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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Pricing TARN options with a stochastic local volatility model

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Abstract

Target Accumulation Redemption Notes (TARNs) are financial derivatives which give their holders the right to receive periodic coupons until the accumulated sum of those ones reaches an agreed target. In this work, we solve a partial differential equations (PDEs) model for pricing TARN options by implementing an alternating-direction implicit finite difference method (ADI method). We combine the numerical solution with a stochastic local volatility (SLV) technique and show the numerical results for a particular example.

1. Introduction

European and American options are the most well-known derivative products, and have been widely studied from the financial and mathematical points of view. For these options, jointly known as vanilla options, their price depends mainly on the value of the underlying asset.

On the opposite side, exotic options present different features which also have an effect on the price. For example, Bermudan options offer multiple exercise dates, the pay-off of Asian options depends on the average price of the underlying asset and the pay-off of barrier options depends on whether or not the price of the underlying asset reaches an agreed value during the option's lifetime. Other examples of exotic products are the target redemption products, whose notional amount increases until a certain target is reached [5]. In particular, the value of a Target Accumulation Redemption Note (TARN) depends on an accumulated amount: if the sum of coupons reaches an agreed target before the maturity date, the holder of the note receives a final payment, also known as knockout, and the contract terminates. These products are usually traded in foreign exchange (FX) markets.

2. Mathematical model

We propose a stochastic local volatility model based on Heston model [4]:

$$\begin{cases} dS_t = (r_d(t) - r_f(t))S_t\,dt + L(S_t,t)\sqrt{V_t}S_t\,dW_t^1 \\ dV_t = \kappa(\theta - V_t)\,dt + \lambda\sqrt{V_t}\,dW_t^2 \\ dW_t^1\,dW_t^2 = \rho\,dt\,, \end{cases} \tag{2.1}$$

where L is the leverage function, which represents the contribution of the local volatility and will be calibrated with the help of market data, S_t is the underlying asset, V_t is the stochastic variance, W_t^1 and W_t^2 are two Brownian motions, r_d and r_f are the domestic and foreign interest rates, respectively, and κ, θ, λ and ρ are the Heston parameters.

Let $r(t) = r_d(t) - r_f(t)$. Let us assume $2\kappa\theta \leq \lambda^2$ and $V_0 > 0$. Then, Feller condition states that $V_t > 0$ for every $t > 0$. Moreover, we will assume S_0, κ, θ and λ are strictly positive, $-1 < \rho < 1$ and the leverage function is positive and bounded. Under these hypotheses, it is proven [4] that there exists a unique solution of model (2.1) and there also exists a function $p := p(S,V,t)$, called transition probability function, solution of the Fokker-Planck (FP) equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial S}\big(r(t)Sp\big) - \frac{\partial}{\partial V}\big(\kappa(\theta - V)p\big) + \frac{1}{2}\frac{\partial^2}{\partial S^2}\big(L^2S^2Vp\big) + \frac{\partial^2}{\partial S\partial V}\big(\lambda\rho LSVp\big) + \frac{1}{2}\frac{\partial^2}{\partial V^2}\big(\lambda^2Vp\big)\,, \tag{2.2}$$

such that the leverage function can be written as [4]:

$$L(S,t) = \sigma_{LV}(S,t)\sqrt{\frac{\int_{\mathbb{R}} p(S,V,t)\,dV}{\int_{\mathbb{R}} V\,p(S,V,t)\,dV}}\,. \tag{2.3}$$

Let $S(t)$ be the FX rate at time t, t_0 the actual date and $t_1, t_2, \ldots, t_K$ the so-called fixing dates. Moreover, let E be the strike, U the target accrual level and $A(t)$ the accumulated amount at time t [3]. On each fixing date, t_k, there is a cash flow payment:

$$\widetilde{C}_k \equiv \beta(S(t_k) - E) \times 1_{\beta\times S(t_k)\geq\beta\times E}\,,$$

where β is a strategy on foreign exchange ($\beta = 1$ for a call option and $\beta = -1$ for a put option), until the accumulated amount A reaches the target U. Let $t_{\widetilde{K}}$ be the first fixing date when the target is breached:

$$\widetilde{K} = \min_{k=1,\dots,K} \{k : A(t_k) \geq U\}$$

and let $\widetilde{K} = K$ if the target is not breached, the payment can be written as:

$$C_k(S, A) = \begin{cases} \widetilde{C}_k \times \left(1_{A(t_{k-1})+\widetilde{C}_k < U} + W_k \times 1_{A(t_{k-1})+\widetilde{C}_k \geq U} \right), & \text{if } t_k \leq t_{\widetilde{K}}, \\ 0, & \text{if } t_k > t_{\widetilde{K}}, \end{cases} \tag{2.4}$$

where $A(t_{k-1})$ is the accumulated amount immediately after t_{k-1}. This magnitude is given by a piecewise constant function:

$$A(t) = \begin{cases} A(t_{k-1}), & \text{if } t_{k-1} \leq t < t_k, \\ A(t_{k-1}) + C_k(S(t_k), A(t_{k-1})), & \text{if } t = t_k. \end{cases}$$

Moreover, W_k is a weight that depends on the knockout when the target is breached. We will consider three types of knockout, such that the weight can be written as:

$$W_k = \begin{cases} 1, & \text{in the case of full gain,} \\ \dfrac{U - A(t_{k-1})}{\beta \times (S(t_k) - E)}, & \text{in the case of part gain,} \\ 0, & \text{in the case of no gain.} \end{cases}$$

Finally, let $u := u(S, V, t, A)$ be the value of a TARN option, the following SLV option pricing PDE:

$$\frac{\partial u}{\partial t} + r(t) S \frac{\partial u}{\partial S} + (\kappa(\theta - V)) \frac{\partial u}{\partial V} + \frac{1}{2} L^2 V S^2 \frac{\partial^2 u}{\partial S^2} + \rho L \lambda S V \frac{\partial^2 u}{\partial S \partial V} + \frac{1}{2} \lambda^2 V \frac{\partial^2 u}{\partial V^2} - r_d(t) u = 0$$

is valid between fixing dates. Furthermore, for each fixing date we can pose:

$$u\big(S, V, t_k^-, A(t_k^-)\big) = u\big(S, V, t_k, A(t_k^-) + C_k(S, A(t_k^-))\big) + C_k\big(S, A(t_k^-)\big),$$

where C_k is given by (2.4) and t_k^- is the time infinitesimally before t_k.

2.1. Numerical methods

We compute the optimal Heston parameters $(\kappa, \theta, \lambda, \rho)$ for the calibration of the SLV model. For each maturity, the COS method is developed to price European options under the Heston model, $[w^{\text{Hes}}]_i^m := w^{\text{Hes}}(E_i, T^m)$ and the Levenberg-Marquardt non-linear least squares algorithm is applied to find the optimal parameters by minimizing

$$\min_{\kappa,\theta,\lambda,\rho} \sum_{i=1}^{N_E} \big([w^{\text{Hes}}(\kappa, \theta, \lambda, \rho)]_i^m - w_i^m \big)^2,$$

where $w_i^m := w(E_i, T^m)$ are the market data for different strikes and maturities.

As Feller condition does not always hold in real markets, we propose a logarithmic change of variable [2]. Thus, let S_0 and V_0 be the initial values of the underlying and the variance, and let

$$X_t = \log(S_t/S_0), \qquad Z_t = \log(V_t/V_0).$$

In the new domain $(-\infty, \infty) \times (-\infty, \infty)$, model (2.1) is rewritten as:

$$\begin{cases} dX_t = \big(r(t) - \frac{1}{2} L(X_t, t)^2 V_0 e^{Z_t}\big) dt + L(X_t, t) \sqrt{V_0 e^{Z_t}}\, dW_t^1 \\ dZ_t = \big((\kappa\theta - \frac{1}{2}\lambda^2) \frac{1}{V_0 e^{Z_t}} - \kappa\big) dt + \lambda \frac{1}{\sqrt{V_0 e^{Z_t}}}\, dW_t^2 \\ dW_t^1 dW_t^2 = \rho dt. \end{cases}$$

A previous step to apply numerical methods is the truncation of the new unbounded domain to a bounded one. Thus, we consider the fixed domain $\Omega = (X_{\min}, X_{\max}) \times (Z_{\min}, Z_{\max})$, which is finer around $(X_0 = 0, Z_0 = 0)$, and pose the FP equation (2.2):

$$\begin{aligned} \frac{\partial p}{\partial t} = &-\frac{\partial}{\partial X}\big((r(t) - \frac{1}{2} L^2 V_0 e^Z) p\big) - \frac{\partial}{\partial Z}\big(((\kappa\theta - \frac{1}{2}\lambda^2)\frac{1}{V_0 e^Z} - \kappa) p\big) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial X^2}\big(L^2 V_0 e^Z p\big) + \frac{\partial^2}{\partial X \partial Z}\big(\lambda \rho L p\big) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}\big(\lambda^2 \frac{1}{V_0 e^Z} p\big), \end{aligned} \tag{2.5}$$

with the initial condition:

$$p(X,Z,0) = \delta(X)\delta(Z), \tag{2.6}$$

where δ is the Dirac function. Additionally, we impose the boundary conditions:

$$\frac{\partial^2 p}{\partial X^2}(X_{\min},Z,t)=0, \quad \frac{\partial^2 p}{\partial X^2}(X_{\max},Z,t)=0, \quad \frac{\partial^2 p}{\partial Z^2}(X,Z_{\min},t)=0, \quad \frac{\partial^2 p}{\partial Z^2}(X,Z_{\max},t)=0. \tag{2.7}$$

Moreover, the leverage function (2.3) is given by:

$$L(X,t) = \sigma_{LV}(X,t)\sqrt{\frac{\int_{\mathbb{R}^+} p(X,Z,t)dZ}{\int_{\mathbb{R}^+} V_0 e^Z p(X,Z,t)dZ}}. \tag{2.8}$$

As we can see in (2.5) and (2.8), the computing of the transition probability function needs the leverage function and reciprocally. Therefore, we propose a fixed point scheme to solve the problem at each time step, in which an alternating directions implicit (ADI) method is developed to solve the FP problem and the trapezoidal rule is applied to approximate the leverage function. The modified Douglas ADI scheme at each step can be written as:

$$A = p^{n-1} + \Delta t^n \left[F_0(p^{n-1},t^{n-1}) + F_1(p^{n-1},t^{n-1}) + F_2(p^{n-1},t^{n-1})\right],$$

$$B - \alpha\Delta t^n F_1(B,t^n) = A - \alpha\Delta t^n F_1(p^{n-1},t^{n-1}), \tag{2.9}$$

$$C - \alpha\Delta t^n F_2(C,t^n) = B - \alpha\Delta t^n F_2(p^{n-1},t^{n-1}), \tag{2.10}$$

$$p^n = C,$$

for $n = 1, 2, \ldots, N_T$, where

$$F_0(p,t) = \frac{\partial^2}{\partial X \partial Z}\big(\lambda\rho L p\big),$$

$$F_1(p,t) = -\frac{\partial}{\partial Z}\Big(\big((\kappa\theta - \frac{1}{2}\lambda^2)\frac{1}{V_0 e^Z} - \kappa\big)p\Big) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}\Big(\frac{\lambda^2}{V_0 e^Z}p\Big),$$

$$F_2(p,t) = -\frac{\partial}{\partial X}\Big(\big(r(t) - \frac{1}{2}L^2 V_0 e^Z\big)p\Big) + \frac{1}{2}\frac{\partial^2}{\partial X^2}\big(L^2 V_0 e^Z p\big).$$

In addition, a mixing fraction parameter, η, is applied to the volatility of the volatility, λ:

$$dV_t = \kappa(\theta - V_t)dt + \eta\lambda\sqrt{V_t}dW_t^2.$$

For each maturity, the ADI method is developed to price options under the SLV model, $[y^{\mathrm{SLV}}]_i^m := y^{\mathrm{SLV}}(E_i, T^m)$ and the golden section search algorithm is applied to determine the optimal parameter by minimizing

$$\min_{\eta} \sum_{i=1}^{N_E} \big([y^{\mathrm{SLV}}(\eta)]_i^m - y_i^m\big)^2,$$

where $y_i^m := y(E_i, T^m)$ are the market data for different strikes and maturities.

Finally, we introduce the time-to-maturity variable ($\tau = T - t$) and deduce the TARN price PDE in terms of X and Z:

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \big(r(\tau) - \frac{1}{2}L^2 V_0 e^Z\big)\frac{\partial u}{\partial X} + \frac{1}{2}L^2 V_0 e^Z \frac{\partial^2 u}{\partial X^2} + \lambda\eta\rho L \frac{\partial^2 u}{\partial X \partial Z} \\ &+ \big((\kappa\theta - \frac{1}{2}\lambda^2\eta^2)\frac{1}{V_0 e^Z} - \kappa\big)\frac{\partial u}{\partial Z} + \frac{1}{2}\frac{\lambda^2\eta^2}{V_0 e^Z}\frac{\partial^2 u}{\partial Z^2} - r_d(\tau)u,\end{aligned} \tag{2.11}$$

with the initial condition:

$$u(X,Z,0,A) = 0. \tag{2.12}$$

Additionaly, we assume the boundary conditions [4]:

$$\begin{aligned}&\frac{1}{S_0^2}e^{-2X}\Big(\frac{\partial^2 u}{\partial X^2}(X_{\min},Z,t) - \frac{\partial u}{\partial X}(X_{\min},Z,t)\Big) = 0, \quad \frac{1}{V_0^2}e^{-2Z}\Big(\frac{\partial^2 u}{\partial Z^2}(X,Z_{\min},t) - \frac{\partial u}{\partial Z}(X,Z_{\min},t)\Big) = 0, \\ &\frac{1}{S_0^2}e^{-2X}\Big(\frac{\partial^2 u}{\partial X^2}(X_{\max},Z,t) - \frac{\partial u}{\partial X}(X_{\max},Z,t)\Big) = 0, \quad \frac{1}{V_0^2}e^{-2Z}\Big(\frac{\partial^2 u}{\partial Z^2}(X,Z_{\max},t) - \frac{\partial u}{\partial Z}(X,Z_{\max},t)\Big) = 0.\end{aligned} \tag{2.13}$$

As for the numerical solution of the FP problem, we also propose the use of the ADI algorithm to solve $(2.11 - 2.13)$, jointly with the jump condition for each fixing date τ_k:

$$u\big(X,Z,\tau_k^+,A(\tau_k^+)\big) = u\big(X,Z,\tau_k,A(\tau_k^+) + C_k(X,A(\tau_k^+))\big) + C_k\big(X,A(\tau_k^+)\big),$$

where τ_k^+ is the time infinitesimally after τ_k. Fig. 1 shows a sketch of the scheme.

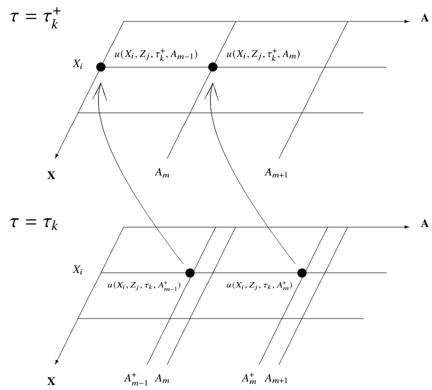

Fig. 1 Jump condition

2.2. Numerical results

We present a numerical test for the valuation of a TARN call option in the frame of foreign exchange. For this aim, we have considered the US dollar and British pound as domestic and foreign currencies, respectively, an initial underlying $S_0 = 1.320$, an initial variance $V_0 = 0.004$ and a strike $E = 1.283$. Furthermore, the domestic and foreign interest rates are shown in Tab. 1 of [1], the maturity period is $T = 12$ months and the fixing dates are taken every 30 days, thus $K = 12$.

Fig. 2 shows the market implied volatility (left) and the local volatility (right), which is computed by means of Dupire's formula. Thus, we apply the previous techniques to compute the Heston parameters, which are shown in Tab. 3 (left) of [1].

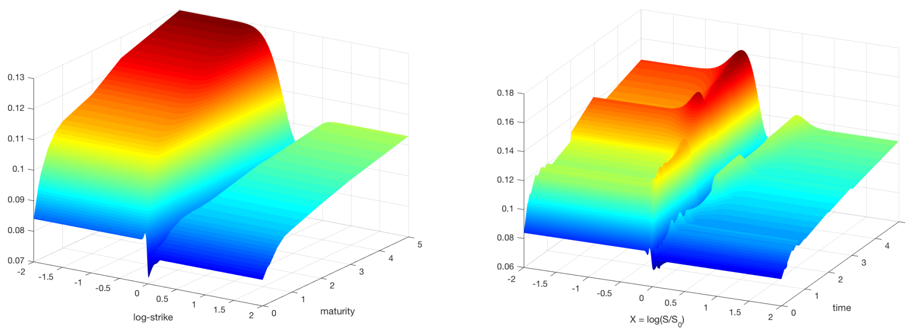

Fig. 2 The implied volatility (left) and the local volatility (right)

Next, we have approximated the (X, Z) domain with a mesh similar to the one plotted in Fig. 3, consisting of 400×100 nodes. As we have previously detailed, the mesh is finer around the point $(X_0 = 0, Z_0 = 0)$. We have also refined the mesh for values close to $Z_{\min}$ in order to minimize the errors arising from the fact that the Feller condition may be not accurate for these values of the volatility. Moreover, we have used 180 time steps and the parameter $\alpha = 0.5$ in the ADI method. Fig. 4 shows the computed solution of the FP problem at the maturity (left) and the leverage function (right).

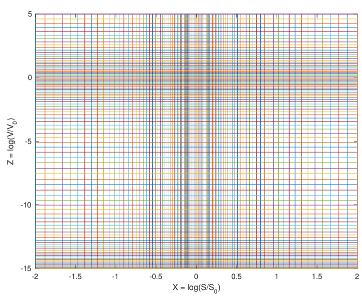

Fig. 3 Mesh

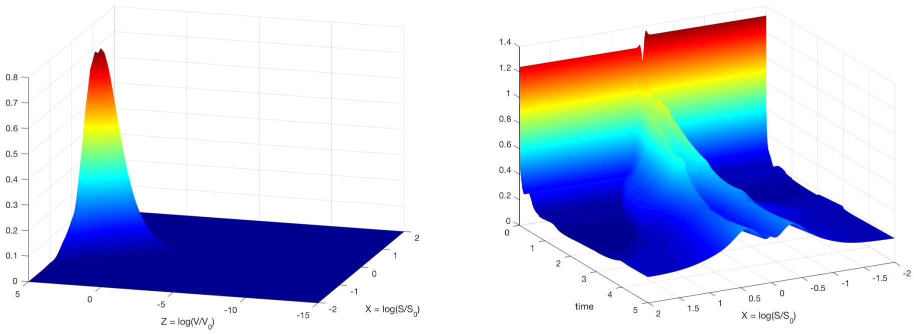

Fig. 4 Numerical solution of the FP equation (left) and the leverage function (right)

Finally, we calibrate the mixing fraction parameter η, which is shown in Tab. 3 (right) of [1], and show the numerical approximation of the TARN price for different knockouts in Fig. 5. More details and results are available in [1].

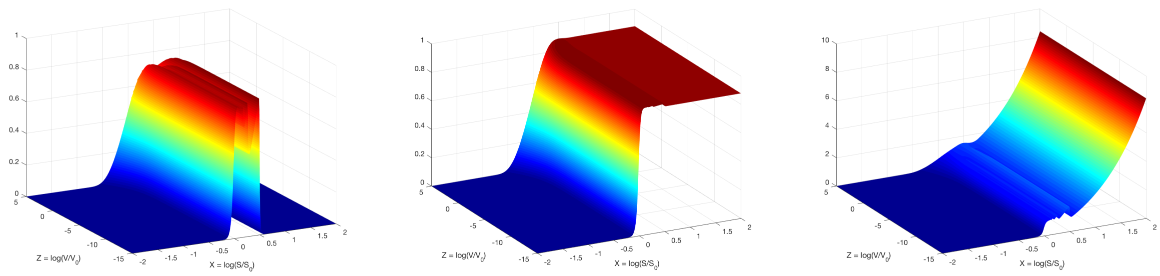

Fig. 5 TARN option price: no gain (left), part gain (center) and full gain (right) knockouts for $U = 0.90$.

2.3. Conclusions

We solve a partial differential equations model to price TARN options. We have improved previous results by introducing a SLV technique in order to better reflect the market volatilities (taking advantage of local volatility methods) on a path dependent derivative product (for which stochastic volatility methods are more convenient). This SLV approach can be extended to other kinds of exotic options.

An alternating directions method (ADI) is implemented and the volatility surfaces, transition probability function, leverage function and option price are computed. Therefore, we have a tool to valuate this kind of exotic options.

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