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Research paper

Equilibrium models with heterogeneous agents under rational expectations and its numerical solution

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A R T I C L E I N F O

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A B S T R A C T

In this work we assume rational expectations to pose general equilibrium models with heterogeneous firms that can enter or exit the industry. More precisely, we assume a general Ito process for the dynamics of the agents productivity, including the main dynamics in the literature. A Hamilton-Jacobi-Bellman (HJB) formulation models the endogenous decision of firms to remain or exit the industry. All firms that exit are immediately replaced by a group of new ones, so that the probability density function of firms satisfies an appropriate Kolmogorov-Fokker-Plank (KFP) equation with source term. Equilibrium models are completed with the household problem formulation and the feasibility conditions. In the evolutive and general stationary settings, analytical or semi-analytical formulas are not available, so that appropriate numerical methods are required. We propose a Crank-Nicolson scheme for the time discretization of the evolutive problems. Moreover, we use an augmented Lagrangian active set (ALAS) method combined with a finite difference discretization for the HJB formulation and a suitable finite differences discretization for the KFP problem. For the global equilibrium problem we propose a Steffensen algorithm. Numerical examples illustrate the performance of the proposed numerical methodologies as well as the expected behaviours of the computed economic variables.

1. Introduction

In the present work we study a class of continuous time heterogeneous agent models with idiosyncratic shocks and incomplete markets under rational expectations. As in Hopenhayn [1], we consider a continuous time formulation in which firms face with idiosyncratic productivity shocks. The productivity evolution is the only underlying source of uncertainty, which is modelled by a stochastic Ito process. We assume that the industry is operated by heterogeneous agents to relate (expected) future opportunity benefits for firms. Moreover, as in Weninger and Just [2], we assume that the abilities of individual firms follow a stochastic process and that there is a fixed operating cost for firms must to remain in the industry. These two last assumptions generate the evolution of firm dynamics over time. Therefore, individual rational decisions are not only based on current profits and the whole transitional dynamics must be computed to capture the behavior of firms and its consequences on economic variables. Also, in view of their shocks, firms decide to remain or exit. Exit decisions are

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endogenously determined by the optimal decision where the opportunity cost of remaining is taken into account by firms through the existence of a scrap value at which firms exit the industry. This decision is formulated in terms of an optimal stopping time problem. Moreover, exit firms are replaced by the entry of new firms such that exit and entry rates are balanced. In this way, the rate of the entry of firms is also endogenous at prescribed levels of productivity. In this respect, we consider two examples, one case where entry takes place at a given productivity above the optimal exit productivity in line with Luttmer [3], and a second one where firms enter at a finite set of productivities with their corresponding probabilities. More general cases could be incorporated by using suitable probability distributions for the productivity of entry firms. Also, in some industries, these entry productivities are exogeneously provided by suitable models to incorporate the income, and therefore the wages and labor (see [4], for an example in the case of fisheries). Unlike in the work by Guzzini and Pallestrini [5], we do not consider the interaction between the idiosyncratic shock of firms and their network structure, which would require to use adjacency matrices from graph theory to represent links betwen firms.

As in Achdou et al. [6], we assume that individuals interact in markets and make choices for given prices. These prices are determined in general equilibrium and depend on the entire distribution of individuals in the economy and its evolution. Individuals choices jointly with idiosyncratic shocks in turn determine the evolution of this distribution. Mathematical models for such economies can be mainly formulated in terms of a system of two coupled partial differential equations (PDEs). More precisely, they involve a Hamilton-Jacobi-Bellman (HJB) PDE for the incumbents problem. This backward in time HJB equation characterizes the optimal value of the utility function associated with the profit of the firm over time with a given stochastic process evolution for labor productivity. Additionally, a forward in time Kolmogorov-Fokker-Planck (KFP) PDE governs the evolution of the firms distribution. Also note that the coupling of these two equations for modelling heterogeneous agents behaviour can be framed into the mathematical theory of Mean Field Games (MFG) developed by Lasry and Lions in [7] and the coupled system is know as a *backward-forward MFG system*. For example, this coupling also arises in the Aiyagari-Bewley-Hugget model to characterize the optimal saving and consumption behaviour of individuals with a stochastic income (see Hugget [8] and Aiyagari [9], for example). Numerical methods for the MFG system are proposed in [10].

In our model, individual firms assess the expected value of remaining in the industry at each time moment and compare it to the present discounted value of profits associated with exiting the industry. Based on this comparison, individual firms decide to stay in or exit the industry. Note that these models with endogenous firms' exit are also investigated in the so called real options framework (see Dixit et al. [11], for example). Analogous problems arise in mathematical finance, when pricing financial derivatives with early exercise opportunities, such as the American options (see Wilmott et al. [12], for example).

In this work, we make progress by first adding new modelling aspects in the problem. Thus a first innovative aspect comes from the consideration that entry of new establishments is allowed and takes place in the same quantity as firms exit due that their low productivity cannot compensate the fixed costs. The entry of new firms is taken into account by means of a source term in the KFP equation. In previous works, such as Luttmer [3] or Da-Rocha and Sempere [4], it was assumed that the stochastic productivity process follows a geometric Brownian motion and new firms enter at a given productivity above the exit productivity, so that in the steady state we can adapt our model to get an analytical solution for the KFP equation. Here we present a more general setting, in which we incorporate randomness in the productivity at which new establishments start to operate by considering a probability function associated with a discrete random variable, thus opening the possibility to consider any variable with the support of the associated probability function inside the interval of operating productivities. In this new setting, analytical solutions for the steady state KFP equation are no more available and appropriate numerical methods are required. Furthermore, numerical methods are also mandatory when considering a general Ito process instead of a Brownian motion for the evolution of the productivity. In Achdou et al. [13], the authors point out that the development of numerical methods for solving both stationary and time-dependent equilibria is an open question when more general stochastic processes than geometric Brownian motion are considered.

In this new more general setting which includes possible random entry productivities, a relevant innovative aspect comes from the consideration of appropriate numerical methods. Thus, in the incumbents' problem we propose the use of the augmented Lagrangian active set (ALAS) algorithm developed in Kärkkäinen et al. [14] for solving the linear complementarity problem associated with the HJB equation. For example, this algorithm has been used in Calvo et al. [15] for pricing pension plans with early retirement and in Calvo et al. [16] for the pricing of swing options in electricity markets. For the KFP PDE we propose a suitable numerical method and for the equilibrium problem we consider a Steffensen speed up of the fixed point iteration, which provides second order convergence in the global algorithm. Also evolutive models require suitable additional numerical methods, so we incorporate a Crank-Nicolson for the time discretization to be combined with the previously indicated methods for the steady state models (see Strikwerda [17], for example).

The rest of the paper is organized as follows. In Section 2 we pose the models involved in the different problems that are the building block of the proposed more general equilibrium model, thus including the incumbent problem governed by the HJB equation and firms distribution problem governed by the KFP equation. Section 3 is devoted to a particular model where the stochastic dynamics of productivity follows a geometric Brownian motion and entry of new firms takes place at a given productivity, so that an analytical solution is available for the KFP equation and a semianalytical one for the equilibrium problem. Section 4 describes the numerical methods for the different subproblems in our more general model. Section 5 contains the numerical results for steady state problems. Section 6 describes the additional numerical techniques for the evolutive models, as well as the numerical results mainly to illustrate the convergence of the numerical solutions of the evolutive model to the steady state ones. Finally, some conclusions are presented in Section 7 and a list of the main used notations.

2. The mathematical models

First, we assume that there are two markets in the economy: final goods and labor markets. Labor is used to produce the final goods. By choosing the output price as the numeraire, we denote the wages at time t by $\omega(t)$. We also assume that there exists a continuum of identical households who own the firms, consume the final goods and supply labor by solving a consumption-leisure maximization problem.

Secondly, we assume that firms are heterogeneous. Let $g(z,t)$ denote the probability distribution of firms with productivity z at time t. The decision rules of incumbent firms at time t depend on the productivity z. For a given productivity z and time t, we denote the optimal choices of output and labor by $y(z,t)$ and $l(z,t)$, respectively.

As in Weninger and Just [2], we assume that the uncertain productivity of firms evolves in time according to a stochastic process z_t and that firms must pay the fixed operating cost c_f to remain active in the industry. These two assumptions imply that individual firms change over time in the transition problem. At each particular time t, some of them expand production, hiring staff, while others contract production, firing staff, and others exit the industry. When a firms exits, it receives a scrap value s that may depend on time, but it can never re-entry in the future. The firms that exit are instantaneously replaced by a group of entry firms, in such a way that the exit and entry rates are balanced. In the present work, the productivity of firms that enter the industry is assumed to be random and governed by a probability function g^e. For example, if this probability function is defined by a Dirac delta centered at a particular productivity z_0^e, then all firms enter with this productivity at a rate that balances the exit rate.

The decision problem of incumbent firms involves two types of decision rules. There are continuous decision rules for the optimal choice of output $y(z,t)$ and labor $l(z,t)$, and there is a discrete decision rule for the optimal stay/exit decision that takes into account the scrap value which involves the opportunity cost of remaining in operation.

Therefore, we have an endogenous exit decision. This decision depends on $l(z,t)$ and $y(z,t)$ in each period. Conditioned by the choices of functions l and y in each period, the firms must compare the expected value of remaining in the industry with the discounted value of profits associated with exiting the industry. Note that unlike to the standard framework, the distribution of the productivity of firms is not exogenous. In our model it is endogenously determined by their decisions on exiting. Therefore, the distribution g changes over time in the transition model. In the steady state case, the balance between entry and exit of firms allows to understand entry and exit as job creation and destruction. In this case, we obtain stationary probability functions of firms size and values, as well as in the other equilibrium magnitudes.

Next, we analyze the coupled model in three steps. First, we consider the individual problems of firms and households, which establishes the optimal stay/exit decisions. Secondly, we specify the dynamics of the distribution of firms and the feasibility conditions. Finally, we define the equilibrium coupled problem. In all cases, the transition and steady state problems are presented.

2.1. The problem of incumbent firms

In the first continuous decision rule inside this problem, we assume that firms maximize at each time $t \in [0,T]$ their profits, $\pi(z,t)$, with respect to the labor force $l(z,t)$ and the revenues $y(z,t)$ provided by their available technology, which satisfy the relation $y(z,t)=\sqrt{zl(z,t)}$. Therefore, at each time $t \in [0,T]$, the profit maximization problem is formulated in the form

$$\begin{aligned} \max_{l(z,t),y(z,t)} \quad & \pi(z,t) = y(z,t) - \omega(t) l(z,t) - c_f, \\ \text{s.t.} \quad & y(z,t) = \sqrt{zl(z,t)}, \end{aligned} \tag{1}$$

where profits are given as the difference between revenues $y(z,t)$ and the sum of labor costs $\omega(t)l(z,t)$ plus fixed operation costs c_f.

By analytically solving problem (1) for each time $t \in [0,T]$, we find that the labor force and revenues that maximize the profit are given by

$$l(z,t) = \frac{z}{4\omega(t)^2}, \quad y(z,t) = \frac{z}{2\omega(t)}, \tag{2}$$

so that the optimal profit is given by

$$\pi(z,t) = \frac{z}{4\omega(t)} - c_f . \tag{3}$$

Next, we assume that the productivity z is a stochastic process which satisfies the following stochastic differential equation (SDE):

$$dz_t = \mu(z_t,t)dt + \sigma(z_t,t)dW_t, \tag{4}$$

where μ and σ represent the drift and volatility of the productivity, while W_t denotes a standard Brownian motion. Thus, the productivity follows a Ito process, where μ and σ satisfy the conditions that guarantee the existence and uniqueness of solution of Eq. (4), see [18] for example. As we mainly are interested in the convergence of evolutive problems to the corresponding steady state ones, we assume the following SDE:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t. \tag{5}$$

Note that the geometric Brownian motion is a particular case for (4) and (5). Also Ornstein-Uhlenbeck processes with mean reversion to a long term productivity are included in (4) or (5), see some models in Stokey [19]. However, if we would like to incorporate jumps in productivity then a jump-diffusion models need to be considered, where a Poisson process or a more general Levy process must be introduced (see Pascucci [20], for example). For example, jump-diffusion models have been included in Gabaix et al. [21] for income dynamics in the modelling of random growth theories for income inequalities.

For a time horizon $T > 0$ and the set of positive productivities, we consider the domain $D_T = \mathbb{R}^+ \times [0, T]$. Assuming the option of firms to exit, the dynamic incumbents problem is posed as the optimal stopping time problem:

$$v_t = \max_{\tau_l} \mathbb{E}_0\left[\int_0^{\tau_l} \pi(z_t, t)e^{-\rho t}dt + s(\tau_l)e^{-\rho\tau_l}\right],$$
$$\text{s.t.} \;\; dz_t = \mu(z_t, t)dt + \sigma(z_t, t)dW_t,$$

where $v_t = v(z_t, t)$ is the optimal firm value for the productivity z_t and time t, τ_l is a stopping time, $\mathbb{E}_0$ is the expectation conditional on the initial state, $\pi(z, t)$ is given by (3) from the solution of maximization problem (1) and ρ is a discount rate. By using the tools of stochastic control, we obtain that the function v satisfies the following HJB equation in D_t:

$$\min\left\{\rho v - \pi(z, t) - \mu(z, t)\partial_z v - \frac{1}{2}\sigma^2(z, t)\partial^2_{zz}v - \partial_t v, v - s\right\} = 0. \tag{6}$$

Problem (6) and analogous incumbent problems will be referred as HJB problems. Moreover, it can be written as the linear complementarity problem:

$$\mathcal{L}^p(v) \ge \pi, \quad v \ge s, \quad (\mathcal{L}^p(v) - \pi)\cdot(v - s) = 0, \tag{7}$$

with the involved parabolic differential operator given by

$$\mathcal{L}^p(v) = -\partial_t v - \mu(z, t)\partial_z v - \frac{1}{2}\sigma^2(z, t)\partial^2_{zz}v + \rho v. \tag{8}$$

Note that (7) can also be formulated as a parabolic variational inequality of obstacle type, see Kinderlehrer-Stampacchia [22] for example. Therefore, we are handling the concept of weak solution of a variational inequality, whose regularity can be further studied. Note that the concept of viscosity solution for HJB equation has been also handled in the literature (see [7] and references therein, for example).

Moreover, we can identify the exit and stay regions defined by:

$$D_{Ex} = \{(z, t) \in D_t | v(z, t) = s(t)\}, \quad D_{St} = \{(z, t) \in D_t | v(z, t) > s(t)\},$$

which are also unknowns of the incumbents problem. The following smooth pasting conditions hold at the unknown boundary separating these regions:

$$v = s, \quad \partial_z v = 0, \quad \text{on } \partial D_{Ex} \cap \partial D_{St}. \tag{9}$$

Note that $\partial D_{Ex} \cap \partial D_{St}$ can be understood as an optimal exit boundary and can be parameterized in the zt-plane as $\partial D_{Ex} \cap \partial D_{St} = \{(\underline{z}(t), t),\ t \in [0, T]\}$. Therefore, the function v satisfies the equations:

$$\mathcal{L}^p(v) = \pi \;\text{ in } D_{St}, \quad v = s \;\text{ in } D_{Ex}, \tag{10}$$

jointly with the smooth pasting conditions (9).

By considering the dynamics in (5), the steady state version of (7) for the incumbents problem is:

$$\mathcal{L}^e(v) \ge \pi, \quad v \ge s, \quad (\mathcal{L}^e(v) - \pi)\cdot(v - s) = 0, \tag{11}$$

with the involved differential operator given by

$$\mathcal{L}^e(v) = -\mu(z)\partial_z v - \frac{1}{2}\sigma^2(z)\partial^2_{zz}v + \rho v. \tag{12}$$

Moreover, the steady state formulation analogous to (10) reads:

$$\mathcal{L}^e(v) = \pi(z) \;\text{ in } \Omega_{St}, \quad v = s \;\text{ in } \Omega_{Ex}, \tag{13}$$

jointly with the smooth pasting conditions:

$$v(z) = s, \quad \partial_z v(z) = 0, \;\text{ on } \partial\Omega_{Ex} \cap \partial\Omega_{St}, \tag{14}$$

where $\Omega_{Ex} = \{z \in \mathbb{R}^+ | v(z) = s\}$, $\Omega_{St} = \{z \in \mathbb{R}^+ | v(z) > s\}$ and $\partial\Omega_{Ex} \cap \partial\Omega_{St} = \{\underline{z}\}$. Note that in the steady state problem, the optimal exit boundary is given by the productivity $\underline{z}$, which separates the region $\Omega_{St} = (\underline{z}, +\infty)$ where is optimal to remain from the region $\Omega_{Ex} = [0, \underline{z}]$ where it is optimal to exit.

2.2. The household problem

Each representative household supplies labor (L) and consumes goods (C). If we consider given prices as numeraire and denote wages by ω, each household at each time t solves a static consumption-leisure maximization problem:

$$\max_{L(t),C(t)} \left[\log C(t) - eL(t)\right],$$

where e is a utility parameter, subject to the constraint $C(t) = \omega(t)L(t) + \Pi(t)$, where the budget C is the sum of wage income ωL and the aggregated profits of operating firms, Π. If we replace the constraint in the function to be maximized and apply the first order condition for the maximum we get the equation

$$\omega(t) = e\Big[\omega(t)L(t) + \Pi(t)\Big]. \tag{15}$$

2.3. The problem of the dynamics of firms distribution

In order to compute the prices, the dynamics of firms must be obtained. For this purpose, the probability distribution of firms is denoted by g, which is endogenously determined by stay/exit decisions made by firms and the distribution of entry firms. The function g satisfies the following Kolmogorov-Fokker-Planck (KFP) equation with a source term related to the entry of firms:

$$\partial_t g = -\partial_z(\mu(z,t)g) + \frac{1}{2}\partial^2_{zz}(\sigma^2(z,t)g) + \alpha(t)g^e(z,t), \quad \text{in } D_{St}, \tag{16}$$

where $\alpha(t)$ represents the new establishment entry rate and g^e denotes the probability function of the new establishments.

More precisely, g^e is a probability density (pdf) or a probability mass (pmf) function for a continuous or discrete random variable, respectively. We assume that the support of $g^e(\cdot,t)$ is contained in $(\underline{z}(t), +\infty)$ to be consistent with the solution of the incumbents problem. Although alternative probability functions could be considered, here we consider that new firms enter either at a given productivity z^e_0, so that $g^e(z) = \delta(z - z^e_0)$ with δ being the Dirac function centered at $z = 0$, or randomly among a set of given productivities with their associated probabilities, so that

$$g^e(z) = \sum_{i=1}^{I} p^e_i \delta(z - z^e_i), \qquad \text{with} \quad z^e_i > \underline{z}, \quad p^e_i \geq 0, \quad \sum_{i=1}^{I} p^e_i = 1. \tag{17}$$

Note that the productivity process has a reflecting barrier at the optimal productivity boundary, so that Eq. (16) is satisfied on D_{St}. Moreover, we consider the boundary conditions at the spatial boundaries of D_{St}:

$$g(\underline{z}(t), t) = 0 \qquad \text{and} \qquad \lim_{z\to+\infty} g(z,t) = 0. \tag{18}$$

By integrating (16) between $\underline{z}(t)$ and $+\infty$, applying boundary conditions (18) and using the Leibniz integral rule, we obtain

$$\alpha(t) = \frac{1}{2}\sigma^2(\underline{z}(t), t)\partial_z g(\underline{z}(t), t),$$

which states that the entry rate $\alpha(t)$ is equal to the exit rate at point $z = \underline{z}(t)$, so that the entry rate is endogenous. In Gabaix et al. [21], the same argument is used to pose a KFP equation to model income inequality.

Moreover, we can write the steady state version of Eq. (16) as follows:

$$-\partial_z(\mu(z)g) + \frac{1}{2}\partial^2_{zz}(\sigma^2(z)g) + \frac{1}{2}\sigma^2(\underline{z})\partial_z g(\underline{z})\, g^e(z) = 0, \quad \text{in } \Omega_{St}, \tag{19}$$

where the entry of firms does not depend on time.

2.4. Feasibility conditions

In order to close the model we need to define feasibility conditions in the labor and output markets. These conditions are respectively given by:

$$L(t) = \int_{\underline{z}(t)}^{+\infty} l(z,t)g(z,t)dz, \quad Q(t) = \int_{\underline{z}(t)}^{+\infty} y(z,t)g(z,t)dz, \tag{20}$$

where $l(z,t)$ and $y(z,t)$ are given by (2).

Note that in [4], the quantity $Q(t)$ represents the fishing opportunities and it is exogenously obtained from a stocks dynamics model. Moreover, the household budget constraint implies that the final output market is in equilibrium, i.e.

$$\begin{aligned} C(t) &= \omega(t)L(t) + \Pi(t) \\ &= \omega(t)\int_{\underline{z}(t)}^{+\infty} l(z,t)g(z,t)dz + \int_{\underline{z}(t)}^{+\infty}\big(y(z,t) - \omega(t)l(z,t) - c_f\big)g(z,t)\,dz \\ &= \int_{\underline{z}(t)}^{+\infty} y(z,t)g(z,t)dz - c_f\,. \end{aligned} \tag{21}$$

After some calculus, we can write (15) in the form

$$\omega(t) = e\left(\int_{\underline{z}(t)}^{+\infty}\frac{z}{2\omega(t)}g(z,t)dz - c_f\right), \tag{22}$$

where $\underline{z}(t)$ depends on ω.

2.5. Equilibrium models

In a general dynamic equilibrium framework, the economy can be represented by the time-varying solution $\underline{z}(t)$, $v(z,t)$, $g(z,t)$ and $\omega(t)$ to the HJB and KFP Eqs. (7) and (16), we rewrite as follows:

$$\mathcal{L}^p(v) \ge \pi, \quad v \ge s, \quad (\mathcal{L}^p(v) - \pi)\cdot(v - s) = 0, \tag{23}$$

$$\partial_t g = -\partial_z\big(\mu(z,t)g\big) + \frac{1}{2}\partial^2_{zz}\big(\sigma^2(z,t)g\big) + \frac{1}{2}\sigma^2\big(\underline{z}(t),t\big)\partial_z g\big(\underline{z}(t),t\big)\,g^e(z,t), \tag{24}$$

jointly with (22). Moreover, domains for (23) and (24) are D_T and D_{St}, respectively, with appropriate boundary and final or initial conditions.

A stationary equilibrium is represented by the time independent solution $v(z)$, $\underline{z}$, $g(z)$ and ω to the corresponding steady state HJB and KFP PDEs, jointy with (22). In this case, the domains of the HJB and KFP PDEs are the productivity intervals $[0,+\infty)$ and $(\underline{z},+\infty)$, respectively. Note that the steady state HJB problem admits an equivalent formulation as an elliptic variational inequality (see [22], for example).

3. A particular stationary model for productivity evolution

In this section, we assume that the productivity follows a geometric Brownian motion, so that Eq. (5) with $\mu(z_t) = \mu z_t$ and $\sigma(z_t) = \sigma z_t$ is satisfied, where constants $\mu < 0$ and σ are the expected growth rate and the per-unit time volatility, respectively. Moreover, we consider that all new establishments enter with a given productivity $z_0^e > \underline{z}$, so that $g^e(z) = \delta(z - z_0^e)$.

3.1. The particular steady state incumbent firms problem

The steady state incumbent firms problem can be formulated by Eqs. (13) and (14), with the particular choices of μ and σ. By using analogous techniques to the ones in Da-Rocha and Sempere [23], the exact solution is given in Proposition 1.

Proposition 1. *Given wages w, the exit threshold $\underline{z}$ and the value function v(z) that solve the particular incumbent firms problem are given by*

$$\underline{z} = \frac{4\omega\beta(\rho-\mu)}{1+\beta}\left(s + \frac{c_f}{\rho}\right), \tag{25}$$

and

$$v(z) = \begin{cases} s, & z \le \underline{z}, \\ \dfrac{s + \frac{c_f}{\rho}}{1+\beta}\left(\dfrac{\underline{z}}{z}\right)^{\beta} + \dfrac{z}{4\omega(\rho-\mu)} - \dfrac{c_f}{\rho}, & z > \underline{z}, \end{cases} \tag{26}$$

where $\beta = \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$.

Proof. First, assuming a solution of the form $v(z) = A_1 z^{-\beta} + A_2 z + A_3$ in the domain Ω_{St}, its substitution in the HJB Eq. (13) leads to

$$\rho\big(A_1 z^{-\beta} + A_2 z + A_3\big) - \pi(z) - \mu z\big(-\beta A_1 z^{-\beta-1} + A_2\big) - \frac{\sigma^2}{2}z^2\big(\beta(\beta+1)A_1 z^{-\beta-2}\big) = 0,$$

where $\pi(z) = \frac{z}{4w} - c_f$. If we rearrange terms, we get the following equation for β: $\sigma^2\beta^2 - (2\mu - \sigma^2)\beta - 2\rho = 0$, the positive solution of which is given by $\beta = \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$. Moreover, we also obtain $A_2 = (4w(\rho-\mu))^{-1}$ and $A_3 = -c_f/\rho$.

Next, we use the smooth pasting conditions (14), so that

$$v(\underline{z}) = A_1\underline{z}^{-\beta} + \frac{1}{4w(\rho-\mu)}\underline{z} - \frac{c_f}{\rho} = s, \;\; \partial_z v(\underline{z}) = -\beta A_1 \underline{z}^{-\beta-1} + \frac{1}{4w(\rho-\mu)} = 0,$$

from which we obtain expression (25) for $\underline{z}$ and $A_1 = \frac{\underline{z}^{\beta+1}}{4w\beta(\rho-\mu)}$. Finally, expression (26) for v is obtained. □

3.2. The particular steady state problem of firms distribution

As we have chosen $g^e(z) = \delta(z - z_0)$, we can rewrite the steady state version of the KFP Eq. (19) as follows:

$$-\overline{\mu} z \partial_z g(z) + \frac{\sigma^2 z^2}{2}\partial^2_{zz} g(z) - \overline{\lambda} g(z) + \frac{\sigma^2 \underline{z}^2}{2}\partial_z g(\underline{z})\delta(z - z_0^e) = 0, \quad \text{in } \Omega_{St}, \tag{27}$$

where $\overline{\mu} = \mu - 2\sigma^2$ and $\overline{\lambda} = \mu - \sigma^2$.

Luttmer [24] shows that Eq. (27) can be solved explicitly for all z, except at the entry point $z = z_0^e$. The solution is continuous and there is a kink that reflects the entry of firms at $z = z_0^e$. In Proposition 2 we characterize the stationary distribution g that solves (27).

Proposition 2. *Given the exit and entry productivities $\underline{z}$ and z_0^e, respectively, the stationary size distribution in $z \in [\underline{z}, z_0^e) \cup (z_0^e, +\infty)$ is given by*

$$g(z) = c \min\left\{ \left(\frac{z}{\underline{z}}\right)^{\zeta+\zeta_*} - 1, \left(\frac{z_0^e}{\underline{z}}\right)^{\zeta+\zeta_*} - 1 \right\} \times \left(\frac{z}{\underline{z}}\right)^{-\zeta}, \tag{28}$$

where $c = \frac{\underline{z}^{\zeta_*}}{\underline{z}^{\zeta_*+1} - (z_0^e)^{\zeta_*+1}} \frac{(1-\zeta)(1+\zeta_*)}{\zeta+\zeta_*}$ and the tail indexes $-\zeta$ and ζ_* are the roots of the characteristic equation

$$\frac{\sigma^2}{2}r^2 - \left(\overline{\mu} - \frac{\sigma^2}{2}\right)r - \overline{\lambda} = 0.$$

Proof. Let $x = \log(z/\underline{z})$ be the logarithmic normalized productivity and let $x_0 = \log(z_0^e/\underline{z})$. For $x \in [0, x_0) \cup (x_0, +\infty)$, the KFP Eq. (27) can be equivalently written in the form

$$\partial^2_{xx}\hat{g}(x) - \gamma_1 \partial_x \hat{g}(x) - \gamma_2 \hat{g}(x) = 0, \tag{29}$$

where $\gamma_1 = (2\overline{\mu} - \sigma^2)/\sigma^2$ and $\gamma_2 = 2\overline{\lambda}/\sigma^2$. Moreover, the boundary conditions are transformed into

$$\hat{g}(0) = 0, \qquad \lim_{x \to +\infty} \hat{g}(x) = 0. \tag{30}$$

By applying Laplace transform in Eq. (29), we obtain

$$(u^2 - \gamma_1 u - \gamma_2)\mathcal{L}[\hat{g}](u) - (u - \gamma_1)\hat{g}(0) - \partial_x \hat{g}(0) = 0,$$

so that we have

$$\mathcal{L}[\hat{g}](u) = \frac{\partial_x \hat{g}(0) + (u - \gamma_1)\hat{g}(0)}{(u^2 - \gamma_1 u - \gamma_2)}. \tag{31}$$

As $\gamma_1 < 0$, $\gamma_2 < 0$ and $\gamma_1^2 + 4\gamma_2 > 0$, the two negative roots of the characteristic equation are $r_\pm = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 + 4\gamma_2}}{2}$ and expression (31) turns into

$$\mathcal{L}[\hat{g}](u) = \frac{\partial_x \hat{g}(0) + (u - \gamma_1)\hat{g}(0)}{(u - r_+)(u - r_-)}.$$

Next, we obtain the following Laplace inverses

$$\mathcal{L}^{-1}\left[\frac{1}{(u - r_+)(u - r_-)}\right] = \frac{1}{r_+ - r_-}\left(e^{r_+ x} - e^{r_- x}\right),$$

$$\mathcal{L}^{-1}\left[\frac{u - \gamma_1}{(u - r_+)(u - r_-)}\right] = \frac{1}{r_+ - r_-}\left((r_+ - \gamma_1)e^{r_+ x} - (r_- - \gamma_1)e^{r_- x}\right).$$

From them, for $x \in [0, x_0) \cup (x_0, +\infty)$ we get

$$\hat{g}(x) = \frac{\partial_x \hat{g}(0)}{r_+ - r_-}\left(e^{r_+ x} - e^{r_- x}\right) + \frac{\hat{g}(0)}{r_+ - r_-}\left((r_+ - \gamma_1)e^{r_+ x} - (r_- - \gamma_1)e^{r_- x}\right).$$

Next, we use the boundary conditions (30) and obtain that

$$\hat{g}(x) = \frac{\partial_x \hat{g}(0)}{r_+ - r_-}\left(e^{r_+ x} - e^{r_- x}\right) = \frac{\partial_x \hat{g}(0)}{r_+ - r_-}\left(e^{(r_+ - r_-)x} - 1\right) \cdot e^{r_- x}. \tag{32}$$

Therefore, we can get the expression

$$g(z) = c\left[\left(\frac{z}{\underline{z}}\right)^{\zeta+\zeta_*} - 1\right] \cdot \left(\frac{z}{\underline{z}}\right)^{-\zeta},$$

where $\zeta = -r_-$ and $\zeta_* = r_+$.

By applying that $\int_{\underline{z}}^{+\infty} g(z)dz = 1$, we get

$$c\left[\frac{1}{\underline{z}^{\zeta_*}} \lim_{z\to+\infty} \frac{z^{\zeta_*+1}}{\zeta_*+1} - \frac{\underline{z}}{\zeta_*+1} - \frac{1}{\underline{z}^{-\zeta}} \lim_{z\to+\infty} \frac{z^{-\zeta+1}}{-\zeta+1} + \frac{\underline{z}}{-\zeta+1}\right] = 1. \tag{33}$$

The condition for the expression (33) to be finite is $\zeta_* < -1$ and $\zeta > 1$. As in Luttmer [24], we assume that $\zeta > 1$, but we cannot guarantee $\zeta_* < -1$. Thus, we can narrow $\left(\frac{z}{\underline{z}}\right)^{\zeta+\zeta_*} - 1$ with the expression

$$\min\left\{\left(\frac{z}{\underline{z}}\right)^{\zeta+\zeta_*} - 1, \left(\frac{z_0^e}{\underline{z}}\right)^{\zeta+\zeta_*} - 1\right\}.$$

Therefore, we finally obtain expression (28) and conclude the proof. □

In view of the result in the previous proposition, we note that the solution of the steady state KFP equation with source term exhibits a power law behaviour in right hand side tail, which actually corresponds to a Pareto distribution. This is in agreement with previous results from Luttmer [24] and the review article of Gabaix [25] which highlights the presence of power laws in the solution of many problems arising in economics and finance. This behaviour will be better illustrated by the graphical representations in Section 3.5.

Note that in the KFP models we assume the existence of a weak solution g, i.e. a function g such that g, $\partial_x g$ and $\partial_{xx} g$ are square integrable in variable z. This is in line with the kind of solutions considered in [21] for income inequality problems, for example. In a more general MFG setting, an alternative concept of measure-valued solution for the KFP PDE has been introduced in [7].

3.3. The particular steady state equilibrium problem

In this particular stationary equilibrium problem, we aim to find the functions v and g, and the values $\underline{z}$ and ω that solve the system of equations

$$\min\{\rho v(z) - \pi(z) - \mu z \partial_z v(z) - \frac{\sigma^2 z^2}{2} \partial^2_{zz} v(z), v(z) - s\} = 0, \tag{34}$$

$$-\overline{\mu} z \partial_z g(z) + \frac{\sigma^2 z^2}{2} \partial^2_{zz} g(z) - \overline{\lambda} g(z) + \frac{\sigma^2 \underline{z}^2}{2} \partial_z g(\underline{z}) \delta(z - z_0^e) = 0, \tag{35}$$

$$\omega = e\left[\int_{\underline{z}}^{+\infty} \frac{z g(z)}{2\omega} dz - c_f\right]. \tag{36}$$

From the previous computations, the wage ω that solves (36) also satisfies

$$\omega = e\left[\frac{\left((z_0^e)^{\zeta_*+2} - \underline{z}(\omega)^{\zeta_*+2}\right)(1-\zeta)(1+\zeta_*)}{2\omega\left((z_0^e)^{\zeta_*+1} - \underline{z}(\omega)^{\zeta_*+1}\right)(2-\zeta)(2+\zeta_*)} - c_f\right].$$

In Proposition 3 we formally characterize the solution of (34)–(36).

Proposition 3. *The solution of the stationary equilibrium problem defined by Eqs. (34)–(36) is given by:*

$$\underline{z}(\omega) = \frac{4\omega\beta(\rho-\mu)}{1+\beta}\left(s + \frac{c_f}{\rho}\right),$$

$$v(z) = \begin{cases} s, & z \le \underline{z}, \\ \dfrac{s + \frac{c_f}{\rho}}{1+\beta}\left(\dfrac{\underline{z}}{z}\right)^{\beta} + \dfrac{z}{4\omega(\rho-\mu)} - \dfrac{c_f}{\rho}, & z > \underline{z}, \end{cases}$$

$$g(z) = c\min\left\{\left(\frac{z}{\underline{z}}\right)^{\zeta+\zeta_*} - 1, \left(\frac{z_0^e}{\underline{z}}\right)^{\zeta+\zeta_*} - 1\right\} \times \left(\frac{z}{\underline{z}}\right)^{-\zeta}, \qquad z \ge \underline{z},$$

Table 1
Macroeconomic parameters.

Symbol	Value	Description
e	1.53	Utility parameter
ρ	0.05	Discount rate
μ	−0.04	Productivity drift
σ^2	0.01	Productivity volatility
s	0	Scrap value
c_f	0.31	Fixed cost

Table 2
Wages approximations, absolute errors and experimental order of convergence for $z_0^e = 3$ obtained with the semianalytical solution.

m	ω^m	$E^m(\omega)$	$p^m(\omega)$
0	1	–	–
1	1.090294666372	0.090294666372	–
2	1.087567512305	0.002501300016	–
3	1.087565071168	0.000002244584	1.956364106225
4	1.087565071166	0.000000000002	2.000629506675

and the implicit equation for ω:

$$\omega = e\left[\frac{\left((z_0^e)^{\zeta_*+2} - \left[\frac{4\omega\beta(\rho-\mu)}{1+\beta}(s+\frac{c_f}{\rho})\right]^{\zeta_*+2}\right)(1-\zeta)(1+\zeta_*)}{2\omega\left((z_0^e)^{\zeta_*+1} - \left[\frac{4\omega\beta(\rho-\mu)}{1+\beta}(s+\frac{c_f}{\rho})\right]^{\zeta_*+1}\right)(2-\zeta)(2+\zeta_*)} - c_f\right]. \tag{37}$$

3.4. Numerical method for the particular productivity model

Note that (37) allows to characterize ω as a fixed point of the function

$$f(\omega) = e\left[\frac{\left((z_0^e)^{\zeta_*+2} - \left[\frac{4\omega\beta(\rho-\mu)}{1+\beta}(s+\frac{c_f}{\rho})\right]^{\zeta_*+2}\right)(1-\zeta)(1+\zeta_*)}{2\omega\left((z_0^e)^{\zeta_*+1} - \left[\frac{4\omega\beta(\rho-\mu)}{1+\beta}(s+\frac{c_f}{\rho})\right]^{\zeta_*+1}\right)(2-\zeta)(2+\zeta_*)} - c_f\right],$$

so that $\omega = f(\omega)$ is equivalent to (37).

Therefore, a fixed-point iteration technique $\omega^{m+1} = f(\omega^m)$ seems a straightforward numerical method. Furthermore, we propose a Steffensen method [26], which is specifically designed to speed up the convergence of fixed-point algorithms with linear convergence. Steffensen method starts with ω^0 given and considers the updating formula:

$$\omega^{m+1} = \omega^m - \frac{(f(\omega^m) - \omega^m)^2}{f \circ f(\omega^m) - 2f(\omega^m) + \omega^m}.$$

When the difference between two consecutive values is below a prescribed tolerance, we consider ω^m jointly with $\underline{z}(\omega^m)$, $v(\omega^m)$ and $g(\omega^m)$ given by Proposition 3 as the semianalytical solution of the steady state equilibrium problem.

3.5. Semianalytical solution of the particular model

In this section we present an example with parameters in Table 1 to illustrate the results provided by the model and the proposed numerical methods.

More precisely, we take μ and σ^2 from [2] and [23], respectively. Once the geometric Brownian motion parameters have been fixed, it remains to provide s, c_f, e and ρ. We select the annual discount rate $\rho = 0.05$, which is standard in the macroeconomic literature and assume that there are no decommissioning schemes, so that we choose the scrap value $s = 0$. Finally, we obtain the utility parameter e by solving the model when the economy is non-distorted in order to match a labor supply of $L = 1/3$, which is a standard normalization in macroeconomic literature. We consider fixed costs $c_f = 0.31$. Concerning the entry productivity, z_0^e, we will consider different values throughout this example.

Once the Steffensen numerical method has been applied, we consider the absolute error and the experimental order of convergence

$$E^m(\omega) = |\omega^m - \omega^{m-1}|, \quad p^m(\omega) = \frac{\log\left(\frac{E^m(\omega)}{E^{m-1}(\omega)}\right)}{\log\left(\frac{E^{m-1}(\omega)}{E^{m-2}(\omega)}\right)}. \tag{38}$$

In Table 2 we can see how the computed values of p^m tend to the expected order of convergence equal two for the case $z_0^e = 3$. The same order of convergence has been observed in all choices of z_0^e.

Table 3
Entry productivity, wages and optimal exit productivity obtained with the semianalytical solution.

z_0^e	ω	$\underline{z}$
2	0.904284	1.009181
3	1.087565	1.213723
4	1.237616	1.381179

Next, we illustrate the behaviour of the solution in terms of the productivity at which entry of firms takes place. In Table 3 we observe how the values of ω and $\underline{z}$ increase with respect to z_0^e.

Fig. 1 shows the value function and firms distribution for the values of z_0^e in Table 3. Note that the value function decreases when z_0^e increases. As pointed in Gabaix [25], the empirical regularity of distribution of firms fails at the entry productivity z_0^e and exhibits a singular behaviour at that point, where a kind of kink is observed. At the right tail (i.e. for $z > z_0^e$) an expected power law behaviour is observed. As soon as we increase the value of z_0^e, the firms distribution is more spread up along a larger set of productivities, thus reaching a lower maximum. Actually, the kink is better observed for the higher values of z_0^e.

4. Numerical methods for the general steady state problems

As the semianalytical solution is not available for the general steady state problem, in this section we propose a set of numerical techniques to approximate the solution. The main point is the numerical discretization of the HJB and KFP equations governing the incumbents and firms distribution problems.

4.1. Problem of stationary incumbent firms

A previous step to apply numerical methods is the truncation of the unbounded domain to a bounded one. Thus, we choose a fixed large enough value $Z > z_0^e$ and consider the fixed domain $\Omega_Z = [0, Z]$ to pose the stationary incumbent's problem (11) jointly with the boundary conditions:

$$v(0) = s \quad \text{and} \quad \partial^2_{zz} v(Z) = 0. \tag{39}$$

Among the different alternatives for solving linear complementarity problems or its equivalent formulation in terms of variational inequalities (see Glowinski et al. [27], for example), we have chosen one proposed in Kärkkäinen et al. [14] which is known as augmented Lagrangian active set (ALAS) method and we apply it to the fully discretized problem. For the spatial discretization, we use a uniform grid, with stepsize $\Delta z = Z/N_z$, for a given natural number N_z. The grid nodes are $z_k = k\Delta z$, for $k = 0, ..., N_z$, so that we denote $v_k \approx v(z_k)$ the approximation to the solution.

Next, in order to approximate problem (11) at the grid point z_k, $k = 1, \ldots, N_z - 1$, we consider the approximations:

$$\partial_z v(z_k) \approx \frac{v_k - v_{k-1}}{\Delta z} \qquad \text{and} \qquad \partial^2_{zz} v(z_k) \approx \frac{v_{k+1} - 2v_k + v_{k-1}}{(\Delta z)^2}.$$

After some reordering of terms, we get for $k = 1, \ldots, N_z - 1$ the expressions

$$\mathcal{L}_k^e(v) \geq \pi_k, \quad v_k \geq s, \quad \left(\mathcal{L}_k^e(v) - \pi_k\right) \cdot (v_k - s) = 0,$$

where

$$\mathcal{L}_k^e(v) = \left[\frac{\mu(z_k)}{\Delta z} - \frac{\sigma^2(z_k)}{2(\Delta z)^2}\right] v_{k-1} + \left[\rho - \frac{\mu(z_k)}{\Delta z} + \frac{\sigma^2(z_k)}{(\Delta z)^2}\right] v_k - \frac{\sigma^2(z_k)}{2(\Delta z)^2} v_{k+1}.$$

Moreover, boundary conditions (39) are approximated by

$$v_0 = s, \quad \text{and} \quad \frac{2v_{N_z} - 5v_{N_z-1} + 4v_{N_z-2} - v_{N_z-3}}{(\Delta z)^2} = 0.$$

Thus, the fully discretized linear complementarity problem can be written in matrix form:

$$AV \geq f, \quad V \geq S, \quad (AV - f)^t \cdot (V - S) = 0, \tag{40}$$

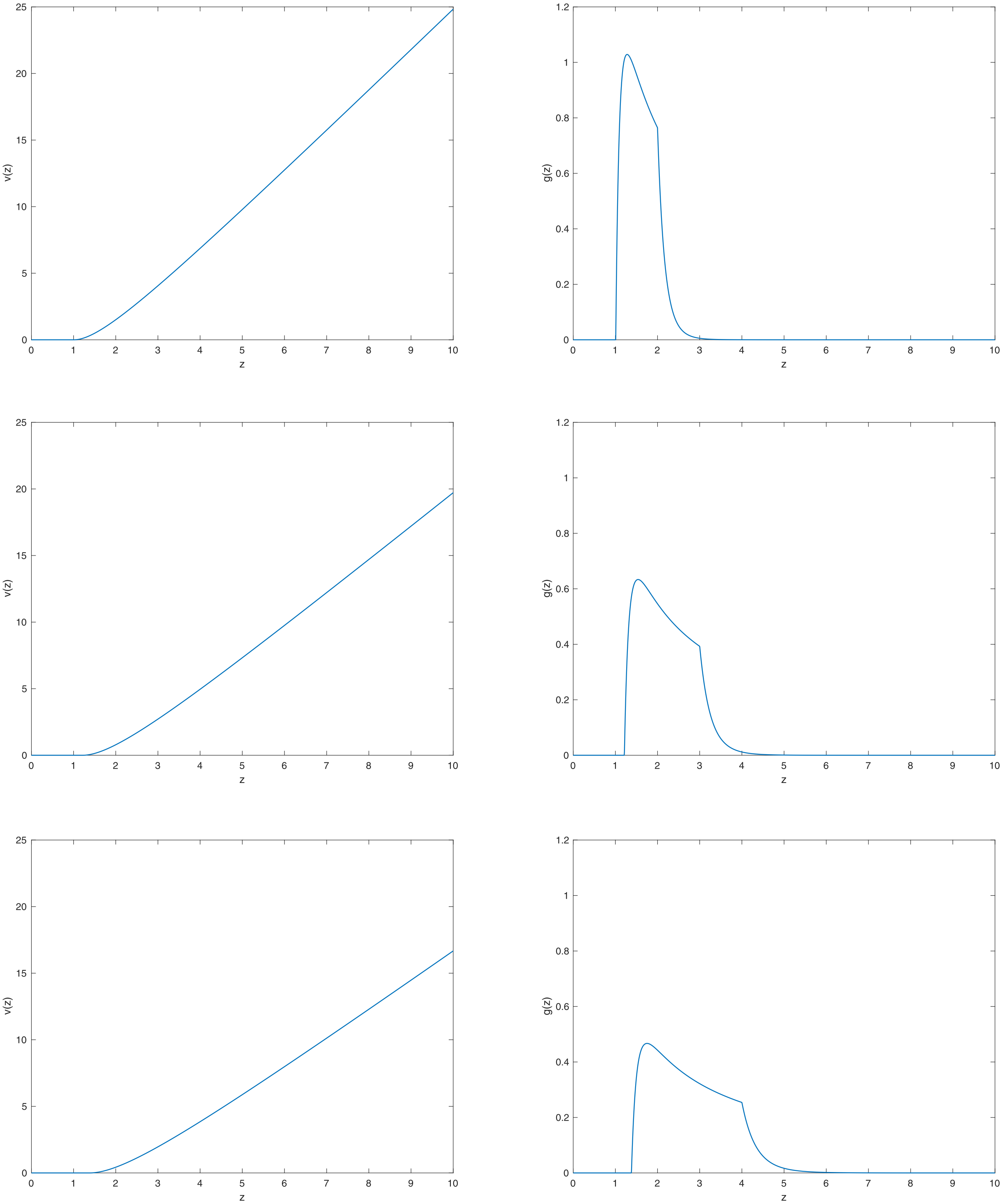


Fig. 1. The value function (left) and the firm distribution (right) with $z_0^e = 2$ (top), $z_0^e = 3$ (middle) and $z_0^e = 4$ (bottom), obtained with the semianalytical solution.

where $V = (v_0, \ldots, v_{N_z})^t$ is the unknown vector of nodal values of v, $S \in \mathbb{R}^{N_z+1}$ is the vector with all components equal to the scrap value s and vector $f \in \mathbb{R}^{N_z+1}$, such that $f_0 = s$ and $f_{N_z} = 0$, while $f_k = \pi_k, \quad k = 1, \ldots, N_z - 1$. Matrix A is given by

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
a_1 & b_1 & c_1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
0 & a_2 & b_2 & c_2 & \ddots & & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & c_{N_z-4} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \ddots & b_{N_z-3} & c_{N_z-3} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & a_{N_z-2} & b_{N_z-2} & c_{N_z-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & a_{N_z-1} & b_{N_z-1} & c_{N_z-1} \\
0 & 0 & 0 & 0 & \cdots & \cdots & e_{N_z} & d_{N_z} & a_{N_z} & b_{N_z}
\end{pmatrix}$$

such that

$$a_k = \left[\frac{\mu(z_k)}{\Delta z} - \frac{\sigma^2(z_k)}{2(\Delta z)^2} \right], \;\; b_k = -\frac{\mu(z_k)}{\Delta z} + \frac{\sigma^2(z_k)}{(\Delta z)^2} + \rho, \;\; c_k = -\frac{\sigma^2(z_k)}{2(\Delta z)^2},$$

for $k = 1, \ldots, N_z - 1$, and,

$$a_{N_z} = \frac{-5}{(\Delta z)^2}, \;\; b_{N_z} = \frac{2}{(\Delta z)^2}, \;\; d_{N_z} = \frac{4}{(\Delta z)^2}, \;\; e_{N_z} = \frac{-1}{(\Delta z)^2}.$$

As previously indicated, we propose the ALAS algorithm from Kärkkäinen et al. [14] to solve (40), which can be written with the equivalent mixed formulation

$$AV + P = f,$$

where P denotes the vector of the multiplier values associated with the inequality constraint. The basic iteration of the ALAS algorithm consists of two steps. In the first one, the nodes are decomposed into active and inactive nodes (depending on whether the constraints are active or not), and in the second step, a reduced linear system associated with the inactive nodes is solved.

First, for any decomposition of nodes $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} = \{0, 1, 2, \ldots, N_z\}$, let us denote by $[A]_{\mathcal{I}\mathcal{I}}$ the main diagonal block of matrix A indexed by $\mathcal{I}$ and let us denote by $[A]_{\mathcal{I}\mathcal{J}}$ the codiagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, the ALAS algorithm computes not only V and P, but also updates the decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$ such that

$$\begin{aligned}
&AV + P = f, \\
&P_j + \beta(V_j - S_j) \le 0, \qquad \text{for all } j \in \mathcal{J}, \\
&P_i = 0, \qquad \text{for all } i \in \mathcal{I},
\end{aligned}$$

for a given positive parameter β. In the above equations, $\mathcal{I}$ and $\mathcal{J}$ are the inactive and the active sets, respectively. More precisely, the iterative algorithm builds sequences $\{V^m\}$, $\{P^m\}$, $\{\mathcal{I}^m\}$ and $\{\mathcal{J}^m\}$, converging to V, P, $\mathcal{I}$ and $\mathcal{J}$, by means of the following steps:

1. Initialize $V^0 = S$ and $P^0 = \min\{f - AV^0, 0\}$. Choose $\beta > 0$ and set $m = 0$.
2. Determine the active/inactive partition,

$$\begin{aligned}
&\mathcal{Q}^m = \min\{P^m + \beta(V^m - S), 0\}, \\
&\mathcal{J}^m = \{n \in \mathcal{N} : [\mathcal{Q}^m]_n < 0\}, \\
&\mathcal{I}^m = \{n \in \mathcal{N} : [\mathcal{Q}^m]_n = 0\}.
\end{aligned}$$

3. If $m \ge 1$ and $\mathcal{J}^m = \mathcal{J}^{m-1}$, then convergence is achieved.
4. Determine (V, P) as the unique solution of the linear system

$$\begin{aligned}
&AV + P = f, \\
&\quad P = 0, \qquad \text{on } \mathcal{I}^m, \\
&\quad V = S, \qquad \text{on } \mathcal{J}^m.
\end{aligned} \tag{41}$$

Set $V^{m+1} = V$, $P^{m+1} = \min\{P, 0\}$, $m = m + 1$ and go to Step 2.

It is important to notice that actually (41) reduces to solving

$$[A]_{\mathcal{I}\mathcal{I}}[V]_{\mathcal{I}} = [f]_{\mathcal{I}} - [A]_{\mathcal{I}\mathcal{J}}[S]_{\mathcal{J}},$$
$$[V]_{\mathcal{J}} = [S]_{\mathcal{J}},$$
$$P = f - AV,$$

where we have denoted $\mathcal{I} = \mathcal{I}^m$ and $\mathcal{J} = \mathcal{J}^m$.

4.2. Problem of stationary firm distribution

In this problem, the computational domain will be $\Omega_Z = \Omega_Z^1 \cup \Omega_Z^2$, where $\Omega_Z^1 = [0, \underline{z}]$ and $\Omega_Z^2 = (\underline{z}, Z]$. Moreover, we will consider $g(z) = 0$ on Ω_Z^1, pose Eq. (19) on Ω_Z^2 and the boundary condition $g(Z) = 0$.

Let us denote by $g_k \approx g(z_k)$ the approximation obtained with the numerical method. Let $z_{\tilde{k}}$ be the approximation of the optimal exit productivity, where $\tilde{k} = \max\{j : j \in \mathcal{J}\}$ is obtained in terms of the sets defined in the numerical solution of the incumbents problem. Next, we discretize Eq. (19) at the grid points z_k, $k = \tilde{k}+1, \ldots, N_z - 1$, by introducing the approximations

$$\partial_z\big(\mu(z_k)g(z_k)\big) \approx \frac{[\mu(z_k)]^+ g_k - [\mu(z_{k-1})]^+ g_{k-1}}{\Delta z} + \frac{[\mu(z_{k+1})]^- g_{k+1} - [\mu(z_k)]^- g_k}{\Delta z}$$
$$\partial_{zz}^2\big(\sigma^2(z_k)g(z_k)\big) \approx \frac{\sigma^2(z_{k+1})g_{k+1} - 2\sigma^2(z_k)g_k + \sigma^2(z_{k-1})g_{k-1}}{(\Delta z)^2}.$$

Thus, for $k = \tilde{k}+1, \ldots, N_z - 1$, we get the equations

$$-\left[\frac{[\mu(z_k)]^+ g_k - [\mu(z_{k-1})]^+ g_{k-1}}{\Delta z} + \frac{[\mu(z_{k+1})]^- g_{k+1} - [\mu(z_k)]^- g_k}{\Delta z}\right]$$
$$+\frac{1}{2}\left[\frac{\sigma^2(z_{k+1})g_{k+1} - 2\sigma^2(z_k)g_k + \sigma^2(z_{k-1})g_{k-1}}{(\Delta z)^2}\right] + \frac{\sigma^2(z_{\tilde{k}})}{2}\partial_z g(z_{\tilde{k}})g^e(z_k) = 0.$$

In order to maintain the structure of matrices to make an efficient solution of tridiagonal systems, we make a fixed point iteration in the term involving $\partial_z g(z_{\tilde{k}})$ to approximate $\partial_z g(z_{\tilde{k}}) \approx \partial_z g^p(z_{\tilde{k}})$, where g^p comes from the previous fixed point iteration. Thus, previous equations are replaced by

$$\frac{1}{\Delta z}\left[\frac{\sigma^2(z_{k-1})}{2\Delta z} + [\mu(z_{k-1})]^+\right]g_{k-1} - \frac{1}{\Delta z}\left[\frac{\sigma^2(z_k)}{\Delta z} + [\mu(z_k)]^+ - [\mu(z_k)]^-\right]g_k$$
$$+\frac{1}{\Delta z}\left[\frac{\sigma^2(z_{k+1})}{2\Delta z} - [\mu(z_{k+1})]^-\right]g_{k+1} = -\frac{\sigma^2(z_{\tilde{k}})}{2}\partial_z g^p(z_{\tilde{k}})g^e(z_k),$$

which are completed with $g_{N_z} = 0$ provided by the boundary condition and $g_k = 0$, for $k = 0, \ldots, \tilde{k}$ as $g = 0$ in Ω_Z^1.

Therefore, the discretized problem can be written as the linear system:

$$BG = b, \tag{42}$$

where $G = (g_0, g_1, \ldots, g_{N_z})^t$ is the solution vector and $b \in \mathbb{R}^{N_z+1}$, such that $b_k = 0$, $k = 0, \ldots, \tilde{k}$ and $b_{N_z} = 0$, while

$$b_k = -\frac{\sigma^2(z_{\tilde{k}})}{2}\partial_z g^p(z_{\tilde{k}})g^e(z_k), \quad k = \tilde{k}+1, \ldots, N_z - 1.$$

In order to impose that $g_k = 0$ for $k = 0, \ldots, \tilde{k}$ and $k = N_z$, we consider

$$B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \hat{a}_{\tilde{k}+1} & \hat{b}_{\tilde{k}+1} & \hat{c}_{\tilde{k}+1} & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \hat{a}_{N_z-2} & \hat{b}_{N_z-2} & \hat{c}_{N_z-2} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \hat{a}_{N_z-1} & \hat{b}_{N_z-1} & \hat{c}_{N_z-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix},$$

where for $k = \tilde{k}+1, \ldots, N_z-1$, we have

$$\hat{a}_k = \frac{1}{\Delta z}\left[\frac{\sigma^2(z_{k-1})}{2\Delta z} + [\mu(z_{k-1})]^+\right], \ \hat{b}_k = -\frac{1}{\Delta z}\left[\frac{\sigma^2(z_k)}{\Delta z} + [\mu(z_k)]^+ - [\mu(z_k)]^-\right],$$
$$\hat{c}_k = \frac{1}{\Delta z}\left[\frac{\sigma^2(z_{k+1})}{2\Delta z} - [\mu(z_{k+1})]^-\right].$$

As g is a probability density function, its integral must be equal to one, so that discretizing this condition with the composed trapezoidal rule we get:

$$\sum_{i=0}^{N_z-1} \frac{g_i + g_{i+1}}{2} \Delta z = 1.$$

Moreover, we take into account that the probability density function of a $\mathcal{N}(z_i^e, \kappa^2)$ random variable tends to $\delta(z - z_i^e)$ when $\kappa \to 0$, so that we consider the approximation

$$\delta(z - z_i^e) \approx \frac{1}{\kappa\sqrt{2\pi}} e^{-\frac{(z-z_i^e)^2}{2\kappa^2}}.$$

in the expression of g^e by choosing κ small enough. In all examples we verified that the choice $\kappa = 10^{-6}$ is suitable.

4.3. Stationary equilibrium problem

In this section we briefly describe how we calculate the steady state equilibrium given specified functions $\mu(z)$ and $\sigma(z)$; and values for the parameters s, ρ, e and c_f. The algorithm is the natural generalization of that used to compute the equilibrium for the particular productivity model. We start with ω^0 given and for $m = 0, 1, 2, \ldots$ we follow:

1. Given the wage ω^m, solve the stationary HJB equation to obtain v^m and $\underline{z}^m$ with the method described in Section 4.1.
2. Given $\underline{z}^m$, solve the stationary KFP equation for g^m using the method detailed in Section 4.2.
3. Given g^m, we compute the wages ω_1^m.
4. Given the wages ω_1^m, we repeat Steps 1 to 3 for calculating ω_2^m and generate a new guess using the Steffensen formula:

$$\omega^{m+1} = \omega^m - \frac{(\omega_1^m - \omega^m)^2}{\omega_2^m - 2\omega_1^m + \omega^m}. \tag{43}$$

Remark 1. In [6], for a PDE system of HJB and KFP equations, the authors use that differential operators associated with HJB and KFP PDEs are selfadjoint, so that they propose a numerical method where the matrix associated with the discretized KFP PDE is the trasposed of the one associated with the HJB problem. In our approach, the discretization of the boundary conditions and the blocking of the KFP solution to zero at the nodes in the exit region implies that associated matrices to HJB and KFP PDEs are not trasposed each other.

5. Numerical results

In this section we present two examples to illustrate the performance of the proposed numerical methods, as well as the qualitative and quantitative properties of the computed solutions. After a first example with semianalytical solution, the second example includes randomness in the entry productivities by means of a discrete random variable, so that its solution requires the use of numerical methods for all the involved subproblems. In all examples we have used the economic data in Table 1.

5.1. Example 1: Numerical solution of the model in Section 3

We considered the particular model described in Section 3, for which a semianalytical solution has been obtained (see Proposition 3). In this section we use the numerical methods described in Section 4 to develop the full numerical solution of all the involved models. Numerical results are validated with the semianalytical solution obtained in Section 3.

5.1.1. Numerical results for the steady state HJB equation

We first show the results for $v_{\Delta z}$ that solves the HJB equation for different mesh steps Δz. Table 4 shows the order of convergence of the proposed numerical strategy. For $z_0^e = 3$ we consider the wages value $\omega = 1.087565$ showed in Table 2 and we use in all forthcoming examples the upper limit $Z = 10$ in the productivity interval.

In order to study convergence, we use the absolute error between the approximations in the common nodes with meshes with stepsize $2\Delta z$ and Δz, i.e. $E_{\Delta z}(v) = ||v_{2\Delta z} - \overline{v}_{\Delta z}||_\infty$, where $\overline{v}_{\Delta z}$ only contains the approximations of $v_{\Delta z}$ at the nodes that also belong to the coarser mesh with step $2\Delta z$. Next, we compute the convergence ratio $R_{\Delta z}(v) = \frac{E_{2\Delta z}(v)}{E_{\Delta z}(v)}$ and the experimental order of convergence $p_{\Delta z}(v) = \log_2(R_{\Delta z}(v))$. Note that the proposed method achieves linear convergence. In

Table 4
Absolute errors, convergence ratios and order of convergence in the value function approximation, jointly with computed optimal exit productivity.

N_z	$E_{\Delta z}(v)$	$R_{\Delta z}(v)$	$p_{\Delta z}(v)$	$\underline{z}$
4000	–	–	–	1.212500
8000	0.000533	–	–	1.213750
16000	0.000266	2.002788	1.002009	1.213750
32000	0.000133	2.000222	1.000160	1.213750
64000	0.000067	2.000168	1.000121	1.213594

Table 5
Absolute errors, convergence ratios and order of convergence in the firm distribution approximation.

N_z	$E_{\Delta z}(g)$	$R_{\Delta z}(g)$	$p_{\Delta z}(g)$
4000	–	–	–
8000	0.001780	–	–
16000	0.000883	2.002788	1.010898
32000	0.000440	2.000222	1.005370
64000	0.000220	2.000168	1.002679

Table 6
Computed wages, absolute errors and experimental order of convergence for $z_0^e = 3$.

m	ω^m	$E^m(\omega)$	$p^m(\omega)$
0	1	–	–
1	1.090007559278	0.090007559278	–
2	1.087563949290	0.002443609989	–
3	1.087547943192	0.000016006098	1.394254099212
4	1.087547942992	0.000000000199	2.245891561217

Table 7
Equilibrium wages and optimal exit productivity for different entry productivities.

z_0^e	ω	$\underline{z}$
2	0.904223	1.009063
3	1.087548	1.213594
4	1.237583	1.381094

Table 4 we can see how the computed values of p tend to the expected order of convergence one. Also the obtained optimal exit productivity for each mesh is shown. Note its convergence to the value obtained from the semianalytical solution in Section 3.5.

5.1.2. Numerical solution of the steady state KFP equation

Let $g_{\Delta z}$ be the solution of the KFP equation with mesh step Δz. Using analogous notations as in previous section, Table 5 shows the order of convergence of the proposed numerical strategy for solving the KFP model. Note that the proposed method achieves linear convergence. Table 5 shows that the computed values of p tend to the expected order of convergence one.

5.1.3. Numerical results for the steady state equilibrium

Once the algorithm in Section 4.3 has been applied, Table 6 shows the order of convergence for the entry productivity $z_0^e = 3$. For this purpose, we consider the absolute error and the experimental order of convergence in (38). Table 6 shows that the computed values of p tend to the expected order of convergence equal two for $z_0^e = 3$ (the same order has been observed for all choices of z_0^e).

Next, we choose three values of z_0^e to illustrate the behaviour of the solution in terms of the productivity at which entry of firms takes place. Table 7 shows the different values we obtain for the equilibrium ω and $\underline{z}$. Note that all results are very close to the ones obtained from the semianalytical solution in Section 3.5 (see Table 3).

Fig. 2 shows the corresponding value function and density of firms for $z_0^e = 3$. In order to illustrate the behavior of the proposed numerical strategy, we also show the semianalytical solution obtained in Section 3.5.

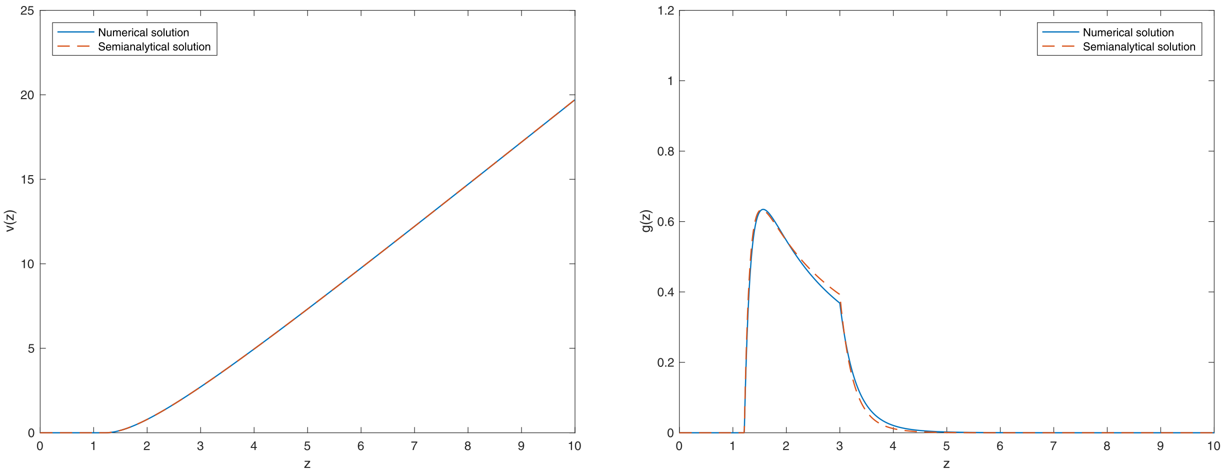


Fig. 2. The value function (left) and the firm distribution (right) for $z_0^e = 3$.

Table 8
Wages approximations, absolute errors and experimental order of convergence in ω for $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$.

m	ω^m	$E^m(\omega)$	$p^m(\omega)$
0	1	–	–
1	1.121165977662	0.121165977662	–
2	1.116987872017	0.004178105645	–
3	1.117000129288	0.000012257271	1.731799380392
4	1.117000129176	0.000000000112	1.988838781379

Table 9
Probabilities, wages and optimal exit productivities for $\mathbf{z^e} = (2, 3, 4)$.

$\mathbf{p^e}$	ω	$\underline{z}$
(0.25, 0.5, 0.25)	1.117000	1.246563
(0.1, 0.8, 0.1)	1.098930	1.226406
(0.1, 0.1, 0.8)	1.213797	1.354531

5.2. Example 2: Numerical solution of the steady state equilibrium with random entry productivities

In this example, we incorporate to the model in Section 3 the randomness in the entry productivity by means of a discrete random variable. For this purpose, we consider a probability function of new establishments given by expression (17) with $I = 3$. In this case, we can rewrite the steady state version of the KFP Eq. (27) as follows:

$$-\overline{\mu} z \partial_z g(z) + \frac{\sigma^2 z^2}{2} \partial_{zz}^2 g(z) - \overline{\lambda} g(z) + \frac{\sigma^2 \underline{z}^2}{2} \partial_z g(\underline{z}) \sum_{i=1}^{3} p_i^e \delta(z - z_i^e) = 0, \quad \text{in } \Omega_{St}.$$

For this equation, an analytical solution is not available as in the particular model of Section 3. In this setting we will consider different choices for the productivities z_i^e and the associated probabilities p_i^e. Although we have solved the different subproblems separately to validate the different numerical methods, we prefer to present just those results corresponding to the equilibrium problem, which requires the solution of each subproblem at each iteration of the fixed point algorithm. Moreover, for an easier presentation of the results we will consider a vector notation for entry productivities $\mathbf{z^e} = (z_1^e, z_2^e, z_3^e)$ with the associated probabilities $\mathbf{p^e} = (p_1^e, p_2^e, p_3^e)$.

As a first data set, we use $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$. In this way, we assume that entry productivity is random and assign probabilities to each of the entry productivities considered in Example 1. Once the Steffensen numerical method has been applied to solve the equilibrium problem, Table 8 clearly shows how the order of convergence given by (38) tends to the expected order of convergence equal two for the case $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$. The same order of convergence has been observed in all choices of $\mathbf{z^e}$ and $\mathbf{p^e}$ we show in this section.

Next, we have chosen two additional data sets for $\mathbf{p^e}$ to illustrate the behaviour of the solution of the equilibrium problem in terms of the productivity at which entry of firms takes place and their associated probabilities. These computations

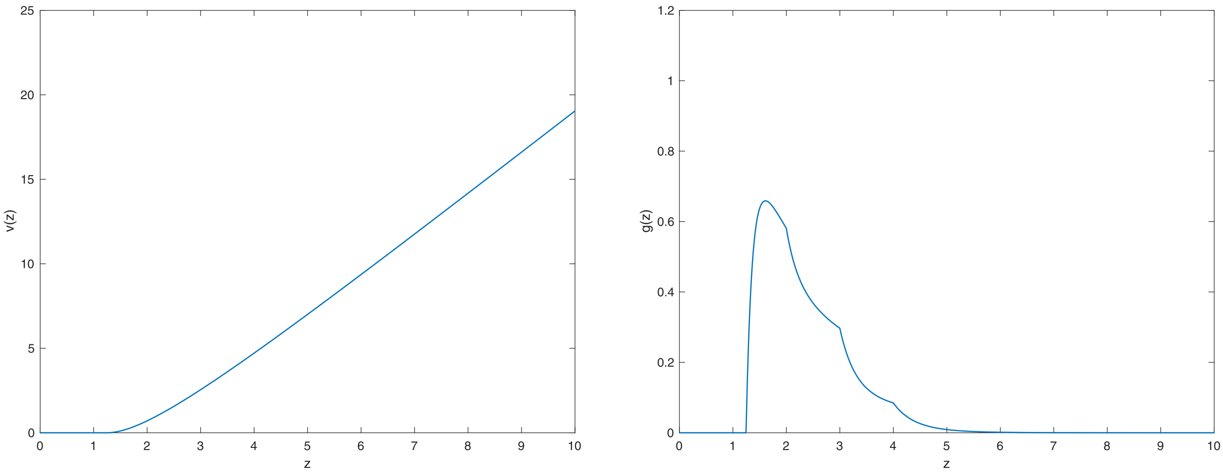


Fig. 3. The value function (left) and the firm distribution (right) with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$.

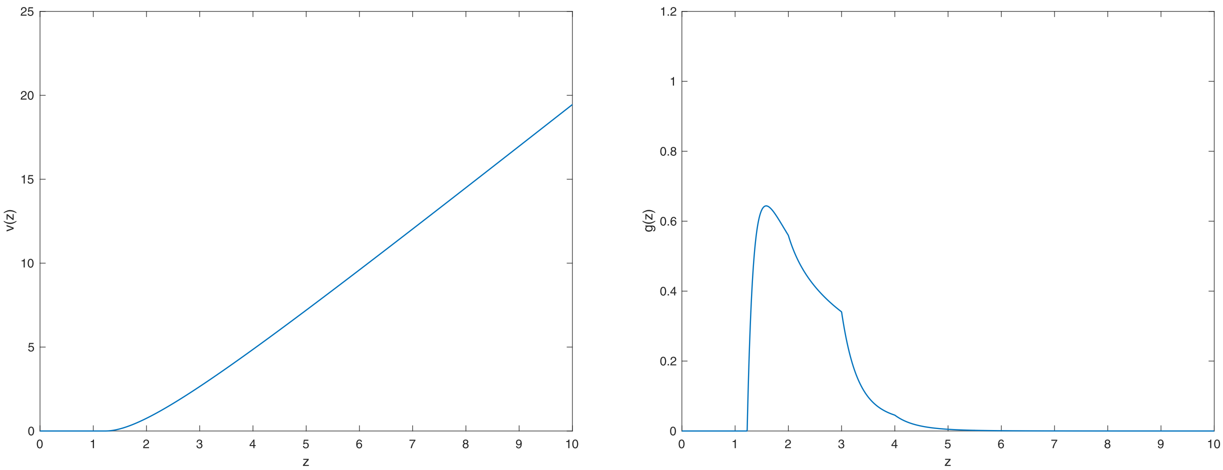


Fig. 4. The value function (left) and the firm distribution (right) with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.1, 0.8, 0.1)$.

allow us to obtain the equilibrium values of ω and $\underline{z}$ for these sets of parameters. In Table 9 these equilibrium values are shown.

Next, in Figs. 3–5 we show the corresponding value functions and probability densities of firms for the different sets $\mathbf{p^e}$ that appear in Table 9.

First, Fig. 3 shows the computed value function and firms distribution when we assign probabilities $\mathbf{p^e} = (0.25, 0.5, 0.25)$ to the three cases of entry productivity we considered in Section 3. In the probability density of firms we observe a kink at each possible entry productivity, three kinks in total, being more steep the one at the productivity with higher probability. If we compare with Fig. 2, we observe that the value function decreases and a change in the distribution of firms. Note that we could consider the example in Fig. 2 as associated with the case with $\mathbf{p^e} = (0, 1, 0)$ instead of $\mathbf{p^e} = (0.25, 0.5, 0.25)$.

In Table 9, for the fixed set $\mathbf{z^e} = (2,\ 3,\ 4)$, in the second row we change the probabilities to make $z_2^e = 3$ much more probable than the others, thus giving rise to Fig. 4. Next, we make $z_3^e = 4$ more probable than the others and obtain results in Fig. 5. Figs. 4 and 5 illustrate how the kink in the probability density function is more steep at the entry productivity where the associated probability is higher, as expected. Moreover, it seems that the parts of the density functions at the right tails of the different entry productivities z_i^e exhibit a power law behavior with a different exponent in each case. We conjecture that these power laws correspond to Pareto distributions. Also at each z_i^e the density function becomes singular, which is a consequence of using combinations of delta functions in the source term g^e.

In Example 2 the semianalytical solution of the equilibrium problem is not available so that the numerical solution of all involved PDEs is mandatory.

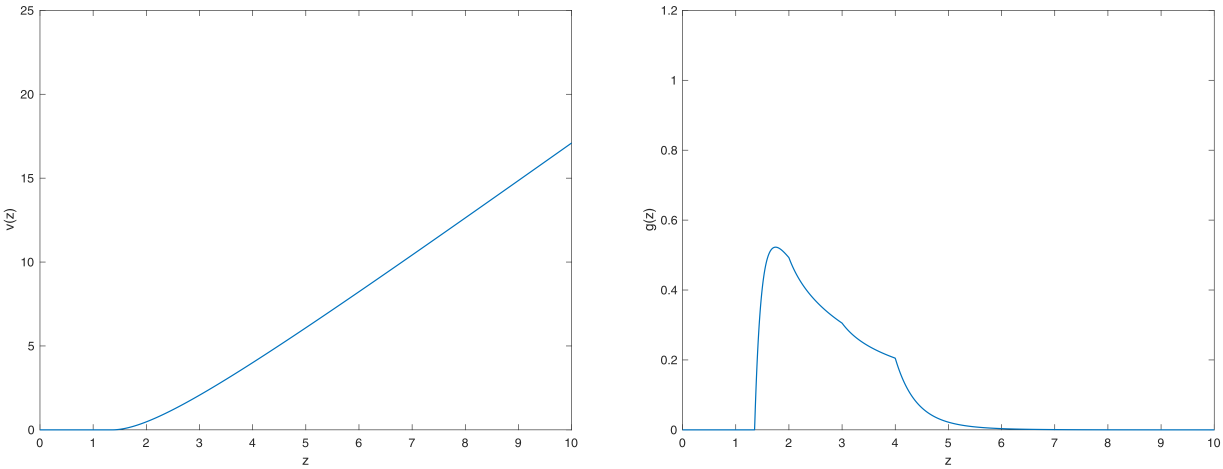


Fig. 5. The value function (left) and the firm distribution (right) with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.1, 0.1, 0.8)$.

6. Numerical solution for the time-dependent equilibrium problem

In this section we address the numerical solution of the time dependent equilibrium problem, which is mainly defined by Eqs. (23), (24) and (22). As we aim to show the convergence of the solution of the time-dependent problem to the solution of the corresponding steady state one, we consider as final condition for (23) the solution of the steady state problem. Concerning the initial condition to complete Eq. (24), we assume that it is related to the productivities at which new establishments enter the industry.

As in the steady state problem, in the time-dependent equilibrium we pose a fixed point iteration between the numerical solution of problems defined by Eqs. (23), (24) and (22). Note that the problem associated with (23) is backward in time with final condition, while the problem associated with (24) is forward in time with initial condition.

We first describe the additional numerical methods we incorporate in the time-dependent problems and secondly we show some numerical examples to illustrate the performance of the proposed methodology.

6.1. Numerical methods

For the numerical solution of problems defined by Eqs. (23) and (24), we propose a Crank-Nicolson scheme for the time discretization, which will be combined with the finite differences methods we used for the spatial discretization in the steady state problems. At each time step, the fully discretized problems will be solved with the same techniques as in the discretized steady state ones.

6.1.1. Problem of evolutionary incumbent firms

The evolutionary incumbent's problem is posed in terms of the complementarity problem (23) on the computational domain $D_Z^T = [0, Z] \times [0, T]$, where Z is a large enough productivity and T is a large enough time horizon to enter in the steady state regime at $t = T$. Additionally, we assume the following final and boundary conditions:

$$v(z, T) = v_T(z), \quad v(0, t) = s(t) \quad \text{and} \quad \partial^2_{zz} v(Z, t) = 0, \tag{44}$$

where $v_T(z)$ is the solution of the steady state problem.

For the time discretization we introduce a uniform mesh, with time step $\Delta t = T/N_t$ for a given natural number N_t, so that the nodes are given by $t^n = n\Delta t$, with $n = 0, 1, ..., N_t$. For the productivities we use a uniform mesh with the same notation as in the steady state case, so that we define v_k^n as the approximation of the solution of problem (23) at $(k\Delta z, n\Delta t)$, i.e. $v_k^n \approx v(z_k, t^n)$.

Next, in order to approximate problem (23) at the point (z_k, t_θ^n), with $k = 1, \ldots, N_z - 1$ and $n = 0, 1, \ldots, N_t - 1$, we introduce the following approximations of $\partial_t v$, $\partial_z v$ and $\partial_{zz} v$ at (z_k, t_θ^n), where $t_\theta^n = \theta t^n + (1-\theta) t^{n+1}$, with $0 \le \theta \le 1$:

$$\partial_t v(z_k, t_\theta^n) \approx \frac{v_k^{n+1} - v_k^n}{\Delta t}, \quad \partial_z v(z_k, t_\theta^n) \approx \theta \frac{v_k^n - v_{k-1}^n}{\Delta z} + (1-\theta) \frac{v_k^{n+1} - v_{k-1}^{n+1}}{\Delta z},$$
$$\partial^2_{zz} v(z_k, t_\theta^n) \approx \theta \frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{(\Delta z)^2} + (1-\theta) \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{(\Delta z)^2}.$$

Previous approximations correspond to the so called θ-method ($0 \le \theta \le 1$) applied to the backward in time HJB PDE, where $\theta = 0$ and $\theta = 1$ correspond to the explicit and implicit methods, respectively. In practice we consider $\theta = 0.5$, which defines the Crank-Nicolson scheme and exhibits second order convergence in time and space for linear PDEs [17]. Once expressions (45) are introduced in the first inequality of (23) at points (z_k, t_θ^n), we obtain the following inequalities:

$$-\frac{v_k^{n+1}-v_k^n}{\Delta t} - \left[\theta\mu(z_k,t^n)\frac{v_k^n - v_{k-1}^n}{\Delta z} + (1-\theta)\mu(z_k,t^{n+1})\frac{v_k^{n+1}-v_{k-1}^{n+1}}{\Delta z}\right]$$
$$-\left[\theta\frac{\sigma^2(z_k,t^n)}{2}\frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{(\Delta z)^2} + (1-\theta)\frac{\sigma^2(z_k,t^{n+1})}{2}\frac{v_{k+1}^{n+1}-2v_k^{n+1}+v_{k-1}^{n+1}}{(\Delta z)^2}\right]$$
$$+\rho\left[\theta v_k^n + (1-\theta)v_k^{n+1}\right] - \left[\theta\pi(z_k,t^n) + (1-\theta)\pi(z_k,t^{n+1})\right] \geq 0,$$

for $n = N_t-1, N_t-2, \dots, 1, 0$ and $k = 1, \dots, N_z-1$, or equivalently,

$$\Delta t\theta\left[\frac{\mu(z_k,t^n)}{\Delta z} - \frac{\sigma^2(z_k,t^n)}{2(\Delta z)^2}\right]v_{k-1}^n + \left[1 + \Delta t\theta\left[\rho - \frac{\mu(z_k,t^n)}{\Delta z} + \frac{\sigma^2(z_k,t^n)}{(\Delta z)^2}\right]\right]v_k^n$$
$$-\Delta t\theta\frac{\sigma^2(z_k,t^n)}{2(\Delta z)^2}v_{k+1}^n \;\geq\; \Delta t(1-\theta)\left[-\frac{\mu(z_k,t^{n+1})}{\Delta z} + \frac{\sigma^2(z_k,t^{n+1})}{2(\Delta z)^2}\right]v_{k-1}^{n+1}$$
$$+\left[1 + \Delta t(1-\theta)\left[\frac{\mu(z_k,t^{n+1})}{\Delta z} - \frac{\sigma^2(z_k,t^{n+1})}{(\Delta z)^2} - \rho\right]\right]v_k^{n+1}$$
$$+\Delta t(1-\theta)\frac{\sigma^2(z_k,t^{n+1})}{2(\Delta z)^2}v_{k+1}^{n+1} + \theta\pi(z_k,t^n) + (1-\theta)\pi(z_k,t^{n+1}).$$

Also, the second inequality and the third complementarity condition in (23) can be discretized accordingly. Moreover, the final and boundary conditions (44) are approximated by

$$v_k^{N_t} = v_T(z_k), \qquad \theta v_0^n + (1-\theta)v_0^{n+1} = \theta s(t^n) + (1-\theta)s(t^{n+1}),$$
$$\theta\frac{2v_{N_z}^n - 5v_{N_z-1}^n + 4v_{N_z-2}^n - v_{N_z-3}^n}{(\Delta z)^2} + (1-\theta)\frac{2v_{N_z}^{n+1} - 5v_{N_z-1}^{n+1} + 4v_{N_z-2}^{n+1} - v_{N_z-3}^{n+1}}{(\Delta z)^2} = 0.$$

Therefore, the previous discretization leads to the sequence of discrete complementarity problems:

$$A^nV^n \geq f^{n+1}, \quad V^n \geq S^n, \quad (A^nV^n - f^{n+1})^t \cdot (V^n - S^n) = 0, \tag{45}$$

which are sequentially solved for $n = N_t-1, N_t-2, \dots, 1, 0$. Note that the components of vector V^n are the value function at the productivity nodes for time t^n, the matrix A^n only depends on n when the functions μ and σ depend on time, $S^n = S(t^n)$ for a time dependent scrap value, and f^{n+1} depends on the previously computed vector V^{n+1}.

For the numerical solution of problem (45) we also propose the use of the ALAS algorithm previously described for the steady state problem.

6.1.2. Problem of evolutionary firm distribution

In this problem, we consider the computational domain $D_Z^T = [D_Z^T]^1(t) \cup [D_Z^T]^2(t)$, where $[D_Z^T]^1(t) = [0, \underline{z}(t)] \times [0, T]$ and $[D_Z^T]^2(t) = (\underline{z}(t), Z] \times [0, T]$. Note that both subdomains depend on t, as the optimal exit productivity $\underline{z}(t)$ depends on t in the evolutive problem. Thus, we impose $g(z,t) = 0$ on $[D_Z^T]^1(t)$ and pose Eq. (24) on $[D_Z^T]^2$, jointly with the initial and boundary conditions

$$g(z,0) = g_0(z) \quad \text{and} \quad g(Z,t) = 0, \tag{46}$$

where $g_0(z)$ is the initial probability function, which is given in all examples by:

$$g_0(z) = \frac{1}{\nu\sqrt{2\pi}}\sum_{i=1}^{I} p_i e^{-\frac{(z-z_i^e)^2}{2\nu^2}}, \quad \text{with} \quad \sum_{i=1}^{I} p_i = 1. \tag{47}$$

Thus, the initial condition comes from a weighted sum of Gaussian random variables, each one with mean and variance equal to z_i^e and ν^2, respectively.

Let us denote by g_k^n the approximation to the solution at the point $(k\Delta z, n\Delta t)$, i.e. $g_k^n \approx g(z_k, t^n)$, and by $z_{\tilde{k}}^n$ the approximation of the optimal exit productivity at time t^n, i.e. $z_{\tilde{k}}^n \approx \underline{z}(t^n)$.

As for the backward in time HJB PDE, we consider the θ-method for the time discretization, so that for $\theta = 1$ and $\theta = 0$ we recover the implicit and explicit methods respectively, although in practice we use Crank-Nicolson ($\theta = 0.5$). Thus, we discretize (24) at the grid point (z_k, t_θ^n), with $t_\theta^n = (1-\theta)t^n + \theta t^{n+1}$, for $k = \tilde{k}+1, \dots, N_z-1$ and $n = 0, 1, \dots, N_t-1$, using the approximations:

$$\partial_z\Big(\mu(z_k,t_\theta^n)g(z_k,t_\theta^n)\Big) \approx (1-\theta)\left[\frac{[\mu(z_k,t^n)]^+g_k^n-[\mu(z_{k-1},t^n)]^+g_{k-1}^n}{\Delta z}+\frac{[\mu(z_{k+1},t^n)]^-g_{k+1}^n-[\mu(z_k,t^n)]^-g_k^n}{\Delta z}\right]$$
$$+\theta\left[\frac{[\mu(z_k,t^{n+1})]^+g_k^{n+1}-[\mu(z_{k-1},t^{n+1})]^+g_{k-1}^{n+1}}{\Delta z}+\frac{[\mu(z_{k+1},t^{n+1})]^-g_{k+1}^{n+1}-[\mu(z_k,t^{n+1})]^-g_k^{n+1}}{\Delta z}\right], \tag{48}$$

$$\partial_{zz}^2\Big(\sigma^2(z_k,t_\theta^n)g(z_k,t_\theta^n)\Big) \approx (1-\theta)\frac{\sigma^2(z_{k+1},t^n)g_{k+1}^n-2\sigma^2(z_k,t^n)g_k^n+\sigma^2(z_{k-1},t^n)g_{k-1}^n}{(\Delta z)^2}$$
$$+\theta\frac{\sigma^2(z_{k+1},t^{n+1})g_{k+1}^{n+1}-2\sigma^2(z_k,t^{n+1})g_k^{n+1}+\sigma^2(z_{k-1},t^{n+1})g_{k-1}^{n+1}}{(\Delta z)^2}. \tag{49}$$

Thus, replacing these approximations in (24) we obtain the equations:

$$-(1-\theta)\left[\frac{[\mu(z_k,t^n)]^+g_k^n-[\mu(z_{k-1},t^n)]^+g_{k-1}^n}{\Delta z}+\frac{[\mu(z_{k+1},t^n)]^-g_{k+1}^n-[\mu(z_k,t^n)]^-g_k^n}{\Delta z}\right]$$
$$-\theta\left[\frac{[\mu(z_k,t^{n+1})]^+g_k^{n+1}-[\mu(z_{k-1},t^{n+1})]^+g_{k-1}^{n+1}}{\Delta z}+\frac{[\mu(z_{k+1},t^{n+1})]^-g_{k+1}^{n+1}-[\mu(z_k,t^{n+1})]^-g_k^{n+1}}{\Delta z}\right]$$
$$+(1-\theta)\,\frac{\sigma^2(z_{k+1},t^n)g_{k+1}^n-2\sigma^2(z_k,t^n)g_k^n+\sigma^2(z_{k-1},t^n)g_{k-1}^n}{2(\Delta z)^2}$$
$$+\theta\,\frac{\sigma^2(z_{k+1},t^{n+1})g_{k+1}^{n+1}-2\sigma^2(z_k,t^{n+1})g_k^{n+1}+\sigma^2(z_{k-1},t^{n+1})g_{k-1}^{n+1}}{2(\Delta z)^2}$$
$$+\frac{\sigma^2(z_{\tilde{k}}^n,t^n)}{2}\partial_z g(z_{\tilde{k}}^n,t^n)g^e(z_k,t^n)=\frac{g_k^{n+1}-g_k^n}{\Delta t},$$

or equivalently,

$$\frac{\Delta t\theta}{\Delta z}\left[-\frac{\sigma^2(z_{k-1},t^{n+1})}{2\Delta z}-[\mu(z_{k-1},t^{n+1})]^+\right]g_{k-1}^{n+1}$$
$$+\left[1+\frac{\Delta t\theta}{\Delta z}\left(\frac{\sigma^2(z_k,t^{n+1})}{\Delta z}+[\mu(z_k,t^{n+1})]^+-[\mu(z_k,t^{n+1})]^-\right)\right]g_k^{n+1}$$
$$+\frac{\Delta t\theta}{\Delta z}\left[-\frac{\sigma^2(z_{k+1},t^{n+1})}{2\Delta z}+[\mu(z_{k+1},t^{n+1})]^-\right]g_{k+1}^{n+1} \quad =$$
$$\frac{\Delta t(1-\theta)}{\Delta z}\left[\frac{\sigma^2(z_{k-1},t^n)}{2\Delta z}+[\mu(z_{k-1},t^n)]^+\right]g_{k-1}^n$$
$$+\left[1+\frac{\Delta t(1-\theta)}{\Delta z}\left(-\frac{\sigma^2(z_k,t^n)}{\Delta z}-[\mu(z_k,t^n)]^++[\mu(z_k,t^n)]^-\right)\right]g_k^n$$
$$+\frac{\Delta t(1-\theta)}{\Delta z}\left[\frac{\sigma^2(z_{k+1},t^n)}{2\Delta z}-[\mu(z_{k+1},t^n)]^-\right]g_{k+1}^n$$
$$+\frac{\sigma^2(z_{\tilde{k}}^n,t^n)}{2}\partial_z g(z_{\tilde{k}}^n,t^n)g^e(z_k,t^n),$$

for $k=\tilde{k}+1,\ldots,N_z-1$ and $n=0,1,\ldots,N_t-1$.

Previous equations are completed with equations $g_k^{n+1}=0$, for $k=0,\ldots\tilde{k}$.

Finally, the initial and boundary conditions (46) are approximated by

$$g_k^0=g_0(z_k) \quad \text{and} \quad (1-\theta)g_{N_z}^n+\theta g_{N_z}^{n+1}=0.$$

After the previous discretization, the fully discretized problem can be written in matrix form and the resulting linear system at each time step is solved.

6.1.3. Evolutionary equilibrium problem

In this section we describe the numerical algorithm to compute the evolutionary equilibrium for given specified functions $\mu(z,t)$, $\sigma(z,t)$ and $s(t)$, as well as given values for the parameters ρ, e and c_f. The algorithm is the natural generalization of the one used for the stationary equilibrium. Thus, we start with $\omega^0(t)$ given and for $m=0,1,2,\ldots$ we follow:

Table 10
Wages and optimal exit productivity for $z_0^e = 3$ in the time dependent Example 1.

t	$\omega(t)$	$\underline{z}(t)$
0	1.296225	1.379375
2.5	1.224418	1.316250
5	1.162491	1.266250
7.5	1.118295	1.233125
10	1.094643	1.216875
12.5	1.087246	1.213125
SS	1.087548	1.213594

1. Given the wages $\omega^m(t)$ at the mesh points in time, we solve the backward in time HJB Eq. (23) with the method described in Section 6.1.1 using Crank-Nicolson ($\theta = 0.5$). Thus, we obtain the value function approximation v^m at the mesh points and the approximation of the optimal exit boundary $\underline{z}^m(t)$ at t^n, with $n = 0, 1, \ldots, N_t$.
2. Given the approximation of $\underline{z}^m(t)$, we solve the forward in time KFP problem with the method detailed in Section 6.1.2 for $\theta = 0.5$. Thus, we obtain the approximation of g^m at the mesh points.
3. Given the approximation of g^m, we compute the wages $\omega_1^m(t)$.
4. Given the wages $\omega_1^m(t)$, we repeat Steps 1 to 3 for calculating $\omega_2^m(t)$ and update the wages using (43).

When the difference between two consecutive approximations of $\omega^m(t)$ is below a prescribed tolerance, we consider $\omega^m(t)$, $\underline{z}\big(\omega^m(t)\big)$, $v\big(\omega^m(t)\big)$ and $g(\omega^m(t))$ as the solution of the evolutionary equilibrium problem.

6.2. Numerical results

We present two examples related to the ones considered in the steady state case. As the main objective is to illustrate the convergence of the solutions of the evolutionary equilibrium problem to corresponding ones of the steady state, we assume that the productivity dynamics follows a geometric Brownian motion, with parameters μ and σ^2 given in Table 1.

6.2.1. Example 1: numerical solution for the time-dependent equilibrium with a prescribed entry productivity

In this example we consider the time-dependent equilibrium model associated with the particular steady state one presented in Section 3, for which we could obtain a semianalytical solution and we also validated the proposed numerical solution with the examples in Section 5.1. Although we have addressed the tests with all the values of z_0^e considered in Section 5.1, we just report here the results for $z_0^e = 3$. We use the same parameters as in the steady state case and the initial condition for the distribution of firms problem is given by the choice $I = 1$, $p_1 = 1$ and $\nu = 0.44$ in expression (47), so that firms initially distribute around z_0^e with a certain spread associated with the value of ν.

Table 10 shows the computed equilibrium values of the wages and optimal exit productivity, for times $t = 0, 2.5, 5, 7.5, 10$ and 12.5, jointly with the values obtained for the steady state (SS) equilibrium model in the last row.

Next, in Figs. 6 and 7 we show the time evolution of the value function and the probability density function of firms, so that we can observe how the graphics for $T = 12.5$ are in agreement with the computed ones in Section 3 with the semianalytical solution and in Section 5.1 with the numerical methods for the steady state problem with $z_0^e = 3$.

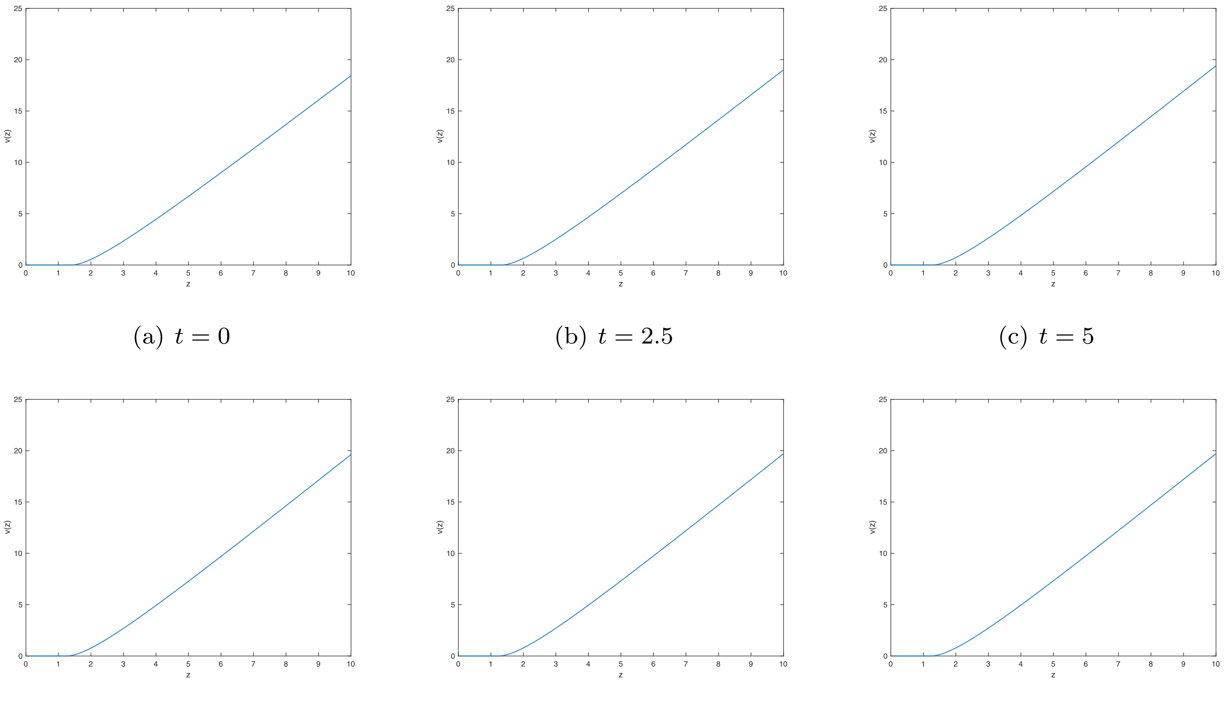
6.2.2. Example 2: numerical solution of the time-dependent equilibrium with random entry productivities

We consider the time-dependent equilibrium model associated with the steady state one posed in Section 5.2. Although we have addressed the tests with all values considered in Section 5.2, we just report here the results for the choice $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$ for the entry productivities and probabilities. We use the same parameters as in the steady state case. As initial distribution of firms we use $I = 3$, $p_1 = 0.25$, $p_2 = 0.5$, $p_3 = 0.25$ and $\nu = 0.13$ in expression (47), so that they initially distribute around the entry productivities following a weighted combination of Gaussian random variables.

Table 11 shows the computed wages and optimal exit productivity for specific times $t = 0, 2.5, 5, 7.5, 10$ and 12.5, jointly with the values obtained for the steady state (SS) equilibrium model in the last row.

Next, in Figs. 8 and 9 we show the time evolution of the value function and the probability density function of firms, so that we can observe how the graphics for $T = 12.5$ are in agreement with the ones in Section 5.2 which are obtained with the numerical methods for the steady state problem. Note that in Fig. 9 we use a different scale for $t = 0$.

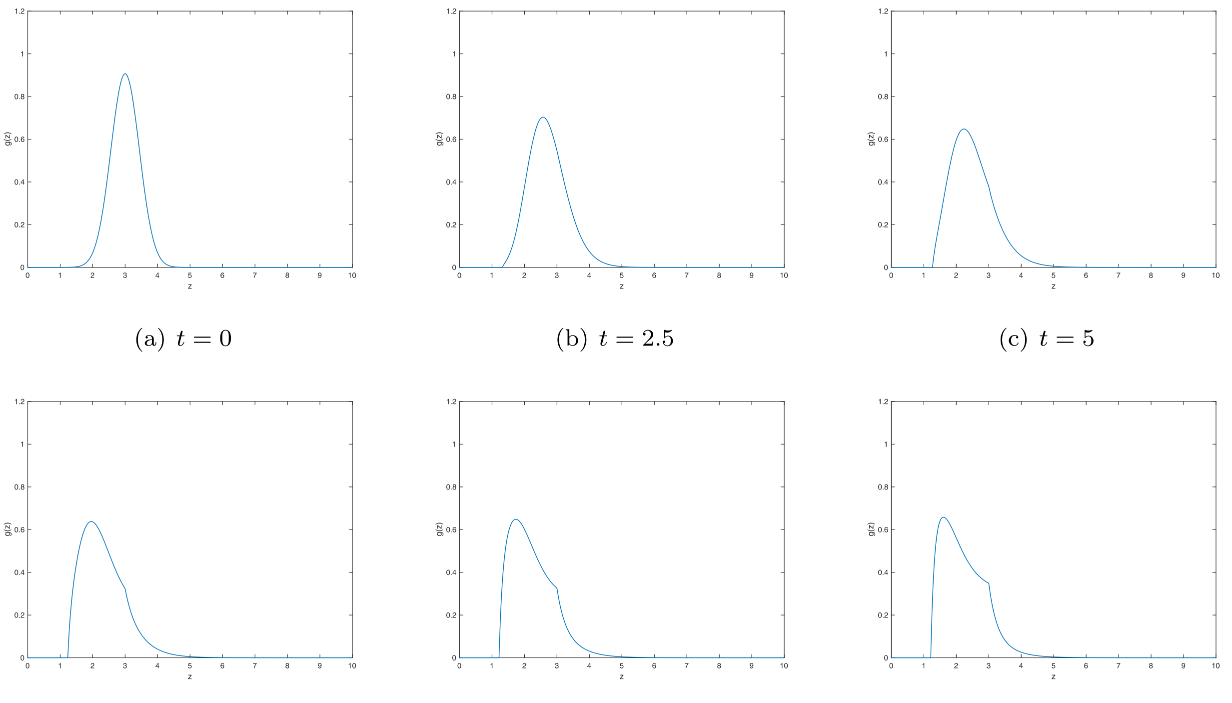
Remark 2. All computations in this article are obtained with a MATLAB implementation of the algorithms. The numerical solution of the evolutionary equilibrium problem with 16001 × 10001 mesh takes 1034.91 s, while the stationary equilibrium problem with 64001 mesh points takes 95.48 s on a MacBook Pro laptop. The CPU we use is a Intel(R) Core(TM) i5 at 2,7 GHz with 8 GB 1867 MHz DDR3 RAM.



(a) $t = 0$ (b) $t = 2.5$ (c) $t = 5$

(d) $t = 7.5$ (e) $t = 10$ (f) $t = 12.5$

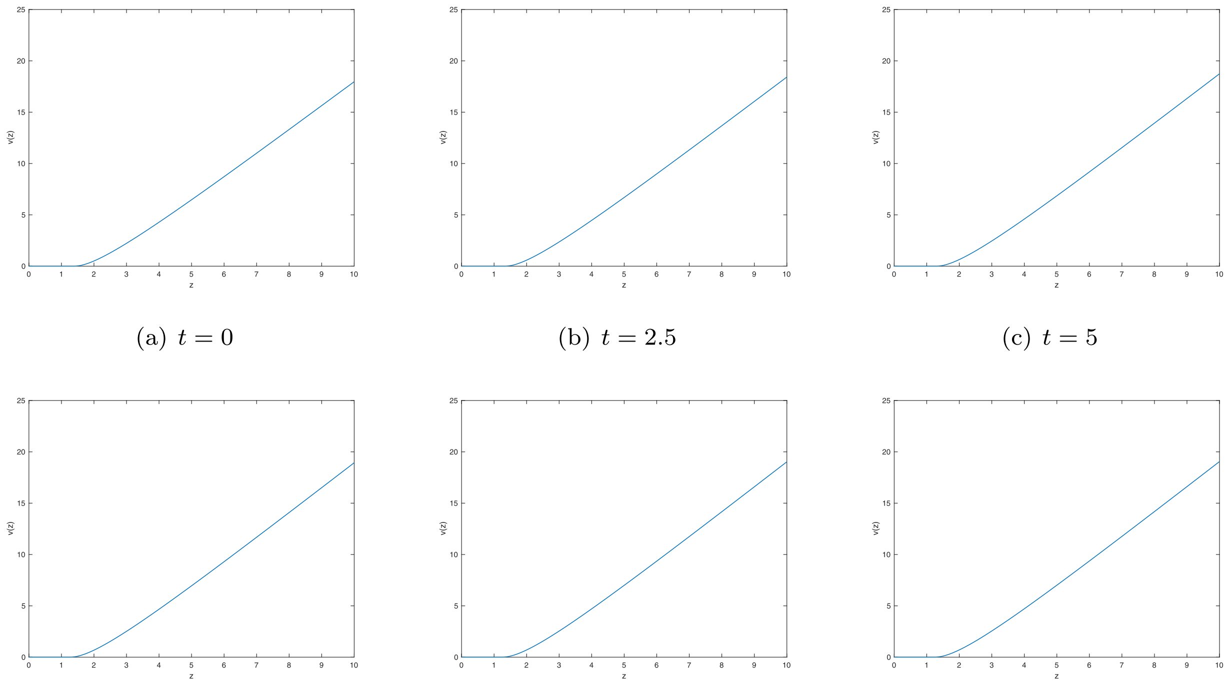
Fig. 6. The time-dependent value function for $z_0^e = 3$ in Example 1.



(a) $t = 0$ (b) $t = 2.5$ (c) $t = 5$

(d) $t = 7.5$ (e) $t = 10$ (f) $t = 12.5$

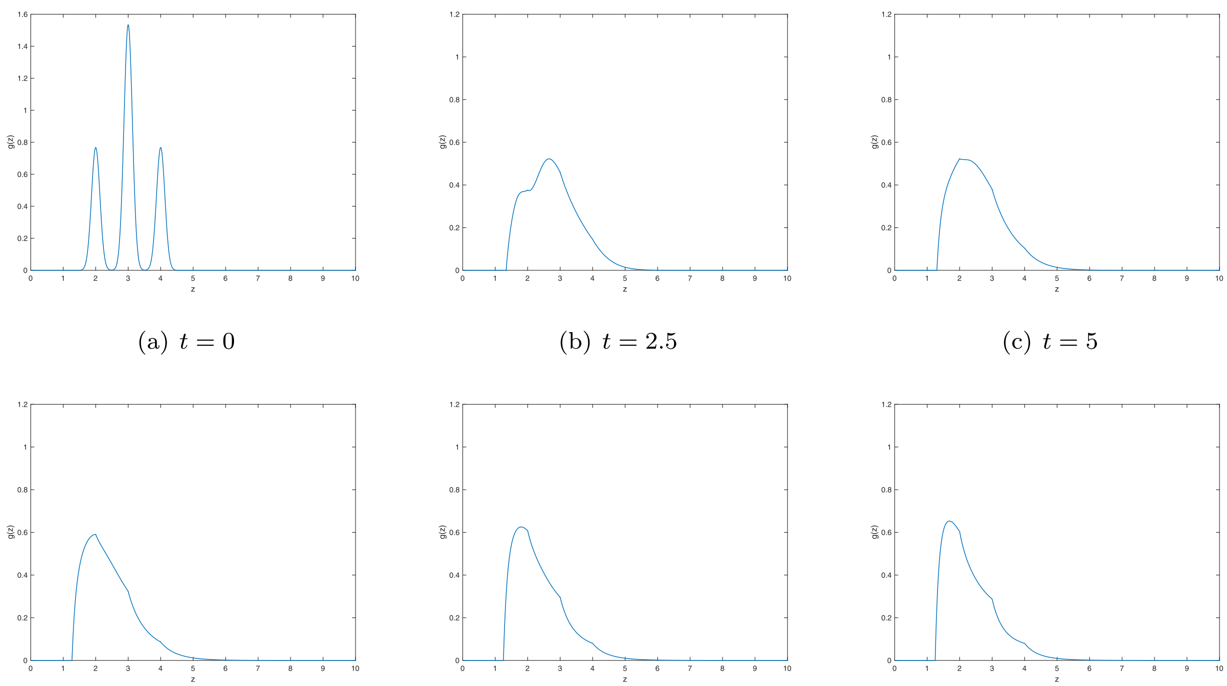
Fig. 7. The time-dependent firm distribution for $z_0^e = 3$ in Example 1.



(a) $t = 0$ (b) $t = 2.5$ (c) $t = 5$

(d) $t = 7.5$ (e) $t = 10$ (f) $t = 12.5$

Fig. 8. The time-dependent value function with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$ in Example 2.



(a) $t = 0$ (b) $t = 2.5$ (c) $t = 5$

(d) $t = 7.5$ (e) $t = 10$ (f) $t = 12.5$

Fig. 9. The time-dependent firm distribution with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$ in Example 2.

Table 11
Wages and optimal exit productivity with $\mathbf{z^e} = (2, 3, 4)$ and $\mathbf{p^e} = (0.25, 0.5, 0.25)$ in the time dependent Example 2.

t	$\omega(t)$	$\underline{z}(t)$
0	1.296225	1.390625
2.5	1.231462	1.336875
5	1.186020	1.298125
7.5	1.150821	1.269375
10	1.127934	1.252500
12.5	1.116721	1.246250
SS	1.117000	1.246563

7. Conclusions

In this article we propose a more general equilibrium model for heterogeneous agents under rational expectations, with the possibility of exit and entry of new establishments. The equilibrium model mainly involves a problem of incumbent firms which is governed by a HJB equation, a household problem, a distribution of firms evolution problem which is governed by a KFP equation, and a formulation of feasibility conditions.

A first innovative aspect in the modelling comes from the consideration of a more general stochastic process for the uncertain productivity evolution, namely a Ito process. In this way, we generalize previous choices in the literature based on Geometric Brownian motion or Ornstein Uhlenbeck processes. Secondly, besides the endogenous exit decision, we incorporate the entry of new establishments at a rate which balances the exit rate. We formulate the firms distribution problem for any probability density of entries with support contained in the range of productivities with operating firms. Besides the case where entry of all firms takes place at a specific productivity already treated in the literature, we propose an example where the entry of firms takes place at a finite set of productivities with their associated probabilities, i.e. the probability density function of new establishments is governed by a discrete random variable. In view of this innovative modelling aspects, even for the more general steady state equilibrium models we cannot obtain a semianalytical solution, so that numerical discretization of the involved differential equation problems is mandatory. This is also the case when the time-dependent equilibrium models are considered.

Thus, another relevant original achievement comes from the proposal of suitable numerical methods to solve the different subproblems that are involved in the global equilibrium problem. For the incumbents problem, we propose a combination of a finite differences scheme for the discretization with an augmented lagrangian active set (ALAS) algorithm for the nonlinear aspect related to the complementarity formulation associated with the endogenous exit decision. For the firms distribution problem, we propose a suitable discretization of the Fokker-Plank equation. For the time dependent equilibrium problem we additionally incorporate a Crank-Nicolson scheme for the time discretization of the evolutive HJB and KFP equations. All in all, we propose a fixed point iteration for the equilibrium problems, with the sequential solution of the involved subproblems at each fixed point iteration.

We first validate the proposed numerical methodology with a steady state equilibrium problem for which we can obtain a semianalytical solution. This semianalytical solution is obtained from a fixed point iteration in which we can obtain analytical solution for the different subproblems. A second order of convergence is observed in the Steffensen acceleration of the classical fixed point iteration. Next, we consider an example without analytical solution, where we check the convergence of the different parts of the solution. We also validate the numerical techniques for the time dependent equilibrium problem and show the convergence of its solution to the steady state ones.

Main conclusion is that as soon as we incorporate more general modelling issues, the solution of the equilibrium problem requires the full discretization and the use of appropriate numerical methods. The proposed set of numerical methods allows to obtain approximated solutions of the different models which provide expected qualitative and quantitative results. Note that computational times are very reasonable, so that numerical algorithms seem to be fast and reliable.

Future research work will be devoted to the mathematical analysis of the equilibrium models to obtain existence and uniqueness of solution, as well as to the theoretical analysis of convergence of the numerical methods. On the other hand, we will address the study of new more general modelling settings and its numerical solution.

Main notations

- t: Time
- z: Productivity (z_t: stochastic productivity process at time t)
- l: Labor force
- y: Revenues/output
- π: Optimal profit
- c_f: Fixed operating costs

- ω: Wages
- μ: Drift of the productivity
- σ: Volatility of the productivity
- ρ: Discount rate
- τ_l: Stopping times
- v: Firms value (v_t: stochastic firms value process at time t)
- s: Scrap value
- $\underline{z}$: Optimal exit productivity
- L: Labor supply by householders
- C: Goods consumption by householders
- Π: Aggregated profits of firms
- e: Utility parameter
- g: Probability function of firms in the industry
- g^e: Probability function of new establishments
- α: Entry rate of new firms (balances exit rate)
- δ: Dirac function centered at $z = 0$
- z_i^e: Entry productivities of new establishments

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Jonatan Ráfales: Formal analysis, Methodology, Software, Validation, Visualization, Writing - original draft, Writing - review & editing. **Carlos Vázquez:** Formal analysis, Methodology, Software, Validation, Visualization, Writing - original draft, Writing - review & editing.

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