CRITICAL METRICS AND MASSIVE GRAVITY SOLUTIONS ON THREE-DIMENSIONAL BRINKMANN WAVES

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Abstract. Three-dimensional Brinkmann waves which are critical for quadratic curvature functionals are determined. Generically, if the metric is critical for some functional then it is critical for all of them. In contrast, there are four special functionals that do not share critical metrics with any other quadratic functional. It is also shown that these metrics provide explicit solutions for different massive gravity models.

1. Introduction

Generalizations of three-dimensional general relativity, allowing propagating degrees of freedom in spacetimes, have gained increasing interest during last years. Topologically massive gravity, which complements the Einstein-Hilbert functional with a Lorentz Chern-Simons term, yields a third order field equation [12]. Massive gravity introduces quadratic curvature invariants into the action, giving rise to fourth order equations [3, 4]. Although special classes of solutions already exist in the literature (see, for example, [1, 10, 19, 24]), the main purpose of this work is to investigate a family of local solutions generalizing the classical *pp*-waves.

Brinkmann waves, characterized by the existence of a parallel null line field [5], have been extensively studied and they appear as the underlying structure in a number of interesting situations. Brinkmann metrics have vanishing scalar invariants (VSI) in some special cases which were considered in [24]. Our work extends those results to arbitrary Brinkmann waves.

1.1. **Quadratic curvature functionals.** Let M be a closed oriented manifold and let $\mathcal{M}_1$ denote the space of Lorentzian metrics of volume one on M. The scalar curvature generates all curvature invariants of order one. The Einstein-Hilbert functional, given by the total scalar curvature: $g \mapsto \int_M d^3x\sqrt{|g|}\;\tau$, has been widely studied and it is well-known that critical metrics for this functional restricted to $\mathcal{M}_1$ are the Einstein ones. Natural generalizations of this functional are built using scalar curvature invariants of order two, which are generated by $\{\tau^2, \|\rho\|^2, \|R\|^2, \Delta\tau\}$ (see [2]). These functionals have been investigated both from a purely geometric point of view and from a physics perspective, as in conformal gravity (see, for example, [16, 20, 25]).

In dimension three the curvature tensor R is determined by the Ricci tensor ρ so that $\|R\|^2 = 2\|\rho\|^2 - \frac{1}{2}\tau^2$. Hence, the space of quadratic curvature functionals is generated by

$$\mathcal{S} : g \mapsto \int_M d^3x\sqrt{|g|}\,\tau^2, \quad \text{and} \quad \mathcal{T} : g \mapsto \int_M d^3x\sqrt{|g|}\,\|\rho\|^2.$$

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Thus, every quadratic curvature functional is a multiple of $\mathcal{S}$ or $\mathcal{F}_t = \mathcal{T} + t\,\mathcal{S}$ for some $t \in \mathbb{R}$. Euler-Lagrange equations corresponding to critical metrics for these functionals restricted to $\mathcal{M}_1$ were given by Berger in [2] for closed manifolds. Thus, critical metrics for the functional $\mathcal{S}$ in dimension three are characterized by

$$-2\nabla^2\tau + \tfrac{2}{3}\Delta\tau g + 2\tau\left(\rho - \tfrac{1}{3}\tau g\right) = 0, \tag{1}$$

and critical metrics for the functional $\mathcal{T}$ in dimension three satisfy

$$\Delta\rho - \nabla^2\tau + 2\left(R[\rho] - \tfrac{1}{3}\|\rho\|^2 g\right) = 0, \tag{2}$$

where $R[\rho]$ denotes the action of the curvature tensor on the Ricci tensor ($R[\rho]_{ij} = R_{ikjl}\rho^{kl}$). The Euler-Lagrange equations for the functionals $\mathcal{F}_t$ have the following expression:

$$\Delta\rho - (1+2t)\nabla^2\tau + \tfrac{2}{3}t\Delta\tau g + 2(R[\rho] - \tfrac{1}{3}\|\rho\|^2 g) + 2t\tau(\rho - \tfrac{1}{3}\tau g) = 0. \tag{3}$$

The functionals above can also be considered for non-compact manifolds, which play a role in General Relativity. In this case one must assume that the corresponding integrals exist. The Euler-Lagrange equations, which are the same as in the compact case, are obtained by considering variations of the metrics with compact support (see, for example, the discussions in [14, 16]).

It follows directly from equation (1) that metrics with $\tau = 0$ are $\mathcal{S}$-critical. Also, from equation (2), it follows that Einstein metrics are $\mathcal{T}$-critical. If a metric is $\mathcal{S}$ and $\mathcal{F}_t$-critical for some $t \in \mathbb{R}$, then it follows from equations (1), (2) and (3) that it is also critical for $\mathcal{T}$ and, hence, for all $\mathcal{F}_t$. Moreover, if a metric is critical for $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ with $t_1 \neq t_2$, then it is critical for all quadratic curvature functionals.

1.2. **Brinkmann waves.** A Lorentzian manifold admitting a parallel vector field U (i.e., $\nabla U = 0$) splits locally as a metric product when U is non-null. If U is null (lightlike), however, the previous splitting result does not hold, although one has still a special situation. Spacetimes admitting such a null U have been widely studied in General Relativity, where they are called *pp-waves* (plane-fronted waves with parallel rays) in the transversally flat case, i.e., if the curvature endomorphism satisfies $R(U^\perp, U^\perp) = 0$ [13]. Furthermore, the spacetime is a *plane wave* if the curvature tensor is transversally parallel (i.e., $\nabla_{U^\perp} R = 0$). *Plane gravitational waves*, being Ricci flat plane waves, play a special role in Relativity (see [21, 22, 23] for further details). Recent detections of gravitational waves have increased the interest on these classes of spacetimes.

More generally, a Lorentzian manifold is said to be a *Brinkmann wave* if it admits a parallel null line field. Coordinates (u, x, y) may be chosen for a three-dimensional Brinkmann wave g so that (see [5])

$$g = 2du dy + dx^2 + \varphi(u, x, y) dy^2. \tag{4}$$

The parallel null line field is locally generated by a null recurrent vector field (i.e., $\nabla U = \omega \otimes U$ for some 1-form ω). For a metric (4), the vector field ∂_u is null and recurrent. The generating null vector field may be chosen to be parallel if and only if the Ricci operator is two-step nilpotent, in which case coordinates may be further specialized so that the defining function $\varphi(u, x, y)$ does not depend on the null coordinate and (M, g) is a three-dimensional *pp*-wave. Moreover, in this case (M, g) is a plane wave if and only if the function φ is a quadratic polynomial on the coordinate x, which, after an appropriate change of coordinates, reduces to $\varphi(x, y) = a(y)x^2$.

To fix notation, we use subscripts to denote partial derivatives, thus φ_ℓ denotes the partial derivative $\partial_\ell\varphi$. The Ricci tensor of a Brinkmann metric as in (4) is given by the following non-zero components:

$$\rho(\partial_u, \partial_y) = \tfrac{1}{2}\varphi_{uu}, \qquad \rho(\partial_x, \partial_y) = \tfrac{1}{2}\varphi_{ux}, \qquad \rho(\partial_y, \partial_y) = \tfrac{1}{2}\left(\varphi\varphi_{uu} - \varphi_{xx}\right). \tag{5}$$

Hence it has Ricci curvatures 0 and $\frac{1}{2}\varphi_{uu}$, the later with multiplicity two Thus, the scalar curvature is given by $\tau = \varphi_{uu}$ and the norm of the Ricci tensor by $\|\rho\|^2 = \frac{1}{2}\varphi_{uu}^2$. The integrand of $\mathcal{S}$ is always non-negative, hence metrics with $\tau = 0$ are $\mathcal{S}$-critical, as can be checked directly in equation (1). Also, for Brinkmann waves the integrand of $\mathcal{T}$ is non-negative, but this is not a general fact in Lorentzian geometry. As a consequence, we will see in Section 3 that there exist metrics with $\|\rho\|^2 = 0$ which are not $\mathcal{T}$-critical. Furthermore, the functional $\mathcal{F}_{-1/2}$ has zero energy in the Brinkmann context. This makes the value $t = -1/2$ special, as will be shown in Lemma 3.1.

1.3. **Outline of the paper.** In this work we analyze equations (1) and (3) to classify three-dimensional Brinkmann waves which are critical for quadratic curvature functionals. We begin by characterizing critical metrics for the functional $\mathcal{S}$ as those with vanishing scalar curvature in Section 2. We identify metrics that are critical for all quadratic curvature functionals by studying the generic functional $\mathcal{F}_t$ in Section 3 and we obtain three special cases: $\mathcal{F}_{-1/3}$, $\mathcal{F}_{-1/4}$ and $\mathcal{F}_{-1/2}$, that admit critical metrics which are not critical for another quadratic curvature functional. The distinguished functionals $\mathcal{F}_{-1/3}$, $\mathcal{F}_{-1/4}$ and $\mathcal{F}_{-1/2}$ have a clear geometric meaning, since they correspond to the functionals given by the L^2-norms of the trace-free Ricci tensor $\rho_0 = \rho - \frac{1}{3}\tau g$ and the curvature tensor R, and the mean distance of Brownian motion, respectively. These are examined in detail in Section 4. Special classes of Brinkmann metrics are considered in Section 5, namely manifolds with constant scalar curvature, locally symmetric, locally conformally flat and conformally symmetric. Finally, in Section 6, we turn our attention to massive gravity models. We use Brinkmann metrics to construct explicit solutions corresponding to topologically massive gravity and new massive gravity actions.

2. $\mathcal{S}$-CRITICAL METRICS

We already know that metrics with vanishing scalar curvature are critical for the functional $\mathcal{S}$, the following result shows that these are indeed the only $\mathcal{S}$-critical Brinkmann metrics.

Theorem 2.1. *A three-dimensional Brinkmann metric g is critical for the functional $\mathcal{S}$ if and only if the scalar curvature vanishes, i.e. there exist coordinates (u, x, y) so that g has the form of expression* (4) *with $\varphi(u, x, y) = f(x, y)u + h(x, y)$.*

Proof. A metric is critical for the functional $\mathcal{S}$ if and only if it satisfies equation (1). This is, the symmetric tensor field $\mathfrak{S} = \nabla^2\tau - \frac{1}{3}\Delta\tau g - \tau(\rho - \frac{1}{3}\tau g)$ vanishes. A direct calculation using coordinates as in (4) gives:

$$\begin{aligned}
&\mathfrak{S}(\partial_u, \partial_u) = \varphi_{uuuu}, \qquad \mathfrak{S}(\partial_u, \partial_x) = \varphi_{uuux},\\
&\mathfrak{S}(\partial_x, \partial_y) = -\tfrac{1}{2}\varphi_{ux}\varphi_{uu} + \varphi_{uuxy} - \tfrac{1}{2}\varphi_x\varphi_{uuu},\\
&\mathfrak{S}(\partial_x, \partial_x) = -2\mathfrak{S}(\partial_u, \partial_y) = \tfrac{1}{3}\left(\varphi_{uu}^2 + 2\varphi_{uuxx} + \varphi_u\varphi_{uuu} - 2\varphi_{uuuy} - 2\varphi\varphi_{uuuu}\right),\\
&\mathfrak{S}(\partial_y, \partial_y) = \tfrac{1}{6}\big(3\varphi_{xx}\varphi_{uu} + 3\varphi_u\varphi_{uuy} + 6\varphi_{uuyy} + 3\varphi_x\varphi_{uux} - 3\varphi_y\varphi_{uuu} - \varphi\varphi_{uu}^2\\
&\qquad\qquad -2\varphi\varphi_{uuxx} - \varphi\varphi_u\varphi_{uuu} - 4\varphi\varphi_{uuuy} + 2\varphi^2\varphi_{uuuu}\big).
\end{aligned}$$

From $\mathfrak{S}(\partial_u, \partial_u) = 0$ and $\mathfrak{S}(\partial_u, \partial_x) = 0$ one obtains that φ has the form

$$\varphi(u, x, y) = f_3(y)u^3 + f_2(x, y)u^2 + f_1(x, y)u + f_0(x, y).$$

Using this expression $\mathfrak{S}(\partial_u, \partial_y)$ reduces to

$$\mathfrak{S}(\partial_u, \partial_y) = \tfrac{1}{6}\left(-6f_3\left(f_1 + 2uf_2 + 3u^2f_3\right) - 4f_{2xx} - (2f_2 + 6uf_3)^2 + 12f_3'\right).$$

Now, differentiating twice with respect to u gives

$$\mathfrak{S}(\partial_u, \partial_y)_{uu} = -18{f_3}^2,$$

so $f_3 = 0$ and $\varphi(u,x,y) = f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$. Simplifying again, we get

$$\mathfrak{S}(\partial_x,\partial_y) = -f_2\left(f_{1x} + 2uf_{2x}\right) + 2f_{2xy} \quad \text{and} \quad \mathfrak{S}(\partial_x,\partial_y)_u = -2f_2f_{2x},$$

so f_2 does not depend on x. Hence $\mathfrak{S}(\partial_u,\partial_y) = -\frac{2}{3}f_2^2$ and $f_2 = 0$. We conclude that $\varphi(u,x,y) = f_1(x,y)u + f_0(x,y)$ and check that $\mathfrak{S}$ vanishes identically. □

3. $\mathcal{F}_t$-critical metrics: the generic case

It follows from (3) that a metric is $\mathcal{F}_t$-critical if and only if the tensor field $\mathfrak{F}_t = \Delta\rho - (1+2t)\nabla^2\tau + \frac{2}{3}t\Delta\tau g + 2(R[\rho] - \frac{1}{3}\|\rho\|^2 g) + 2t\tau(\rho - \frac{1}{3}\tau g)$ vanishes identically. Considering Brinkmann coordinates as in (4), a long but straightforward calculation shows that $\mathcal{F}_t$ is determined by

$$\begin{aligned}
&\mathfrak{F}_t(\partial_u,\partial_u) = -(1+2t)\varphi_{uuuu},\\
&\mathfrak{F}_t(\partial_u,\partial_x) = -(1+2t)\varphi_{uuux}, \qquad \mathfrak{F}_t(\partial_x,\partial_x) = -2\mathfrak{F}_t(\partial_u,\partial_y),\\
&\mathfrak{F}_t(\partial_u,\partial_y) = \tfrac{1}{6}\left((1+2t)\varphi_{uu}^2 - 4t\varphi_{uuuy} + (3+4t)\varphi_{uuxx} + 2t\varphi_u\varphi_{uuu}\right.\\
&\qquad\qquad\qquad \left. -(3+4t)\varphi\varphi_{uuuu}\right),\\
&\mathfrak{F}_t(\partial_x,\partial_y) = \tfrac{1}{2}\left(\varphi_{uxxx} - 4t\varphi_{uuxy} + (1+2t)\varphi_{ux}\varphi_{uu} + 2t\varphi_x\varphi_{uuu} - \varphi\varphi_{uuux}\right),\\
&\mathfrak{F}_t(\partial_y,\partial_y) = \tfrac{1}{6}\left((1+2t)\varphi\varphi_{uu}^2 - 3\varphi_{xxxx} - 3\varphi_{ux}^2 - 6\varphi_{uxxy} - 6t\varphi_{xx}\varphi_{uu}\right.\\
&\qquad\qquad\qquad + 2(3+2t)\varphi\varphi_{uuxx} - 3\varphi_u\varphi_{uxx} - 3(1+2t)\varphi_u\varphi_{uuy}\\
&\qquad\qquad\qquad + 2t\varphi\varphi_u\varphi_{uuu} - 6(1+2t)\varphi_{uuyy} + 3(1-2t)\varphi_x\varphi_{uux}\\
&\qquad\qquad\qquad \left. + 3(1+2t)\varphi_y\varphi_{uuu} + 2(3+4t)\varphi\varphi_{uuuy} - (3+4t)\varphi^2\varphi_{uuuu}\right).
\end{aligned} \tag{6}$$

Although there is a generic behavior for the different values of the parameter t, there are three exceptional cases corresponding to $-\frac{1}{2}$, $-\frac{1}{3}$ and $-\frac{1}{4}$.

Lemma 3.1. *If a three-dimensional Brinkmann metric is $\mathcal{F}_t$-critical for some t different from $-\frac{1}{2}$, $-\frac{1}{3}$ and $-\frac{1}{4}$, then the scalar curvature vanishes and (M,g) is critical for all quadratic curvature functionals.*

Proof. We assume that $t \neq -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$ and that the metric g is given by (4). Since $t \neq -\frac{1}{2}$, it follows from $\mathfrak{F}_t(\partial_u,\partial_u) = 0$ and $\mathfrak{F}_t(\partial_u,\partial_x) = 0$ that $\varphi(u,x,y) = f_3(y)u^3 + f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$. We simplify expressions in (6) to obtain

$$\mathfrak{F}_t(\partial_u,\partial_y)_{uu} = -12(1+3t)f_3^2.$$

Since $t \neq -\frac{1}{3}$, $f_3 = 0$ and $\varphi(u,x,y) = f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$. Now, we compute

$$2\mathfrak{F}_t(\partial_x,\partial_y)_u - 3\mathfrak{F}_t(\partial_u,\partial_y)_x = (1+4t)f_{2xxx}.$$

Since $t \neq -\frac{1}{4}$, we conclude $f_{2xxx} = 0$. Moreover, this implies that

$$\mathfrak{F}_t(\partial_x,\partial_y)_u = -2(1+2t)f_2f_{2x}.$$

Since $t \neq -\frac{1}{2}$, it follows that f_2 does not depend on x and $\varphi(u,x,y) = f_2(y)u^2 + f_1(x,y)u + f_0(x,y)$. This expression leads to

$$\mathfrak{F}(\partial_u,\partial_y) = \tfrac{2}{3}(1+2t)f_2^2.$$

Hence $f_2 = 0$ and $\varphi(u,x,y) = f_1(x,y)u + f_0(x,y)$, which shows that the scalar curvature vanishes and the metric is $\mathcal{S}$-critical by Theorem 2.1. Since the metric is critical for two functionals, it is critical for all quadratic curvature functionals. □

Theorem 3.2. *A three-dimensional Brinkmann metric* (4) *is critical for all quadratic curvature functionals if and only if* $\varphi(u,x,y) = f_1(x,y)u + f_0(x,y)$ *with*

$$\begin{aligned} f_1(x,y) =& A(y)x^2 + B(y)x + C(y), \\ f_0(x,y) =& -\tfrac{1}{60}A(y)^2x^6 - \tfrac{1}{20}A(y)B(y)x^5 - \tfrac{1}{24}\left(B(y)^2 + 2A(y)C(y) + 4A'(y)\right)x^4 \\ &+ D(y)x^3 + E(y)x^2 + F(y)x + G(y). \end{aligned}$$

Proof. Since g is $\mathcal{S}$-critical, the scalar curvature vanishes and thus $\varphi(u,x,y) = f_1(x,y)u + f_0(x,y)$. Now, since g is $\mathcal{T}$-critical, the tensor field $\mathfrak{F}_0$ in (6) vanishes. The only possibly non-zero components are $\mathfrak{F}_0(\partial_x,\partial_y)$ and $\mathfrak{F}_0(\partial_y,\partial_y)$. We compute

$$\mathfrak{F}_0(\partial_x,\partial_y) = -\tfrac{1}{2}f_{1xxx},$$

from where $f_{1xxx} = 0$ and hence $f_1(x,y) = A(y)x^2 + B(y)x + C(y)$. Now, we compute

$$2\mathfrak{F}_0(\partial_y,\partial_y) = 6A^2x^2 + 6ABx + B^2 + 2AC + 4A' + f_{0xxxx},$$

to obtain that f_0 is given by

$$\begin{aligned} f_0(x,y) =& -\tfrac{1}{60}A(y)^2x^6 - \tfrac{1}{20}A(y)B(y)x^5 - \tfrac{1}{24}(B(y)^2 + 2A(y)C(y) + 4A'(y))x^4 \\ &+ D(y)x^3 + E(y)x^2 + F(y)x + G(y), \end{aligned}$$

which completes the proof . □

It follows from Theorem 2.1 that *pp*-waves, having vanishing scalar curvature, are $\mathcal{S}$-critical. Any *pp*-wave can be described in local Brinkmann coordinates (4) by a function $\varphi = \varphi(x,y)$. A direct consequence of Theorem 3.2 is that *pp*-waves are critical for all quadratic curvature functionals if and only if $\varphi(x,y) = D(y)x^3 + E(y)x^2 + F(y)x + G(y)$. A standard argument shows that in this case one may specialize the local Brinkmann coordinates so that $\varphi(x,y) = \kappa(y)x^3 + a(y)x^2$.

Let $S = \rho - \frac{\tau}{4}g$ be the Schouten tensor of (M,g). The Cotton tensor $\mathcal{C}_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik}$ measures the failure of the Schouten tensor to be Codazzi. Since M is assumed to be of dimension three, we define the $(0,2)$-Cotton tensor $\mathcal{C}_{ij} = \frac{1}{2\sqrt{|g|}}\mathcal{C}_{nmi}\epsilon^{nm\ell}g_{\ell j}$, where $\epsilon^{123} = 1$ is the anti-symmetric symbol. A straightforward calculation from (4) shows that the only non-zero component of the divergence of the Cotton tensor of a Brinkmann metric is $\operatorname{div}\mathcal{C}(\partial_y,\partial_y) = -\frac{1}{2}\varphi_{xxxx}$. Hence one has

Corollary 3.3. *A three-dimensional pp-wave is critical for some quadratic curvature functional* $\mathcal{F}_t$ *if and only if the Cotton tensor is divergence-free, and hence it is critical for all quadratic curvature functionals. Moreover, in such a case there exist Brinkmann coordinates as in* (4) *so that* $\varphi(x,y) = \kappa(y)x^3 + a(y)x^2$.

4. $\mathcal{F}_t$-critical metrics: special cases

From the proof of Lemma 3.1 we distinguish two situations: vanishing and non-vanishing scalar curvature. If a metric with vanishing scalar curvature is $\mathcal{F}_t$-critical for some t, then it is critical for all quadratic curvature functionals, whereas a metric with non-vanishing scalar curvature cannot be critical for any functional but, perhaps, for $\mathcal{F}_t$ with $t = -\frac{1}{2}, -\frac{1}{3}$, or $-\frac{1}{4}$. The analysis of these last three cases is the objective of this section.

4.1. $\mathcal{F}_{-1/3}$**-critical metrics.** The norm of the trace-free Ricci tensor $\rho_0 = \rho - \frac{\tau}{3}g$ is given by $\|\rho_0\|^2 = \|\rho\|^2 - \frac{1}{3}\tau^2$. Hence the functional $\mathcal{F}_{-1/3}$ is precisely the functional given by the L^2-norm of the trace-free Ricci tensor in dimension three.

Theorem 4.1. *Let g be a Brinkmann metric* (4) *with non-zero scalar curvature. If g is $\mathcal{F}_{-1/3}$-critical, then the scalar curvature has the form*

$$\tau = 6f_3(y)u + 2\lambda x^2 + 2f_{21}(y)x + 2f_{20}(y) \tag{7}$$

where $\lambda \in \mathbb{R}$. Conversely, given a function $\tilde{\tau}$ as in (7)*, there exists a $\mathcal{F}_{-1/3}$-critical Brinkmann metric given by* (4) *with scalar curvature $\tilde{\tau}$.*

Proof. A metric is critical for the functional $\mathcal{F}_{-1/3}$ if and only if the tensor $\mathfrak{F}_{-1/3}$ given in equation (6) vanishes. We use that $\mathfrak{F}_{-1/3}(\partial_u, \partial_u) = 0$ and $\mathfrak{F}_{-1/3}(\partial_u, \partial_x) = 0$, to see that $\varphi(u, x, y) = f_3(y)u^3 + f_2(x, y)u^2 + f_1(x, y)u + f_0(x, y)$.

If $f_3(y) = 0$, then $27\mathfrak{F}_{-1/3}(\partial_x, \partial_x)_x + 30\mathfrak{F}_{-1/3}(\partial_x, \partial_y)_u = 4f_2 f_{2x}$, so f_2 does not depend on x. Now, we compute again $\mathfrak{F}_{-1/3}(\partial_x, \partial_x) = \frac{4}{9}(f_2)^2$ to see that $f_2 = 0$, which contradicts the assumption $\tau \neq 0$. Hence, we assume $f_3(y) \neq 0$ henceforth.

From equation (6), we compute $9\mathfrak{F}_{-1/3}(\partial_u, \partial_y) = -2f_2^2 + 6f_1 f_3 - 12f_3' - 5f_{2xx}$. Since $\mathfrak{F}_{-1/3}(\partial_u, \partial_y) = 0$, it follows that

$$f_1(x, y) = \tfrac{1}{6f_3(y)}\left(2f_2(x, y)^2 + 12f_3'(y) + 5f_{2xx}(x, y)\right).$$

We use this expression to substitute f_1 in $\mathfrak{F}_{-1/3}(\partial_x, \partial_y)$ and obtain

$$\mathfrak{F}_{-1/3}(\partial_x, \partial_y)_u = -\tfrac{1}{6}f_{2xxx}.$$

Hence $f_{2xxx} = 0$ and f_2 has the form $f_2(x, y) = f_{22}(y)x^2 + f_{21}(y)x + f_{20}(y)$. Now, we simplify $\mathfrak{F}_{-1/3}(\partial_y, \partial_y)$ to compute

$$\begin{aligned}\mathfrak{F}_{-1/3}(\partial_y, \partial_y)_u =&\tfrac{1}{9f_3}\left(20f_{22}^3x^4 + 40f_{21}f_{22}^2x^3 + 24f_{22}(f_{21}^2 + f_{20}f_{22})x^2\right.\\&+ 4(f_{21}^3 + 6f_{20}f_{21}f_{22})x + 4f_{20}f_{21}^2 + 4f_{20}^2f_{22} + 36f_{22}^2 + 21f_3f_{22}'\\&\left.-18f_3^2f_{0xx}\right).\end{aligned}$$

From where it follows that

$$\begin{aligned}f_0(x, y) =&\tfrac{1}{27f_3(y)^2}\left(f_{22}(y)^3x^6 + 3f_{22}(y)^2f_{21}(y)x^5 + 3f_{22}(y)(f_{22}(y)f_{20}(y) + f_{21}(y)^2)x^4\right.\\&+ (6f_{22}(y)f_{20}(y)f_{21}(y) + f_{21}(y)^3)x^3\\&\left.+(27f_{22}(y)^2 + 3f_{22}(y)f_{20}(y)^2 + 3f_{20}(y)f_{21}(y)^2 + \tfrac{63}{4}f_3(y)f_{22}'(y))x^2\right)\\&+ f_{01}(y)x + f_{00}(y).\end{aligned}$$

Assuming that f_0 is as above, we obtain that $\mathfrak{F}_{-1/3}(\partial_x, \partial_y)_x = -\frac{1}{3}f_{22}'$, so $f_{22} = \lambda$ is constant. We further simplify as follows

$$\mathfrak{F}_{-1/3}(\partial_x, \partial_y) = -\tfrac{2}{9f_3}\left((9\lambda + f_{20}^2)f_{21} - 9f_{01}f_3^2 + 6f_3f_{21}'\right),$$

and from $\mathfrak{F}_{-1/3}(\partial_x, \partial_y) = 0$ we obtain

$$f_{01}(y) = \tfrac{1}{9f_3(y)^2}\left(9\lambda f_{21}(y) + f_{20}(y)^2f_{21}(y) + 6f_3(y)f_{21}'(y)\right).$$

Finally, the remaining term of $\mathfrak{F}_{-1/3}$ reduces to

$$\begin{aligned}\mathfrak{F}_{-1/3}(\partial_y, \partial_y) =&\tfrac{1}{27f_3^2}\left(-f_3f_{00}' - 2(\lambda + f_3')f_{00} + \tfrac{2}{3}f_{20}''\right.\\&+ \tfrac{1}{9f_3}(f_{20}^2f_{20}' + 2f_{21}f_{21}' + 17\lambda f_{20}' + 6f_{20}'f_3')\\&\left.+\tfrac{1}{27f_3^2}2\lambda(15f_{20} + f_{20}^3 + 3f_{21}^2)\right).\end{aligned}$$

In summary, the function φ has the form $\varphi(u,x,y) = f_3(y)u^3 + f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$ with

$$\begin{aligned} f_3(y) \neq& 0, \\ f_2(x,y) =& \lambda x^2 + f_{21}(y)x + f_{20}(y), \\ f_1(x,y) =& \tfrac{1}{3f_3(y)}\left((\lambda x^2 + f_{21}(y)x + f_{20}(y))^2 + 5\lambda + 6f_3'(y)\right), \\ f_0(x,y) =& \tfrac{1}{27f_3(y)^2}\big(\lambda^3x^6 + 3\lambda f_{21}(y)x^5 + 3\lambda(\lambda f_{20}(y) + f_{21}(y)^2)x^4 \\ &+ (6\lambda f_{20}(y)f_{21}(y) + f_{21}(y)^3)x^3 \\ &+ (27\lambda^2 + 3\lambda f_{20}(y)^2 + 3f_{20}(y)f_{21}(y)^2)x^2 \\ &+3((9\lambda + f_{20}(y)^2)f_{21}(y) + 6f_3(y)f_{21}'(y))x\big) + f_{00}(y), \end{aligned}$$

where f_{00} is a solution of the linear ODE

$$\begin{aligned} &- f_3f_{00}' - 2(\lambda + f_3')f_{00} + \tfrac{2}{3}f_{20}'' + \tfrac{1}{9f_3}(f_{20}^2f_{20}' + 2f_{21}f_{21}' + 17\lambda f_{20}' + 6f_{20}'f_3') \\ &+ \tfrac{1}{27f_3^2}2\lambda(15f_{20} + f_{20}^3 + 3f_{21}^2) = 0. \end{aligned} \tag{8}$$

We directly compute the scalar curvature to see that it is given by (7). Conversely, given a function $\tilde\tau$ as in (7), it prescribes functions f_3, f_{21}, f_{20} and a constant λ that determine functions f_3, f_2 and f_0 through the expressions above and the ODE (8). With these functions, the Brinkmann metric given by $\varphi(u,x,y) = f_3(y)u^3 + f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$ is $\mathcal{F}_{-1/3}$-critical with scalar curvature $\tilde\tau$. □

4.2. **$\mathcal{F}_{-1/4}$-critical metrics.** Since the curvature tensor is determined by the Ricci tensor in dimension three, $\|R\|^2 = 2\|\rho\|^2 - \frac{1}{2}\tau^2$, the functional $\mathcal{F}_{-1/4}$ is equivalent to that one defined by the L^2-norm of the curvature tensor.

Theorem 4.2. *Let g be a Brinkmann metric* (4) *with non-zero scalar curvature. If g is $\mathcal{F}_{-1/4}$-critical, then the scalar curvature satisfies*

$$\tau_u = 0, \qquad 4\tau_{xx} + \tau^2 = 0. \tag{9}$$

Conversely, for any solution $\tilde\tau$ of the equations (9)*, there exists a $\mathcal{F}_{-1/4}$-critical Brinkmann metric with scalar curvature $\tilde\tau$.*

Proof. Let g be a Brinkmann metric as in (4). Considering the tensor field $\mathfrak{F}_{-1/4}$ given by (6), we proceed as in Lemma 3.1 to see that $\varphi(u,x,y) = f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$. Hence the scalar curvature $\tau = \varphi_{uu} = 2f_2(x,y)$ does not depend on the coordinate u and the first equation in (9) follows. Moreover

$$\mathfrak{F}_{-1/4}(\partial_u,\partial_y) = -\tfrac{1}{3}\left(f_2^2 + 2f_{2xx}\right) = -\tfrac{1}{3}(4\tau_{xx} + \tau^2),$$

which gives the second equation in (9). Now, we use that $2f_{2xx} = -f_2^2$ to compute

$$f_{2xxx} = -f_2f_{2x}, \quad f_{2xxy} = -f_2f_{2y}, \quad f_{2xxxx} = \tfrac{1}{2}{f_2}^3 - f_{2x}^2,$$

and simplify $\mathfrak{F}_{-1/4}(\partial_x,\partial_y) = -\frac{1}{2}\left(f_{1xxx} + f_2f_{1x} + 2f_{2xy}\right)$ in (6) to obtain the equation

$$f_{1xxx} + f_2f_{1x} + 2f_{2xy} = 0. \tag{10}$$

Once f_2 (or, equivalently, τ) is settled, equation (10) provides the only relation to be satisfied by f_1. Assuming that f_1 is a solution of (10), we use

$$f_{1xxxx} = 2f_2f_{2y} - f_{2x}f_{1x} - f_2f_{1xx},$$

to simplify $2\mathfrak{F}_{-1/4}(\partial_y,\partial_y)$ in (6) and obtain the following equation:

$$f_{0xxxx} - f_2f_{0xx} - 3f_{2x}f_{0x} + {f_2}^2f_0 + 2f_{2yy} + {f_1}_x^2 + f_1(f_{2y} + f_{1xx}) + 2f_{1xxy} = 0. \tag{11}$$

Once again, if f_2 and f_1 are settled, then (11) is the only equation determining f_0.

The previous analysis shows that the scalar curvature of a $\mathcal{F}_{-1/4}$-critical Brinkmann metric (4) satisfies equation (9). Conversely, if (9) is satisfied, then there exist solutions f_1 and f_0 of the linear PDEs (10) and (11), respectively, so that the corresponding Brinkmann metric determined by $\varphi(u,x,y)=f_2(x,y)u^2+f_1(x,y)u+f_0(x,y)$ is $\mathfrak{F}_{-1/4}$-critical. □

Remark 4.3. Solutions of the nonlinear wave equation $2f_{2xx}+f_2^2=0$ are given by $f_2(x,y)=-(-2)^{2/3}\sqrt[3]{3}\,\mathcal{P}((x+\alpha(y))\frac{\sqrt[3]{-\frac{1}{3}}}{2^{2/3}};g_2,g_3)$, where $\mathcal{P}(-;g_2,g_3)$ denotes the Weierstrass elliptic function with invariants $g_2=0$, $g_3=\beta(y)$, and α and β are arbitrary functions (see, for example, [9, 18]).

4.3. **$\mathcal{F}_{-1/2}$-critical metrics.** For any three-dimensional Brinkmann wave we have that $\|\rho\|^2-\frac{1}{2}\tau^2=0$. Moreover, the term of degree three in the asymptotic expansion of the mean distance for the Brownian motion on a Riemannian manifold is determined by the quadratic expression $\mathcal{E}=-6\Delta\tau-\|R\|^2+\|\rho\|^2$ (see [17]). Hence, the associated quadratic curvature functional is equivalent to $\mathcal{F}_{-1/2}$ in the three-dimensional setting.

Lemma 4.4. *If a Brinkmann metric* (4) *is $\mathcal{F}_{-1/2}$-critical, then the scalar curvature is a harmonic function and, moreover,*

$$\Delta_g\varphi+\tfrac{3}{2}\varphi_u^2=C_1(y)u+C_2(y)x+C_3(y), \tag{12}$$

for some functions C_1, C_2, C_3. Conversely, any Brinkmann metric (4) *determined by a function φ satisfying* (12) *is $\mathcal{F}_{-1/2}$-critical.*

Proof. We fix $t=-\frac{1}{2}$ and work with the expressions in (6). The possibly non-vanishing terms are

$$\begin{aligned}
\mathfrak{F}_{-1/2}(\partial_x,\partial_x)=&-2\mathfrak{F}_{1/2}(\partial_u,\partial_y)=\tfrac{1}{3}(\varphi_{uuxx}-\varphi_u\varphi_{uuu}+2\varphi_{uuuy}-\varphi\varphi_{uuuu})=\tfrac{1}{3}\Delta_g\tau,\\
\mathfrak{F}_{-1/2}(\partial_x,\partial_y)=&\tfrac{1}{2}(-\varphi_{uxxx}-2\varphi_{uuxy}+\varphi_x\varphi_{uuu}+\varphi\varphi_{uuux}),\\
\mathfrak{F}_{-1/2}(\partial_y,\partial_y)=&\tfrac{1}{6}(3\varphi_{xxxx}+3\varphi_{ux}^2+6\varphi_{uxxy}-3\varphi_{xx}\varphi_{uu}-6\varphi_x\varphi_{uux}\\
&+\varphi_u(3\varphi_{uxx}+\varphi\varphi_{uuu})+\varphi(-4\varphi_{uuxx}-2\varphi_{uuuy}+\varphi\varphi_{uuuu})).
\end{aligned}$$

Using that $\mathfrak{F}_{-1/2}(\partial_x,\partial_x)=\frac{1}{3}\Delta_g\tau=0$, we simplify $\mathfrak{F}_{-1/2}(\partial_y,\partial_y)$ to see that

$$\mathfrak{F}_{-1/2}(\partial_y,\partial_y)=\tfrac{1}{2}(\varphi_{xxxx}+\varphi_{ux}^2+\varphi_u\varphi_{uxx}+2\varphi_{uxxy}-\varphi_{xx}\varphi_{uu}-2\varphi_x\varphi_{uux}-\varphi\varphi_{uuxx}),$$

and the previous three expressions reduce to

$$\begin{aligned}
\mathfrak{F}_{-1/2}(\partial_u,\partial_y)=&-\tfrac{1}{6}(\varphi_{xx}+\tfrac{1}{2}\varphi_u^2+2\varphi_{uy}-\varphi\varphi_{uu})_{uu}=-\tfrac{1}{6}(\Delta_g\varphi+\tfrac{3}{2}\varphi_u^2)_{uu},\\
\mathfrak{F}_{-1/2}(\partial_x,\partial_y)=&-\tfrac{1}{2}(\varphi_{xx}+\tfrac{1}{2}\varphi_u^2+2\varphi_{uy}-\varphi\varphi_{uu})_{ux}=-\tfrac{1}{2}(\Delta_g\varphi+\tfrac{3}{2}\varphi_u^2)_{ux},\\
\mathfrak{F}_{-1/2}(\partial_y,\partial_y)=&\tfrac{1}{2}(\varphi_{xx}+\tfrac{1}{2}\varphi_u^2+2\varphi_{uy}-\varphi\varphi_{uu})_{xx}=\tfrac{1}{2}(\Delta_g\varphi+\tfrac{3}{2}\varphi_u^2)_{xx}.
\end{aligned}$$

Hence, a Brinkmann metric (4) is $\mathcal{F}_{-1/2}$-critical if and only if the function φ satisfies equation (12) for some functions C_1, C_2, C_3. □

Next we use the Cauchy-Kovalevskaya Theorem to construct local solutions of (12). Let (M,g) be a three-dimensional Brinkmann wave as in (4), and let Σ be the hyperplane $x=0$ with the induced Brinkmann metric $g_\Sigma=2dudy+\tilde{\varphi}(u,y)dy^2$. A straightforward calculation shows that the second fundamental form of $\Sigma\subset M$ is given by $\mathbb{I}=-\frac{1}{2}\varphi_x dy\otimes dy\otimes\partial_x$, since ∂_u, ∂_y are tangent to Σ and ∂_x is normal to Σ. Hence (Σ,g_Σ) is totally geodesic if and only if $\varphi_x=0$.

Theorem 4.5. *Let (Σ,g_Σ) be a two-dimensional analytic Brinkmann manifold. Then it can be extended to a three-dimensional $\mathcal{F}_{-1/2}$-critical analytic Brinkmann wave (M,g) such that (Σ,g_Σ) is a totally geodesic submanifold of (M,g).*

Proof. We consider a two-dimensional Brinkmann metric g_Σ given in local coordinates (u, y) by $g_\Sigma = 2dudy + \tilde{\varphi}(u, y)dy^2$. Let g be a three-dimensional Brinkmann metric as in (4) so that g_Σ corresponds to the induced metric on the plane $x = 0$.

Lemma 4.4 shows that g is $\mathcal{F}_{-1/2}$-critical if and only if

$$\varphi_{xx} + \tfrac{1}{2}\varphi_u^2 + 2\varphi_{uy} - \varphi\varphi_{uu} = C_1(y)u + C_2(y)x + C_3(y) \tag{13}$$

for arbitrary functions C_1, C_2 and C_3. We choose these functions to be analytic and note that $x = 0$ is a non-characteristic surface for this PDE (see, for example, [15]). Now, we set $\varphi_{|x=0} = \tilde{\varphi}$ and $\varphi_{x_{|x=0}}=0$ as initial data and use the Cauchy-Kovalevskaya Theorem to conclude that there exists an analytic solution φ to equation (13). This solution allows to extend g_Σ to g so that the plane $x = 0$ is a totally geodesic submanifold of g using the local coordinates in (4). □

Remark 4.6. Theorem 4.5 shows the possibility of generating examples of $\mathcal{F}_{-1/2}$-critical metrics with a clear geometric interpretation. Furthermore, one can also consider equation (12) with other non-characteristics surfaces, different from the plane $x = 0$, and produce different families of $\mathcal{F}_{-1/2}$-critical metrics with other initial data.

5. Special classes of Brinkmann metrics

In this section we consider some special families of Brinkmann waves motivated by geometric conditions which are related to homogeneity and local conformal flatness. For each special class we determine all critical metrics.

5.1. **Brinkmann metrics with constant scalar curvature.**

Brinkmann metrics with vanishing scalar curvature were discussed in Section 3. The case of non-zero constant scalar curvature reduces to the functional $\mathcal{F}_{-1/2}$ and it is covered by the following result.

Theorem 5.1. *If a Brinkmann metric with non-zero constant scalar curvature is $\mathcal{F}_t$-critical, then $t = -\frac{1}{2}$ and, moreover, for any $\kappa \in \mathbb{R}$, there exist three-dimensional Brinkmann $\mathcal{F}_{-1/2}$-critical metrics with $\tau = \kappa$.*

Proof. If a Brinkmann metric with non-zero constant scalar curvature is critical for a quadratic curvature functional, then by Theorem 2.1 and Lemma 3.2 it cannot be critical for $\mathcal{S}$ or $\mathcal{F}_t$ with $t \notin \{-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}\}$. Theorem 4.1 shows that the scalar curvature of a $\mathcal{F}_{-1/3}$-critical Brinkmann metric has the form $\tau(u, x, y) = 6f_3(y)u + 2f_2(x, y)$, with $f_3(y) \neq 0$ unless $\tau = 0$. Then this functional does not have critical metrics with non-zero constant scalar curvature. It follows from Theorem 4.2 that the scalar curvature of a $\mathcal{F}_{-1/4}$-critical Brinkmann metric has the form $\tau(u, x, y) = 2f_2(x, y)$, where f_2 is an Weierstrass elliptic function. Therefore the only quadratic curvature functional which may admit Brinkmann critical metrics of non-zero constant scalar curvature is $\mathcal{F}_{-1/2}$.

We set Brinkmann coordinates and use Lemma 4.4 to identify $\mathcal{F}_{-1/2}$-critical metrics by equation (12). If $\tau = k$, then $\varphi_{uu} = k$ and the function φ has the form $\varphi(u, x, y) = \frac{k}{2}u^2 + \alpha(x, y)u + \beta(x, y)$. Hence, equation (12) is expressed as

$$\alpha_{xx}u + \beta_{xx} + \tfrac{1}{2}\alpha^2 + 2\alpha_y - k\beta = C_1(y)u + C_2(y)x + C_3(y).$$

Differentiating with respect to u, we see that $\alpha_{xx} = C_1(y)$, so

$$\alpha(x, y) = \tfrac{1}{2}C_1(y)x^2 + \xi(y)x + \eta(y).$$

Hereafter we remove the dependence of y on the functions to simplify notation. Equation (12) now reads as

$$\begin{aligned}&C_1u + \tfrac{1}{8}C_1^2x^4 + \tfrac{1}{2}C_1\xi x^3 + \tfrac{1}{2}(C_1\eta + \xi^2 + 2C_1')x^2 + (\eta\xi + 2\xi')x\\&+ \tfrac{1}{2}\eta^2 + 2\eta' - k\beta + \beta_{xx} = C_1u + C_2x + C_3.\end{aligned}$$

Differentiating with respect to x we obtain

$$\tfrac{1}{2}C_1^2x^3 + \tfrac{3}{2}C_1\xi x^2 + (C_1\eta + \xi^2 + 2C_1')x + (\eta\xi + 2\xi') - k\beta_x + \beta_{xxx} = C_2,$$

so β has the following form:

$$\begin{aligned}\beta(x,y) =&\tfrac{1}{8k}C_1(y)^2x^4 + \tfrac{1}{2k}C_1(y)\xi(y)x^3\\&+ \tfrac{1}{2k^2}(3C_1(y)^2 + kC_1(y)\eta(y) + k\xi(y)^2 + 2kC_1'(y))x^2\\&+ \tfrac{1}{k^2}(-kC_2(y) + 3C_1(y)\xi(y) + k\eta(y)\xi(y) + 2k\xi'(y))x\\&+ \Xi(x,y) + \epsilon(y),\end{aligned}$$

with

$$\begin{aligned}&\Xi(x,y) = \tfrac{\sqrt{k}}{k}(\gamma(y)e^{\sqrt{k}x} - \delta(y)e^{-\sqrt{k}x}) \quad \text{if } k>0, \text{ and}\\&\Xi(x,y) = \tfrac{\sqrt{-k}}{k}(\gamma(y)\sin(\sqrt{-k}x) - \delta(y)\cos(\sqrt{-k}x)) \quad \text{if } k<0.\end{aligned}$$

Finally, we check that equation (12) has solutions for

$$\begin{aligned}C_3(y) =&\tfrac{1}{2k^2}(6C_1(y)^2 + 2kC_1(y)\eta(y)\\&+ k(-2k^2\epsilon(y) + k\eta(y)^2 + 2\xi(y)^2 + 4C_1'(y) + 4k\eta'(y))).\end{aligned}$$

Hence there exists a family of $\mathcal{F}_{-1/2}$-critical metrics with $\tau = k$ for arbitrary functions C_1, ξ, η, γ, δ, ϵ defining φ as above. □

Remark 5.2. Note that the proof of Theorem 5.1 goes through if, using Brinkmann coordinates as in (4), we assume that τ is only a function of the variable y. Hence the theorem above extends to this setting.

Remark 5.3. As natural generalizations of homogeneous geometries, Lorentz metrics with vanishing scalar curvature invariants (VSI) or constant scalar curvature invariants (CSI) have been extensively studied [11, 24]. Three-dimensional VSI spacetimes split into the families $\mathcal{A}.1$ and $\mathcal{B}.1$, following notation in [11]. Metrics within the family $\mathcal{A}.1$ are Brinkmann with vanishing scalar curvature and correspond to the discussion in Sections 2 and 3. The family $\mathcal{B}.1$ can be described in local coordinates (u,v,x) as

$$g = -2du\left[dv + \frac{1}{2}\left\{-\frac{v^2}{x^2} + vf_1(u,x) + f_0(u,x)\right\}du - \frac{2v}{x}dx\right] + dx^2,$$

for arbitrary functions $f_0(u,x)$ and $f_1(u,x)$. A straightforward calculation shows that the symmetric tensor $\mathfrak{F}_t$ is completely determined by the components $\mathfrak{F}_t(\partial_u,\partial_u)$ and $\mathfrak{F}_t(\partial_u,\partial_x) = -\frac{1}{2}f_{1_{xxx}} - \frac{1}{x}f_{1_{xx}}$. Hence $f_1(u,x) = -\alpha(u)\log(x) + x\gamma(u) + \beta(u)$ and metrics in family $\mathcal{B}.1$ which are $\mathcal{F}_t$-critical are determined by the linear fourth-order PDE

$$\begin{aligned}2x^4\,\mathfrak{F}_t(\partial_u,\partial_u) =& x^4\left(f_{0_{xxxx}}(u,x) + \gamma(u)^2\right) + x^3\left(4f_{0_{xxx}}(u,x) - \alpha(u)\gamma(u)\right)\\&+ x^2\left(\alpha(u)\beta(u) + 2\left(\alpha'(u) - 6f_{0_{xx}}(u,x)\right) + \alpha(u)^2(1-\log(x))\right)\\&+ 24x\,f_{0_x}(u,x) - 24f_0(u,x)\,.\end{aligned}$$

Hence a VSI metric in family $\mathcal{B}.1$ is $\mathcal{F}_t$-critical for some $t \in \mathbb{R}$ if and only if it is critical for all quadratic curvature functionals, in which case

$$\begin{aligned} f_1(u,x) &= -\alpha(u)\log(x) + x\gamma(u) + \beta(u), \\ f_0(u,x) &= x^4\left\{A_4(u) + \tfrac{1}{36}\gamma(u)^2(6\log(x)-11)\right\} \\ &\quad + x^3\left\{A_3(u) + \tfrac{1}{4}\alpha(u)\gamma(u)(2\log(x)-1)\right\} \\ &\quad + x^2\left\{A_2(u) + \tfrac{1}{2}\alpha'(u)(2\log(x)+1) + \tfrac{1}{4}\alpha(u)\beta(u)(2\log(x)+1)\right. \\ &\qquad \left. + \tfrac{1}{8}\alpha(u)^2\left(2\log(x) - 2\log^2(x) - 3\right)\right\} + xA_1(u). \end{aligned}$$

5.2. **Locally symmetric metrics.** A three-dimensional Lorentzian metric is locally symmetric if it is Einstein, a product $\mathbb{R}\times N(c)$ (where N is a surface of constant Gauss curvature), or a Cahen-Wallach symmetric space, which is a particular case of plane wave expressed in Brinkmann coordinates (4) with $\varphi(u,x,y) = \kappa x^2$ [7].

As already mentioned, Einstein manifolds are critical for all quadratic curvature functionals. However, non-flat products of the form $\mathbb{R}\times N(c)$ are critical only for the functional $\mathcal{F}_{-1/2}$. In the Riemannian setting, these product metrics are the only homogeneous metrics which are critical for this particular functional (see [6]). The next result shows that in Lorentzian signature, also Cahen-Wallach symmetric spaces are critical for $\mathcal{F}_{-1/2}$, indeed they are critical for all quadratic curvature functionals by Theorem 3.2.

Corollary 5.4. *Any three-dimensional locally symmetric Lorentzian metric is critical for the functional $\mathcal{F}_{-1/2}$. Moreover, Cahen-Wallach metrics are critical for all quadratic curvature functionals.*

5.3. **Locally conformally flat Brinkmann metrics.** Local conformal flatness in dimension $n \geq 4$ is characterized by the vanishing of the Weyl tensor. Hence any locally conformally flat metric is critical for the L^2-norm of the Weyl conformal tensor which, by the Gauss-Bonnet Theorem, is equivalent to the functional $\mathcal{F}_{-1/3}$ in dimension four. The situation is quite different in dimension three, where local conformal flatness is characterized by the vanishing of the Cotton tensor. There are locally conformally flat Brinkmann metrics which are not critical for any quadratic curvature functional. Indeed, locally conformally flat critical metrics are given by Corollary 3.3 and Corollary 5.4 as follows.

Theorem 5.5. *Let (M,g) be a three-dimensional locally conformally flat Brinkmann wave which is critical for a quadratic curvature functional. Then, one of the following two possibilities holds:*

1. *(M,g) is a plane-wave.*
2. *(M,g) is a locally symmetric product $\mathbb{R}\times N(c)$, where $N(c)$ is a Lorentzian surface of constant Gauss curvature.*

Proof. A three-dimensional metric is locally conformally flat if and only if the Cotton tensor vanishes. For manifolds in local coordinates as in (4), the $(0,2)$-Cotton tensor is determined by

$$\begin{aligned} &\mathcal{C}(\partial_u,\partial_x) = -\tfrac{1}{4}\varphi_{uuu}, \qquad \mathcal{C}(\partial_u,\partial_y) = -\tfrac{1}{2}\mathcal{C}(\partial_x,\partial_x) = \tfrac{1}{4}\varphi_{uux}, \\ &\mathcal{C}(\partial_x,\partial_y) = \tfrac{1}{2}\varphi_{uxx} + \tfrac{1}{4}\varphi_{uuy} - \tfrac{1}{4}\varphi\varphi_{uuu}, \\ &\mathcal{C}(\partial_y,\partial_y) = -\tfrac{1}{2}\varphi_{xxx} - \tfrac{1}{4}\varphi_u\varphi_{ux} - \tfrac{1}{2}\varphi_{uxy} + \tfrac{1}{4}\varphi_x\varphi_{uu} + \tfrac{1}{2}\varphi\varphi_{uux}. \end{aligned} \tag{14}$$

From (14) we obtain that φ has the following form:

$$\varphi(u,x,y) = A(y)u^2 - \left(\frac{1}{2}x^2A'(y) - B(y)x - C(y)\right)u + Q(x,y).$$

It follows from (6) that $\mathfrak{F}_t(\partial_x, \partial_x) = \frac{4}{3}(1+2t)A(y)^2$ and, thus, either $A(y) = 0$ or $t = -\frac{1}{2}$.

If $A(y) = 0$, then $\tau = 0$ and a direct analysis of the equations (14) shows that $\varphi(u,x,y) = (Bx+C)u - \frac{1}{48}B^2x^4 - \frac{1}{12}(2B' + BC)x^3 + Dx^2 + Ex + F$ for some functions B, C, D, E and F on the variable y. Now, $\mathfrak{F}_t(\partial_y, \partial_y) = \frac{B^2}{4}$, so $B = 0$. Therefore, the Ricci operator is two-step nilpotent and the metric is a *pp*-wave. Now, in appropriate coordinates we have $\varphi(x,y) = D(y)x^2 + E(y)x + F(y)$. Hence the metric is a plane wave and coordinates may be further specialized so that $\varphi(x,y) = a(y)x^2$.

If $A(y) \neq 0$ and $t = -\frac{1}{2}$, we work with the term $\mathfrak{F}_{-1/2}(\partial_y, \partial_y)$ and (14) to see that $A(y)$ is a constant and φ reduces to $\varphi(u,x,y) = \kappa u^2 + (B(y)x + C(y))u + \frac{1}{4\kappa}B(y)^2x^2 + \frac{1}{2\kappa}(B(y)C(y) + 2B'(y))x + D(y)$. Hence (M,g) is locally symmetric and the unit spacelike vector field $X = -\frac{1}{2\kappa}B(y)\partial_u + \partial_x$ is parallel, from where it follows that the metric splits locally as a product $\mathbb{R} \times N(c)$ where N is a Lorentz surface of constant Gauss curvature. □

5.4. **Conformally symmetric Brinkmann metrics.** A three-dimensional manifold is conformally symmetric if the Cotton tensor is parallel. Clearly, locally symmetric and locally conformally flat manifolds are conformally symmetric. It was shown in [8] that any other example is locally a Brinkmann metric (4) determined by a function $\varphi(u,x,y) = x^3 + \alpha(y)x$. The following is an immediate consequence of Theorem 3.2.

Corollary 5.6. *A three-dimensional conformally symmetric manifold which is neither locally conformally flat nor locally symmetric is critical for all quadratic curvature functionals.*

6. Brinkmann metric solutions in massive gravity

In this final section we use Brinkmann metrics to construct new solutions in massive gravity. Brinkmann metrics with vanishing scalar curvature have VSI and one gets the solutions previously obtained in [24]. It is worth emphasizing, however, the existence of additional solutions with non vanishing scalar curvature (see Theorem 6.3 below).

6.1. **Topologically massive gravity functional.** The *topologically massive gravity* functional, $S_{TMG} = S_{EH} + \frac{1}{\omega}S_{CS}$, is defined by adding the gravitational Chern-Simons term $S_{CS} = \frac{1}{2}\int d^3x\sqrt{|g|}\varepsilon^{ijk}\Gamma^r_{is}\left(\partial_j\Gamma^s_{rk} + \frac{2}{3}\Gamma^s_{jv}\Gamma^v_{kr}\right)$ to the Einstein-Hilbert functional, where ε^{ijk} is the fully anti-symmetric symbol in three dimensions with $\varepsilon^{123} = 1$. The Euler-Lagrange equations for the functional S_{TMG} are given by (see [1] and references therein)

$$\rho - \tfrac{1}{3}\tau g + \tfrac{1}{\omega}\mathcal{C} = 0, \tag{15}$$

where $\mathcal{C}$ denotes the Cotton tensor.

The next result shows that the only Brinkmann metric solutions of topologically massive gravity have vanishing scalar curvature and they reduce to those previously investigated in [24].

Theorem 6.1. *A Brinkmann metric* (4) *is a solution for the topologically massive gravity functional if and only if* $\varphi(u,x,y) = f_1(x,y)u + f_0(x,y)$, *with*

$$\begin{aligned} f_1(x,y) &= -\tfrac{1}{\omega}\alpha(y)e^{-\omega x} + \beta(y),\\ f_0(x,y) &= -\tfrac{1}{8\omega^4}\alpha(y)^2e^{-2\omega x} - \tfrac{1}{2\omega^3}\left\{(\omega x + 2)(\alpha(y)\beta(y) - 2\alpha'(y)) - 2\omega\gamma(y)\right\}e^{-\omega x}\\ &\quad + \xi(y)x + \eta(y), \end{aligned}$$

where α, β, γ, ξ and η are arbitrary functions.

Proof. We consider the symmetric tensor field $\mathfrak{T} = \rho - \frac{1}{3}\tau g + \frac{1}{\omega}C$ on a Brinkmann manifold (4). It follows from expressions (5) and (14) that $\mathfrak{T}(\partial_u, \partial_x) = -\frac{1}{4\omega}\varphi_{uuu}$, which shows that φ has the form

$$\varphi(u, x, y) = f_2(x, y)u^2 + f_1(x, y)u + f_0(x, y).$$

Then $\mathfrak{T}(\partial_u, \partial_y) = \frac{1}{3}f_2 + \frac{1}{2\omega}f_{2x}$, so $f_2(x, y) = A(y)e^{-\frac{2x\omega}{3}}$. Now, we have $\mathfrak{T}(\partial_x, \partial_y) = \frac{1}{18\omega}\{9\,(\omega f_{1x} + f_{1xx}) + e^{-\frac{2x\omega}{3}}\left(9A'(y) - 4u\omega^2 A(y)\right)\}$. Since the last expression vanishes identically, we conclude $A(y) = 0$. Hence, $f_2 = 0$ and, thus, $\tau = 0$.

It now follows that $\mathfrak{T}(\partial_x, \partial_y) = \frac{1}{2}f_{1x} + \frac{1}{2\omega}f_{1xx}$, from where $f_1(x, y) = -\frac{1}{\omega}\alpha(y)e^{-\omega x} + \beta(y)$ for some functions α and β. Finally the remaining term of $\mathfrak{T}$ reduces to

$$\mathfrak{T}(\partial_y, \partial_y) = \frac{1}{4\omega^2}\alpha(y)^2 e^{-2\omega x} - \frac{1}{4\omega}\left\{\alpha(y)\beta(y) + 2\alpha'(y)\right\}e^{-\omega x} - \frac{1}{2\omega}\left(f_{0xxx} + \omega f_{0xx}\right),$$

which determines the function f_0. □

Remark 6.2. Three-dimensional Brinkmann solutions for the topologically massive gravity functional S_{TMG} are critical for the functional $\mathcal{S}$, since its scalar curvature vanishes. Moreover, they are critical for some other quadratic curvature functional only if the metric is flat.

6.2. **New massive gravity functional.** The *new massive gravity* functional is defined by adding a multiple of the quadratic curvature functional $\mathcal{F}_{-3/8}$ to the Einstein-Hilbert functional: $S_{NMG} = S_{EH} - \frac{1}{m^2}\mathcal{F}_{-3/8}$. The Euler-Lagrange equations for the functional S_{NMG} are given by (see[3, 4])

$$\rho - \tfrac{1}{3}\tau g - \tfrac{1}{2m^2}(K - \tfrac{1}{3}(|\rho|^2 - \tfrac{3}{8}\tau^2)g) = 0, \tag{16}$$

where $K = 2\Delta\rho - \frac{1}{2}\nabla^2\tau - \frac{3}{2}\tau\rho + 4R[\rho] - (\frac{1}{2}\Delta\tau + |\rho|^2 - \frac{3}{8}\tau^2)g$ is a symmetric $(0, 2)$-tensor field.

Theorem 6.3. *A Brinkmann metric is a solution for the new massive gravity functional if and only if one of the following holds*

(1) *The scalar curvature vanishes, and the metric* (4) *is determined by a function* $\varphi(u, x, y) = f_1(x, y)u + f_0(x, y)$ *with*

$$\begin{aligned} f_1(x, y) &= \tfrac{1}{m}A(y)e^{mx} - \tfrac{1}{m}B(y)e^{-mx} + C(y),\\ f_0(x, y) &= -\tfrac{1}{6m^4}A(y)^2e^{2mx} - \tfrac{1}{6m^4}B(y)^2e^{-2mx} - \tfrac{1}{4m^3}H_1(x, y)e^{mx}\\ &\quad - \tfrac{1}{4m^3}H_2(x, y)e^{-mx} + \xi(y)x + \eta(y), \end{aligned}$$

where the functions H_1 *and* H_2 *are given by,*

$$\begin{aligned} H_1(x, y) &= 2m(A(y)C(y) + 2A'(y))x - (5A(y)C(y) + 4m\alpha(y) + 10A'(y)),\\ H_2(x, y) &= 2m(B(y)C(y) + 2B'(y))x + (5B(y)C(y) - 4m\beta(y) + 10B'(y)). \end{aligned}$$

(2) *The scalar curvature is constant* $\tau = 4m^2$*, and the metric* (4) *is determined by a function* $\varphi(u, x, y) = 2m^2u^2 + f_1(x, y)u + f_0(x, y)$ *with*

$$\begin{aligned} f_1(x, y) &= A(y)x^2 + B(y)x + C(y),\\ f_0(x, y) &= \tfrac{1}{8m^2}A(y)^2x^4 + \tfrac{1}{4m^2}A(y)B(y)x^3\\ &\quad + \tfrac{1}{8m^4}\left\{m^2B(y)^2 + 2m^2A(y)C(y) + 3A(y)^2 + 4m^2A'(y)\right\}x^2\\ &\quad + \tfrac{1}{4m^2}\left\{\alpha(y)e^{2mx} + \beta(y)e^{-2mx}\right\} + \xi(y)x + \eta(y). \end{aligned}$$

Proof. The non-zero components of the tensor field $\mathfrak{N} = \rho - \frac{1}{3}\tau g - \frac{1}{m^2}\mathfrak{F}_{-3/8}$ for a metric (4) are given by

$$\mathfrak{N}(\partial_u,\partial_u) = \tfrac{\varphi_{uuuu}}{4m^2}, \quad \mathfrak{N}(\partial_u,\partial_x) = \tfrac{\varphi_{uuux}}{4m^2}, \quad \mathfrak{N}(\partial_x,\partial_x) = -2\mathfrak{N}(\partial_u,\partial_y) + \varphi\mathfrak{N}(\partial_u,\partial_u),$$

$$\mathfrak{N}(\partial_u,\partial_y) = -\tfrac{1}{24m^2}\left(\varphi_{uu}^2 - 4m^2\varphi_{uu} + 6\varphi_{uuxx} - 3\varphi_u\varphi_{uuu} + 6\varphi_{uuuy} - 6\varphi\varphi_{uuuu}\right),$$

$$\mathfrak{N}(\partial_x,\partial_y) = \tfrac{1}{8m^2}\left((4m^2 - \varphi_{uu})\varphi_{ux} - 4\varphi_{uxxx} - 6\varphi_{uuxy} + 3\varphi_x\varphi_{uuu} + 4\varphi\varphi_{uuux}\right),$$

$$\begin{aligned}\mathfrak{N}(\partial_y,\partial_y) = \tfrac{1}{24m^2}\big(&12\varphi_{xxxx} + 12\varphi_{ux}^2 + 12\varphi_u\varphi_{uxx} + 24\varphi_{uxxy} + 4m^2\varphi\varphi_{uu} - \varphi\varphi_{uu}^2\\ &- 3(4m^2 + 3\varphi_{uu})\varphi_{xx} + 3\varphi_u\varphi_{uuy} - 12\varphi\varphi_{uuuy} + 3\varphi\varphi_u\varphi_{uuu}\\ &+6\varphi^2\varphi_{uuuu} - 21\varphi_x\varphi_{uux} - 18\varphi\varphi_{uuxx} - 3\varphi_y\varphi_{uuu} + 6\varphi_{uuyy}\big).\end{aligned}$$

From $\mathfrak{N}(\partial_u,\partial_u) = 0$ and $\mathfrak{N}(\partial_u,\partial_x) = 0$ one obtains that φ has the form

$$\varphi(u,x,y) = f_3(y)u^3 + f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y).$$

The component $\mathfrak{N}(\partial_u,\partial_y)$ reduces to

$$\mathfrak{N}(\partial_u,\partial_y) = \tfrac{1}{12m^2}\left\{9f_3^2u^2 + 6(2m^2 + f_2)f_3u + 4m^2f_2 - 2{f_2}^2 + 9f_1f_3 - 18f_3' - 6f_{2xx}\right\}.$$

Differentiating twice with respect to u gives $\mathfrak{N}(\partial_u,\partial_y)_{uu} = \frac{3}{2m^2}{f_3}^2$. Hence $f_3 = 0$ and $\varphi(u,x,y) = f_2(x,y)u^2 + f_1(x,y)u + f_0(x,y)$. Now we compute the derivatives

$$2\mathfrak{N}(\partial_u,\partial_y)_x - \mathfrak{N}(\partial_x,\partial_y)_u = -\tfrac{1}{6m^2}(f_2 + 2m^2)f_{2x},$$

so f_2 does not depend on the coordinate x. Hence $\varphi(u,x,y) = f_2(y)u^2 + f_1(x,y)u + f_0(x,y)$ and one has that $\mathfrak{N}(\partial_x,\partial_x) = \frac{1}{3m^2}(f_2 - 2m^2)f_2$. This shows that either $f_2 = 0$ or $f_2 = 2m^2$, that correspond to Assertions (1) and (2), respectively.

If $f_2 = 0$, then $\mathfrak{N}(\partial_x,\partial_y) = \frac{1}{2}f_{1x} - \frac{1}{2m^2}f_{1xxx}$ and the function f_1 has the form

$$f_1(x,y) = \tfrac{1}{m}A(y)e^{mx} - \tfrac{1}{m}B(y)e^{-mx} + C(y).$$

The remaining term $\mathfrak{N}(\partial_y,\partial_y)$ now reduces to

$$\begin{aligned}\mathfrak{N}(\partial_y,\partial_y) = \tfrac{1}{2m^2}\big\{&2A^2e^{2mx} + 2B^2e^{-2mx} + m(AC + 2A')e^{mx}\\ &-m(BC + 2B')e^{-mx} - m^2f_{0xx} + f_{0xxxx}\big\},\end{aligned}$$

from where Assertion (1) follows.

If $f_2 = 2m^2$, then $\mathfrak{N}(\partial_x,\partial_y) = -\frac{1}{2m^2}f_{1xxx}$ and, hence, $f_1(x,y) = A(y)x^2 + B(y)x + C(y)$. Using this expression the remaining term $\mathfrak{N}(\partial_y,\partial_y)$ reduces to

$$\mathfrak{N}(\partial_y,\partial_y) = \frac{1}{2m^2}\left\{6A^2x^2 + 6ABx + B^2 + 2AC + 4A' - 4m^2f_{0xx} + f_{0xxxx}\right\},$$

from where Assertion (2) follows. □

Remark 6.4. Metrics in Assertion (1) of Theorem 6.3 are critical for the functional $\mathcal{S}$ and correspond to those in [24], while metrics in Assertion (2) are critical for the functional $\mathcal{F}_{-1/2}$.

7. Conclusions

Motivated by new gravitational theories like *topologically massive gravity* and *new massive gravity*, whose solutions correspond to critical metrics of curvature functionals which involve quadratic terms, we develop a systematic study of critical Brinkmann waves for all possible quadratic curvature functionals.

A different behavior is observed depending on whether the scalar curvature vanishes or not. We showed that Brinkmann waves are critical for the functional determined by the L^2-norm of the scalar curvature if and only if the scalar curvature vanishes. If the scalar curvature is zero, then a Brinkmann wave is critical for a quadratic curvature functional $\mathcal{F}_t$ if and only if it is critical for all quadratic functionals simultaneously. We emphasize that these metrics, which are given explicitly in Theorem 3.2, are not necessarily Ricci flat nor *pp*-waves. Generically they have three-step nilpotent Ricci operator.

A Brinkmann wave with non-zero scalar curvature is critical for a quadratic curvature functional if and only if it is critical for $\mathcal{F}_t$ with $t \in \{-\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\}$. The quadratic functionals $\mathcal{F}_{-1/4}$, $\mathcal{F}_{-1/3}$, and $\mathcal{F}_{-1/2}$ behave differently, and there exist Brinkmann waves which are critical for $\mathcal{F}_{-1/4}$, $\mathcal{F}_{-1/3}$, or $\mathcal{F}_{-1/2}$ without being critical for any other quadratic curvature functional. A explicit description of these metrics is given in Theorem 4.2, Theorem 4.1, and Theorem 4.4, respectively.

It is shown that if a Brinkmann wave with non-zero constant scalar curvature is critical for some quadratic curvature functional, then it is $\mathcal{F}_{-1/2}$-critical. Moreover we show that any two-dimensional Brinkmann wave may be embedded as a totally geodesic hypersurface in a three-dimensional Brinkmann wave which is $\mathcal{F}_{-1/2}$-critical.

As an application of these results we construct new explicit solutions in massive gravity theories. Firstly, we show that Brinkmann wave solutions of topologically massive gravity have vanishing scalar curvature and they reduce to the examples found in the work of Siampos and Spindel [24]. Secondly, we determine all Brinkmann waves which provide solutions to new massive gravity, i.e., critical metrics for the functional $S_{NMG} = S_{EH} - \frac{1}{m^2}\mathcal{F}_{-3/8}$. The scalar curvature is necessarily constant and we obtain the solutions in [24] if it vanishes (these solutions are critical for the L^2-norm of the scalar curvature). In the case of non-zero scalar curvature, Theorem 6.3 provides new explicit solutions which are critical for the functional $\mathcal{F}_{-1/2}$.

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