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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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XVA for American options with two stochastic factors: modelling, mathematical analysis and numerical methods

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Abstract

In this work, we derive new linear and nonlinear partial differential equations (PDEs) models for pricing American options and total value adjustment in the presence of counterparty risk. Moreover, stochastic spreads are considered, which increases the dimension of the problem.

1. Introduction

Counterparty risk can be understood as the risk to each party of a contract from a future situation in which one of the counterparties cannot live up its contractual obligations. Since the last financial, crisis when several institutions went bankrupt, a relevant effort in quantitative finance research concerns to the consideration of counterparty risk in financial contracts, specially in the pricing of derivatives As a consequence, different adjustments on the value of the derivative without counterparty risk (hereafter referred as risk–free derivative) are being included in the derivative pricing. For example, the credit value adjustment (CVA) refers to the variation on the price of a contract due to the possibility of default of one (or both) of the counterparties. Adjustments on debit (DVA) and funding (FVA) are also important issues included in the so called total value adjustment (XVA). The XVA incorporates the sum of all the adjustments related to counterparty risk.

In a previous work [2], European and American options have been priced considering corunterparty risk. In such model, constant intensities of default for both counterparties have been assumed. So that a model depending on just one underlying stochastic factor (the underlying asset) is posed to price XVA. However, the intensity of default is not always constant, then stochastic intensities of default has been assumed in [3] as a result a model depending on two stochastic factors (the asset price and the spread from the investor) was deduced to price European options. In the current work, as we have done in [3], we consider that only the investor is defaultable and presents a stochastic intensity of default. Moreover, similar hypotheses as in the European options model introduced in [3] are assumed. Then, we extend the models introduced in [2], [3] to price the American options considering counterparty risk and compute the associated total value adjustment when stochastic intensity of default is assumed. So, we deduce a two dimensional PDE model for the American risky derivative value with stochastic intensity of deafult. The plan of the chapter is the following. In Section 1 we pose the complementarity problems deduced from the hedging arguments. In Section 2 we present the mathematical analysis of the previous problems. Section 3 presents the numerical methods and Section 4 shows some illustrative numerical results.

2. Mathematical model

In this section, we deduce the models for American options considering counterparty risk. With this aim, we consider self–financing portfolio and non–arbitrage scenarios. Moreover, we assume an investor as a risky counterparty and consider that the issuer's intensity of default is null. Thus, the underlying asset price S, and the short term CDS spread of the investor h are modelled by the following system of stochastic differential equations:

$$
\begin{aligned}
dS_t &= (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^S\,,\\
dh_t &= (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^h\,,
\end{aligned}
$$

where $(r(t) - q(t))$ and $(\mu^h(t) - M^h(t)\sigma^h(t))$ are the (respective) drifts of the processes. Moreover, $r(t)$ denotes the risk-free interest rate, $q(t)$ is the asset dividend yield rate, $M^h(t)$ is the market price of investor's credit risk, $\sigma^S(t, S)$ and $\sigma^h(t, h)$ are the volatility functions, and W_t^S and W_t^h are two correlated Wiener processes (i.e., $\rho\, dt = dW_t^S\, dW_t^h$) so that ρ is the instantaneous correlation between S_t and h_t.

Thus, we consider a derivative trade between a hedger and an investor, where only the investor has probability of default. The risky derivative value from the point of view of the investor, at time t, is denoted by $\widehat{V}(t, S_t, h_t, J_t^I)$, and depends on the spot value of the asset (S_t), on the spread of the investor (h_t) and on the investor's default state at time t (J_t^I). Remind that $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The risk–free

American option value, corresponding to the same contract between two free–bankruptcy counterparties, is denoted by $\widehat{V}(t, S_t)$ and does not include any counterparty risk adjustment, whereas the risky derivative price $\widehat{V}_t$ includes total value adjustment.

The risky derivative price in case of the investor makes default is given by:

$$\widehat{V}(t, S_t, h_t, 1) = RM^+(t, S_t, h_t) + M^-(t, S_t, h_t), \tag{2.1}$$

where $M(t, S_t, h_t)$ denotes the mark–to–market price, $M^+ = \max(M, 0)$ and $M^- = \min(0, M)$. In terms of the mark-to-market condition (2.1), we introduce $\Delta\widehat{V}$ as the variation of $\widehat{V}$ at default, which is given by:

$$\Delta\widehat{V}_t = RM_t^+ + M_t^- - \widehat{V}_t\,, \tag{2.2}$$

where $M_t = M(t, S_t, h_t)$. As it is usually assumed in the literature [4], and as we did in [2] and [3], we consider two possibilities for the mark–to–market value: either the risk–free value, either the derivative value including counterparty risk.

The hedger will trade with different financial instruments to hedge the market risk, the spread risk and the investor's default risk. Thus, in order to derive the value of American options with counterparty risk, we consider the same self–financing portfolio built for European options in [3], Π_t, which is designed to hedge all underlying risk factors:

$$\Pi_t = \alpha(t)H(t) + \beta(t) + \gamma(t)\mathrm{CDS}(t, T) + \varepsilon(t)\mathrm{CDS}(t, t + dt) + \Omega(t)B(t, t + dt)\,. \tag{2.3}$$

Furthermore, in order to avoid arbitrage opportunities we introduce the following hedging inequality:

$$d\widehat{V}_t \leq d\Pi_t\,. \tag{2.4}$$

Next, by applying Itô's Lemma for jump diffusion processes, we obtain the variation $d\widehat{V}_t$ of the derivative value $\widehat{V}_t$. Thus, replacing the change of the portfolio and the change of the derivative in (2.4), the hedging equation is transformed into:

$$\begin{aligned}
&\frac{\partial\widehat{V}}{\partial t} + \frac{1}{2}(\sigma^S)^2S^2\frac{\partial^2\widehat{V}}{\partial S^2} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2\widehat{V}}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2\widehat{V}}{\partial S\partial h}\\
&\quad\leq \frac{\partial\widehat{V}/\partial S}{\partial H/\partial S}\left(cH - (r-q)S\frac{\partial H}{\partial S}\right) + \frac{\partial\widehat{V}/\partial S}{\partial H/\partial S}(-fH)\\
&\quad+ \frac{\partial\widehat{V}/\partial h}{\partial\mathrm{CDS}(t,T)/\partial h}\left(-\frac{h}{1-R}\Delta\mathrm{CDS}(t,T) - \left(\mu^h - M\sigma^h\right)\frac{\partial\mathrm{CDS}(t,T)}{\partial h}\right)\\
&\quad+ \left(\frac{\partial\widehat{V}/\partial h}{\partial\mathrm{CDS}(t,T)/\partial h}\frac{\Delta\mathrm{CDS}(t,T)}{1-R} - \frac{\Delta\widehat{V}}{1-R}\right)h + f\widehat{V}\,,
\end{aligned} \tag{2.5}$$

in $[0, T) \times (0, \infty) \times (0, \infty)$. Then, the American option value when considering counterparty risk is modelled by the following complementarity problem:

$$\begin{cases}
\mathcal{L}(\widehat{V}) = \dfrac{\partial\widehat{V}}{\partial t} + \tilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta\widehat{V}}{1-R}h - f\widehat{V} \leq 0\\
\widehat{V}(t, S, h) \geq G(S)\\
\mathcal{L}(\widehat{V})(\widehat{V} - G) = 0\\
\widehat{V}(T, S, h) = G(S)\,,
\end{cases} \tag{2.6}$$

where $G(S)$ represents the option payoff and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is

$$\mathcal{L}_{Sh}V \equiv \frac{1}{2}(\sigma^S)^2S^2\frac{\partial^2V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2V}{\partial h\partial S} + (r-q)S\frac{\partial V}{\partial S} - \frac{\kappa}{1-R}h\frac{\partial V}{\partial h}\,.$$

According to the mark-to–market choices, two alternative linear complementarity problems are deduced:

- If $M = \widehat{V}$, we deduce the nonlinear complementarity problem:

$$\begin{cases}
\mathcal{L}_1(\widehat{V}) = \dfrac{\partial\widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} - h\widehat{V}^+ \leq 0, \quad \text{in } [0, T) \times (0, \infty) \times (0, \infty)\\
\widehat{V}(t, S, h) \geq G(S)\\
\mathcal{L}_1(\widehat{V})(\widehat{V} - G) = 0\\
\widehat{V}(T, S, h) = G(S)\,.
\end{cases} \tag{2.7}$$

- If $M = V$, the following linear complementarity problem is derived:

$$\begin{cases} \mathcal{L}_2(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R} + f\right)\widehat{V} \\ \qquad -((1-R)V^+ - V)\dfrac{h}{1-R} \leq 0, \quad \text{in } [0,T) \times (0,\infty) \times (0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_2(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{2.8}$$

Moreover, to compute the XVA value, we consider that $\widehat{V} = V + U$ where U denotes the XVA, then the adjustments can be obtained as the difference of the risky derivative value, $\widehat{V}$, and the risk–free derivative value, V, which is the solution of the classical Black-Scholes American problem:

$$\begin{cases} \mathcal{L}_3(V) = \dfrac{\partial V}{\partial t} + \mathcal{L}_S V - fV \leq 0\,, \quad \text{in } [0,T) \times (0,\infty) \\ V(t,S) \geq G(S) \\ \mathcal{L}_3(V)(V - G) = 0 \\ V(T,S) = G(S)\,, \end{cases} \tag{2.9}$$

where the operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S}\,.$$

In order to numerically solve problems (2.7) and (2.8) by a finite element method, we proceed to localize the problems on a bounded domain. For this purpose, let us consider $\Omega = (0, S_\infty) \times (0, h_\infty)$ for large enough values of S_∞ and h_∞, so that their choice does not affect the solution in the domain of financial interest. We need to impose appropiate boundary conditions on the risky derivative value problem in the bounded domain. For this purpose, we consider the same boundary conditions than for V and $\widehat{V}$ as in the case of European options in [3]. Then, at $S = 0$ and $S = S_\infty$, the derivative value is given by:

$$\begin{cases} \widehat{V}(t,0,h) = V(t,0) = V_0(t)\,, \\ \widehat{V}(t,S_\infty,h) = V(t,S_\infty) = V_\infty(t)\,, \end{cases}$$

where the values of $V_0(t)$ and $V_\infty(t)$ are respectively given by:

$$V_0(t) = \begin{cases} 0\,, & \text{for a call option,} \\ K\exp(-f(T-t))\,, & \text{for a put option,} \end{cases} \tag{2.10}$$

$$V_\infty(t) = \begin{cases} S_\infty - K\,, & \text{for a call option,} \\ 0\,, & \text{for a put option.} \end{cases} \tag{2.11}$$

In the next section, the existence and uniqueness of solution of problem (2.7) are studied. For this purpose, we introduce the problem which models the XVA in order to obtain a problem with homogeneous boundary conditions. Then, we split up the risky derivative value, $\widehat{V}$, as the sum of the XVA, U, plus the total value adjustment, V, i.e. $\widehat{V} = V + U$. Introducing this breakdown in (2.7), the following nonlinear complementarity problem is deduced:

$$\begin{cases} \mathcal{L}_t(U) = \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU - h(U+V)^+ \leq -\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\,, \qquad t \in [0,T)\,, \quad (S,h) \in \Omega \\ U(t,S,h) \geq G(S) - V(t,S) \\ \left[\mathcal{L}_t(U) - \left(-\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\right)\right]\left[U - (G(S) - V(t,S))\right] = 0 \\ U(T,S,h) = 0 \\ U(t,0,h) = 0 \\ U(t,S_\infty,h) = 0 \\ U(t,S,0) = 0 \\ (A\nabla U \cdot \vec{n})(\tau,S,h_\infty) = 0\,. \end{cases} \tag{2.12}$$

For the linear problem (2.8), the same boundary conditions are considered.

3. Mathematical analysis

In this section we prove the existence and uniqueness of solution for the XVA problem (2.12) for a given function V. Then, taking into account the existence and uniqueness of solution V for the classical Black-Scholes problem, we obtain the existence and uniqueness of solution for problem (2.7). Introducing the time to maturity variable, $\tau = T - t$, as well as the new variables and unknown:

$$x = \ln \frac{S}{K}, \qquad u(\tau, x, h) = U(t, S, h), \qquad v(\tau, x) = V(t, S)\,.$$

we pose the nonlinear complementarity problem (2.12) as follows:

$$\begin{cases} \mathcal{L}_\tau(u) = \dfrac{\partial u}{\partial \tau} + \mathcal{A}u - \Phi(\tau, u) \geq \ell\,, \quad (x, h) \in \widehat{\Omega}, \quad \tau \in (0, T] \\ u \geq \psi \\ [\mathcal{L}_\tau(u) - \ell]\ [u - \psi] = 0 \\ u(0, S, h) = 0 \\ u(\tau, x_0, h) = 0 \\ u(\tau, x_\infty, h) = 0 \\ u(\tau, x, 0) = 0 \\ (\widehat{A}\nabla u \cdot \vec{n})(\tau, x, h_\infty) = 0\,, \end{cases} \tag{3.1}$$

Theorem 3.1 *The following statements are satisfied*:

1. *The continuous operator $\mathcal{A}$ satisfies Gårding's inequality, i.e.*:
$$(\mathcal{A}z, z) \geq \omega_1 \|z\|^2_{H^1_\Gamma(\widehat{\Omega})} - \omega_2 \|z\|^2_{L^2(\widehat{\Omega})}\,, \quad \forall z \in H^1_\Gamma(\widehat{\Omega})\,, \tag{3.2}$$
with $\omega_1 > 0$ and $\omega_2 \in \mathbb{R}$.
2. $\ell \in L^2(0, T; L^2(\widehat{\Omega})) \subset L^2(0, T; W^*)$.
3. *Let* $D(\phi) = \left\{ z \in H^1_\Gamma(\widehat{\Omega}) \ /\ \phi(z) < \infty \right\}$ *and* $u_0 = u(0, x, h)$. *Then,* $u_0 \in \overline{D(\phi)}$.
4. $\Phi(\tau, \varphi)$ *is Lipschitz continuous on variable φ, i.e.*
$$\|\Phi(\tau, \varphi_1) - \Phi(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_G \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}\,.$$

Therefore, the nonlinear variational inequality (3.1) *has a unique solution* $u \in L^2(0, T; H^1_\Gamma(\widehat{\Omega})) \cap C([0, T]; L^2(\widehat{\Omega}))$; *in particular* $u \in W^{1,2}(0, T; L^2(\widehat{\Omega}))$ *and satisfies*

$$\|u\|_{W^{1,2}(0,T;L^2(\widehat{\Omega}))} \leq C_1 \left(1 + \|u_0\|_{L^2(\widehat{\Omega})} + \|\ell\|_{L^2(0,T;H^1_\Gamma(\widehat{\Omega}))}\right)\,. \tag{3.3}$$

4. Numerical simulation

The numerical approximation is mainly based on finite element methods combined with the method of characteristics. Moreover, a fixed point scheme is implemented for the nonlinear complementarity problem.

4.1. The method of characteristics

More precisely, taking into account the advective term, the risky derivative problem is approximated by

$$\begin{cases} \mathcal{L}_1^n(\widehat{V}^{n+1}) = \dfrac{\widehat{V}^{n+1} - \widehat{V}^n \circ \chi^n}{\Delta \tau^n} - \operatorname{div}(A\nabla \widehat{V}^{n+1}) + f\widehat{V}^{n+1} + h(\widehat{V}^{n+1})^+ \geq 0\,, \\ \widehat{V}^0(S, h) = 0\,, \\ \widehat{V}^{n+1}(S, h) \geq G(S)\,, \\ \mathcal{L}_1^n(\widehat{V}^{n+1})(\widehat{V}^{n+1} - G) = 0\,, \end{cases} \tag{4.1}$$

for $n = 0, 1, 2 \ldots, N_T - 1$, where $\widehat{V}^n(\cdot) \approx \widehat{V}(\tau^n, \cdot)$ and $\chi^n = \chi(\tau^n) = \chi((S, h), \tau^{n+1}; \tau^n)$ represents the characteristic curve passing through point (S, h) at time τ^{n+1}. Then function χ is the solution of the final value ODE problem:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \left((\sigma^S)^2 - (r - q)\right)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1 - R}\chi_2, \\ \chi_2(\tau^{n+1}) = h\,, \end{cases} \tag{4.2}$$

The components of χ^n can thus be deduced and are given by:

$$\chi_1^n = S\exp\left(-((\sigma^S)^2 - r + q)(\tau^{n+1} - \tau^n)\right),$$
$$\chi_2^n = -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}(\tau^{n+1} - \tau^n)\right).$$

4.2. Fixed point scheme

In order to solve the nonlinearity of problem (4.1), a fixed point scheme is proposed at each iteration of the characteristics method. Thus, the global scheme is shown in Algorithm 1.

Algorithm 1

Let $N_T > 1$, $n = 0$, $\varepsilon > 0$ and $\widehat{V}^0$ given
For $n = 1, 2, \ldots, N_T - 1$:

1. Let $\widehat{V}^{n+1,0} = \widehat{V}^n$, $k = 0$, $e = \varepsilon + 1$
2. For $k = 0, 1, \ldots$
   - Search $\widehat{V}^{n+1,k+1}$ solution of:

$$\begin{aligned}
&(1 + \Delta\tau^n f)\,\widehat{V}^{n+1,k+1} - \Delta\tau^n \operatorname{div}(A\nabla\widehat{V}^{n+1,k+1}) \\
&\qquad\qquad \geq \widehat{V}^n \circ \chi^n - \Delta\tau^n\, h\,(\widehat{V}^{n+1,k})^+ \\
&\widehat{V}^{n+1,k+1}(S,h) \geq G(S) \\
&\mathcal{L}_1^n(\widehat{V}^{n+1,k+1})(\widehat{V}^{n+1,k+1} - G) = 0
\end{aligned} \tag{4.3}$$

   - Compute the relative error $e = \dfrac{\|\widehat{V}^{n+1,k+1} - \widehat{V}^{n+1,k}\|}{\|\widehat{V}^{n+1,k+1}\|}$

   until $e < \varepsilon$.

4.3. Finite element method

For the spatial discretization of (4.3) a triangular mesh of Ω and the associated finite element space of piecewise linear Lagrange polynomials are considered. For fixed natural numbers $N_S > 0$ and $N_h > 0$, we consider a uniform mesh of the computational domain Ω, the nodes of which are (S_i, h_j), with $S_i = i\Delta S$ ($i = 0, \ldots, N_S + 1$) and $h_j = j\Delta h$ ($j = 0, \ldots, N_h + 1$), where $\Delta S = S_\infty/(N_S + 1)$ and $\Delta h = h_\infty/(N_h + 1)$ denote the constant mesh steps in each coordinate. Associated to this uniform mesh, a piecewise linear Lagrange finite element discretization is considered. More precisely, we introduce the finite element spaces

$$W_h = \{\varphi_h \in C(\Omega) \,/\, \widehat{\varphi}|_{T_j} \in \mathcal{P}_1\,,\ \forall T_j \in \mathcal{T}\}\,,$$
$$\mathcal{K}_h = \{\varphi_h \in W_h \,/\, \varphi_h = \widehat{V} \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \text{and } \varphi_h \geq G(S)\}\,,$$

in order to find $\widehat{V}_h^{n+1,k+1} \in \mathcal{K}_h$ satisfying the boundary conditions and such that:

$$\begin{aligned}
&\int_\Omega (1 + \Delta\tau^n f)\widehat{V}_h^{n+1,k+1}\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\,dh \\
&\quad + \Delta\tau^n \int_\Omega A\nabla\widehat{V}_h^{n+1,k+1}\, \nabla\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\,dh \\
&\quad - \Delta\tau^n \int_{\Gamma_2^{*,+}} (A\nabla V_h^{n+1,k+1}, n)(\varphi_h - \widehat{V}_h^{n+1,k+1})\partial\gamma \\
&\quad \geq \int_\Omega \left(\widehat{V}_h^n \circ \chi^n\right)\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\,dh - \Delta\tau^n \int_\Omega h\left(\widehat{V}_h^{n+1,k}\right)^+\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\,dh\,,
\end{aligned}$$

for all $\varphi_h \in \mathcal{K}_h$. Quadrature formula based on the midpoints of the edges of the triangles has been used to obtain the coefficients of the matrix and the right hand side vector which define the linear system associated to the discretized problem.

After the time discretization with the method of characteristics and the spatial discretization with finite elements, the fully discretized problem can be written in the form:

$$\begin{cases} A_h \widehat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1}\,, \\ \widehat{V}_h^{n+1,k+1} \geq \Psi_h\,, \\ (A_h \widehat{V}_h^{n+1,k+1} - b_h^{n+1,k+1})(\widehat{V}_h^{n+1,k+1} - \Psi_h) = 0\,, \end{cases} \tag{4.4}$$

where Ψ_h denotes the discretized exercise value, $G(S)$, which also coincides with the value at maturity.

In order to solve problem (4.4), the augmented Lagrangian active set (ALAS) algorithm is employed.

5. Numerical results

Finally, in order to show the relevance of incorporating counterparty risk pricing derivatives we show some numerical results to understand the behaviour of the total value adjustment for American options. We focus on an American put option sold by the investor. The maturity time is $T = 0.5$ years and is discretized with $N_T = 700$ time steps. Firstly, we plot the risky and risk-free derivative value and the XVA. Moreover, we present the exercise region for both derivative value in order to show how affects the counterparty risk in the early exercise.

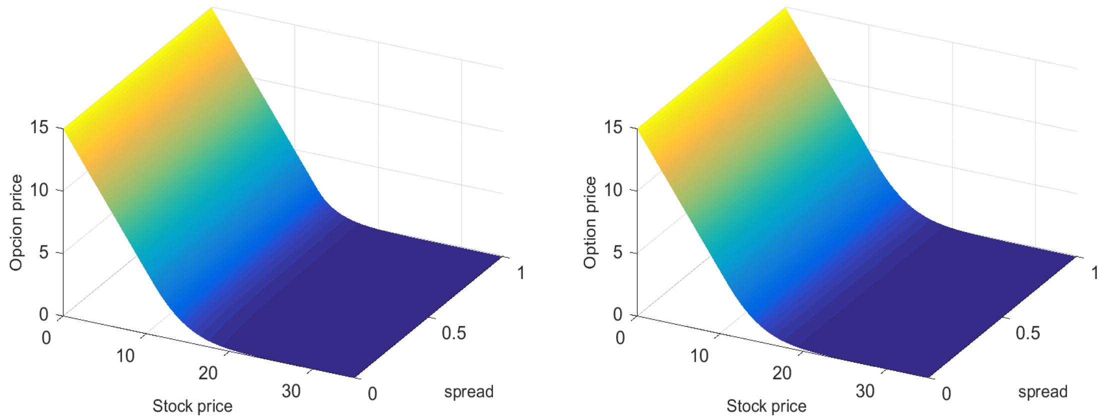

Fig. 1 American put option value risky valur (left), risk–free value (right)

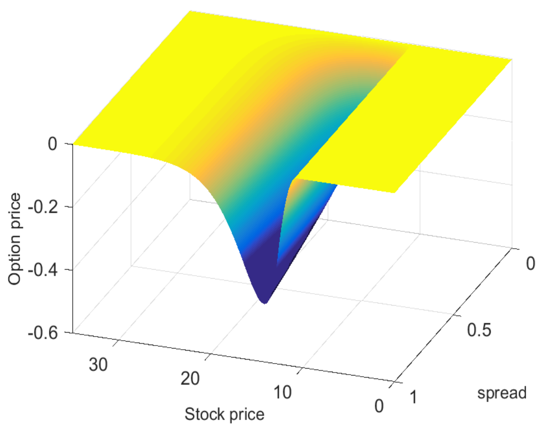

Fig. 2 Total value adjustment

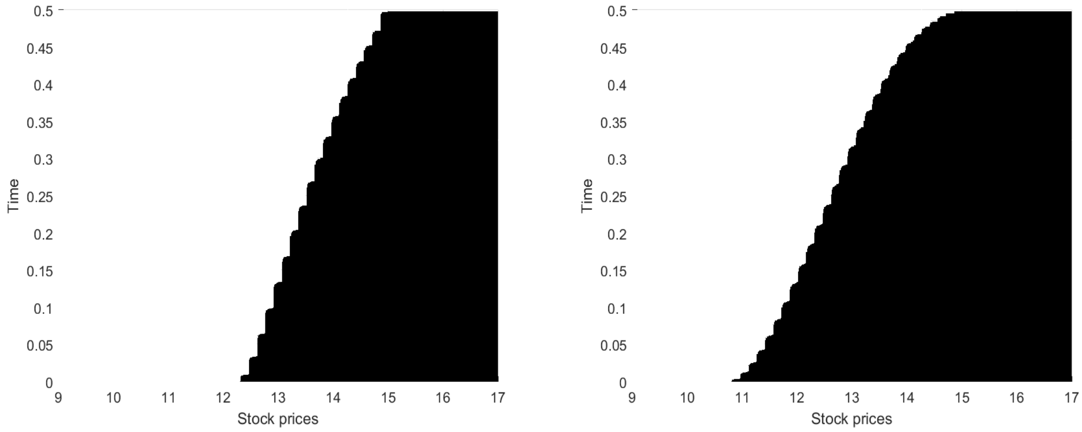

Fig. 3 Exercise regions (white) risky value (left) risk–free value (right)

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