with $r = \sqrt{x_1^2 + x_2^2}$ and $u(x) = u(r, x_3)$ to reduce the curl-curl operator to the vector Laplacian; at the same time we consider an isometric isomorphism between $\mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$ and $H^1(\mathbb{S}^3, \mathbb{R}^3)$ to recover compactness.

Asymptotic study of a thin layer of viscous fluid between two surfaces

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In this work, we are interested in studying the behavior of an incompressible viscous fluid moving between two closely spaced surfaces, also in motion.

We consider a three-dimensional thin domain, Ω_t^ε, filled by a fluid, that varies with time $t \in [0, T]$, given by

$$\begin{aligned} \Omega_t^\varepsilon \;=\; & \big\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in R^3 : \\ & x_i(\xi_1, \xi_2, t) \le x_i^\varepsilon \le x_i(\xi_1, \xi_2, t) + h^\varepsilon(\xi_1, \xi_2, t) N_i(\xi_1, \xi_2, t), \\ & (i = 1, 2, 3),\ (\xi_1, \xi_2) \in D \subset \mathbb{R}^2\big\} \end{aligned} \tag{1}$$

where $\vec{X}(\xi_1, \xi_2, t)$ is the lower bound surface parametrization, $\vec{N}(\xi_1, \xi_2, t)$ is the unit normal vector and $h^\varepsilon(\xi_1, \xi_2, t)$ is the gap between the two surfaces in motion assumed to be small with regard to the dimension of the bound surfaces. We take into account that the fluid film between the surfaces is thin by introducing a small non-dimensional parameter ε, and setting that

$$h^\varepsilon(\xi_1, \xi_2, t) = \varepsilon h(\xi_1, \xi_2, t) \tag{2}$$

We assume that the fluid motion is governed by Navier-Stokes equations and using the asymptotic development technique, the following lubrication model in a thin domain with curved mean surface has been obtained:

$$\begin{aligned} \frac{1}{\sqrt{A^0}} \operatorname{div}\left(\frac{(h^\varepsilon)^3}{\sqrt{A^0}} M \nabla p^{-2,\varepsilon}\right) &= 12\mu \frac{\partial h^\varepsilon}{\partial t} + 12\mu \frac{h^\varepsilon A^1}{A^0}\left(\frac{\partial \vec{X}}{\partial t} \cdot \vec{N}\right) \\ &- 6\mu \nabla h^\varepsilon \cdot (\vec{W}^0 - \vec{V}^0) + \frac{6\mu h^\varepsilon}{\sqrt{A^0}} \operatorname{div}(\sqrt{A^0}(\vec{W}^0 + \vec{V}^0)) \end{aligned} \tag{3}$$

It is a new generalized Reynolds equation where the pressure, p^ε, is approximated by $p^{-2,\varepsilon} = \varepsilon^{-2} p^{-2}$. The fluid velocities inside the domain are subsequently approximated from the pressure using the equations

$$u_1^0 = \frac{h^2(\xi_3^2 - \xi_3)}{2\mu A^0}\left(G \frac{\partial p^{-2}}{\partial \xi_1} - F \frac{\partial p^{-2}}{\partial \xi_2}\right) + \xi_3(W_1^0 - V_1^0) + V_1^0 \tag{4}$$

$$u_2^0 = \frac{h^2(\xi_3^2 - \xi_3)}{2\mu A^0}\left(E \frac{\partial p^{-2}}{\partial \xi_2} - F \frac{\partial p^{-2}}{\partial \xi_1}\right) + \xi_3(W_2^0 - V_2^0) + V_2^0 \tag{5}$$

$$u_3^0 = \frac{\partial \vec{X}}{\partial t} \cdot \vec{N} \tag{6}$$

where the velocity on the lower surface, $\vec{V}^0$, and on the upper surface, $\vec{W}^0$, are known. We denote by

$$A^0 = EG - F^2 \tag{7}$$

$$A^1 = -eG - gE + 2fF \tag{8}$$

$$M = \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \tag{9}$$

where E, F, G and e, f, g are the coefficients of the first and the second (respectively) fundamental forms of the surface parametrized by $\vec{X}$.

We have observed that, depending on the boundary conditions, other models can be obtained. We derive a shallow water model changing the boundary conditions that we had imposed: instead of assuming that we know the velocities on the upper and lower boundaries of the domain, we assume that we know the tractions on these upper and lower boundaries. We yield:

$$\begin{aligned}
&\frac{\partial V_i^0}{\partial t} + \sum_{l=1}^{2} \left(V_l^0 - C_l^0\right) \frac{\partial V_i^0}{\partial \xi_l} + \sum_{k=1}^{2} \left(R_{ik}^0 + \sum_{l=1}^{2} H_{ilk}^0 V_l^0 \right) V_k^0 \\
&= -\frac{1}{\rho_0} \left(\alpha_i^0 \frac{\partial \pi_0^0}{\partial \xi_1} + \beta_i^0 \frac{\partial \pi_0^0}{\partial \xi_2} \right) \\
&+ \nu \left\{ \sum_{m=1}^{2} \sum_{l=1}^{2} \frac{\partial^2 V_i^0}{\partial \xi_m \partial \xi_l} J_{lm}^0 + \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial V_k^0}{\partial \xi_l} (L_{kli}^0 + \psi(h)_{ikl}^0) \right. \\
&\left. + \sum_{k=1}^{2} V_k^0 (S_{ik}^0 + \chi(h)_{ik}^0) + \hat{\kappa}(h)_i^0 \right\} + F_i^0(h) - Q_{i3}^0 \left(\frac{\partial \vec{X}}{\partial t} \cdot \vec{N} \right) \\
&\qquad (i = 1, 2)
\end{aligned} \tag{10}$$

$$\frac{\partial h}{\partial t} + \frac{h}{\sqrt{A^0}} \mathrm{div} \left(\sqrt{A^0} \vec{V}^0 \right) + \frac{h A^1}{A^0} \left(\frac{\partial \vec{X}}{\partial t} \cdot \vec{N} \right) = 0 \tag{11}$$

where α_i^0, β_i^0, C_l^0, H_{ilk}^0, J_{lm}^0, L_{kli}^0, Q_{i3}^0, R_{ik}^0, S_{ik}^0 depend only on the parametrization $\vec{X}$ and $F_i^0(h)$, $\psi(h)_{ikl}^0$, $\chi(h)_{ik}^0$, $\kappa(h)_i^0$ depend on the parametrization $\vec{X}$ and on the gap h. The exact definition of these coefficients can be found in [5], where the complete derivation of both models is presented.

Once V_1^0, V_2^0 and π_0^0 (the approximation of the pressure on the lower bound) are calculated we have the following approximation of the velocities and the pressure

$$u_i^0 = W_i^0 = V_i^0 \quad i = 1, 2 \tag{12}$$

$$u_3^0 = \frac{\partial \vec{X}}{\partial t} \cdot \vec{N} \tag{13}$$

$$p^0 = \frac{2\mu}{h} \frac{\partial h}{\partial t} + \pi_0^0 \tag{14}$$

These models can not be found in the literature, as far as we know. We reach the conclusion that the magnitude of the pressure differences at the lateral boundary of the domain is key when deciding which of the two models best describes the fluid behavior.

Boundary conditions tell us which of the two models should be used when simulating the flow of a thin fluid layer between two surfaces: if the fluid pressure is dominant (that is, it is of order $O(\varepsilon^{-2})$), and the fluid velocity is known on the upper and lower surfaces, we must use the lubrication model; if the fluid pressure is not dominant (that is, it is of order $O(1)$), and the tractions are known on the upper and lower surfaces, we must use the shallow water model. In the first case we will say that the fluid is "driven by the pressure" and in the second that it is "driven by the velocity".

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Method of energy estimates for studying of singular boundary regimes in quasilinear parabolic equations

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In the cylindrical domain $Q = (0,T) \times \Omega,\ 0 < T < \infty$, where $\Omega \subset R^n$ is a bounded domain such that $\partial\Omega \in C^2$, the following problem is considered:

$$\begin{aligned} &(|u|^{q-1}u)_t - \Delta_p u = 0, \quad p \geqslant q > 0, \\ &u(0,x) = u_0 \ \text{ in } \Omega, \ \ u_0 \in L^{q+1}(\Omega), \\ &u(t,x)\Big|_{\partial\Omega} = f(t,x), \end{aligned} \tag{1}$$

where f generates boundary regime with singular peaking, namely,

$$f(t,x) \to \infty \quad \text{as } t \to T, \quad \forall\, x \in\ K \subset \partial\Omega, K \neq \emptyset. \tag{2}$$

Function f is called a localized boundary regime (S-regime) if

$$\overline{\Omega} \setminus \Omega_0 \neq \emptyset, \quad \text{where } \Omega_0 := \left\{ x \in \overline{\Omega} : \sup_{t\to T} u(t,x) = \infty \right\}$$

for an arbitrary weak solution u of problem (1). Sharp conditions of localization of boundary regime were obtained by some version of local energy estimates (see [1] and references therein).