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CURVATURE HOMOGENEOUS CRITICAL METRICS IN DIMENSION THREE

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Abstract. We study curvature homogeneous three-manifolds modeled on a symmetric space which are critical for some quadratic curvature functional. If the Ricci operator is diagonalizable, critical metrics are 1-curvature homogeneous Brinkmann waves and are critical for one specific functional. Otherwise, critical metrics are modeled on Cahen-Wallach symmetric spaces and they are Kundt spacetimes which are critical for all quadratic curvature functionals.

1. Introduction

1.1. **Curvature homogeneous Lorentzian metrics.** A pseudo-Riemannian manifold (M, g) is said to be *k-curvature homogeneous* if for any pair of points $p,\ q \in M$ there exists a linear isometry $\Phi_{pq} : T_pM \to T_qM$ satisfying $\Phi_{pq}^* \nabla^i R_q = \nabla^i R_p$ for all $0 \leq i \leq k$. For $k = 0$, we simply say that the manifold is *curvature homogeneous*. Clearly, any locally homogeneous pseudo-Riemannian manifold is k-curvature homogeneous for all $k \geq 0$. However, a manifold can be k-curvature homogeneous for some k even if it is not homogeneous. A three-dimensional Riemannian manifold is curvature homogeneous if the Ricci operator has constant eigenvalues, and 1-curvature homogeneity implies local homogeneity [26]. In contrast to the Riemannian setting, the Ricci operator of a Lorentzian manifold is not always diagonalizable. Thus, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, it corresponds to one of the following Jordan normal forms (see for example [22]):

$$\underset{\text{Type I.a}}{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}}, \quad \underset{\text{Type I.b}}{\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}}, \quad \underset{\text{Type II}}{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & -1 & b \pm 2 \end{pmatrix}}, \quad \underset{\text{Type III}}{\begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}}.$$

Hence a three-dimensional Lorentzian manifold is curvature homogeneous if and only if the Ricci curvatures and the Jordan normal form of the Ricci operator do not change from point to point (see [20] for more information). Their study naturally splits into four general cases corresponding to the algebraic possibilities above, all of which are geometrically realizable as shown in [14]. Furthermore, while 2-curvature homogeneity guarantees local homogeneity, there are exactly two families of 1-curvature homogeneous Lorentzian three-manifolds which are not locally homogeneous (see [10]).

Locally symmetric manifolds are an important class of homogeneous manifolds. Three-dimensional Lorentzian symmetric manifolds are of constant sectional curvature, a direct product $N(\lambda) \times \mathbb{R}$ (where $N(\lambda)$ is a surface of constant Gauss

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curvature) or a Cahen-Wallach symmetric space [13]. A pseudo-Riemannian manifold is said to be *semi-symmetric* if the curvature tensor at each point coincides with that of a symmetric space (but possibly changing from point to point). Non-homogeneous three-dimensional curvature homogeneous pseudo-Riemannian manifolds modeled on a symmetric space either have diagonalizable Ricci operator, $\mathrm{Ric} = \mathrm{diag}[\lambda, \lambda, 0]$, as in direct products $N(\lambda) \times \mathbb{R}$, or the Ricci operator is two-step nilpotent, as in Cahen-Wallach symmetric spaces. They constitute the simplest three-dimensional Lorentzian manifolds besides the symmetric ones mentioned above.

1.2. **Quadratic curvature functionals.** Let (M, g) be a pseudo-Riemannian manifold. The Einstein-Hilbert functional given by the average of the scalar curvature τ, $g \mapsto \int_M d^3x\sqrt{|g|}\,\tau$, has been widely investigated. Einstein metrics correspond to critical metrics of this functional constrained to constant volume. Second order scalar curvature invariants naturally lead to quadratic curvature functionals, which have been considered both in geometry and physics (see, for example, [3, 4, 15, 16, 21, 28]). In dimension three any quadratic curvature functional is a multiple of

$$\mathcal{S} : g \mapsto \int_M d^3x\sqrt{|g|}\,\tau^2, \quad \text{or} \quad \mathcal{F}_t : g \mapsto \int_M d^3x\sqrt{|g|}\,\{\|\rho\|^2 + t\tau^2\},$$

for some $t \in \mathbb{R}$. Euler-Lagrange equations corresponding to critical metrics of the functionals above constrained to constant volume were given in [2]. Thus, critical metrics for the functional $\mathcal{S}$ in dimension three are characterized by

$$\nabla^2\tau - \tfrac{1}{3}\Delta\tau g - \tau\left(\rho - \tfrac{1}{3}\tau g\right) = 0, \tag{1}$$

and critical metrics for a functional $\mathcal{F}_t$ satisfy

$$\Delta\rho - (1+2t)\nabla^2\tau + \tfrac{2}{3}t\Delta\tau g + 2(R[\rho] - \tfrac{1}{3}\|\rho\|^2 g) + 2t\tau(\rho - \tfrac{1}{3}\tau g) = 0, \tag{2}$$

where $R[\rho]_{ij} = R_{ikjl}\rho^{kl}$ is the contraction of the curvature and the Ricci tensors.

It follows directly from equation (1) that metrics with $\tau = 0$ are $\mathcal{S}$-critical. Also, Einstein metrics are $\mathcal{S}$-critical and, from equation (2), it follows that three-dimensional Einstein metrics are $\mathcal{F}_t$-critical for all $t \in \mathbb{R}$.

Any curvature homogeneous manifold has constant scalar curvature, so equation (1) reduces to $\tau\left(\rho - \frac{1}{3}\tau g\right) = 0$. Therefore, a curvature homogeneous metric is $\mathcal{S}$-critical if and only if it is Einstein or its scalar curvature vanishes. Equation (2) also simplifies if τ is constant and a three-dimensional curvature homogeneous metric is $\mathcal{F}_t$-critical if and only if it satisfies

$$\Delta\rho + 2(R[\rho] - \tfrac{1}{3}\|\rho\|^2 g) + 2t\tau(\rho - \tfrac{1}{3}\tau g) = 0. \tag{3}$$

Three-dimensional homogeneous Lorentzian manifolds which are critical for some quadratic curvature functional have been classified in [8]. The homogeneity assumption allows to describe (1)-(2) as a system of polynomial equations whose analysis is algebraic in nature. Semi-symmetric curvature homogeneous manifolds, being non-homogeneous relatives of symmetric spaces, are as homogeneous as they can be from the point of view of their curvature. However, the Euler-Lagrange equations (1)-(2) now become equivalent to a (generically overdetermined) system of nonlinear PDEs, so the present situation is substantially different from the homogeneous one.

1.3. **Kundt spacetimes.** A Lorentzian manifold (M, g) is said to be a *Kundt spacetime* if it admits a null geodesic vector field ℓ which is shear-free, twist-free and expansion-free. Since in dimension three both the twist and the shear vanish, a spacetime is Kundt if it admits an expansion-free null geodesic vector field (i.e.,

$\|\ell\|^2 = 0$, $\nabla_\ell \ell = 0$, and $\operatorname{Tr} \nabla \ell = 0$). Besides its physical significance, Kundt spacetimes are also geometrically relevant since any Lorentzian metric with vanishing scalar curvature invariants (VSI) is necessarily Kundt [17, 18].

The general form of a three-dimensional Kundt spacetime in local coordinates (v, u, x) is

$$g = \frac{1}{P(u,x)^2} dx^2 + 2dudv + f(v,u,x)du^2 + 2W(v,u,x)dudx,$$

and one may assume $P(u, x) = 1$ by considering the coordinate transformation $\tilde{x} = \int \frac{1}{P} dx$ (see, for example, the discussion in [16]). Hence a three-dimensional Kundt spacetime locally takes the form

$$g = dx^2 + 2dudv + f(v,u,x)du^2 + 2W(v,u,x)dudx. \tag{4}$$

A Kundt metric is said to be degenerate if $\partial_{vv} W = \partial_{vvv} f = 0$. We refer to [17, 25] for more information on the geometry of Kundt spacetimes.

A Kundt spacetime is said to be a *Brinkmann wave* if $\mathcal{L} = \operatorname{span}\{\ell\}$ is a parallel null line field. In dimension three, the metric takes the form (4) when expressing in Kundt coordinates (v, u, x). Moreover, these may be further specialized so that the metric takes the form (see [5])

$$g = dx^2 + 2dudv + f(v,u,x)du^2. \tag{5}$$

The parallel null line field is locally generated by the null recurrent vector field ∂_v. If the null vector field may be re-scaled to be parallel (which occurs if and only if the Ricci operator is two-step nilpotent), then coordinates may be chosen so that the defining function $f(v, u, x)$ does not depend on the coordinate v and the three-dimensional manifold (M, g) is a *pp*-wave. Three-dimensional Brinkmann waves which are critical for some quadratic curvature functional have been investigated in [7] (see also [15]).

1.4. **Main results.** Our aim is to characterize semi-symmetric curvature homogeneous metrics which are critical for some quadratic curvature functional and provide local coordinates for all of them. Semi-symmetric curvature homogeneous manifolds behave differently with respect to curvature functionals depending on the structure of the Ricci operator. If the Ricci operator is diagonalizable and the metric is not symmetric, then it can only be critical for the functional $\mathcal{F}_{-1/2}$, in which case it is a Brinkmann wave as follows

Theorem 1.1. *Let (M, g) be a three-dimensional semi-symmetric curvature homogeneous manifold with diagonalizable Ricci operator which is not homogeneous. If (M, g) is critical for some quadratic curvature functional, then it is $\mathcal{F}_{-1/2}$-critical and it is a 1-curvature homogeneous Brinkmann wave.*

Furthermore, there exist local coordinates (v, u, x) so that the metric tensor is of the form $g = dx^2 + 2dudv + f(v, u, x)du^2$, where

$$f(v,u,x) = \lambda v^2 + v(\alpha(u) + x\beta(u)) + \frac{x^2\beta(u)^2}{4\lambda} + x\delta(u) + \gamma(u)$$

for smooth functions α, β, γ, δ and a constant $\lambda \neq 0$.

It is worth emphasizing that semi-symmetric curvature homogeneous critical Riemannian metrics are locally symmetric (see Remark 2.7). Metrics in Theorem 1.1 are locally conformally flat if and only if they are locally symmetric (hence locally isometric to a product $N(\lambda) \times \mathbb{R}$). Moreover, they are steady gradient Cotton solitons (see Section 2.1).

On the other hand, if the Ricci operator is two-step nilpotent, then the scalar curvature vanishes and metrics are $\mathcal{S}$-critical. Hence, they are critical for some other quadratic curvature functional if and only if they are $\mathcal{F}_t$-critical for all $t \in \mathbb{R}$, and one has

Theorem 1.2. *Let (M, g) be a three-dimensional semi-symmetric curvature homogeneous manifold with two-step nilpotent Ricci operator which is not homogeneous. If (M, g) is critical for some quadratic curvature functional $\mathcal{F}_t$, then it is critical for all quadratic curvature functionals and it is a degenerate Kundt spacetime.*

Furthermore, there exist local coordinates (v, u, x) so that

$$g = dx^2 + 2dudv + \left(\frac{v^2}{x^2} + vf_1(u) + f_0(u,x)\right) du^2 - 4\frac{v}{x}dudx, \tag{6}$$

where $f_1(u)$ is an arbitrary function and $f_0(u, x)$ is given by a 4^{th}-degree polynomial in x: $f_0(u, x) = \alpha_4(u)x^4 + \alpha_3(u)x^3 + \alpha_2(u)x^2 + \alpha_1(u)x$, with $\alpha_3(u)^2 + \alpha_4(u)^2 \neq 0$.

The Cotton tensor of Kundt metrics in Theorem 1.2 vanishes if and only if $\alpha_4(u) = 0$. Moreover Kundt metrics above are locally symmetric if and only if $\alpha_4(u) = \alpha_3(u) = 0$, in which case they are flat (see Remark 3.5). Furthermore, all metrics (6) have vanishing scalar curvature invariants (see Remark 3.6).

Theorems 1.1 and 1.2 will be proved in Sections 2 and 3, respectively, where we analyze curvature homogeneous critical metrics in more detail. Finally, in Section 4, we use results in Sections 2 and 3 to study critical metrics for massive gravity functionals. This provides new solutions for the corresponding gravity theories.

2. Semi-symmetric curvature homogeneous metrics with diagonalizable Ricci operator

The purpose of this section is to analyze criticallity of curvature homogeneous metrics modeled on a product $N(\lambda)\times\mathbb{R}$. The functional $\mathcal{F}_{-1/2}$ given by $\mathcal{F}_{-1/2}(g) = \int_M d^3x\sqrt{|g|}\,\{\|\rho\|^2 - \frac{1}{2}\tau^2\}$ plays a distinguished role in the symmetric setting [8], as all symmetric manifolds are $\mathcal{F}_{-1/2}$-critical.

Let (M, g) be a three-dimensional Lorentzian manifold whose Ricci operator is of the form $\mathrm{Ric} = \mathrm{diag}[\lambda, \lambda, 0]$. Since the Ricci operator is self-adjoint, the distribution $\ker \mathrm{Ric}$ cannot be null, so it is either timelike or spacelike. We follow the work in [12] and let $\{E_1, E_2, E_3\}$ be a local orthonormal frame diagonalizing the Ricci operator with $\ker \mathrm{Ric} = \mathrm{span}\{E_3\}$. Let $\tilde\sigma$ be the associated shear operator (i.e., the self-adjoint component of the traceless tensor $S^0 = S - \frac{1}{3}(\mathrm{tr}\, S)Id$ associated to $S(X) = \nabla_X E_3$). The second Bianchi identity shows that $\tilde\sigma(E_3) = 0$ and hence the shear operator has Jordan normal form

$$\underset{\text{Type I.a}}{\begin{pmatrix} a & 0 & 0\\ 0 & -a & 0\\ 0 & 0 & 0\end{pmatrix}}, \qquad \underset{\text{Type I.b}}{\begin{pmatrix} 0 & -a & 0\\ a & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}, \qquad \text{or} \qquad \underset{\text{Type II}}{\begin{pmatrix} \varepsilon & -1 & 0\\ 1 & -\varepsilon & 0\\ 0 & 0 & 0\end{pmatrix}},$$

where $\varepsilon = \pm 1$. Furthermore the norm of the shear tensor is positive for Type I.a, negative for Type I.b and zero for Type II. The semi-symmetric curvature homogeneous manifolds with diagonalizable Ricci operator are described as follows.

Lemma 2.1. *Let (M, g) be a three-dimensional semi-symmetric curvature homogeneous Lorentzian manifold with diagonalizable Ricci operator. Then, (M, g) is locally symmetric or it is locally isometric to one of the following models:*

(1) *Diagonalizable shear operator. For a local orthonormal frame $\{E_1, E_2, E_3\}$ of signature $(+ + -)$ and smooth functions θ, φ, Lie brackets are given by*

$$[E_1, E_2] = \varepsilon\theta E_1 - \theta E_2 + 2\varphi E_3,$$

$$[E_1, E_3] = \varepsilon\varphi E_1 + \varphi E_2, \quad [E_2, E_3] = -\varphi E_1 - \varepsilon\varphi E_2, \qquad (\varepsilon = \pm 1),$$

where θ and φ are smooth functions satisfying the differential equations $E_3(\theta) = E_3(\varphi) = 0$, and

$$E_1(\theta) + \varepsilon E_2(\theta) + 2\theta^2 + \lambda = 0, \qquad E_1(\varphi) + \varepsilon E_2(\varphi) + 2\theta\varphi = 0.$$

(2) *Shear operator with complex eigenvalues. For a local orthonormal frame $\{E_1, E_2, E_3\}$ of signature $(-++)$, Lie brackets are given by*

(2.i) $[E_1, E_2] = \theta E_1 - 2\varphi E_3, \quad [E_1, E_3] = 2\varphi E_2, \quad [E_2, E_3] = 0,$

where the functions θ, φ satisfy the differential equations $E_3(\theta) = E_3(\varphi) = 0$, $E_2(\theta) + \theta^2 + \lambda = 0$, and $E_2(\varphi) + \theta\varphi = 0$.

(2.ii) $[E_1, E_2] = -\theta E_2 - 2\varphi E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 2\varphi E_1,$

where the functions θ, φ satisfy the differential equations $E_3(\theta) = E_3(\varphi) = 0$, $E_1(\theta) + \theta^2 - \lambda = 0$, and $E_1(\varphi) + \theta\varphi = 0$.

(3) *Two-step nilpotent shear operator. For a local orthonormal frame $\{E_1, E_2, E_3\}$ of signature $(-++)$, Lie brackets are given by*

$$[E_1, E_2] = -\theta(E_1 + E_2), \quad [E_1, E_3] = E_1 + E_2, \quad [E_2, E_3] = -E_1 - E_2,$$

where θ satisfies the differential equations $E_3(\theta) = 0$ and $E_1(\theta) + E_2(\theta) = \lambda$.

Proof. Let (M, g) be a curvature homogeneous Lorentzian three-manifold with diagonalizable Ricci operator $\mathrm{Ric} = \mathrm{diag}[\lambda, \lambda, \alpha]$. If the shear operator is of Type I.a, then it was shown in [12, Theorem 1] that there exists an orthonormal basis $\{E_1, E_2, E_3\}$ such that $g(E_1, E_1) = \kappa = \pm 1$, $g(E_2, E_2) = 1$, $g(E_3, E_3) = -\kappa$, $\rho(E_1, E_1) = \kappa\lambda$, $\rho(E_2, E_2) = \lambda$ and $\rho(E_3, E_3) = -\kappa\alpha$, and there exist functions θ_1, θ_2, ω and σ satisfying

$$[E_1, E_2] = \theta_2 E_1 - \theta_1 E_2 + 2\kappa\omega E_3,$$
$$[E_1, E_3] = \sigma E_1 + \omega E_2, \quad [E_2, E_3] = -\kappa\omega E_1 - \sigma E_2, \qquad \alpha = 2(\kappa\sigma^2 - \omega^2),$$
$$\begin{array}{ll} E_3(\omega) = E_3(\sigma) = 0, & \kappa E_1(\theta_1) + E_2(\theta_2) + \kappa\theta_1^2 + \theta_2^2 + \lambda = 0, \\ E_3(\theta_1) - \sigma\theta_1 + \omega\theta_2 = 0, & E_3(\theta_2) - \kappa\omega\theta_1 + \sigma\theta_2 = 0, \\ E_1(\sigma) + E_2(\omega) + 2\theta_1\sigma = 0, & \kappa E_1(\omega) + E_2(\sigma) + 2\theta_2\sigma = 0. \end{array}$$

Setting $\alpha = 0$, one has three possible cases: $\kappa = 1$ and $\sigma = \pm\omega$, or $\kappa = -1$ and $\sigma = \omega = 0$. If $\kappa = -1$ and $\sigma = \omega = 0$, then the manifold is locally symmetric. If $\kappa = 1$ and $\sigma = \omega$, then a straightforward calculation on the differential equations above shows that, in addition, $\omega = 0$, and the manifold is locally symmetric, or $\theta_1 = \theta_2$, which corresponds to Assertion (1) with $\varepsilon = 1$. If $\kappa = 1$ and $\sigma = -\omega$, then the differential equations above show that, in addition, $\omega = 0$, and the manifold is locally symmetric, or $\theta_1 = -\theta_2$, which corresponds to the case $\varepsilon = -1$ in Assertion (1).

If the shear operator is of Type I.b, then one has from [12, Theorem 2] that there exists an orthonormal basis $\{E_1, E_2, E_3\}$ such that $g(E_1, E_1) = -1$, $g(E_2, E_2) = 1$, $g(E_3, E_3) = 1$, $\rho(E_1, E_1) = -\lambda$, $\rho(E_2, E_2) = \lambda$ and $\rho(E_3, E_3) = \alpha$, and there exist functions θ_1, θ_2, ω and σ satisfying

$$[E_1, E_2] = \theta_2 E_1 - \theta_1 E_2 - 2\omega E_3,$$
$$[E_1, E_3] = (\sigma + \omega)E_2, \quad [E_2, E_3] = -(\sigma - \omega)E_1, \qquad \alpha = 2(\sigma^2 - \omega^2),$$
$$\begin{array}{ll} E_3(\omega) = E_3(\sigma) = 0, & E_1(\theta_1) - E_2(\theta_2) + \theta_1^2 - \theta_2^2 - \lambda = 0, \\ E_3(\theta_1) - \theta_2(\sigma - \omega) = 0, & E_3(\theta_2) + \theta_1(\sigma + \omega) = 0, \\ E_2(\sigma + \omega) + 2\theta_2\sigma = 0, & E_1(\sigma - \omega) + 2\theta_1\sigma = 0. \end{array}$$

Setting $\alpha = 0$, one has $\sigma = \pm\omega$. If $\sigma = \omega$, then a straightforward calculation shows that, in addition, $\omega = 0$, and the manifold is locally symmetric, or $\theta_1 = 0$, which corresponds to Assertion (2.i). If $\sigma = -\omega$, then the differential equations above show that $\omega = 0$, and the manifold is locally symmetric, or $\theta_2 = 0$, which corresponds to Assertion (2.ii) of the lemma.

Finally, if the shear operator is of Type II, then it follows from [12, Theorem 3] that there exists an orthonormal basis $\{E_1, E_2, E_3\}$ such that $g(E_1, E_1) = -1$, $g(E_2, E_2) = 1$, $g(E_3, E_3) = 1$, $\rho(E_1, E_1) = -\lambda$, $\rho(E_2, E_2) = \lambda$ and $\rho(E_3, E_3) = \alpha$, and there exist functions θ and ω satisfying

$$[E_1, E_2] = -\theta E_1 - \theta E_2 - 2\omega E_3,$$
$$[E_1, E_3] = E_1 + (\omega + 1)E_2, \quad [E_2, E_3] = (\omega - 1)E_1 - E_2, \qquad \alpha = -2\omega^2,$$
$$E_i(\omega) = 0, \quad E_3(\theta) - \omega\theta = 0, \quad E_1(\theta) + E_2(\theta) - \lambda = 0.$$

Setting $\alpha = 0$, one has that $\omega = 0$, which corresponds to Assertion (3). □

Remark 2.2. Manifolds corresponding to Assertions (1)-(2) are locally symmetric if and only if $\varphi = 0$. On the other hand, manifolds in Assertion (3) are never locally symmetric.

Now, we turn back our attention to critical metrics. First of all, observe that the Euler-Lagrange equation (2) strongly simplifies in the semi-symmetric setting with diagonalizable Ricci operator as follows.

Lemma 2.3. *A three-dimensional metric with diagonalizable Ricci operator of the form* $\mathrm{Ric} = \mathrm{diag}[\lambda, \lambda, 0]$ *for* λ *constant is* $\mathcal{F}_{-1/2}$*-critical if and only if* $\Delta\rho = 0$.

Proof. We assume $\mathrm{Ric} = \mathrm{diag}[\lambda, \lambda, 0]$ with λ constant. Then $\tau = 2\lambda$ is constant too. Now, a straightforward calculation shows that $R[\rho] = \check{\rho}$, where $\check{\rho}(X, Y) = g(\mathrm{Ric}^2 X, Y)$, and (3) reduces to

$$\Delta\rho + \frac{2}{3}\lambda^2(1 + 2t)\,\mathrm{diag}[1, 1, -2]g = 0.$$

Hence, it follows that $t = -\frac{1}{2}$ if and only if $\Delta\rho = 0$. □

The next result shows that only metrics with nilpotent shear operator (i.e. those described in Lemma 2.1-(3)) are critical for some quadratic curvature functional within the non locally homogeneous setting.

Lemma 2.4. *Let* (M, g) *be a three-dimensional semi-symmetric curvature homogeneous Lorentzian manifold with diagonalizable Ricci operator which is not homogeneous. If* (M, g) *is critical for some quadratic curvature functional, then it is locally given by*

$$[E_1, E_2] = -\theta(E_1 + E_2), \quad [E_1, E_3] = E_1 + E_2, \quad [E_2, E_3] = -E_1 - E_2,$$

for some function θ *satisfying the differential equations* $E_3(\theta) = 0$ *and* $E_1(\theta) + E_2(\theta) = \lambda$, *where* $\{E_1, E_2, E_3\}$ *is a local orthonormal frame of signature* $(-++)$. *Furthermore, these metrics are* $\mathcal{F}_{-1/2}$*-critical and* 1*-curvature homogeneous.*

Proof. We analyze the different possibilities in Lemma 2.1. Attending to Equation (3), (M, g) is $\mathcal{F}_t$-critical if and only if the symmetric $(0, 2)$-tensor field $\mathfrak{F}_t = \Delta\rho + 2(R[\rho] - \frac{1}{3}\|\rho\|^2 g) + 2t\tau(\rho - \frac{1}{3}\tau g)$ vanishes identically.

Tensors ρ and $R[\rho]$ for manifolds in Lemma 2.1 (1) are given by $\rho(E_1, E_1) = \rho(E_2, E_2) = \lambda$ and $R[\rho](E_1, E_1) = R[\rho](E_2, E_2) = \lambda^2$, respectively (other components being zero). Moreover, the components of the Laplacian of the Ricci tensor are given by

$$\Delta\rho(E_1, E_1) = \Delta\rho(E_2, E_2) = \varepsilon\Delta\rho(E_1, E_2) = 4\lambda\varphi^2, \qquad \Delta\rho(E_3, E_3) = 8\lambda\varphi^2,$$
$$\Delta\rho(E_1, E_3) = -2\varepsilon\lambda(2\theta\varphi + E_1(\varphi)), \qquad \Delta\rho(E_2, E_3) = -2\lambda E_1(\varphi).$$

Hence, $\mathfrak{F}_t(E_1, E_2) = 4\varepsilon\lambda\varphi^2$ and so, necessarily, $\varphi = 0$. Therefore, $\Delta\rho = 0$ and, by Lemma 2.3, these metrics are critical for $t = -\frac{1}{2}$. Moreover, from Remark 2.2, we have that all critical metrics in this case are locally symmetric, so we conclude that metrics in Lemma 2.1-(1) are $\mathcal{F}_t$-critical if and only if they are locally isometric to a product $N(\lambda)\times\mathbb{R}$, in which case they are only $\mathcal{F}_{-1/2}$-critical (see [8]).

The Ricci tensor and the curvature contraction $R[\rho]$ of manifolds in Lemma 2.1-(2) have non-zero components $\rho(E_1,E_1)=-\rho(E_2,E_2)=-\lambda$, and $R[\rho](E_1,E_1)=-R[\rho](E_2,E_2)=-\lambda^2$. Moreover, $\Delta\rho$ is determined by the non-zero terms

$$\begin{aligned}&\Delta\rho(E_2,E_2)=-\Delta\rho(E_3,E_3)=8\lambda\varphi^2,\\ &\Delta\rho(E_1,E_3)=-2\lambda\theta\varphi, \qquad\qquad \Delta\rho(E_2,E_3)=2\lambda E_1(\varphi),\end{aligned}$$

for manifolds in Lemma 2.1-(2.i), and

$$\begin{aligned}&\Delta\rho(E_1,E_1)=\Delta\rho(E_3,E_3)=-8\lambda\varphi^2,\\ &\Delta\rho(E_1,E_3)=2\lambda E_2(\varphi), \qquad\qquad \Delta\rho(E_2,E_3)=-2\lambda\theta\varphi,\end{aligned}$$

for manifolds in Lemma 2.1-(2.ii). Thus, for manifolds in Lemma 2.1-(2.i) we have $\mathfrak{F}_t(E_1,E_1)=-\frac{2}{3}(1+2t)\lambda^2$ and, for manifolds in Lemma 2.1-(2.ii), we have $\mathfrak{F}_t(E_2,E_2)=\frac{2}{3}(1+2t)\lambda^2$. Hence $t=-\frac{1}{2}$ in both cases. Now, by Lemma 2.3, these metrics are critical if and only if $\Delta\rho=0$, which occurs only if $\varphi=0$. Hence, by Remark 2.2, the only critical metrics in these two families are locally isometric to a product $N(\lambda)\times\mathbb{R}$ and are critical for the functional $\mathcal{F}_{-1/2}$.

Finally, let (M,g) be as in Lemma 2.1-(3). Then $\rho(E_1,E_1)=-\rho(E_2,E_2)=-\lambda$, and $R[\rho](E_1,E_1)=-R[\rho](E_2,E_2)=-\lambda^2$. Furthermore the Laplacian of the Ricci tensor vanishes identically so, by Lemma 2.3, all curvature homogeneous metrics in Lemma 2.1-(3) are $\mathcal{F}_{-1/2}$-critical. Moreover, the non-zero components of the covariant derivative of the Ricci tensor are given by

$$(\nabla_{E_1}\rho)(E_1,E_3)=-(\nabla_{E_2}\rho)(E_1,E_3)=(\nabla_{E_2}\rho)(E_2,E_3)=-(\nabla_{E_1}\rho)(E_2,E_3)=\lambda,$$

which shows that (M,g) is indeed 1-curvature homogeneous. Results in [8] show that metrics in Lemma 2.1-(3) cannot be locally homogeneous. □

Proof of Theorem 1.1. It follows from Lemma 2.4 that a non-Einstein semi-symmetric curvature homogeneous critical metric with diagonalizable Ricci operator is necessarily 1-curvature homogeneous and it is critical for $t=-\frac{1}{2}$. Furthermore, the non-zero components of the Levi-Civita connection for metrics in Lemma 2.1-(3) are

$$\begin{aligned}&\nabla_{E_1}E_1=E_3-E_2\theta, \quad \nabla_{E_1}E_2=-E_1\theta-E_3, \quad \nabla_{E_1}E_3=E_1+E_2,\\ &\nabla_{E_2}E_1=-E_3+E_2\theta, \quad \nabla_{E_2}E_2=E_1\theta+E_3, \quad \nabla_{E_2}E_3=-E_1-E_2.\end{aligned}$$

Hence $\ell=E_1+E_2$ is a recurrent null vector field, so $\mathcal{L}=\operatorname{span}\{\ell\}$ is a parallel null line field and the manifold is a Brinkmann wave.

We consider a Brinkmann wave in local coordinates (v,u,x) of the form $g=dx^2+2dudv+f(v,u,x)du^2$. The eigenvalues of the Ricci operator are $\{\frac{1}{2}\partial_{vv}f,\frac{1}{2}\partial_{vv}f,0\}$, so $\frac{1}{2}\partial_{vv}f=\lambda$. Hence $f(v,u,x)=\lambda v^2+a(u,x)v+b(u,x)$ for some functions a and b. Now, we have

$$\operatorname{Ric}(\partial_v)=\lambda\partial_v,\ \operatorname{Ric}(\partial_u)=-\frac{1}{2}\left(\partial_{xx}b+v\partial_{xx}a\right)\partial_v+\lambda\partial_u+\frac{1}{2}\partial_x a\partial_x,\ \operatorname{Ric}(\partial_x)=\frac{1}{2}\partial_x a\partial_v.$$

From these expressions it is straightforward to see that, for the Ricci operator to be diagonalizable, one has $f(v,u,x)=\lambda v^2+v(\alpha(u)+x\beta(u))+\frac{x^2\beta(u)^2}{4\lambda}+x\delta(u)+\gamma(u)$. It follows from [7] that all these metrics are $\mathfrak{F}_{-1/2}$-critical. □

Remark 2.5. Note that the energy $\|\rho\|^2-\frac{1}{2}\tau^2$ vanishes for all metrics in Theorem 1.1.

Remark 2.6. Three-dimensional 1-curvature homogeneous critical Lorentzian metrics which are not homogeneous have been classified in [8]. Thus, notice that the family of metrics given in Theorem 1.1 corresponds to that in Theorem 2.1-(1) in [8], although the description is given with respect to a different frame.

Remark 2.7. Let (M, g) be a three-dimensional semi-symmetric curvature homogeneous Riemannian manifold. Then, (M, g) is locally symmetric or it admits a local orthonormal frame $\{E_1, E_2, E_3\}$ so that [9]

$$[E_1, E_2] = \theta(\varepsilon E_1 - E_2) - 2\varphi E_3, \quad [E_1, E_3] = \varphi(\varepsilon E_1 + E_2), \quad [E_2, E_3] = -\varphi(E_1 + \varepsilon E_2),$$

for some smooth functions θ, φ satisfying $E_3(\theta) = E_3(\varphi) = 0$, and

$$E_1(\theta) + \varepsilon E_2(\theta) + 2\theta^2 + \lambda = 0, \quad E_1(\varphi) + \varepsilon E_2(\varphi) + 2\theta\varphi = 0.$$

A similar argument to that given in Lemma 2.4 shows that if one of the metrics in these manifolds is critical, then either $\lambda = 0$ and the metric is flat, or $\varphi = 0$ and the manifolds is locally isometric to a product $N(c) \times \mathbb{R}$. Moreover, in this case the metric is $\mathcal{F}_{-1/2}$-critical. Hence, we conclude that

> *Three-dimensional semi-symmetric curvature homogeneous Riemannian manifolds are critical for some quadratic curvature functional if and only if they are locally symmetric.*

Alternatively, it was shown in [6] that any non-Einstein curvature homogeneous semi-symmetric Riemannian manifold admits local coordinates (v, u, x) such that the metric is homothetic to

$$g = (\cosh u - h(x)\sinh u)^2 dx^2 + (du - f(x)v\, dx)^2 + (dv + f(x)u\, dx)^2. \tag{7}$$

Straightforward calculations show that metrics above are critical for some quadratic curvature functional if and only if $f = 0$, in which case (M, g) is locally symmetric.

2.1. **Cotton Solitons.** Let $S_{ij} = \rho_{ij} - \frac{\tau}{2(n-1)} g_{ij}$ denote the Schouten tensor. The *Cotton tensor*, $C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik}$, is the unique conformal invariant in dimension three. Moreover, C vanishes if and only if the manifold is locally conformally flat. The associated $(0,2)$-Cotton tensor, which is trace-free and divergence-free, is defined in dimension three by

$$C_{ij} = \frac{1}{2\sqrt{|g|}} C_{nmi} \epsilon^{nm\ell} g_{\ell j}, \tag{8}$$

where $\epsilon^{123} = 1$. The Cotton flow, which was introduced in [24], is determined by the evolution equation $\partial_t g(t) = \kappa\, C(t)$, where $C(t)$ is the $(0,2)$-Cotton tensor of $(M, g(t))$ and κ is a constant (see also [23]). Self-similar solutions of the flow, i.e. solutions which remain fixed up to scalings and diffeomorphisms, are the Cotton solitons. A *gradient Cotton soliton* is a triple (M, g, ϕ), where (M, g) is a pseudo-Riemannian manifold and ϕ is a smooth function satisfying

$$\nabla^2\phi + C = \mu\, g \tag{9}$$

for $\mu \in \mathbb{R}$. A Cotton soliton is said to be *shrinking*, *steady* or *expanding* if $\mu > 0$, $\mu = 0$ or $\mu < 0$, respectively. Also note that, taking traces in (9), one has $\mu = \frac{1}{3}\Delta\phi$.

We consider the metric $g = dx^2 + 2dudv + f(v, u, x)du^2$ with

$$f(v, u, x) = \lambda v^2 + v(\alpha(u) + x\beta(u)) + \frac{x^2\beta(u)^2}{4\lambda} + x\delta(u) + \gamma(u)$$

as in Theorem 1.1. The only non-zero component of the $(0,2)$-Cotton tensor is $C(\partial_u, \partial_u) = \frac{1}{4}\{\alpha(u)\beta(u) - 2\lambda\delta(u) + 2\beta'(u)\}$, which vanishes if and only if (M, g) is locally symmetric, indeed locally isometric to a product $N(\lambda) \times \mathbb{R}$.

Let $\phi(u, x, y)$ be a smooth function. In order to analyze the gradient Cotton soliton equation, consider the symmetric $(0,2)$-tensor field $\mathfrak{C} = \nabla^2\phi + C - \mu g$. A direct calculation gives the following expressions

$$\begin{aligned}
&\mathfrak{C}(\partial_v,\partial_v)=\partial_{vv}\phi, \qquad \mathfrak{C}(\partial_v,\partial_x)=\partial_{vx}\phi, \qquad \mathfrak{C}(\partial_x,\partial_x)=\partial_{xx}\phi-\mu,\\
&\mathfrak{C}(\partial_v,\partial_u)=\partial_{vu}\phi-\tfrac{1}{2}(2v\lambda+\alpha(u)+x\beta(u))\partial_v\phi-\mu,\\
&\mathfrak{C}(\partial_u,\partial_x)=\partial_{xu}\phi-\tfrac{1}{4\lambda}(2v\lambda\beta(u)+x\beta(u)^2+2\lambda\delta(u))\partial_v\phi.
\end{aligned}$$

The first three equations give $\phi(v,u,x)=\phi_2(u)v+\frac{\mu}{2}x^2+\phi_1(u)x+\phi_0(u)$, and using that $\mathfrak{C}(\partial_v,\partial_u)=0$, one has that $\phi_2(u)=0$ and that $\mu=0$. Therefore, the gradient Cotton soliton is necessarily steady. Moreover, the component $\mathfrak{C}(\partial_u,\partial_x)$ now reduces to $\mathfrak{C}(\partial_u,\partial_x)=\phi_1'(u)$. Hence the function ϕ_1 is constant and the potential function becomes $\phi(v,u,x)=\kappa\,x+\varphi(u)$. The remaining component now reduces to

$$\begin{aligned}
\mathfrak{C}(\partial_u,\partial_u) \;=\; & \tfrac{1}{4\lambda}\left\{(2\lambda v+x\beta(u))\left(\kappa\beta(u)+2\lambda\varphi'(u)\right)\right.\\
&\left.+\lambda\alpha(u)(\beta(u)+2\varphi'(u))+2\lambda\beta'(u)+2\lambda(\kappa-\lambda)\delta(u)+4\lambda\varphi''(u)\right\}.
\end{aligned}$$

Hence $\varphi'(u)=-\frac{\kappa}{2\lambda}\beta(u)$, and the expression above simplifies to

$$\mathfrak{C}(\partial_u,\partial_u)=\frac{1}{4\lambda}(\lambda-\kappa)(\alpha(u)\beta(u)-2\lambda\delta(u)+2\beta'(u)).$$

Now one has that $\kappa=\lambda$ (unless the manifold is Cotton-flat and locally symmetric). This shows that for $\phi(v,u,x)=\lambda x-\frac{1}{2}\int\beta(u)du$, these Brinkmann waves are steady gradient Cotton solitons. Summarizing the above, one has that

> *Any three-dimensional curvature homogeneous semi-symmetric manifold with diagonalizable Ricci operator which is critical for a quadratic curvature functional is a steady gradient Cotton soliton.*

3. Semi-symmetric curvature homogeneous metrics with two-step nilpotent Ricci operator

Next we consider semi-symmetric manifolds modeled on a Cahen-Wallach symmetric space. The Ricci operator of these manifolds is two-step nilpotent, hence the scalar curvature vanishes and they are critical for the functional $\mathcal{S}$. If, moreover, a metric is $\mathcal{F}_t$-critical for some $t\in\mathbb{R}$, then it is critical for all quadratic curvature functionals [7]. This family of metrics provides examples of non-Einstein metrics which are critical for all quadratic curvature functionals.

Lemma 3.1. *A 3-dimensional manifold with two-step nilpotent Ricci operator is $\mathcal{F}_t$-critical for some t (and, hence, for all $t\in\mathbb{R}$) if and only if $\Delta\rho=0$.*

Proof. Since $\operatorname{Ric}^2=0$, we have that $\tau=0$ and $\|\rho\|^2=0$. Hence, $R[\rho]=-2\check{\rho}=0$ and the Euler-Lagrange equations (3) reduce to $\Delta\rho=0$. □

Lemma 3.2. *Let (M,g) be a semi-symmetric curvature homogeneous Lorentzian manifold with two-step nilpotent Ricci operator which is critical for some quadratic curvature functional. Then there exists a local pseudo-orthonormal frame $\{E_1,E_2,E_3\}$ with $g(E_1,E_1)=g(E_2,E_3)=1$ such that*

$$[E_1,E_2]=0,\qquad [E_1,E_3]=-AE_1-GE_2-(C+H)E_3,\qquad [E_2,E_3]=(H-C)E_1-IE_2,$$

for some smooth functions A,C,G,H,I satisfying the differential equations

$$\begin{array}{ll}
E_2(A)=-C(C+2H), & \\
E_1(C)=-2C(C+H), & E_2(C)=0,\\
E_1(G)=E_3(A)+A(I-A)-G(3C+H)+2\varepsilon, & E_2(G)-E_3(C)=A(H-C)\\
E_1(H)=-H^2, & E_2(H)=0,\\
E_1(I)=E_2(G)-I(C+H), & E_2(I)=C(2H-C),
\end{array}$$

where $\varepsilon=\pm1$. Moreover, these metrics are critical for all quadratic curvature functionals.

Proof. It follows from the work of Bueken [11] that a curvature homogeneous Lorentzian three-manifold with nilpotent Ricci operator admits a local pseudo-orthonormal frame $\{E_1, E_2, E_3\}$ with $g(E_1, E_1) = g(E_2, E_3) = 1$ so that

$$\begin{aligned} [E_1, E_2] &= -2FE_1, \\ [E_1, E_3] &= -AE_1 - GE_2 - (C + H)E_3, \\ [E_2, E_3] &= (H - C)E_1 - IE_2 - FE_3, \end{aligned} \tag{10}$$

with smooth functions A, C, F, G, H, I satisfying

$$\begin{array}{ll} E_1(C) - E_2(A) = -C^2 - AF, & E_2(F) = 3F^2, \\ E_1(H) - 2E_3(F) = -H^2 - 2F(I + A), & E_1(F) - E_2(C) = -4CF, \\ E_2(I) - E_3(F) = C(2H - C) - 2FI, & E_2(H) = 2F(H - C), \\ E_1(I) - E_2(G) = -FG - I(C + H), & \\ E_2(G) - E_3(C) = A(H - C) - 2FG, & \\ E_1(G) - E_3(A) = 2\varepsilon + A(I - A) - G(3C + H), & \varepsilon = \pm 1. \end{array} \tag{11}$$

The only non-zero component of the Ricci operator is $\mathrm{Ric}(E_3) = -2\varepsilon E_2$. Lemma 3.1 shows that a metric as above is critical for a quadratic curvature functional $\mathcal{F}_t$ if and only if $\Delta\rho = 0$. The non-zero components of the Laplacian of the Ricci tensor are given by

$$\begin{aligned} \Delta\rho(E_1, E_1) &= -16\varepsilon F^2, \\ \Delta\rho(E_1, E_3) &= 4\varepsilon(E_2(C) + F(3H - C)), \\ \Delta\rho(E_2, E_3) &= 8\varepsilon F^2, \\ \Delta\rho(E_3, E_3) &= -4\varepsilon(E_2(A) + 2E_3(F) + C(C + 2H) - 4AF). \end{aligned}$$

It follows that $\Delta\rho = 0$ if and only if $F = 0$, $E_2(C) = 0$ and $E_2(A) = -C(C + 2H)$. Now the result follows by simplifying the relations in (10) and (11). □

Remark 3.3. Let (M, g) be a metric as in Lemma 3.2. The non-zero components of $\nabla\rho$ are given by

$$(\nabla_{E_3}\rho)(E_1, E_3) = 2\varepsilon H, \quad (\nabla_{E_1}\rho)(E_3, E_3) = -4\varepsilon C, \quad (\nabla_{E_3}\rho)(E_3, E_3) = -4\varepsilon I.$$

Hence, metrics in Lemma 3.2 are locally symmetric if and only if the functions C, H and I vanish identically. Moreover if C, H and I are constant, then the metrics are 1-curvature homogeneous and results in [8, 10] show that they are indeed locally homogeneous.

Remark 3.4. The covariant derivative of the null vector field E_2 for any metric in Lemma 3.2 satisfies

$$\nabla_{E_1}E_2 = CE_2, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_3}E_2 = -HE_1 + IE_2.$$

If the function H vanishes identically, then $\mathcal{L} = \mathrm{span}\{E_2\}$ is a parallel null line field and, since the Ricci operator is two-step nilpotent, the underlying structure is a *pp*-wave. Hence there exist local coordinates (v, u, x) where the metric tensor expresses as $g = 2dvdu + dx^2 + (\alpha(u)x^3 + \beta(u)x^2)du^2$, for functions $\alpha(u)$, $\beta(u)$ (see [7, Corollary 3.3]).

Proof of Theorem 1.2. It follows from the expressions of the covariant derivative in Remark 3.4 that $\ell = E_2$ is an expansion-free null geodesic vector field (i.e., $\nabla_\ell \ell = 0$, $\|\ell\|^2 = 0$, and $\mathrm{Tr}\,\nabla\ell = 0$). Hence the underlying manifold is a Kundt spacetime.

We consider local coordinates (v, u, x) where the metric expresses as in (4). Since the Ricci operator of any metric in Lemma 3.2 is two-step nilpotent, $g(\mathrm{Ric}^2\,\partial_u, \partial_v) = \frac{1}{4}(\partial_{vv}W(v, u, x))^2 = 0$, so one has that

$$W(v, u, x) = vW_1(u, x) + W_0(u, x).$$

Now, the scalar curvature becomes $\tau = \partial_{vv} f + 2\partial_x W_1 - \frac{3}{2} W_1^2$ and, since $\tau = 0$, one obtains that

$$f(v,u,x) = v^2\left(\tfrac{3}{4} W_1(u,x)^2 - \partial_x W_1(u,x)\right) + v f_1(u,x) + f_0(u,x).$$

Then $g(\operatorname{Ric}^2 \partial_x, \partial_x) = (\partial_x W_1 - \frac{1}{2} W_1^2)^2 = 0$, and thus

$$W_1(u,x) = -\frac{2}{x+\omega(u)},$$

for an arbitrary smooth function $\omega(u)$. At this point, the form of the metric is preserved by a change of coordinates (see, for example, [16]) given by

$$\tilde{v} = v + F(u,x), \quad \tilde{u} = u, \quad \tilde{x} = x + \omega(u),$$

where F is an arbitrary function. An appropriate choice of F results in W having the form $W(\tilde{u}, \tilde{x}) = -2\frac{\tilde{v}}{\tilde{x}}$ in the new coordinates system. Thus, the metric tensor becomes

$$g = dx^2 + 2dudv + \left(\frac{v^2}{x^2} + v f_1(u,x) + f_0(u,x)\right) du^2 - 4\frac{v}{x} du dx.$$

For this form of the metric one has $g(\operatorname{Ric}^2 \partial_u, \partial_u) = \frac{1}{4}(\partial_x f_1)^2 = 0$, so f_1 does not depend on $\tilde{x}$. Now $\operatorname{Ric}^2 = 0$.

Since, by Lemma 3.1, critical metrics with 2-step nilpotent Ricci operator have harmonic Ricci tensor, an explicit calculation of $\Delta\rho$ shows that the only non-zero component is given by

$$\Delta\rho(\partial_u, \partial_u) = -\tfrac{1}{2}\partial_x^{(4)} f_0 + \tfrac{2}{x}\partial_x^{(3)} f_0 - \tfrac{6}{x^2}\partial_x^{(2)} f_0 + \tfrac{12}{x^3}\partial_x f_0 - \tfrac{12}{x^4} f_0.$$

Solving the equation $\Delta\rho = 0$, one has that the function f_0 is given by a 4^{th}-degree polynomial on the coordinate x without independent term: $f_0(u,x) = \alpha_4(u)x^4 + \alpha_3(u)x^3 + \alpha_2(u)x^2 + \alpha_1(u)x$, with $\alpha_3(u)^2 + \alpha_4(u)^2 \neq 0$, otherwise the metric is flat. □

Remark 3.5. A Kundt metric as in Theorem 1.2 is locally conformally flat if and only if the component $C(\partial_v, \partial_v) = 3x\alpha_4(u)$ vanishes identically, this is, if and only if $\alpha_4(u) = 0$. Moreover a straightforward calculation shows that it is locally symmetric if and only if $\alpha_3(u) = \alpha_4(u) = 0$, in which case the metric is flat.

Remark 3.6. Given the form of the metrics in Theorem 1.2, they are degenerate Kundt metrics. Furthermore, it follows from [18] that they are VSI spacetimes.

4. Solutions in massive gravity theories

The field equations in General Relativity, $\rho - \frac{1}{2}\tau g = 8\pi G T - \Lambda g$, are established in terms of the Einstein tensor $\rho - \frac{1}{2}\tau g$, the energy–momentum tensor T, the Newton's constant G and the cosmological constant Λ. The field equations are derived from the variation of the Einstein-Hilbert action $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}(\tau - 2\Lambda)$, which is based on the total scalar curvature. Since the very early days of General Relativity, different alternatives have been proposed to be incorporated into a larger unified theory of gravity. While General Relativity involves second-order derivatives of the metric, some extensions permit the field equations to be higher than second order. Other extensions allow the action to be a function not only of the scalar curvature but a function of the quadratic contractions of the curvature tensor: τ^2, $\|\rho\|^2$ and $\|R\|^2$. In this final section we consider solutions to various massive gravity theories within the framework of semi-symmetric curvature homogeneous Lorentzian three-manifolds. Note that homogeneous solutions have been previously described in [1].

4.1. **Topologically massive gravity functional.** Adding the gravitational Chern-Simons term $S_{CS} = \frac{1}{2}\int d^3x\sqrt{|g|}\varepsilon^{ijk}\Gamma^r_{is}\left(\partial_j\Gamma^s_{rk} + \frac{2}{3}\Gamma^s_{jv}\Gamma^v_{kr}\right)$ to the Einstein-Hilbert functional results in the *topologically massive gravity* functional, $S_{TMG} = S_{EH} + \frac{1}{\omega}S_{CS}$, where ω is a mass parameter [19]. The Euler-Lagrange equations of this functional restricted to metrics of constant volume are $\rho - \frac{1}{2}\tau g + \frac{1}{\omega}\mathcal{C} = \Lambda g$ (see [19]), where $\mathcal{C}$ denotes the Cotton tensor defined in (8). The value of Λ is determined by taking traces in this equation, so $\Lambda = -\frac{1}{6}\tau$, and we consider the symmetric $(0,2)$-tensor field given by $\mathfrak{T}_{TMG} = \rho - \frac{1}{3}\tau g + \frac{1}{\omega}\mathcal{C}$.

4.1.1. *Curvature homogeneous semi-symmetric TMG solutions with diagonalizable Ricci operator.* Let (M,g) be a curvature homogeneous manifold as in Lemma 2.1. If (M,g) corresponds to Lemma 2.1-(1), then a straightforward calculation shows that $\mathfrak{T}_{TMG}(E_1,E_1) = \lambda(\frac{\varphi}{\omega} + \frac{1}{3})$ and $\mathfrak{T}_{TMG}(E_1,E_2) = \frac{\varepsilon\lambda}{\omega}\varphi$, so $\varphi = \lambda = 0$ and any TMG solution is flat. We have a similar result for metrics in Lemma 2.1-(2) since $\mathfrak{T}_{TMG}(E_1,E_1) = -\frac{1}{3}\lambda$ (resp., $\mathfrak{T}_{TMG}(E_2,E_2) = \frac{1}{3}\lambda$) for metrics in Assertion (2.i) (resp., Assertion (2.ii)), which shows that solutions in these cases are necessarily flat. Also, metrics in Lemma 2.1-(3) do not provide nontrivial TMG solutions, since $\mathfrak{T}_{TMG}(E_3,E_3) = -\frac{2}{3}\lambda$ in this case.

4.1.2. *Curvature homogeneous semi-symmetric TMG solutions with nilpotent Ricci operator.* We proceed as in Lemma 3.2 and consider a curvature homogeneous metric with two-step nilpotent Ricci operator as in (10)-(11). A straightforward calculation shows that $\mathfrak{T}_{TMG}$ is determined by the nonzero terms

$$\mathfrak{T}_{TMG}(E_1,E_3) = -\frac{4\varepsilon}{\omega}F, \qquad \mathfrak{T}_{TMG}(E_3,E_3) = \frac{2\varepsilon}{\omega}(2C + H - \omega).$$

Hence a metric (10)-(11) is a solution in TMG if and only if $F = 0$ and $H = \omega - 2C$. Since $F = 0$ and $H = \omega - 2C$, it follows from (11) that $E_2(C) = 0$ and $E_2(A) = \frac{1}{2}(6C^2 - 4\omega C + \omega^2)$. Now a direct calculation shows that $\Delta\rho(E_3,E_3) = -2\varepsilon\omega^2 \neq 0$ and thus, by Lemma 3.1, no metric above can be critical for a quadratic curvature functional.

Since the function $F = 0$ in (10)-(11), E_2 is an expansion-free null geodesic vector field (see Remark 3.4), therefore the underlying structure is a Kundt spacetime. From the proof of Theorem 1.2, it follows that the metric is given in local coordinates as

$$g = dx^2 + 2dudv + \left(\frac{v^2}{x^2} + f_1(u)v + f_0(u,x)\right)du^2 - 4\frac{v}{x}dudx.$$

A straightforward calculation shows that $\mathfrak{T}_{TMG}$ is determined by the non-zero term

$$\mathfrak{T}_{TMG}(\partial_u,\partial_u) = \tfrac{1}{2\omega}\left(\partial_x^{(3)}f_0 - \frac{3+\omega x}{x}\partial_x^{(2)}f_0 + 2\frac{3+\omega x}{x^2}\partial_x f_0 - 2\frac{3+\omega x}{x^3}f_0\right).$$

Solving the equation $\mathfrak{T}_{TMG}(\partial_u,\partial_u) = 0$, the above is summarized as follows:

> *A semi-symmetric curvature homogeneous TMG solution which is not locally symmetric is a Kundt spacetime*
>
> $$g = dx^2 + 2dudv + \left(\frac{v^2}{x^2} + f_1(u)v + f_0(u,x)\right)du^2 - 4\frac{v}{x}dudx$$
>
> *with* $f_0(u,x) = \beta_3(u)xe^{x\omega} + \beta_2(u)x^2 + \beta_1(u)x$, *for arbitrary functions* $\beta_1(u), \beta_2(u)$, *and* $\beta_3(u) \neq 0$.

4.2. **New massive gravity functional.** The *new massive gravity* proposed by Bergshoeff, Hohm, and Townsen is a three-dimensional modification of General Relativity which complements the Einstein-Hilbert action by a quadratic term, $\|\rho\|^2 - \frac{3}{8}\tau^2$, which adds a conserved term to the field equations (see [3, 4]). The action $S_{NMG} = S_{EH} - \frac{1}{m^2}\mathcal{F}_{-3/8}$, where m is the mass of the graviton, yields the field equations

$$\rho - \tfrac{1}{3}\tau g - \tfrac{1}{2m^2}(K - \tfrac{1}{3}(|\rho|^2 - \tfrac{3}{8}\tau^2)g) = 0, \tag{12}$$

where $K = 2\Delta\rho - \frac{1}{2}\nabla^2\tau - \frac{3}{2}\tau\rho + 4R[\rho] - (\frac{1}{2}\Delta\tau + |\rho|^2 - \frac{3}{8}\tau^2)g$.

The curvature homogeneity assumption reduces significantly the field equations so that they are equivalent to $\mathfrak{T}_{NMG} = 0$, where $\mathfrak{T}_{NMG}$ is the symmetric $(0,2)$-tensor field given by $\mathfrak{T}_{NMG} = \rho - \frac{1}{3}\tau g - \frac{1}{2m^2}(K - \frac{1}{3}(|\rho|^2 - \frac{3}{8}\tau^2)g)$.

4.2.1. *Curvature homogeneous semi-symmetric NMG solutions with diagonalizable Ricci operator.* We proceed as in Section 4.1.1 and consider the different possibilities in Lemma 2.1. If (M,g) corresponds to Lemma 2.1-(1), then $\mathfrak{T}_{NMG}(E_1,E_2) = \frac{4\varepsilon\lambda\varphi^2}{m^2}$, and thus any NMG solution is locally symmetric by Remark 2.2. For metrics in Lemma 2.1-(2.i) since $\mathfrak{T}_{NMG}(E_1,E_1) = \frac{\lambda(\lambda-2m^2)}{6m^2}$ and $\mathfrak{T}_{NMG}(E_2,E_2) = -\frac{\lambda(\lambda-2m^2+48\varphi^2)}{6m^2}$, one has that $\lambda = 2m^2$ and $\varphi = 0$, so (M,g) is also locally symmetric by Remark 2.2. Metrics in Lemma 2.1-(2.ii) behave similarly, since $\mathfrak{T}_{NMG}(E_2,E_2) = -\frac{\lambda(\lambda-2m^2)}{6m^2}$ and $\mathfrak{T}_{NMG}(E_1,E_1) = \frac{\lambda(\lambda-2m^2+48\varphi^2)}{6m^2}$.

The non-zero components of $\mathfrak{T}_{NMG}$ for metrics in Lemma 2.1-(3) are given by

$$\mathfrak{T}_{NMG}(E_1,E_1) = -\mathfrak{T}_{NMG}(E_2,E_2) = \frac{1}{2}\mathfrak{T}_{NMG}(E_3,E_3) = \frac{\lambda(\lambda-2m^2)}{6m^2}\lambda.$$

Hence manifolds with $\lambda = 2m^2 > 0$ are 1-curvature homogeneous solutions in NMG modeled on $N(\lambda)\times\mathbb{R}$. Furthermore, note that these manifolds are Brinkmann waves (see Theorem 1.1) and are also $\mathcal{F}_{-1/2}$-critical (see Lemma 2.4). Summarizing the above, one has:

> *A semi-symmetric curvature homogeneous NMG solution with diagonalizable Ricci operator is symmetric or it corresponds to a Brinkmann wave $g = dx^2 + 2dudv + f(v,u,x)du^2$, determined by*
>
> $$f(v,u,x) = 2m^2v^2 + v(\alpha(u) + x\beta(u)) + \frac{1}{8m^2}x^2\beta(u)^2 + x\delta(u) + \gamma(u),$$
>
> *for smooth functions $\alpha(u)$, $\beta(u)$, $\gamma(u)$, $\delta(u)$.*

4.2.2. *Curvature homogeneous semi-symmetric NMG solutions with nilpotent Ricci operator.* We proceed as in Section 4.1.2 and consider a curvature homogeneous metric with two-step nilpotent Ricci operator described by (10)-(11). A straightforward calculation shows that $\mathfrak{T}_{NMG}$ is determined by the nonzero terms

$$\begin{aligned}
&\mathfrak{T}_{NMG}(E_1,E_1) = -2\mathfrak{T}_{NMG}(E_2,E_3) = \tfrac{16\varepsilon}{m^2}F^2,\\
&\mathfrak{T}_{NMG}(E_1,E_3) = \tfrac{4\varepsilon}{m^2}\{F(C-3H) - E_2(C)\},\\
&\mathfrak{T}_{NMG}(E_3,E_3) = \tfrac{2\varepsilon}{m^2}\{2C^2 - 8AF + 4CH - m^2 + 2E_2(A) + 4E_3(F)\}.
\end{aligned}$$

Hence a metric (10)-(11) is a solution in NMG if and only if $F = 0$, $E_2(C) = 0$, and $E_2(A) = -C^2 - 2CH + \frac{1}{2}m^2$. Observe from Lemma 3.2 that these metrics are not critical for any quadratic curvature functional $\mathcal{F}_t$.

As for TMG solutions, since $F = 0$ in (10)-(11), the underlying structure is a Kundt spacetime. Using the expression (6), the metric is given in local coordinates by

$$g = dx^2 + 2dudv + \left(\frac{v^2}{x^2} + f_1(u)v + f_0(u,x)\right)du^2 - 4\frac{v}{x}dudx.$$

A straightforward calculation shows that $\mathfrak{T}_{NMG}$ is determined by the non-zero term

$$\mathfrak{T}_{NMG}(\partial_u,\partial_u)=\frac{1}{2m^2}\left(\partial_x^{(4)}f_0-\frac{4}{x}\partial_x^{(3)}f_0+\frac{12-m^2x^2}{x^2}\left(\partial_x^{(2)}f_0-\frac{2}{x}\partial_x f_0+\frac{2}{x^2}f_0\right)\right).$$

We solve the equation $\mathfrak{T}_{NMG}(\partial_u,\partial_u)=0$ to obtain the following.

A semi-symmetric curvature homogeneous NMG solution with non-diagonalizable Ricci operator is a Kundt spacetime

$$g=dx^2+2dudv+\left(\frac{v^2}{x^2}+f_1(u)v+f_0(u,x)\right)du^2-4\frac{v}{x}dudx$$

with $f_0(u,x)=\beta_4(u)xe^{mx}+\beta_3(u)xe^{-mx}+\beta_2(u)x^2+\beta_1(u)x$*, for arbitrary functions* $\beta_1(u),\beta_2(u),\beta_3(u),\beta_4(u)$*, with* $\beta_3(u)^2+\beta_4(u)^2\neq 0$.

4.3. **General massive gravity functional.** The *general massive gravity* action is defined by adding the Chern-Simons and the new massive gravity deformation terms to the Einstein-Hilbert action, thus resulting into a combination of the topological and new massive gravity theories $S_{GMG}=S_{EH}+\frac{1}{\omega}S_{CS}-\frac{1}{m^2}\mathcal{F}_{-3/8}$. The field equations are given by $\rho-\frac{1}{3}\tau g+\frac{1}{\omega}\mathcal{C}-\frac{1}{2m^2}(K-\frac{1}{3}(|\rho|^2-\frac{3}{8}\tau^2)g)=0$ (see [27]).

The case of diagonalizable Ricci operator is very rigid. Proceeding as in the previous subsections one has that metrics corresponding to Assertions (1) and (2) in Lemma 2.1 provide GMG solutions only if they are products $N(2m^2)\times\mathbb{R}$, while metrics in Lemma 2.1-(3) are never GMG solutions.

If the Ricci operator is nilpotent, then a metric described by (10)-(11) is a GMG solution if and only if the symmetric $(0,2)$-tensor field $\mathfrak{T}_{GMG}=\rho-\frac{1}{3}\tau g+\frac{1}{\omega}\mathcal{C}-\frac{1}{2m^2}(K-\frac{1}{3}(|\rho|^2-\frac{3}{8}\tau^2)g)$ vanishes identically. Now, the tensor field $\mathfrak{T}_{GMG}$ of a curvature homogeneous metric with nilpotent Ricci operator described by (10)-(11) is determined by

$$\begin{aligned}\mathfrak{T}_{GMG}(E_1,E_1)&=-2\mathfrak{T}_{GMG}(E_2,E_3)=\tfrac{16\varepsilon}{m^2}F^2,\\ \mathfrak{T}_{GMG}(E_1,E_3)&=-\tfrac{4\varepsilon}{m^2\omega}\left\{F(m^2+(3H-C)\omega)+E_2(C)\omega)\right\},\\ \mathfrak{T}_{GMG}(E_3,E_3)&=\tfrac{2\varepsilon}{m^2\omega}\left\{(2C+H)m^2+(2C(C+2H)-8AF-m^2)\omega\right.\\ &\qquad\left.+2E_2(A)\omega+4E_3(F)\omega\right\},\end{aligned}$$

the other components being zero. Hence $\mathfrak{T}_{GMG}=0$ if and only if $F=0$, $E_2(C)=0$, and $E_2(A)=\frac{1}{2}(m^2-2C(C+2H))-\frac{m^2}{2\omega}(H+2C)$. Moreover, these metrics are critical for some (hence for all) quadratic curvature functional if and only if $H+2C=\omega$, i.e., if and only if they are also TMG solutions.

Since the function $F=0$ in (10)-(11), the underlying structure is a Kundt metric. Using the expression (6), the metric is given in local coordinates as

$$g=dx^2+2dudv+\left(\frac{v^2}{x^2}+f_1(u)v+f_0(u,x)\right)du^2-4\frac{v}{x}dudx.$$

A straightforward calculation shows that $\mathfrak{T}_{GMG}$ is determined by the non-zero term

$$\begin{aligned}\mathfrak{T}_{GMG}(\partial_u,\partial_u)=\tfrac{1}{2m^2}\Big(&\partial_x^{(4)}f_0-\tfrac{4\omega-m^2x}{\omega x}\partial_x^{(3)}f_0\\ &+\tfrac{12-m^2x(3+\omega x)}{x^2}\left(\partial_x^{(2)}f_0-\tfrac{2}{x}\partial_x f_0+\tfrac{2}{x^2}f_0\right)\Big).\end{aligned}$$

Solving the equation $\mathfrak{T}_{GMG}(\partial_u,\partial_u)=0$, the above is summarized as follows:

A semi-symmetric curvature homogeneous GMG solution which is not locally symmetric is a Kundt spacetime

$$g=dx^2+2dudv+\left(\frac{v^2}{x^2}+f_1(u)v+f_0(u,x)\right)du^2-4\frac{v}{x}dudx$$

with

$$f_0(u,x) = \beta_4(u)xe^{-\frac{m}{2\omega}(m+\sqrt{m^2+4\omega^2})x} + \beta_3(u)xe^{-\frac{m}{2\omega}(m-\sqrt{m^2+4\omega^2})x} + \beta_2(u)x^2 + \beta_1(u)x,$$

for arbitrary functions $\beta_1(u), \beta_2(u), \beta_3(u), \beta_4(u)$, *with* $\beta_3(u)^2+\beta_4(u)^2 \neq 0$.

References

[1] I. Bakas and Ch. Sourdis, Homogeneous vacua of (generalized) new massive gravity, *Class. Quantum Grav.* **28** (1) (2011), 015012, 20 pp.

[2] M. Berger, Quelques formules de variation pour une structure riemannienne, *Ann. Sci. École Norm. Sup.* **3** (1970), no. 4, 285–294.

[3] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Massive gravity in three dimensions, *Phys. Rev. Lett.* **102** (20) (2009), 201301, 4 pp.

[4] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, More on massive 3D gravity, *Phys. Rev. D* **79** (2009), 124042.

[5] H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, *Math. Ann.* **94** (1925), 119–145.

[6] T. G. Brooks, 3-manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$. arXiv:2111.15499v1

[7] M. Brozos-Vázquez, S. Caeiro-Oliveira, and E. García-Río, Critical metrics and massive gravity solutions on three-dimensional Brinkmann waves, *Class. Quantum Grav.* **39** (2022), 015007 (20pp).

[8] M. Brozos-Vázquez, S. Caeiro-Oliveira, and E. García-Río, Homogeneous and curvature homogeneous Lorentzian critical metrics, preprint 2021.

[9] P. Bueken, Three-dimensional Riemannian manifolds with constant principal Ricci curvatures $\rho 1 = \rho 2 \neq \rho 3$, *J. Math. Phys.* **37** (1996), 4062 .

[10] P. Bueken and M. Djorić, Three-dimensional Lorentz metrics and curvature homogeneity of order one, *Ann. Global Anal. Geom.* **18** (2000), 85–103.

[11] P. Bueken, On curvature homogeneous three-dimensional Lorentzian manifolds, *J. Geom. Phys.* **22** (1997), 349–362.

[12] P. Bueken, Three-dimensional Lorentzian manifolds with constant principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3$, *J. Math. Phys.* **38** (1997), 1000–1013.

[13] M. Cahen, J. Leroy, M. Parker, F. Tricerri, and L. Vanhecke, Lorentz manifolds modelled on a Lorentz symmetric space, *J. Geom. Phys.* **7** (1990), 571–581.

[14] G. Calvaruso, Curvature homogeneous Lorentzian three-manifolds, *Ann. Glob. Anal. Geom.* **36** (2009), 1–17.

[15] D. D. K. Chow, C. N. Pope, and E. Sezgin, Classification of solutions in topologically massive gravity, *Class. Quantum Grav.* **27** (2010), 105001 (26pp)

[16] D. D. K. Chow, C. N. Pope, and E. Sezgin, Kundt spacetimes as solutions of topologically massive gravity, *Class. Quantum Grav.* **27** (2010), 105002, 19 pp.

[17] A. Coley, S. Hervik, G. Papadopoulos, and N. Pelavas, Kundt spacetimes, *Class. Quantum Grav.* **26** (2009), no. 10, 105016, 34 pp.

[18] A. Coley, S. Hervik, and N. Pelavas, Lorentzian spacetimes with constant curvature invariants in three dimensions, *Class. Quantum Grav.* **25** (2008), 025008 (14pp)

[19] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, *Ann. Phys.* **140** (1982), 372–411.

[20] P. B. Gilkey, *The geometry of curvature homogeneous pseudo-Riemannian manifolds*, volume 2 of *Advanced Texts in Mathematics.* Imperial College Press, London 2007.

[21] M. J. Gursky and J. A. Viaclovsky, A new variational characterization of three-dimensional space forms, *Invent. Math.* **145** (2001), 251–278.

[22] G. Hall, T. Morgan, Z. Perjes, Three-dimensional space-times, *Gen. Rel. Grav.* **19** (1987), 1137–1147.

[23] E. Kilicarslan, S. Dengiz, B. Tekin, More on Cotton flow, *J. High Energy Phys.* (2015), no. 6, 136, 24 pp.

[24] A. U. O. Kisisel, O. Sarioglu and B. Tekin, Cotton flow, *Class. Quantum Grav.* **25** (2008), 165019.

[25] J. Podolský, M. Žofka, General Kundt spacetimes in higher dimensions, *Class. Quantum Grav.* **26** (2009), no. 10, 105008, 18 pp.

[26] K. Sekigawa, On some 3-dimensional curvature homogeneous spaces, *Tensor N.S.* **31** (1977), 87–97.

[27] M. R. Setare, On the generalized minimal massive gravity, *Nuclear Phys. B* **898** (2015), 259–275

[28] J. A. Viaclovsky, Critical metrics for Riemannian curvature functionals, *Geometric analysis*, 197–274, IAS/Park City Math. Ser., **22**, Amer. Math. Soc., Providence, RI, 2016.

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