0\frac{\partial H}{\partial y}\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)\qquad \text{en } z=0$$

$$T_{23}^2=f_{R_2}^2+\nu\rho_0\frac{\partial H}{\partial x}\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)+2\nu\rho_0\frac{\partial H}{\partial y}\left(2\frac{\partial v^1}{\partial y}+\frac{\partial u^1}{\partial x}\right)\qquad \text{en } z=0$$

Entonces, obtenemos:

$$\frac{\partial u^1}{\partial t}+u^0\frac{\partial u^1}{\partial x}+u^1\frac{\partial u^0}{\partial x}+v^0\frac{\partial u^1}{\partial y}+v^1\frac{\partial u^0}{\partial y}-2\phi\left(\operatorname{sen}\varphi\right)v^1$$

$$-\nu\left(2\frac{\partial^2 u^1}{\partial x^2}+\frac{\partial^2 u^1}{\partial y^2}+\frac{\partial^2 v^1}{\partial y\partial x}\right)=-\frac{\partial s}{\partial x}g+\frac{1}{h\rho_0}(f_{W_1}^2-f_{R_1}^2)$$

$$+\frac{\nu}{h}\left[2\frac{\partial h}{\partial x}\left(2\frac{\partial u^1}{\partial x}+\frac{\partial v^1}{\partial y}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)\right]$$

$$\frac{\partial v^1}{\partial t}+u^0\frac{\partial v^1}{\partial x}+u^1\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^1}{\partial y}+v^1\frac{\partial v^0}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^1$$

$$- \nu \left(\frac{\partial^2 u^1}{\partial y \partial x} + \frac{\partial^2 v^1}{\partial x^2} + 2\frac{\partial^2 v^1}{\partial y^2} \right) = -\frac{\partial s}{\partial y} g + \frac{1}{h\rho_0}(f_{W_2}^2 - f_{R_2}^2)$$

$$+ \frac{\nu}{h} \left[2\frac{\partial h}{\partial y} \left(\frac{\partial u^1}{\partial x} + 2\frac{\partial v^1}{\partial y} \right) + \frac{\partial h}{\partial x} \left(\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right) \right]$$

que también se puede escribir:

$$\frac{\partial u^1}{\partial t} + u^0 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^1}{\partial y} + v^1 \frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v^1 - \nu \left\{ \Delta_{xy} u^1 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} \right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] + \frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right] \right\} = -\frac{\partial s}{\partial x} g + \frac{1}{h\rho_0}(f_{W_1}^2 - f_{R_1}^2) \tag{5.8.14}$$

$$\frac{\partial v^1}{\partial t} + u^0 \frac{\partial v^1}{\partial x} + u^1 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^1}{\partial y} + v^1 \frac{\partial v^0}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^1 - \nu \left\{ \Delta_{xy} v^1 + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^1}{\partial y} \right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] + \frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right] \right\} = -\frac{\partial s}{\partial y} g + \frac{1}{h\rho_0}(f_{W_2}^2 - f_{R_2}^2) \tag{5.8.15}$$

Ahora, usando (5.3.131) y (5.8.7) se llega a:

$$\tilde{p}(\varepsilon) = p_s - \varepsilon \rho_0 h g (z - 1) \tag{5.8.16}$$

De igual modo, por (5.3.126), (5.3.135) y (5.3.146), sabemos que:

$$\tilde{w}(\varepsilon) = \varepsilon \left[\tilde{u}(\varepsilon)\frac{\partial H}{\partial x} + \tilde{v}(\varepsilon)\frac{\partial H}{\partial y} - hz\left(\frac{\partial \tilde{u}(\varepsilon)}{\partial x} + \frac{\partial \tilde{v}(\varepsilon)}{\partial y} \right) \right] \tag{5.8.17}$$

Se deshace el cambio de variable y, así, se obtiene la siguiente aproximación de la solución en el dominio de partida:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t, x, y, z) = u^0(t, x, y) + \varepsilon u^1(t, x, y)$$

$$v^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t, x, y, z) = v^0(t, x, y) + \varepsilon v^1(t, x, y)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t, x, y, z) = \varepsilon w^1(t, x, y, z) + \varepsilon^2 w^2(t, x, y, z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t, x, y, z) = p^0(t, x, y) + \varepsilon p^1(t, x, y, z)$$

Definimos también

$$\tilde{T}^{\varepsilon}_{ij} = \tilde{T}_{ij}(\varepsilon) = T^0_{ij} + \varepsilon T^1_{ij} \quad (i,j=1,2)$$

$$\tilde{T}^{\varepsilon}_{i3} = \tilde{T}_{i3}(\varepsilon) = \varepsilon T^1_{i3} + \varepsilon^2 T^2_{i3} \quad (i=1,2)$$

$$\tilde{T}^{\varepsilon}_{33} = \tilde{T}_{33}(\varepsilon) = T^0_{33} + \varepsilon T^1_{33}$$

$$\tilde{f}^{\varepsilon}_{R_i} = \tilde{f}_{R_i}(\varepsilon) = \varepsilon f^1_{R_i} + \varepsilon^2 f^2_{R_i},$$

$$\tilde{f}^{\varepsilon}_{W_i} = \tilde{f}_{W_i}(\varepsilon) = \varepsilon f^1_{W_i} + \varepsilon^2 f^2_{W_i} \quad (i=1,2)$$

Al igual que en la sección 5.5 se verifica

$$\frac{\partial \tilde{u}^{\varepsilon}}{\partial z^{\varepsilon}} = \frac{\partial \tilde{v}^{\varepsilon}}{\partial z^{\varepsilon}} = 0 \tag{5.8.18}$$

y la aproximación de la componente vertical de la velocidad que se obtiene es:

$$\tilde{w}^{\varepsilon} = \tilde{u}^{\varepsilon}\frac{\partial H^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon}\frac{\partial H^{\varepsilon}}{\partial y^{\varepsilon}} - (z^{\varepsilon} - H^{\varepsilon})\left(\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}}\right) \tag{5.8.19}$$

Si se realiza el cambio de variable en (5.8.16), obtenemos la aproximación de la presión en Ω^{ε}:

$$\tilde{p}^{\varepsilon} = p^{\varepsilon}_s + \rho_0(s^{\varepsilon} - z^{\varepsilon})g \tag{5.8.20}$$

Teniendo en cuenta (5.4.12) y (5.8.14) obtenemos la siguiente ecuación para el cálculo de $\tilde{u}^{\varepsilon}$:

$$\begin{aligned}
&\frac{\partial \tilde{u}^{\varepsilon}}{\partial t^{\varepsilon}} + \tilde{u}^{\varepsilon}\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon}\frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} - 2\phi\,(\operatorname{sen}\varphi^{\varepsilon})\,\tilde{v}^{\varepsilon} - \nu\left\{\Delta_{x^{\varepsilon}y^{\varepsilon}}\tilde{u}^{\varepsilon} + \frac{2}{h^{\varepsilon}}\frac{\partial h^{\varepsilon}}{\partial x^{\varepsilon}}\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}}\right.\\
&\quad\left. + \frac{1}{(h^{\varepsilon})^2}\frac{\partial}{\partial x^{\varepsilon}}\left[(h^{\varepsilon})^2\left(\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}}\right)\right] + \frac{1}{h^{\varepsilon}}\frac{\partial h^{\varepsilon}}{\partial y^{\varepsilon}}\left[\frac{\partial \tilde{u}^{\varepsilon}}{\partial y^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial x^{\varepsilon}}\right]\right\}\\
&\quad= \frac{\partial u^0}{\partial t} + \varepsilon\frac{\partial u^1}{\partial t} + (u^0 + \varepsilon u^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x}\right) + (v^0 + \varepsilon v^1)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial y}\right)\\
&\quad - 2\phi\,(\operatorname{sen}\varphi)\,(v^0 + \varepsilon v^1) - \nu\left\{\Delta_{xy}u^0 + \varepsilon\Delta_{xy}u^1 + \frac{2}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x}\right)\right.\\
&\quad + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y}\right)\right]\\
&\quad\left. + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x}\right)\right\}
\end{aligned}$$

$$= \frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v^0 - \nu \left\{ \Delta_{xy} u^0 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right\}$$

$$+ \varepsilon \left(\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^0}{\partial x} + u^0 \frac{\partial u^1}{\partial x} + v^1 \frac{\partial u^0}{\partial y} + v^0 \frac{\partial u^1}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\, v^1 \right.$$

$$\left. - \nu \left\{ \Delta_{xy} u^1 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right] \right\} \right)$$

$$+ \varepsilon^2 \left(u^1 \frac{\partial u^1}{\partial x} + v^1 \frac{\partial u^1}{\partial y} \right)$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} + \frac{1}{h\rho_0}(f^1_{W_1} - f^1_{R_1}) + \varepsilon \left[-\frac{\partial s}{\partial x} g + \frac{1}{h\rho_0}(f^2_{W_1} - f^2_{R_1}) \right] + O(\varepsilon^2)$$

$$= -\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1}) + O(\varepsilon^2)$$

De manera análoga, empleando (5.4.13) y (5.8.15) se obtiene una ecuación para el cálculo de $\tilde{v}^\varepsilon$. Se tiene por tanto:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, \tilde{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_1} - \tilde{f}^\varepsilon_{R_1}) + O(\varepsilon^2) \qquad (5.8.21)$$

$$\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, \tilde{u}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p^\varepsilon_s}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}^\varepsilon_{W_2} - \tilde{f}^\varepsilon_{R_2}) + O(\varepsilon^2) \qquad (5.8.22)$$

y también:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = O(\varepsilon^2) \qquad (5.8.23)$$

Si en (5.8.21)-(5.8.23) se desprecian los términos de orden $O(\varepsilon^2)$ se obtiene el siguiente modelo de aguas someras:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (\tilde{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (\tilde{v}^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0$$

$$\begin{aligned}
&\frac{\partial \tilde{u}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} \right.\\
&\quad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}\\
&\quad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon)\\
&\frac{\partial \tilde{v}^\varepsilon}{\partial t^\varepsilon} + \tilde{u}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \tilde{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right.\\
&\quad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \tilde{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial x^\varepsilon} \right] \right\}\\
&\quad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - \frac{\partial s^\varepsilon}{\partial y^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(\tilde{f}_{W_2}^\varepsilon - \tilde{f}_{R_2}^\varepsilon)
\end{aligned} \tag{5.8.24}$$

$$\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon) g$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right)$$

donde $\tilde{u}^\varepsilon$, $\tilde{v}^\varepsilon$ y h^ε no dependen de z^ε.

Si se compara el modelo clásico de “shallow waters”((1.2.23), ver por ejemplo [5] (pág. 3) o en [101] (pág. 38)) con éste, se puede apreciar que la diferencia está esencialmente en el término de viscosidad.

El orden de precisión con el que se verifican las ecuaciones de partida es exactamente el mismo que el comprobado para la aproximación de primer orden sin imponer que la aceleración de Coriolis sea (1.1.9).

5.8.6. Aproximación de segundo orden

Se considera la aproximación de segundo orden en ε:

$$\tilde{u}(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2$$

$$\tilde{v}(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2$$

$$\tilde{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3$$

$$\tilde{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2$$

$$\tilde{T}_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 + \varepsilon^2 T_{ij}^2 \quad (i,j = 1,2)$$

$$\tilde{T}_{i3}(\varepsilon) = \varepsilon^{-1} T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 + \varepsilon^2 T_{i3}^2 + \varepsilon^3 T_{i3}^3 \quad (i = 1,2)$$

$$\tilde{T}_{33}(\varepsilon) = \varepsilon^{-1} T_{33}^{-1} + T_{33}^0 + \varepsilon T_{33}^1 + \varepsilon^2 T_{33}^2$$

$$\tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^1 + \varepsilon^2 f_{R_i}^2 + \varepsilon^3 f_{R_i}^3 \quad (i = 1,2)$$

$$\tilde{f}_{W_i}(\varepsilon) = \varepsilon f_{W_i}^1 + \varepsilon^2 f_{W_i}^2 + \varepsilon^3 f_{W_i}^3 \quad (i = 1,2)$$

Los términos w^0, p^0, $T_{i3}^k \quad (k = -1, 0, i = 1,2)$, T_{33}^{-1}, u^0, v^0, h, w^1, p^1, u^1, v^1 y w^2 se calculan del mismo modo que en la sección anterior para la aproximación de primer orden a partir de (5.3.121)-(5.3.133), (5.3.135), (5.3.136), (5.3.146) y (5.8.7)-(5.8.8).

Buscamos, ahora, p^2, para ello seguimos el mismo proceso de la sección 5.5 pero partimos de la ecuación (5.8.10):

$$D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0}\left(D_x T_{13}^1 + D_y T_{23}^1\right) = 0 \tag{5.8.25}$$

Comenzamos por despejar $D_z^2 w^3$ de la expresión (5.3.158) y sustituirlo en (5.8.25). A continuación, como u^0 y v^0 son conocidos, calculamos $T_{i3}^1 \quad (i = 1,2)$ a partir de (5.3.132)-(5.3.133), obteniendo (5.6.3)-(5.6.4). Sustituimos entonces T_{i3}^1 $(i = 1,2)$ por las expresiones anteriores y w^1 por (5.3.135). Se integra respecto a z y se impone la condición de contorno $p^2(t,x,y,1) = 0$ ((5.3.162)). La expresión para p^2 que resulta es la siguiente:

$$\begin{aligned}
p^2 = {} & \rho_0 h(1-z)\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0 v^0 \frac{\partial^2 H}{\partial y \partial x} + (v^0)^2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^1}{\partial x} + \frac{\partial f_{R_2}^1}{\partial y}\right)\right. \\
& - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\, v^0\right) - \frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\, u^0\right) \\
& \left. -2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0 \frac{\partial^2 H}{\partial y \partial x}\right) + \frac{\partial}{\partial y}\left(u^0 \frac{\partial^2 H}{\partial y \partial x} + v^0 \frac{\partial^2 H}{\partial y^2}\right)\right]\right\} \\
& + \frac{1}{2}\rho_0 h^2 (1-z^2)\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right. \\
& \left. + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\frac{\partial v^0}{\partial x} + 2\phi \frac{\partial}{\partial y}((\operatorname{sen}\varphi)\, u^0)\right]
\end{aligned} \tag{5.8.26}$$

Los términos u^2 y v^2 vienen dados por (5.6.6) y (5.6.7):

$$u^2 = u_0^2 + hz\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}\right]$$

$$+ \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right) - \Delta_{xy}u^0\right]$$

$$v^2 = v_0^2 + hz\left[\frac{1}{\nu\rho_0}f_{R_2}^1 + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x \partial y} - v^0\frac{\partial^2 H}{\partial y^2}\right]$$

$$+ \frac{1}{2}z^2h^2\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - \Delta_{xy}v^0\right]$$

donde $u_0^2(t,x,y) = u^2(t,x,y,0)$, $v_0^2(t,x,y) = v^2(t,x,y,0)$ están determinados por (5.8.11) y (5.8.12). En estas ecuaciones aparecen T_{13}^3 y T_{23}^3 que sólo son conocidos en $z=0$ y $z=1$. Es por ello que se integran las ecuaciones respecto de z entre 0 y 1. Para realizar la integración es necesario conocer explícitamente la dependencia de z de los diferentes términos que intervienen en las ecuaciones. Por eso sustituimos u^2, v^2, p^2, w^1 y w^2 por las expresiones vistas en (5.6.6), (5.6.7), (5.8.26) (5.3.135) y (5.3.146). Obtenemos finalmente las siguientes ecuaciones:

$$\frac{\partial u_0^2}{\partial t} + u^0\frac{\partial u_0^2}{\partial x} + u^1\frac{\partial u^1}{\partial x} + u_0^2\frac{\partial u^0}{\partial x} + v^0\frac{\partial u_0^2}{\partial y} + v^1\frac{\partial u^1}{\partial y} + v_0^2\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v_0^2$$

$$- \nu\left[2\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \frac{\partial^2 v_0^2}{\partial x \partial y} + \frac{2}{h}\frac{\partial h}{\partial x}\left(2\frac{\partial u_0^2}{\partial x} + \frac{\partial v_0^2}{\partial y}\right) + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u_0^2}{\partial y} + \frac{\partial v_0^2}{\partial x}\right)\right]$$

$$= -\frac{1}{\rho_0}\left.D_x p^2\right|_{z=0} - \left\{\nu\left(2\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{4}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + \frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial y}\right)\right.$$

$$\left.- \frac{h}{2}\frac{\partial v^0}{\partial y}\right\}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}\right]$$

$$- \left(-4\nu\frac{\partial h}{\partial x} + \frac{h}{2}u^0\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2}\right.$$

$$\left.- v^0\frac{\partial^2 H}{\partial y \partial x}\right] - \left(\nu\frac{\partial H}{\partial y} - \nu\frac{\partial h}{\partial y} + \frac{h}{2}v^0\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right.$$

$$\left.+ \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}\right]$$

$$- \frac{h}{2}\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f_{R_1}^1 + \frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y} - u^0\frac{\partial^2 H}{\partial x^2} - v^0\frac{\partial^2 H}{\partial y \partial x}\right]$$

$$+ \nu h \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right]$$

$$+ \nu \frac{h}{2} \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu \rho_0} f^1_{R_1} + \frac{\partial H}{\partial x} \left(2 \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial y} \frac{\partial u^0}{\partial y} - u^0 \frac{\partial^2 H}{\partial x^2} - v^0 \frac{\partial^2 H}{\partial y \partial x} \right]$$

$$- \left\{ \nu \left(\frac{\partial^2 H}{\partial x \partial y} + \frac{2}{h} \frac{\partial h}{\partial x} \frac{\partial H}{\partial y} + \frac{1}{h} \frac{\partial h}{\partial y} \frac{\partial H}{\partial x} \right) + \frac{h}{2} \frac{\partial u^0}{\partial y} - \phi \left(\operatorname{sen} \varphi \right) h \right\} \left[\frac{1}{\nu \rho_0} f^1_{R_2} \right.$$

$$\left. + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$+ \nu \frac{\partial h}{\partial y} \frac{\partial}{\partial x} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} - v^0 \frac{\partial^2 H}{\partial y^2} \right]$$

$$+ \nu \left(\frac{\partial H}{\partial x} + 2 \frac{\partial h}{\partial x} \right) \frac{\partial}{\partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] + \nu \frac{h}{2} \frac{\partial^2}{\partial x \partial y} \left[\frac{1}{\nu \rho_0} f^1_{R_2} + \frac{\partial H}{\partial y} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) + \frac{\partial H}{\partial x} \frac{\partial v^0}{\partial x} - u^0 \frac{\partial^2 H}{\partial x \partial y} \right.$$

$$\left. - v^0 \frac{\partial^2 H}{\partial y^2} \right] - \left\{ \nu \left[2 \left(\frac{\partial H}{\partial x} \right)^2 + h \frac{\partial^2 H}{\partial x^2} + \frac{h}{2} \frac{\partial^2 H}{\partial y^2} + 4 \frac{\partial H}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial h}{\partial y} \right] \right.$$

$$\left. - \frac{h^2}{6} \left(\frac{\partial u^0}{\partial x} + 2 \frac{\partial v^0}{\partial y} \right) \right\} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) \right.$$

$$\left. - \Delta_{xy} u^0 \right] - \left(-2 \nu h \frac{\partial h}{\partial x} + \frac{h^2}{6} u^0 \right) \frac{\partial}{\partial x} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} \right. \right.$$

$$\left. \left. - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] - \left\{ \nu \frac{h}{2} \left(\frac{\partial H}{\partial y} - \frac{\partial h}{\partial y} \right) + \frac{h^2}{6} v^0 \right\} \frac{\partial}{\partial y} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} \right. \right.$$

$$\left. \left. + v^0 \frac{\partial u^0}{\partial y} + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] - \frac{h^2}{6} \frac{\partial}{\partial t} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} \right. \right.$$

$$\left. \left. + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] + \nu \frac{h^2}{3} \frac{\partial^2}{\partial x^2} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} \right. \right.$$

$$\left. \left. + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] + \nu \frac{h^2}{6} \frac{\partial^2}{\partial y^2} \left[\frac{1}{\nu} \left(\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} \right. \right.$$

$$\left. \left. + \frac{1}{\rho_0} \frac{\partial p_s}{\partial x} - 2 \phi \left(\operatorname{sen} \varphi \right) v^0 \right) - \Delta_{xy} u^0 \right] - \left\{ \nu \left(\frac{h}{2} \frac{\partial^2 H}{\partial x \partial y} + 2 \frac{\partial s}{\partial x} \frac{\partial H}{\partial y} + \frac{\partial h}{\partial y} \frac{\partial H}{\partial x} \right) \right.$$

$$+\frac{h^2}{6}\frac{\partial u^0}{\partial y}-\frac{h^2}{3}\phi\left(\operatorname{sen}\varphi\right)\bigg\}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$\left.-\Delta_{xy}v^0\right]+\nu\frac{h}{2}\frac{\partial h}{\partial y}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$\left.-\Delta_{xy}v^0\right]-\nu\left(\frac{h}{2}\frac{\partial H}{\partial x}-h\frac{\partial s}{\partial x}\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.\right.$$

$$\left.\left.+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]+\nu\frac{h^2}{6}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t}+u^0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^0}{\partial y}+\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}\right.\right.$$

$$\left.\left.+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-\Delta_{xy}v^0\right]+\frac{h}{2}\left\{\frac{\partial}{\partial x}\left((u^0)^2\frac{\partial^2 H}{\partial x^2}+2u^0v^0\frac{\partial^2 H}{\partial y\partial x}+(v^0)^2\frac{\partial^2 H}{\partial y^2}\right.\right.$$

$$\left.+\frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x}+\frac{\partial f^1_{R_2}}{\partial y}\right)-\frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}-2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$-\frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+2\phi\left(\operatorname{sen}\varphi\right)u^0\right)-2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2}+v^0\frac{\partial^2 H}{\partial y\partial x}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x}+v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right)-\frac{\partial H}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left.\left.+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}(\left(\operatorname{sen}\varphi\right)u^0)\right]\right\}$$

$$+\frac{h^2}{6}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2+2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y}+2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}+2\left(\frac{\partial v^0}{\partial y}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right.$$

$$\left.+\frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2}-2\phi\left(\operatorname{sen}\varphi\right)\frac{\partial v^0}{\partial x}+2\phi\frac{\partial}{\partial y}(\left(\operatorname{sen}\varphi\right)u^0)\right]$$

$$+\frac{1}{\rho_0 h}\left\{\left(\frac{\partial s}{\partial x}\right)^2\left[2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\right.$$

$$-\left(\frac{\partial H}{\partial x}\right)^2\left[2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+\frac{\partial s}{\partial x}\frac{\partial s}{\partial y}\left[f^1_{W_2}+2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$
\begin{aligned}
&-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left[f^1_{R_2}+2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x}\right)\right]\\
&+f^3_{W_1}-f^3_{R_1}+f^1_{W_1}\left(\frac{3}{2}\left(\frac{\partial s}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial s}{\partial y}\right)^2\right)-f^1_{R_1}\left(2\left(\frac{\partial H}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial H}{\partial y}\right)^2\right)\Bigg\}
\end{aligned}
\tag{5.8.27}
$$

$$
\begin{aligned}
&\frac{\partial v^2_0}{\partial t}+u^0\frac{\partial v^2_0}{\partial x}+u^1\frac{\partial v^1}{\partial x}+u^2_0\frac{\partial v^0}{\partial x}+v^0\frac{\partial v^2_0}{\partial y}+v^1\frac{\partial v^1}{\partial y}+v^2_0\frac{\partial v^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u^2_0\\
&-\nu\left[\frac{\partial^2 u^2_0}{\partial x\partial y}+\frac{\partial^2 v^2_0}{\partial x^2}+2\frac{\partial^2 v^2_0}{\partial y^2}+\frac{2}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^2_0}{\partial x}+2\frac{\partial v^2_0}{\partial y}\right)+\frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^2_0}{\partial y}+\frac{\partial v^2_0}{\partial x}\right)\right]\\
&=-\frac{1}{\rho_0}\left.D_y p^2\right|_{z=0}-\Bigg\{\nu\left(\frac{\partial^2 H}{\partial x\partial y}+\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial y}+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial x}\right)+\frac{h}{2}\frac{\partial v^0}{\partial x}\\
&+\phi\,(\operatorname{sen}\varphi)\,h\Bigg\}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]\\
&+\nu\frac{\partial h}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}-v^0\frac{\partial^2 H}{\partial y\partial x}\right]\\
&+\nu\left(\frac{\partial s}{\partial y}+\frac{\partial h}{\partial y}\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]+\nu\frac{h}{2}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_1}+\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial y}\frac{\partial u^0}{\partial y}-u^0\frac{\partial^2 H}{\partial x^2}\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y\partial x}\right]-\Bigg\{\nu\left(\frac{\partial^2 H}{\partial x^2}+2\frac{\partial^2 H}{\partial y^2}+\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial H}{\partial x}+\frac{4}{h}\frac{\partial h}{\partial y}\frac{\partial H}{\partial y}\right)-\frac{h}{2}\frac{\partial u^0}{\partial x}\Bigg\}\left[\frac{1}{\nu\rho_0}f^1_{R_2}\right.\\
&\left.+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]-\Bigg\{\nu\left(\frac{\partial H}{\partial x}-\frac{\partial h}{\partial x}\right)\\
&+\frac{h}{2}u^0\Bigg\}\frac{\partial}{\partial x}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}-v^0\frac{\partial^2 H}{\partial y^2}\right]\\
&-\left(\frac{h}{2}v^0-4\nu\frac{\partial h}{\partial y}\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.\\
&\left.-v^0\frac{\partial^2 H}{\partial y^2}\right]-\frac{h}{2}\Bigg\{\frac{\partial}{\partial t}\left[\frac{1}{\nu\rho_0}f^1_{R_2}+\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x}+2\frac{\partial v^0}{\partial y}\right)+\frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x}-u^0\frac{\partial^2 H}{\partial x\partial y}\right.
\end{aligned}
$$

$$- v^0\frac{\partial^2 H}{\partial y^2}\Bigg] - \nu\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu\rho_0}f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left. - v^0\frac{\partial^2 H}{\partial y^2}\right] - 2\nu\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu\rho_0}f^1_{R_2} + \frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \frac{\partial H}{\partial x}\frac{\partial v^0}{\partial x} - u^0\frac{\partial^2 H}{\partial x\partial y}\right.$$

$$\left. - v^0\frac{\partial^2 H}{\partial y^2}\right]\Bigg\} - \left\{\nu\left(\frac{\partial H}{\partial y}\frac{\partial h}{\partial x} + 2\frac{\partial s}{\partial y}\frac{\partial H}{\partial x} + \frac{h}{2}\frac{\partial^2 H}{\partial x\partial y}\right) + \frac{h^2}{6}\frac{\partial v^0}{\partial x}\right.$$

$$\left. + \frac{h^2}{3}\phi\left(\operatorname{sen}\varphi\right)\right\}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right) - \Delta_{xy}u^0\right]$$

$$- \nu\left(\frac{h}{2}\frac{\partial H}{\partial y} - h\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left. - \Delta_{xy}u^0\right] + \nu\frac{h}{2}\frac{\partial h}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left. - \Delta_{xy}u^0\right] + \nu\frac{h^2}{6}\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{\nu}\left(\frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\left(\operatorname{sen}\varphi\right)v^0\right)\right.$$

$$\left. - \Delta_{xy}u^0\right] - \left\{\nu\left[\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + 2\left(\frac{\partial H}{\partial y}\right)^2 + 4\frac{\partial h}{\partial y}\frac{\partial H}{\partial y} + \frac{h}{2}\left(\frac{\partial^2 H}{\partial x^2} + 2\frac{\partial^2 H}{\partial y^2}\right)\right]\right.$$

$$\left. - \frac{h^2}{6}\left(\frac{\partial v^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\right)\right\}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right)\right.$$

$$\left. - \Delta_{xy}v^0\right] - \left\{\nu\frac{h}{2}\left(\frac{\partial H}{\partial x} - \frac{\partial h}{\partial x}\right) + \frac{h^2}{6}u^0\right\}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y}\right.\right.$$

$$\left.\left. + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right] + \left(2\nu h\frac{\partial h}{\partial y} - \frac{h^2}{6}v^0\right)\frac{\partial}{\partial y}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right] - \frac{h^2}{6}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right] - \nu\left(\frac{\partial^2}{\partial x^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x}\right.\right.\right.$$

$$\left.\left. + v^0\frac{\partial v^0}{\partial y} + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right] + 2\frac{\partial^2}{\partial y^2}\left[\frac{1}{\nu}\left(\frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y}\right.\right.$$

$$\left.\left.\left.\left. + \frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0\right) - \Delta_{xy}v^0\right]\right)\right\} + \frac{h}{2}\frac{\partial}{\partial y}\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x}\right.$$

$$+ (v^0)^2\frac{\partial^2 H}{\partial y^2} + \frac{1}{\rho_0}\left(\frac{\partial f^1_{R_1}}{\partial x} + \frac{\partial f^1_{R_2}}{\partial y}\right) - \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right)$$

$$- \frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) - 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0\frac{\partial^2 H}{\partial y \partial x}\right)\right.$$

$$\left.\left. + \frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x} + v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\} - \frac{h}{2}\frac{\partial H}{\partial y}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y}\right.$$

$$\left. + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+ \frac{h^2}{6}\frac{\partial}{\partial y}\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2\right.$$

$$\left. + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right]$$

$$+ \frac{1}{\rho_0 h}\left\{2\frac{\partial s}{\partial y}\frac{\partial s}{\partial x}\left[f^1_{W_1} + 2\mu\frac{\partial s}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]\right.$$

$$- 2\frac{\partial H}{\partial y}\frac{\partial H}{\partial x}\left[f^1_{R_1} + 2\mu\frac{\partial H}{\partial x}\left(2\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$+ \left(\frac{\partial s}{\partial y}\right)^2\left[f^1_{W_2} + 2\mu\frac{\partial s}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial s}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$- \left[\left(\frac{\partial H}{\partial y}\right)^2 - \left(\frac{\partial H}{\partial x}\right)^2\right]\left[f^1_{R_2} + 2\mu\frac{\partial H}{\partial y}\left(\frac{\partial u^0}{\partial x} + 2\frac{\partial v^0}{\partial y}\right) + \mu\frac{\partial H}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right]$$

$$+ f^3_{W_2} - f^3_{R_2} + \frac{1}{2}f^1_{W_2}\left(\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right) - f^1_{R_2}\left(\left(\frac{3}{2}\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2\right)$$

$$\left. - \mu\frac{\partial H}{\partial x}\left(2\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\left(\frac{\partial v^0}{\partial y} - \frac{\partial u^0}{\partial x}\right) + \left[\left(\frac{\partial H}{\partial x}\right)^2 - \left(\frac{\partial H}{\partial y}\right)^2\right]\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right)\right\} \tag{5.8.28}$$

Una vez calculados u^2 y v^2, w^3 se calcula integrando respecto a z (5.3.161)

$$D_z w^3 = -D_x u^2 - D_x v^2$$

Para ello, en primer lugar se sustituyen las expresiones obtenidas para u^2 y v^2 en (5.6.6) y (5.6.7), integramos respecto a z e imponemos la condición (derivada de (5.3.8)), $w^3 = u_0^2\dfrac{\partial H}{\partial x} + v_0^2\dfrac{\partial H}{\partial y}$ en $z = 0$, para llegar a la misma expresión de w^3 en función de u_0^2, v_0^2, u^0 y v^0 que en la sección 5.5 ((5.6.22)).

Usando las expresiones encontradas para p^0, p^1 y p^2, (5.3.131), (5.8.7) y (5.8.26) respectivamente, tenemos la siguiente aproximación de la presión en el dominio de referencia

$$
\begin{aligned}
\tilde{p}(\varepsilon) &= p^0 + \varepsilon p^1 + \varepsilon^2 p^2 = p_s + \varepsilon\rho_0 h(1-z)g \\
&+ \varepsilon^2\rho_0 h(1-z)\left\{(u^0)^2\frac{\partial^2 H}{\partial x^2} + 2u^0v^0\frac{\partial^2 H}{\partial y\partial x} + (v^0)^2\frac{\partial^2 H}{\partial y^2} + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^1}{\partial x} + \frac{\partial f_{R_2}^1}{\partial y}\right)\right. \\
&- \frac{\partial H}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial x} - 2\phi\,(\operatorname{sen}\varphi)\,v^0\right) - \frac{\partial H}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p_s}{\partial y} + 2\phi\,(\operatorname{sen}\varphi)\,u^0\right) \\
&\left.- 2\nu\left[\frac{\partial}{\partial x}\left(u^0\frac{\partial^2 H}{\partial x^2} + v^0\frac{\partial^2 H}{\partial y\partial x}\right) + \frac{\partial}{\partial y}\left(u^0\frac{\partial^2 H}{\partial y\partial x} + v^0\frac{\partial^2 H}{\partial y^2}\right)\right]\right\} \\
&+ \varepsilon^2\frac{1}{2}\rho_0 h^2(1-z^2)\left[2\left(\frac{\partial u^0}{\partial x}\right)^2 + 2\frac{\partial v^0}{\partial x}\frac{\partial u^0}{\partial y} + 2\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + 2\left(\frac{\partial v^0}{\partial y}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial x^2}\right. \\
&\left.+ \frac{1}{\rho_0}\frac{\partial^2 p_s}{\partial y^2} - 2\phi\,(\operatorname{sen}\varphi)\,\frac{\partial v^0}{\partial x} + 2\phi\frac{\partial}{\partial y}((\operatorname{sen}\varphi)\,u^0)\right] \qquad (5.8.29)
\end{aligned}
$$

De igual modo, a partir de las expresiones vistas para w^k $(k = 0, 1, 2, 3)$ ((5.3.126), (5.3.135), (5.3.146) y (5.6.22)) obtenemos la siguiente aproximación de la velocidad vertical en Ω:

$$
\tilde{w}(\varepsilon) = \varepsilon\frac{\partial H}{\partial x}\tilde{u}(\varepsilon) + \varepsilon\frac{\partial H}{\partial y}\tilde{v}(\varepsilon) - \varepsilon hz\left(D_x\tilde{u}(\varepsilon) + D_y\tilde{v}(\varepsilon)\right) + O(\varepsilon^3) \qquad (5.8.30)
$$

Deshacemos ahora el cambio de variable, volviendo al dominio original, y obtenemos la siguiente aproximación de la solución en Ω^ε:

$$\tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{u}(\varepsilon)(t,x,y,z) = u^0(t,x,y) + \varepsilon u^1(t,x,y) + \varepsilon^2 u^2(t,x,y,z)$$

$$\tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{v}(\varepsilon)(t,x,y,z) = v^0(t,x,y) + \varepsilon v^1(t,x,y) + \varepsilon^2 v^2(t,x,y,z)$$

$$\tilde{w}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{w}(\varepsilon)(t,x,y,z) = \varepsilon w^1(t,x,y,z) + \varepsilon^2 w^2(t,x,y,z) + \varepsilon^3 w^3(t,x,y,z)$$

$$\tilde{p}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{p}(\varepsilon)(t,x,y,z) = p^0(t,x,y) + \varepsilon p^1(t,x,y,z) + \varepsilon^2 p^2(t,x,y,z)$$

$$\tilde{f}_{R_i}^\varepsilon = \tilde{f}_{R_i}(\varepsilon) = \varepsilon f_{R_i}^1 + \varepsilon^2 f_{R_i}^2 + \varepsilon^3 f_{R_i}^3 \quad (i = 1, 2)$$

$$\tilde{f}_{W_i}^\varepsilon = \tilde{f}_{W_i}(\varepsilon) = \varepsilon f_{W_i}^1 + \varepsilon^2 f_{W_i}^2 + \varepsilon^3 f_{W_i}^3 \quad (i = 1, 2)$$

La expresión obtenida para la presión en el dominio original se obtiene deshaciendo el cambio de variable en (5.8.29):

$$\begin{aligned}\tilde{p}^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon) \Bigg\{ & g + (u^{0,\varepsilon})^2 \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + 2u^{0,\varepsilon} v^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + (v^{0,\varepsilon})^2 \frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2} \\ & + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon} \right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, v^{0,\varepsilon} \right) \\ & - \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0} \frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\, u^{0,\varepsilon} \right) - 2\nu \left[\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} + v^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} \right) \right. \\ & \left. + \frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + v^{0,\varepsilon} \frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2} \right) \right] \Bigg\} \\ & + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \right] \left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \right)^2 + 2 \frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + 2 \frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon} \right. \\ & \left. + 2\left(\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon} \right)^2 + \frac{1}{\rho_0} \frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{\rho_0} \frac{\partial^2 p_s^\varepsilon}{\partial (y^\varepsilon)^2} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon) \frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + 2\phi \frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\, u^{0,\varepsilon}) \right]\end{aligned} \tag{5.8.31}$$

donde $u^{0,\varepsilon} = u^0$, $v^{0,\varepsilon} = v^0$, $f_{R_i}^{1,\varepsilon} = \varepsilon f_{R_i}^1$ $(i = 1, 2)$, tras el cambio de variable de Ω a Ω^ε.

De forma análoga, realizando el cambio de variable de Ω a Ω^ε en (5.8.30), llegamos a la siguiente expresión de la velocidad vertical:

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon} \right) + O(\varepsilon^3) \tag{5.8.32}$$

El modelo que vamos a proponer requiere que definamos

$$\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{u}(\varepsilon)(t, x, y) = u^0(t, x, y) + \varepsilon u^1(t, x, y) + \varepsilon^2 u_0^2(t, x, y)$$

$$\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \hat{v}(\varepsilon)(t, x, y) = v^0(t, x, y) + \varepsilon v^1(t, x, y) + \varepsilon^2 v_0^2(t, x, y)$$

(es decir, $\hat{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \tilde{u}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, 0)$, $\hat{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \tilde{v}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, 0)$).

Observamos que:

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu \left\{ \Delta_{x^\varepsilon y^\varepsilon} \hat{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2 \left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} \right) \right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} \right) \right\}$$

$$= \frac{\partial \hat{u}(\varepsilon)}{\partial t} + \hat{u}(\varepsilon)\frac{\partial \hat{u}(\varepsilon)}{\partial x} + \hat{v}(\varepsilon)\frac{\partial \hat{u}(\varepsilon)}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,\hat{v}(\varepsilon) - \nu \left\{ \Delta_{xy}\hat{u}(\varepsilon) + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial \hat{u}(\varepsilon)}{\partial x} \right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial x^\varepsilon}\left[h^2 \left(\frac{\partial \hat{u}(\varepsilon)}{\partial x} + \frac{\partial \hat{v}(\varepsilon)}{\partial y} \right) \right] + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial \hat{u}(\varepsilon)}{\partial y} + \frac{\partial \hat{v}(\varepsilon)}{\partial x} \right) \right\}$$

$$= \frac{\partial u^0}{\partial t} + \varepsilon\frac{\partial u^1}{\partial t} + \varepsilon^2\frac{\partial u_0^2}{\partial t} + (u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2)\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x} \right)$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y} \right) - 2\phi\,(\operatorname{sen}\varphi)\,(v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)$$

$$- \nu \left\{ \Delta_{xy}u^0 + \varepsilon\Delta_{xy}u^1 + \varepsilon^2\Delta_{xy}u_0^2 + \frac{2}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x} \right) \right.$$

$$+ \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + \varepsilon^2\frac{\partial v_0^2}{\partial y} \right) \right]$$

$$\left. + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y} + \frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x} + \varepsilon^2\frac{\partial v_0^2}{\partial x} \right) \right\}$$

$$= \frac{\partial u^0}{\partial t} + u^0\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^0 - \nu \left\{ \Delta_{xy}u^0 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^0}{\partial x} \right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \right] + \frac{1}{h}\frac{\partial h}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right) \right\}$$

$$+ \varepsilon \left(\frac{\partial u^1}{\partial t} + u^0\frac{\partial u^1}{\partial x} + u^1\frac{\partial u^0}{\partial x} + v^0\frac{\partial u^1}{\partial y} + v^1\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v^1 \right.$$

$$\left. - \nu \left\{ \Delta_{xy}u^1 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u^1}{\partial x} + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2 \left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \right) \right] + \frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} \right] \right\} \right)$$

$$+ \varepsilon^2 \left(\frac{\partial u_0^2}{\partial t} + u^0\frac{\partial u_0^2}{\partial x} + u^1\frac{\partial u^1}{\partial x} + u_0^2\frac{\partial u^0}{\partial x} + v^0\frac{\partial u_0^2}{\partial y} + v^1\frac{\partial u^1}{\partial y} + v_0^2\frac{\partial u^0}{\partial y} - 2\phi\,(\operatorname{sen}\varphi)\,v_0^2 \right.$$

$$- \nu \left\{ \Delta_{xy} u_0^2 + \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial u_0^2}{\partial x} + \frac{1}{h^2}\frac{\partial}{\partial x}\left[h^2\left(\frac{\partial u_0^2}{\partial x} + \frac{\partial v_0^2}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial y}\left[\frac{\partial u_0^2}{\partial y} + \frac{\partial v_0^2}{\partial x}\right]\right\}\Bigg)$$

$$+ O(\varepsilon^3)$$

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\hat{u}^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$= \frac{\partial \hat{v}(\varepsilon)}{\partial t^\varepsilon} + \hat{u}(\varepsilon)\frac{\partial \hat{v}(\varepsilon)}{\partial x^\varepsilon} + \hat{v}(\varepsilon)\frac{\partial \hat{v}(\varepsilon)}{\partial y^\varepsilon} + 2\phi\left(\operatorname{sen}\varphi\right)\hat{u}(\varepsilon) - \nu\left\{\Delta_{xy}\hat{v}(\varepsilon) + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial \hat{v}(\varepsilon)}{\partial y}\right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial \hat{u}(\varepsilon)}{\partial x} + \frac{\partial \hat{v}(\varepsilon)}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial \hat{u}(\varepsilon)}{\partial y} + \frac{\partial \hat{v}(\varepsilon)}{\partial x}\right]\right\}$$

$$= \frac{\partial v^0}{\partial t} + \varepsilon\frac{\partial v^1}{\partial t} + \varepsilon^2\frac{\partial v_0^2}{\partial t} + (u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2)\left(\frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x} + \varepsilon^2\frac{\partial v_0^2}{\partial x}\right)$$

$$+ (v^0 + \varepsilon v^1 + \varepsilon^2 v_0^2)\left(\frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + \varepsilon^2\frac{\partial v_0^2}{\partial y}\right) + 2\phi\left(\operatorname{sen}\varphi\right)(u^0 + \varepsilon u^1 + \varepsilon^2 u_0^2)$$

$$- \nu\left\{\Delta_{xy}v^0 + \varepsilon\Delta_{xy}v^1 + \varepsilon^2\Delta_{xy}v_0^2 + \frac{2}{h}\frac{\partial h}{\partial y}\left(\frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + \varepsilon^2\frac{\partial v_0^2}{\partial y}\right)\right.$$

$$+ \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^0}{\partial x} + \varepsilon\frac{\partial u^1}{\partial x} + \varepsilon^2\frac{\partial u_0^2}{\partial x} + \frac{\partial v^0}{\partial y} + \varepsilon\frac{\partial v^1}{\partial y} + \varepsilon^2\frac{\partial v_0^2}{\partial y}\right)\right]$$

$$\left. + \frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y} + \varepsilon\frac{\partial u^1}{\partial y} + \varepsilon^2\frac{\partial u_0^2}{\partial y} + \frac{\partial v^0}{\partial x} + \varepsilon\frac{\partial v^1}{\partial x} + \varepsilon^2\frac{\partial v_0^2}{\partial x}\right)\right\}$$

$$= \frac{\partial v^0}{\partial t} + u^0\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^0 - \nu\left\{\Delta_{xy}v^0 + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^0}{\partial y}\right.$$

$$\left. + \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right\}$$

$$+ \varepsilon\Bigg(\frac{\partial v^1}{\partial t} + u^0\frac{\partial v^1}{\partial x} + u^1\frac{\partial v^0}{\partial x} + v^0\frac{\partial v^1}{\partial y} + v^1\frac{\partial v^0}{\partial y} + 2\phi\left(\operatorname{sen}\varphi\right)u^1$$

$$- \nu\left\{\Delta_{xy}v^1 + \frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v^1}{\partial y} + \frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}\right)\right] + \frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x}\right]\right\}\Bigg)$$

$$+\varepsilon^2\left(\frac{\partial v_0^2}{\partial t}+u^0\frac{\partial v_0^2}{\partial x}+u^1\frac{\partial v^1}{\partial x}+u_0^2\frac{\partial v^0}{\partial x}+v^0\frac{\partial v_0^2}{\partial y}+v^1\frac{\partial v^1}{\partial y}+v_0^2\frac{\partial v^0}{\partial y}+2\phi\,(\operatorname{sen}\varphi)\,u_0^2\right.$$
$$\left.-\nu\left\{\Delta_{xy}v_0^2+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial v_0^2}{\partial y}+\frac{1}{h^2}\frac{\partial}{\partial y}\left[h^2\left(\frac{\partial u_0^2}{\partial x}+\frac{\partial v_0^2}{\partial y}\right)\right]+\frac{1}{h}\frac{\partial h}{\partial x}\left[\frac{\partial u_0^2}{\partial y}+\frac{\partial v_0^2}{\partial x}\right]\right\}\right)$$
$$+O(\varepsilon^3)$$

y teniendo en cuenta (5.4.12), (5.4.13), (5.8.14), (5.8.15), (5.8.27) y (5.8.28) resulta:

$$\frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\hat{v}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}\right.$$
$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\right\}$$
$$=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial x}+\frac{1}{h\rho_0}(f_{W_1}^1-f_{R_1}^1)+\varepsilon\left[-\frac{\partial s}{\partial x}g+\frac{1}{h\rho_0}(f_{W_1}^2-f_{R_1}^2)\right]$$
$$+\varepsilon^2\left[-\frac{1}{\rho_0}\left.D_x p^2\right|_{z=0}+\frac{1}{h\rho_0}(f_{W_1}^3-f_{R_1}^3)+\Upsilon_1(h,u^0,v^0)\right]+O(\varepsilon^3)$$
$$=-\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon-\tilde{f}_{R_1}^\varepsilon)+\Upsilon_1^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})+O(\varepsilon^3) \quad (5.8.33)$$

$$\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon}+\hat{u}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}+\hat{v}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right.$$
$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$
$$=-\frac{1}{\rho_0}\frac{\partial p_s}{\partial y}+\frac{1}{h\rho_0}(f_{W_2}^1-f_{R_2}^1)+\varepsilon\left[-\frac{\partial s}{\partial y}g+\frac{1}{h\rho_0}(f_{W_2}^2-f_{R_2}^2)\right]$$
$$+\varepsilon^2\left[-\frac{1}{\rho_0}\left.D_y p^2\right|_{z=0}+\frac{1}{h\rho_0}(f_{W_2}^3-f_{R_2}^3)+\Upsilon_2(h,u^0,v^0)\right]+O(\varepsilon^3)$$
$$=-\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}+\frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_2}^\varepsilon-\tilde{f}_{R_2}^\varepsilon)+\Upsilon_2^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})+O(\varepsilon^3) \quad (5.8.34)$$

donde

$$\Upsilon_i^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon})=\varepsilon^2\Upsilon_i(h,u^0,v^0)\quad(i=1,2) \quad (5.8.35)$$

y $\Upsilon_i(h,u^0,v^0)\quad(i=1,2)$ vienen dados por (5.6.36) y (5.6.37).

Una vez conocidos $\hat{u}^\varepsilon$ y $\hat{v}^\varepsilon$, $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ vienen dados por:

$$\begin{aligned}\tilde{u}^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu}f_{R_1}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon}\right.\\ \left. - u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2} - v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon}\right.\\ \left. + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^{0,\varepsilon} - \nu\Delta_{x^\varepsilon y^\varepsilon}u^{0,\varepsilon}\right)\end{aligned} \tag{5.8.36}$$

$$\begin{aligned}\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu}f_{R_2}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + 2\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\right.\\ \left. - u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon} - v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{(y^\varepsilon)^2}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial v^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right.\\ \left. + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,u^{0,\varepsilon} - \nu\Delta_{x^\varepsilon y^\varepsilon}v^{0,\varepsilon}\right)\end{aligned} \tag{5.8.37}$$

A continuación proponemos un modelo de aguas someras resultado de despreciar los términos en ε^3 de (5.8.32)-(5.8.34). Se calculan en primer lugar $u^{0,\varepsilon}$, $v^{0,\varepsilon}$ y h^ε, y $\tilde{p}^\varepsilon$ se calcula a partir de ellos usando la expresión (5.8.31):

$$\begin{aligned}&\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}u^{0,\varepsilon} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right.\\ &\quad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\right)\right\}\\ &\quad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^{0,\varepsilon} + \frac{1}{h^\varepsilon\rho_0}\left(f_{W_1}^{1,\varepsilon} - f_{R_1}^{1,\varepsilon}\right)\\ &\frac{\partial v^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon} - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}v^{0,\varepsilon} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right.\\ &\quad \left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left(\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\right)\right\}\\ &\quad = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,u^{0,\varepsilon} + \frac{1}{h^\varepsilon\rho_0}(f_{W_2}^{1,\varepsilon} - f_{R_2}^{1,\varepsilon})\\ &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^{0,\varepsilon})}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon v^{0,\varepsilon})}{\partial y^\varepsilon} = 0\end{aligned}$$

$$
\begin{aligned}
\tilde{p}^\varepsilon = {} & p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\Bigg\{g + (u^{0,\varepsilon})^2\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + 2u^{0,\varepsilon}v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + (v^{0,\varepsilon})^2\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2} \\
& + \frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon}\right) - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^{0,\varepsilon}\right) \\
& - \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,u^{0,\varepsilon}\right) - 2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}\right)\right. \\
& \left.\left. + \frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon \partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial(y^\varepsilon)^2}\right)\right]\right\} + \frac{\rho_0}{2}\left[(h^\varepsilon)^2\right. \\
& \left. - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2 + 2\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + 2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon} + 2\left(\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)^2\right. \\
& \left. + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(y^\varepsilon)^2} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon} + 2\phi\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,u^{0,\varepsilon})\right]
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial \hat{u}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu\Bigg\{\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} \\
& + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right)\Bigg\} = -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} \\
& + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_1}^\varepsilon - \tilde{f}_{R_1}^\varepsilon) + \Upsilon_1^\varepsilon(h^\varepsilon, u^{0,\varepsilon}, v^{0,\varepsilon})
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon - \nu\Bigg\{\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} \\
& + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \hat{u}^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon}\right]\Bigg\} \\
& = -\frac{1}{\rho_0}\left.\frac{\partial \tilde{p}^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon = H^\varepsilon} + \frac{1}{h^\varepsilon\rho_0}(\tilde{f}_{W_2}^\varepsilon - \tilde{f}_{R_2}^\varepsilon) + \Upsilon_2^\varepsilon(h^\varepsilon, u^{0,\varepsilon}, v^{0,\varepsilon})
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}^\varepsilon = {} & \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu}f_{R_1}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + \frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} - u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2}\right. \\
& \left. - v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2\left(\frac{\partial u^{0,\varepsilon}}{\partial t^\varepsilon} + u^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon} + v^{0,\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right.
\end{aligned}
$$

$$- 2\phi \left(\operatorname{sen} \varphi^{\varepsilon}\right) v^{0,\varepsilon} - \nu \Delta_{x^{\varepsilon} y^{\varepsilon}} u^{0,\varepsilon} \Bigg)$$

$$\tilde{v}^{\varepsilon} = \hat{v}^{\varepsilon} + (z^{\varepsilon} - H^{\varepsilon}) \left[\frac{1}{\rho_0 \nu} f^{1,\varepsilon}_{R_2} + \frac{\partial H^{\varepsilon}}{\partial y^{\varepsilon}} \left(\frac{\partial u^{0,\varepsilon}}{\partial x^{\varepsilon}} + 2 \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}} \right) + \frac{\partial H^{\varepsilon}}{\partial x^{\varepsilon}} \frac{\partial v^{0,\varepsilon}}{\partial x^{\varepsilon}} - u^{0,\varepsilon} \frac{\partial^2 H^{\varepsilon}}{\partial x^{\varepsilon} \partial y^{\varepsilon}} \right.$$

$$\left. - v^{0,\varepsilon} \frac{\partial^2 H^{\varepsilon}}{(y^{\varepsilon})^2} \right] + \frac{1}{2\nu} (z^{\varepsilon} - H^{\varepsilon})^2 \Bigg(\frac{\partial v^{0,\varepsilon}}{\partial t^{\varepsilon}} + u^{0,\varepsilon} \frac{\partial v^{0,\varepsilon}}{\partial x^{\varepsilon}} + v^{0,\varepsilon} \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}} + \frac{1}{\rho_0} \frac{\partial p^{\varepsilon}_s}{\partial x^{\varepsilon}}$$

$$+ 2\phi \left(\operatorname{sen} \varphi^{\varepsilon}\right) u^{0,\varepsilon} - \nu \Delta_{x^{\varepsilon} y^{\varepsilon}} v^{0,\varepsilon} \Bigg)$$

$$\tilde{w}^{\varepsilon} = \tilde{u}^{\varepsilon} \frac{\partial H^{\varepsilon}}{\partial x^{\varepsilon}} + \tilde{v}^{\varepsilon} \frac{\partial H^{\varepsilon}}{\partial y^{\varepsilon}} - (z^{\varepsilon} - H^{\varepsilon}) \left(\frac{\partial \tilde{u}^{\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial \tilde{v}^{\varepsilon}}{\partial y^{\varepsilon}} \right) \tag{5.8.38}$$

donde $\Upsilon^{\varepsilon}_i(h^{\varepsilon}, u^{0,\varepsilon}, v^{0,\varepsilon})$ $(i = 1, 2)$, vienen dados por (5.8.35), (5.6.36)-(5.6.37). En estas expresiones está claro que $\Upsilon^{\varepsilon}_i(h^{\varepsilon}, u^{0,\varepsilon}, v^{0,\varepsilon}) = O(\varepsilon^2)$ $(i = 1, 2)$, por lo que el modelo (5.8.38) es equivalente al siguiente (en el sentido de que tan solo se diferencian en términos $O(\varepsilon^3)$):

$$\frac{\partial u^{0,\varepsilon}}{\partial t^{\varepsilon}} + u^{0,\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial x^{\varepsilon}} + v^{0,\varepsilon} \frac{\partial u^{0,\varepsilon}}{\partial y^{\varepsilon}} - \nu \Bigg\{ \Delta_{x^{\varepsilon} y^{\varepsilon}} u^{0,\varepsilon} + \frac{2}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial x^{\varepsilon}} \frac{\partial u^{0,\varepsilon}}{\partial x^{\varepsilon}}$$

$$+ \frac{1}{(h^{\varepsilon})^2} \frac{\partial}{\partial x^{\varepsilon}} \left[(h^{\varepsilon})^2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}} \right) \right] + \frac{1}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial y^{\varepsilon}} \left(\frac{\partial u^{0,\varepsilon}}{\partial y^{\varepsilon}} + \frac{\partial v^{0,\varepsilon}}{\partial x^{\varepsilon}} \right) \Bigg\}$$

$$= - \frac{1}{\rho_0} \frac{\partial p^{\varepsilon}_s}{\partial x^{\varepsilon}} + 2\phi \left(\operatorname{sen} \varphi^{\varepsilon}\right) v^{0,\varepsilon} + \frac{1}{h^{\varepsilon} \rho_0} \left(f^{1,\varepsilon}_{W_1} - f^{1,\varepsilon}_{R_1} \right) \tag{5.8.39}$$

$$\frac{\partial v^{0,\varepsilon}}{\partial t^{\varepsilon}} + u^{0,\varepsilon} \frac{\partial v^{0,\varepsilon}}{\partial x^{\varepsilon}} + v^{0,\varepsilon} \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}} - \nu \Bigg\{ \Delta_{x^{\varepsilon} y^{\varepsilon}} v^{0,\varepsilon} + \frac{2}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial y^{\varepsilon}} \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}}$$

$$+ \frac{1}{(h^{\varepsilon})^2} \frac{\partial}{\partial y^{\varepsilon}} \left[(h^{\varepsilon})^2 \left(\frac{\partial u^{0,\varepsilon}}{\partial x^{\varepsilon}} + \frac{\partial v^{0,\varepsilon}}{\partial y^{\varepsilon}} \right) \right] + \frac{1}{h^{\varepsilon}} \frac{\partial h^{\varepsilon}}{\partial x^{\varepsilon}} \left(\frac{\partial u^{0,\varepsilon}}{\partial y^{\varepsilon}} + \frac{\partial v^{0,\varepsilon}}{\partial x^{\varepsilon}} \right) \Bigg\}$$

$$= - \frac{1}{\rho_0} \frac{\partial p^{\varepsilon}_s}{\partial y^{\varepsilon}} - 2\phi \left(\operatorname{sen} \varphi^{\varepsilon}\right) u^{0,\varepsilon} + \frac{1}{h^{\varepsilon} \rho_0} (f^{1,\varepsilon}_{W_2} - f^{1,\varepsilon}_{R_2}) \tag{5.8.40}$$

$$\frac{\partial h^{\varepsilon}}{\partial t^{\varepsilon}} + \frac{\partial (h^{\varepsilon} u^{0,\varepsilon})}{\partial x^{\varepsilon}} + \frac{\partial (h^{\varepsilon} v^{0,\varepsilon})}{\partial y^{\varepsilon}} = 0 \tag{5.8.41}$$

$$\tilde{p}^{\varepsilon} = p^{\varepsilon}_s + \rho_0 (s^{\varepsilon} - z^{\varepsilon}) \Bigg\{ g + (u^{0,\varepsilon})^2 \frac{\partial^2 H^{\varepsilon}}{\partial (x^{\varepsilon})^2} + 2 u^{0,\varepsilon} v^{0,\varepsilon} \frac{\partial^2 H^{\varepsilon}}{\partial y^{\varepsilon} \partial x^{\varepsilon}} + (v^{0,\varepsilon})^2 \frac{\partial^2 H^{\varepsilon}}{\partial (y^{\varepsilon})^2}$$

$$+\frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon}+\frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon}\right)-\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)v^{0,\varepsilon}\right)$$

$$-\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)u^{0,\varepsilon}\right)-2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}+v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}\right)\right.$$

$$\left.+\frac{\partial}{\partial y^\varepsilon}\left(u^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}+v^{0,\varepsilon}\frac{\partial^2 H^\varepsilon}{\partial (y^\varepsilon)^2}\right)\right]\Bigg\}$$

$$+\frac{\rho_0}{2}\left[(h^\varepsilon)^2-(z^\varepsilon-H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\right)^2+2\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial u^{0,\varepsilon}}{\partial y^\varepsilon}+2\frac{\partial u^{0,\varepsilon}}{\partial x^\varepsilon}\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right.$$

$$+2\left(\frac{\partial v^{0,\varepsilon}}{\partial y^\varepsilon}\right)^2+\frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (x^\varepsilon)^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial (y^\varepsilon)^2}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\frac{\partial v^{0,\varepsilon}}{\partial x^\varepsilon}$$

$$\left.+2\phi\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,u^{0,\varepsilon})\right] \tag{5.8.42}$$

$$\frac{\partial \hat u^\varepsilon}{\partial t^\varepsilon}+\hat u^\varepsilon\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}+\hat v^\varepsilon\frac{\partial \hat u^\varepsilon}{\partial y^\varepsilon}-2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\hat v^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat u^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat v^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat u^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat v^\varepsilon}{\partial x^\varepsilon}\right)\right\}=-\frac{1}{\rho_0}\left.\frac{\partial \tilde p^\varepsilon}{\partial x^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}$$

$$+\frac{1}{h^\varepsilon\rho_0}(\tilde f_{W_1}^\varepsilon-\tilde f_{R_1}^\varepsilon)+\Upsilon_1^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon}) \tag{5.8.43}$$

$$\frac{\partial \hat v^\varepsilon}{\partial t^\varepsilon}+\hat u^\varepsilon\frac{\partial \hat v^\varepsilon}{\partial x^\varepsilon}+\hat v^\varepsilon\frac{\partial \hat v^\varepsilon}{\partial y^\varepsilon}+2\phi\left(\operatorname{sen}\varphi^\varepsilon\right)\hat u^\varepsilon-\nu\left\{\Delta_{x^\varepsilon y^\varepsilon}\hat v^\varepsilon+\frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat v^\varepsilon}{\partial y^\varepsilon}\right.$$

$$\left.+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat v^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial \hat u^\varepsilon}{\partial y^\varepsilon}+\frac{\partial \hat v^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$=-\frac{1}{\rho_0}\left.\frac{\partial \tilde p^\varepsilon}{\partial y^\varepsilon}\right|_{z^\varepsilon=H^\varepsilon}+\frac{1}{h^\varepsilon\rho_0}(\tilde f_{W_2}^\varepsilon-\tilde f_{R_2}^\varepsilon)+\Upsilon_2^\varepsilon(h^\varepsilon,u^{0,\varepsilon},v^{0,\varepsilon}) \tag{5.8.44}$$

$$\tilde u^\varepsilon=\hat u^\varepsilon+(z^\varepsilon-H^\varepsilon)\left[\frac{1}{\rho_0\nu}f_{R_1}^{1,\varepsilon}+\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(2\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial \hat v^\varepsilon}{\partial y^\varepsilon}\right)+\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\frac{\partial \hat u^\varepsilon}{\partial y^\varepsilon}-\hat u^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial (x^\varepsilon)^2}\right.$$

$$\left.-\hat v^\varepsilon\frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon\partial y^\varepsilon}\right]+\frac{1}{2\nu}(z^\varepsilon-H^\varepsilon)^2\left(\frac{\partial \hat u^\varepsilon}{\partial t^\varepsilon}+\hat u^\varepsilon\frac{\partial \hat u^\varepsilon}{\partial x^\varepsilon}+\hat v^\varepsilon\frac{\partial \hat u^\varepsilon}{\partial y^\varepsilon}+\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right.$$

$$- 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{v}^\varepsilon - \nu\Delta_{x^\varepsilon y^\varepsilon}\hat{u}^\varepsilon \Bigg) \tag{5.8.45}$$

$$\tilde{v}^\varepsilon = \hat{v}^\varepsilon + (z^\varepsilon - H^\varepsilon)\left[\frac{1}{\rho_0\nu} f_{R_2}^{1,\varepsilon} + \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{\partial \hat{u}^\varepsilon}{\partial x^\varepsilon} + 2\frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon}\right) + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} - \hat{u}^\varepsilon \frac{\partial^2 H^\varepsilon}{\partial x^\varepsilon \partial y^\varepsilon}\right.$$

$$\left. - \hat{v}^\varepsilon \frac{\partial^2 H^\varepsilon}{(y^\varepsilon)^2}\right] + \frac{1}{2\nu}(z^\varepsilon - H^\varepsilon)^2 \Bigg(\frac{\partial \hat{v}^\varepsilon}{\partial t^\varepsilon} + \hat{u}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial x^\varepsilon} + \hat{v}^\varepsilon \frac{\partial \hat{v}^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}$$

$$+ 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\hat{u}^\varepsilon - \nu\Delta_{x^\varepsilon y^\varepsilon}\hat{v}^\varepsilon \Bigg) \tag{5.8.46}$$

$$\tilde{w}^\varepsilon = \tilde{u}^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \tilde{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} - (z^\varepsilon - H^\varepsilon)\left(\frac{\partial \tilde{u}^\varepsilon}{\partial x^\varepsilon} + \frac{\partial \tilde{v}^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.8.47}$$

El orden de precisión con el que se verifican las ecuaciones de partida, al igual que sucedía en la aproximación de orden uno, es el mismo que el comprobado para la aproximación de segundo orden sin realizar la hipótesis de la oceanografía dinámica.

5.8.7. Modelo propuesto

De nuevo el modelo (5.8.39)-(5.8.47) es más preciso (al menos formalmente) que el modelo (5.8.24), pero el esfuerzo para resolver el sistema (5.8.39)-(5.8.47) es mucho mayor que el necesario para resolver el modelo de orden uno. Por ello concluimos que la supuesta mejora en el orden de precisión que introduce este modelo no justifica la complejidad que presenta su resolución. Se propone finalmente, suprimiendo ˜ para simplificar la notación:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (u^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (v^\varepsilon h^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{5.8.48}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon} - 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,v^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}u^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right.$$

$$\left. + \frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\right] + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\left[\frac{\partial u^\varepsilon}{\partial y^\varepsilon} + \frac{\partial v^\varepsilon}{\partial x^\varepsilon}\right]\right\}$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g + \frac{1}{h^\varepsilon \rho_0}(f_{W_1}^\varepsilon - f_{R_1}^\varepsilon) \tag{5.8.49}$$

$$\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + 2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,u^\varepsilon - \nu\left\{\Delta_{x^\varepsilon y^\varepsilon}v^\varepsilon + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y^\varepsilon}\frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right.$$

$$+\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial y^\varepsilon}\left[(h^\varepsilon)^2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\right]+\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\left[\frac{\partial u^\varepsilon}{\partial y^\varepsilon}+\frac{\partial v^\varepsilon}{\partial x^\varepsilon}\right]\Bigg\}$$

$$=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon}-\frac{\partial s^\varepsilon}{\partial y^\varepsilon}g+\frac{1}{h^\varepsilon\rho_0}(f_{W_2}^\varepsilon-f_{R_2}^\varepsilon) \tag{5.8.50}$$

$$p^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)g \tag{5.8.51}$$

$$w^\varepsilon=u^\varepsilon\frac{\partial H^\varepsilon}{\partial x^\varepsilon}+v^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}-(z^\varepsilon-H^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon}+\frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right) \tag{5.8.52}$$

que se puede escribir en forma vectorial como sigue:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\operatorname{div}(h^\varepsilon\vec{\mathbf{u}}^\varepsilon)=0$$

$$\frac{\partial\vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon}+\nabla\vec{\mathbf{u}}^\varepsilon\cdot\vec{\mathbf{u}}^\varepsilon-\nu\left\{\Delta\vec{\mathbf{u}}^\varepsilon+\frac{1}{h^\varepsilon}\left[(\nabla\vec{\mathbf{u}}^\varepsilon)^T+\nabla\vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon+\frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\operatorname{div}\vec{\mathbf{u}}^\varepsilon)\right]\right\}$$

$$+g\nabla h^\varepsilon=-\frac{1}{\rho_0}\nabla p_s^\varepsilon-g\nabla H^\varepsilon+2\phi\vec{\mathbf{F}}_C^\varepsilon+\frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon-\vec{\mathbf{f}}_R^\varepsilon\right)$$

$$w^\varepsilon=\vec{\mathbf{u}}^\varepsilon\cdot\nabla H^\varepsilon+(H^\varepsilon-z^\varepsilon)\operatorname{div}\vec{\mathbf{u}}^\varepsilon$$

$$p^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)g \tag{5.8.53}$$

donde

$$\vec{\mathbf{F}}_C^\varepsilon=(\operatorname{sen}\varphi)\begin{pmatrix} v^\varepsilon\\ -u^\varepsilon\end{pmatrix}$$

y h^ε y $\vec{\mathbf{u}}^\varepsilon$ son independientes de z^ε.

Se podría emplear la expresión de $\tilde{p}^\varepsilon$ escrita en términos de $\tilde{u}^\varepsilon$ y de $\tilde{v}^\varepsilon$ en la aproximación de orden dos como una mejora de (5.8.20):

$$\tilde{p}^\varepsilon=p_s^\varepsilon+\rho_0(s^\varepsilon-z^\varepsilon)\left\{g+(\tilde{u}^\varepsilon)^2\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+2\tilde{u}^\varepsilon\tilde{v}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}+(\tilde{v}^\varepsilon)^2\frac{\partial^2H^\varepsilon}{\partial(y^\varepsilon)^2}\right.$$

$$+\frac{1}{\rho_0}\left(\frac{\partial f_{R_1}^{1,\varepsilon}}{\partial x^\varepsilon}+\frac{\partial f_{R_2}^{1,\varepsilon}}{\partial y^\varepsilon}\right)-\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{v}^\varepsilon\right)$$

$$-\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\left(\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon}+2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon\right)-2\nu\left[\frac{\partial}{\partial x^\varepsilon}\left(\tilde{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(x^\varepsilon)^2}+\tilde{v}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}\right)\right.$$

$$\left.\left.+\frac{\partial}{\partial y^\varepsilon}\left(\tilde{u}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial y^\varepsilon\partial x^\varepsilon}+\tilde{v}^\varepsilon\frac{\partial^2H^\varepsilon}{\partial(y^\varepsilon)^2}\right)\right]\right\}$$

$$+\frac{\rho_0}{2}\left[(h^\varepsilon)^2-(z^\varepsilon-H^\varepsilon)^2\right]\left[2\left(\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}\right)^2+2\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\tilde{u}^\varepsilon}{\partial y^\varepsilon}+2\frac{\partial\tilde{u}^\varepsilon}{\partial x^\varepsilon}\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}+2\left(\frac{\partial\tilde{v}^\varepsilon}{\partial y^\varepsilon}\right)^2\right.$$

$$+\frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2}+\frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(y^\varepsilon)^2}-2\phi\,(\operatorname{sen}\varphi^\varepsilon)\,\frac{\partial\tilde{v}^\varepsilon}{\partial x^\varepsilon}+2\phi\frac{\partial}{\partial y^\varepsilon}((\operatorname{sen}\varphi^\varepsilon)\,\tilde{u}^\varepsilon)\Big] \qquad (5.8.54)$$

5.9. Conclusiones

En este capítulo hemos obtenido distintos modelos de aguas someras bidimensionales con viscosidad. Hemos trabajado, en primer lugar, a partir de la expresión completa de la aceleración de Coriolis, sin aplicar ninguna de las hipótesis que hemos llamado de la Oceanografía Dinámica y, en segundo lugar, hemos realizado dichas simplificaciones.

En el primer caso, el modelo de orden cero ((5.4.27)-(5.4.31)) y el modelo de orden uno ((5.5.20)) se diferencian, además de en la expresión para la presión, en algunos términos que aparecen en los segundos miembros de las ecuaciones de $\tilde{u}^\varepsilon$ y $\tilde{v}^\varepsilon$ que provienen del término $-2\phi\,(\cos\varphi)\,w$ que forma parte de la aceleración de Coriolis. Esta diferencia nos permite suponer que el modelo de primer orden es una mejor aproximación que el modelo de orden cero. Este modelo es comparable a los modelos de aguas poco profundas que aparecen en la literatura (como veremos en el capítulo 6) con un nuevo término de viscosidad y con una velocidad vertical que no se supone nula.

Al menos teóricamente, la aproximación de segundo orden es la que permite una mayor precisión. Las ecuaciones de Navier-Stokes para las componentes de la velocidad horizontal se verifican con un error de $O(\varepsilon^3)$, la tercera ecuación (para la velocidad vertical) y la condición de incompresibilidad se verifican con un error de $O(\varepsilon^2)$, las condiciones de contorno (5.1.4) y (5.1.5) se verifican de forma exacta, mientras que (5.1.7)-(5.1.8) se verifican con un error de $O(\varepsilon^4)$.

A partir de esta aproximación se ha propuesto el modelo (5.6.44)-(5.6.51) que mejora el modelo propuesto para el primer orden, al menos formalmente, salvo para la ecuación necesaria para el cálculo de h^ε, ya que en este caso el error que se comete sigue siendo de orden $O(\varepsilon^2)$ (es $u^{0,\varepsilon}$ y no la aproximación $\tilde{u}^\varepsilon$ quien verifica la ecuación de continuidad, lo que puede considerarse un defecto de este modelo) y la ecuación de incompresibilidad (que en este modelo no se verifica de forma exacta y en los de orden cero y orden uno sí). Además, el esfuerzo de cálculo es el doble del necesario para resolver el modelo (5.5.20).

Teniendo en cuenta la discusión anterior proponemos del modelo (5.7.1)-(5.7.5) (o (5.7.6) en su versión vectorial).

En el caso de que impongamos las simplificaciones de la Oceanografía Dinámica, las diferencias significativas comienzan con la aproximación de primer orden en ε, cuando las hipótesis realizadas permiten obtener un modelo más sencillo de resolver ((5.8.24)) debido fundamentalmente a la supresión del término $-2\phi\,(\cos\varphi)\,w$ en la primera ecuación de Navier-Stokes. De nuevo, en este caso, el modelo de segundo orden propuesto ((5.8.39)-(5.8.47)) requiere un esfuerzo de cálculo mucho mayor que el modelo (5.8.24) para obtener una pequeña mejora en la precisión (teóricamente del orden de ε^2), por lo que proponemos para este caso el modelo (5.8.48)-(5.8.52) (o (5.8.53) en su versión vectorial).

Capítulo 6

Comparación del nuevo modelo de aguas someras con viscosidad con otros modelos de aguas someras

En este capítulo pretendemos comparar el modelo de aguas someras con viscosidad propuesto en el capítulo 3 para dimensión uno (véase (3.7.1)-(3.7.4)) y en el capítulo 5 para dimensión dos (véase (5.7.3)) con los modelos de aguas someras que se pueden encontrar en la literatura. En primer lugar compararemos los distintos modelos analíticamente, deteniéndonos en qué términos son diferentes en las ecuaciones de los distintos modelos, y en segundo lugar los compararemos numéricamente, resolviendo los modelos para diferentes ejemplos. Observaremos que el nuevo término de viscosidad que incorpora el modelo que proponemos supone una mejora respecto a los otros modelos.

6.1. Comparación analítica del modelo propuesto con otros modelos de aguas someras

6.1.1. Dimensión uno

Compararemos ahora el modelo propuesto en el capítulo 3 ((3.7.1)-(3.7.4)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.1}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - 2\nu \left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right)$$

$$= -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon\right) \tag{6.1.2}$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)g \tag{6.1.3}$$

$$w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial u^\varepsilon}{\partial x^\varepsilon} \tag{6.1.4}$$

con otros modelos de aguas someras que aparecen en la literatura, y que ya han sido descritos en el capítulo 1, de manera que podamos observar lo que tiene de novedoso el modelo propuesto. Comenzamos por reescribir los diferentes modelos presentados en el capítulo 1 en dimension uno usando nuestra notación de modo que la comparación con nuestro modelo sea más sencilla.

En primer lugar, en [101] (página 38) se escribe el modelo clásico de aguas someras sin viscosidad como sigue (véase (1.2.28)-(1.2.29)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon v^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{6.1.5}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon} + g\frac{\partial h^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_{W_1}^\varepsilon - f_{R_1}^\varepsilon\right) + F_x^\varepsilon \tag{6.1.6}$$

$$\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon} + g\frac{\partial h^\varepsilon}{\partial y^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial y^\varepsilon} - g\frac{\partial H^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_{W_2}^\varepsilon - f_{R_2}^\varepsilon\right) + F_y^\varepsilon \tag{6.1.7}$$

Reescribimos este modelo en dimensión uno y despreciamos los efectos de las fuerzas externas:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.8}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - g\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right) + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon\right) \tag{6.1.9}$$

Si comparamos (6.1.8)-(6.1.9) con (6.1.1)-(6.1.2) observamos que ambos son el mismo modelo, donde (6.1.2) añade un nuevo término de viscosidad.

Tenemos otro ejemplo del modelo clásico de aguas someras sin viscosidad en [5] (página 3):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0 \tag{6.1.10}$$

$$\frac{\partial (h^\varepsilon \vec{\mathbf{u}}^\varepsilon)}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon \otimes \vec{\mathbf{u}}^\varepsilon) + \nabla\left(\frac{g}{2}(h^\varepsilon)^2\right) + g h^\varepsilon \nabla H^\varepsilon = 0 \tag{6.1.11}$$

donde $\vec{\mathbf{u}}^\varepsilon = (u^\varepsilon, v^\varepsilon)$. La ecuación (6.1.10) en dimensión uno es exactamente la misma que (6.1.1):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.12}$$

Escribamos (6.1.11) en dimensión uno:

$$\frac{\partial(h^\varepsilon u^\varepsilon)}{\partial t^\varepsilon} + \frac{\partial(h^\varepsilon (u^\varepsilon)^2)}{\partial x^\varepsilon} + \frac{\partial}{\partial x^\varepsilon}\left(\frac{g}{2}(h^\varepsilon)^2\right) + gh^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} = 0 \Longleftrightarrow$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} u^\varepsilon + h^\varepsilon \frac{\partial u^\varepsilon}{\partial t^\varepsilon} + \frac{\partial h^\varepsilon}{\partial x^\varepsilon}(u^\varepsilon)^2 + 2h^\varepsilon u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + gh^\varepsilon \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + gh^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} = 0 \Longleftrightarrow$$

$$u^\varepsilon \left(\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} u^\varepsilon + h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) + h^\varepsilon \left[\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + g\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right)\right] = 0 \tag{6.1.13}$$

Expandiendo (6.1.12) vemos que $\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial h^\varepsilon}{\partial x^\varepsilon} u^\varepsilon + h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} = 0$, por tanto (6.1.13) se reduce a:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} = -g\left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\right) \tag{6.1.14}$$

Si suponemos en (6.1.1)-(6.1.2) que $\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} = 0$, no tenemos en cuenta los efectos del viento o la fricción y despreciamos los términos viscosos, obtenemos exactamente (6.1.12) y (6.1.14).

Algunos autores prefieren expresar el modelo de aguas someras en términos del flujo. Un ejemplo puede encontrarse en [21] (página 117)

$$\frac{\partial \eta^\varepsilon}{\partial t^\varepsilon} + \operatorname{div} \mathbf{Q}^\varepsilon = 0 \tag{6.1.15}$$

$$\frac{\partial \mathbf{Q}^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}\left\{(\vec{\mathbf{u}}^\varepsilon \otimes \mathbf{Q}^\varepsilon) + \frac{1}{2} g\left[(\eta^\varepsilon)^2 + 2\eta^\varepsilon B^\varepsilon\right]\delta\right\}$$

$$= g\eta^\varepsilon \nabla B^\varepsilon + F^\varepsilon + \frac{1}{\rho_0}(\vec{f}_W^\varepsilon - \vec{f}_R^\varepsilon) \tag{6.1.16}$$

donde $\vec{\mathbf{u}}^\varepsilon = (u^\varepsilon, v^\varepsilon)$, $\mathbf{Q}^\varepsilon = h^\varepsilon \vec{\mathbf{u}}^\varepsilon$, $\eta^\varepsilon = s^\varepsilon - A$, $B^\varepsilon = A - H^\varepsilon$, A es un nivel de referencia constante y δ es el tensor unidad.

De nuevo la ecuación (6.1.15) escrita en dimensión uno es la ecuación (6.1.1) porque $\frac{\partial \eta^\varepsilon}{\partial t^\varepsilon} = \frac{\partial s^\varepsilon}{\partial t^\varepsilon} = \frac{\partial h^\varepsilon}{\partial t^\varepsilon}$. Escribimos (6.1.16) en dimensión uno (sin tener en cuenta en este caso el término F^ε, que suele incorporar los efectos de la aceleración de Coriolis):

$$\frac{\partial(h^\varepsilon u^\varepsilon)}{\partial t^\varepsilon} + \frac{\partial(h^\varepsilon (u^\varepsilon)^2)}{\partial x^\varepsilon} + g\left[(s^\varepsilon - A)\frac{\partial s^\varepsilon}{\partial x^\varepsilon} + (A - H^\varepsilon)\frac{\partial s^\varepsilon}{\partial x^\varepsilon}\right] = \frac{1}{\rho_0}(f_W^\varepsilon - f_R^\varepsilon)$$

Ahora, expandimos esta expresión, razonamos como en (6.1.13) y tenemos presente que $h^\varepsilon = s^\varepsilon - H^\varepsilon$ para obtener:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} = -g\frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}(f_W^\varepsilon - f_R^\varepsilon) \tag{6.1.17}$$

Podemos deducir la ecuación (6.1.17) a partir de (6.1.2) si suponemos $\dfrac{\partial p_s^\varepsilon}{\partial x^\varepsilon} = 0$ y despreciamos los términos de viscosidad.

Estudiemos, ahora, los modelos que incluyen los efectos de la viscosidad. En primer lugar presentamos el modelo que se recoge en [100] (página 1137) (véase (1.2.67)-(1.2.68))

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon)) = 0 \tag{6.1.18}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon + f^\varepsilon \vec{\mathbf{k}} \times \vec{\mathbf{u}}^\varepsilon = \frac{\nu}{h^\varepsilon}\operatorname{div}(h^\varepsilon \nabla \vec{\mathbf{u}}^\varepsilon) \tag{6.1.19}$$

donde f^ε es el parámetro de Coriolis y $\vec{\mathbf{k}} = (0,0,1)$. En dimensión uno este modelo (sin tener en cuenta f^ε como hemos hecho en (6.1.1)-(6.1.4) y en (6.1.17)) resulta:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.20}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{1}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) = -g\frac{\partial h^\varepsilon}{\partial x^\varepsilon} \tag{6.1.21}$$

La ecuación (6.1.20) es exactamente (6.1.1). Si suponemos en (6.1.2) que $\dfrac{\partial p_s^\varepsilon}{\partial x^\varepsilon} = 0$ y que el fondo es $H^\varepsilon = 0$ y despreciamos los efectos del viento y del rozamiento entonces se obtiene (6.1.21) con la única diferencia de los términos de viscosidad.

Otro modelo con el mismo tipo de términos viscosos ((1.2.67)-(1.2.68)) aparece en [53] (página 302):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \frac{\partial (h^\varepsilon v^\varepsilon)}{\partial y^\varepsilon} = 0 \tag{6.1.22}$$

$$h^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon}\right) - \nu\left[\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) + \frac{\partial}{\partial y^\varepsilon}\left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial y^\varepsilon}\right)\right] + gh^\varepsilon \frac{\partial h^\varepsilon}{\partial x^\varepsilon} = h^\varepsilon F_x^\varepsilon \tag{6.1.23}$$

$$h^\varepsilon\left(\frac{\partial v^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right) - \nu\left[\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon \frac{\partial v^\varepsilon}{\partial x^\varepsilon}\right) + \frac{\partial}{\partial y^\varepsilon}\left(h^\varepsilon \frac{\partial v^\varepsilon}{\partial y^\varepsilon}\right)\right] + gh^\varepsilon \frac{\partial h^\varepsilon}{\partial y^\varepsilon} = h^\varepsilon F_y^\varepsilon \tag{6.1.24}$$

Reescribimos el modelo en dimensión uno y dividimos la ecuación (6.1.23) por h^ε:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.25}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{1}{h^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right) = -g \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + F_x^\varepsilon \tag{6.1.26}$$

Basta despreciar F_x^ε para obtener (6.1.20)-(6.1.21).

En [11] (páginas 60-61) se sugieren tres términos de viscosidad diferentes. El modelo es el siguiente:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(\mathbf{u}^\varepsilon h^\varepsilon) = 0 \tag{6.1.27}$$

$$\frac{\partial (h^\varepsilon \mathbf{u}^\varepsilon)}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + c\nabla \left(h^\varepsilon\right)^2 - < \nu \Delta \mathbf{v}^\varepsilon >= \mathbf{f}^\varepsilon \tag{6.1.28}$$

donde $\mathbf{u}^\varepsilon = \dfrac{< \mathbf{v}^\varepsilon >}{h^\varepsilon}$ con $< \mathbf{v}^\varepsilon >= \displaystyle\int_{H^\varepsilon}^{s^\varepsilon} \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix} dz^\varepsilon$ y $\mathbf{v}^\varepsilon$ es la velocidad de Navier-Stokes, $\mathbf{f}^\varepsilon$ incluye los efectos del viento, el rozamiento y la aceleración de Coriolis y c es una constante proporcional a g. Los autores dan estas tres formulaciones diferentes del término de viscosidad:

$$- < \nu \Delta \mathbf{v}^\varepsilon >= \begin{cases} -\nu h^\varepsilon \Delta \mathbf{u}^\varepsilon \\ \\ -\nu \Delta (h^\varepsilon \mathbf{u}^\varepsilon) \\ \\ -\nu \operatorname{div}(|h^\varepsilon| \nabla \mathbf{u}^\varepsilon) \end{cases} \tag{6.1.29}$$

Reescribimos el modelo (6.1.27)-(6.1.28) en dimensión uno. La ecuación (6.1.27) se transforma en (6.1.1) y (6.1.28), usando el razonamiento empleado a partir de (6.1.13) para obtener (6.1.14), eligiendo $c = \dfrac{g}{2}$ y considerando la igualdad (6.1.29.a), se llega a:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.30}$$

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} = -g \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon} f^\varepsilon \tag{6.1.31}$$

Si suponemos que el término de viscosidad es (6.1.29.b), la ecuación (6.1.28) podemos expresarla como sigue:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{1}{h^\varepsilon} \frac{\partial^2 (h^\varepsilon u^\varepsilon)}{\partial (x^\varepsilon)^2} = -g \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon} f^\varepsilon \tag{6.1.32}$$

Aceptando que la expresión para los términos viscosos es (6.1.29.c) se obtiene la siguiente ecuación:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{1}{h^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right) = -g \frac{\partial h^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon} f^\varepsilon \tag{6.1.33}$$

Señalamos que a partir de (6.1.30)-(6.1.33) podemos reobtener (6.1.1)-(6.1.2) con la única diferencia de los términos de viscosidad, y que H^ε y p_s^ε se suponen constantes.

En [11], podemos encontrar que (6.1.27)-(6.1.28) está escrito también en términos del flujo como sigue:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \text{div}\mathbf{Q}^\varepsilon = 0 \tag{6.1.34}$$

$$\frac{\partial \mathbf{Q}^\varepsilon}{\partial t^\varepsilon} + \text{div} \left(\frac{1}{h^\varepsilon} \mathbf{Q}^\varepsilon \otimes \mathbf{Q}^\varepsilon \right) + 2c\nabla h^\varepsilon - \nu \Delta \mathbf{Q}^\varepsilon = \mathbf{f}^\varepsilon \tag{6.1.35}$$

Dado que (6.1.34)-(6.1.35) es el mismo modelo que (6.1.27)-(6.1.28) no lo compararemos de nuevo con (6.1.1)-(6.1.2).

En [43] (página 89) se propone el siguiente modelo unidimensional:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \tag{6.1.36}$$

$$\frac{\partial (h^\varepsilon u^\varepsilon)}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon (u^\varepsilon)^2)}{\partial x^\varepsilon} + \frac{g}{2} \frac{\partial (h^\varepsilon)^2}{\partial x^\varepsilon} = -\frac{k u^\varepsilon}{1 + \dfrac{k h^\varepsilon}{2\nu}} + 4\nu \frac{\partial}{\partial x^\varepsilon} \left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right) \tag{6.1.37}$$

donde k es un coeficiente de rozamiento y $H^\varepsilon = 0$. De nuevo argumentamos como al obtener (6.1.13) y se llega a:

$$\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - 4\nu \frac{1}{h^\varepsilon} \frac{\partial}{\partial x^\varepsilon} \left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} \right) = -g \frac{\partial h^\varepsilon}{\partial x^\varepsilon} - \frac{k u^\varepsilon}{1 + \dfrac{k h^\varepsilon}{2\nu}} \tag{6.1.38}$$

Hemos comparado nuestro modelo (6.1.1)-(6.1.4) con diferentes modelos que se encuentran en la literatura. Estos modelos, una vez reescritos en dimensión uno y con nuestra notación, son (6.1.8)-(6.1.9), (6.1.12) y (6.1.14), (6.1.1) y (6.1.17), (6.1.20)-(6.1.21), (6.1.25)-(6.1.26), (6.1.30)-(6.1.31), (6.1.30) y (6.1.32), (6.1.30) y (6.1.33), (6.1.36) y (6.1.38).

Hemos observado que nuestro modelo incluye términos que algunos otros modelos no consideran: $\dfrac{\partial p_s^\varepsilon}{\partial x^\varepsilon}$ cuando p_s^ε (presión atmosférica) no es constante (este término puede ser muy importante en la simulación del oleaje de una tormenta (véase [106], página 27)), las fuerzas del viento y de rozamiento ($f_W^\varepsilon, f_R^\varepsilon$), fondo no constante

($H^\varepsilon \neq 0$) y el término de viscosidad (la viscosidad no se considera en algunos de los modelos y en otros, el término propuesto es diferente).

Además, sugerimos en (6.1.3) una expresión para la presión que mejora la precisión del modelo (al menos formalmente) respecto a suponer que la presión es la hidrostática y en (6.1.4) una formulación explícita de la velocidad vertical w^ε que, en gran número de modelos en la literatura, no aparece explícitamente (bien porque se supone implícitamente o bien porque se desprecia).

Por tanto, consideramos que la mayor contribución de nuestro modelo es que se trata de un modelo general (incluye términos que otros modelos desprecian) y el nuevo término de viscosidad. Alguno de los modelos mencionados antes ((6.1.8)-(6.1.9), (6.1.12) y (6.1.14), (6.1.1) y (6.1.17)) no incluyen ningún término de viscosidad. Los diferentes términos de viscosidad propuestos en los modelos (6.1.20)-(6.1.21), (6.1.25)-(6.1.26), (6.1.30)-(6.1.31), (6.1.30) y (6.1.32), (6.1.30) y (6.1.33), (6.1.36) y (6.1.38) son

$$-\nu\frac{1}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) = -\nu\left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) \tag{6.1.39}$$

$$-\nu\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} \tag{6.1.40}$$

$$-\nu\frac{1}{h^\varepsilon}\frac{\partial^2 (h^\varepsilon u^\varepsilon)}{\partial (x^\varepsilon)^2} = -\nu\left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon} + \frac{1}{h^\varepsilon}\frac{\partial^2 h^\varepsilon}{\partial (x^\varepsilon)^2}u^\varepsilon\right) \tag{6.1.41}$$

$$-4\nu\frac{1}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) = -4\nu\left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) \tag{6.1.42}$$

mientras que el término que incluye los efectos viscosos en nuestro modelo es

$$-2\nu\frac{1}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) = -2\nu\left(\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} + \frac{2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) \tag{6.1.43}$$

Comparemos ahora los términos (6.1.39)-(6.1.42) con el término de viscosidad de nuestro modelo. Consideramos en primer lugar $-\nu\frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2}$ que aparece en (6.1.31). Es obvio que este término es el mismo que aparece en las ecuaciones de Navier-Stokes ($-\nu\Delta^\varepsilon u^\varepsilon$) pero en dimensión uno.

El término de viscosidad (6.1.39), que incluyen los modelos (6.1.21), (6.1.26) y (6.1.33), añade al término de viscosidad "tipo Navier-Stokes" el término $-\nu\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}$. El modelo (6.1.38) incorpora el término (6.1.42) que es precisamente (6.1.39) multiplicado por 4.

La expresión (6.1.41), presente en el modelo (6.1.32), agrega a los términos vistos antes uno nuevo, $-\nu\frac{1}{h^\varepsilon}\frac{\partial^2 h^\varepsilon}{\partial (x^\varepsilon)^2}u^\varepsilon$, y el término $-\nu\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}$ aparece multiplicado por 2.

Finalmente, en nuestro modelo, al estudiar el término (6.1.43) encontramos que en este caso el efecto de la viscosidad "tipo Navier-Stokes" está duplicado mientras que el término de difusión $-\nu\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}$ visto en (6.1.39), (6.1.41) y (6.1.42) está multiplicado por 4.

Así, la diferencia entre los modelos que hemos presentado en cuanto a la viscosidad es el hecho de que aparezcan o no y el número de veces que lo hacen los términos $-\nu\frac{\partial^2 u^\varepsilon}{\partial(x^\varepsilon)^2}$ y $-\nu\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\frac{\partial u^\varepsilon}{\partial x^\varepsilon}$ (en (6.1.41) también encontramos el término $-\nu\frac{1}{h^\varepsilon}\frac{\partial^2 h^\varepsilon}{\partial(x^\varepsilon)^2}u^\varepsilon$), y si observamos la expresión a la izquierda de las igualdades (6.1.39)-(6.1.43), nos damos cuenta de que la diferencia se sustancia en cómo la profundidad h^ε "corrige" los efectos de la difusión.

A la vista de la gran variedad de términos de viscosidad que aparecen en la literatura, está claro que su justificación carece del rigor necesario. Consideramos que el método de desarrollos asintóticos puede aportar dicho rigor, por lo que proponemos el término ((6.1.43)) como el correcto. En la sección 6.2 intentaremos confirmar, mediante simulaciones numéricas, que esta es la elección adecuada.

6.1.2. Dimensión dos

Recordamos ahora los modelos de aguas someras bidimensionales propuestos en el capítulo 5 ((5.7.6), (5.8.53)). Compararemos estos modelos con los que se pueden encontrar en la literatura, algunos de los cuales se han presentado en el capítulo 1.

En primer lugar comparamos los dos modelos propuestos en el capítulo 5. Escribimos el modelo (5.7.6) y señalamos en rojo los términos que no aparecen en (5.8.53):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0 \tag{6.1.44}$$

$$\begin{aligned}\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon\cdot\vec{\mathbf{u}}^\varepsilon - \nu\left\{\Delta\vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla\vec{\mathbf{u}}^\varepsilon)^T + \nabla\vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\operatorname{div}\vec{\mathbf{u}}^\varepsilon)\right]\right\}&\\ + g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + 2\phi\vec{\mathbf{F}}_C^\varepsilon + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon\right)&\end{aligned} \tag{6.1.45}$$

$$w^\varepsilon = \vec{\mathbf{u}}^\varepsilon\cdot\nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon)\operatorname{div}\vec{\mathbf{u}}^\varepsilon \tag{6.1.46}$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)u^\varepsilon\right] \tag{6.1.47}$$

donde

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon)v^\varepsilon + \cos\varphi^\varepsilon\left(\frac{\partial(h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \frac{h^\varepsilon}{2}\frac{\partial v^\varepsilon}{\partial y^\varepsilon} - v^\varepsilon\frac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\operatorname{sen}\varphi^\varepsilon)u^\varepsilon + \frac{h^\varepsilon}{2}\frac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon] + \frac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon]\end{pmatrix}$$

Los términos en rojo en (6.1.44)-(6.1.47) no aparecen en (5.8.53) debido a que al realizar la hipótesis oceanográfica sobre la aceleración de Coriolis (1.1.9) se desprecia el término $-2\phi(\cos\varphi)w$ y se incorpora $-2\phi(\cos\varphi)u$ a la gravedad.

Reescribiremos ahora los modelos bidimensionales del capítulo 1 utilizando nuestra notación.

En primer lugar, en [101] (página 38) se escribe el modelo clásico de aguas someras sin viscosidad ((1.2.28)-(1.2.29)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0 \tag{6.1.48}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon\right) + \vec{\mathbf{F}}^\varepsilon \tag{6.1.49}$$

donde $\vec{\mathbf{F}}^\varepsilon$ representa la fuerza por unidad de masa que actúa sobre el sistema (además de la gravedad y las fuerzas del viento y de rozamiento, que ya han sido tenidas en cuenta; típicamente será la aceleración de Coriolis).

Si comparamos (6.1.48)-(6.1.49) con (6.1.44)-(6.1.47) observamos que (6.1.44)-(6.1.47) añade a (6.1.48)-(6.1.49) los términos de viscosidad.

Veamos otro ejemplo de modelo clásico de aguas someras sin viscosidad en [5] (página 3) ((1.2.23)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.50}$$

$$\frac{\partial (h^\varepsilon \vec{\mathbf{u}}^\varepsilon)}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon \otimes \vec{\mathbf{u}}^\varepsilon) + \nabla\left(\frac{g}{2}(h^\varepsilon)^2\right) + gh^\varepsilon\nabla H^\varepsilon = 0 \tag{6.1.51}$$

donde $\vec{\mathbf{u}}^\varepsilon = (u^\varepsilon, v^\varepsilon)$. Si desarrollamos (6.1.51) teniendo en cuenta (6.1.50) obtenemos que (6.1.50)-(6.1.51) es equivalente a (véase (1.2.23)):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.52}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla s^\varepsilon = 0 \tag{6.1.53}$$

que se obtiene a partir de (6.1.44)-(6.1.47) si despreciamos la aceleración de Coriolis, los términos de viscosidad, los efectos del viento y el rozamiento y suponemos que $\nabla p_s^\varepsilon = \vec{0}$.

De entre los modelos que incluyen los efectos de la viscosidad, presentamos en primer lugar el modelo que se recoge en [100] (página 1137) o en [53] (página 302) (véase también (1.2.67)-(1.2.68))

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}\left(h^\varepsilon \vec{\mathbf{u}}^\varepsilon\right) = 0 \tag{6.1.54}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon + f^\varepsilon \vec{\mathbf{k}} \times \vec{\mathbf{u}}^\varepsilon = \frac{\nu}{h^\varepsilon}\mathrm{div}\left(h^\varepsilon \nabla\vec{\mathbf{u}}^\varepsilon\right) \tag{6.1.55}$$

Se observa que si en el modelo (6.1.44)-(6.1.47) suponemos que el fondo y la presión superficial son constantes y despreciamos los efectos del viento y del rozamiento entonces se obtiene (6.1.54)-(6.1.55) con la única diferencia de los términos de viscosidad.

En [11] (páginas 60-61) se sugiere el siguiente modelo

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \text{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.56}$$

$$\frac{\partial (h^\varepsilon \vec{\mathbf{u}}^\varepsilon)}{\partial t^\varepsilon} + \text{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon \otimes \vec{\mathbf{u}}^\varepsilon) + c\nabla (h^\varepsilon)^2 - < \nu \Delta \vec{\mathbf{v}}^\varepsilon >= \vec{\mathbf{f}}^\varepsilon \tag{6.1.57}$$

donde $\vec{\mathbf{u}}^\varepsilon = \dfrac{< \vec{\mathbf{v}}^\varepsilon >}{h^\varepsilon}$ con $< \vec{\mathbf{v}}^\varepsilon >= \displaystyle\int_{H^\varepsilon}^{s^\varepsilon} \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix} dz^\varepsilon$ y $\vec{\mathbf{v}}^\varepsilon$ es la velocidad de Navier-Stokes, $\vec{\mathbf{f}}^\varepsilon$ incluye los efectos del viento, el rozamiento y la aceleración de Coriolis y c es una constante proporcional a g. Los autores dan tres posibles formulaciones diferentes del término de viscosidad:

$$- < \nu \Delta \vec{\mathbf{v}}^\varepsilon >= \begin{cases} -\nu h^\varepsilon \Delta \vec{\mathbf{u}}^\varepsilon \\ \\ -\nu \Delta (h^\varepsilon \vec{\mathbf{u}}^\varepsilon) \\ \\ -\nu \text{div}(|h^\varepsilon| \nabla \vec{\mathbf{u}}^\varepsilon) \end{cases} \tag{6.1.58}$$

Si en (6.1.56)-(6.1.57) consideramos $c = \dfrac{g}{2}$ y las tres diferentes opciones de (6.1.58) obtenemos (teniendo en cuenta (6.1.56) y desarrollando (6.1.57)) los siguientes modelos:

- Eligiendo la primera expresión de (6.1.58) para la difusión,

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \text{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.59}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon - \nu \Delta \vec{\mathbf{u}}^\varepsilon = \frac{1}{h^\varepsilon} \vec{\mathbf{f}}^\varepsilon \tag{6.1.60}$$

- Si suponemos que el término de viscosidad es (6.1.58.b),

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \text{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.61}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon - \frac{\nu}{h^\varepsilon} \Delta (h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = \frac{1}{h^\varepsilon} \vec{\mathbf{f}}^\varepsilon \tag{6.1.62}$$

- Aceptando que la expresión para los términos viscosos es (6.1.58.c),

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \text{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0 \tag{6.1.63}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon - \frac{\nu}{h^\varepsilon}\text{div}(|h^\varepsilon|\nabla\vec{\mathbf{u}}^\varepsilon) = \frac{1}{h^\varepsilon}\vec{\mathbf{f}}^\varepsilon \tag{6.1.64}$$

Observamos que estos tres modelos, si obviamos la diferencia en los términos de viscosidad, se obtienen a partir de (6.1.44)-(6.1.47) suponiendo que H^ε y p_s^ε son constantes.

Finalmente y resumiendo, se observa que nuestro modelo (6.1.44)-(6.1.47) incluye términos que otros modelos no consideran: ∇p_s^ε cuando p_s^ε no es constante, las fuerzas del viento y de rozamiento, fondo no constante, una expresión para la presión ((6.1.47)) que añade términos a la presión hidrostática, una formulación explícita de la velocidad vertical w^ε ((6.1.46)) y un término de viscosidad diferente. Respecto al término de viscosidad cabe señalar que los modelos (6.1.48)-(6.1.49), (6.1.50)-(6.1.51) y (6.1.52)-(6.1.53) no incluyen términos de viscosidad, que el modelo (6.1.54)-(6.1.55) incluye el término de viscosidad:

$$-\frac{\nu}{h^\varepsilon}\text{div}(|h^\varepsilon|\nabla\vec{\mathbf{u}}^\varepsilon) = -\frac{\nu}{h^\varepsilon}\text{div}(h^\varepsilon\nabla\vec{\mathbf{u}}^\varepsilon) = -\nu\left(\Delta\vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\nabla\vec{\mathbf{u}}^\varepsilon \cdot \nabla h^\varepsilon\right) \tag{6.1.65}$$

El modelo (6.1.63)-(6.1.64) presenta el término de viscosidad (6.1.65), mientras que (6.1.59)-(6.1.60) y(6.1.61)- (6.1.62) incluyen, respectivamente, los siguientes términos:

$$-\nu\Delta u^\varepsilon \tag{6.1.66}$$

$$-\frac{\nu}{h^\varepsilon}\Delta(h^\varepsilon\vec{\mathbf{u}}^\varepsilon) \tag{6.1.67}$$

El término que introduce los efectos viscosos en nuestro modelo es

$$-\nu\left\{\Delta\vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla\vec{\mathbf{u}}^\varepsilon)^T + \nabla\vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\text{div}\,\vec{\mathbf{u}}^\varepsilon)\right]\right\} \tag{6.1.68}$$

Observamos que la principal diferencia entre los términos de viscosidad radica en cómo h^ε influye en los términos difusivos (6.1.65)-(6.1.68). En todos ellos se reobtiene el término "tipo Navier-Stokes" ((6.1.66)) si h^ε es constante, pero difieren bastante cuando el calado sufre grandes variaciones.

Consideramos que los modelos que se encuentran en la literatura no justifican suficientemente la elección del término de viscosidad y de ahí la gran variedad de términos propuestos (véase (6.1.65)-(6.1.67)). En la sección siguiente veremos que el término obtenido mediante el uso del análisis asintótico ((6.1.68)) ofrece los mejores resultados en las comparaciones numéricas, avalando que esta técnica es una buena herramienta para encontrar el término de viscosidad más preciso.

6.2. Comparación numérica

En esta sección compararemos numéricamente los diferentes modelos que ya hemos comparado analíticamente en la sección anterior.

Primero recordaremos los modelos que vamos a comparar, a continuación presentaremos el método numérico que vamos a utilizar y finalmente compararemos los resultados numéricos obtenidos por cada modelo para diferentes casos.

6.2.1. Modelos que se van a comparar

Relacionamos a continuación los modelos que vamos a comparar numéricamente, tanto en dimensión uno como en dimensión dos.

6.2.1.1. Dimensión uno

Escogemos algunos de los modelos unidimensionales presentados en 6.1.1, centrándonos en la diferencia en el término de viscosidad, de modo que los reescribimos incluyendo los efectos más generales (presión y fondo no constantes).

En primer lugar consideramos el modelo sin viscosidad que se ha propuesto en (2.9.1):

$$\begin{aligned} &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \\ &\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0} g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} \end{aligned} \tag{6.2.1}$$

A continuación consideramos el modelo con viscosidad tipo Navier-Stokes visto en (6.1.30)-(6.1.31):

$$\begin{aligned} &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \\ &\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \nu \frac{\partial^2 u^\varepsilon}{\partial (x^\varepsilon)^2} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0} g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon\right) \end{aligned} \tag{6.2.2}$$

En tercer lugar presentamos un modelo de aguas someras que, en lugar del término de viscosidad clásico, incluye el término $-\nu \frac{1}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right)$ (por ejemplo, (6.1.30) con (6.1.33)):

$$\begin{aligned} &\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} = 0 \\ &\frac{\partial u^\varepsilon}{\partial t^\varepsilon} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon} - \frac{\nu}{h^\varepsilon}\frac{\partial}{\partial x^\varepsilon}\left(h^\varepsilon \frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right) = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{1}{\rho_0} g \frac{\partial s^\varepsilon}{\partial x^\varepsilon} + \frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon - f_R^\varepsilon\right) \end{aligned} \tag{6.2.3}$$

Finalmente escribimos el modelo unidimensional con viscosidad que se propuso en el capítulo 3 (véase (3.7.1)-(3.7.2)):

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\frac{\partial (h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon}=0\\
&\frac{\partial u^\varepsilon}{\partial t^\varepsilon}+u^\varepsilon\frac{\partial u^\varepsilon}{\partial x^\varepsilon}-\frac{2\nu}{(h^\varepsilon)^2}\frac{\partial}{\partial x^\varepsilon}\left((h^\varepsilon)^2\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right)=-\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}-\frac{1}{\rho_0}g\frac{\partial s^\varepsilon}{\partial x^\varepsilon}+\frac{1}{\rho_0 h^\varepsilon}\left(f_W^\varepsilon-f_R^\varepsilon\right)
\end{aligned}
\tag{6.2.4}
$$

En lo sucesivo, y para hacer más sencillas las referencias en el texto, tablas y figuras, nos referiremos a los modelos que acabamos de presentar como sigue: el modelo de dimensión uno sin viscosidad ((6.2.1)) lo llamaremos **M1SV**, nos referiremos a los modelos (6.2.2) y (6.2.3) como **M1V1** y **M1V2**, respectivamente, y al nuevo modelo que proponemos ((6.2.4)) lo llamaremos **M1VN**.

6.2.1.2. Dimensión dos

Los modelos que se van a comparar en dimensión dos son los propuestos en 6.1.2. En primer lugar consideramos el modelo sin viscosidad (6.1.48)-(6.1.49):

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\operatorname{div}(h^\varepsilon\vec{\mathbf{u}}^\varepsilon)=0\\
&\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon}+\nabla\vec{\mathbf{u}}^\varepsilon\cdot\vec{\mathbf{u}}^\varepsilon=-\frac{1}{\rho_0}\nabla p_s^\varepsilon-g\nabla s^\varepsilon+2\phi\vec{\mathbf{F}}_{C_1}^\varepsilon
\end{aligned}
\tag{6.2.5}
$$

con

$$
\vec{\mathbf{F}}_{C_1}^\varepsilon=\begin{pmatrix}(\operatorname{sen}\varphi^\varepsilon)v^\varepsilon\\-(\operatorname{sen}\varphi^\varepsilon)u^\varepsilon\end{pmatrix}
$$

A continuación consideramos los modelos con viscosidad (6.1.59)-(6.1.60), (6.1.61)-(6.1.62) y (6.1.63)-(6.1.64), que tienen en cuenta los tres diferentes tipos de términos de viscosidad que se encuentran en la literatura:

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\operatorname{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon)=0\\
&\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon}+\nabla\vec{\mathbf{u}}^\varepsilon\cdot\vec{\mathbf{u}}^\varepsilon-\nu\Delta\vec{\mathbf{u}}^\varepsilon=-\frac{1}{\rho_0}\nabla p_s^\varepsilon-g\nabla s^\varepsilon+2\phi\vec{\mathbf{F}}_{C_1}^\varepsilon+\frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon-\vec{\mathbf{f}}_R^\varepsilon\right)
\end{aligned}
\tag{6.2.6}
$$

$$
\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon}+\operatorname{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon)=0\\
&\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon}+\nabla\vec{\mathbf{u}}^\varepsilon\cdot\vec{\mathbf{u}}^\varepsilon-\frac{\nu}{h^\varepsilon}\operatorname{div}(|h^\varepsilon|\nabla\vec{\mathbf{u}}^\varepsilon)=-\frac{1}{\rho_0}\nabla p_s^\varepsilon-g\nabla s^\varepsilon+2\phi\vec{\mathbf{F}}_{C_1}^\varepsilon
\end{aligned}
$$

$$+\frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}^\varepsilon_W - \vec{\mathbf{f}}^\varepsilon_R\right) \tag{6.2.7}$$

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(\vec{\mathbf{u}}^\varepsilon h^\varepsilon) = 0$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - \frac{\nu}{h^\varepsilon}\Delta(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = -\frac{1}{\rho_0}\nabla p^\varepsilon_s - g\nabla s^\varepsilon + 2\phi \vec{\mathbf{F}}^\varepsilon_{C_1}$$

$$+\frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}^\varepsilon_W - \vec{\mathbf{f}}^\varepsilon_R\right) \tag{6.2.8}$$

Finalmente consideramos los modelos bidimensionales con viscosidad que hemos propuesto en el capítulo 5. El modelo obtenido sin imponer la llamada hipótesis de la oceanografía dinámica es:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - \nu\left\{\Delta \vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla \vec{\mathbf{u}}^\varepsilon)^T + \nabla \vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\mathrm{div}\,\vec{\mathbf{u}}^\varepsilon)\right]\right\}$$

$$= -\frac{1}{\rho_0}\nabla p^\varepsilon_s - g\nabla s^\varepsilon + 2\phi \vec{\mathbf{F}}^\varepsilon_{C_2} + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}^\varepsilon_W - \vec{\mathbf{f}}^\varepsilon_R\right) \tag{6.2.9}$$

con

$$\vec{\mathbf{F}}^\varepsilon_{C_2} = \begin{pmatrix} (\mathrm{sen}\,\varphi^\varepsilon)v^\varepsilon + \cos\varphi^\varepsilon\left(\dfrac{\partial(h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \dfrac{h^\varepsilon}{2}\dfrac{\partial v^\varepsilon}{\partial y^\varepsilon} - v^\varepsilon \dfrac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\mathrm{sen}\,\varphi^\varepsilon)u^\varepsilon + \dfrac{h^\varepsilon}{2}\dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon] + \dfrac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon] \end{pmatrix}$$

y el modelo que se obtiene al imponer la mencionada hipótesis:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - \nu\left\{\Delta \vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla \vec{\mathbf{u}}^\varepsilon)^T + \nabla \vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\mathrm{div}\,\vec{\mathbf{u}}^\varepsilon)\right]\right\}$$

$$= -\frac{1}{\rho_0}\nabla p^\varepsilon_s - g\nabla s^\varepsilon + 2\phi \vec{\mathbf{F}}^\varepsilon_{C_1} + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}^\varepsilon_W - \vec{\mathbf{f}}^\varepsilon_R\right) \tag{6.2.10}$$

De ahora en adelante, nos referiremos al modelos de dimensión dos sin viscosidad ((6.2.5)) como **M2SV**, a los modelos (6.2.6), (6.2.7) y (6.2.8) como **M2V1**, **M2V2** y **M1V3**, respectivamente, y a los nuevos modelos que proponemos ((6.2.9) y (6.2.10)) como **M2VN1** y **M2VN2**.

6.2.2. Esquema de MacCormack

Nuestra intención es resolver numéricamente los modelos (6.2.1)-(6.2.4) en dimensión uno y los modelos (6.2.5)-(6.2.10) en dimensión dos, para diversos ejemplos. Debemos aplicar el mismo esquema numérico a todos los modelos para que así, las diferencias que se puedan observar se deban a los modelos y no al método numérico. Este método numérico ha de ser, además, sencillo de implementar y robusto (no estamos buscando el esquema numérico más eficiente para cada modelo, sino uno que sea efectivo con todos ellos). Es por ello que, tras consultar la bibliografía existente (véase por ejemplo [3, 26, 35, 37, 38, 50, 77, 78]), nos hemos decidido por utilizar el esquema numérico de MacCormack.

El método de MacCormack es una variación sencilla e interesante del esquema de Lax-Wendroff que está construido a partir de series de Taylor de segundo orden. Este método se ha aplicado con éxito a la resolución de numerosos problemas de dinámica de fluidos (ver por ejemplo [8, 10, 26, 41, 42, 51]). Se trata de un esquema de orden 2 de precisión en espacio y tiempo. Es consistente y estable si se verifica la condición CFL de estabilidad[1].

MacCormack desarrolló un esquema en diferencias finitas de dos pasos no centrado (véase [69]). La derivada espacial se sustituye por diferencias backward o forward. Si el sistema a resolver se puede representar en la forma conservativa

$$\partial_t U + \partial_x F(U) = T \tag{6.2.11}$$

entonces, conocida la solución en el instante $n\Delta t$, la solución en el instante $(n+1)\Delta t$ para el punto i se obtiene aplicando el esquema de MacCormack predictor-corrector de la forma siguiente:

$$U_i^p = U_i^n - \frac{\Delta t}{\Delta x}\left[(1-\alpha)F_{i+1}^n - (1-2\alpha)F_i^n - \alpha F_{i-1}^n\right] + \Delta t\; T_i^n \tag{6.2.12}$$

$$U_i^{n+1} = \frac{1}{2}\left\{U_i^n + U_i^p - \frac{\Delta t}{\Delta x}\left[\alpha F_{i+1}^p + (1-2\alpha)F_i^p + (\alpha-1)F_{i-1}^p\right] + \Delta t\; T_i^p\right\}$$

donde el superíndice n indica el paso de tiempo y el subíndice i el punto de la malla, y F_i^n, T_i^n, F_i^p, T_i^p representan, respectivamente, $F(U_i^n)$, $T(U_i^n)$, $F(U_i^p)$, $T(U_i^p)$ en el

[1] La condición CFL para la ecuación lineal $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$ es la bien conocida condición

$$\left| c\frac{\Delta t}{\Delta x} \right| < 1$$

donde Δt es el paso temporal y Δx el paso espacial. La condición CFL para la ecuación lineal $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = \nu\dfrac{\partial^2 u}{\partial x^2}$ es (véase [107])

$$\left(c\frac{\Delta t}{\Delta x}\right)^2 + 2\nu\frac{\Delta t}{\Delta x^2} \le 1$$

paso de tiempo $n\Delta t$ y en el punto x_i. El valor de α varía de una iteración a otra tomando los valores 0 y 1 alternativamente, consiguiendo así que el esquema sea forward y backward sucesivamente, lo que mejora la estabilidad.

Por ejemplo, el modelo M1SV ((6.2.1)) se puede escribir como en (6.2.11) con

$$U = \begin{pmatrix} h \\ u \end{pmatrix}, \quad F(U) = \begin{pmatrix} uh \\ \frac{1}{2}u^2 + gh \end{pmatrix}, \quad T = \begin{pmatrix} 0 \\ -(\rho_0)^{-1}\partial_x p_s - gH' \end{pmatrix}$$

y entonces (6.2.12) resulta:

$$h_i^p = h_i^n - \frac{\Delta t}{\Delta x}\left[(1-\alpha)(hu)_{i+1}^n - (1-2\alpha)(hu)_i^n - \alpha\ (hu)_{i-1}^n\right]$$

$$u_i^p = u_i^n - \frac{\Delta t}{\Delta x}\left[u_i^n((1-\alpha)u_{i+1}^n - (1-2\alpha)u_i^n - \alpha\ u_{i-1}^n)\right.$$
$$\left. + g((1-\alpha)h_{i+1}^n - (1-2\alpha)h_i^n - \alpha\ h_{i-1}^n)\right] - \frac{\Delta t}{\rho_0}(\partial_x p_s(x_i, n\Delta t) + gH'(x_i))$$

$$h_i^{n+1} = \frac{1}{2}\left\{h_i^n + h_i^p - \frac{\Delta t}{\Delta x}\left[\alpha\ (hu)_{i+1}^p + (1-2\alpha)(hu)_i^p + (\alpha-1)(hu)_{i-1}^p\right]\right\}$$

$$u_i^{n+1} = \frac{1}{2}\left\{u_i^n + u_i^p - \frac{\Delta t}{\Delta x}\left[u_i^p(\alpha\ u_{i+1}^p + (1-2\alpha)u_i^p + (\alpha-1)u_{i-1}^p)\right.\right.$$
$$\left.\left. + g(\alpha\ h_{i+1}^p + (1-2\alpha)h_i^p + (\alpha-1)h_{i-1}^p)\right] - \frac{\Delta t}{\rho_0}(\partial_x p_s(x_i, n\Delta t) + gH'(x_i))\right\}$$

Observación 6.1 *Se ha aplicado el esquema de MacCormack al término*

$$\frac{\partial F(U)}{\partial x} = \begin{pmatrix} \dfrac{\partial (hu)}{\partial x} \\ u\dfrac{\partial u}{\partial x} + g\dfrac{\partial h}{\partial x} \end{pmatrix}$$

en lugar de a

$$\frac{\partial F(U)}{\partial x} = \begin{pmatrix} \dfrac{\partial (hu)}{\partial x} \\ \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}u^2 + gh\right) \end{pmatrix}$$

De forma similar se aplica el esquema de MacCormack a los otros modelos cuyo comportamiento vamos a comparar.

6.2.3. Comparación con soluciones analíticas

6.2.3.1. Dimensión uno

En primer lugar comparamos los cuatro modelos (M1SV, M1V1, M1V2, M1VN) tratando de aproximar soluciones analíticas conocidas de las ecuaciones de Navier-Stokes bidimensionales ((3.1.1)-(3.1.3)), como por ejemplo:

$$\begin{aligned}
&u = A,\\
&w = A\left(\frac{A}{\nu}ae^{Ax/\nu} + 2bx + c\right),\\
&h = B,\\
&H = ae^{Ax/\nu} + bx^2 + cx + d,\\
&p = p_s + \rho_0(g + 2A^2b)(h + H - z),\\
&p_s = K(t) - \rho_0(g + 2A^2b)\left[ae^{Ax/\nu} + bx^2 + cx\right],\\
&f_W = f_R = \mu A\left(\frac{A^2}{\nu^2}ae^{Ax/\nu} + 2b\right)\frac{1 - (H')^2}{1 + (H')^2}
\end{aligned} \tag{6.2.13}$$

con a, b, c, d, A, B constantes, $K(t)$ una función cualquiera de t, ν coeficiente de viscosidad cinemática, ρ_0 densidad, g aceleración de la gravedad y μ coeficiente de viscosidad dinámica.

El valor de las constantes a y b determina si la solución anterior lo es también de los modelos de aguas someras y si la viscosidad se tiene en cuenta. Si $b = 0$, (6.2.13) es solución también de los modelos M1SV, M1V1, M1V2 y M1VN ((6.2.1)-(6.2.4)). En este caso, los cuatro modelos aproximan de forma exacta la solución (esto sucede con todas las soluciones constantes para h y u).

Si $b \neq 0$ (6.2.13) es solución de las ecuaciones bidimensionales de Navier-Stokes pero no de los modelos de aguas someras. En este caso, si $a = 0$, los términos en los que aparece la viscosidad (ν) desaparecen y los cuatro modelos cometen los mismos errores al aproximar esta solución. Por ejemplo, si se toman

$$a = 0, \;\; b = -0.02, \;\; c = 0.2, \;\; d = 0, \;\; A = B = 1, \;\; K = 101325, \;\; g = 9.8$$

(en este ejemplo y los que siguen, la aceleración de la gravedad tomará el valor $g = 9.8$) y los coeficientes de viscosidad y la densidad correspondientes al agua a 20^o C ($\nu = 1.02 \cdot 10^{-6}$ y $\rho = 998.2$, en los ejemplos que siguen ρ_0 permanecerá constante pero permitiremos a ν tomar otros valores), resolviendo en el intervalo espacial $[0, 10]$ con $\Delta x = 0.1$ y en el mismo intervalo para el tiempo pero con $\Delta t = 0.01$, los errores, en norma 2, que se cometen al aproximar la altura del agua (h) están acotados por $5.3 \cdot 10^{-3}$ y los cometidos al aproximar la velocidad horizontal (u) por $7.8 \cdot 10^{-3}$.

En cambio, si $a \neq 0$, como la viscosidad entra en juego, se aprecian diferencias, aunque pequeñas, entre los cuatro modelos. Considerando los mismos intervalos espacial y temporal y los mismos pasos de tiempo y espacio, eligiendo las constantes y coeficientes como antes salvo $a = 1. \cdot 10^{-5}$, $A = 0.1$ y $\nu = 1.02 \cdot 10^{-1}$ obtenemos las acotaciones para los errores en norma 2 que se indican en el cuadro 6.1.

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.8 \cdot 10^{-5}$	$7.1 \cdot 10^{-5}$
M1V1	$4.0 \cdot 10^{-5}$	$6.0 \cdot 10^{-5}$
M1V2	$4.0 \cdot 10^{-5}$	$6.0 \cdot 10^{-5}$
M1VN	$3.6 \cdot 10^{-5}$	$5.7 \cdot 10^{-5}$

Cuadro 6.1: Acotación errores ejemplo (6.2.13)

Se aprecia que el modelo que proponemos (M1VN) aproxima de forma más precisa la solución de las ecuaciones de Navier-Stokes.

Veamos, ahora, otra solución analítica (la velocidad horizontal depende en este caso de la variable espacial x y la altura de x y del tiempo t) para la que se aprecia más la diferente precisión de los modelos de aguas someras.

$$
\begin{aligned}
&u = \frac{-6\nu b}{a+bx}, \\
&w = -\frac{6\nu b^2}{(a+bx)^2} z, \\
&h = A e^{B\left[12\nu b^2 t + (a+bx)^2\right]}(a+bx), \\
&H = 0, \\
&p = p_s + \rho_0 (h-z) g, \\
&p_s = K + \rho_0 \left(\frac{-12\nu^2 b^2}{(a+bx)^2} - gh \right), \\
&f_W = \mu \frac{12\nu b^3 h \left[1 - \left(\frac{\partial h}{\partial x} \right)^2 \right] - 24\nu b^2 \frac{\partial h}{\partial x}(a+bx)}{\left[1 + \left(\frac{\partial h}{\partial x} \right)^2 \right] (a+bx)^3}, \quad f_R = 0
\end{aligned}
\tag{6.2.14}
$$

donde de nuevo a, b, A, B son constantes, ν es el coeficiente de viscosidad cinemática, ρ_0 es la densidad, g es la aceleración de la gravedad y μ es el coeficiente de viscosidad dinámica.

Ninguna de estas soluciones de las ecuaciones de Navier-Stokes lo es de las de aguas someras.

Tomamos

$$a = 11, \;\; b = -1, \;\; A = 0.1, \;\; B = 2. \cdot 10^{-3}, \;\; K = 101325, \;\; \nu = 1.02 \cdot 10^{-1}$$

y resolvemos en el intervalo espacial $[0, 10]$ con $\Delta x = 0.1$ y en el mismo intervalo para el tiempo pero con $\Delta t = 0.01$, los errores en norma 2 están acotados por las cantidades que se indican en el cuadro 6.2.

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.0 \cdot 10^{-3}$	$2.8 \cdot 10^{-2}$
M1V1	$4.1 \cdot 10^{-3}$	$2.6 \cdot 10^{-2}$
M1V2	$2.2 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$
M1VN	$3.3 \cdot 10^{-4}$	$1.1 \cdot 10^{-3}$

Cuadro 6.2: Acotación errores ejemplo (6.2.14).a

En el cuadro anterior se aprecia que el modelo obtenido mediante el método de los desarrollos asintóticos aproxima de forma más precisa que los demás la solución exacta (6.2.14) de las ecuaciones de Navier-Stokes. Además, si reducimos los pasos espacial y temporal ($\Delta x = 5. \cdot 10^{-2}$, $\Delta t = 5. \cdot 10^{-3}$) los errores únicamente se reducen en el caso del modelo M1VN (véase cuadro 6.3). Y para $\Delta x = 2. \cdot 10^{-2}$ y $\Delta t = 5. \cdot 10^{-4}$ la situación se repite como se ve en el cuadro 6.4.

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.0 \cdot 10^{-3}$	$2.9 \cdot 10^{-2}$
M1V1	$4.2 \cdot 10^{-3}$	$2.5 \cdot 10^{-2}$
M1V2	$2.2 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$
M1VN	$8.6 \cdot 10^{-5}$	$2.7 \cdot 10^{-4}$

Cuadro 6.3: Acotación errores ejemplo (6.2.14).b

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.0 \cdot 10^{-3}$	$2.9 \cdot 10^{-2}$
M1V1	$5.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-2}$
M1V2	$2.9 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$
M1VN	$1.9 \cdot 10^{-5}$	$4.0 \cdot 10^{-5}$

Cuadro 6.4: Acotación errores ejemplo (6.2.14).c

Este comportamiento se repite para otras soluciones, como por ejemplo:

$$
\begin{aligned}
&u = \frac{-6\nu b}{a+bx},\\
&w = -\frac{6\nu b^2}{(a+bx)^2}z + \frac{12\nu b^2 B}{(a+bx)^3},\\
&h = A(a+bx),\\
&H = \frac{B}{a+bx} + C(a+bx),\\
&p = p_s + \rho_0(s-z)g, \quad p_s = K + \rho_0\left(\frac{-12\nu^2 b^2}{(a+bx)^2} - gs\right),\\
&f_W = 12\nu\mu b^2 \frac{b\left[1-\left(\frac{\partial s}{\partial x}\right)^2\right]\left[s\,(a+bx) - 3B\right] - 2\frac{\partial s}{\partial x}(a+bx)^2}{\left[1+\left(\frac{\partial s}{\partial x}\right)^2\right](a+bx)^4}\\
&f_R = 12\nu\mu b^2 \frac{b\left[1-(H')^2\right]\left[H\,(a+bx)-3B\right] - 2H'(a+bx)^2}{\left[1+(H')^2\right](a+bx)^4}
\end{aligned}
\tag{6.2.15}
$$

Tomando los valores siguientes para las constantes,

$$A = B = 0.1,\ \ C = 0.01,\ \ K = 101325,\ \ a = 11,\ \ b = -1,\ \ \nu = 1.02 \cdot 10^{-1}$$

y si se toma como intervalo espacial $[0, 10]$ con $\Delta x = 0.1$ y el mismo intervalo para el tiempo pero con $\Delta t = 0.01$, los errores en norma 2 están acotados por las cantidades indicadas en el cuadro 6.5.

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.1 \cdot 10^{-3}$	$2.8 \cdot 10^{-2}$
M1V1	$4.2 \cdot 10^{-3}$	$2.5 \cdot 10^{-2}$
M1V2	$2.2 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$
M1VN	$3.6 \cdot 10^{-4}$	$9.9 \cdot 10^{-4}$

Cuadro 6.5: Acotación errores ejemplo (6.2.15).a

Cuando refinamos la malla tomando $\Delta x = 2 \cdot 10^{-2}$ y $\Delta t = 5 \cdot 10^{-4}$ los errores cometidos por nuestro modelo se reducen mientras que el resto de los modelos mantiene el orden de precisión (véase el cuadro 6.6).

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$4.2 \cdot 10^{-3}$	$3.0 \cdot 10^{-2}$
M1V1	$5.8 \cdot 10^{-3}$	$2.4 \cdot 10^{-2}$
M1V2	$3.0 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$
M1VN	$1.6 \cdot 10^{-5}$	$3.9 \cdot 10^{-5}$

Cuadro 6.6: Acotación errores ejemplo (6.2.15).b

Por último vamos a considerar la siguiente solución de las ecuaciones de Navier-Stokes en la que la velocidad horizontal varía con el tiempo y la variable espacial x, y la profundidad depende únicamente de x:

$$
\begin{aligned}
&u = Ce^{-A^2\nu t}\,\mathrm{sen}(A(x+B)),\\
&w = -ACe^{-A^2\nu t}\cos(A(x+B))z,\\
&h = \frac{D}{\mathrm{sen}(A(x+B))},\\
&H = 0,\\
&p = p_s + \rho_0 g(h-z) + \frac{1}{2}\rho_0 A^2C^2e^{-2A^2\nu t}(h^2-z^2),\\
&p_s = K - \rho\left\{\frac{C^2}{2}e^{-2A^2\nu t}\left[\mathrm{sen}^2(A(x+B)) + A^2h^2\right] + gh\right\},\\
&f_W = \mu A^2DCe^{-A^2\nu t}\frac{\mathrm{sen}^2(A(x+B))\left[1+3\cos^2(A(x+B))\right] - A^2D^2\cos^2(A(x+B))}{\mathrm{sen}^4(A(x+B)) + A^2d^2\cos^2(A(x+B))},\\
&f_R = 0
\end{aligned}
\tag{6.2.16}
$$

donde hemos utilizado la misma notación que en las soluciones anteriores. Los resultados que se muestran en el cuadro 6.7 corresponden a los valores siguientes de las constantes:

$$A = 0.2,\ \ B = 2,\ \ C = 1,\ \ D = 0.5,\ \ K = 101325,\ \ \nu = 1.02 \cdot 10^{-1}$$

Se ha resuelto en el intervalo espacial $[0, 10]$ con $\Delta x = 0.1$ y el mismo intervalo temporal con $\Delta t = 0.01$. En este caso se aprecia como el modelo M1V1 mejora la aproximación del modelo sin viscosidad, el modelo M1V2 es ligeramente más preciso que M1V1, pero el modelo que proporciona la aproximación más exacta es, de nuevo, el propuesto por nosotros.

Observación 6.2 *Los cuatro modelos han sido programados en términos del flujo y de la velocidad horizontal, obteniéndose resultados muy similares salvo para algunos*

Modelo	Acotación error al calcular h	Acotación error al calcular u
M1SV	$1.9 \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$
M1V1	$1.1 \cdot 10^{-2}$	$3.9 \cdot 10^{-2}$
M1V2	$8.4 \cdot 10^{-3}$	$3.2 \cdot 10^{-2}$
M1VN	$4.5 \cdot 10^{-3}$	$7.5 \cdot 10^{-3}$

Cuadro 6.7: Acotación errores ejemplo (6.2.16)

casos puntuales en los que la formulación en flujo presenta algunos problemas debido a la forma particular de la solución.

Presión de segundo orden

En las secciones 3.7 y 5.7, al proponer los nuevos modelos con viscosidad, se señaló la posibilidad de utilizar en lugar de las expresiones de orden uno para la presión ((3.5.9), (5.7.4)) las mejoras obtenidas en las aproximaciones de orden dos ((3.7.5) para dimensión uno y (5.7.7) para dimensión dos), dado que no es necesario conocer la velocidad de segundo orden para ello. La presión de orden dos en el caso de dimensión uno ((3.7.5)) es:

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g + (u^\varepsilon)^2 \frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} + \frac{1}{\rho_0}\frac{\partial f_R^\varepsilon}{\partial x^\varepsilon} - 2\nu \frac{\partial}{\partial x^\varepsilon}\left(\frac{\partial^2 H^\varepsilon}{\partial(x^\varepsilon)^2} u^\varepsilon\right)\right.$$
$$\left. - \frac{\partial H^\varepsilon}{\partial x^\varepsilon}\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}\right] + \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\left(\frac{\partial u^\varepsilon}{\partial x^\varepsilon}\right)^2 + \frac{1}{\rho_0}\frac{\partial^2 p_s^\varepsilon}{\partial(x^\varepsilon)^2}\right] \quad (6.2.17)$$

En esta sección deseamos mostrar que esta corrección de la presión proporciona una mejor aproximación de la misma cuando ésta no se parece a la presión hidrostática. Para ello volvemos sobre la solución de las ecuaciones de Navier-Stokes (6.2.16) y escogemos los mismos intervalos y pasos (espacial $[0, 10]$ con $\Delta x = 0.1$, temporal $[0, 10]$ con $\Delta t = 0.01$) y los mismos valores para las constantes que en la sección anterior:

$$A = 0.2, \ B = 2, \ C = 1, \ D = 0.5, \ K = 101325, \ \nu = 1.02 \cdot 10^{-1}$$

Calculamos la presión hidrostática (p_h) y la presión de segundo orden dada por (6.2.17) (p_2) a partir de los calados y velocidades horizontales obtenidos al resolver el modelo M1VN. Las acotaciones para los errores en norma dos que obtenemos se muestran en el cuadro 6.8, donde se aprecia que la presión p_2 aproxima mejor la presión exacta que la presión hidrostática.

Profundidad (z)	Acotación error p_h	Acotación error p_2
0	$2.0 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$
0.25	$2.1 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$
0.5	$2.2 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$

Cuadro 6.8: Acotación errores de la presión ejemplo (6.2.16)

Otra solución de las ecuaciones bidimensionales de Navier-Stokes para la que sucede algo similar es la siguiente:

$$\begin{aligned} &u = a + bz, \quad w = B(a + b(A + Bx)),\\ &h = -2(a/b + A + Bx), \quad H = A + Bx, \quad s = h + H\\ &p = p_s + \rho_0(g + abB^2)(s - z) + \frac{1}{2}\rho_0 b^2 B^2 (s^2 - z^2), \qquad (6.2.18)\\ &p_s = K - \rho(g + abB^2)s - \frac{1}{2}\rho_0 b^2 B^2 s^2 - \rho Bb(a + cA_1)x - \frac{1}{2}\rho_0 B^2 b^2 x^2,\\ &f_W = f_R = \mu b(1 - B^2) \end{aligned}$$

Por ejemplo, si se toman:

$$a = -3, \quad b = 2, \quad A = 1, \quad B = -0.1, \quad \rho_0 = 998.2, \quad K = 101325, \quad \nu = 1.02 \cdot 10^{-1}$$

y calculamos la presión hidrostática y la presión de segundo orden dada por (6.2.17), de nuevo los errores en norma dos que obtenemos son menores para p_2 que para p_h, como se muestra en el cuadro 6.9.

Profundidad (z)	Acotación error p_h	Acotación error p_2
1	$7.4 \cdot 10^{-2}$	$6.9 \cdot 10^{-2}$
1.5	$7.8 \cdot 10^{-2}$	$7.3 \cdot 10^{-2}$
2	$8.3 \cdot 10^{-2}$	$7.8 \cdot 10^{-2}$

Cuadro 6.9: Acotación errores de la presión ejemplo (6.2.18)

Los resultados que observamos en los cuadros 6.8-6.9 nos indican que aunque la presión de segundo orden aproxima de forma más precisa la presión exacta que la presión hidrostática, la mejora no es considerable.

6.2.3.2. Dimensión dos

Vamos ahora a comparar el comportamiento de los modelos M2SV, M2V1, M2V2, M2V3, M2VN1 y M2VN2 ((6.2.5)-(6.2.10)) al aproximar soluciones analíticas conocidas de las ecuaciones tridimensionales de Navier-Stokes (véase la sección 5.1.1).

1. Comenzamos por considerar una solución que depende de x y z:

$$u = A + Bx, \quad v = C, \quad w = D(A + 2Bx) - Bz,$$

$$H = Dx, \quad h = \frac{E}{A + Bx}, \quad \phi = 0,$$

$$p = p_s + \rho \left[(s - z)(g + ABD) + \frac{B^2}{2}(s^2 - z^2) \right], \tag{6.2.19}$$

$$p_s = K - \rho \left[s(g + ABD) + \frac{B^2}{2} s^2 + ABx + \frac{B^2}{2} x^2 \right],$$

$$f_{R_1} = -2\mu BD, \quad f_{R_2} = 0,$$

$$f_{W_1} = 2\mu B \frac{\dfrac{2BE}{(A+Bx)^2} - D - \left(D - \dfrac{BE}{(A+Bx)^2} \right)^2}{\sqrt{1 + \left(D - \dfrac{BE}{(A+Bx)^2} \right)^2}}, \quad f_{W_2} = 0$$

donde se utiliza la notación ya mencionada. Se asignan los siguientes valores a las constantes:

$$A = 2, \;\; B = -0.1, \;\; C = 0.5, \;\; D = 0.05, \;\; E = 1, \;\; K = 101325, \;\; \nu = 1.02 \cdot 10^{-1}$$

Resolvemos en el dominio $[0, 10] \times [0, 2]$ con pasos $\Delta x = \Delta y = 0.1$. El intervalo temporal que se toma es $[0, 10]$ con $\Delta t = 0.01$. Se imponen condiciones de contorno Dirichlet en $x = 0$ y $x = 10$ y obtenemos las acotaciones para los errores en norma 2 que se indican en el cuadro 6.10.

Observamos que como sucedía en dimensión uno la mejor aproximación se obtiene con nuestro modelo.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.6 \cdot 10^{-2}$	$7.0 \cdot 10^{-2}$	0.0
M2V1	$1.4 \cdot 10^{-2}$	$5.3 \cdot 10^{-2}$	0.0
M2V2	$1.1 \cdot 10^{-2}$	$3.9 \cdot 10^{-2}$	0.0
M2V3	$1.6 \cdot 10^{-2}$	$6.4 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$
M2VN1[1]	$6.6 \cdot 10^{-4}$	$2.3 \cdot 10^{-3}$	0.0

Cuadro 6.10: Acotación errores ejemplo (6.2.19)

2. En segundo lugar consideraremos una solución que depende de x, y y t:

$$
\begin{aligned}
&u = A + Bx - \frac{B^2}{C}y, \quad v = D + Cx - By, \quad w = 0, \\
&H = 0, \quad h = E + F\left[-\frac{C}{B}x + y + \left(\frac{C}{B}A - D\right)t\right], \quad \phi = 0, \\
&p = ps + \rho(s - z)g, \qquad (6.2.20) \\
&p_s = K + \rho\left[-gh - \left(AB - \frac{B^2}{C}D\right)x - (AC - DB)y\right], \\
&f_{R_1} = f_{R_2} = 0, \\
&f_{W_1} = \mu F \frac{C^2 + B^2}{C\sqrt{1 + F^2\left(1 + \frac{C^2}{B^2}\right)}}, \quad f_{W_2} = \mu F \frac{C^2 + B^2}{B\sqrt{1 + F^2\left(1 + \frac{C^2}{B^2}\right)}}
\end{aligned}
$$

En este caso, si elegimos para las distintas constantes los siguientes valores

$$A = 1,\ B = 0.1,\ C = 0.05,\ D = 0.2,\ E = 1,\ F = 0.1,\ \nu = 1.02 \cdot 10^{-6}$$

al resolver los modelos de aguas someras en el dominio $[0, 10] \times [0, 2]$ con pasos $\Delta x = \Delta y = 0.1$, en el intervalo temporal $[0, 10]$ con $\Delta t = 0.01$ e imponiendo condiciones de contorno Dirichlet en $x = 0$, $x = 10$, $y = 0$ e $y = 2$ obtenemos las acotaciones para los errores en norma 2 que se encuentran en el cuadro 6.11.

Podemos apreciar como el modelo sin viscosidad (M2SV), el modelo con viscosidad "tipo Navier-Stokes" (M2V1) y el modelo M2V3 proporcionan aproximaciones de la misma precisión mientras que el modelo que proponemos en el capítulo 5 (M2VN1, M2VN2) y el modelo M2V2 mejoran esa aproximación cometiendo errores 10^2 veces menores.

[1] Como se considera $\phi = 0$ los modelos M2VN1 y M2VN2 coinciden

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$2.4 \cdot 10^{-8}$	$2.0 \cdot 10^{-7}$	$4.9 \cdot 10^{-8}$
M2V1	$2.4 \cdot 10^{-8}$	$2.0 \cdot 10^{-7}$	$4.9 \cdot 10^{-8}$
M2V2	$1.5 \cdot 10^{-10}$	$1.2 \cdot 10^{-9}$	$3.1 \cdot 10^{-10}$
M2V3	$2.4 \cdot 10^{-8}$	$2.0 \cdot 10^{-7}$	$5.0 \cdot 10^{-8}$
M2VN1[1]	$1.5 \cdot 10^{-10}$	$1.2 \cdot 10^{-9}$	$3.1 \cdot 10^{-10}$

Cuadro 6.11: Acotación errores ejemplo (6.2.20).a

Si se aumenta la viscosidad, tomando $\nu = 1.02 \cdot 10^{-1}$, vemos que la situación se repite. Apreciamos que el modelo M2VN1 encuentra la solución más aproximada a la exacta (véase el cuadro 6.12).

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$2.4 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$	$4.9 \cdot 10^{-3}$
M2V1	$2.0 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$	$2.5 \cdot 10^{-3}$
M2V2	$1.3 \cdot 10^{-5}$	$7.7 \cdot 10^{-5}$	$1.6 \cdot 10^{-5}$
M2V3	$2.1 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$	$2.6 \cdot 10^{-3}$
M2VN1[1]	$1.4 \cdot 10^{-5}$	$7.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$

Cuadro 6.12: Acotación errores ejemplo (6.2.20).b

3. Otra solución de las ecuaciones tridimensionales de Navier-Stokes es la siguiente:

$$
\begin{aligned}
&u = \frac{-6\nu A}{Ax+B}, \quad v = C, \quad w = \frac{-6\nu A^2}{(Ax+B)^2} z, \\
&H = 0, \quad \phi = 0, \qquad (6.2.21) \\
&p = p_s + \rho(h-z)g, \quad p_s = -\rho\left[gh + \frac{12A^2\nu^2}{(Ax+B)^2}\right] + f(t) \\
&f_{R_1} = f_{R_2} = 0
\end{aligned}
$$

Hay varias posibilidades para h:

a) El calado es lineal en x:

$$
\begin{aligned}
&h = -\frac{Ax+B}{6\nu}, \\
&f_{W_1} = \mu \frac{2A^3(12(Ax+B) + (36\nu^2 - A^2)h)}{(Ax+B)^3\sqrt{36\nu^2 + A^2}}, \quad f_{W_2} = 0 \qquad (6.2.22)
\end{aligned}
$$

Resolvemos una vez más en $[0,10]\times[0,2]$ con pasos $\Delta x=\Delta y=0.1$, en el intervalo temporal $[0,10]$ con $\Delta t=0.01$ dando los siguientes valores:

$$A=-1,\ \ B=10.5,\ \ C=0.5,\ \ D=0.2,\ \ \nu=1.02\cdot 10^{-1}$$

e imponiendo condiciones de contorno Dirichlet en $x=0$, $x=10$, $y=0$ e $y=2$. Obtenemos las acotaciones para los errores que muestra el cuadro 6.13. En este caso los mejores resultados se logran, una vez más, con el modelo que proponemos. Lo mismo sucede si se imponen condiciones de contorno Dirichlet únicamente en $x=0$ y $x=10$, como se ve en el cuadro 6.14.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$8.6\cdot 10^{-4}$	$9.0\cdot 10^{-3}$	$3.5\cdot 10^{-3}$
M2V1	$7.1\cdot 10^{-3}$	$8.4\cdot 10^{-2}$	$1.2\cdot 10^{-2}$
M2V2	$3.6\cdot 10^{-3}$	$4.3\cdot 10^{-2}$	$5.5\cdot 10^{-3}$
M2V3	$1.2\cdot 10^{-3}$	$1.2\cdot 10^{-2}$	$4.2\cdot 10^{-3}$
M2VN1[1]	$2.4\cdot 10^{-4}$	$1.7\cdot 10^{-3}$	$3.2\cdot 10^{-4}$

Cuadro 6.13: Acotación errores ejemplo ((6.2.21)-(6.2.22)).a

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.9\cdot 10^{-3}$	$8.6\cdot 10^{-3}$	0.0
M2V1	$1.8\cdot 10^{-2}$	$1.2\cdot 10^{-1}$	0.0
M2V2	$8.7\cdot 10^{-3}$	$5.9\cdot 10^{-2}$	0.0
M2V3	$4.8\cdot 10^{-3}$	$3.3\cdot 10^{-2}$	$9.2\cdot 10^{-3}$
M2VN1[1]	$3.9\cdot 10^{-4}$	$2.3\cdot 10^{-3}$	0.0

Cuadro 6.14: Acotación errores ejemplo ((6.2.21)-(6.2.22)).b

b) La profundidad del agua depende de x y de y:

$$h=D(Ax+B)e^{\left(\dfrac{(Ax+B)^2}{4A^2}+\dfrac{3\nu y}{C}\right)},$$

$$f_{W_1}=\mu\frac{\dfrac{\partial h}{\partial x}\left(-4u'+\dfrac{\partial h}{\partial x}u''h\right)-u''h}{\sqrt{1+(\dfrac{\partial h}{\partial x})^2+\left(\dfrac{\partial h}{\partial y}\right)^2}},\tag{6.2.23}$$

$$f_{W_2} = \mu \frac{\dfrac{\partial h}{\partial y}\left(-2u' + \dfrac{\partial h}{\partial x} u'' h\right)\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2}}{\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2}}$$

Elegimos los siguientes valores de las constantes:

$$A = -0.5, \ \ B = 2.55, \ \ C = 0.5, \ \ D = 10^{-3}, \ \ \nu = 1.02 \cdot 10^{-1}$$

para resolver en $[0,5] \times [0,1]$ con $\Delta x = \Delta y = 0.1$ en el intervalo temporal $[0,10]$ con $\Delta t = 0.01$, imponiendo condiciones de contorno Dirichlet en $x = 0$, $x = 5$, $y = 0$ e $y = 1$. Los errores que cometen los modelos M2SV y M2V3 se disparan. Los otros tres modelos cometen errores del orden reflejado en el cuadro 6.15.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2V1	$1.4 \cdot 10^{-2}$	$1.8 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$
M2V2	$1.3 \cdot 10^{-2}$	$1.7 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$
M2VN1[1]	$7.7 \cdot 10^{-3}$	$1.2 \cdot 10^{-1}$	$2.8 \cdot 10^{-2}$

Cuadro 6.15: Acotación errores ejemplo ((6.2.21)-(6.2.23)).a

Si se refina la malla reduciendo los pasos espaciales a la mitad ($\Delta x = \Delta y = 0.05$) y el paso temporal a la cuarta parte ($\Delta t = 2.5 \cdot 10^{-3}$) no logramos controlar los errores que se cometen al resolver M2SV y M2V3, los errores obtenidos con el modelo M2V1 no se reducen, la aproximación obtenida con M2V2 mejora ligeramente y finalmente los errores cometidos al resolver el modelo M2VN1 se reducen más de la mitad como podemos ver en el cuadro 6.16.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2V1	$5.3 \cdot 10^{-3}$	$2.5 \cdot 10^{-1}$	$9.8 \cdot 10^{-2}$
M2V2	$4.3 \cdot 10^{-3}$	$1.2 \cdot 10^{-1}$	$9.3 \cdot 10^{-2}$
M2VN1[1]	$1.9 \cdot 10^{-3}$	$3.0 \cdot 10^{-2}$	$6.5 \cdot 10^{-3}$

Cuadro 6.16: Acotación errores ejemplo ((6.2.21)-(6.2.23)).b

c) También es posible que el calado dependa de x y t:

$$h = D(Ax+B)e^{\dfrac{(Ax+B)^2}{4a^2}+3\nu t},$$

$$f_{W_1} = \mu \frac{\dfrac{\partial h}{\partial x}\left(-4u' + \dfrac{\partial h}{\partial x}u''h\right) - u''h}{\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2}}, \quad f_{W_2} = 0 \qquad (6.2.24)$$

Los resultados obtenidos en este caso son muy similares a los que se obtienen en la solución anterior (véanse cuadros 6.15 y 6.16).

4. Consideramos la siguiente solución en la que una de las componentes de la velocidad horizontal depende de la variable y mientras que la otra es constante y el calado depende de x e y:

$$u = A\frac{\nu^2}{B^2}e^{\frac{B}{\nu}y}, \quad v = B, \quad w = 0,$$

$$h = Ce^{x - \dfrac{A\nu^3 e^{\frac{B}{\nu}y}}{B^4}}, \quad H = 0, \quad \phi = 0,$$

$$p = p_s + \rho(h-z)g, \quad p_s = -\rho g h + f(t) \qquad (6.2.25)$$

$$f_{R_1} = f_{R_2} = 0,$$

$$f_{W_1} = \mu \frac{-\dfrac{\partial h}{\partial y}u'}{\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2}}, \quad f_{W_2} = \mu \frac{-\dfrac{\partial h}{\partial x}u'}{\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2}}$$

Estudiamos el caso en el que

$$A = 10^{-1}, \;\; B = 2.5\cdot 10^{-1}, \;\; C = 0.5\, e^{0.1}, \;\; \nu = 1.02\cdot 10^{-1}$$

en el dominio $[0,10]\times[0,2]$ con pasos $\Delta x = \Delta y = 0.1$, en el intervalo temporal $[0,10]$ con $\Delta t = 0.01$. Imponemos condiciones de contorno Dirichlet en $x = 0$, $x = 10$, $y = 0$ e $y = 2$ y obtenemos las acotaciones para los errores en norma 2 que pueden verse en el cuadro 6.17.

La mejor aproximación de esta solución la proporciona el modelo M2V2, si bien los errores que se cometen al utilizar nuestro modelo son del mismo orden de precisión. Los modelos M2V1 y M2V3 proporcionan aproximaciones 10

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.1 \cdot 10^{-1}$	3.0	$4.0 \cdot 10^{-1}$
M2V1	$1.4 \cdot 10^{-2}$	$1.3 \cdot 10^{-1}$	$2.7 \cdot 10^{-2}$
M2V2	$2.7 \cdot 10^{-3}$	$1.6 \cdot 10^{-2}$	$5.5 \cdot 10^{-3}$
M2V3	$2.6 \cdot 10^{-2}$	$2.6 \cdot 10^{-1}$	$5.9 \cdot 10^{-2}$
M2VN1[1]	$4.0 \cdot 10^{-3}$	$1.7 \cdot 10^{-2}$	$5.7 \cdot 10^{-3}$

Cuadro 6.17: Acotación errores ejemplo (6.2.25).a

veces peores y el modelo sin viscosidad comete errores enormes. Si se reducen los pasos espaciales y temporal de modo que $\Delta x = \Delta y = 0.05$ y $\Delta t = 0.0025$ apreciamos que el único modelo que reduce los errores es el M2VN1 (ver cuadro 6.18).

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.1 \cdot 10^{-1}$	3.0	$4.2 \cdot 10^{-1}$
M2V1	$1.7 \cdot 10^{-2}$	$1.3 \cdot 10^{-1}$	$2.6 \cdot 10^{-2}$
M2V2	$3.0 \cdot 10^{-3}$	$9.0 \cdot 10^{-3}$	$6.9 \cdot 10^{-3}$
M2V3	$3.0 \cdot 10^{-2}$	$2.4 \cdot 10^{-1}$	$5.4 \cdot 10^{-2}$
M2VN1[1]	$1.3 \cdot 10^{-3}$	$9.7 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$

Cuadro 6.18: Acotación errores ejemplo (6.2.25).b

5. Veamos un ejemplo más en el que al refinar la malla nuestro modelo es el único que reduce los errores que se cometen:

$$
\begin{aligned}
& u = Axy + Bx + Cy + \frac{BC}{A}, \quad v = -\frac{(Ay+B)^2}{A}, \quad w = (Ay+B)z, \\
& h = \frac{D}{Ay+B}, \quad H = 0, \quad \phi = 0, \\
& p = p_s + \rho(h-z)g, \quad p_s = -\rho\left[gh + \frac{(Ay+B)^4}{2A^2} + 2A\nu y\right] + f(t), \\
& f_{R_1} = f_{R_2} = 0, \qquad (6.2.26) \\
& f_{W_1} = \mu \frac{-\dfrac{\partial h}{\partial y}\left(Ax + C + \dfrac{\partial h}{\partial x} Ah\right)}{\sqrt{1 + (\dfrac{\partial h}{\partial x})^2 + \left(\dfrac{\partial h}{\partial y}\right)^2}},
\end{aligned}
$$

$$f_{W_2} = \mu \frac{-\frac{\partial h}{\partial x}(Ax+C) + 6\frac{\partial h}{\partial y}(Ay+B) + Ah\left(1 - \left(\frac{\partial h}{\partial y}\right)^2\right)}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}$$

Trabajamos con los siguientes valores

$$A = 5 \cdot 10^{-1},\ \ B = 10^{-1},\ \ C = 2 \cdot 10^{-1},\ \ D = 10^{-1},\ \ \nu = 1.02 \cdot 10^{-1}$$

con $(x, y) \in [0, 10] \times [0, 2]$, $t \in [0, 10]$, $\Delta x = \Delta y = 0.1$ y $\Delta t = 0.01$. Se imponen condiciones de contorno tipo Dirichlet. Los resultados obtenidos (ver cuadro 6.19) muestran de nuevo que el modelo M2VN1 es el más preciso.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.4 \cdot 10^{-2}$	$1.3 \cdot 10^{-1}$	$4.3 \cdot 10^{-2}$
M2V1	$1.5 \cdot 10^{-2}$	$6.9 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$
M2V2	$5.5 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	$1.8 \cdot 10^{-2}$
M2V3	$1.6 \cdot 10^{-2}$	$9.1 \cdot 10^{-2}$	$3.5 \cdot 10^{-2}$
M2VN1[1]	$5.5 \cdot 10^{-3}$	$4.2 \cdot 10^{-3}$	$6.4 \cdot 10^{-3}$

Cuadro 6.19: Acotación errores ejemplo (6.2.26).a

Al refinar la malla ($\Delta x = \Delta y = 0.05$ y $\Delta t = 0.0025$), como comentamos más arriba, el único modelo que mejora la aproximación es el M2VN1 (véase cuadro 6.20) y los errores que se cometen con el modelo M2V3 se disparan.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2SV	$1.5 \cdot 10^{-2}$	$1.2 \cdot 10^{-1}$	$4.3 \cdot 10^{-2}$
M2V1	$1.9 \cdot 10^{-2}$	$6.3 \cdot 10^{-2}$	$2.9 \cdot 10^{-2}$
M2V2	$6.6 \cdot 10^{-3}$	$3.7 \cdot 10^{-3}$	$1.6 \cdot 10^{-2}$
M2VN1[1]	$6.2 \cdot 10^{-4}$	$2.0 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$

Cuadro 6.20: Acotación errores ejemplo (6.2.26).b

Comparación de los términos de Coriolis $\vec{\mathbf{F}}^{\varepsilon}_{C_1}$ y $\vec{\mathbf{F}}^{\varepsilon}_{C_2}$

Consideramos ahora algunas soluciones analíticas de Navier-Stokes con $\phi \neq 0$ (para todos los ejemplos de esta sección se considerará $\phi = 7.29 \cdot 10^{-5}$ y $\varphi = 0.76$) de modo que podamos comparar los términos de Coriolis $\vec{\mathbf{F}}^{\varepsilon}_{C_1}$ y $\vec{\mathbf{F}}^{\varepsilon}_{C_2}$ (véase la sección 6.2.1.2). Para ello compararemos los modelos M2VN1 y M2VN2 que tienen el mismo término de viscosidad y sólo se diferencian precisamente en el término de Coriolis.

1. En primer lugar tomamos:

$$
\begin{aligned}
&u = A, \quad v = B, \quad w = 0,\\
&H = 0, \quad h = C + D((A + B)t - x - y),\\
&p = p_s + \rho(z - s)(2A\phi\cos\varphi - g), \qquad (6.2.27)\\
&p_s = K(t) + \rho\left\{2\phi[\operatorname{sen}\varphi(Bx - Ay) + As\cos\varphi] - gs\right\},\\
&f_R = 0, \quad f_W = 0
\end{aligned}
$$

con los valores siguientes para las constantes

$$A = 2, \quad B = 0.5, \quad C = 1, \quad D = 0.05, \quad \nu = 1.02 \cdot 10^{-6}$$

Las diferencias que se aprecian entre los errores cometidos con uno y otro término de Coriolis son del orden de 10^9 como se puede apreciar en el cuadro 6.21.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2VN2	$1.0 \cdot 10^{-5}$	$7.2 \cdot 10^{-5}$	$1.6 \cdot 10^{-5}$
M2VN1	$1.1 \cdot 10^{-14}$	$7.3 \cdot 10^{-14}$	$1.6 \cdot 10^{-14}$

Cuadro 6.21: Acotación errores ejemplo (6.2.27)

2. La segunda solución que tratamos de aproximar es la siguiente:

$$
\begin{aligned}
&u = A, \quad v = B, \quad w = AC,\\
&H = Cx + Dy,\\
&h = E + \frac{2\phi(Bx - Ay)[\operatorname{sen}\varphi + D\cos\varphi] + g[C(At - x) + D(Bt - y)]}{g - 2\phi A\cos\varphi},\\
&p = ps + \rho(z - s)(2A\phi\cos\varphi - g), \quad p_s = K, \qquad (6.2.28)\\
&f_R = 0, \quad f_W = 0
\end{aligned}
$$

Escogemos los siguientes valores:

$$A = 2, \quad B = 0.5, \quad C = 0.05, \quad D = 0.02, \quad E = 1, \quad \nu = 1.02 \cdot 10^{-6}$$

y observamos (véase cuadro 6.22) que de nuevo los errores que se cometen con el modelo M2VN1 son mucho menores que los obtenidos con el término clásico de Coriolis.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2VN2	$9.5 \cdot 10^{-6}$	$8.1 \cdot 10^{-5}$	$1.7 \cdot 10^{-5}$
M2VN1	$1.4 \cdot 10^{-14}$	$9.0 \cdot 10^{-14}$	$2.2 \cdot 10^{-14}$

Cuadro 6.22: Acotación errores ejemplo (6.2.28)

3. De la familia de soluciones de las ecuaciones de Navier-Stokes:

$$
\begin{aligned}
&u = A, \quad v = 0, \quad w = 0, \\
&H = By^2 + Cy, \quad h = D + \frac{2\phi A y \operatorname{sen}\varphi}{2\phi A \cos\varphi - g} - (By^2 + Cy), \\
&p = p_s + \rho(z - s)(2A\phi\cos\varphi - g), \quad p_s = K, \\
&f_R = 0, \quad f_W = 0
\end{aligned}
\tag{6.2.29}
$$

nos quedamos con la que resulta de sustituir las constantes por los siguientes valores:

$$A = 2, \quad B = 0.5, \quad C = -1, \quad D = 0.5, \quad \nu = 1.02 \cdot 10^{-2}$$

En este caso el nuevo término de Coriolis que proponemos permite reducir los errores al menos 10^6 veces como se ve en el cuadro 6.23.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2VN2	$2.0 \cdot 10^{-9}$	$5.0 \cdot 10^{-9}$	$4.3 \cdot 10^{-9}$
M2VN1	$1.6 \cdot 10^{-16}$	$1.4 \cdot 10^{-16}$	$3.2 \cdot 10^{-15}$

Cuadro 6.23: Acotación errores ejemplo (6.2.29)

4. La última familia de soluciones que consideramos introduce la dependencia de t en la componente u de la velocidad horizontal:

$$
\begin{aligned}
&u = a + bt, \quad v = c, \quad w = 0, \\
&H = 0, \quad h = A(y - ct) + B\left(x - at - b/2t^2\right) + C \\
&p = p_s + \rho(s - z)(g - 2\phi\cos\varphi u), \\
&p_s = K(t) + \rho(-gs + (2c\phi\operatorname{sen}\varphi - b)x + 2\phi u(\cos\varphi h - \operatorname{sen}\varphi y)) \\
&f_{R_1} = f_{R_2} = f_{W_1} = f_{W_2} = 0
\end{aligned}
\tag{6.2.30}
$$

Estudiamos el caso en el que

$$a = 2, \quad b = 0.1, \quad c = 0.5,$$

$$A = 0.5, \quad B = -0.1, \quad C = 1.2, \quad D = 0.1, \quad \nu = 1.02 \cdot 10^{-6}$$

En este ejemplo las diferencias entre los errores cometidos por los dos modelos son menores, pero todavía considerables (al menos $6 \cdot 10^3$ veces mayores los cometidos por M2VN2) como se aprecia en el cuadro 6.24.

Modelo	Acotación error h	Acotación error u	Acotación error v
M2VN2	$6.0 \cdot 10^{-5}$	$2.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$
M2VN1	$1.0 \cdot 10^{-8}$	$3.8 \cdot 10^{-8}$	$2.7 \cdot 10^{-8}$

Cuadro 6.24: Acotación errores ejemplo (6.2.30)

Todas las ejecuciones se han realizado en el dominio $[0, 10] \times [0, 2]$ con $\Delta x = \Delta y = 10^{-1}$ y considerando el intervalo temporal $[0, 10]$ con $\Delta t = 10^{-2}$.

6.2.4. Otras comparaciones

6.2.4.1. Dimensión uno

Una vez visto que el modelo de aguas someras que aproxima mejor el modelo bidimensional es M1VN, vamos a comparar cómo se comportan los cuatro modelos (M1SV, M1V1, M1V2, M1VN) a la hora de resolver algunos tests.

1. Consideramos una ola inicial elevada 0.5 metros sobre una altura inicial de 1 metro situada entre $x = 2$m y $x = 3$m en un canal de 20 metros de largo, con velocidad inicial de 1 m/s, el fondo plano, las condiciones de contorno periódicas y sin fuerzas de rozamiento o del viento actuando sobre el fluido.

   Al transcurrir 10 segundos, considerando un coeficiente de viscosidad $\nu = 1.02 \cdot 10^{-1}$, $\Delta x = 10^{-1}$ y $\Delta t = 10^{-2}$, la diferencia entre las velocidades de los cuatro modelos llega a ser de $5. \cdot 10^{-1}$ y entre las alturas de $3. \cdot 10^{-1}$ como se ve en los cuadros 6.25 y 6.26.

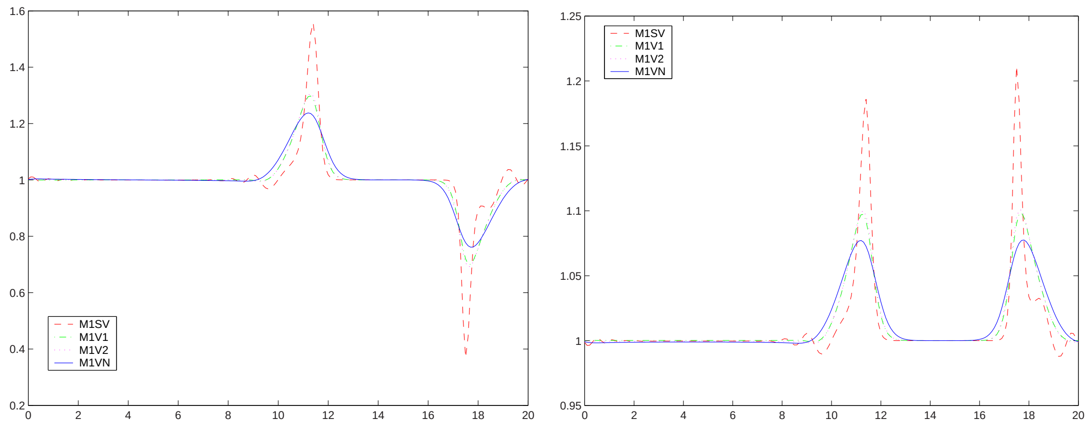
Modelos	M1V1	M1V2	M1VN
M1SV	$4.6 \cdot 10^{-1}$	$4.5 \cdot 10^{-1}$	$5.0 \cdot 10^{-1}$
M1V1		$1.9 \cdot 10^{-2}$	$9.1 \cdot 10^{-2}$
M1V2			$9.6 \cdot 10^{-2}$

Cuadro 6.25: Diferencias máximas entre las velocidades horizontales (test 1)

Modelos	M1V1	M1V2	M1VN
M1SV	$2.3 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	$2.6 \cdot 10^{-1}$
M1V1		$1.1 \cdot 10^{-2}$	$3.9 \cdot 10^{-2}$
M1V2			$4.6 \cdot 10^{-2}$

Cuadro 6.26: Diferencias máximas entre los calados (test 1)

Como se puede ver en las figuras 6.1-6.3 el modelo M1VN es el que aporta una solución más "suavizada". Los efectos de la viscosidad son más patentes en este modelo.



(a) Velocidades horizontales (b) Calados

Figura 6.1: Resultados en el instante $t = 2$ s (test 1)

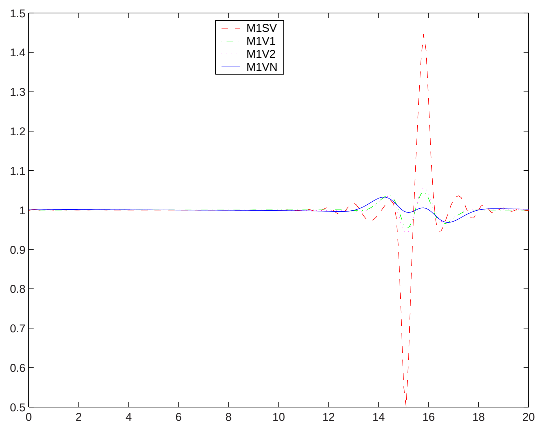
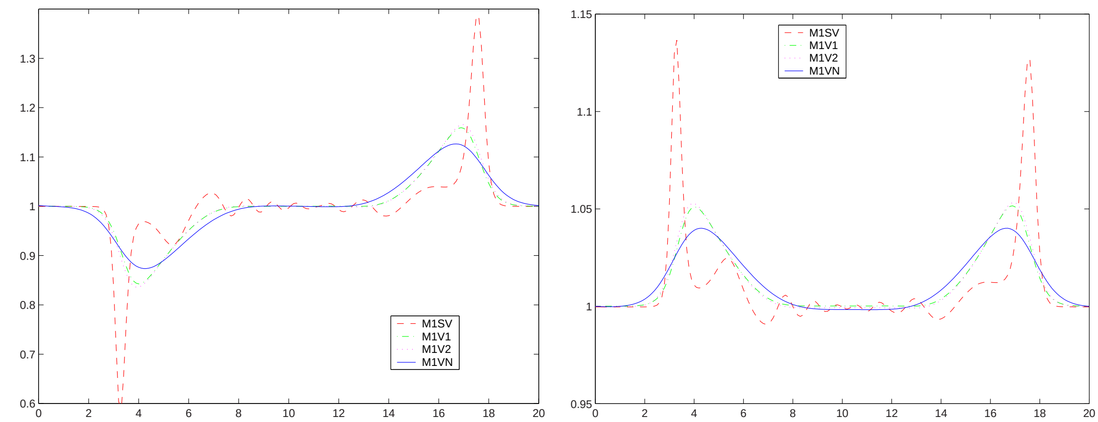


Figura 6.2: Velocidades horizontales instante t=3 s (test 1)

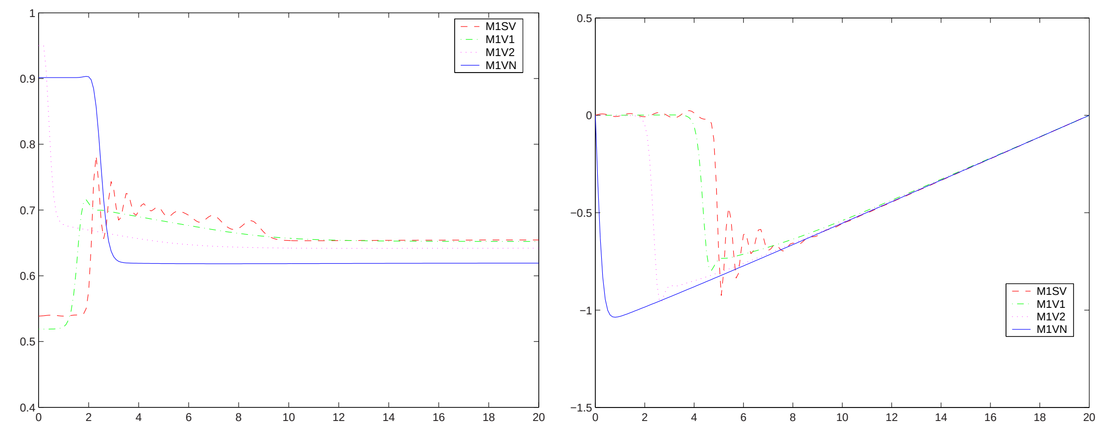


(a) Velocidades horizontales (b) Calados

Figura 6.3: Resultados en el instante $t = 8$ s (test 1)

2. Estudiamos ahora un canal de 20 m de largo en el que se coloca una barrera en $x = 2$. La altura del agua a la izquierda de la barrera es 2 m mientras que a la derecha es de 0.5 m. En el instante inicial se retira la barrera. El fondo es plano y se desprecian las fuerzas del viento y de rozamiento. El fluido inicialmente está en reposo. Las condiciones de contorno que se imponen son de tipo pared.

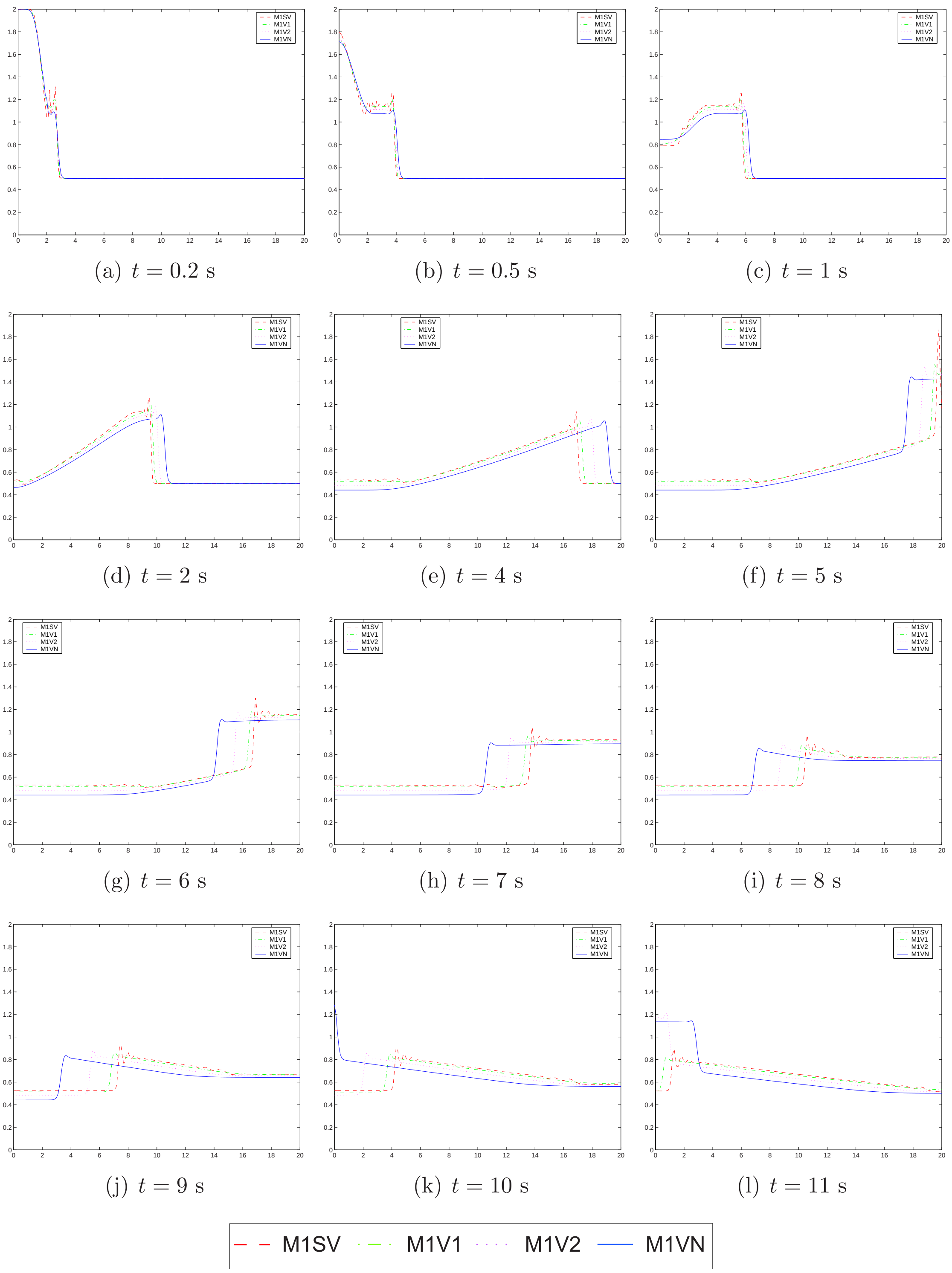
   En las figuras 6.5-6.6 se aprecia la evolución de los calados en el tiempo para los cuatro modelos que estamos comparando, mientras que en 6.7-6.8 se aprecia cómo van variando las velocidades horizontales. La figura 6.4 muestra los calados y velocidades en el instante $t = 24$ s.



(a) Calados (b) Velocidades horizontales

Figura 6.4: Instante t=24 s (test 2)

En primer lugar, se aprecia que el modelo sin viscosidad (M1SV) presenta muchas oscilaciones mientras que los que incluyen un término viscoso no. Y



(a) $t = 0.2$ s (b) $t = 0.5$ s (c) $t = 1$ s

(d) $t = 2$ s (e) $t = 4$ s (f) $t = 5$ s

(g) $t = 6$ s (h) $t = 7$ s (i) $t = 8$ s

(j) $t = 9$ s (k) $t = 10$ s (l) $t = 11$ s

Figura 6.5: Calados tiempo 0.2 - 11 s (test 2)

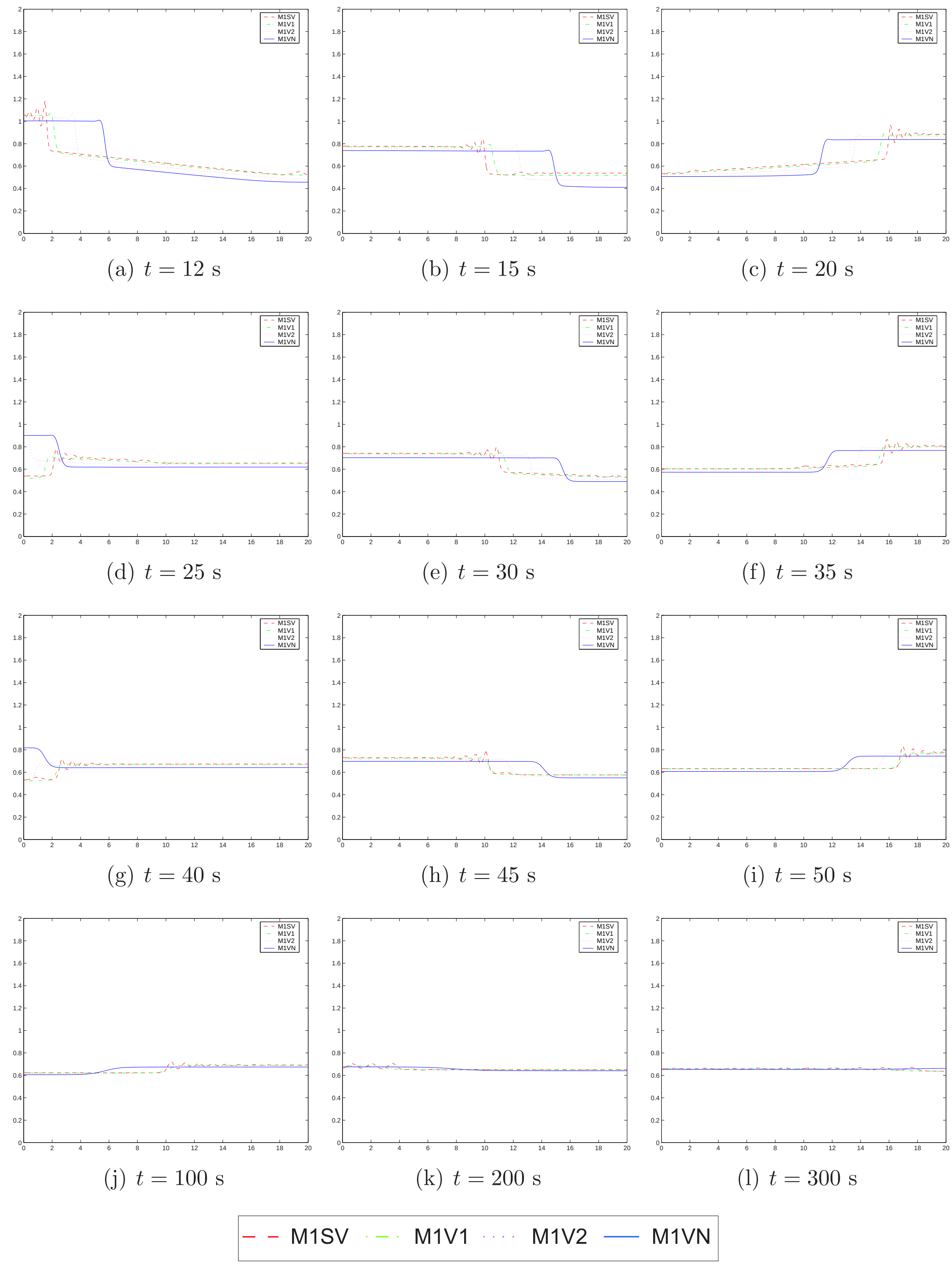
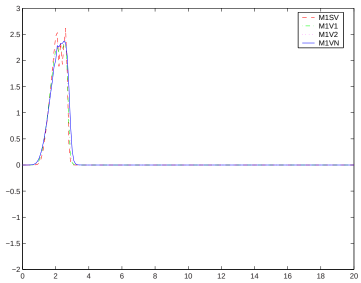
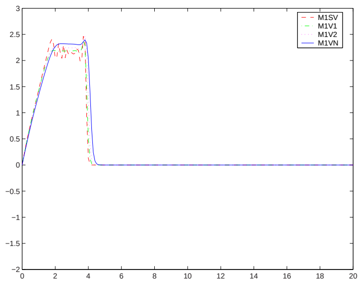


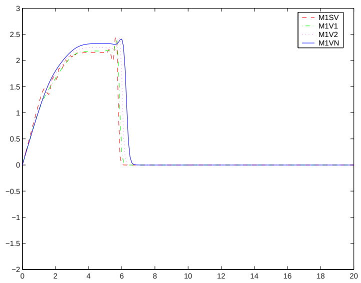
Figura 6.6: Calados tiempo 12 - 300 s (test 2)



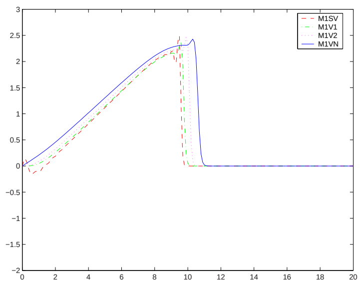
(a) $t = 0.2$ s



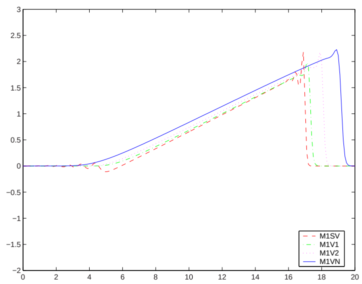
(b) $t = 0.5$ s



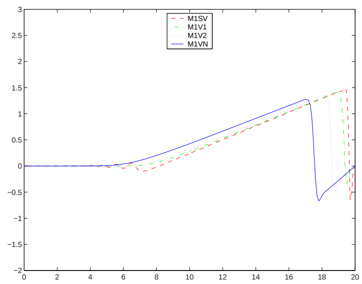
(c) $t = 1$ s



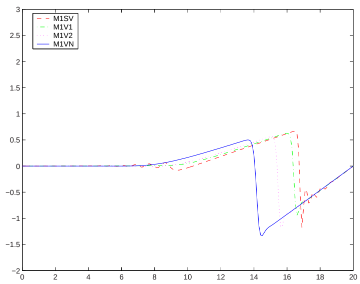
(d) $t = 2$ s



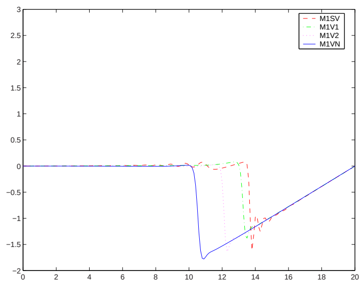
(e) $t = 4$ s



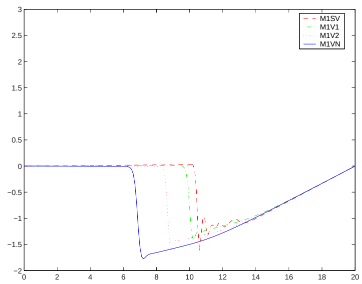
(f) $t = 5$ s



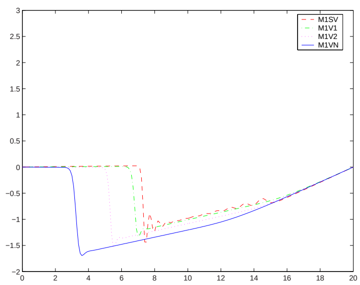
(g) $t = 6$ s



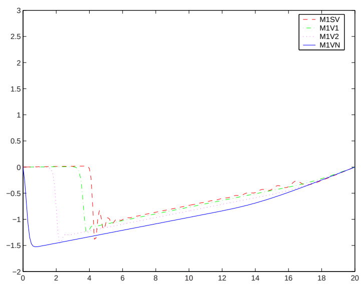
(h) $t = 7$ s



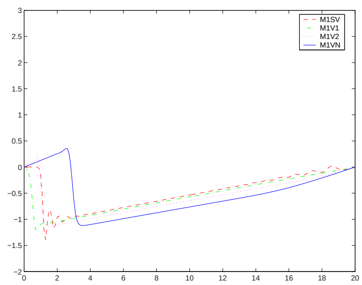
(i) $t = 8$ s



(j) $t = 9$ s



(k) $t = 10$ s



(l) $t = 11$ s

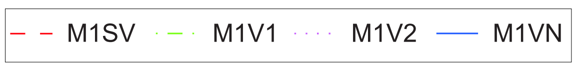
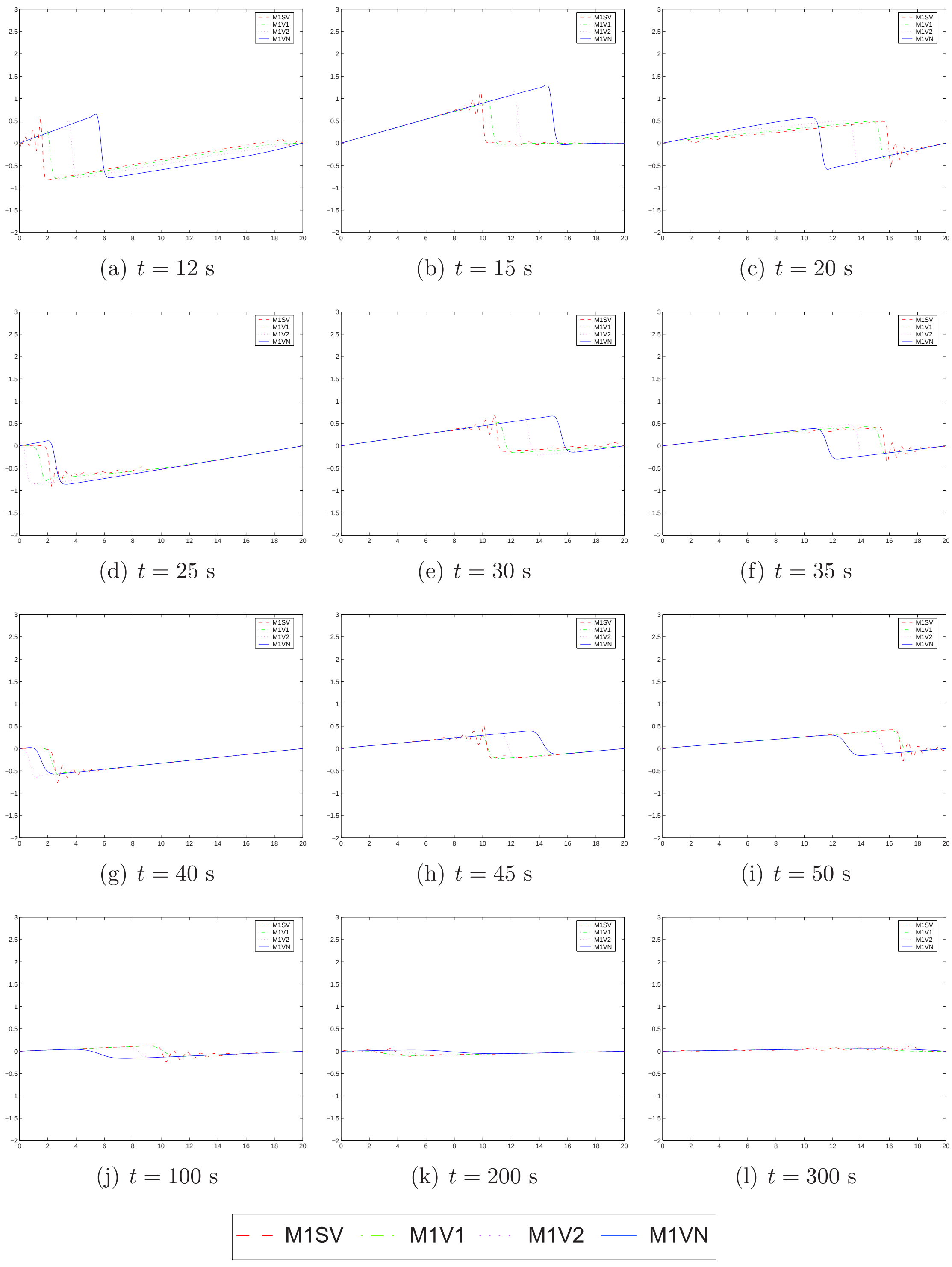


Figura 6.7: Velocidades horizontales tiempo 0.2 - 11 s (test 2)



(a) $t = 12$ s (b) $t = 15$ s (c) $t = 20$ s

(d) $t = 25$ s (e) $t = 30$ s (f) $t = 35$ s

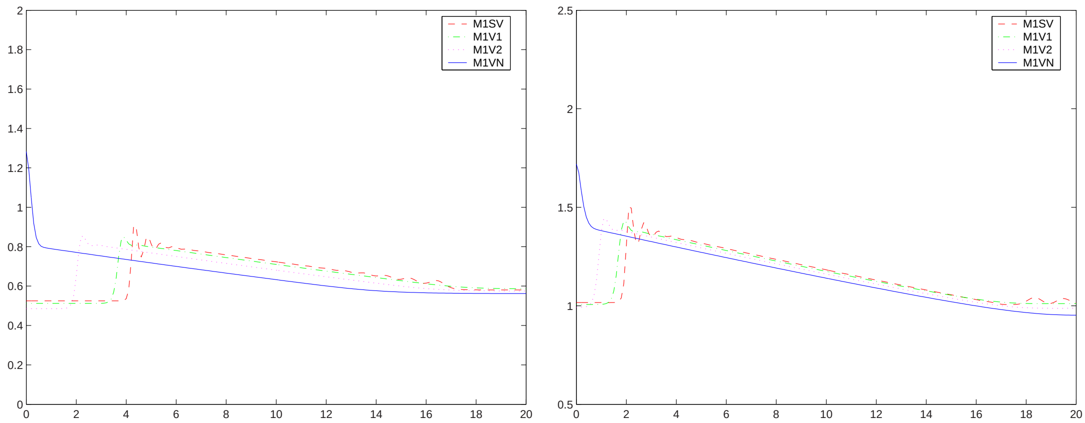
(g) $t = 40$ s (h) $t = 45$ s (i) $t = 50$ s

(j) $t = 100$ s (k) $t = 200$ s (l) $t = 300$ s

Figura 6.8: Velocidades horizontales tiempo 12-300 s (test 2) en segundo, debido a que en los términos de viscosidad de los modelos M1V2 y M1VN ((6.1.39) y (6.1.43)) interviene la inversa del calado y la derivada de éste, la viscosidad de estos modelos aumenta o disminuye en función del tamaño del calado y de cuánto varía éste.

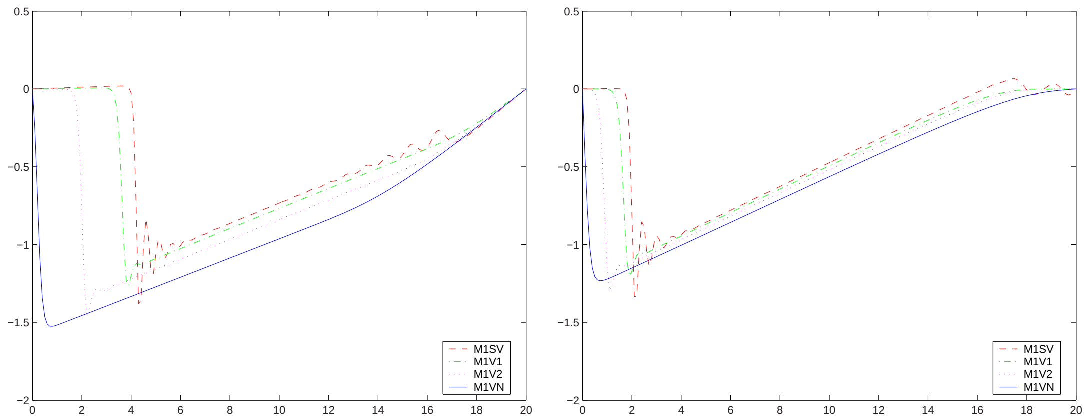
Para tratar de ver cómo afectan estos dos factores (poca profundidad del agua y variación rápida de calado) realizamos los siguientes test:

a) Mantenemos la diferencia de calado pero incrementando la altura de agua en 0.5 m a ambos lados de la barrera, apreciamos que las diferencias entre los modelos disminuyen y que en este caso el frente de agua se mueve más rápidamente (ver figuras 6.9 y 6.10).



(a) Altura de agua 2 m - 0.5 m $t = 10$ s (b) Altura de agua 2.5 m - 1 m $t = 8.6$ s

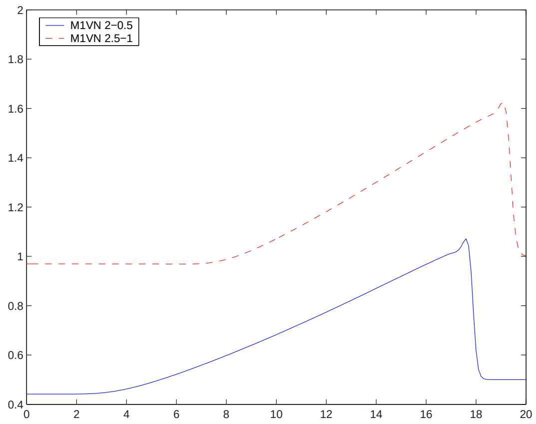
Figura 6.9: Comparación calados (test 2.a)



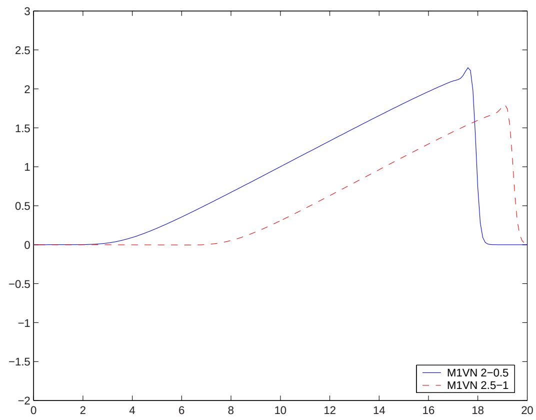
(a) Altura de agua 2 m - 0.5 m $t = 10$ s (b) Altura de agua 2.5 m - 1 m $t = 8.6$ s

Figura 6.10: Comparación velocidades horizontales (test 2.a)

En la figura 6.11 comparamos el comportamiento del modelo M1VN cuando se aumenta en 0.5 m la altura de agua.



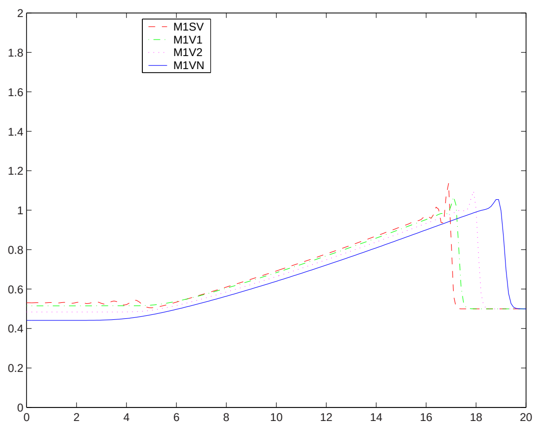
(a) Calados $t = 3.7$ s



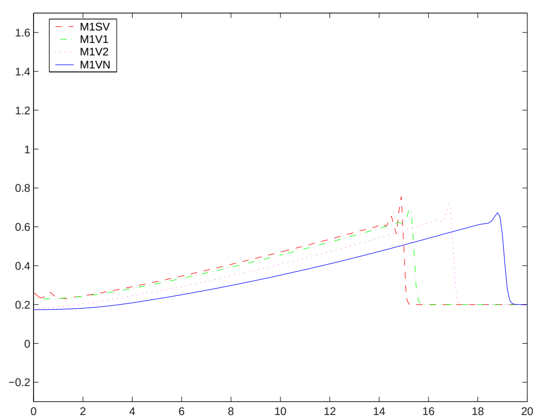
(b) Velocidades horizontales $t = 3.7$ s

Figura 6.11: Comparación modelo M1VN con distintos calados (test 2.a)

b) De nuevo se mantiene la diferencia de calado pero ahora el agua a la izquierda de la barrera tiene una altura de 1.7 m y a la derecha 0.2 m. En este caso se produce la situación inversa (ver figuras 6.12 y 6.13).

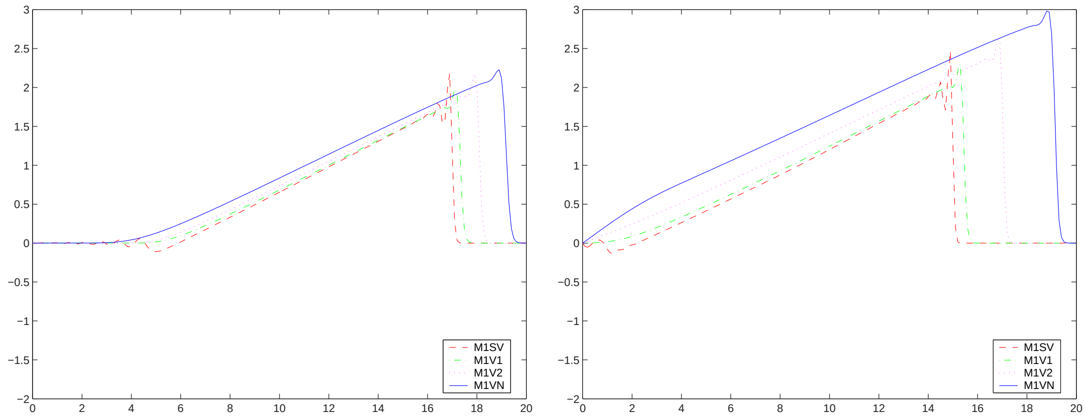


(a) Altura de agua 2 m - 0.5 m $t = 4$ s



(b) Altura de agua 1.7 m - 0.2 m $t = 4$ s

Figura 6.12: Comparación calados (test 2.b)

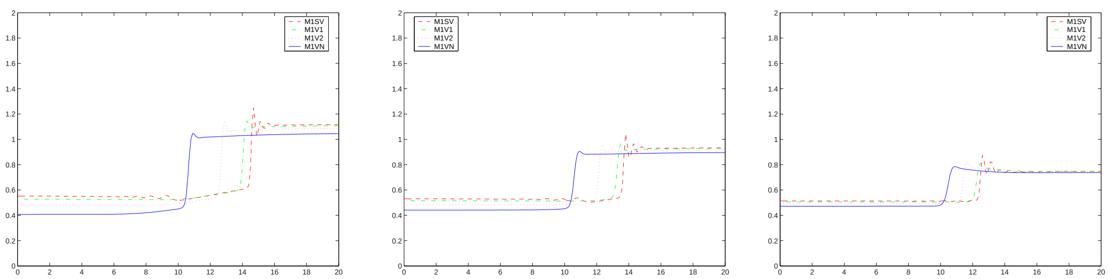


(a) Altura de agua 2 m - 0.5 m $t = 4$ s

(b) Altura de agua 1.7 m - 0.2 m $t = 4$ s

Figura 6.13: Comparación velocidades horizontales (test 2.b)

c) Variamos ahora la diferencia de calado pero manteniendo 0.5 m de agua a la derecha de la barrera y apreciamos (figuras 6.14 y 6.15) que cuanto mayor es la diferencia inicial de calado más diferencias aparecen entre los modelos. Los modelos M1VN y M1V2 se aceleran más con esta diferencia.



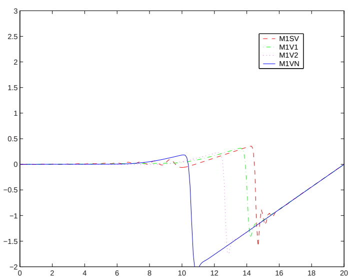
(a) Altura de agua 2.5 m - 0.5 m $t = 6.3$ s

(b) Altura de agua 2 m - 0.5 m $t = 7$ s

(c) Altura de agua 1.5 m - 0.5 m $t = 8$ s

Figura 6.14: Comparación calados (test 2.c)

Todos los test se han realizado imponiendo una viscosidad muy elevada ($\nu = 1.02 \cdot 10^{-1}$) para exagerar las diferencias entre los distintos modelos. Si resolvemos con la viscosidad del agua ($\nu = 1.02 \cdot 10^{-6}$) las diferencias son mucho menores estando acotadas superiormente por $2.3 \cdot 10^{-3}$ para las velocidades y $8.8 \cdot 10^{-4}$ para los calados.

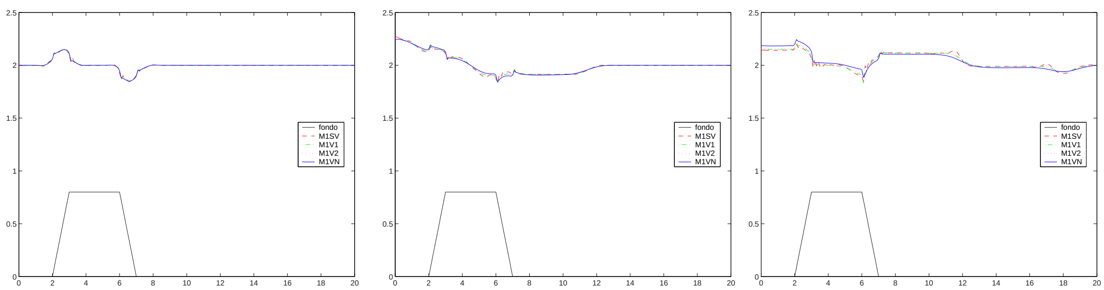


(a) Altura de agua 2.5 m - 0.5 m $t = 6.3$ s (b) Altura de agua 2 m - 0.5 m $t = 7$ s (c) Altura de agua 1.5 m - 0.5 m $t = 8$ s

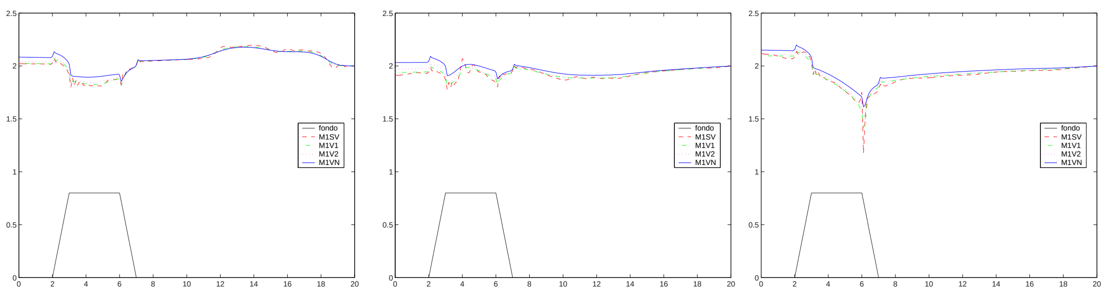
Figura 6.15: Comparación velocidades horizontales (test 2.c)

3. Se considera que en el fondo de un canal de 20 m de largo con una altura inicial del agua de 2 m se tiene una plataforma situada ente $x = 2$ y $x = 7$ que tiene una altura máxima de 0.8 m. El flujo entra por la izquierda del canal de forma constante con una velocidad de 1 m/s. Se impone el calado aguas abajo.

   En las figuras 6.16-6.19 se aprecia la evolución de los calados y las velocidades horizontales en el tiempo para los cuatro modelos que estamos comparando.



(a) $t = 0.2$ s (b) $t = 1$ s (c) $t = 3$ s



(d) $t = 5$ s (e) $t = 8$ s (f) $t = 10$ s

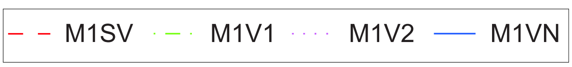


Figura 6.16: Calados tiempo 0.2 - 10 s (test 3)

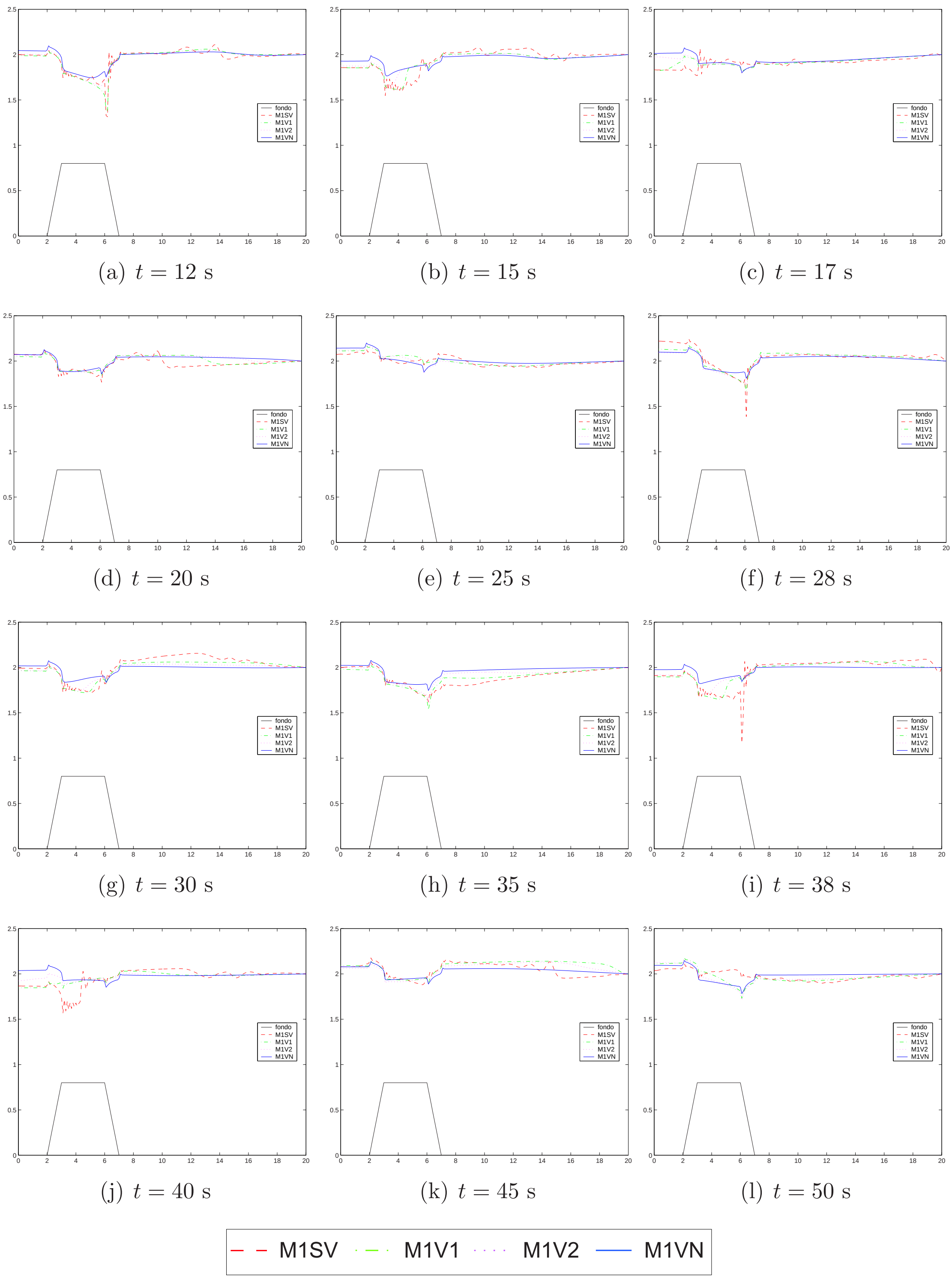


Figura 6.17: Calados tiempo 12 - 50 s (test 3)

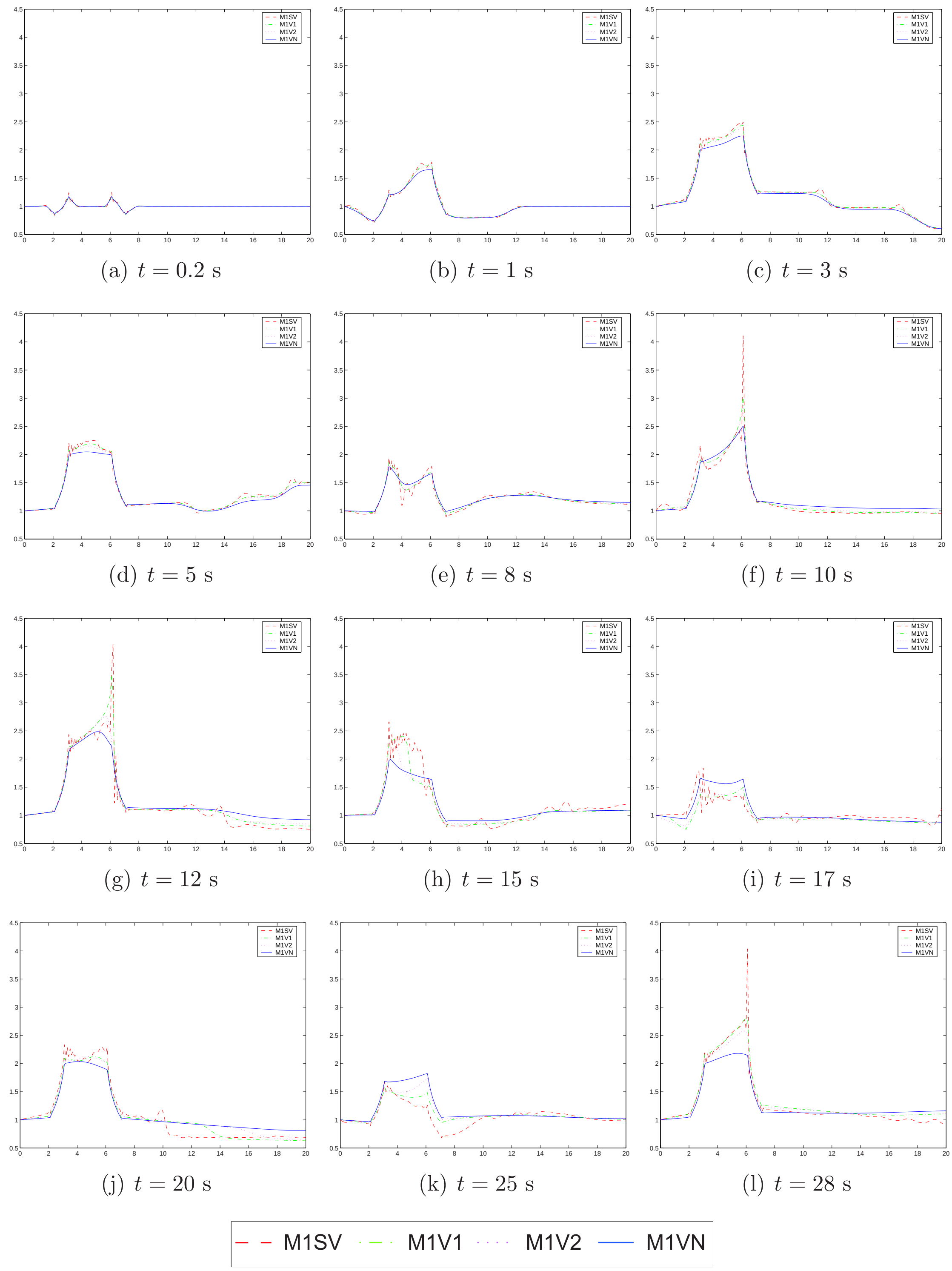
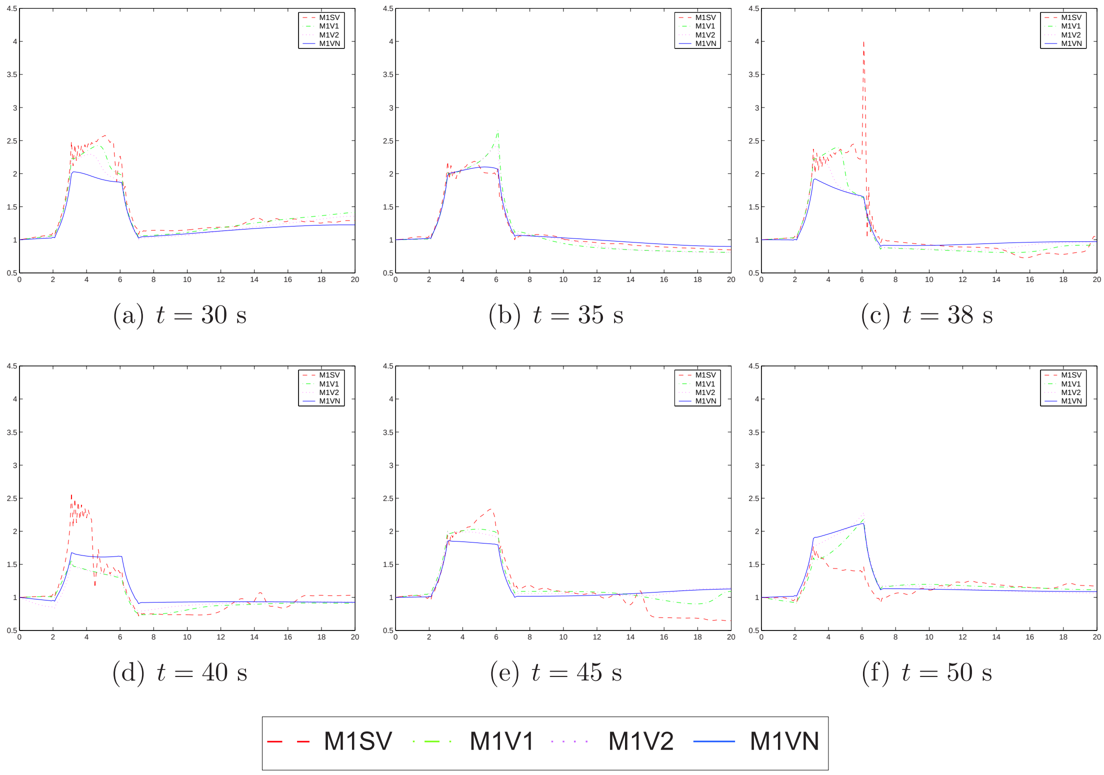


Figura 6.18: Velocidades tiempo 0.2 - 28 s (test 3)

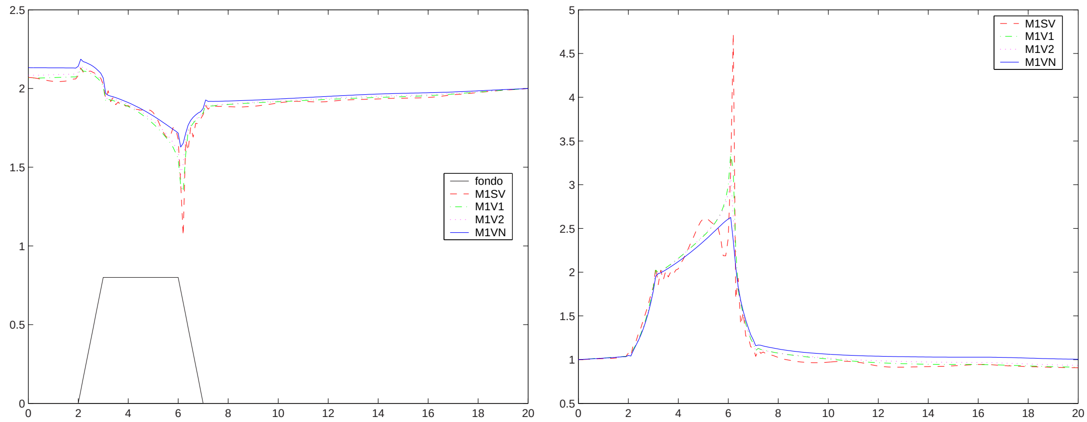


(a) $t = 30$ s (b) $t = 35$ s (c) $t = 38$ s

(d) $t = 40$ s (e) $t = 45$ s (f) $t = 50$ s

Figura 6.19: Velocidades tiempo 30 - 50 s (test 3)

En la figura 6.20 se ven los calados y velocidades en el instante $t = 10.4$ s.



(a) Calados (b) Velocidades horizontales

Figura 6.20: Instante t=10.4 s (test 3)

De nuevo los test se han realizado imponiendo una viscosidad muy elevada ($\nu = 1.02 \cdot 10^{-1}$) para exagerar las diferencias entre los distintos modelos. Con la viscosidad del agua ($\nu = 1.02 \cdot 10^{-6}$) las diferencias son del orden de 10^{-3} tanto para las velocidades como para los calados.

En los cuadros 6.27-6.28 se pueden ver las diferencias máximas entre las soluciones calculadas usando los distintos modelos con tres discretizaciones diferentes identificadas con (1), (2) y (3). Se observa que si se refina la malla y se reduce convenientemente el paso temporal las diferencias más pequeñas entre las soluciones de un mismo modelo con las distintas discretizaciones son las que corresponden al modelo M1VN.

		Discretización											
		$\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$ (1)				$\Delta x = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$ (2)				$\Delta x = 2 \cdot 10^{-2}$, $\Delta t = 1 \cdot 10^{-3}$ (3)			
		M1SV	M1V1	M1V2	M1VN	M1SV	M1V1	M1V2	M1VN	M1SV	M1V1	M1V2	M1VN
(1)	M1SV	—	25	30	44	51	28	34	47	26	30	35	48
	M1V1		—	8.6	20	32	6.5	14	22	32	10	17	23
	M1V2			—	14	33	6.2	5.8	16	34	7.2	8.8	18
	M1VN				—	44	17	12	3.8	46	17	13	5.8
(2)	M1SV					—	33	37	46	29	35	39	47
	M1V1						—	7.9	19	32	3.8	11	21
	M1V2							—	13	37	5.4	3.0	14
	M1VN								—	48	18	12	2.1
(3)	M1SV									—	34	39	49
	M1V1										—	7.2	20
	M1V2											—	12
	M1VN												—

Cuadro 6.27: Diferencias máximas entre calados (test 3) ($\times 10^{-2}$)

4. Se considera un canal de 10 metros de largo con un montículo situado en el fondo entre $x = 4$ y $x = 6$, alcanzando su máxima altura (0.4 metros) en $x = 5$. La altura inicial de agua en el canal es de 1 m y la velocidad inicial de 1 m/s. Al comienzo de la simulación, consideramos que hay una ola, elevada 0.5 metros sobre el nivel inicial de agua, situada entre $x = 1.5$ y $x = 2.5$. El coeficiente de viscosidad cinemática se toma $\nu = 1.02 \cdot 10^{-1}$. La fuerza de rozamiento en el fondo y del viento en la superficie se consideran nulas. Las condiciones de contorno son periódicas. Los pasos espacial y temporal que se han utilizado en este caso son $\Delta x = 5 \cdot 10^{-2}$ y $\Delta t = 2 \cdot 10^{-3}$.

		Discretización											
		$\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$ (1)				$\Delta x = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$ (2)				$\Delta x = 2 \cdot 10^{-2}$, $\Delta t = 1 \cdot 10^{-3}$ (3)			
		M1SV	M1V1	M1V2	M1VN	M1SV	M1V1	M1V2	M1VN	M1SV	M1V1	M1V2	M1VN
(1)	M1SV	—	110	130	160	180	120	140	170	100	130	140	170
	M1V1		—	23	52	160	19	39	58	160	28	45	61
	M1V2			—	34	170	16	15	40	170	21	22	43
	M1VN				—	190	46	28	8.2	200	43	32	12
(2)	M1SV					—	170	180	200	81	180	190	200
	M1V1						—	21	52	170	9.7	27	55
	M1V2							—	30	180	18	6.8	34
	M1VN								—	210	48	29	4.2
(3)	M1SV									—	170	190	210
	M1V1										—	23	51
	M1V2											—	28
	M1VN												—

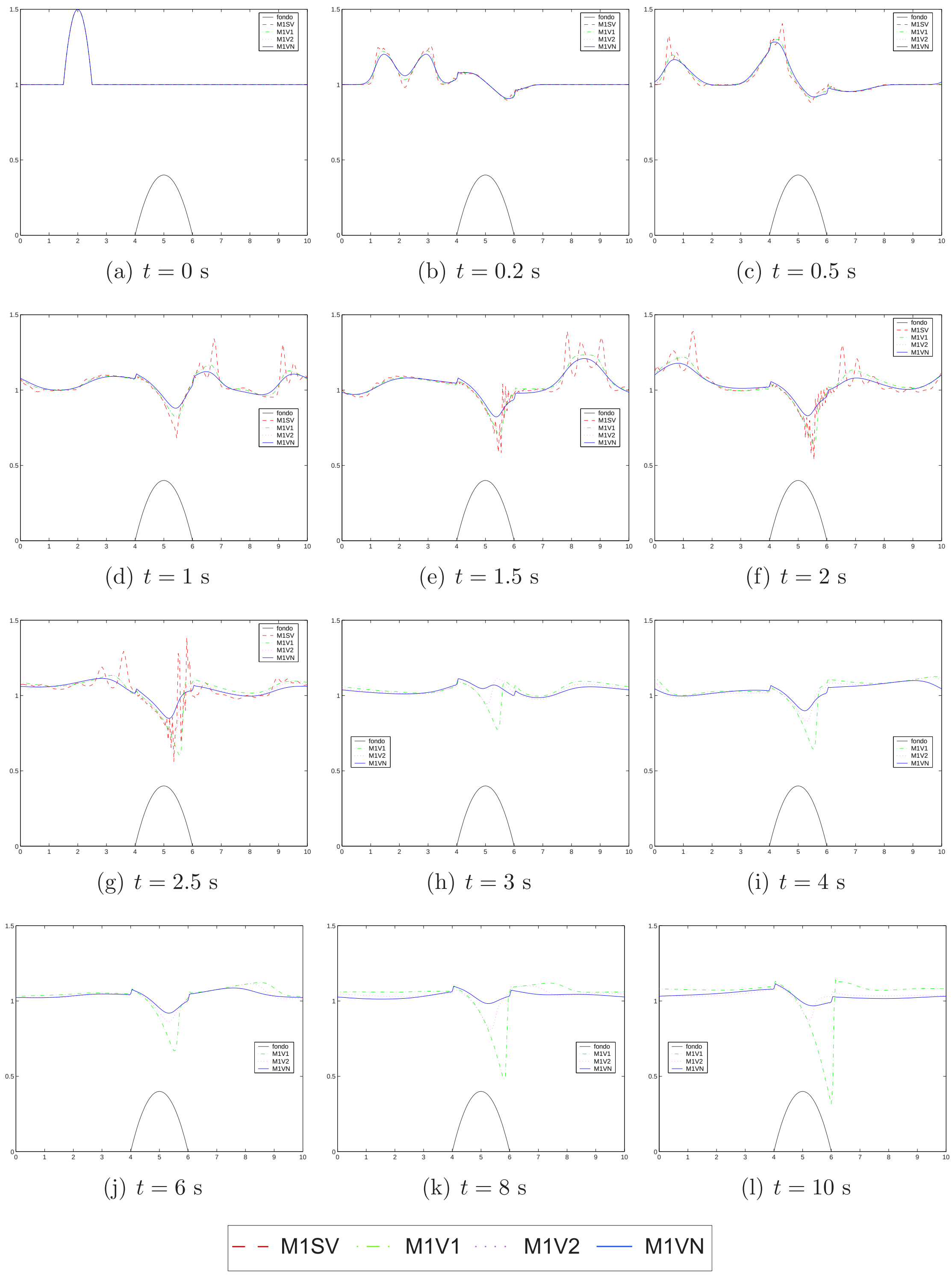
Cuadro 6.28: Diferencias máximas entre velocidades (test 3) ($\times 10^{-2}$)

En las figuras 6.21-6.22 podemos observar los calados y las velocidades horizontales obtenidos con los tres modelos con viscosidad que estamos comparando durante los 10 primeros segundos. El comportamiento del modelo sin viscosidad aparece únicamente hasta el instante $t = 2.5$ s debido a que como se puede apreciar en las figuras 6.21-6.22 las oscilaciones cada vez son mayores llegando a tomar valores negativos para h. Este test es un ejemplo en el que la profundidad (h) que se hace pequeña en algunos puntos propiciando que los términos de viscosidad ((6.1.39), (6.1.40) y (6.1.43)) sean bastante diferentes y provocando que el comportamiento de los tres modelos con viscosidad sea distinto, como puede verse en las figuras mencionadas.

En este caso las diferencias máximas entre los calados son las que se recogen en el cuadro 6.29 y entre las velocidades en el cuadro 6.30.

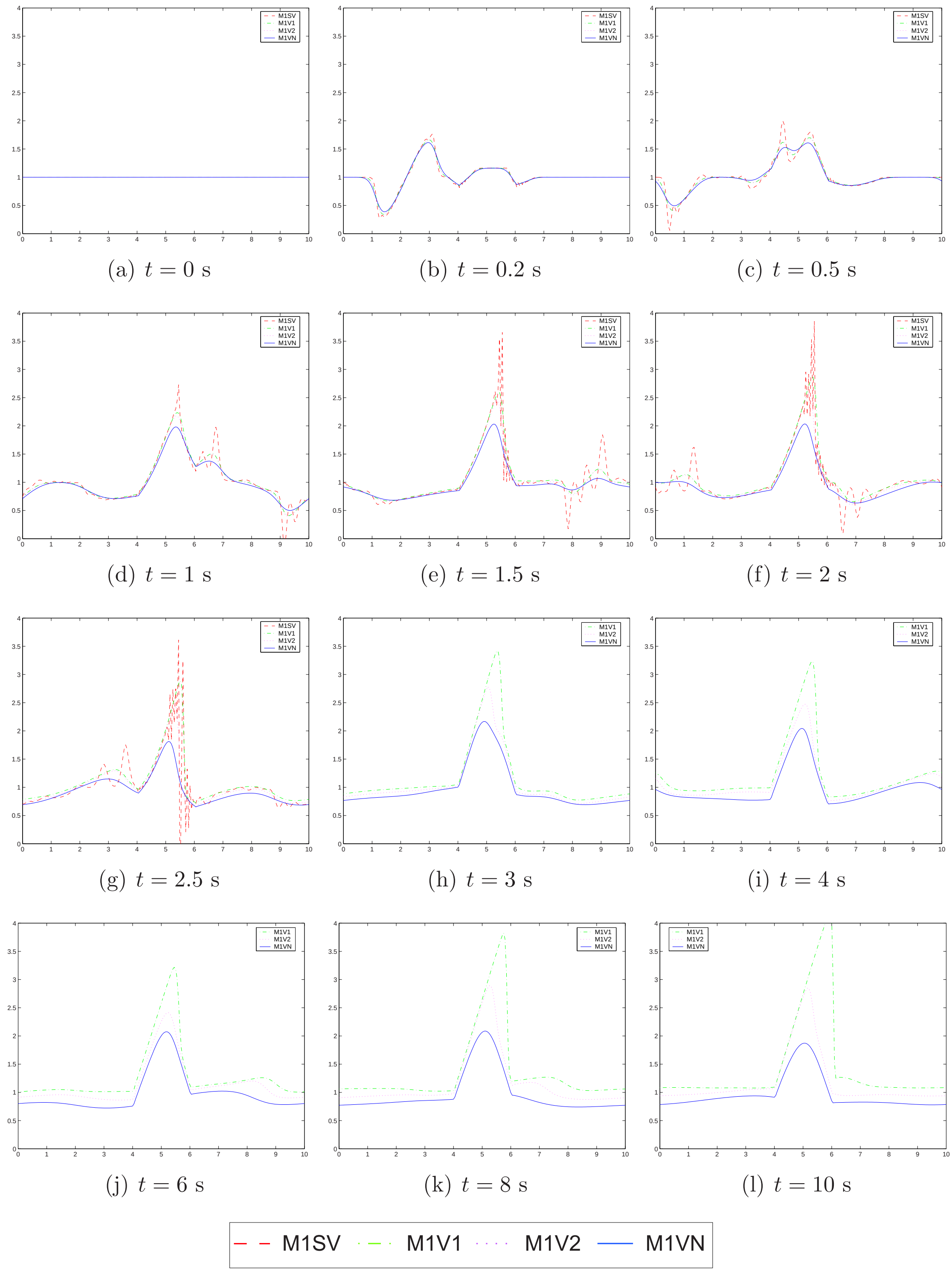
Modelos	M1V2	M1VN
M1V1	$7.2 \cdot 10^{-1}$	$6.9 \cdot 10^{-1}$
M1V2		$1.9 \cdot 10^{-1}$

Cuadro 6.29: Diferencias máximas entre los calados (test 4)



(a) $t = 0$ s
(b) $t = 0.2$ s
(c) $t = 0.5$ s
(d) $t = 1$ s
(e) $t = 1.5$ s
(f) $t = 2$ s
(g) $t = 2.5$ s
(h) $t = 3$ s
(i) $t = 4$ s
(j) $t = 6$ s
(k) $t = 8$ s
(l) $t = 10$ s

Figura 6.21: Calados (test 4)



(a) $t = 0$ s (b) $t = 0.2$ s (c) $t = 0.5$ s

(d) $t = 1$ s (e) $t = 1.5$ s (f) $t = 2$ s

(g) $t = 2.5$ s (h) $t = 3$ s (i) $t = 4$ s

(j) $t = 6$ s (k) $t = 8$ s (l) $t = 10$ s

Figura 6.22: Velocidades horizontales (test 4)

Modelos	M1V2	M1VN
M1V1	3.2	3.3
M1V2		1.2

Cuadro 6.30: Diferencias máximas entre las velocidades horizontales (test 4)

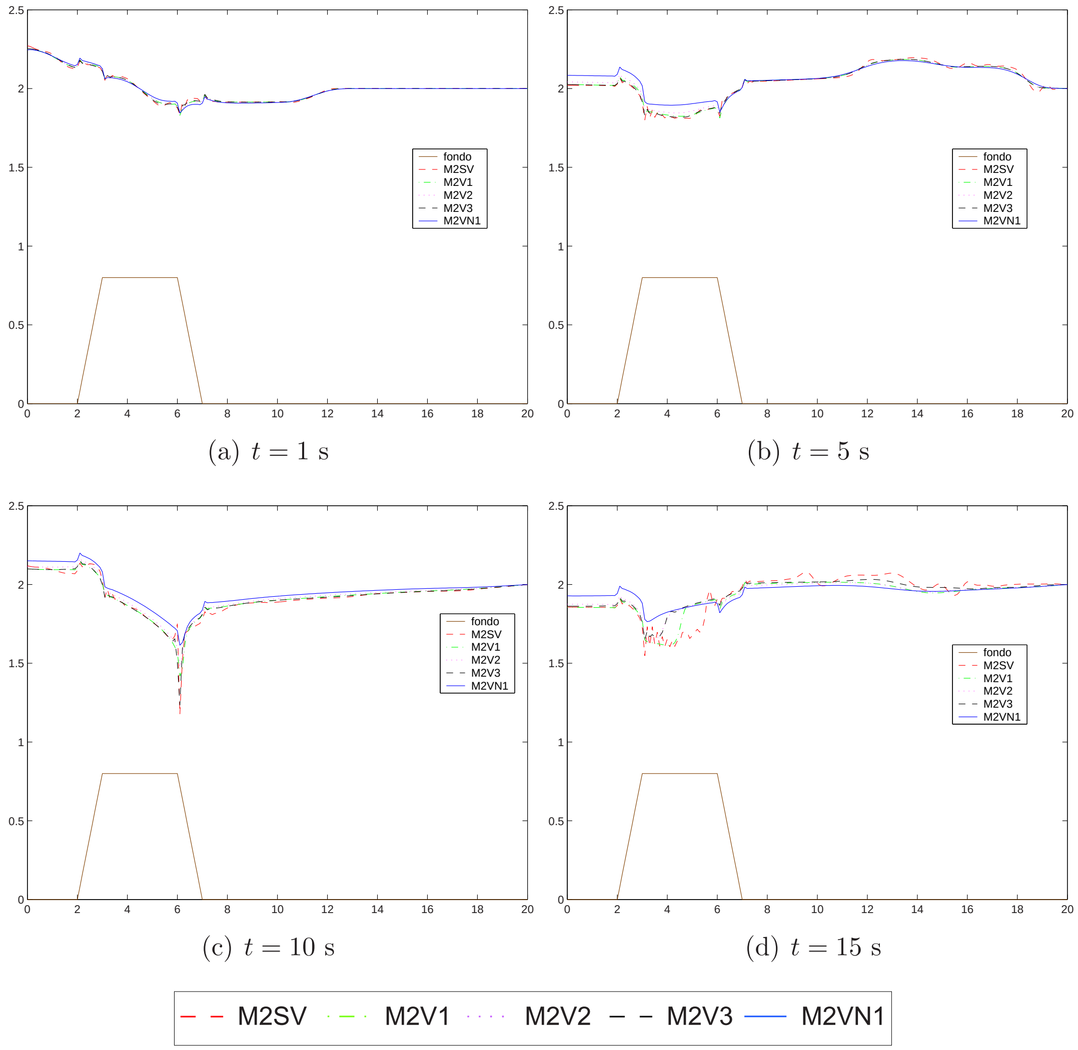
6.2.4.2. Dimensión dos

De forma similar a lo que acabamos de hacer con los modelos de dimensión uno, intentaremos ahora comparar el comportamiento numérico de los modelos de dimensión dos M2SV, M2V1, M2V2, M2V3, M2VN1 y M2VN2 ((6.2.5)-(6.2.10)). Para ello consideraremos algunos ejemplos que analizamos a continuación.

1. Se considera en primer lugar un canal de 20 m de largo por 2 m de ancho. Situada en el fondo del canal se encuentra una plataforma u obstáculo.

   a) Comenzamos por considerar una situación como la del test 3 (1D), es decir, la altura inicial del agua es de 2 m, la plataforma está situada ente $x = 2$ y $x = 7$ tiene una altura máxima de 0.8 m. El flujo entra por $x = 0$ de forma constante con una velocidad de 1 m/s ($u = 1$, $v = 0$) y se impone el calado aguas abajo.

   Por ser la velocidad en dirección y nula, el calado y la velocidad en dirección x no dependen de y. Realizamos un corte longitudinal por $y = 1$ y obtenemos los calados y velocidades horizontales (en la dirección del eje OX) que se ven en las figuras 6.23 - 6.24. Se puede apreciar que el resultado es exactamente el mismo que en el caso unidimensional comparando con las figuras 6.16, 6.17 y 6.18.

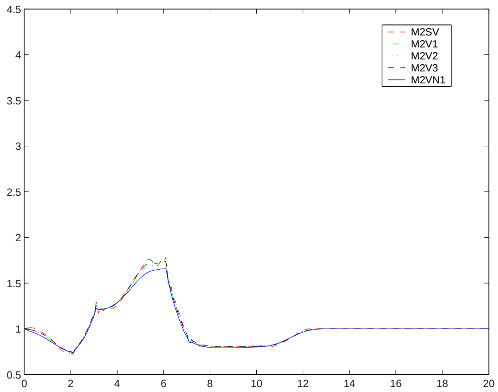


(a) $t = 1$ s (b) $t = 5$ s (c) $t = 10$ s (d) $t = 15$ s

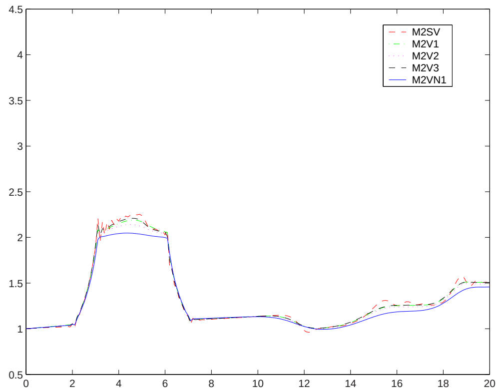
Figura 6.23: Calados test 1.a (2D)

b) Supongamos ahora que en lugar de la plataforma en el fondo del canal se encuentra una pirámide (ver figura 6.25) de altura 0.2 m (seguimos considerando que la altura inicial del agua es de 2 m, que el flujo entra por $x = 0$ de forma constante con una velocidad de 1 m/s ($u = 1$, $v = 0$) y se impone el calado aguas abajo).

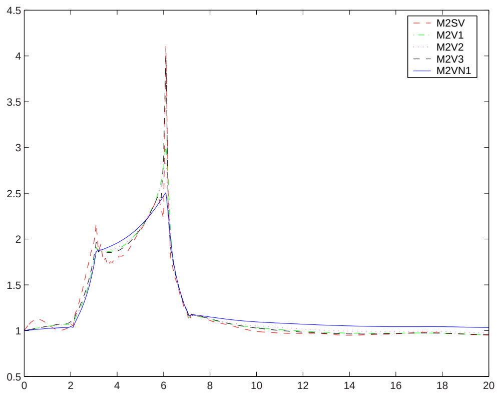
Si imponemos una condición de no penetración en las paredes laterales del canal ($v = 0$) los resultados que se obtienen son similares a los obtenidos con los test 3 y 4 de dimensión uno y con el test 1.a de dimensión dos. Si, sin embargo, no imponemos dicha condición, obtenemos unos resultados diferentes que pasamos a estudiar.



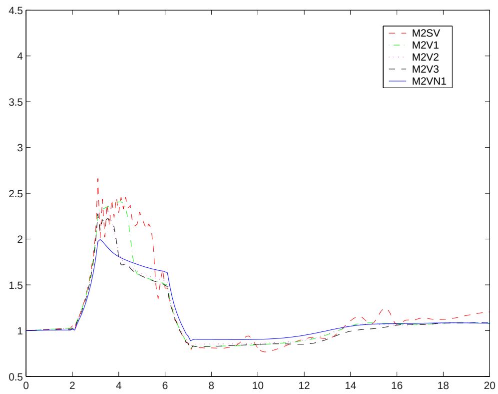
(a) $t = 1$ s



(b) $t = 5$ s



(c) $t = 10$ s



(d) $t = 15$ s

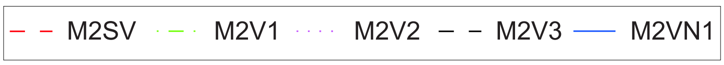
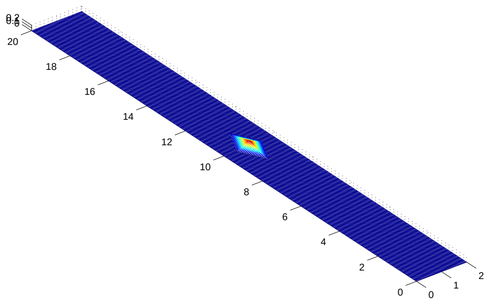
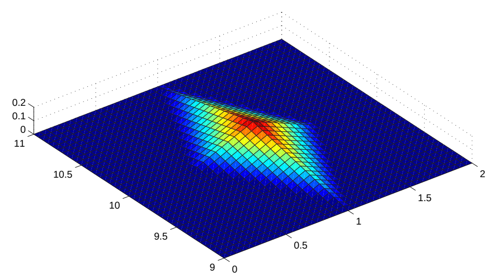


Figura 6.24: Velocidades horizontales (componente x) test 1.a (2D)



(a)



(b)

Figura 6.25: Dominio test 1.b

El vértice de la pirámide es un punto muy singular y complica la resolución del problema de modo que con una discretización con $\Delta x = \Delta y = 10^{-1}$ y $\Delta t = 5 \cdot 10^{-3}$ el único modelo para el que h no llega a anularse y proporciona una solución aceptable es M2VN1.

Por este motivo, se refina la discretización ($\Delta x = \Delta y = 5 \cdot 10^{-2}$ y $\Delta t = 2 \cdot 10^{-3}$) para comparar el comportamiento de los distintos modelos. Podemos ver los calados obtenidos con el modelo M2VN1 en el instante $t = 15$ s con la discretización anterior de forma bidimensional en la figura 6.26.

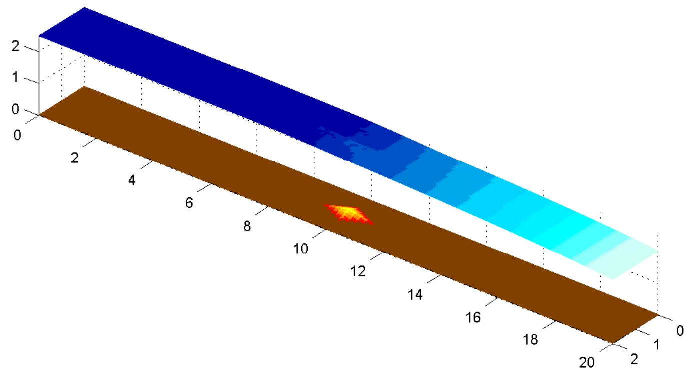
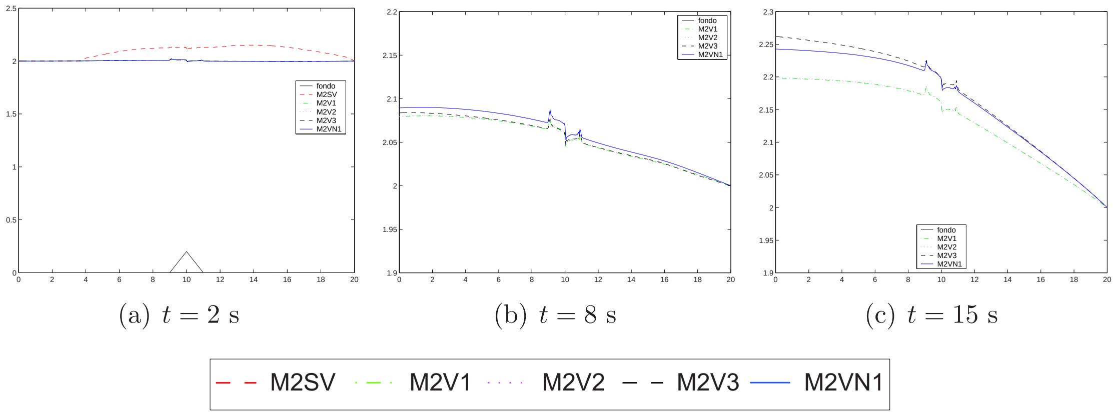


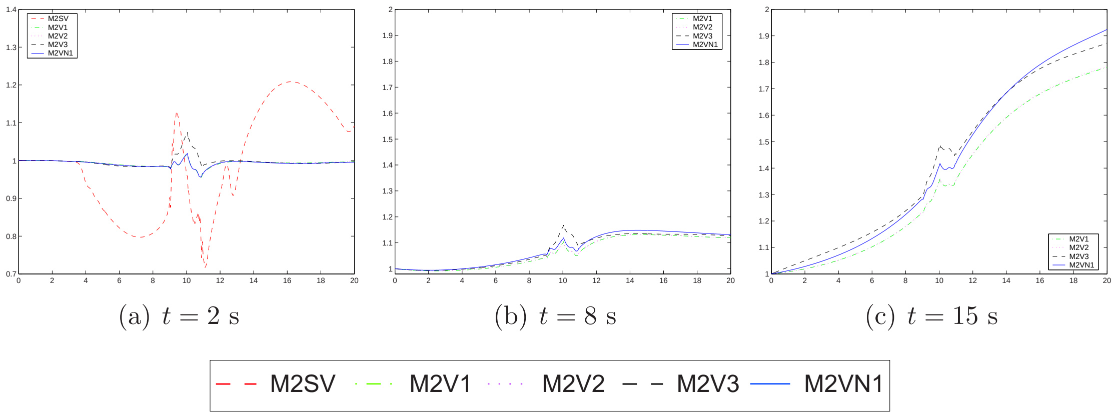
Figura 6.26: Calados obtenidos con el modelo M2VN1 en el instante $t = 15$ s (test 1.b 2D)

Realizamos un corte longitudinal para $y = 1$ y representamos para distintos instantes de tiempo los calados (figura 6.27) y las dos componentes de la velocidad horizontal (figuras 6.28-6.29). El modelo M2SV únicamente aparece en las gráficas correspondientes a $t = 2$ s, pues para este modelo el calado se hace negativo más adelante. Se aprecia en las figuras 6.27-6.29 que el comportamiento de los cuatro modelos con viscosidad es muy similar pero que comienza a diferenciarse según avanza el tiempo.



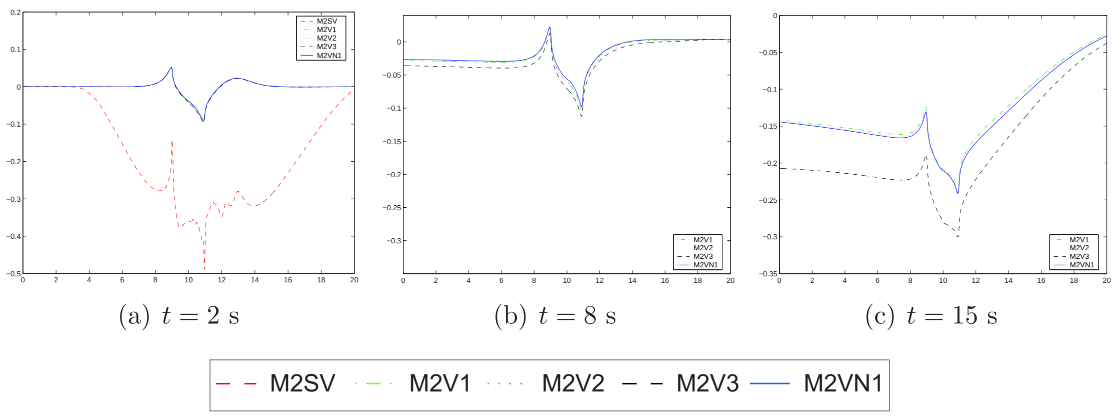
(a) $t = 2$ s (b) $t = 8$ s (c) $t = 15$ s

Figura 6.27: Calados corte longitudinal en $y = 1$



(a) $t = 2$ s (b) $t = 8$ s (c) $t = 15$ s

Figura 6.28: Componente x de la velocidad horizontal corte longitudinal en $y = 1$



(a) $t = 2$ s (b) $t = 8$ s (c) $t = 15$ s

Figura 6.29: Componente y de la velocidad horizontal corte longitudinal en $y = 1$

En la figura 6.30 se puede apreciar la evolución de los calados para el corte transversal $x = 10$.

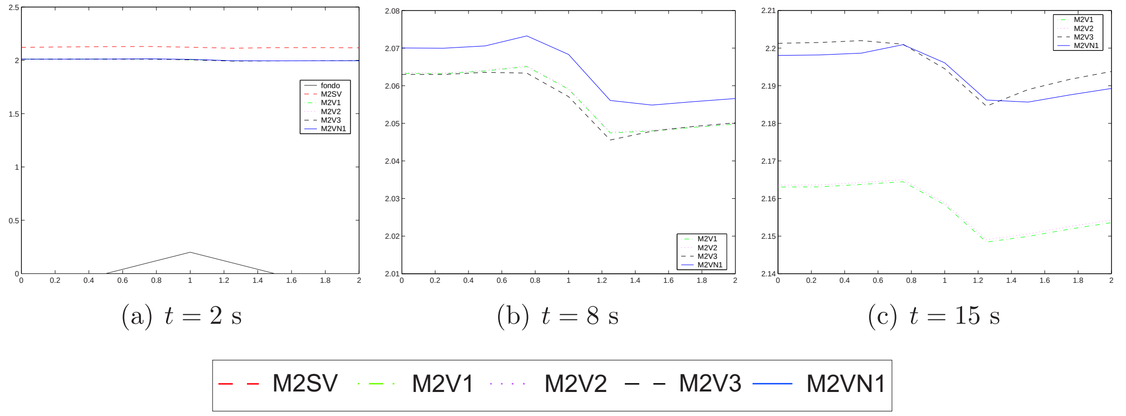


Figura 6.30: Calados corte transversal en $x = 10$ ($\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$)

La tabla 6.31 muestra las diferencias máximas entre calados, mientras que la tabla 6.32 recoge las diferencias medias entre los calados y velocidades horizontales. Se comparan los modelos con viscosidad para las dos discretizaciones antes mencionadas.

		Discretización							
		$\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$ (1)				$\Delta x = \Delta x = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$ (2)			
		M2V1	M2V2	M2V3	M2VN1	M2V1	M2V2	M2V3	M2VN1
(1)	M2V1	—	$7.6 \cdot 10^{-4}$	$6.5 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$	$9.3 \cdot 10^{-1}$	$9.3 \cdot 10^{-1}$	$9.6 \cdot 10^{-1}$	$3.3 \cdot 10^{-1}$
	M2V2		—	$6.5 \cdot 10^{-2}$	$4.4 \cdot 10^{-2}$	$9.3 \cdot 10^{-1}$	$9.3 \cdot 10^{-1}$	$9.6 \cdot 10^{-1}$	$3.3 \cdot 10^{-1}$
	M2V3			—	$2.0 \cdot 10^{-2}$	$9.4 \cdot 10^{-1}$	$9.4 \cdot 10^{-1}$	$9.7 \cdot 10^{-1}$	$2.9 \cdot 10^{-1}$
	M2VN1				—	$9.3 \cdot 10^{-1}$	$9.3 \cdot 10^{-1}$	$9.6 \cdot 10^{-1}$	$2.9 \cdot 10^{-1}$
(2)	M2V1					—	$2.9 \cdot 10^{-2}$	$8.5 \cdot 10^{-1}$	$9.1 \cdot 10^{-1}$
	M2V2						—	$8.5 \cdot 10^{-1}$	$9.3 \cdot 10^{-1}$
	M2V3							—	$9.4 \cdot 10^{-1}$
	M2VN1								—

Cuadro 6.31: Diferencias máximas entre los calados (test 1.b (2D))

En estas tablas se aprecia que con la discretización más fina el comportamiento de todos los modelos que incluyen viscosidad es muy parecido. Se observa que no sucede lo mismo con la discretización más grosera y que, si se comparan los resultados obtenidos con las dos discretizaciones, se comprueba que el modelo M2VN1 proporciona un resultado, con la discretización peor, muy similar al obtenido con cualquiera de los modelos

		Discretización							
		$\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$ (1)				$\Delta x = \Delta x = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$ (2)			
		M2V1	M2V2	M2V3	M2VN1	M2V1	M2V2	M2V3	M2VN1
		Calados							
(1)	M2V1	—	$1.8 \cdot 10^{-4}$	$5.6 \cdot 10^{-3}$	$8.4 \cdot 10^{-3}$	$2.2 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	$2.3 \cdot 10^{-1}$	$1.8 \cdot 10^{-2}$
	M2V2		—	$5.5 \cdot 10^{-3}$	$8.2 \cdot 10^{-3}$	$2.2 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	$2.3 \cdot 10^{-1}$	$1.8 \cdot 10^{-2}$
	M2V3			—	$3.5 \cdot 10^{-3}$	$2.2 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	$2.3 \cdot 10^{-1}$	$1.9 \cdot 10^{-2}$
	M2VN1				—	$2.2 \cdot 10^{-1}$	$2.2 \cdot 10^{-1}$	$2.4 \cdot 10^{-1}$	$2.0 \cdot 10^{-2}$
(2)	M2V1					—	$9.8 \cdot 10^{-5}$	$1.3 \cdot 10^{-2}$	$2.0 \cdot 10^{-1}$
	M2V2						—	$1.3 \cdot 10^{-2}$	$2.0 \cdot 10^{-1}$
	M2V3							—	$2.2 \cdot 10^{-1}$
	M2VN1								—
		Componente x de las velocidades horizontales							
(1)	M2V1	—	$4.0 \cdot 10^{-4}$	$1.4 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$4.4 \cdot 10^{-1}$	$4.4 \cdot 10^{-1}$	$4.7 \cdot 10^{-1}$	$3.9 \cdot 10^{-2}$
	M2V2		—	$1.3 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$4.4 \cdot 10^{-1}$	$4.4 \cdot 10^{-1}$	$4.6 \cdot 10^{-1}$	$3.9 \cdot 10^{-2}$
	M2V3			—	$9.2 \cdot 10^{-3}$	$4.4 \cdot 10^{-1}$	$4.4 \cdot 10^{-1}$	$4.6 \cdot 10^{-1}$	$3.9 \cdot 10^{-2}$
	M2VN1				—	$4.4 \cdot 10^{-1}$	$4.4 \cdot 10^{-1}$	$4.6 \cdot 10^{-1}$	$3.9 \cdot 10^{-2}$
(2)	M2V1					—	$7.8 \cdot 10^{-4}$	$3.0 \cdot 10^{-2}$	$4.1 \cdot 10^{-1}$
	M2V2						—	$3.0 \cdot 10^{-2}$	$4.1 \cdot 10^{-1}$
	M2V3							—	$4.3 \cdot 10^{-1}$
	M2VN1								—
		Componente y de las velocidades horizontales							
(1)	M2V1	—	$3.6 \cdot 10^{-4}$	$1.1 \cdot 10^{-2}$	$1.7 \cdot 10^{-3}$	$1.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$3.0 \cdot 10^{-2}$
	M2V2		—	$1.1 \cdot 10^{-2}$	$1.7 \cdot 10^{-3}$	$1.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$3.0 \cdot 10^{-2}$
	M2V3			—	$1.2 \cdot 10^{-2}$	$1.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$2.8 \cdot 10^{-2}$
	M2VN1				—	$1.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$3.0 \cdot 10^{-2}$
(2)	M2V1					—	$4.2 \cdot 10^{-4}$	$4.6 \cdot 10^{-2}$	$9.2 \cdot 10^{-2}$
	M2V2						—	$4.7 \cdot 10^{-2}$	$9.2 \cdot 10^{-2}$
	M2V3							—	$1.4 \cdot 10^{-1}$
	M2VN1								—

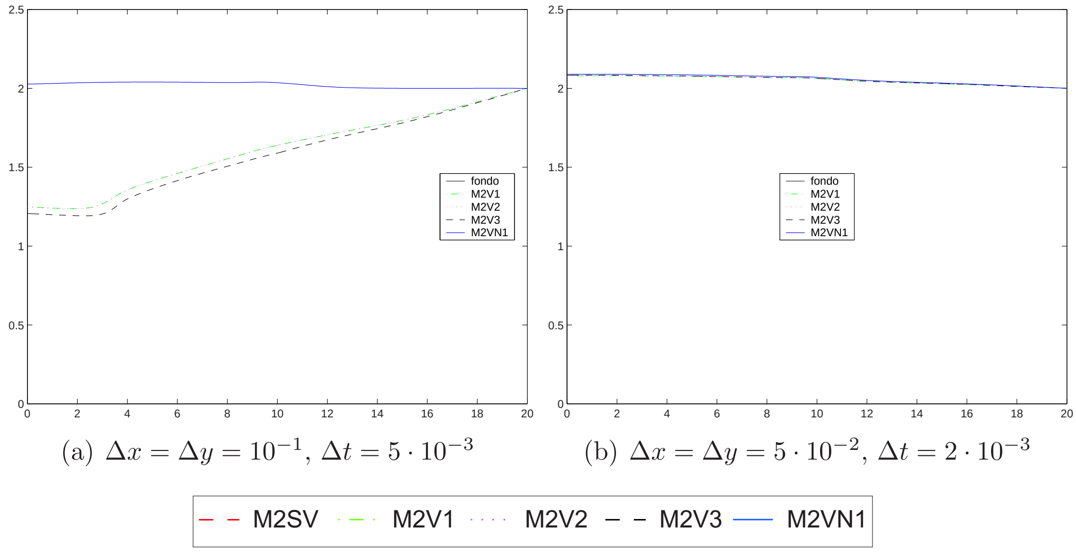
Cuadro 6.32: Diferencias medias entre los calados y las velocidades horizontales (test 1.b (2D))

con la malla más fina. Las diferencias medias entre los calados obtenidos con nuestro modelo con la discretización $\Delta x = \Delta x = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$ y los otros modelos con la discretización $\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$ son en torno a $2 \cdot 10^{-2}$ mientras que las diferencias con el resto de modelos es 10 veces mayor.

El hecho de que el modelo M2VN1 proporcione resultados razonables con una discretización más gruesa representa un resultado muy interesante, ya que el paso de una discretización a otra supone que el tiempo de cálculo se multiplique por 10.

En las figuras 6.31-6.36 se comparan los resultados obtenidos al resolver los cuatro modelos con viscosidad con las dos discretizaciones ya mencionadas.

Comenzamos por comparar los calados en $y = 0$ en el instante $t = 8$ s. En la figura 6.31 aparecen los resultados con cada discretización por separado mientras que en la figura 6.32 se representan los resultados obtenidos con las dos discretizaciones sobre la misma gráfica (los correspondientes a la más fina en trazo continuo y a la más grosera en trazo discontinuo, las líneas continuas correspondientes a los modelos M2V1 y M2V2 a penas pueden verse pues están bajo el trazo continuo negro del modelo M2V3, algo similar sucede con el trazo discontinuo verde que corresponde al modelo M2V1, en este caso está bajo la línea magenta del modelo M2V2).



(a) $\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$

(b) $\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$

Figura 6.31: Calados en el instante $t = 8$ s con distintas discretizaciones en $y = 0$ (test 1.b 2D)

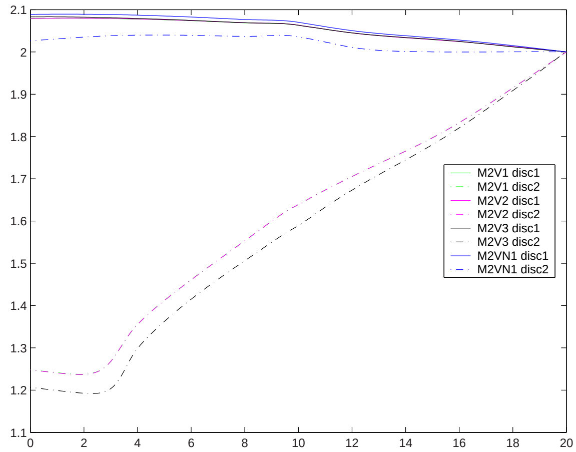
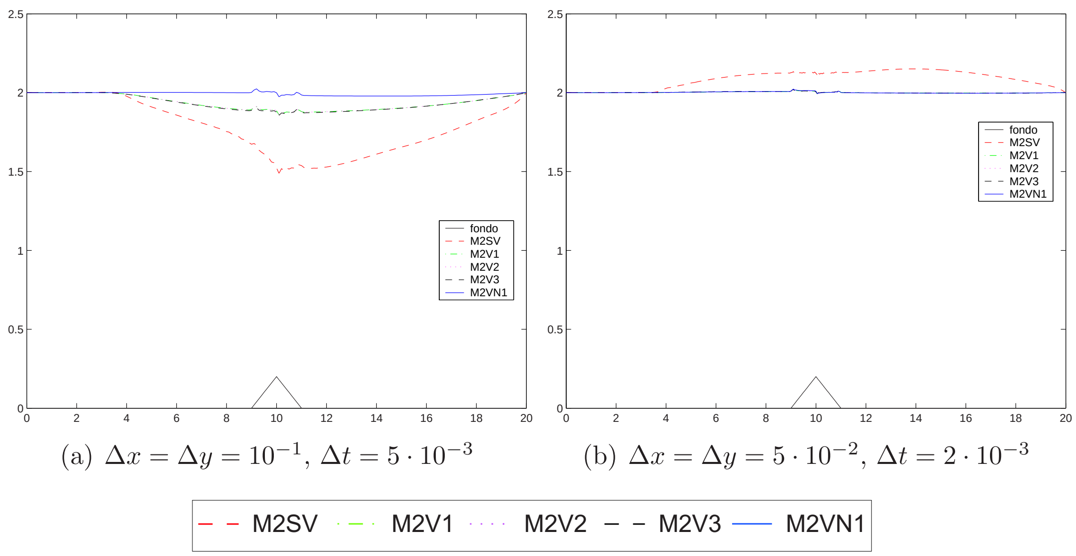


Figura 6.32: Comparación de calados en el instante t=8 s en $y = 0$ con las discretizaciones: disc1 ($\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$) y disc2 ($\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$) (test 1.b 2D)

El mismo tipo de comparación se plantea en las figuras 6.33-6.34 pero a lo largo del corte transversal $y = 1$ en el instante $t = 2$ s (de nuevo las líneas verdes y magentas correspondientes a los modelos M2V1 y M2V2 a penas se distinguen al estar bajo las líneas negras correspondientes al modelo M2V3).



(a) $\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$ (b) $\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$

Figura 6.33: Calados en el instante $t = 2$ s con distintas discretizaciones en $y = 1$

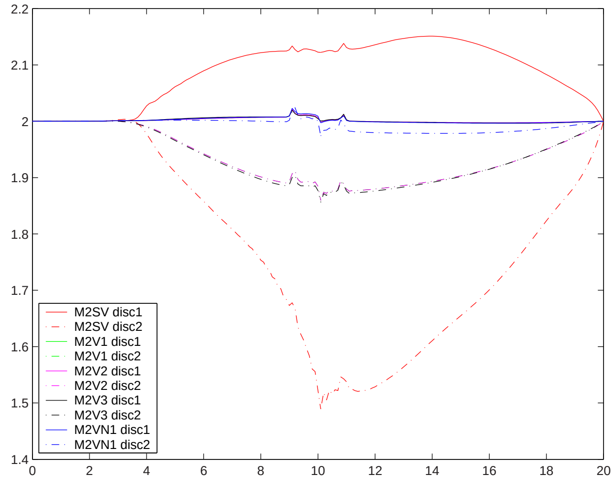
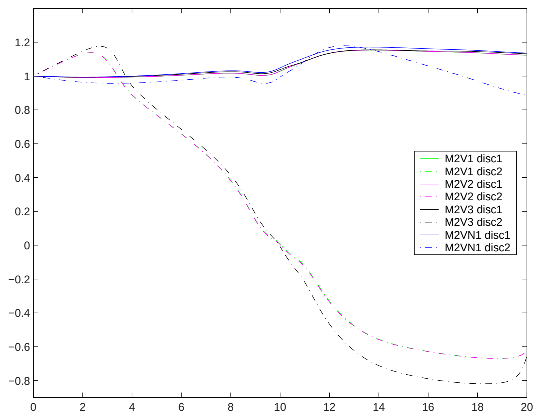


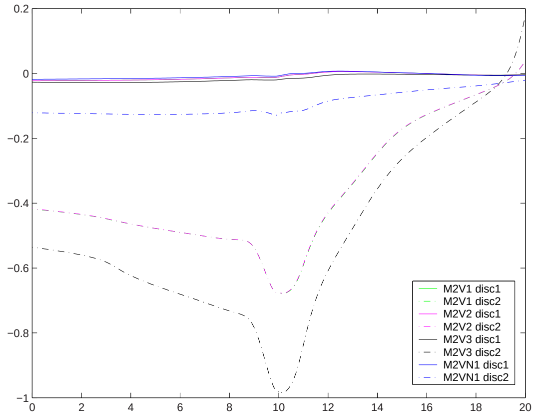
Figura 6.34: Comparación de calados a lo largo de $y = 1$ con las discretizaciones: disc1 ($\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$) y disc2 ($\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$) en el instante t=2 s (test 1.b 2D)

Para las dos componentes de la velocidad horizontal tenemos las figuras 6.35-6.36, en las que aparecen las dos discretizaciones a la vez (en la figura 6.35 para el instante $t = 0$ s en $y = 0$ y la figura 6.36 para $t = 2$ s en $y = 1$).

Como ya habíamos avanzado, el modelo M2VN1 obtiene para la discretización gruesa resultados razonables, mientras que el resto de los modelos necesitan de la discretización fina para ello.

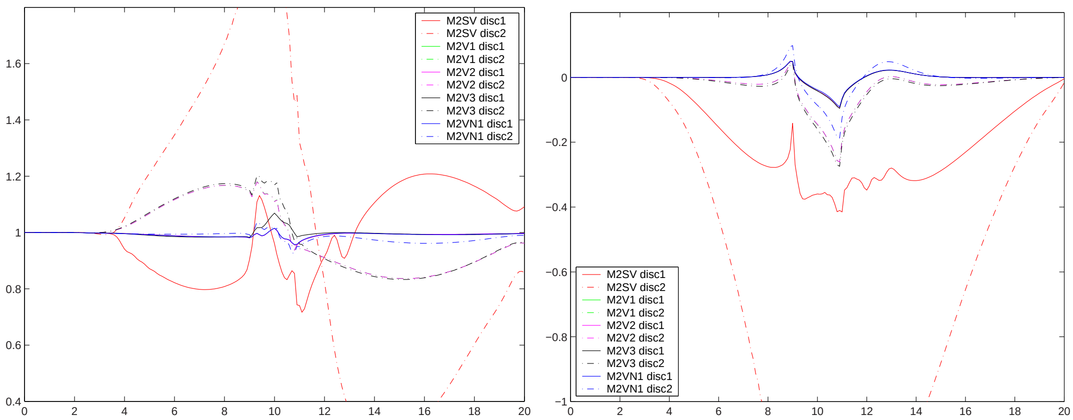


(a) Componente x (u)



(b) Componente y (v)

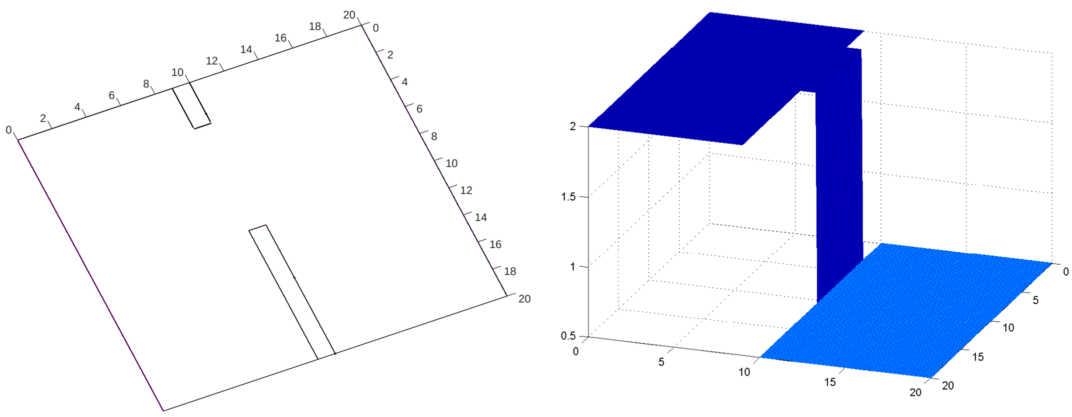
Figura 6.35: Comparación de velocidades horizontales a lo largo del corte longitudinal $y = 0$ con las discretizaciones: disc1 ($\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$) y disc2 ($\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$) en el instante t=8 s (test 1.b 2D)



(a) Componente x (u) (b) Componente y (v)

Figura 6.36: Comparación de velocidades horizontales a lo largo del corte longitudinal $y = 1$ con las discretizaciones: disc1 ($\Delta x = \Delta y = 5 \cdot 10^{-2}$, $\Delta t = 2 \cdot 10^{-3}$) y disc2 ($\Delta x = \Delta y = 10^{-1}$, $\Delta t = 5 \cdot 10^{-3}$) en el instante t=2 s (test 1.b 2D)

2. Consideramos ahora que se produce de forma instantánea una rotura de presa o la apertura de unas compuertas. En las simulaciones que presentamos, el canal aguas abajo de la presa o la compuerta tiene una altura de agua de 0.5 metros mientras que aguas arriba de la presa es de 2 metros. El dominio computacional consiste en un canal de 20 metros de largo por 20 metros de ancho. La grieta o las compuertas no simétricas tienen un ancho de 7.5 metros, y la presa tiene 1 metro de ancho en la dirección del flujo (véase la figura 6.37). La cuadrícula que se utiliza es de 201 puntos por 201 puntos, es decir, el paso de la malla es de 0.1 m en ambas direcciones.



(a) (b)

Figura 6.37: Dominio y condiciones iniciales (test 2)

Transcurridos 2 segundos desde la rotura de presa o de la apertura de compuertas el modelo M2VN1 proporciona la solución que mostramos en la figura 6.38 (para los calados) y en 6.39.a (campo de velocidades en ese instante). La figura 6.39 incluye el campo de velocidades obtenido con el modelo M2VN1 para $t = 2$ s y $t = 6$ s.

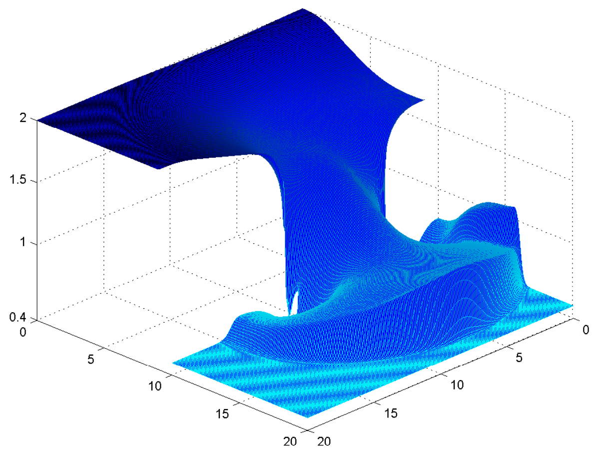
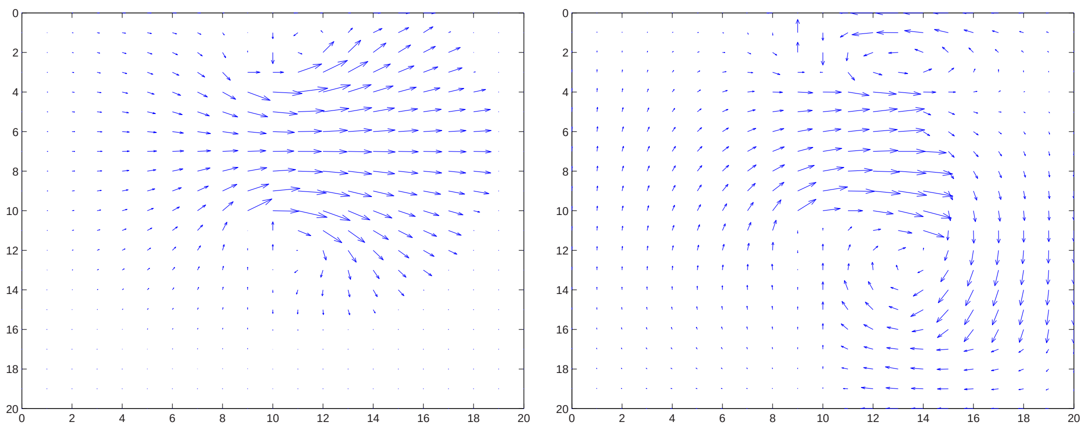


Figura 6.38: Calados obtenidos con el modelo M2VN1 en el instante $t = 2$ s (test 2 2D)



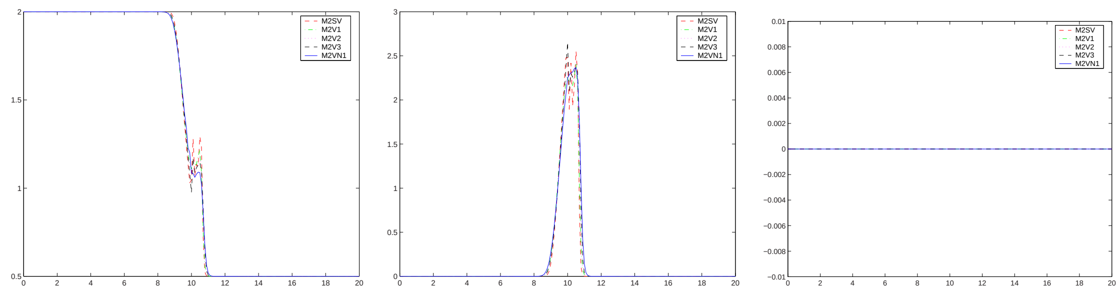
(a) $t = 2$ s

(b) $t = 6$ s

Figura 6.39: Campos de velocidades obtenidos con el modelo M2VN1

Los resultados obtenidos con los distintos modelos se pueden apreciar en las figuras 6.40-6.46 donde se muestran distintos cortes longitudinales y transversales.

Comenzamos por estudiar qué sucede a lo largo del corte longitudinal $y = 5$ m. En las figuras 6.40-6.42 observamos en tres instantes diferentes ($t = 0.2$ s, $t = 2$ s y $t = 6$ s) los calados y las velocidades horizontales calculados con los cinco modelos que tratamos de comparar.



(a) Calados (b) Velocidad horizontal (u) (c) Velocidad horizontal (v)

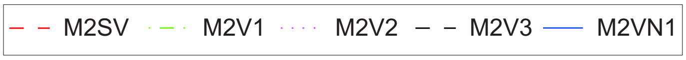
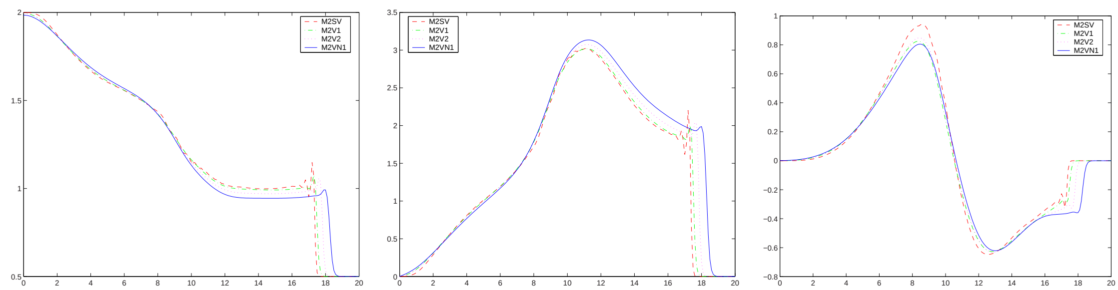


Figura 6.40: Corte longitudinal en $y = 5$ en el instante $t = 0.2$ s (test 2 2D)



(a) Calados (b) Velocidad horizontal (u) (c) Velocidad horizontal (v)

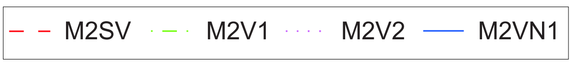
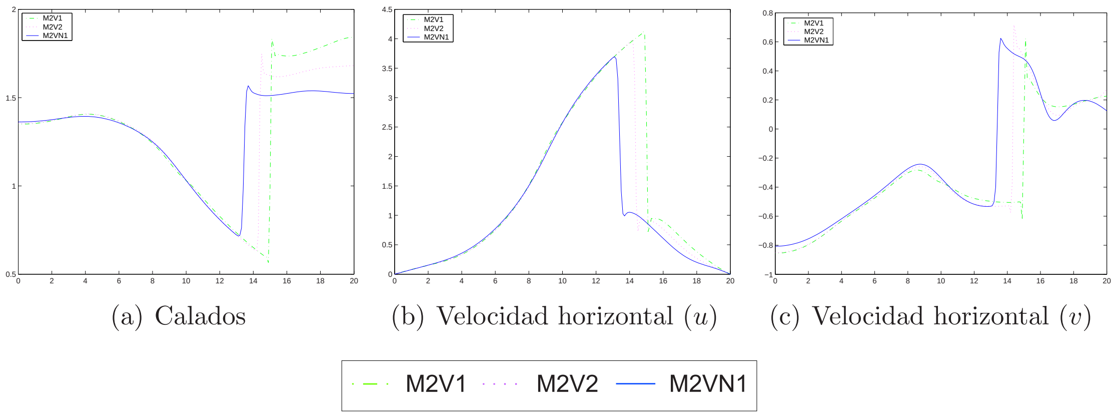


Figura 6.41: Corte longitudinal en $y = 5$ en el instante $t = 2$ s (test 2 2D)

El modelo M2V3 únicamente aparece en las gráficas correspondientes a $t = 0.2$ s (figura 6.40), pues para este modelo el calado se hace negativo a partir de ese instante (esto se debe a que el término de viscosidad incluye un sumando en el que aparece Δh, que aumenta mucho con este mallado ($\Delta x = \Delta y = 0.1$) y el paso de tiempo escogido ($\Delta t = 0.01$)). También se hace negativo el calado obtenido con el modelo M2SV a partir del instante $t = 2.4$ s, por lo que en la gráfica correspondiente a $t = 6$ s (figura 6.42) este modelo tampoco está presente.

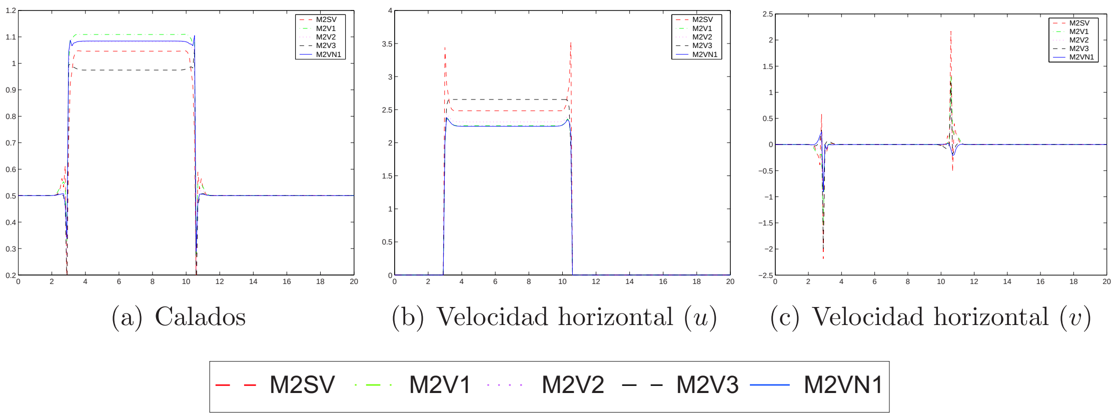


(a) Calados (b) Velocidad horizontal (u) (c) Velocidad horizontal (v)

Figura 6.42: Corte longitudinal en $y = 5$ en el instante $t = 6$ s (test 2 2D)

En las figuras 6.40-6.42 se aprecia cómo según transcurre el tiempo las diferencias entre modelos son mayores y que el comportamiento de todos los modelos es similar al visto en el test 2 unidimensional.

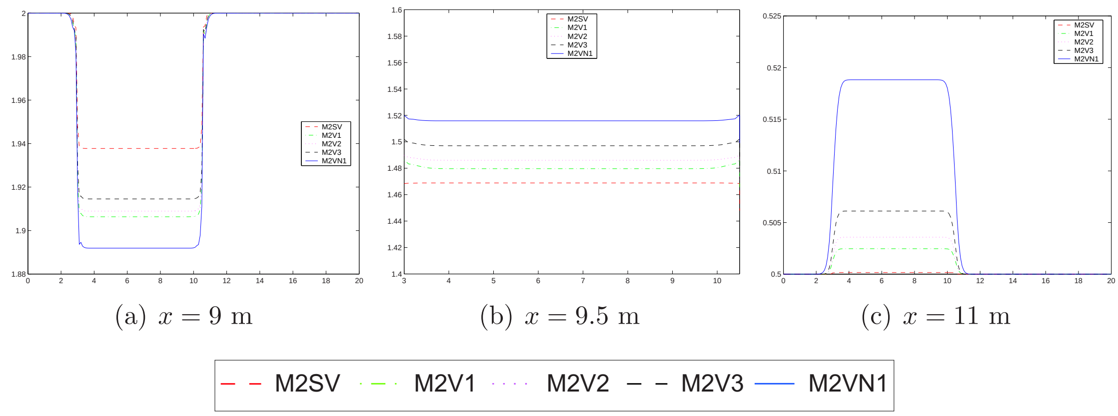
Realizamos a continuación algunos cortes transversales. Comenzamos por estudiar el calado y la velocidad horizontal en el instante $t = 0.2$ s a lo largo de $x = 10$ m (véase figura 6.43).



(a) Calados (b) Velocidad horizontal (u) (c) Velocidad horizontal (v)

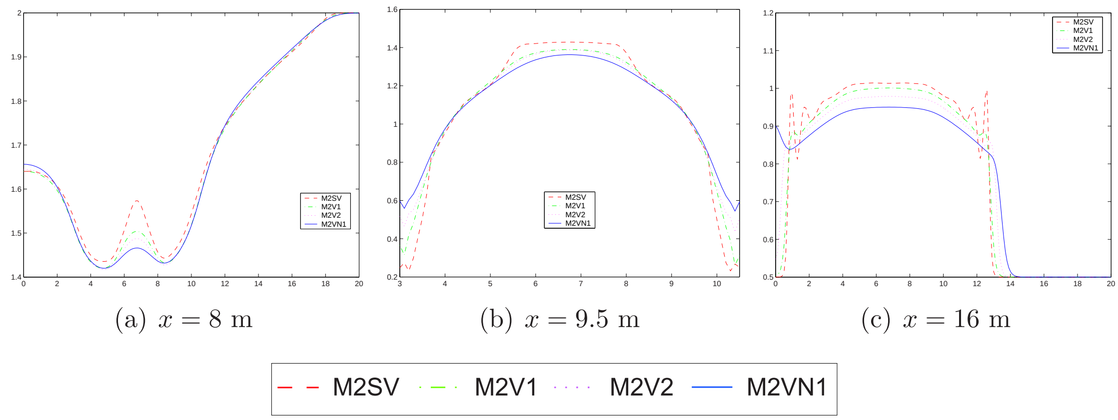
Figura 6.43: Corte transversal en $x = 10$ en el instante $t = 0.2$ s (test 2 2D)

En las figuras 6.44-6.46 presentamos en tres instantes distintos ($t = 0.2$ s, $t = 2$ s y $t = 6$ s) tres cortes transversales que nos permiten ver la evolución del frente de agua a lo largo del dominio. En el instante $t = 0.2$ s, la rotura de presa o apertura de compuertas se acaba de producir y los calados apenas varían respecto a la situación inicial salvo en los puntos de abertura de la presa y los más cercanos a éstos, por este motivo los cortes de la figura 6.44 se han realizado para $x = 9$ m, $x = 9.5$ m y $x = 11$ m.



(a) $x = 9$ m (b) $x = 9.5$ m (c) $x = 11$ m

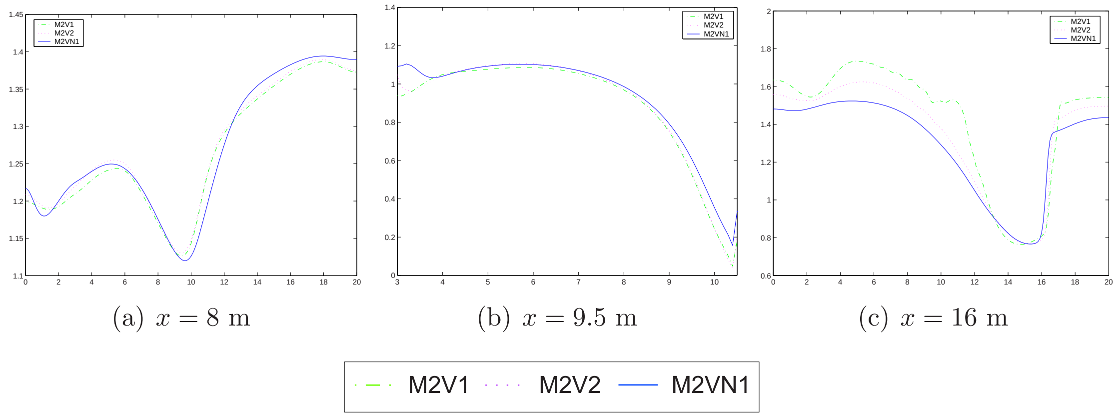
Figura 6.44: Calados cortes transversales en el instante $t = 0.2$ s (test 2 2D)



(a) $x = 8$ m (b) $x = 9.5$ m (c) $x = 16$ m

Figura 6.45: Calados cortes transversales en el instante $t = 2$ s (test 2 2D)

Para los instantes $t = 2$ s y $t = 6$ s (figuras 6.45 y 6.46) los cortes transversales corresponden a $x = 8$ m (corte en el canal aguas arriba de la presa), $x = 9.5$ m (corte en el estrechamiento de la presa) y $x = 16$ m (corte en el canal aguas abajo de la presa). Cuando han transcurrido 2 segundos el frente de agua avanza hacia la pared situada en $x = 20$ m mientras que a los 6 segundos la ola ya ha chocado contra esa pared, lo que explica la diferencia de perfiles que se pueden observar en las figuras 6.45 y 6.46. En ambas figuras se aprecia que el modelo M2VN1 es el que calcula una solución más “suave” y “regular” y que, como cabía esperar, el modelo M2V2 es el que se comporta de una manera más similar al modelo que proponemos dado que su término de viscosidad es el más parecido al nuestro.



(a) $x = 8$ m (b) $x = 9.5$ m (c) $x = 16$ m

Figura 6.46: Calados cortes transversales en el instante $t = 6$ s (test 2 2D)

6.3. Conclusiones

Hemos comparado nuestro modelo analítica y numéricamente con otros modelos que se pueden encontrar en la literatura. En el caso unidimensional, en la sección 6.1.1, el modelo M1VN se ha contrastado con otros cinco modelos: un modelo sin viscosidad (M1SV), un modelo con el mismo término de viscosidad que las ecuaciones de Navier-Stokes (M1V1), un modelo con un término de viscosidad similar (aunque distinto) al que nosotros proponemos (M1V2), un modelo con término de viscosidad $-\dfrac{\nu}{h^\varepsilon}\dfrac{\partial^2(h^\varepsilon u^\varepsilon)}{\partial(x^\varepsilon)^2}$ ((6.1.32)) y un modelo con el término de viscosidad de M1V2 multiplicado por cuatro ((6.1.38)).

La gran variedad de términos de viscosidad que aparecen en la literatura (aquí se han recogido cuatro además del que proponemos) sugiere que la justificación de cuál es el más adecuado carece del rigor necesario. Las simulaciones numéricas vistas en la sección 6.2, como comentaremos a continuación, confirman que el método de desarrollos asintóticos aporta dicho rigor y nos permiten proponer el término (6.1.43) como el correcto.

Al realizar comparaciones con soluciones exactas de las ecuaciones bidimensionales de Navier-Stokes (sección 6.2.3), hemos comprobado que nuestro modelo (M1VN) siempre obtiene resultados mejores (por ejemplo (6.2.13) con $a \neq 0$ y $b \neq 0$, (6.2.14)-(6.2.16)) o similares (por ejemplo (6.2.13) con $a = 0$ o $b = 0$) que los otros modelos, y que en varios ejemplos ((6.2.14)-(6.2.15)) el modelo M1VN llega a cometer errores cien veces menores que el resto de modelos, pues al refinar el mallado la aproximación que calcula nuestro modelo es más precisa, mientras que no mejora para los modelos M1SV, M1V1 y M1V2.

También se ha comprobado numéricamente que las expresiones obtenidas para la presión en los modelos de segundo orden ((3.7.5) y (5.7.7)) proporcionan una mejor aproximación de la presión exacta que la presión hidrostática en los casos en los que la presión exacta se aleja más de la presión hidrostática, aunque la mejora conseguida no es, en general, muy considerable.

Hemos resuelto con los modelos M1SV, M1V1, M1V2 y M1VN cuatro problemas (sección 6.2.4) que nos permiten analizar el comportamiento cualitativo de los mismos. Se observa que nuestro modelo reduce las oscilaciones en mayor grado que los otros, obteniéndose así soluciones más “suaves” y permitiendo resolver mejor los casos en que h se hace muy pequeña.

Hemos procedido del mismo modo en el caso bidimensional. Comenzamos por comparar los modelos bidimensionales que hemos obtenido (M2VN1 y M2VN2) con cuatro modelos diferentes: el modelo clásico sin viscosidad (M2SV) y las versiones bidimensionales de los modelos M1V1, M1V2 y (6.1.32) vistos en 6.1.1, es decir, los modelos M2V1, M2V2 y M2V3, observando que también en este caso existe gran variedad de términos difusivos en la literatura y comprobando con las comparaciones numéricas (véase las secciones 6.2.3 y 6.2.4) que también en el caso bidimensional el término de viscosidad más preciso resulta ser el que hemos obtenido utilizando el método de los desarrollos asintóticos.

Dentro de la sección 6.2.3.2, hemos dedicado un apartado al estudio de los dos distintos términos de Coriolis que hemos obtenido en el capítulo 5, encontrando que el nuevo término que obtenemos al no imponer la hipótesis de oceanografía dinámica permite obtener aproximaciones mucho más precisas.

Para finalizar, queremos señalar que nuestro modelo junto con el modelo M2V2 (o M1V2 en su versión unidimensional)) introduce un término de viscosidad cuya versión discretizada es muy similar a la viscosidad numérica que se introduce en algunos casos para resolver el modelo sin viscosidad (véanse por ejemplo las páginas 362-363 de [26]).

Capítulo 7

Comparación del nuevo modelo de aguas someras sin viscosidad con el clásico

En este capítulo pretendemos comparar los modelos de aguas someras sin viscosidad más generales propuestos en los capítulo 2 y 4 para dimensión uno y dos respectivamente (véanse (2.10.111) y (4.9.2)) con los modelos de aguas someras clásicos (sin viscosidad) que se pueden encontrar en la literatura. En primer lugar haremos una comparación analítica, observando en qué términos se diferencian las ecuaciones de unos y otros modelos y, en segundo lugar, los compararemos numéricamente, resolviendo los modelos para diferentes ejemplos.

7.1. Comparación analítica del modelo propuesto con el modelo clásico de aguas someras

En el capítulo 2 obtuvimos dos modelos. El primero de ellos, suponiendo que la vorticidad inicial era nula (véase (2.9.1)), coincidía con el modelo clásico de aguas someras sin viscosidad (véase (1.2.24) o en la página 581 de [56]):

$$\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial s}{\partial x} = 0 \end{aligned} \tag{7.1.1}$$

Al suponer que la vorticidad inicial no es nula el modelo obtenido es más general, permitiendo que la velocidad horizontal dependa de la profundidad a través de la vorticidad (véase (2.10.111)):

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial(\bar{u}^\varepsilon h^\varepsilon)}{\partial x^\varepsilon} = 0 \\
&\frac{\partial \bar{u}^\varepsilon}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x^\varepsilon} = -\frac{1}{\rho_0}\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon} - \frac{\partial s^\varepsilon}{\partial x^\varepsilon} g \\
&\frac{\partial \gamma^{0,\varepsilon}}{\partial t^\varepsilon} + \bar{u}^\varepsilon \frac{\partial \gamma^{0,\varepsilon}}{\partial x^\varepsilon} = 0 \\
&\tilde{u}^\varepsilon = \bar{u}^\varepsilon + (z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2})\,\gamma^{0,\varepsilon} \qquad (7.1.2) \\
&p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon) g \\
&w^\varepsilon = u^\varepsilon \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + (H^\varepsilon - z^\varepsilon)\frac{\partial u^\varepsilon}{\partial x^\varepsilon}
\end{aligned}$$

Al comparar (7.1.1) (que hemos escogido entre los modelos de aguas someras sin viscosidad que aparecen en la literatura, por ser el más habitual) y (7.1.2), observamos que nuestro modelo incluye el término $\frac{\partial p_s^\varepsilon}{\partial x^\varepsilon}$ (interesante cuando la presión atmosférica (p_s^ε) no es constante), introduce el cálculo de la vorticidad de modo que, si ésta no es nula, la velocidad horizontal depende de la profundidad a través precisamente de la vorticidad (no coincide con la velocidad media como sucede en los modelos clásicos) e incorpora una formulación explícita de la velocidad vertical w^ε que en el modelo clásico no siempre se indica.

Consideramos que la mayor contribución de nuestro modelo es que se trata de un modelo general (incluye términos que otros modelos desprecian) y que incluye la posibilidad de que la velocidad horizontal dependa de la profundidad en caso de que la vorticidad no sea nula.

En el caso de que el dominio de partida sea tridimensional en lugar de bidimensional, en el capítulo 4, hemos propuesto también dos modelos (véanse (4.9.2) y (4.10.100)). El primero de ellos generaliza el modelo (7.1.2):

$$\begin{aligned}
&\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \mathrm{div}(h^\varepsilon \vec{\bar{\mathbf{u}}}^\varepsilon) = \mathbf{0} \\
&\frac{\partial \vec{\bar{\mathbf{u}}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\bar{\mathbf{u}}}^\varepsilon \cdot \vec{\bar{\mathbf{u}}}^\varepsilon + g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon \\
&\frac{\partial \vec{\gamma}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\gamma}^\varepsilon \cdot \vec{\bar{\mathbf{u}}}^\varepsilon - (\nabla \vec{\bar{\mathbf{u}}}^\varepsilon)^T \cdot \vec{\gamma}^\varepsilon = 2\phi \vec{\mathbf{F}}_V^\varepsilon \\
&u^\varepsilon = \bar{u}^\varepsilon + \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_2^\varepsilon, \quad v^\varepsilon = \bar{v}^\varepsilon - \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_1^\varepsilon
\end{aligned}$$

$$\vec{\bar{\mathbf{u}}}^{\varepsilon} = \vec{\mathbf{u}}^{\varepsilon}|_{z^{\varepsilon}=H^{\varepsilon}}$$

$$p^{\varepsilon} = p_s^{\varepsilon} + \rho_0(s^{\varepsilon} - z^{\varepsilon})\left[g - 2\phi\left(\cos\varphi^{\varepsilon}\right)\bar{u}^{\varepsilon}\right]$$

$$w^{\varepsilon} = \vec{\bar{\mathbf{u}}}^{\varepsilon} \cdot \nabla H^{\varepsilon} + (H^{\varepsilon} - z^{\varepsilon})(\operatorname{div}\vec{\bar{\mathbf{u}}}^{\varepsilon} - \vec{\gamma}^{\varepsilon} \cdot \operatorname{rot} H^{\varepsilon}) + \frac{1}{2}(H^{\varepsilon} - z^{\varepsilon})^2 \operatorname{rot}\vec{\gamma}^{\varepsilon} \qquad (7.1.3)$$

donde $\operatorname{rot}\vec{\alpha}^{\varepsilon} = \dfrac{\partial\alpha_2^{\varepsilon}}{\partial x^{\varepsilon}} - \dfrac{\partial\alpha_1^{\varepsilon}}{\partial y^{\varepsilon}}$, $\operatorname{rot}\alpha = \left(\dfrac{\partial\alpha^{\varepsilon}}{\partial y^{\varepsilon}}, -\dfrac{\partial\alpha^{\varepsilon}}{\partial x^{\varepsilon}}\right)$,

$$\vec{\mathbf{F}}_C^{\varepsilon} = \begin{pmatrix} (\operatorname{sen}\varphi^{\varepsilon})\bar{v}^{\varepsilon} + \cos\varphi^{\varepsilon}\left(\dfrac{\partial(h^{\varepsilon}\bar{u}^{\varepsilon})}{\partial x^{\varepsilon}} + \dfrac{h^{\varepsilon}}{2}\dfrac{\partial\bar{v}^{\varepsilon}}{\partial y^{\varepsilon}} - \bar{v}^{\varepsilon}\dfrac{\partial H^{\varepsilon}}{\partial y^{\varepsilon}}\right) \\ -(\operatorname{sen}\varphi^{\varepsilon})\bar{u}^{\varepsilon} + \dfrac{h^{\varepsilon}}{2}\dfrac{\partial}{\partial y^{\varepsilon}}[(\cos\varphi^{\varepsilon})\bar{u}^{\varepsilon}] + \dfrac{\partial s^{\varepsilon}}{\partial y^{\varepsilon}}[(\cos\varphi^{\varepsilon})\bar{u}^{\varepsilon}] \end{pmatrix}$$

y

$$\vec{\mathbf{F}}_V^{\varepsilon} = \begin{pmatrix} \dfrac{\partial}{\partial y^{\varepsilon}}[(\cos\varphi^{\varepsilon})\bar{u}^{\varepsilon}] + (\operatorname{sen}\varphi^{\varepsilon})\gamma_2^{\varepsilon} \\ -(\operatorname{sen}\varphi^{\varepsilon})\gamma_1^{\varepsilon} + (\cos\varphi^{\varepsilon})\dfrac{\partial\bar{v}^{\varepsilon}}{\partial y^{\varepsilon}} \end{pmatrix}$$

Observamos que las componentes de la velocidad horizontal dependen de la profundidad si la vorticidad es no nula.

Para obtener el segundo modelo ((4.10.100)) hemos impuesto la hipótesis oceanográfica sobre la aceleración de Coriolis ((1.1.9)), lo que provoca que las dos primeras componentes de la vorticidad se anulen, por lo que las velocidades horizontales no dependen de la profundidad (y son, por tanto, iguales a las velocidades medias) y recuperamos de ese modo el modelo clásico de aguas someras (véase (1.2.23), [5] (pág. 3) o [108] (pág. 456)):

$$\frac{\partial h^{\varepsilon}}{\partial t^{\varepsilon}} + \operatorname{div}(h^{\varepsilon}\vec{\mathbf{u}}^{\varepsilon}) = 0$$

$$\frac{\partial\vec{\mathbf{u}}^{\varepsilon}}{\partial t^{\varepsilon}} + \nabla\vec{\mathbf{u}}^{\varepsilon} \cdot \vec{\mathbf{u}}^{\varepsilon} + g\nabla h^{\varepsilon} = -\frac{1}{\rho_0}\nabla p_s^{\varepsilon} - g\nabla H^{\varepsilon} + 2\phi\vec{\mathbf{F}}_C^{\varepsilon}$$

$$w^{\varepsilon} = \vec{\mathbf{u}}^{\varepsilon} \cdot \nabla H^{\varepsilon} + (H^{\varepsilon} - z^{\varepsilon})\operatorname{div}\vec{\mathbf{u}}^{\varepsilon}$$

$$p^{\varepsilon} = p_s^{\varepsilon} + \rho_0(s^{\varepsilon} - z^{\varepsilon})g \qquad (7.1.4)$$

donde

$$\vec{\mathbf{F}}_C^{\varepsilon} = (\operatorname{sen}\varphi)\begin{pmatrix} v^{\varepsilon} \\ -u^{\varepsilon} \end{pmatrix}$$

Se observa que el modelo que proponemos (7.1.3) (al igual que el correspondiente en dimensión uno) incluye términos que otros modelos no consideran: ∇p_s^{ε} cuando p_s^{ε} no es constante, una formulación explícita de la velocidad vertical w^{ε} y velocidad horizontal dependiente de la profundidad si la vorticidad es no nula.

A continuación compararemos las ecuaciones de Euler (véase (4.1.2)-(4.1.5)) y los modelos (7.1.3) y (7.1.4) para un ejemplo sencillo que, en nuestra opinión, ilustra en qué sentido el modelo (7.1.3) mejora el modelo clásico (7.1.4).

Si escogemos

$$u = u_0 + (z - A/2)\gamma_2, \quad v = v_0 - (z - A/2)\gamma_1, \quad w = 0,$$
$$H = 0, \quad s = h = A > 0 \tag{7.1.5}$$
$$p = p_s + \rho(A - z)g$$

con $u_0, v_0, A, \gamma_1, \gamma_2, p_s \in \mathbb{R}$, y si suponemos $\phi = 0$, entonces (7.1.5) es solución de las ecuaciones de Euler y de (7.1.3), pero no es solución de (7.1.4) (pues la solución de (7.1.4) no puede depender de z) salvo que $\gamma_1 = \gamma_2 = 0$. Es decir, los modelos (7.1.3) y (7.1.4) capturan la solución exacta de las ecuaciones de Euler cuando las velocidades son constantes, pero sólo el modelo (7.1.3) es capaz de calcular la solución de las ecuaciones de Euler (7.1.5), ya que en ese caso (7.1.4) únicamente obtiene las velocidades medias.

7.2. Comparación numérica

En esta sección compararemos numéricamente el modelo clásico de aguas someras sin viscosidad (**M2SV**) y el modelo que hemos propuesto en el capítulo 4 ((7.1.3)), que denominaremos **M2SVN**. Para ello utilizaremos, como hicimos en el capítulo 6, el esquema de MacCormack descrito en la sección 6.2.2.

7.2.1. Comparación con soluciones analíticas

Si tratamos de aproximar la solución (7.1.5) con $\gamma_1 = \gamma_2 = 0$, ambos modelos (M2SV y M2SVN) la calculan de forma exacta, por lo que probamos a resolver (7.1.5) para los siguientes valores:

$$A = 1, \ u_0 = 6, \ v_0 = 1/2, \ \gamma_1 = 1, \ \gamma_2 = 2,$$
$$p_s = 91542.64, \ \rho_0 = 998.2, \ g = 9.8$$

Si resolvemos los modelos M2SV y M2SVN para el ejemplo anterior en el dominio $[0, 10] \times [0, 2]$ con pasos $\Delta x = \Delta y = 0.1$, en el intervalo temporal $[0, 10]$ con $\Delta t = 0.01$ e imponiendo condiciones de contorno Dirichlet en $x = 0$ y $x = 10$, obtenemos las acotaciones para los errores en norma 2 que se muestran en el cuadro 7.1.

Se puede apreciar para este ejemplo como el modelo M2SVN aproxima exactamente la solución de las ecuaciones de Euler, mientras que el modelo clásico únicamente lo hace en el punto medio del calado (0.5 m), cometiendo errores de un tamaño

Profundidad (z)	Modelo	Acotación error h	Acotación error u	Acotación error v
0	M2SV	0.0	4.6	2.3
	M2SVN	0.0	0.0	0.0
0.25	M2SV	0.0	2.3	1.2
	M2SVN	0.0	0.0	0.0
0.5	M2SV	0.0	0.0	0.0
	M2SVN	0.0	0.0	0.0
1	M2SV	0.0	4.6	2.3
	M2SVN	0.0	0.0	0.0

Cuadro 7.1: Acotación errores ejemplo (7.1.5)

considerable para el resto de los valores de z (mayores cuanto más nos alejamos de $z = 0.5$).

Consideraremos ahora otro ejemplo en el que la solución exacta de las ecuaciones de Euler (suponiendo $\phi = 0$) verifica que:

$$
\begin{aligned}
&u = -4 + \frac{99}{100}x + \frac{99}{50}y + 2z, \\
&v = 2 - \frac{99}{200}x - \frac{99}{100}y - z, \\
&w = 0, \\
&H = 0, \quad h = 2 + \frac{x}{100} + \frac{y}{50}, \\
&p = p_s + \rho_0(h - z)g, \quad p_s = K(t) - \rho_0 g h
\end{aligned}
\tag{7.2.1}
$$

En este caso M2SVN nos da la solución exacta (con $\gamma_1 = 1$ y $\gamma_2 = 2$) y M2SV no. Resolvemos numéricamente ambos modelos en el dominio $[0, 10] \times [0, 2]$ con pasos $\Delta x = \Delta y = 0.2$, y el intervalo temporal que se toma es $[0, 10]$, con $\Delta t = 5. \cdot 10^{-3}$. Se imponen condiciones de contorno Dirichlet en $x = 0$, $x = 10$, $y = 0$ e $y = 2$ y obtenemos las acotaciones para los errores en norma 2 que se indican en el cuadro 7.2. En este cuadro se aprecia de nuevo cómo el modelo M2SVN aproxima la solución con mucha más precisión que el modelo clásico, que comete errores que llegan a ser 10^4 veces mayores que los obtenidos con nuestro modelo.

7.2.1.1. Presión de segundo orden

En las secciones 2.10.4 y 4.9, al proponer los nuevos modelos sin viscosidad, se comentó la posibilidad de utilizar, en lugar de la expresiones de orden uno para

Profundidad (z)	Modelo	Acotación error h	Acotación error u	Acotación error v
0	M2SV	$3.2 \cdot 10^{-3}$	9.8	4.9
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$
0.25	M2SV	$3.2 \cdot 10^{-3}$	7.4	3.7
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$
0.5	M2SV	$3.2 \cdot 10^{-3}$	5.1	2.5
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$
1	M2SV	$3.2 \cdot 10^{-3}$	$3.6 \cdot 10^{-1}$	$1.8 \cdot 10^{-1}$
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$
1.5	M2SV	$3.2 \cdot 10^{-3}$	4.4	2.2
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$
2	M2SV	$3.2 \cdot 10^{-3}$	9.1	4.6
	M2SVN	$3.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$

Cuadro 7.2: Acotación errores ejemplo (7.2.1)

la presión ((2.7.4), (4.7.65)) (y dado que no es necesario conocer la velocidad de segundo orden para ello) las mejoras obtenidas en las aproximaciones de orden dos (2.10.112) para dimensión uno y (4.9.3) para dimensión dos, que escribimos en forma vectorial a continuación:

$$
\begin{aligned}
p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon) &\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon + \frac{\partial}{\partial t^\varepsilon}\left(\vec{\mathbf{u}}^\varepsilon \cdot \nabla H^\varepsilon\right) + \vec{\mathbf{u}}^\varepsilon \cdot \nabla\left(\vec{\mathbf{u}}^\varepsilon \cdot \nabla H^\varepsilon\right)\right] \\
&- \frac{\rho_0}{2}\left[(h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2\right]\left[2\phi\left(\cos\varphi^\varepsilon\right)\gamma_2^{0,\varepsilon} + \frac{\partial}{\partial t^\varepsilon}\left(\operatorname{div}\vec{\mathbf{u}}^\varepsilon\right)\right. \\
&\left. + \vec{\mathbf{u}}^\varepsilon \cdot \nabla\left(\operatorname{div}\vec{\mathbf{u}}^\varepsilon\right) - \left(\operatorname{div}\vec{\mathbf{u}}^\varepsilon\right)^2\right] \qquad (7.2.2)
\end{aligned}
$$

En esta sección deseamos mostrar que esta corrección de la presión proporciona una mejor aproximación cuando la presión exacta no se parece a la hidrostática. Para ello consideramos la siguiente solución de las ecuaciones de Euler:

$$
\begin{aligned}
&u = u_0, \quad v = v_0, \quad w = u_0(2ax + by + c) + v_0(bx + 2dy + e), \\
&H = ax^2 + bxy + cx + dy^2 + ey + f, \quad h = A \qquad (7.2.3) \\
&p = p_s + \rho_0(s - z)(g + 2u_0^2 a + 2u_0 v_0 b + 2v_0^2 d), \\
&p_s = K - \rho_0 s(g + 2u_0^2 a + 2u_0 v_0 b + 2v_0^2 d)
\end{aligned}
$$

que no es solución de los modelos de aguas someras que estamos considerando si a, b o d son distintos de cero. Escogemos los siguientes valores para las constantes:

$$A = 1, \quad u_0 = 5, \quad v_0 = 1,$$

$$a = -4 \cdot 10^{-2}, \quad b = 0, \quad c = 4 \cdot 10^{-1}, \quad d = 0, \quad e = 0, \quad f = 0$$

$$K = 101325, \quad \rho_0 = 998.2, \quad g = 9.8$$

Calculamos la presión hidrostática (p_h) y la presión de segundo orden dada por (7.2.2) (p_2) a partir de los calados y velocidades horizontales exactos y calculados. Las acotaciones para los errores en norma dos que obtenemos, se muestran en el cuadro 7.3, donde se aprecia que la presión p_2 calculada a partir del calado y la velocidad horizontal exactos aproxima mucho mejor la presión exacta que la presión hidrostática calculada de la misma forma.

Profundidad (z)	Acotación error p_h exacta	Acotación error p_h calculada	Acotación error p_2 exacta	Acotación error p_2 calculada
0	1.2	1.6	$2.7 \cdot 10^{-16}$	$6.8 \cdot 10^{-1}$
0.5	1.2	1.8	$3.8 \cdot 10^{-16}$	$8.3 \cdot 10^{-1}$
0.75	1.2	1.9	$4.0 \cdot 10^{-16}$	$9.6 \cdot 10^{-1}$

Cuadro 7.3: Acotación errores de la presión ejemplo (7.2.3)

Los errores cometidos al calcular p_2 a partir de los calados y velocidades horizontales que se obtienen al resolver M2SVN son grandes (sobre todo al tener en cuenta que si se calcula p_2 de forma exacta los errores entonces son muy pequeños) debido a que los errores para h, u y v también lo son.

Observación 7.1 *En otros ejemplos en los que hemos comparado la presión de orden dos (p_2) con la presión hidrostática (p_h), aunque las diferencias no son tan significativas como en el ejemplo del cuadro 7.3, se sigue observando que cuando la presión exacta se aleja de la hidrostática, p_2 obtiene una mejor aproximación.*

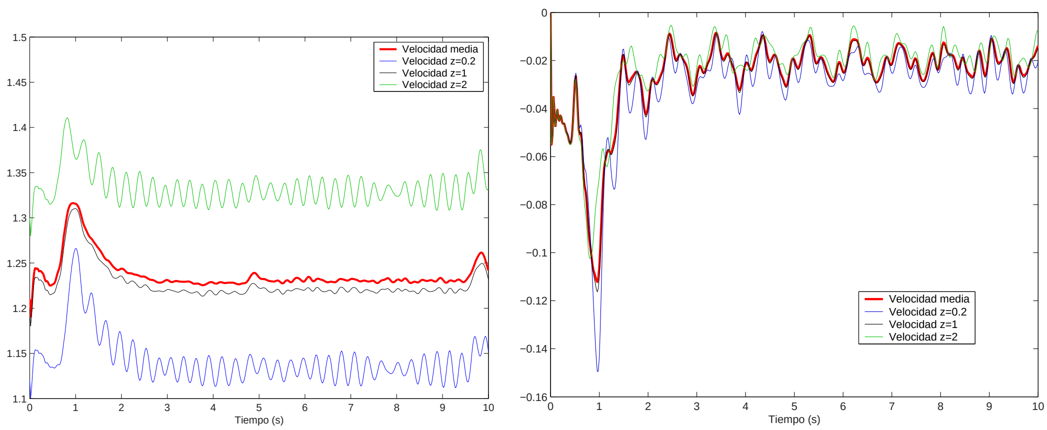
7.2.2. Otras comparaciones

Intentaremos ahora comparar el comportamiento numérico de los modelos M2SV y M2SVN, volviendo para ello sobre el test 1 de la sección 6.2.4.2. Se considera un canal de 20 m de largo por 2 m de ancho; situada en el fondo del canal se encuentra una pirámide (ver figura 6.25) de altura 0.2 m, la altura inicial del agua es de 2 m, el flujo entra por $x = 0$ de forma constante con una velocidad de 1 m/s ($u = 1$, $v = 0$) y se impone el calado aguas abajo y la condición de no penetración para las paredes laterales del canal ($v = 0$); la vorticidad aguas arriba (por donde entra el flujo) se mantiene constante tomando los valores $\gamma_1 = 0$ y $\gamma_2 = 0.1$.

Para la resolución de este problema hemos utilizado un mallado con pasos $\Delta x = \Delta y = 5 \cdot 10^{-2}$ y paso temporal $\Delta t = 2.5 \cdot 10^{-3}$, pues con mallas más gruesas se producen muchas oscilaciones.

Nuestro interés se centra en estudiar las diferencias entre las velocidades que calcula el modelo M2SV y el nuestro, dado que los calados que obtienen ambos modelos coinciden. Por ello elegimos algunos puntos del dominio y estudiamos cómo evoluciona en esos puntos la velocidad media (que es la que calcula el modelo M2SV) y la velocidad que calcula el modelo M2SVN en $z = 0$ ó $z = 0.2$ (fondo), $z = 1$ y $z = 2$.

Consideramos en primer lugar el punto (10,1) situado en el vértice de la pirámide. En la figura 7.1 podemos ver la evolución en el tiempo de las dos componentes de la velocidad horizontal en ese punto. Se aprecia que la componente x de la velocidad horizontal (u) en el fondo ($z = 0.2$ en este punto) y en $z = 2$ (prácticamente la superficie) oscilan mucho.



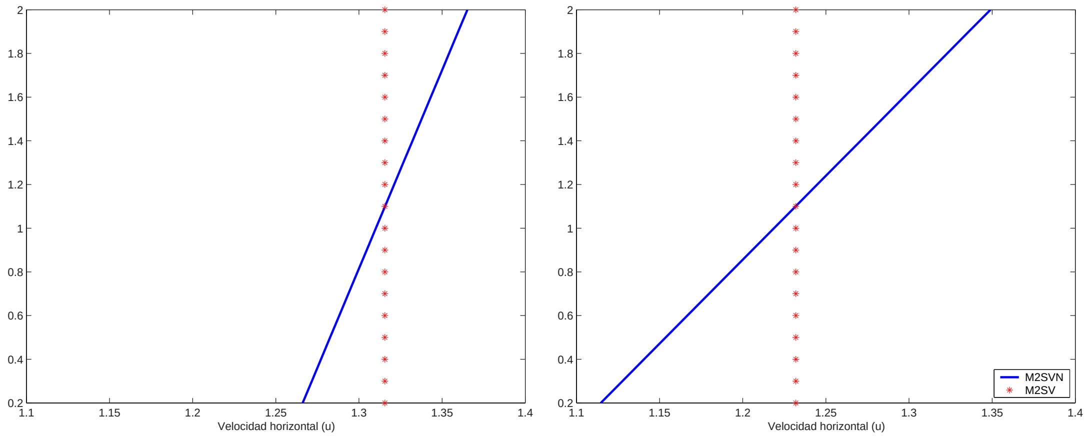
(a) Componente x (u) (b) Componente y (v)

Figura 7.1: Evolución en el tiempo de la velocidad horizontal en el punto (10,1)

En la figura 7.2 tenemos la velocidad media y la velocidad calculada por el modelo M2SVN en el punto (10,1) para dos instantes, $t = 1$ s y $t = 9$ s. Se observa que la pendiente del perfil de velocidades horizontales (u) para $t = 1$ s es más del doble que para $t = 9$ s.

En la figura 7.1 también podemos ver la evolución de la componente y de la velocidad horizontal (v). En este caso advertimos que aunque sigue habiendo muchas oscilaciones, las variaciones son mucho menores que en el caso de la componente x de la velocidad, por lo que únicamente incluimos (en la figura 7.3) la comparación entre la velocidad media y la velocidad del modelo M2SVN en el instante en el que la pendiente de esta velocidad es mayor, es decir, en $t = 1$.

Estudiamos ahora lo que sucede en dos puntos menos singulares que el vértice de la pirámide. Escogemos el punto (9,0.5) (situado en el canal antes de que el flujo



(a) $t = 1$ s (b) $t = 9$ s

Figura 7.2: Variación de la componente x de la velocidad horizontal (u) con la profundidad en el punto (10,1)

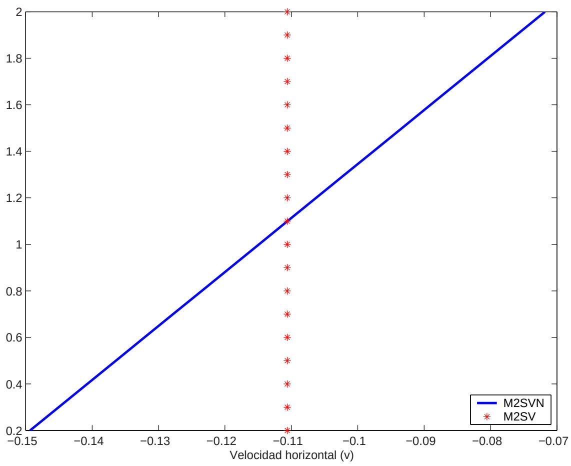
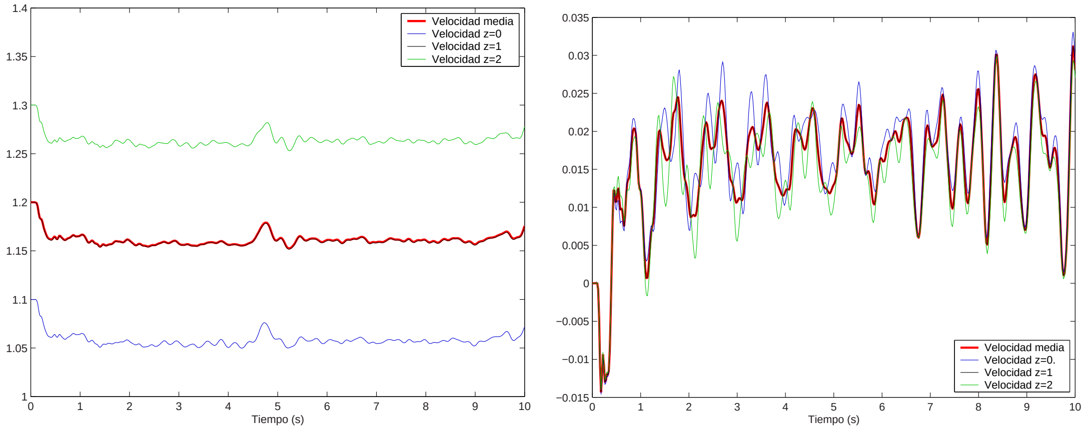


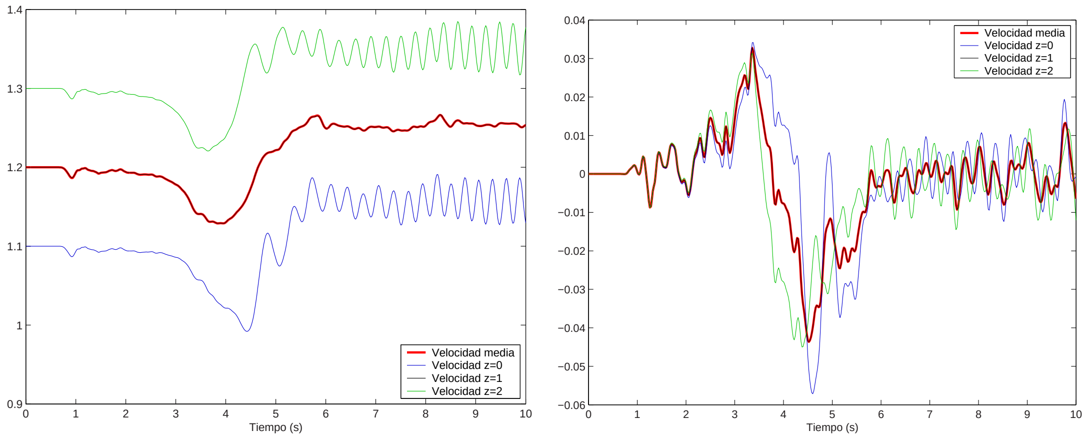
Figura 7.3: Variación de la componente y de la velocidad horizontal (v) con la profundidad en el punto (10,1) en el instante $t = 1$ s

pase sobre la pirámide pero muy cercano a ésta) y el punto (15,1.5) (situado en el canal después de la pirámide más alejado de ésta). La evolución en el tiempo de la velocidad horizontal para estos puntos puede verse en las figuras 7.4-7.5. En ambos casos se aprecia que las velocidades medias prácticamente coinciden con la velocidad para $z = 1$, pues en estos casos la profundidad media es aproximadamente $z = 1$. Observamos que en el punto anterior a la pirámide ((9,0.5)) la velocidad oscila mucho menos que en los puntos (10,1) y (15,1.5), que las oscilaciones mayores se producen en el punto (15,1.5), pero que éstas no comienzan hasta el instante $t = 5$ s.



(a) Componente x (u) (b) Componente y (v)

Figura 7.4: Evolución en el tiempo de la velocidad horizontal en el punto (9,0.5)



(a) Componente x (u) (b) Componente y (v)

Figura 7.5: Evolución en el tiempo de la velocidad horizontal en el punto (15,1.5)

7.3. Conclusiones

Hemos comparado nuestro modelo sin viscosidad analítica y numéricamente con el modelo clásico sin viscosidad.

Hemos visto que los modelos M2SV y M2SVN capturan la solución exacta de las ecuaciones de Euler cuando las velocidades son constantes, pero sólo el modelo M2SVN es capaz de calcular la solución de las ecuaciones de Euler (7.1.5) si γ_1 y γ_2 son distintas de cero, ya que en ese caso M2SV únicamente obtiene las velocidades medias.

Al realizar comparaciones con soluciones exactas de las ecuaciones tridimensionales de Euler (sección 7.2.1) hemos comprobado que nuestro modelo (M2SVN) siempre obtiene resultados mejores (cuadros 7.1 y 7.2) cuando la solución depende de z o iguales (en caso de que la solución sea independiente de z) que el modelo clásico de aguas someras, que llega a cometer errores muy grandes en valores de z alejados del valor de profundidad media.

También hemos comparado numéricamente la expresión de orden dos para la presión ((7.2.2)) con la presión hidrostática y hemos obtenido que, cuando la presión exacta de las ecuaciones de Euler se aleja de la hidrostática, la presión de orden dos ((7.2.2)) ofrece una mejor aproximación que la presión hidrostática.

Además, hemos resuelto con los modelos M2SV y M2SVN uno de los problemas planteados en la sección 6.2.4.2 comparando en distintos puntos la velocidad media que calcula M2SV con la velocidad obtenida con el modelo M2SVN, que permite captar la variación que en función de la profundidad sufre la velocidad horizontal.

Capítulo 8

Conclusiones

Hemos visto a lo largo de los capítulos 2 al 5 como el método de desarrollos asintóticos permite obtener diferentes modelos de aguas someras con y sin viscosidad, unidimensionales y bidimensionales.

En el capítulo 2, al considerar que la vorticidad inicial es nula, recuperamos y generalizamos el modelo clásico de aguas someras sin viscosidad obteniendo el modelo (2.9.1). En el caso de suponer que la vorticidad inicial no es nula, obtenemos una modificación interesante del modelo anterior en la que la velocidad horizontal depende de forma explícita de la variable z^ε a través de la vorticidad ((2.10.111)).

En el capítulo 3 hemos obtenido distintos modelos de aguas someras con viscosidad. Al menos formalmente, al aumentar el orden de aproximación mejoramos la aproximación del modelo, pero de hecho alguna de las ecuaciones (como la de continuidad) no mejora el orden de precisión. Si a ello añadimos que la resolución de los modelos de orden dos es mucho más complicada que la del modelo de orden uno, nos decidimos a proponer el modelo (3.7.1)-(3.7.4).

En el capítulo 4 actuamos de forma similar a como lo hicimos en el capítulo 2, pero partiendo ahora de las ecuaciones tridimensionales de Euler. Se obtienen varios modelos y, mediante razonamientos análogos a los del párrafo anterior (y expuestos con más detalle en la sección 4.9), finalmente proponemos el modelo (4.9.2):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = \mathbf{0}$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon + g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon$$

$$\frac{\partial \vec{\gamma}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\gamma}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - (\nabla \vec{\mathbf{u}}^\varepsilon)^T \cdot \vec{\gamma}^\varepsilon = 2\phi \vec{\mathbf{F}}_V^\varepsilon$$

$$u^\varepsilon = \bar{u}^\varepsilon + \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_2^\varepsilon, \quad v^\varepsilon = \bar{v}^\varepsilon - \left(z^\varepsilon - H^\varepsilon - \frac{h^\varepsilon}{2}\right)\gamma_1^\varepsilon$$

$$\vec{\breve{\mathbf{u}}}^\varepsilon = \vec{\mathbf{u}}^\varepsilon|_{z^\varepsilon = H^\varepsilon}$$

$$w^\varepsilon = \vec{\breve{\mathbf{u}}}^\varepsilon \cdot \nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon)(\operatorname{div} \vec{\breve{\mathbf{u}}}^\varepsilon - \vec{\gamma}^\varepsilon \cdot \operatorname{rot} H^\varepsilon) + \frac{1}{2}(H^\varepsilon - z^\varepsilon)^2 \operatorname{rot} \vec{\gamma}^\varepsilon$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)\bar{u}^\varepsilon\right] \tag{8.1}$$

donde $\operatorname{rot}\vec{\alpha}^\varepsilon = \dfrac{\partial \alpha_2^\varepsilon}{\partial x^\varepsilon} - \dfrac{\partial \alpha_1^\varepsilon}{\partial y^\varepsilon}$, $\operatorname{rot}\alpha^\varepsilon = \left(\dfrac{\partial \alpha^\varepsilon}{\partial y^\varepsilon}, -\dfrac{\partial \alpha^\varepsilon}{\partial x^\varepsilon}\right)$,

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon)\bar{v}^\varepsilon + (\cos\varphi^\varepsilon)\left(\dfrac{\partial(h^\varepsilon \bar{u}^\varepsilon)}{\partial x^\varepsilon} + \dfrac{h^\varepsilon}{2}\dfrac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} - \bar{v}^\varepsilon \dfrac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\operatorname{sen}\varphi^\varepsilon)\bar{u}^\varepsilon + \dfrac{h^\varepsilon}{2}\dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] + \dfrac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] \end{pmatrix}$$

y

$$\vec{\mathbf{F}}_V^\varepsilon = \begin{pmatrix} \dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] + (\operatorname{sen}\varphi^\varepsilon)\gamma_2^\varepsilon \\ -(\operatorname{sen}\varphi^\varepsilon)\gamma_1^\varepsilon + (\cos\varphi^\varepsilon)\dfrac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} \end{pmatrix}$$

Debemos señalar que en el modelo anterior las componentes de la velocidad horizontal dependen de la profundidad si la vorticidad es no nula, lo que supone una interesante novedad respecto a los modelos que se encuentran en la literatura. Observamos también que la hipótesis simplificadora sobre la aceleración de Coriolis usualmente utilizada en oceanografía dinámica (véase la observación 1.1) implica que las dos primeras componentes de la vorticidad se anulan, por lo que las velocidades horizontales no dependen de la profundidad y recuperamos el modelo clásico de aguas profundas (véase (4.10.100)).

En el capítulo 5 partimos de las ecuaciones tridimensionales de Navier-Stokes para obtener distintos modelos bidimensionales de aguas someras con viscosidad. El análisis del orden formal de precisión de dichos modelos y la dificultad de su resolución nos lleva a proponer el modelo (5.7.6):

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \vec{\mathbf{u}}^\varepsilon) = 0$$

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \nabla\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{u}}^\varepsilon - \nu\left\{\Delta\vec{\mathbf{u}}^\varepsilon + \frac{1}{h^\varepsilon}\left[(\nabla\vec{\mathbf{u}}^\varepsilon)^T + \nabla\vec{\mathbf{u}}^\varepsilon\right]\nabla h^\varepsilon + \frac{1}{(h^\varepsilon)^2}\nabla\left[(h^\varepsilon)^2(\operatorname{div}\vec{\mathbf{u}}^\varepsilon)\right]\right\}$$

$$+ g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + 2\phi\vec{\mathbf{F}}_C^\varepsilon + \frac{1}{\rho_0 h^\varepsilon}\left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon\right)$$

$$w^\varepsilon = \vec{\mathbf{u}}^\varepsilon \cdot \nabla H^\varepsilon + (H^\varepsilon - z^\varepsilon)\operatorname{div}\vec{\mathbf{u}}^\varepsilon$$

$$p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon)\left[g - 2\phi\left(\cos\varphi^\varepsilon\right)u^\varepsilon\right] \tag{8.2}$$

donde

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\operatorname{sen}\varphi^\varepsilon)v^\varepsilon + (\cos\varphi^\varepsilon)\left(\dfrac{\partial(h^\varepsilon u^\varepsilon)}{\partial x^\varepsilon} + \dfrac{h^\varepsilon}{2}\dfrac{\partial v^\varepsilon}{\partial y^\varepsilon} - v^\varepsilon \dfrac{\partial H^\varepsilon}{\partial y^\varepsilon}\right) \\ -(\operatorname{sen}\varphi^\varepsilon)u^\varepsilon + \dfrac{h^\varepsilon}{2}\dfrac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon] + \dfrac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)u^\varepsilon] \end{pmatrix}$$

En el caso de que impongamos la hipótesis simplificadora de la oceanografía dinámica sobre la aceleración de Coriolis (véase (1.1.9)), obtenemos un modelo más sencillo debido fundamentalmente a la supresión del término $-2\phi\,(\cos\varphi^\varepsilon)\,w^\varepsilon$ en la primera ecuación de Navier-Stokes (véase (5.8.53)).

En la literatura se encuentran diferentes propuestas para los términos de viscosidad que aparecen en los modelos de aguas someras. Esta variedad demuestra que su justificación carece del rigor necesario. En nuestra opinión el método de desarrollos asintóticos puede aportar la rigurosidad que se echa en falta en los otros métodos empleados para deducir los modelos de aguas someras. Para comprobarlo, en el capítulo 6 comparamos nuestro modelo ((3.7.1)-(3.7.4) en su versión unidimensional y (8.2) en su versión bidimensional) analítica y numéricamente con otros modelos que se pueden encontrar en la literatura, obteniendo que el modelo que proponemos mejora (o en el peor de los casos iguala) los resultados de los otros modelos. Cabe señalar que en varios ejemplos (véanse (6.2.14)-(6.2.15) y los cuadros 6.2-6.6) nuestro modelo llega a cometer errores cien veces menores que el resto, pues al refinar el mallado el modelo que proponemos mejora la aproximación y los otros no. También se observa que en algún caso nuestro modelo obtiene mejores resultados con mallas más groseras (véanse los cuadros 6.31-6.32 y las figuras 6.31-6.36).

Cualitativamente podemos señalar que nuestro modelo es el que reduce las oscilaciones en mayor grado y que la discretización de su término de viscosidad (junto a la del modelo M2V2) recuerda mucho a la viscosidad numérica que se introduce en algunos textos para resolver el modelo sin viscosidad (véanse, por ejemplo, las páginas 362-363 de [26]).

Nuestros modelos recuperan la presión hidrostática sin necesidad de hipótesis a priori. Además, se proponen correcciones de orden 2 para la presión en todos los modelos que, como hemos visto en la sección 6.2.3.1 en el caso con viscosidad y en la sección 7.2.1.1 en el caso sin viscosidad, funcionan mejor que la presión hidrostática cuando la presión "exacta" se aleja de ella.

En el capítulo 6 también hemos estudiado numéricamente los distintos términos de Coriolis que hemos obtenido en el capítulo 5, encontrando que el nuevo término que obtenemos al no imponer la hipótesis de oceanografía dinámica ($\vec{\mathbf{F}}_C^\varepsilon$) permite obtener aproximaciones mucho más precisas.

En el capítulo 7 comparamos el modelo de aguas someras sin viscosidad que proponemos ((2.9.1) en su versión unidimensional y (8.1) en su versión bidimensional) y observamos que nuestro modelo es capaz de calcular las soluciones de las ecuaciones de Euler lineales en z, mientras el modelo clásico únicamente obtiene las velocidades medias.

En definitiva, creemos que los modelos propuestos (el modelo de aguas someras sin viscosidad (8.1) y el modelo de aguas someras con viscosidad (8.2)) suponen una mejora respecto a los modelos que se encuentran en la literatura. El modelo (8.1) incorpora una dependencia de la profundidad que, como hemos visto en el capítulo 7, permite aproximar mejor las ecuaciones de Euler. El modelo (8.2) incorpora un nuevo término de viscosidad, justificado mediante el método de desarrollos asintóticos en el capítulo 5 y numéricamente en el capítulo 6.

Una futura línea de trabajo será incorporar ambas novedades en un único modelo, obteniendo así un modelo de aguas someras con viscosidad que tenga el término de viscosidad del modelo (8.2) y la dependencia de la profundidad del modelo (8.1).

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Apéndice A

Notación

f_W	fuerza del viento en la superficie para el modelo 1D
$\vec{\mathbf{f}}_W = (f_{W_1}, f_{W_2}, 0)$	fuerza del viento en la superficie para el modelo 2D
f_R	fuerza del viento en la superficie para el modelo 1D
$\vec{\mathbf{f}}_R = (f_{R_1}, f_{R_2})$	fuerza de rozamiento en el fondo para el modelo 2D
F_x, F_y, F_z	componentes de las fuerzas externas que actúan sobre el fluido
g	aceleración de la gravedad (que suponemos constante)
h	altura o calado del agua
H	fondo
L	largo del dominio bidimensional
$\vec{\mathbf{n}}$	vector normal exterior unitario
p	presión
p_s	presión en la superficie
p^k	término de orden k ($k = 0, 1, 2, \cdots$) de la presión
s	superficie del agua
T	$T = \mu \left((\nabla \vec{\mathbf{u}})^T + \nabla \vec{\mathbf{u}} \right)$ (véase σ)
T_{ij}^k	término de orden k ($k = -1, 0, 1, 2, \cdots$) de la componente ij (i,j=1,2,3) de T
u	velocidad horizontal dirección eje X
$\bar{u}$	velocidad horizontal dirección eje X promediada en altura
u_0	velocidad horizontal dirección eje X para $t = 0$
u^k, v^k, w^k	términos de orden k ($k = 0, 1, 2, \cdots$) de las distintas componentes de la velocidad
$\vec{\mathbf{u}} = (u, v)$	vector velocidad horizontal en 3D
$\vec{\mathbf{u}} = (u, w)$	vector velocidad en 2D
$\vec{\mathbf{u}} = (u, v, w)$	vector velocidad en 3D
v	velocidad horizontal dirección eje Y
v_0	velocidad horizontal dirección eje Y para $t = 0$
$\bar{v}$	velocidad horizontal dirección eje Y promediada en altura

w	velocidad vertical
w_0	velocidad vertical para $t = 0$
γ, $\vec{\gamma}$	vorticidad
γ^k	término de orden k ($k = -1, 0, 1, 2, \cdots$) de la vorticidad
γ_i^k	término de orden k ($k = -1, 0, 1, 2, \cdots$) de la componente i ($i = 1, 2, 3$) de la vorticidad
Δ	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ en 3D
Δ	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ en 2D
Δ_{xy}	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ en 3D
ε	parámetro adimensional del orden del cociente entre la profundidad media del agua y el diámetro del dominio. Cuando aparece como superíndice indica la dependencia de las funciones de este parámetro
μ	coeficiente de viscosidad dinámica
ν	coeficiente de viscosidad cinemática ($\nu = \mu/\rho_0$)
ρ_0	densidad del fluido (que suponemos constante)
σ	tensor de tensiones, $\sigma = -pI + T$
$\vec{\tau}$	vector unitario tangente a la frontera
φ	latitud Norte (o bien es constante o bien depende sólo de y)
ϕ	velocidad angular de rotación de la Tierra
Ω	dominio en $\mathbb{R}^2$ ó $\mathbb{R}^3$

$$\operatorname{rot} \alpha = \left(\frac{\partial \alpha}{\partial y}, -\frac{\partial \alpha}{\partial x} \right)$$

$$\operatorname{rot} \vec{\alpha} = \frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y} \qquad \text{si } \vec{\alpha} = (\alpha_1, \alpha_2)$$

$$\operatorname{rot} \vec{\alpha} = \left(\frac{\partial \alpha_3}{\partial y} - \frac{\partial \alpha_2}{\partial z}, \frac{\partial \alpha_1}{\partial z} - \frac{\partial \alpha_3}{\partial x}, \frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y} \right) \quad \text{si } \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$$