



UNIVERSIDADE DA CORUÑA

DOCTORAL THESIS

**A unifying formulation for nonlinear solid
mechanics and Finite Element Analysis**

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Abstract

A unifying formulation for nonlinear solid mechanics and Finite Element Analysis

The finite element method is a well-known technology that allows to obtain an approximation to the real structural behaviour of a continuum solid media subjected to external forces. Its use is widely extended in civil engineering and many other fields, such as naval or aerospace engineering.

This formulation can be derived under the linear or the nonlinear analysis framework. If the displacements and their corresponding gradients are assumed to be small, the analysis is considerably simplified, and it turns out to be carried out under the assumptions of the linear theory. However, if the displacements and/or the displacement gradients become large, the nonlinear analysis arises.

As both analyses are based on different assumptions, they lead to completely different structural responses. And the accuracy of the results depends on the precision of the assumptions made. That is, if the structure does not experiment small displacements or small displacement gradients, the linear analysis leads to unacceptable results that significantly differ from the real behaviour.

Before running a structural simulation, the engineer has to decide, based on its experience and intuition, if the linear assumptions are correct. If the real structural response does not verify the linear assumptions, the linear analysis must be discarded and a nonlinear one should be carried out in order to obtain accurate results.

Therefore, the assumptions made about the magnitude of both the displacements and the displacement gradients are quite important, since they define the theoretical framework of the structural analysis. The implications of each assumption have to be clearly defined. Most references in the existing literature do not clearly identify the implications of these assumptions. Therefore, one of the main aims of this work is to clearly identify them and to properly define both the linear and the nonlinear mathematical models that governs the structural behaviour according to each analysis.

To accomplish this goal, a unifying formulation of both the linear and nonlinear solid mechanics complete and detailed is proposed. This formulation allows to completely describe and understand the deformation that an elastic solid experiments over time. A novel, simple, and clear nomenclature is proposed, in order to properly state the solid

mechanics principles and the strictly necessary equations that describe this deformation process.

Once the mathematical models are well-posed, the finite element method can be applied. A complete original derivation in both linear and nonlinear theory is presented. The linear one is also performed in order to compare this well-known derivation with the nonlinear version.

One of the main differences between both formulations lies in the application of the external forces. In general, the linear formulation leads to a linear behaviour, whereas the nonlinear one drives to a nonlinear one. As long as the response is linear, the total load can be applied in only one step, and the load superposition principle usually applied in linear theory holds. However, this principles can no longer be applied when dealing with a nonlinear behaviour. If the response is nonlinear, a given load state has multiple possible solutions. Therefore, the total load can not be applied in only one step, and the load history has to be taken into account to reach the correct solution. To overcome these inconveniences, the external loads are usually applied according to an incremental loading process.

This incremental strategy is actually a suitable procedure, since the structural response corresponding to each load step has to be solved iteratively. This procedure needs to start iterating from a close approximation to the solution. If the incremental loads are small enough, the result of the previous load step can be adopted to start the iterative procedure, and the convergence should be guaranteed.

Many reference textbooks and research papers address the derivation of the nonlinear finite element formulations. Nevertheless, there is no consensus about a common nomenclature and notation. Moreover, the hypotheses made along these derivations are no clearly specified or are not even stated. Therefore, to completely comprehend the underlying physics and the essence of the proposed algorithms, a detailed overview which clarifies this knowledge becomes necessary.

In this thesis, a great effort is made to clearly identify the intermediate hypotheses, and extensively analyse the origin and composition of the matrices that arise in nonlinear analysis. A detailed guideline that facilitates the deep comprehension of this powerful technology is proposed. This work states a unifying, clear and complete formulation for the nonlinear analysis field, so the extension of some research lines that have been carried out in linear theory until now becomes possible.

Resumen

Una formulación unificadora para la mecánica de sólidos no lineal y el análisis por el Método de los Elementos Finitos

El método de los elementos finitos es una tecnología bien conocida que permite obtener una aproximación al comportamiento estructural real de un medio sólido continuo sometido a fuerzas externas. Su uso está ampliamente extendido en ingeniería civil y en muchos otros campos, como la ingeniería naval o la aeronáutica.

Esta formulación puede obtenerse bajo el marco de los análisis lineal o no lineal. Si se supone que los desplazamientos y sus correspondientes gradientes son pequeños, el análisis se simplifica considerablemente, y resulta realizarse bajo los supuestos de la teoría lineal. Sin embargo, si los desplazamientos y/o los gradientes de los desplazamientos se consideran grandes, surge el análisis no lineal.

Como ambos análisis se basan en supuestos diferentes, conducen a respuestas estructurales completamente distintas. Y la exactitud de los resultados depende de la precisión de las hipótesis realizadas. Es decir, si la estructura no experimenta pequeños desplazamientos o pequeños gradientes de desplazamiento, el análisis lineal conduce a resultados inaceptables que difieren significativamente del comportamiento real.

Antes de llevar a cabo una simulación estructural, el ingeniero tiene que decidir, basándose en su experiencia e intuición, si los supuestos lineales son correctos. Si la respuesta estructural real no verifica las hipótesis lineales, hay que descartar el análisis lineal y realizar uno no lineal para obtener resultados precisos.

Por lo tanto, las hipótesis adoptadas acerca de la magnitud tanto de los desplazamientos como de los gradientes de los desplazamientos son muy importantes, ya que definen el marco teórico del análisis estructural. Es necesario definir claramente las implicaciones de cada supuesto. En la literatura existente, la mayoría de las referencias no identifica claramente las implicaciones de estos supuestos. Por lo tanto, uno de los principales objetivos de este trabajo es identificarlas claramente y definir adecuadamente los modelos matemáticos lineales y no lineales que rigen el comportamiento estructural asociado a cada análisis.

Para lograr este objetivo, se propone una formulación unificadora de la mecánica de sólidos lineal y no lineal completa y detallada. Esta formulación permite describir y comprender completamente la deformación que experimenta un sólido elástico a lo largo del tiempo. Se propone una nomenclatura novedosa, sencilla y clara para enunciar

adecuadamente los principios de la mecánica de sólidos y las ecuaciones estrictamente necesarias que describen este proceso de deformación.

Una vez que los modelos matemáticos están bien planteados, se puede aplicar el método de los elementos finitos. Se presenta una obtención original completa tanto en teoría lineal como no lineal. Se presenta también el desarrollo lineal para compararlo con su versión no lineal.

Una de las principales diferencias entre ambas formulaciones radica en la forma de aplicar las fuerzas externas. En general, la formulación lineal conduce a un comportamiento lineal, mientras que la no lineal conduce a uno no lineal. Mientras la respuesta sea lineal, la carga total puede aplicarse en un solo paso, y el principio de superposición de cargas de la teoría lineal puede aplicarse adecuadamente. Sin embargo, este principio ya no se puede aplicar cuando se trata con un comportamiento no lineal. Si la respuesta es no lineal, un estado de carga dado tiene múltiples soluciones posibles. Por lo tanto, la carga total no puede aplicarse en un solo paso, y hay que tener en cuenta el historial de carga para obtener la solución correcta. Para solventar estos inconvenientes, las cargas externas suelen aplicarse según un proceso de carga incremental.

Esta estrategia incremental es en realidad un procedimiento adecuado, ya que la respuesta estructural correspondiente a cada paso de carga debe resolverse de forma iterativa. Este procedimiento necesita empezar a iterar desde una aproximación cercana a la solución. Si las cargas incrementales son lo suficientemente pequeñas, el resultado del paso de carga anterior puede adoptarse para iniciar el procedimiento iterativo, y la convergencia debería estar garantizada.

Numerosos libros de texto de referencia y trabajos de investigación abordan la obtención de las formulaciones de elementos finitos no lineales. Sin embargo, no existe consenso sobre una nomenclatura y notación comunes. Además, las hipótesis formuladas a lo largo de estos desarrollos no se especifican claramente o ni siquiera se enuncian. Por lo tanto, para comprender completamente la física subyacente y la esencia de los algoritmos propuestos, se hace necesaria una visión más detallada que lo aclare.

En esta tesis se hace un gran esfuerzo por identificar claramente las hipótesis intermedias y analizar ampliamente el origen y la composición de las matrices que surgen en el análisis no lineal. Se elabora una guía detallada que facilita el aprendizaje profundo de esta potente tecnología. Este trabajo plantea una formulación unificadora, clara y completa en el campo del análisis no lineal, para que la extensión de algunas líneas de investigación que hasta ahora se han llevado a cabo en teoría lineal sea posible.

Resumo

Unha formulación unificadora para a mecánica de sólidos non lineal e a análise polo Método dos Elementos Finitos

O método dos elementos finitos é unha tecnoloxía ben coñecida que permite obter unha aproximación ao comportamento estrutural real dun medio sólido continuo sometido a forzas externas. O seu uso está amplamente estendido en enxeñería civil e en moitos outros campos, como a enxeñería naval ou a aeronáutica.

Esta formulación pode baixo o marco das análises lineal ou non lineal. Se se supón que os desprazamentos e os seus correspondentes gradientes son pequenos, a análise simplifícase considerablemente, e resulta realizarse baixo os supostos da teoría lineal. Con todo, se os desprazamentos e/o os gradientes dos desprazamentos se consideran grandes, xorde a análise non lineal.

Como as dúas análises baséanse en supostos diferentes, conducen a respostas estruturais completamente distintas. E a exactitude dos resultados depende da precisión das hipóteses realizadas. É dicir, se a estrutura non experimenta pequenos desprazamentos ou pequenos gradientes de desprazamento, a análise lineal conduce a resultados inaceptables que difiren de forma significativa do comportamento real.

Antes de levar a cabo unha simulación estrutural, o enxeñeiro ten que decidir, baseándose na súa experiencia e intuición, se os supostos lineais son correctos. Se a resposta estrutural real non verifica as hipóteses lineais, hai que descartar a análise lineal e realizar unha non lineal para obter resultados precisos.

Por tanto, as hipóteses adoptadas acerca da magnitude tanto dos desprazamentos como dos gradientes dos desprazamentos son moi importantes, xa que definen o marco teórico da análise estrutural. É necesario definir claramente as implicacións de cada suposto. Na literatura existente, a maioría das referencias non identifica claramente as implicacións destes supostos. Por tanto, un dos principais obxectivos deste traballo é identificalas claramente e definir adecuadamente os modelos matemáticos lineais e non lineais que rexen o comportamento estrutural asociado a cada análise.

Para lograr este obxectivo, propónse unha formulación unificadora da mecánica de sólidos lineal e non lineal completa e detallada. Esta formulación permite describir e comprender completamente a deformación que experimenta un sólido elástico ao longo do tempo. Propónse unha nomenclatura nova, sinxela e clara para enunciar adecuada-

mente os principios da mecánica de sólidos e as ecuacións estritamente necesarias que describen este proceso de deformación.

Unha vez que os modelos matemáticos están ben expostos, pódese aplicar o método dos elementos finitos. Preséntase unha obtención orixinal completa tanto en teoría lineal como non lineal. Preséntase tamén o caso lineal para comparalo coa versión non lineal.

Unha das principais diferenzas entre ámbalas formulacións radica na forma de aplicar as forzas externas. En xeral, a formulación lineal conduce a un comportamento lineal, mentres que a non lineal conduce a unha resposta non lineal. Mentres a resposta sexa lineal, a carga total pode aplicarse nun só paso, e mantense o principio de superposición de cargas que adoita aplicarse na teoría lineal. Con todo, este principio non pode aplicarse cando se trata cun comportamento non lineal. Se a resposta é non lineal, un estado de carga dado ten múltiples solucións posibles. Por tanto, a carga total non pode aplicarse nun só paso, e hai que ter en conta o historial de carga para chegar á solución correcta. Para superar estes inconvenientes, as cargas externas adoitan aplicarse segundo un proceso de carga incremental.

Esta estratexia incremental é en realidade un procedemento adecuado, xa que a resposta estrutural correspondente a cada paso de carga debe resolverse de forma iterativa. Este procedemento necesita empezar a iterar desde unha aproximación próxima á solución. Se as cargas incrementais son o suficientemente pequenas, o resultado do paso de carga anterior pode adoptarse para iniciar o procedemento iterativo, e a converxencia debería estar garantida.

Numerosos libros de texto de referencia e traballos de investigación abordan a obtención das formulacións de elementos finitos non lineais. Con todo, non existe consenso sobre unha nomenclatura e notación comúns. Ademais, as hipóteses formuladas ao longo destas derivacións non se especifican claramente ou nin sequera se enuncian. Por tanto, para comprender completamente a física subxacente e a esencia dos algoritmos propostos, faise necesaria unha visión detallada que os aclare.

Nesta tese faise un grande esforzo por identificar claramente as hipóteses intermedias e analizar amplamente a orixe e a composición das matrices que xorden na análise non lineal. Proponse unha guía detallada que facilita a aprendizaxe profunda desta potente tecnoloxía. Este traballo expón unha formulación unificadora, clara e completa no campo da análise non lineal, para que a extensión dalgunhas liñas de investigación que ata o de agora se levaron a cabo en teoría lineal sexa posible.

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Nomenclature

Constants and scalar functions

δu_i	Test function
ϵ_i	Infinitesimal strain tensor eigenvalue
η	Number of trial functions
γ	Shear strain
λ	Lamé's first parameter
λ_i	Right Cauchy-Green tensor eigenvalue
\mathcal{F}	Determinant of the deformation gradient tensor approximation
μ	Lamé's second parameter
ν	Poisson's ratio
ϕ_i	Trial function
Ψ	Work per unit volume developed by the internal forces during the deformation process
ρ	Media density
ρ^0	Initial media density
σ	Normal stress
τ	Shear stress
ε	Normal strain
ε_g	Maximum relative difference between consecutive weak form residuals
ε_i	Biot strain tensor eigenvalue
ε_u	Maximum relative difference between consecutive displacement fields

φ	Mass source per unit volume per unit time
$d\Gamma$	Differential area
$d\Gamma_0$	Initial differential area
$d\Omega$	Differential volume
$d\Omega_0$	Initial differential volume
$d\Psi$	Differential work per unit volume developed by the internal forces during a differential part of the deformation process
dM	Differential mass
dt	Differential time
E	Young's modulus
E_g	Maximum absolute difference between consecutive weak form residuals
E_u	Maximum absolute difference between consecutive displacement fields
F	Determinant of the deformation gradient tensor
f	Weak form left-hand side
G	Shear modulus
g	Weak form residual
M	Mass
n	Number of finite elements that compose the material domain discretization
P	Weak form right-hand side
S	Tetrahedron basis area
S_i	Tetrahedron face area
t	Instant of time
V	Tetrahedron volume

First order tensors

$\bar{\mathbf{0}}$	Zero vector
$\bar{\boldsymbol{\epsilon}}$	Vectorial form of the infinitesimal strain tensor
$\bar{\boldsymbol{\epsilon}}^h$	Vectorial form of the infinitesimal strain tensor approximation

$\bar{\boldsymbol{\varepsilon}}_p$	Initial strain field approximation
$\bar{\boldsymbol{\sigma}}$	Vector expression of the Cauchy stress tensor
$\bar{\boldsymbol{\sigma}}^h$	Vectorial form of the Cauchy stress tensor approximation
$\bar{\boldsymbol{\sigma}}_p$	Initial stress field approximation
$\bar{\boldsymbol{E}}_G$	Vectorial form of the Green-Lagrange strain tensor
$\bar{\boldsymbol{J}}$	Vectorial form of the displacement gradient tensor
$\bar{\boldsymbol{S}}$	Vectorial form of the second Piola-Kirchhoff stress tensor
$\boldsymbol{\alpha}$	Vector composed by the initial nodal displacement approximations
$\boldsymbol{\alpha}_i$	Initial displacement vector approximation corresponding to the node i
$\boldsymbol{\beta}$	Vector composed by the coefficients that define the trial functions linear combination
$\boldsymbol{\beta}_i$	Vector coefficient that defines the trial functions linear combination
$\boldsymbol{\mathcal{L}}_0$	Angular momentum with respect to the origin of coordinates
$\boldsymbol{\omega}$	Vector test function
\boldsymbol{a}	Velocity vector
\boldsymbol{b}	External volumetric force
\boldsymbol{e}_i	Cartesian basis unit vector
\boldsymbol{f}	External forces vector
\boldsymbol{g}	Stress vector defined on the surface where an external surface force is applied
\boldsymbol{g}_0	Differential force acting on the deformed solid per unit initial area
$\boldsymbol{g}_{R,0}$	Differential force acting on the deformed solid surface where the essential boundary condition is applied, per unit initial area
\boldsymbol{g}_R	Stress vector that appears as a reaction on the surface where an essential boundary condition is applied
\boldsymbol{g}_R^h	Approximation to the stress vector that appears as a reaction on the surface where an essential boundary condition is applied
\boldsymbol{l}	Velocity gradient tensor
\boldsymbol{n}	Normal unit vector

\mathbf{n}_0	Initial normal unit vector
\mathbf{p}	Linear momentum
\mathbf{r}	Deformation vector
$\mathbf{r}_{0,i}$	Nodal position vector (global numeration)
\mathbf{r}_0	Initial position vector
\mathbf{R}_r	Natural boundary condition residual
\mathbf{R}_σ	Equilibrium equation residual
\mathbf{t}	Stress vector
\mathbf{u}	Displacement vector
\mathbf{u}^0	Prescribed displacement
\mathbf{u}^h	Displacement vector approximation
\mathbf{u}_i	Right Cauchy-Green tensor (or Biot strain tensor) eigenvector
\mathbf{u}_p	Initial displacement field approximation
\mathbf{v}	Mass source velocity vector
\mathbf{v}_i	Infinitesimal strain tensor eigenvector
$\Delta \bar{\mathcal{E}}$	Vectorial form of the infinitesimal strain tensor increment
$\delta \bar{\mathcal{E}}$	Vectorial form of the infinitesimal strain tensor variation
$\Delta \bar{\mathcal{E}}_p$	Correction of the initial strain field approximation
$\Delta \bar{\sigma}_p$	Correction of the initial stress field approximation
$\Delta \bar{\mathcal{E}}_G$	Vectorial form of the Green-Lagrange strain tensor increment
$\delta \bar{\mathcal{E}}_G$	Vectorial form of the Green-Lagrange strain tensor variation
$\Delta \bar{\mathcal{E}}_G^t$	Vectorial form of the total incremental Green-Lagrange strain tensor
$\Delta \bar{\mathcal{J}}$	Vectorial form of the displacement gradient tensor increment
$\delta \bar{\mathcal{J}}$	Vectorial form of the displacement gradient tensor variation
$\Delta \bar{\mathcal{S}}$	Vectorial form of the second Piola-Kirchhoff stress tensor increment
$\Delta \boldsymbol{\alpha}$	Vector composed by the nodal displacement corrections
$\Delta \boldsymbol{\alpha}_i$	Displacement vector correction corresponding to the node i

$\delta \mathbf{r}$	Relative deformation vector (deformed material vector at a given instant of time t)
$\delta \mathbf{r}_0$	Initial relative deformation vector (initial material vector)
$\Delta \mathbf{u}$	Displacement vector increment
$\delta \mathbf{u}$	Displacement vector variation, or vector test function
$\delta \mathbf{u}^h$	Vector test function approximation
$\Delta \mathbf{u}_p$	Correction of the initial displacement field approximation
$\hat{\mathbf{f}}^e$	Element forces vector (nodal global numeration)
$d\mathbf{\Gamma}$	Differential area vector
$d\mathbf{\Gamma}_0$	Initial differential area vector
$d\mathbf{f}$	Differential force

Second order tensors

$\mathbf{0}$	Zero tensor
$\mathbf{\Lambda}$	Diagonal tensor composed by the right Cauchy-Green tensor eigenvalues
\mathbf{D}	Infinitesimal distortion tensor (infinitesimal isochoric strain tensor)
$\mathbf{\mathcal{E}}$	Infinitesimal strain tensor
$\mathbf{\mathcal{E}}^*$	Diagonal tensor composed by the infinitesimal strain tensor eigenvalues
$\mathbf{\mathcal{E}}_D$	Infinitesimal deviatoric strain tensor
\mathcal{F}	Deformation gradient tensor approximation
\mathcal{H}	Infinitesimal inflation tensor (infinitesimal isotropic strain tensor)
\mathcal{R}	Infinitesimal rotation tensor
\mathcal{U}	Tensor composed by the infinitesimal strain tensor eigenvectors arranged in columns
\mathcal{W}	Component of the infinitesimal rotation tensor
$\mathbf{\Omega}$	Tensor composed by the tensor test functions
$\mathbf{\Omega}_j$	Tensor test function
ϕ	Tensor composed by the tensor trial functions
ϕ_i	Tensor trial function

Ψ	Stress tensor before the application of the rotation tensor
σ	Cauchy stress tensor
A	Tensor that depends on the displacement field and defines \bar{E}_G vs. \bar{J}
A_C	Constant tensor that defines \bar{E}_G vs. \bar{J}
B	Tensor that defines $\delta\bar{E}_G$ vs. β
$B_{L,0}$	Linear component of the tensor B
$B_{N,0}$	Nonlinear component of the tensor B
C_2	Second order constitutive tensor that defines $\Delta\bar{S}$ vs. $\Delta\bar{E}_G$
D	Distortion tensor (isochoric strain tensor)
D_2	Second order constitutive tensor that defines $\bar{\sigma}$ vs. $\bar{\mathcal{E}}$
E	Biot strain tensor
e	Strain-rate tensor
E_D	Deviatoric strain tensor
e_D	Incremental deviatoric strain tensor
E_G	Green-Lagrange strain tensor
F	Deformation gradient tensor
G_0	Tensor that defines $\delta\bar{J}$ vs. β and $\Delta\bar{J}$ vs. $\Delta\alpha$
H	Inflation tensor (isotropic strain tensor)
I	Second order unit tensor (also represented as I_2)
J	Displacement gradient tensor
K	Stiffness matrix
K_G	Geometric stiffness
K_M	Tangent stiffness material component
K_M'''	Third term of the tangent stiffness material component
K_M''	Second term of the tangent stiffness material component
K_M'	First term of the tangent stiffness material component
K_T	Tangent stiffness

L_0	Tensor that defines $\bar{\mathcal{E}}$ vs. \mathbf{u}
M	Right Cauchy-Green tensor
P	First Piola-Kirchhoff stress tensor
R	Rotation tensor
S	Second Piola-Kirchhoff stress tensor
U	Tensor composed by the right Cauchy-Green tensor (or Biot strain tensor) eigenvectors arranged in columns
w	Spin tensor
$\Delta(\delta\mathbf{E}_G)$	Increment of the Green-Lagrange strain tensor variation
$\Delta\mathcal{E}$	Infinitesimal strain tensor increment
$\delta\mathcal{E}$	Infinitesimal strain tensor variation
$\Delta\mathbf{E}_{G,t}^t$	Updated incremental Green-Lagrange strain tensor
$\Delta\mathbf{E}_G$	Green-Lagrange strain tensor increment
$\delta\mathbf{E}_G$	Green-Lagrange strain tensor variation
$\Delta\mathbf{E}_G^t$	Total incremental Green-Lagrange strain tensor
$\Delta\mathbf{F}$	Deformation gradient tensor increment
$\delta\mathbf{F}$	Deformation gradient tensor variation
$\Delta\mathbf{J}$	Displacement gradient tensor increment
$\delta\mathbf{J}$	Displacement gradient tensor variation
$\Delta\mathbf{S}$	Second Piola-Kirchhoff stress tensor increment
$\hat{\mathbf{K}}^e$	Element stiffness matrix (nodal global numeration)
$\hat{\mathbf{S}}$	Tensor composed by a diagonal grouping of the second Piola-Kirchhoff stress tensor
$\tilde{\mathbf{E}}$	Tensor composed by the cartesian unit vectors arranged in columns
$\mathbf{B}_{L,0}^{(i)}$	Component of the tensor $\mathbf{B}_{L,0}$
$\mathbf{B}_{N,0}^{(i)}$	Component of the tensor $\mathbf{B}_{N,0}$
$\mathbf{G}_0^{(i)}$	Component of the tensor \mathbf{G}_0

Higher order tensors

\bar{I}_4	Second fourth order unit tensor
C_4	Fourth order constitutive tensor that defines $\Delta \mathbf{S}$ vs. $\Delta \mathbf{E}_G$
D_4	Fourth order constitutive tensor that defines $\boldsymbol{\sigma}$ vs. $\boldsymbol{\varepsilon}$
I_4	First fourth order unit tensor
I_4^{sym}	Symmetric term of the first fourth order unit tensor

Other symbols

∂_0	Differential operator composed by derivatives taken with respect to the initial material domain
Γ	Solid surface
Γ_0	Initial solid surface
Γ_0^σ	Initial surface where an external surface force is applied
Γ_0^u	Initial surface where the displacement field is prescribed
Ω	Material domain
Ω_0	Initial material domain
Ω_0^e	Initial material domain corresponding to the finite element e
$\overset{\circ}{\Omega}$	Interior of the material domain
$\overset{\circ}{\Omega}_0$	Interior of the initial material domain
$\partial\Omega$	Solid surface
$\partial\Omega_0$	Initial material domain surface
ψ	Scalar, vector or tensor magnitude
ψ_ε	Eulerian description of a given magnitude ψ
$\psi_\mathcal{L}$	Lagrangian description of a given magnitude ψ
δ_{ij}	Kronecker delta
H_{δ_u}	Subspace where the test functions are contained
$H_{\delta_u}^h$	Subspace where the test functions approximations are contained
H_ω	Subspace where the test functions are contained
H_u	Subspace where the displacement field is contained
H_u^h	Subspace where the displacement field approximation is contained

Deformation of a solid media subjected to external forces

1.1. Introduction

Mechanics is the science that studies the behaviour of solids and fluids subjected to external forces. Computational Mechanics is the discipline of Mechanics that deals with the use of methods and computational resources to characterize, simulate, and predict physical phenomena and engineering systems governed by the principles of Mechanics [Oden et al., 2003].

This work is focused on the analysis of solids subjected to external forces. The branch of Mechanics which focuses on this particular problem is the so-called Solid Mechanics. Once the equations that govern the solid behaviour are defined, numerical and computational methods have to be applied in order to obtain the displacement field experimented by the solid, as well as its corresponding strain and stress fields.

In this chapter, the study of the motion of a given particle that belongs to the initial configuration, and the definition of the geometric transformation experimented by a vector that links two close particles, are proposed. The resulting equations define the solid motion, as well as the change of volume, orientation, and shape experimented by the solid due to the application of the external forces. These are the most important equations to characterize the solid behaviour and establish the basic principles of the Lagrangian approach.

A novel, clear and simple nomenclature is defined. Furthermore, the strictly necessary magnitudes and equations are proposed in order to reach a nonlinear solid mechanics formulation complete and detailed that describes the deformation process.

1.2. Magnitude description

There are two different options to define magnitudes, depending on the variables adopted to define them.

On the one hand, they can be defined depending on the reference domain. This is the so-called Lagrangian description, and it is the description usually adopted in Solid Mechanics. As the initial configuration is well-known, it is quite straightforward to refer all magnitudes to the initial configuration.

On the other hand, the deformed configuration can be adopted as the reference one and magnitudes can be defined depending on this domain. This is an Eulerian point of view, and this description is usually adopted in Fluid Dynamics. This approach is not convenient in Solid Mechanics, as the deformed domain an unknown.

1.2.1. Lagrangian description

Let's consider a specific magnitude ψ , that can be a scalar, vector, or tensor magnitude. As mentioned before, the Lagrangian description adopts the initial configuration as the reference one. Thus, the magnitudes are defined depending on the initial position vector of each particle (figure 1.1). If the problem is a dynamic one, the magnitudes also depend on time.

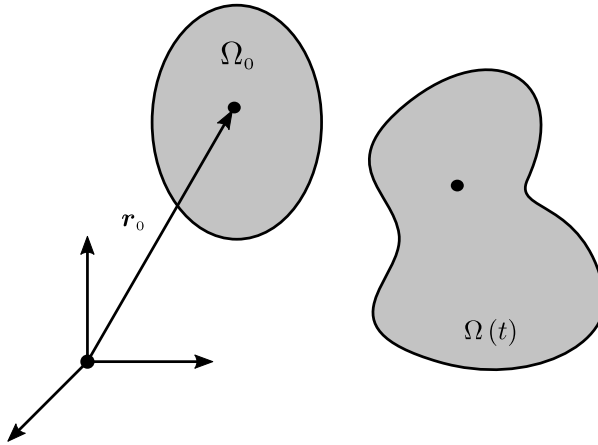


Figure 1.1. Initial position vector of a given particle.

Therefore, the Lagrangian description of a generic magnitude turns out to be:

$$\psi \longrightarrow \psi_{\mathcal{L}}(\mathbf{r}_0, t) \quad (1.1)$$

For instance, let's consider that the studied magnitude is the displacement field. Its Lagrangian description defines the displacement vector that experiment each one of the particles whose initial position is defined by the vector \mathbf{r}_0 , at a specific instant of time t .

$$\mathbf{u} = \mathbf{u}_{\mathcal{L}}(\mathbf{r}_0, t) \quad (1.2)$$

This is the description usually adopted in Solid Mechanics, since the initial domain is known. And the main goal is to compute the displacement, strain and stress fields depending on this reference domain.

1.2.2. Eulerian description

The second option is to describe magnitudes with respect to the deformed configuration. That is, they are defined depending on the vector that defines the position of a particle at a given instant of time t (figure 1.2). Moreover, they also depend on time, if a dynamic problem is faced.

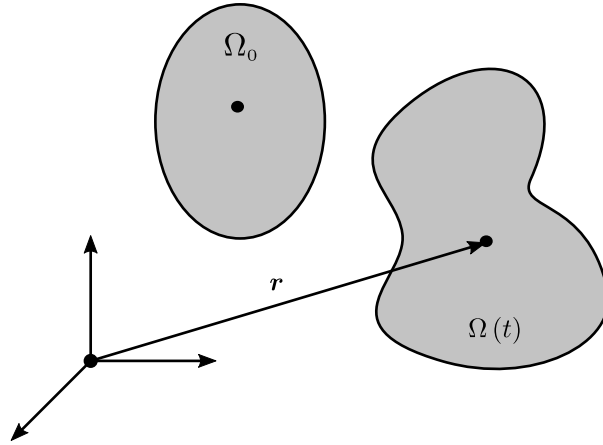


Figure 1.2. Position vector of a given particle at a specific instant of time.

Thus, a given magnitude can be defined according to its Eulerian description as:

$$\psi \longrightarrow \psi_{\varepsilon}(\mathbf{r}, t) \quad (1.3)$$

1.3. Deformation vector

The vector that defines the position of a particle at a specific instant of time, whose initial position is defined by the vector \mathbf{r}_0 , can be defined by the following Lagrangian function:

$$\mathbf{r} = \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) \quad (1.4)$$

The previous vector is the so-called deformation vector. It can be decomposed as the sum of the initial position vector, plus the displacement vector that the particle experiments at time t (figure 1.3). The displacement vector can also be interpreted as the relative position of a given particle with respect to its reference position.

$$\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) = \mathbf{r}_0 + \mathbf{u}_{\mathcal{L}}(\mathbf{r}_0, t) \quad (1.5)$$

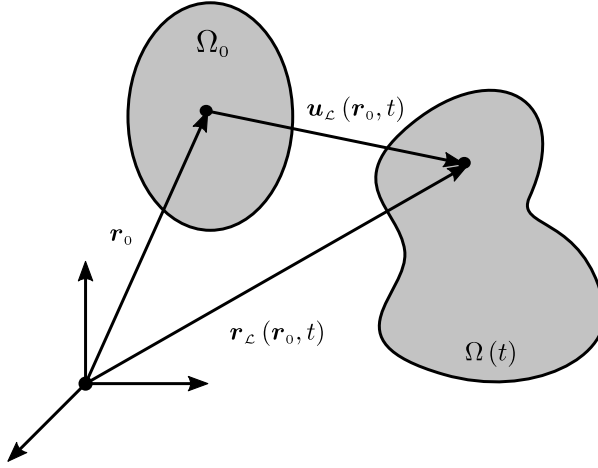


Figure 1.3. Initial position, displacement, and deformation vector of a given particle at a specific instant of time.

The domain occupied by a body at a given instant of time is known as the material domain. Symbolically, it can be mathematically expressed as the result of applying the Lagrangian function defined in (1.4) to the whole reference domain.

$$\Omega(t) = \mathbf{r}_\mathcal{L}(\Omega_0, t) \quad (1.6)$$

1.4. Velocity vector

The time derivative of the deformation vector (1.4) turns out to be the velocity vector of the particle that was located at the position defined by \mathbf{r}_0 at time $t = 0$.

$$\mathbf{a}_\mathcal{L}(\mathbf{r}_0, t) = \frac{\partial \mathbf{r}_\mathcal{L}}{\partial t} \quad (1.7)$$

Furthermore, if the definition of the deformation vector (1.5) is taken into account, the velocity vector becomes the time derivative of the displacement vector, since the initial position vector does not depend on time.

$$\mathbf{a}_\mathcal{L}(\mathbf{r}_0, t) = \frac{\partial}{\partial t}(\mathbf{r}_0 + \mathbf{u}_\mathcal{L}) = \frac{\partial \mathbf{u}_\mathcal{L}}{\partial t} \quad (1.8)$$

1.5. Deformation gradient tensor

As stated in section 1.3, the deformation vector depends on the initial position vector of a given particle, as well as on time. Hence, it is possible to compute the derivative of the deformation vector with respect to the initial position vector. This derivative is the so-called deformation gradient tensor.

$$\mathbf{F}_\mathcal{L} = \frac{\partial \mathbf{r}_\mathcal{L}}{\partial \mathbf{r}_0} \quad (1.9)$$

If the definition of the deformation vector (1.5) is taken into account, the deformation gradient tensor turns out to be the sum of two tensors: the unit tensor and the derivative of the displacement vector with respect to the reference position vector.

$$\mathbf{F}_\mathcal{L} = \frac{\partial}{\partial \mathbf{r}_0} (\mathbf{r}_0 + \mathbf{u}_\mathcal{L}) = \mathbf{I} + \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \quad (1.10)$$

1.6. Displacement gradient tensor

The second term that composes the deformation gradient tensor (1.10) is the so-called displacement gradient tensor. As it will be proved later on, this tensor contains the required information to properly define the displacement, strain, and stress fields of a continuous solid media subjected to external forces.

$$\mathbf{J}_\mathcal{L} = \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \quad (1.11)$$

Consequently, the deformation gradient tensor (1.10), written by means of the displacement gradient tensor, becomes:

$$\mathbf{F}_\mathcal{L} = \mathbf{I} + \mathbf{J}_\mathcal{L} \quad (1.12)$$

1.7. Time derivative of the deformation gradient tensor

As the deformation gradient tensor (1.9) depends on time, as well as on the reference domain, its time derivative can be computed.

$$\frac{\partial \mathbf{F}_\mathcal{L}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_\mathcal{L}}{\partial \mathbf{r}_0} \right) \quad (1.13)$$

The order of the derivatives can be exchanged, which allows the definition of the time derivative by means of the velocity vector (1.7).

$$\frac{\partial \mathbf{F}_\mathcal{L}}{\partial t} = \frac{\partial}{\partial \mathbf{r}_0} \left(\frac{\partial \mathbf{r}_\mathcal{L}}{\partial t} \right) = \frac{\partial \mathbf{a}_\mathcal{L}}{\partial \mathbf{r}_0} \quad (1.14)$$

Therefore, it can be concluded that the time derivative of the deformation gradient tensor is equivalent to the velocity gradient tensor.

1.8. Lagrangian and Eulerian descriptions equivalence

If necessary, the Eulerian description of a given magnitude (1.3) can be switched to its equivalent Lagrangian one (1.1). The equivalence between both descriptions is obtained by replacing the deformation vector (1.4) into the Eulerian description of the corresponding magnitude.

$$\psi_\mathcal{L}(\mathbf{r}_0, t) = \psi_\mathcal{E}(\mathbf{r}, t) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0, t)} \quad (1.15)$$

1.9. Magnitude derivatives

Once the equivalence between both descriptions (1.15) is outlined, the derivatives of a given magnitude defined according to its Lagrangian description can be computed. These derivatives are properly defined by applying the chain rule, as shown in the following subsections.

1.9.1. Time derivative

If the equivalence between both descriptions (1.15) is taken into account, and the chain rule is applied, the time derivative of a given magnitude defined according to its Lagrangian description can be computed as:

$$\frac{\partial \psi_{\mathcal{L}}}{\partial t} = \frac{\partial \psi_{\varepsilon}}{\partial t} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} + \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial t} \quad (1.16)$$

In the above equation, the velocity vector (1.7) appears. Thus, it can be rewritten as:

$$\frac{\partial \psi_{\mathcal{L}}}{\partial t} = \frac{\partial \psi_{\varepsilon}}{\partial t} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} + \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{a}_{\mathcal{L}} \quad (1.17)$$

According to equation (1.15), the Lagrangian description of the velocity vector can be expressed by means of its Eulerian description. Consequently, the above derivative becomes:

$$\frac{\partial \psi_{\mathcal{L}}}{\partial t} = \left(\frac{\partial \psi_{\varepsilon}}{\partial t} + \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \mathbf{a}_{\varepsilon} \right) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \quad (1.18)$$

1.9.2. Derivative with respect to the reference position vector

The derivative of a given magnitude, defined according to its Lagrangian description, can also be computed with respect to the reference position vector.

$$\frac{\partial \psi_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0} \iff \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} = \frac{\partial \psi_{\mathcal{L}}}{\partial \mathbf{r}_0} \left[\frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right]^{-1} \quad (1.19)$$

The deformation gradient tensor (1.9) appears in the above equation. Hence, it can be rewritten as:

$$\frac{\partial \psi_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \iff \frac{\partial \psi_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} = \frac{\partial \psi_{\mathcal{L}}}{\partial \mathbf{r}_0} \mathbf{F}_{\mathcal{L}}^{-1} \quad (1.20)$$

As it will be explained later on, the deformation gradient tensor has to be invertible by definition. Thus, the inverse that appears in (1.20) exists and can be computed.

1.10. Relative deformation vector

The relative deformation vector is the vector that links two material points that belong to the deformed domain at a specific instant of time (figure 1.4). This vector can be obtained as the difference between the two deformation vectors that define the position of both particles in the deformed configuration.

$$\delta \mathbf{r} = \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0 + \delta \mathbf{r}_0, t) - \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) \quad (1.21)$$

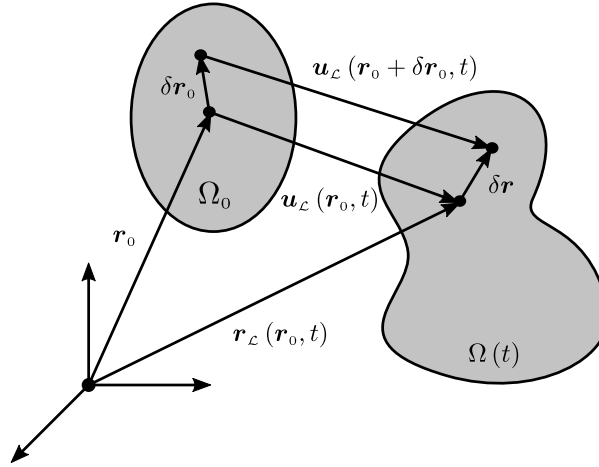


Figure 1.4. Relative deformation vector of a given particle at a specific instant of time.

The main aim is to obtain an equation that defines the relative deformation vector depending on the vector that links these two particles in the reference configuration. This objective is attained, if the first term of equation (1.21) is replaced by the following Taylor series expansion about the vector \mathbf{r}_0 .

$$\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0 + \delta \mathbf{r}_0, t) = \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) + \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0}(\mathbf{r}_0, t) \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (1.22)$$

The substitution of the above series expansion into (1.21) leads to the equation that defines the relative deformation vector over time. This equation defines the geometric transformation that suffers a material vector defined in the reference configuration along the deformation process.

$$\begin{aligned} \delta \mathbf{r} &= \mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \\ \mathbf{F}_{\mathcal{L}} &= \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{I} + \mathbf{J}_{\mathcal{L}} \\ \mathbf{J}_{\mathcal{L}} &= \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \end{aligned} \quad (1.23)$$

From the result obtained in (1.23), it can be concluded that the displacement gradient tensor (1.11) is the one that contains the information related to the change of

volume, orientation, and shape of the solid. This information is required to properly define the displacement, strain, and stress fields of a continuous solid media subjected to external forces.

1.11. Determinant of the deformation gradient tensor

As proved in (1.23), the deformation gradient tensor rules the geometric variation of a material vector.

$$\delta \mathbf{r} = \mathbf{F}_\mathcal{L} \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (1.24)$$

Let's assume that the two material points linked by the material vector are initially very close. This assumption implies that the norm of the original material vector is much smaller than one, and the last term of the above equation can be neglected. Thus, the material vector at a given instant of time can be approximately computed as:

$$\|\delta \mathbf{r}_0\| \ll 1 \implies \delta \mathbf{r} \approx \mathbf{F}_\mathcal{L} \delta \mathbf{r}_0 \quad (1.25)$$

It is also required that the determinant of the deformation gradient tensor is non-equal to zero, so the inverse of the deformation gradient tensor exists and can be computed.

$$F_\mathcal{L} = \det(\mathbf{F}_\mathcal{L}) \neq 0 \implies \exists \mathbf{F}_\mathcal{L}^{-1} \quad (1.26)$$

If the above condition holds, equation (1.25) can be inverted, and the equation that defines the original material vector by means of the deformed one is obtained.

$$\delta \mathbf{r}_0 \approx \mathbf{F}_\mathcal{L}^{-1} \delta \mathbf{r} \quad (1.27)$$

Furthermore, the determinant of the deformation gradient tensor also allows to obtain the evolution of a differential volume over time as:

$$d\Omega = F_\mathcal{L} d\Omega_0 \quad (1.28)$$

The above equation will be demonstrated in the next section. According to (1.28), the determinant of the deformation gradient tensor can be interpreted as the proportion that represents a specific differential volume at a given instant of time, with respect to the original one.

$$F_\mathcal{L} = \frac{d\Omega}{d\Omega_0} \quad (1.29)$$

Since the equation (1.29) only involves volumes, which are positive magnitudes, it can be concluded that the determinant of the deformation gradient tensor has to be greater than zero.

$$F_\mathcal{L} = \det(\mathbf{F}_\mathcal{L}) > 0 \quad (1.30)$$

1.12. Volume variation

Let's consider a material vector that links two material points that belong to the initial reference configuration.

$$\delta \mathbf{r}_0 = \begin{Bmatrix} \delta r_{0,1} \\ \delta r_{0,2} \\ \delta r_{0,3} \end{Bmatrix} \quad (1.31)$$

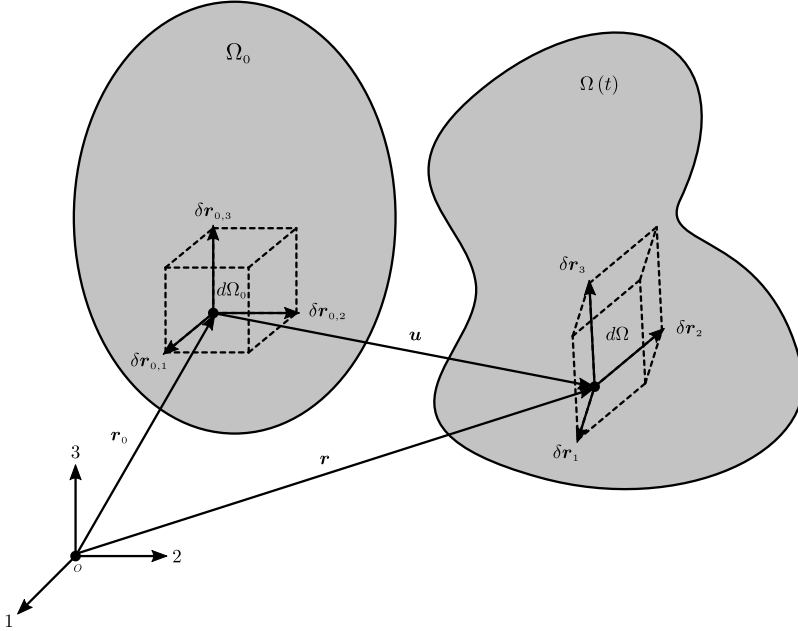


Figure 1.5. Differential volume variation.

On the one hand, the components of the above vector define the following material vectors, that are oriented according to the directions defined by the coordinate axes.

$$\begin{aligned} \delta \mathbf{r}_{0,1} &= \delta r_{0,1} \mathbf{e}_1 = \delta r_{0,1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta r_{0,1} \\ 0 \\ 0 \end{Bmatrix} \\ \delta \mathbf{r}_{0,2} &= \delta r_{0,2} \mathbf{e}_2 = \delta r_{0,2} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta r_{0,2} \\ 0 \end{Bmatrix} \\ \delta \mathbf{r}_{0,3} &= \delta r_{0,3} \mathbf{e}_3 = \delta r_{0,3} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \delta r_{0,3} \end{Bmatrix} \end{aligned} \quad (1.32)$$

The volume of the parallelepiped composed by the above material vectors is:

$$d\Omega_0 = \delta r_{0,1} \delta r_{0,2} \delta r_{0,3} \quad (1.33)$$

On the other hand, these vectors can be defined on the deformed domain. The deformation gradient tensor is the one that transforms the initial material vectors to the deformed configuration.

$$\delta \mathbf{r}_i = \mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_{0,i} \quad i = 1, 2, 3 \quad (1.34)$$

And the components of the deformation gradient tensor are:

$$\mathbf{F}_{\mathcal{L}} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \quad (1.35)$$

Furthermore, the computation of the triple product (A.1.7) between the deformed material vectors leads to the volume of the parallelepiped that they compose.

$$\begin{aligned} d\Omega &= (\delta \mathbf{r}_1 \wedge \delta \mathbf{r}_2) \cdot \delta \mathbf{r}_3 \\ &= \left[(\mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_{0,1}) \wedge (\mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_{0,2}) \right] \cdot (\mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_{0,3}) \\ &= \left[\left(\mathbf{F}_{\mathcal{L}} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \wedge \mathbf{F}_{\mathcal{L}} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \right) \cdot \mathbf{F}_{\mathcal{L}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \right] (\delta r_{0,1} \delta r_{0,2} \delta r_{0,3}) \\ &= \left[\left(\begin{Bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{Bmatrix} \wedge \begin{Bmatrix} F_{12} \\ F_{22} \\ F_{32} \end{Bmatrix} \right) \cdot \begin{Bmatrix} F_{13} \\ F_{23} \\ F_{33} \end{Bmatrix} \right] d\Omega_0 \\ &= \det \left(\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \right) d\Omega_0 \\ &= \det(\mathbf{F}_{\mathcal{L}}) d\Omega_0 \\ &= F_{\mathcal{L}} d\Omega_0 \end{aligned} \quad (1.36)$$

According to the above result, it can be concluded that the determinant of the deformation gradient tensor defines the proportion that represents the final differential volume with respect to the initial one.

$$d\Omega = F_{\mathcal{L}} d\Omega_0 \quad \iff \quad F_{\mathcal{L}} = \frac{d\Omega}{d\Omega_0} \quad (1.37)$$

1.12.1. Novel derivation

An alternative derivation of the above equation can be proposed if the result obtained in appendix D is recalled. In this appendix, a hyper-parallelepiped was defined by a set of vectors that compose a given basis. As these vectors make up a basis, the definition of its corresponding metric tensor can be carried out, as well as the computation of its determinant. Under these assumptions, it has been proven that the

square root of this determinant turns out to be equivalent to the hyper-volume of the hyper-parallelepiped.

In this subsection, let's consider a n -dimensional space where a basis composed by n linearly independent vectors is defined. The vectors that compose this primal basis, and the tensor \mathbf{E} that contains these vectors arranged in columns, are:

$$\{\vec{e}_i\}_{i=1,\dots,n} \quad \longrightarrow \quad \mathbf{E} = [\vec{e}_1 \quad \dots \quad \vec{e}_n] \quad (1.38)$$

Let's also consider two different material vectors that belong to the initial material domain. Their definition with respect to the previous basis is:

$$\left. \begin{array}{l} \delta\vec{r}_0 = \vec{e}_i \delta r_0^i \\ \delta\vec{s}_0 = \vec{e}_i \delta s_0^i \end{array} \right\} \iff \left\{ \begin{array}{ll} \delta\vec{r}_0 = \mathbf{E} \delta\mathbf{r}_0 & \text{with} \quad \delta\mathbf{r}_0 = \{\delta r_0^i\}_{i=1,\dots,n} \\ \delta\vec{s}_0 = \mathbf{E} \delta\mathbf{s}_0 & \text{with} \quad \delta\mathbf{s}_0 = \{\delta s_0^i\}_{i=1,\dots,n} \end{array} \right. \quad (1.39)$$

Where δr_0^i and δs_0^i are their contravariant components.

On the one hand, the scalar product between the above material vectors turns out to be defined by means of the metric tensor \mathbf{G} as:

$$\begin{aligned} \delta\vec{r}_0 \cdot \delta\vec{s}_0 &= \delta\vec{r}_0^T \delta\vec{s}_0 \\ &= (\mathbf{E} \delta\mathbf{r}_0)^T (\mathbf{E} \delta\mathbf{s}_0) \\ &= \delta\mathbf{r}_0^T \underbrace{(\mathbf{E}^T \mathbf{E})}_{\mathbf{G}} \delta\mathbf{s}_0 \\ &= \delta\mathbf{r}_0^T \mathbf{G} \delta\mathbf{s}_0 \end{aligned} \quad (1.40)$$

If the vectors that define the primal basis are considered to compose an infinitesimal parallelepiped embedded into the reference material domain, its corresponding differential volume $d\Omega_0$ can be computed by means of the previous metric tensor. According to the result obtained in appendix D, this volume can be defined as:

$$d\Omega_0 = \sqrt{\det(\mathbf{G})} \quad \text{with} \quad \mathbf{G} = \mathbf{E}^T \mathbf{E} \quad (1.41)$$

On the other hand, the geometric transformation experimented by the material vectors (1.39) is ruled by the deformation gradient tensor $\mathbf{F}_{\mathcal{L}}$. Thus, their definition at a given instant of time turn out to be:

$$\left. \begin{array}{l} \delta\vec{r} = \vec{e}_i \delta r^i \\ \delta r^i = F_j^i \delta r_0^j \end{array} \right\} \iff \left\{ \begin{array}{l} \delta\vec{r}_0 = \mathbf{E} \delta\mathbf{r} \\ \delta\mathbf{r} = \mathbf{F}_{\mathcal{L}} \delta\mathbf{r}_0 = \{\delta r^i\}_{i=1,\dots,n} \end{array} \right. \quad (1.42)$$

$$\left. \begin{array}{l} \delta\vec{s} = \vec{e}_i \delta s^i \\ \delta s^i = F_j^i \delta s_0^j \end{array} \right\} \iff \left\{ \begin{array}{l} \delta\vec{s}_0 = \mathbf{E} \delta\mathbf{s} \\ \delta\mathbf{s} = \mathbf{F}_{\mathcal{L}} \delta\mathbf{s}_0 = \{\delta s^i\}_{i=1,\dots,n} \end{array} \right.$$

Where F_j^i are the contravariant-covariant components of the deformation gradient tensor, defined with respect to the primal basis $\{\vec{e}_i\}_{i=1,\dots,n}$ and its corresponding dual one $\{\vec{e}^j\}_{j=1,\dots,n}$.

$$\mathbf{F}_{\mathcal{L}} \underset{\approx}{=} (\vec{e}_i \otimes \vec{e}^j) F_j^i \quad \implies \quad \mathbf{F}_{\mathcal{L}} = [F_j^i]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (1.43)$$

The deformed material vectors (1.42) can be rewritten as:

$$\begin{aligned}
 \left. \begin{aligned}
 \delta \vec{r} &= \vec{e}_i (F_j^i \delta r_0^j) \\
 &= (\vec{e}_i F_j^i) \delta r_0^j \\
 &= \vec{e}'_j \delta r_0^j
 \end{aligned} \right\} &\iff \left\{ \begin{aligned}
 \delta \vec{r}_0 &= \mathbf{E} (\mathbf{F}_\mathcal{L} \delta \mathbf{r}_0) \\
 &= (\mathbf{E} \mathbf{F}_\mathcal{L}) \delta \mathbf{r}_0 \\
 &= \mathbf{E}' \delta \mathbf{r}_0
 \end{aligned} \right. \\
 \left. \begin{aligned}
 \delta \vec{s} &= \vec{e}_i (F_j^i \delta s_0^j) \\
 &= (\vec{e}_i F_j^i) \delta s_0^j \\
 &= \vec{e}'_j \delta s_0^j
 \end{aligned} \right\} &\iff \left\{ \begin{aligned}
 \delta \vec{s}_0 &= \mathbf{E} (\mathbf{F}_\mathcal{L} \delta \mathbf{s}_0) \\
 &= (\mathbf{E} \mathbf{F}_\mathcal{L}) \delta \mathbf{s}_0 \\
 &= \mathbf{E}' \delta \mathbf{s}_0
 \end{aligned} \right.
 \end{aligned} \tag{1.44}$$

According to the previous definitions, the deformed material vectors can be expressed in terms of the components of their undeformed definition stated in (1.39). To obtain so, the following basis has to be taken as the reference one:

$$\vec{e}'_j = \vec{e}_i F_j^i \iff \mathbf{E}' = \mathbf{E} \mathbf{F}_\mathcal{L} \tag{1.45}$$

And the scalar product between the deformed material vectors is then defined by means of the metric tensor corresponding to the previous basis.

$$\begin{aligned}
 \delta \vec{r} \cdot \delta \vec{s} &= \delta \vec{r}^T \delta \vec{s} \\
 &= (\mathbf{E}' \delta \mathbf{r}_0)^T (\mathbf{E}' \delta \mathbf{s}_0) \\
 &= \delta \mathbf{r}_0^T \underbrace{((\mathbf{E}')^T \mathbf{E}')}_{\mathbf{G}'} \delta \mathbf{s}_0 \\
 &= \delta \mathbf{r}_0^T \mathbf{G}' \delta \mathbf{s}_0
 \end{aligned} \tag{1.46}$$

The metric tensor \mathbf{G}' allows to calculate the volume of the parallelepiped after the deformation process. Its corresponding volume becomes $d\Omega$ and can be computed as:

$$\left. \begin{aligned}
 \mathbf{G}' &= (\mathbf{E}')^T \mathbf{E}' \\
 &= (\mathbf{E} \mathbf{F}_\mathcal{L})^T (\mathbf{E} \mathbf{F}_\mathcal{L}) \\
 &= \mathbf{F}_\mathcal{L}^T \underbrace{(\mathbf{E}^T \mathbf{E})}_{\mathbf{G}} \mathbf{F}_\mathcal{L} \\
 &= \mathbf{F}_\mathcal{L}^T \mathbf{G} \mathbf{F}_\mathcal{L}
 \end{aligned} \right\} \implies d\Omega = \sqrt{\det(\mathbf{G}')} = \sqrt{\det(\mathbf{F}_\mathcal{L}^T \mathbf{G} \mathbf{F}_\mathcal{L})} \tag{1.47}$$

Note that, if an orthonormal reference basis is adopted, \mathbf{G} becomes the second order unit tensor \mathbf{I} , and the definition of \mathbf{G}' becomes equivalent to the right Cauchy-Green tensor $\mathbf{M}_\mathcal{L}$ presented in 4.18.

Finally, the initial differential volume (1.41) and its corresponding deformed configuration (1.47) lead to the definition of the volumetric expansion coefficient, as:

$$\frac{d\Omega}{d\Omega_0} = \sqrt{\frac{\det(\mathbf{F}_\mathcal{L}^T \mathbf{G} \mathbf{F}_\mathcal{L})}{\det(\mathbf{G})}} = \det(\mathbf{F}_\mathcal{L}) = F_\mathcal{L} \tag{1.48}$$

Therefore, it can be concluded that the determinant of the deformation gradient tensor can be interpreted as the proportion that represents the deformed differential volume with respect to its corresponding initial volume.

1.13. Area variation

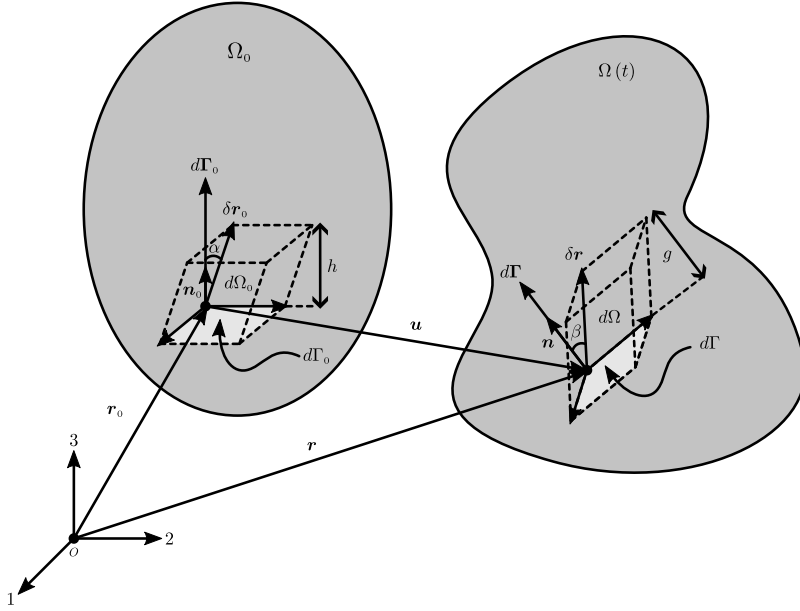


Figure 1.6. Differential area variation.

On the one hand, a vector that represents the initial differential area is defined. This vector depends on the normal unit vector, and its norm turns out to be equivalent to the initial area.

$$\left. \begin{array}{l} d\mathbf{\Gamma}_0 = d\Gamma_0 \mathbf{n}_0 \\ \|\mathbf{n}_0\| = 1 \end{array} \right\} \implies \|d\mathbf{\Gamma}_0\| = \|d\Gamma_0 \mathbf{n}_0\| = d\Gamma_0 \|\mathbf{n}_0\| = d\Gamma_0 \quad (1.49)$$

The scalar product between the above vector and a given material vector leads to the volume of the initial parallelepiped (figure 1.6).

$$\begin{aligned} \delta \mathbf{r}_0 \cdot d\mathbf{\Gamma}_0 &= \|\delta \mathbf{r}_0\| \|d\mathbf{\Gamma}_0\| \cos(\alpha) \\ &= d\Gamma_0 \underbrace{\|\delta \mathbf{r}_0\| \cos(\alpha)}_h = d\Gamma_0 \times h = d\Omega_0 \end{aligned} \quad (1.50)$$

On the other hand, the same magnitudes can be defined in the deformed configuration. Hence, the deformed area vector is:

$$\left. \begin{array}{l} d\mathbf{\Gamma} = d\Gamma \mathbf{n} \\ \|\mathbf{n}\| = 1 \end{array} \right\} \implies \|d\mathbf{\Gamma}\| = \|d\Gamma \mathbf{n}\| = d\Gamma \|\mathbf{n}\| = d\Gamma \quad (1.51)$$

And the scalar product between the deformed material vector and the deformed area vector defines the volume of the parallelepiped they compose.

$$\begin{aligned}\delta\mathbf{r} \cdot d\mathbf{\Gamma} &= \|\delta\mathbf{r}\| \|d\mathbf{\Gamma}\| \cos(\beta) \\ &= d\mathbf{\Gamma} \underbrace{\|\delta\mathbf{r}\| \cos(\beta)}_g = d\mathbf{\Gamma} \times g = d\Omega\end{aligned}\quad (1.52)$$

If the previous results are substituted into the equation that defines the determinant of the deformation gradient tensor (1.37), the determinant becomes:

$$F_{\mathcal{L}} = \frac{d\Omega}{d\Omega_0} = \frac{\delta\mathbf{r}^T d\mathbf{\Gamma}}{\delta\mathbf{r}_0^T d\mathbf{\Gamma}_0} \quad (1.53)$$

Moreover, the substitution of the equation that defines the variation of a material vector over time (1.23) into the above equation, leads to the following result.

$$\begin{aligned}F_{\mathcal{L}} (\delta\mathbf{r}_0^T d\mathbf{\Gamma}_0) &= \delta\mathbf{r}^T d\mathbf{\Gamma} \\ &= (\mathbf{F}_{\mathcal{L}} \delta\mathbf{r}_0)^T d\mathbf{\Gamma} \\ &= \delta\mathbf{r}_0^T \mathbf{F}_{\mathcal{L}}^T d\mathbf{\Gamma} \implies F_{\mathcal{L}} d\mathbf{\Gamma}_0 = \mathbf{F}_{\mathcal{L}}^T d\mathbf{\Gamma}\end{aligned}\quad (1.54)$$

According to the above result, the equation that defines the variation of a differential area over time turns out to be:

$$d\mathbf{\Gamma} = F_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} d\mathbf{\Gamma}_0 \quad (1.55)$$

1.14. Change of the integration domain

The determinant of the deformation gradient tensor ($F_{\mathcal{L}}$) states the relationship between a differential volume at a given instant of time ($d\Omega$) and the initial one ($d\Omega_0$).

If equation (1.28) is taken into account, the differential volume at a given instant of time can be defined depending on the primal one. This equation allows to rewrite an integral expressed over the unknown deformed domain, as an integral defined over the initial configuration. This is a useful mathematical tool, since it allows integrating over a domain that is well-known. In fact, this is the basis of the Lagrangian finite element formulation.

Let's consider the following integral expression defined over the deformed domain.

$$C = \iiint_{\Omega(t)} \psi_{\varepsilon}(\mathbf{r}, t) d\Omega \quad (1.56)$$

To transform this integral and get an equivalent one defined over the reference domain, the change of variable (1.4) has to be applied to the integrand, and the differential volume has to be substituted by the equation defined in (1.28).

$$C = \iiint_{\Omega_0} \psi_{\varepsilon}(\mathbf{r}, t) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t)} F_{\mathcal{L}} d\Omega_0 = \iiint_{\Omega_0} \psi_{\mathcal{L}}(\mathbf{r}_0, t) F_{\mathcal{L}} d\Omega_0 \quad (1.57)$$

1.15. Time derivative of the determinant of the deformation gradient tensor

In this section, the definition of the time derivative of the deformation gradient tensor is stated. This result will be needed later on in this work.

The expression of the derivative of a determinant stated in (A.111) is recalled, such that:

$$\frac{\partial F_{\mathcal{L}}}{\partial t} = F_{\mathcal{L}} \operatorname{Tr} \left(\frac{\partial \mathbf{F}_{\mathcal{L}}}{\partial t} \mathbf{F}_{\mathcal{L}}^{-1} \right) \quad (1.58)$$

In the above equation, the time derivative of the deformation gradient tensor appears. This derivative was already obtained in (1.14). Hence, the substitution of this result into (1.58) leads to the following result:

$$\frac{\partial F_{\mathcal{L}}}{\partial t} = F_{\mathcal{L}} \operatorname{Tr} \left(\frac{\partial \mathbf{a}_{\mathcal{L}}}{\partial \mathbf{r}_0} \mathbf{F}_{\mathcal{L}}^{-1} \right) \quad (1.59)$$

According to the result obtained in (1.20), the matrix whose trace is computed in (1.59) can be replaced by the following term.

$$\frac{1}{F_{\mathcal{L}}} \frac{\partial F_{\mathcal{L}}}{\partial t} = \operatorname{Tr} \left(\frac{\partial \mathbf{a}_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \right) \quad (1.60)$$

Moreover, the trace of the velocity gradient tensor turns out to be equivalent to the divergence, as demonstrated in (B.21). Thus, the time derivative of the determinant can be finally computed as:

$$\frac{1}{F_{\mathcal{L}}} \frac{\partial F_{\mathcal{L}}}{\partial t} = \operatorname{div}(\mathbf{a}_{\varepsilon}) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \quad (1.61)$$

1.16. Overview and conclusions

In this chapter, the motion of a continuous solid media subjected to external forces is studied. As stated before, the Lagrangian approach is adopted. That is, all magnitudes are defined with respect to the initial reference configuration, which is well-known.

Two main equations were obtained: the equation that defines the motion of a given material particle and the equation that describes the geometric transformation of a specific material vector over time. These equations are important since they describe the solid motion, as well as the change of volume, orientation, and shape that the solid experiments due to the external applied forces.

On the one hand, the deformation vector ($\mathbf{r}_{\mathcal{L}}$) is the one that describes the solid motion. This vector describes the position of a material particle over time by means of its initial position (\mathbf{r}_0) and the displacement vector ($\mathbf{u}_{\mathcal{L}}$).

On the other hand, the geometric transformation experimented by a material vector turns out to depend on the deformation gradient tensor ($\mathbf{F}_{\mathcal{L}}$), which, in turn, can be defined by means of the displacement gradient tensor ($\mathbf{J}_{\mathcal{L}}$). According to this result,

it can be concluded that the displacement gradient tensor is the one that contains the required information to define the change of volume, orientation, and shape experienced by the solid. This information is required to properly define the displacement field that the solid undergoes, as well as its corresponding strain and stress fields.

Another main aim of this chapter was to define a novel, simple, and clear nomenclature, as well as to manipulate the strictly necessary magnitudes and equations to describe the deformation process. The final goal is to reach a nonlinear solid mechanics formulation complete and detailed, so a novel subsequent derivation of the nonlinear finite element formulation can be performed.

Balance equations

2.1. Introduction

To solve a Solid Mechanics problem, the first step is to derive the equations that govern the dynamic equilibrium, as well as the equation that rules the value of the media density over time.

On the one hand, the linear momentum and the angular momentum balance equations govern the dynamic equilibrium of forces and torques, respectively. Consequently, they compose the equations that state the dynamic equilibrium of a continuous solid media subjected to external forces. Both equations have to be fulfilled to properly represent the structural behaviour of a solid at equilibrium.

On the other hand, the mass balance equation is the one that governs the value of the density field along the deformation process. As a first approach, the assumption that there are mass sources is adopted. However, dealing with this type of sources is not usual in Solid Mechanics. In further chapters, the mass sources will be neglected, so the hypothesis of mass conservation will be adopted.

2.2. Mass balance

Let's consider that the equation that defines the time variation of a given magnitude M is studied. In this section, the magnitude is the mass. The density of the media is defined by the variable ρ , and the variable φ represents a mass source per unit volume per unit time.

Both the density and the mass source can be expressed according to their Lagrangian (section 1.2.1) or Eulerian descriptions (section 1.2.2).

	Lagrangian	Eulerian
Density	$\rho_{\mathcal{L}}(\mathbf{r}_0, t)$	$\rho_{\mathcal{E}}(\mathbf{r}, t)$
Mass source	$\varphi_{\mathcal{L}}(\mathbf{r}_0, t)$	$\varphi_{\mathcal{E}}(\mathbf{r}, t)$

Table 2.1. Lagrangian and Eulerian descriptions of the density and mass source.

2.2.1. Mass time variation

The mass contained in a particular differential volume of a given deformed solid domain can be computed by means of the media density as:

$$dM = \rho_{\mathcal{E}} d\Omega \quad (2.1)$$

The mass of the whole material domain can be obtained as the sum of the mass contained in all the differential volumes that compose the domain. That is, the mass defined in (2.1) has to be integrated over the whole domain.

$$M = \iiint_{\Omega(t)} \rho_{\mathcal{E}} d\Omega \quad (2.2)$$

This integral is defined over an unknown domain. However, it can be equivalently defined over the initial configuration, which is well-known. To do so, the methodology for changing the integration domain presented in section 1.14 is applied.

$$M = \iiint_{\Omega(t)} \rho_{\mathcal{E}} d\Omega = \iiint_{\Omega_0} \rho_{\mathcal{L}} F_{\mathcal{L}} d\Omega_0 \quad (2.3)$$

On the other hand, the mass time variation depends on the mass source per unit volume per unit time. So, the time variation of the mass contained in the differential volume calculated in (2.1) turns out to depend on the mass source and can be expressed as:

$$\frac{\partial (dM)}{\partial t} = \varphi_{\mathcal{E}} d\Omega \quad (2.4)$$

If the above variation is integrated over the entire domain, the variation of the mass contained in the whole material domain is obtained.

$$\frac{\partial M}{\partial t} = \iiint_{\Omega(t)} \varphi_{\mathcal{E}} d\Omega \quad (2.5)$$

Finally, the integration domain can be changed from the unknown deformed one to the initial configuration. The methodology stated in section (1.14) is again applied. So,

$$\frac{\partial M}{\partial t} = \iiint_{\Omega(t)} \varphi_{\mathcal{E}} d\Omega = \iiint_{\Omega_0} \varphi_{\mathcal{L}} F_{\mathcal{L}} d\Omega_0 \quad (2.6)$$

The above mass time variation allows to obtain the Lagrangian integral form of the mass balance equation.

2.2.2. Lagrangian integral form of the mass balance equation

The Lagrangian integral form of the equation that governs the mass time variation can be obtained by combining equations (2.3) and (2.6).

$$\frac{\partial}{\partial t} \left(\iiint_{\Omega_0} \rho_{\mathcal{L}} F_{\mathcal{L}} d\Omega_0 \right) = \iiint_{\Omega_0} \varphi_{\mathcal{L}} F_{\mathcal{L}} d\Omega_0 \quad (2.7)$$

From this Lagrangian integral form, the obtention of an equivalent differential form is straightforward, if the derivative is taken inside the integral and then the localization theorem is applied.

2.2.3. Lagrangian differential form of the mass balance equation

The differential form of the equation that rules the mass time variation in a given material domain can be obtained from the previous integral form (2.7). As the reference domain does not depend on time, the time derivative can be taken inside the integral.

$$\iiint_{\Omega_0} \left[\frac{\partial}{\partial t} (\rho_{\mathcal{L}} F_{\mathcal{L}}) - \varphi_{\mathcal{L}} F_{\mathcal{L}} \right] d\Omega_0 = 0 \quad (2.8)$$

And the application of the localization theorem (section C.2.1) leads to its corresponding differential form.

$$\frac{\partial}{\partial t} (\rho_{\mathcal{L}} F_{\mathcal{L}}) - \varphi_{\mathcal{L}} F_{\mathcal{L}} = 0 \quad \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0, \forall t \quad (2.9)$$

The above equation can be alternatively rewritten as follows:

$$\frac{\partial \rho_{\mathcal{L}}}{\partial t} + \rho_{\mathcal{L}} \frac{1}{F_{\mathcal{L}}} \frac{\partial F_{\mathcal{L}}}{\partial t} = \varphi_{\mathcal{L}} \quad \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0, \forall t \quad (2.10)$$

2.3. Linear momentum balance

In this section, the equation which defines the linear momentum time variation is derived.

Let's consider a solid media subjected to external forces. The density of the media is defined by the variable ρ , and the external forces per unit mass are represented by the vector \mathbf{b} . Moreover, the existence of a mass source per unit volume per unit time φ is considered, and the vector \mathbf{v} represents its corresponding velocity. The resulting stress field is defined by the Cauchy stress tensor $\boldsymbol{\sigma}$ (section 3.3).

All these magnitudes can be expressed according to their Lagrangian (section 1.2.1) or Eulerian descriptions (section 1.2.2).

2.3.1. Linear momentum time variation

The product between mass and velocity defines the linear momentum. So, the linear momentum corresponding to a given differential volume that belongs to a material

	Lagrangian	Eulerian
Density	$\rho_{\mathcal{L}}(\mathbf{r}_0, t)$	$\rho_{\mathcal{E}}(\mathbf{r}, t)$
External volumetric force	$\mathbf{b}_{\mathcal{L}}(\mathbf{r}_0, t)$	$\mathbf{b}_{\mathcal{E}}(\mathbf{r}, t)$
Mass source	$\varphi_{\mathcal{L}}(\mathbf{r}_0, t)$	$\varphi_{\mathcal{E}}(\mathbf{r}, t)$
Mass source velocity	$\mathbf{v}_{\mathcal{L}}(\mathbf{r}_0, t)$	$\mathbf{v}_{\mathcal{E}}(\mathbf{r}, t)$
Stress field	$\boldsymbol{\sigma}_{\mathcal{L}}(\mathbf{r}_0, t)$	$\boldsymbol{\sigma}_{\mathcal{E}}(\mathbf{r}, t)$

Table 2.2. Lagrangian and Eulerian descriptions of the density, external volumetric force, mass source, mass source velocity and stress field.

domain is:

$$d\mathbf{p} = \mathbf{a}_{\mathcal{E}} \rho_{\mathcal{E}} d\Omega \quad (2.11)$$

The integration of the above magnitude over the whole material domain leads to the linear momentum of the entire solid:

$$\mathbf{p} = \iiint_{\Omega(t)} \mathbf{a}_{\mathcal{E}} \rho_{\mathcal{E}} d\Omega \quad (2.12)$$

The above integral is defined over the deformed material domain, which is unknown. Nevertheless, it can be defined over the initial one if the methodology exposed in section (1.14) is applied. Consequently,

$$\mathbf{p} = \iiint_{\Omega(t)} \mathbf{a}_{\mathcal{E}} \rho_{\mathcal{E}} d\Omega = \iiint_{\Omega_0} \mathbf{a}_{\mathcal{L}} \rho_{\mathcal{L}} F_{\mathcal{L}} d\Omega_0 \quad (2.13)$$

According to Newton's second law, forces are considered as sources of linear momentum. Thus, the time variation of the linear momentum depends on the applied forces, as well as on the mass source that has an associated velocity field.

Besides the forces per unit volume, it is common to deal with forces per unit area applied on the solid surface. These forces are applied on the solid boundary and are represented by the stress vector. According to the result obtained in (3.34), the stress vector can be defined by means of the Cauchy stress tensor and the normal vector perpendicular to the domain surface as:

$$\mathbf{t}_{\mathcal{E}} = \boldsymbol{\sigma}_{\mathcal{E}}^T \mathbf{n} \quad (2.14)$$

Thus, if the effect of the applied forces and the mass source are taken into account, the time variation of the linear momentum turns out to be:

$$\frac{\partial \mathbf{p}}{\partial t} = \iiint_{\Omega} (\mathbf{b}_{\mathcal{E}} \rho_{\mathcal{E}} + \mathbf{v}_{\mathcal{E}} \varphi_{\mathcal{E}}) d\Omega + \iint_{\Gamma=\partial\Omega} \mathbf{t}_{\mathcal{E}} d\Gamma \quad (2.15)$$

The second term of the above equation can be expressed as an integral over the interior of the material domain. According to the divergence theorem stated in (B.44),

this term can be rewritten as:

$$\iint_{\Gamma=\partial\Omega} \mathbf{t}_\varepsilon d\Gamma = \iint_{\Gamma=\partial\Omega} \boldsymbol{\sigma}_\varepsilon^T \mathbf{n} d\Gamma = \iiint_{\Omega} \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) d\Omega \quad (2.16)$$

Hence, the time variation of the linear momentum (2.15) can be defined as the following integral over the material domain:

$$\frac{\partial \mathbf{p}}{\partial t} = \iiint_{\Omega(t)} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \right] d\Omega \quad (2.17)$$

Furthermore, the equivalent integral defined over the reference domain can be stated by taking into account the change of the integration domain exposed in section 1.14 as:

$$\frac{\partial \mathbf{p}}{\partial t} = \iiint_{\Omega_0} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \right] F_\varepsilon d\Omega_0 \quad (2.18)$$

The above time variation allows to derive the Lagrangian integral form of the linear momentum balance equation.

2.3.2. Lagrangian integral form of the linear momentum balance equation

The combination of equations (2.13) and (2.18) leads to the integral equation that defines the time variation of the linear momentum:

$$\frac{\partial}{\partial t} \left(\iiint_{\Omega_0} \mathbf{a}_\varepsilon \rho_\varepsilon F_\varepsilon d\Omega_0 \right) = \iiint_{\Omega_0} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \right] F_\varepsilon d\Omega_0 \quad (2.19)$$

As the reference domain does not depend on time, the time derivative can be taken inside the integral. Thus,

$$\iiint_{\Omega_0} \frac{\partial}{\partial t} (\mathbf{a}_\varepsilon \rho_\varepsilon F_\varepsilon) d\Omega_0 = \iiint_{\Omega_0} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \right] F_\varepsilon d\Omega_0 \quad (2.20)$$

And the application of the localization theorem to the above Lagrangian integral form leads to its equivalent differential form.

2.3.3. Lagrangian differential form of the linear momentum balance equation

The differential form of the equation that defines the time variation of the linear momentum can be obtained by applying the localization theorem (C.2.1) to the previous integral form (2.20).

$$\frac{\partial}{\partial t} (\mathbf{a}_\varepsilon \rho_\varepsilon F_\varepsilon) = \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \right] F_\varepsilon \quad \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0, \forall t \quad (2.21)$$

The above equation can be alternatively written as:

$$\frac{\partial}{\partial t}(\mathbf{a}_\mathcal{L} \rho_\mathcal{L}) + \mathbf{a}_\mathcal{L} \rho_\mathcal{L} \frac{1}{F_\mathcal{L}} \frac{\partial F_\mathcal{L}}{\partial t} = \mathbf{b}_\mathcal{L} \rho_\mathcal{L} + \mathbf{v}_\mathcal{L} \varphi_\mathcal{L} + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0,t)} \quad \forall \mathbf{r}_0 \in \mathring{\Omega}_0, \forall t \quad (2.22)$$

The above equation is composed by magnitudes defined according to their Lagrangian description. Nevertheless, as shown in the following subsection, it can be manipulated to obtain an equivalent form which involves magnitudes expressed according to their Eulerian description.

2.3.4. Eulerian differential form of the linear momentum balance equation

On the one hand, the left-hand side of the Lagrangian differential equation (2.22) can be rewritten by means of the equivalent Eulerian magnitudes. To do so, the time derivative of a Lagrangian magnitude can be replaced by its equivalent Eulerian expression obtained in (1.18). Moreover, the time derivative of the determinant of the deformation gradient tensor was calculated in (1.61).

$$\frac{\partial}{\partial t}(\mathbf{a}_\mathcal{L} \rho_\mathcal{L}) + \mathbf{a}_\mathcal{L} \rho_\mathcal{L} \frac{1}{F_\mathcal{L}} \frac{\partial F_\mathcal{L}}{\partial t} = \left[\frac{\partial}{\partial t}(\mathbf{a}_\varepsilon \rho_\varepsilon) + \frac{\partial}{\partial \mathbf{r}}(\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{div}(\mathbf{a}_\varepsilon) \right] \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0,t)} \quad (2.23)$$

On the other hand, its right-hand side can be rewritten if the equivalence between both descriptions (1.15) is applied.

$$\mathbf{b}_\mathcal{L} \rho_\mathcal{L} + \mathbf{v}_\mathcal{L} \varphi_\mathcal{L} + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0,t)} = \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \right] \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0,t)} \quad (2.24)$$

Thus, the Eulerian differential form of the equation that defines the linear momentum balance is:

$$\frac{\partial}{\partial t}(\mathbf{a}_\varepsilon \rho_\varepsilon) + \frac{\partial}{\partial \mathbf{r}}(\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{div}(\mathbf{a}_\varepsilon) = \mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \quad (2.25)$$

To rewrite this equation, the divergence of the following tensor field is computed. It can be computed according to the result obtained in (B.37).

$$\mathbf{div}\left((\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) = \frac{\partial}{\partial \mathbf{r}}(\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{div}(\mathbf{a}_\varepsilon) \quad (2.26)$$

The above result appears in (2.25). Therefore, the Eulerian differential form of the linear momentum balance equation can be finally expressed as:

$$\frac{\partial}{\partial t}(\mathbf{a}_\varepsilon \rho_\varepsilon) + \mathbf{div}\left(-\boldsymbol{\sigma}_\varepsilon^T + (\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) = \mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon \quad \forall \mathbf{r} \in \mathring{\Omega}, \forall t \quad (2.27)$$

2.3.5. Eulerian integral form of the linear momentum balance equation

Since the equivalent integral form of the above differential one will be also required, its derivation is presented in this subsection.

The linear momentum of the entire solid at a given instant of time was defined in (2.12) as:

$$\mathbf{p} = \iiint_{\Omega(t)} \mathbf{a}_\varepsilon \rho_\varepsilon d\Omega \quad (2.28)$$

And its corresponding time derivative was defined in (2.17) as:

$$\frac{\partial \mathbf{p}}{\partial t} = \iiint_{\Omega(t)} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \right] d\Omega \quad (2.29)$$

Therefore, the combination of the previous equations leads to the Eulerian integral form of the linear momentum balance equation.

$$\frac{\partial}{\partial t} \left(\iiint_{\Omega(t)} \mathbf{a}_\varepsilon \rho_\varepsilon d\Omega \right) = \iiint_{\Omega(t)} \left[\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon + \mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) \right] d\Omega \quad (2.30)$$

2.4. Angular momentum balance

The third magnitude whose time variation is studied is the angular momentum. The same problem as the one exposed in section 2.3 is considered, and the magnitudes involved were already summed up in table 2.2.

2.4.1. Angular momentum time variation

The vector product between the position vector and the linear momentum defines the angular momentum. The linear momentum corresponding to a differential volume that belongs to a given material domain was defined in (2.11). Thus, its corresponding angular momentum at a given instant of time turns out to be:

$$d\mathcal{L}_0 = \mathbf{r} \wedge d\mathbf{p} = \mathbf{r} \wedge (\mathbf{a}_\varepsilon \rho_\varepsilon d\Omega) = (\mathbf{r} \wedge \mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \quad (2.31)$$

The integration of the above magnitude over the material domain leads to the angular momentum of the entire solid:

$$\mathcal{L}_0 = \iiint_{\Omega(t)} (\mathbf{r} \wedge \mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \quad (2.32)$$

Since the angular momentum depends on the linear momentum, the time variation of the angular momentum depends on the sources of linear momentum. As stated before, the external applied forces are sources of linear momentum. Therefore, the volumetric and surface loads, as well as the mass source that has an associated velocity

field, turn out to be sources of angular momentum. And the time variation can be finally defined as:

$$\frac{\partial \mathcal{L}_0}{\partial t} = \iiint_{\Omega} \mathbf{r} \wedge (\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon) d\Omega + \iint_{\Gamma=\partial\Omega} \mathbf{r} \wedge \mathbf{t}_\varepsilon d\Gamma \quad (2.33)$$

Once the angular momentum (2.32) and its corresponding time variation (2.33) are defined, the Eulerian integral form is straightforward to obtain.

2.4.2. Eulerian integral form of the angular momentum balance equation

The Eulerian integral form of the equation that defines the time variation of the angular momentum can be obtained by combining equations (2.32) and (2.33).

$$\frac{\partial}{\partial t} \left[\iiint_{\Omega} (\mathbf{r} \wedge \mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \iiint_{\Omega} \mathbf{r} \wedge (\mathbf{b}_\varepsilon \rho_\varepsilon + \mathbf{v}_\varepsilon \varphi_\varepsilon) d\Omega + \iint_{\Gamma=\partial\Omega} \mathbf{r} \wedge \mathbf{t}_\varepsilon d\Gamma \quad (2.34)$$

2.5. Overview and conclusions

Along this chapter, the equations that state the balance of mass, linear momentum and angular momentum were derived. The balance equations of these three magnitudes rule the structural behaviour of a continuous solid media subjected to external loads.

On the one hand, the mass balance equation is the equation that governs the value of the density field over time. The variation of the media density is caused by the mass sources and the volume variation along the deformation process. As a first approach, the existence of mass sources is considered. However, later on in this work, they will be no longer considered, and the assumption that the mass conservation is fulfilled will be adopted.

On the other hand, the linear momentum balance equation turns out to be the equation that governs the dynamic equilibrium of forces. The time variation of the linear momentum is caused by the external applied forces, as well as by the velocity field of the mass source. The angular momentum balance equation, in turn, is the one that rules the dynamic equilibrium of torques. As the angular momentum is defined by means of the linear momentum, its time variation is originated by the same causes that produce the linear momentum variation.

As shown along this chapter, once the time variation of a given magnitude (mass, linear momentum or angular momentum) is defined, its corresponding balance equation written according to its integral form is straightforward to obtain. And the manipulation of this integral form leads to its equivalent differential form. Moreover, the Lagrangian or the Eulerian versions of these balance equations can be derived, depending on the magnitude description adopted.

Stress field

3.1. Introduction

When facing a continuous solid media subjected to external forces, the main aim is to obtain the displacement field experimented by the solid. As this displacement field originates internal stresses, the definition of the stress field becomes crucial to completely define and interpret the solid structural behaviour.

To characterize this stress field, the first step is to properly define the stress vector acting at a given material particle that belongs to the deformed configuration. As it is a stress magnitude, its definition has to lead to a force per unit area.

Once this basic definition is proposed, the Cauchy stress vector can be derived. This equation defines the stress vector by means of the Cauchy stress tensor, whose components turn out to be the components of the stress vectors defined with respect to the planes that are perpendicular to the Cartesian axis. That is, the Cauchy stress tensor represents the stress field experimented by the solid, defined as internal forces per unit area in the deformed configuration.

From the classical Cauchy's definition, other alternative stress vectors can be derived. These alternative definitions lead to other stress tensors, that hold useful properties to analyse solids which experiment large displacements.

3.2. Stress vector

Let's consider the deformed domain of a solid media subjected to external forces at a given instant of time. A plane defined by its normal vector \mathbf{n} cut this domain, and the area $d\Gamma$ corresponding to a specific material point that belongs to this plane is defined according to figure 3.1.

At this specific material point, the traction force $d\mathbf{f}$ caused by the other part of the

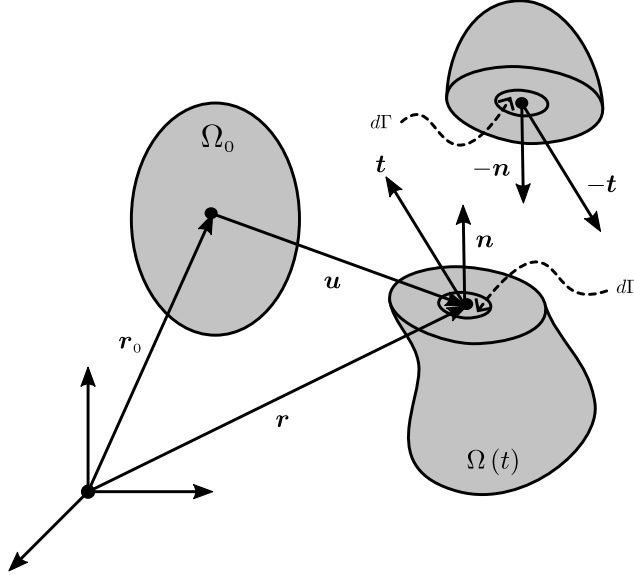


Figure 3.1. Stress vector at a given material point of the deformed domain.

solid on this area is considered. And its corresponding stress vector is defined as:

$$\mathbf{t}_\varepsilon(\mathbf{r}, t, \mathbf{n}) = \frac{d\mathbf{f}_\varepsilon}{d\Gamma} \quad (3.1)$$

According to this definition, the stress vector is a traction force per unit area, and it depends on the material point and time. Moreover, it also depends on the normal vector that defines the plane, since the direction of the traction force depends on the plane taken into account.

Furthermore, according to the Newton's third law, the stress vectors acting at a specific material point on opposite sides of a given plane have the same modulus and orientation, but opposite senses (figure 3.1).

$$\mathbf{t}_\varepsilon(\mathbf{r}, t, -\mathbf{n}) = -\mathbf{t}_\varepsilon(\mathbf{r}, t, \mathbf{n}) \quad (3.2)$$

3.3. Cauchy stress tensor

A tetrahedron is now defined in the deformed domain. Three of its faces are oriented according to the Cartesian planes. The fourth one is defined by an arbitrary plane, whose orientation is defined by its perpendicular unit vector \mathbf{n} . The area of this face is S , and the area of the other ones are S_i (figure 3.2).

According to Newton's second law, forces are sources of linear momentum. Thus, the linear momentum balance in the tetrahedron is defined as stated below.

$$\frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \iiint_V \mathbf{b}_\varepsilon \rho_\varepsilon d\Omega + \iint_S \mathbf{t}_\varepsilon d\Gamma - \sum_{i=1}^3 \iint_{S_i} \mathbf{t}_{\varepsilon,i} d\Gamma \quad (3.3)$$

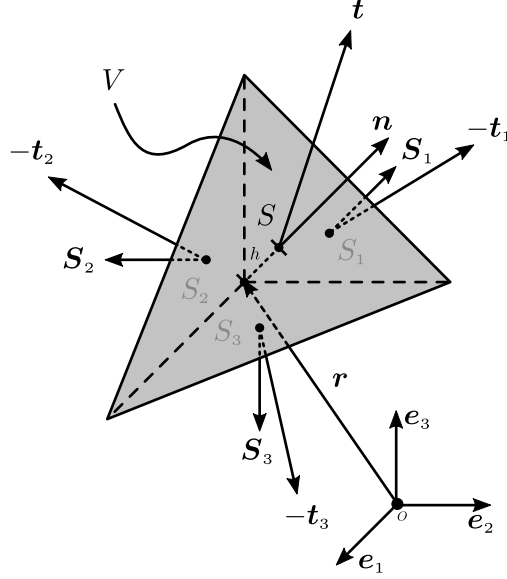


Figure 3.2. Stress vectors on the faces of a tetrahedron.

To manipulate the left-hand side of the above equation, the integration domain can be changed. That is, the change of variable (1.4) is applied to the integrand, and the differential volume is substituted by the equation defined in (1.28). The derivative can now be taken inside the integral, since the reference domain does not depend on time.

$$\begin{aligned}
 \frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] &= \frac{\partial}{\partial t} \left[\iiint_{V_0} (\mathbf{a}_\varepsilon \rho_\varepsilon) F_\varepsilon d\Omega_0 \right] \\
 &= \iiint_{V_0} \frac{\partial}{\partial t} (\mathbf{a}_\varepsilon \rho_\varepsilon F_\varepsilon) d\Omega_0 \\
 &= \iiint_{V_0} \left[\frac{\partial}{\partial t} (\mathbf{a}_\varepsilon \rho_\varepsilon) F_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \frac{\partial F_\varepsilon}{\partial t} \right] d\Omega_0
 \end{aligned} \tag{3.4}$$

In the above equation, the time derivative of a given magnitude defined by its Lagrangian description (1.18) can be substituted, as well as the equation that defines the time derivative of the deformation gradient tensor (1.61). Furthermore, the change of variable is undone, so the integral is again referred to the deformed domain.

$$\begin{aligned}
 \frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] &= \\
 &= \iiint_{V_0} \left[\frac{\partial (\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial t} + \frac{\partial (\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \operatorname{div} (\mathbf{a}_\varepsilon) \right] \Bigg|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0, t)} F_\varepsilon d\Omega_0 \\
 &= \iiint_V \left[\frac{\partial (\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial t} + \frac{\partial (\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \operatorname{div} (\mathbf{a}_\varepsilon) \right] d\Omega
 \end{aligned} \tag{3.5}$$

But, if the result obtained in (B.37) is taken into account:

$$\frac{\partial(\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} \mathbf{a}_\varepsilon + (\mathbf{a}_\varepsilon \rho_\varepsilon) \operatorname{div}(\mathbf{a}_\varepsilon) = \mathbf{div}\left((\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) \quad (3.6)$$

Thus, the time variation of the linear momentum can be rewritten as follows.

$$\frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \iiint_V \left[\frac{\partial \mathbf{a}_\varepsilon}{\partial t} \rho_\varepsilon + \mathbf{a}_\varepsilon \frac{\partial \rho_\varepsilon}{\partial t} + \mathbf{div}\left((\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) \right] d\Omega \quad (3.7)$$

The time derivative of the density field is now computed. The time variation of its Lagrangian description can be obtained from equation (2.10). If no mass sources are considered, and the time derivative of the deformation gradient tensor (1.61) is substituted into this equation, the time derivative of the density field becomes:

$$\frac{\partial \rho_\varepsilon}{\partial t} = -\rho_\varepsilon \frac{1}{F_\varepsilon} \frac{\partial F_\varepsilon}{\partial t} = - \left[\rho_\varepsilon \operatorname{div}(\mathbf{a}_\varepsilon) \right] \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \quad (3.8)$$

This derivative can also be computed according to the equation (1.18) as:

$$\frac{\partial \rho_\varepsilon}{\partial t} = \left(\frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial \rho_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0,t)} \quad (3.9)$$

Hence, if equations (3.8) and (3.9) are compared, the time derivative of the Eulerian density field turns out to be:

$$\frac{\partial \rho_\varepsilon}{\partial t} = -\rho_\varepsilon \operatorname{div}(\mathbf{a}_\varepsilon) - \frac{\partial \rho_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \quad (3.10)$$

The substitution of the above time derivative into the variation of the linear momentum (3.7) leads to:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \\ & = \iiint_V \left[\frac{\partial \mathbf{a}_\varepsilon}{\partial t} \rho_\varepsilon - (\mathbf{a}_\varepsilon \rho_\varepsilon) \operatorname{div}(\mathbf{a}_\varepsilon) - \mathbf{a}_\varepsilon \frac{\partial \rho_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon + \mathbf{div}\left((\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) \right] d\Omega \end{aligned} \quad (3.11)$$

Moreover, according to the divergence computed (B.19), the gradient of the following vector field turns out to be:

$$\frac{\partial(\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} = \mathbf{a}_\varepsilon \frac{\partial \rho_\varepsilon}{\partial \mathbf{r}} + \rho_\varepsilon \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \quad \iff \quad \mathbf{a}_\varepsilon \frac{\partial \rho_\varepsilon}{\partial \mathbf{r}} = \frac{\partial(\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} - \rho_\varepsilon \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \quad (3.12)$$

So, if the above result is taken into account, equation (3.11) becomes:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \\ & = \iiint_V \left[\frac{\partial \mathbf{a}_\varepsilon}{\partial t} \rho_\varepsilon - (\mathbf{a}_\varepsilon \rho_\varepsilon) \operatorname{div}(\mathbf{a}_\varepsilon) - \frac{\partial(\mathbf{a}_\varepsilon \rho_\varepsilon)}{\partial \mathbf{r}} \mathbf{a}_\varepsilon + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} (\mathbf{a}_\varepsilon \rho_\varepsilon) + \mathbf{div}\left((\mathbf{a}_\varepsilon \rho_\varepsilon) \mathbf{a}_\varepsilon^T\right) \right] d\Omega \end{aligned} \quad (3.13)$$

But, if equation (3.6) is recalled, the above equation is reduced to:

$$\frac{\partial}{\partial t} \left[\iiint_V (\mathbf{a}_\varepsilon \rho_\varepsilon) d\Omega \right] = \iiint_V \left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right] d\Omega \quad (3.14)$$

Therefore, the balance of linear momentum in the tetrahedron can be expressed as:

$$\iiint_V \left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right] d\Omega = \iiint_V \mathbf{b}_\varepsilon \rho_\varepsilon d\Omega + \iint_S \mathbf{t}_\varepsilon d\Gamma - \sum_{i=1}^3 \iint_{S_i} \mathbf{t}_{\varepsilon,i} d\Gamma \quad (3.15)$$

The left-hand side of the above equation represents the time variation of the linear momentum in the tetrahedron volume, and the right-hand side gathers the sources of linear momentum (volumetric and surface forces).

If the mean value theorem is applied, the previous balance equation can be rewritten as follows:

$$\left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right]^* V = (\mathbf{b}_\varepsilon \rho_\varepsilon)^* V + \mathbf{t}_\varepsilon^* S - \sum_{i=1}^3 \mathbf{t}_{\varepsilon,i}^* S_i \quad (3.16)$$

Where:

$$\begin{aligned} \left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right]^* &= \left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right] \Big|_{\mathbf{r}=\mathbf{r}^*} & \mathbf{r}^* \in V \\ (\mathbf{b}_\varepsilon \rho_\varepsilon)^* &= \mathbf{b}_\varepsilon(\mathbf{r}_V^*, t) \rho_\varepsilon(\mathbf{r}_V^*, t) & \mathbf{r}_V^* \in V \\ \mathbf{t}_\varepsilon^* &= \mathbf{t}_\varepsilon(\mathbf{r}_S^*, t, \mathbf{n}) & \mathbf{r}_S^* \in S \\ -\mathbf{t}_{\varepsilon,i}^* &= \mathbf{t}_{\varepsilon,i}(\mathbf{r}_{S_i}^*, t, -\mathbf{e}_i) & \mathbf{r}_{S_i}^* \in S_i \end{aligned} \quad (3.17)$$

On the other hand, the volume of the tetrahedron can be defined by means of its height h (figure 3.2).

$$V = \frac{1}{3} S h \quad (3.18)$$

To manipulate the area of each face, a unit vector that is perpendicular to each one of them is defined. Their modulus is forced to be equivalent to the area of each face.

$$\begin{aligned} \mathbf{S} = S \mathbf{n} &\implies \|\mathbf{S}\| = S \|\mathbf{n}\| = S \\ \mathbf{S}_i = -S_i \mathbf{e}_i &\implies \|\mathbf{S}_i\| = S_i \|\mathbf{e}_i\| = S_i \end{aligned} \quad (3.19)$$

According to figure 3.3, surfaces S_1 and S are triangular. So, their corresponding area is given by one-half the base times the height of the surface.

$$\begin{aligned} S_1 &= \frac{1}{2} \overline{ABOC} \\ S &= \frac{1}{2} \overline{ABCD} \end{aligned} \quad (3.20)$$

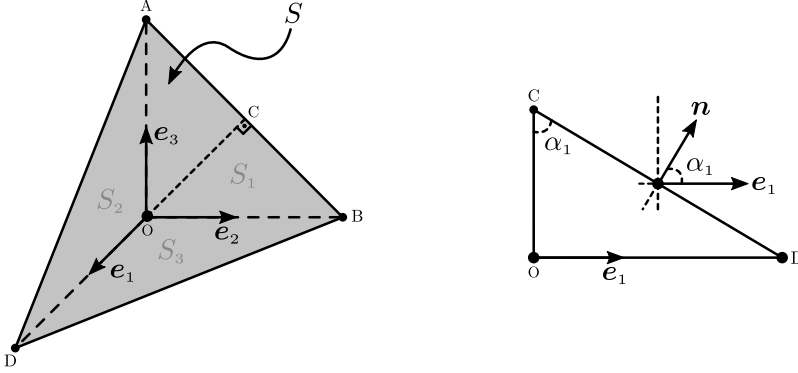


Figure 3.3. Identification of the tetrahedron faces, and relation between the height of the triangular surfaces S_1 and S .

The height of the triangular surface S_1 can be calculated by means of the height of the triangular surface S , as:

$$\begin{aligned}\overline{OC} &= \overline{CD} \cos(\alpha_1) \\ &= \overline{CD} \left(\|\mathbf{e}_1\| \|\mathbf{n}\| \cos(\alpha_1) \right) \\ &= \overline{CD} (\mathbf{e}_1 \cdot \mathbf{n})\end{aligned}\quad (3.21)$$

Where α_1 is the angle between \mathbf{n} and \mathbf{e}_1 .

The substitution of the above height into the definition of the area S_1 exposed in (3.20) leads to:

$$\begin{aligned}S_1 &= \frac{1}{2} \overline{AB} \overline{OC} \\ &= \frac{1}{2} \overline{AB} \left(\overline{CD} (\mathbf{e}_1 \cdot \mathbf{n}) \right) \\ &= \left(\frac{1}{2} \overline{AB} \overline{CD} \right) (\mathbf{e}_1 \cdot \mathbf{n}) \\ &= S (\mathbf{e}_1 \cdot \mathbf{n})\end{aligned}\quad (3.22)$$

An analogous result can be obtained by analysing the other triangular surfaces S_2 and S_3 .

$$\begin{aligned}S_2 &= S (\mathbf{e}_2 \cdot \mathbf{n}) \\ S_3 &= S (\mathbf{e}_3 \cdot \mathbf{n})\end{aligned}\quad (3.23)$$

Therefore, the area of the faces that are parallel to the coordinate planes can be calculated as:

$$S_i = S (\mathbf{e}_i \cdot \mathbf{n}) \quad i = 1, 2, 3 \quad (3.24)$$

If the tetrahedron volume (3.18) and the area of its faces (3.24) are now substituted into (3.16), the balance equation becomes:

$$\frac{1}{3} \left[\left(\frac{\partial \mathbf{a}_\varepsilon}{\partial t} + \frac{\partial \mathbf{a}_\varepsilon}{\partial \mathbf{r}} \mathbf{a}_\varepsilon \right) \rho_\varepsilon \right]^* h = \frac{1}{3} (\mathbf{b}_\varepsilon \rho_\varepsilon)^* h + \mathbf{t}_\varepsilon^* - \sum_{i=1}^3 \mathbf{t}_{\varepsilon,i}^* (\mathbf{e}_i \cdot \mathbf{n}) \quad (3.25)$$

Let's now consider that the tetrahedron height tends to zero. If this happens, the stress vectors that appear in the above equation can be replaced by the following ones.

$$\begin{aligned} \lim_{h \rightarrow 0} (-\mathbf{t}_{\varepsilon,i}^*) &= \lim_{h \rightarrow 0} \mathbf{t}_{\varepsilon,i}(\mathbf{r}_{S_i}^*, t, -\mathbf{e}_i) = -\mathbf{t}_{\varepsilon,i}(\mathbf{r}, t, \mathbf{e}_i) \quad i = 1, 2, 3 \\ \lim_{h \rightarrow 0} \mathbf{t}_{\varepsilon}^* &= \lim_{h \rightarrow 0} \mathbf{t}_{\varepsilon}(\mathbf{r}_S^*, t, \mathbf{n}) = \mathbf{t}_{\varepsilon}(\mathbf{r}, t, \mathbf{n}) \end{aligned} \quad (3.26)$$

And the terms that depend on the tetrahedron height tend to zero:

$$\lim_{h \rightarrow 0} \left[\frac{1}{3} (\mathbf{b}_{\varepsilon} \rho_{\varepsilon})^* h \right] = 0 \quad (3.27)$$

$$\lim_{h \rightarrow 0} \left\{ \frac{1}{3} \left[\left(\frac{\partial \mathbf{a}_{\varepsilon}}{\partial t} + \frac{\partial \mathbf{a}_{\varepsilon}}{\partial \mathbf{r}} \mathbf{a}_{\varepsilon} \right) \rho_{\varepsilon} \right]^* h \right\} = 0 \quad (3.28)$$

Thus, equation (3.25) is reduced to:

$$\mathbf{t}_{\varepsilon}(\mathbf{r}, t, \mathbf{n}) = \sum_{i=1}^3 \mathbf{t}_{\varepsilon,i}(\mathbf{r}, t, \mathbf{e}_i) (\mathbf{e}_i \cdot \mathbf{n}) \quad (3.29)$$

On the other hand, the stress vector corresponding to a plane which is perpendicular to a given unit vector can be defined as follows:

$$\mathbf{t}_{\varepsilon,i}(\mathbf{r}, t, \mathbf{e}_i) = \sum_{k=1}^3 \sigma_{ik} \mathbf{e}_k \quad (3.30)$$

The component σ_{ik} represents the force per unit area, oriented according to the direction k , that act on the plane whose normal vector is \mathbf{e}_i (figure 3.4). And the substitution of the above stress vector into equation (3.29) leads to:

$$\mathbf{t}_{\varepsilon}(\mathbf{r}, t, \mathbf{n}) = \sum_{i=1}^3 \left(\sum_{k=1}^3 \sigma_{ik} \mathbf{e}_k \right) (\mathbf{e}_i \cdot \mathbf{n}) = \sum_{i=1}^3 \sum_{k=1}^3 \sigma_{ik} \mathbf{e}_k (\mathbf{e}_i \cdot \mathbf{n}) \quad (3.31)$$

The vector of each addend can be equivalently expressed as the tensor product

$$\mathbf{e}_k (\mathbf{e}_i \cdot \mathbf{n}) = (\mathbf{e}_k \otimes \mathbf{e}_i) \mathbf{n} \quad (3.32)$$

Therefore, the stress vector acting on the basis of the tetrahedron can be finally defined as:

$$\mathbf{t}_{\varepsilon}(\mathbf{r}, t, \mathbf{n}) = \left[\sum_{k=1}^3 \sum_{i=1}^3 (\mathbf{e}_k \otimes \mathbf{e}_i) \sigma_{ik} \right] \mathbf{n} \quad (3.33)$$

The previous summation defines a tensor with respect to the Cartesian basis. Its components σ_{ik} are the components of the stress vectors defined in (3.30). This is the so-called Cauchy stress tensor, and it represents the stress field experimented by the solid, defined as internal forces per unit area in the deformed configuration.

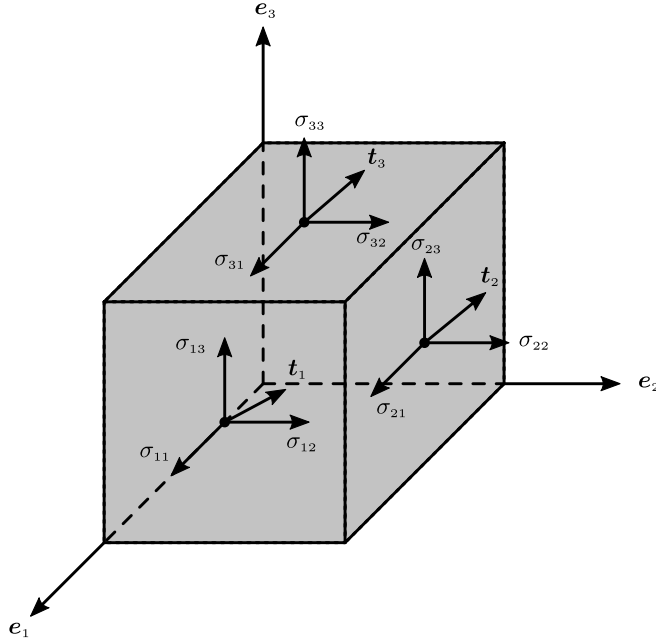


Figure 3.4. Stress vectors on the faces of a cube.

Equation (3.33) is also known as the Cauchy stress vector. It provides the value of the traction force per unit area, depending on the Cauchy stress tensor and the normal vector corresponding to the plane where the stress vector is defined.

$$\mathbf{t}_\varepsilon = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \boldsymbol{\sigma}_\varepsilon^T \mathbf{n} \quad (3.34)$$

It will be demonstrated later on that the stress tensor has to be symmetric, if the conservation of angular momentum is fulfilled (8.35). Hence, the Cauchy stress vector can also be defined as:

$$\mathbf{t}_\varepsilon(\mathbf{r}, t, \mathbf{n}) = \boldsymbol{\sigma}_\varepsilon(\mathbf{r}, t) \mathbf{n} \quad (3.35)$$

If the change of variable (1.4) is applied to the above equation, the Lagrangian description of the stress vector is obtained.

$$\mathbf{t}_\varepsilon \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0, t)} = (\boldsymbol{\sigma}_\varepsilon \mathbf{n}) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0, t)} \iff \mathbf{t}_\mathcal{L}(\mathbf{r}_0, t, \mathbf{n}) = \boldsymbol{\sigma}_\mathcal{L}(\mathbf{r}_0, t) \mathbf{n} \quad (3.36)$$

3.3.1. Effect of a rotation

Let's consider that the stress field experimented by a solid media subjected to external loads at a given load step t is known. The next load step $t + \Delta t$ consists in applying a rotation to the solid. As stated in the previous section, the stress vectors

corresponding to both load steps are defined as:

$$\begin{aligned} \mathbf{t}_{\mathcal{L}}^t &= \boldsymbol{\sigma}_{\mathcal{L}}^t \mathbf{n}_t \\ \mathbf{t}_{\mathcal{L}}^{t+\Delta t} &= \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} \mathbf{n}_{t+\Delta t} \end{aligned} \quad (3.37)$$

Moreover, the stress vector corresponding to the current load step turns out to be the result of rotating the previous stress vector. The rotation tensor applied has to be an orthogonal tensor, as proved in section A.13.

$$\begin{aligned} \mathbf{t}_{\mathcal{L}}^{t+\Delta t} &= \mathbf{R}_{\mathcal{L}} \mathbf{t}_{\mathcal{L}}^t \\ &= \mathbf{R}_{\mathcal{L}} (\boldsymbol{\sigma}_{\mathcal{L}}^t \mathbf{n}_t) \\ &= (\mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t) \mathbf{n}_t \end{aligned} \quad (3.38)$$

And the normal vector can be also obtained by rotating the previous one.

$$\mathbf{n}_{t+\Delta t} = \mathbf{R}_{\mathcal{L}} \mathbf{n}_t \quad \iff \quad \mathbf{n}_t = \mathbf{R}_{\mathcal{L}}^T \mathbf{n}_{t+\Delta t} \quad (3.39)$$

The substitution of the normal vector corresponding to the load step t obtained in the above equation into (3.38), leads to:

$$\begin{aligned} \mathbf{t}_{\mathcal{L}}^{t+\Delta t} &= (\mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t) \mathbf{n}_t \\ &= (\mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t) (\mathbf{R}_{\mathcal{L}}^T \mathbf{n}_{t+\Delta t}) \\ &= \underbrace{(\mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t \mathbf{R}_{\mathcal{L}}^T)}_{\boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t}} \mathbf{n}_{t+\Delta t} \end{aligned} \quad (3.40)$$

Therefore, the Cauchy stress tensor corresponding to the rotated configuration can be computed by means of the previous stress tensor as:

$$\boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} = \mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t \mathbf{R}_{\mathcal{L}}^T \quad (3.41)$$

3.4. First Piola-Kirchhoff stress tensor

As stated before, the differential force acting on a given material point of the deformed configuration can be obtained by means of the stress vector and the differential area where the force is applied. If the stress vector (3.36) is substituted into this definition, and the equation is reorganized, the differential force becomes:

$$\begin{aligned} d\mathbf{f}_{\mathcal{L}}(\mathbf{r}_0, t, \mathbf{n}) &= \mathbf{t}_{\mathcal{L}}(\mathbf{r}_0, t, \mathbf{n}) d\Gamma \\ &= (\boldsymbol{\sigma}_{\mathcal{L}} \mathbf{n}) d\Gamma \\ &= \boldsymbol{\sigma}_{\mathcal{L}} (d\Gamma \mathbf{n}) \\ &= \boldsymbol{\sigma}_{\mathcal{L}} d\mathbf{\Gamma} \end{aligned} \quad (3.42)$$

The equation that defines the deformed differential area by means of the initial one (1.55) can be substituted into the above equation. This substitution leads to the following result.

$$d\mathbf{f}_{\mathcal{L}} = \boldsymbol{\sigma}_{\mathcal{L}} \left(\mathbf{F}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} d\mathbf{\Gamma}_0 \right) \quad (3.43)$$

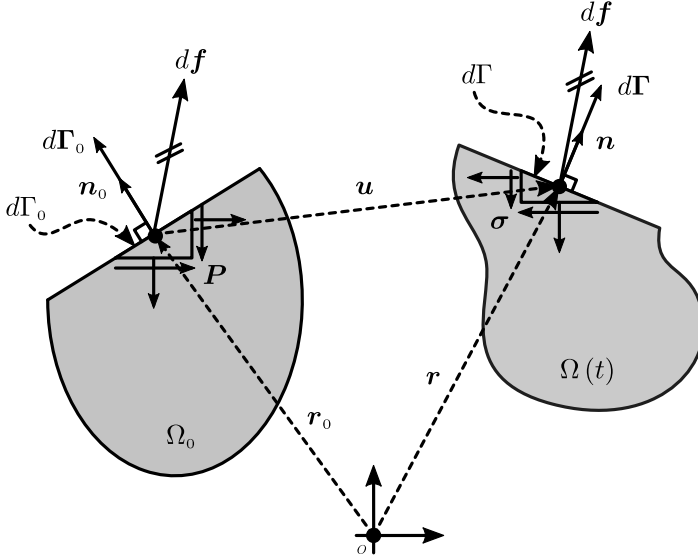


Figure 3.5. Cauchy stress tensor and first Piola-Kirchhoff stress tensor.

The definition of the first Piola-Kirchhoff stress tensor arises by reorganizing the above equation. This tensor allows to calculate the differential force that acts on the deformed configuration, by means of the initial differential area. That is, it represents the traction force acting on the deformed domain, referred to the undeformed area.

$$\begin{aligned}
 d\mathbf{f}_\varepsilon &= \boldsymbol{\sigma}_\varepsilon \left(\mathbf{F}_\varepsilon \mathbf{F}_\varepsilon^{-T} d\Gamma_0 \right) \\
 &= \left(\mathbf{F}_\varepsilon \boldsymbol{\sigma}_\varepsilon \mathbf{F}_\varepsilon^{-T} \right) d\Gamma_0 \\
 &= \mathbf{P}_\varepsilon d\Gamma_0
 \end{aligned} \tag{3.44}$$

According to its definition, the first Piola-Kirchhoff stress tensor turns out to be non-symmetric.

$$\begin{aligned}
 \mathbf{P}_\varepsilon = \mathbf{F}_\varepsilon \boldsymbol{\sigma}_\varepsilon \mathbf{F}_\varepsilon^{-T} &\quad \Rightarrow \quad \mathbf{P}_\varepsilon^T = \left(\mathbf{F}_\varepsilon \boldsymbol{\sigma}_\varepsilon \mathbf{F}_\varepsilon^{-T} \right)^T \\
 &= \mathbf{F}_\varepsilon \mathbf{F}_\varepsilon^{-1} \boldsymbol{\sigma}_\varepsilon^T \neq \mathbf{P}_\varepsilon
 \end{aligned} \tag{3.45}$$

This stress tensor allows to define an alternative stress vector $(\mathbf{t}_{0,\varepsilon})$, that represents the differential force acting on the deformed configuration ($d\mathbf{f}_\varepsilon$) per unit initial differential area ($d\Gamma_0$). The subscript $_0$ is added to indicate that the stress vector is referred to the initial reference configuration, which is well-known.

$$\left. \begin{aligned}
 d\mathbf{f}_\varepsilon &= \mathbf{P}_\varepsilon d\Gamma_0 \\
 &= \mathbf{P}_\varepsilon (d\Gamma_0 \mathbf{n}_0) \\
 &= (\mathbf{P}_\varepsilon \mathbf{n}_0) d\Gamma_0 \\
 &= \mathbf{t}_{0,\varepsilon} d\Gamma_0
 \end{aligned} \right\} \quad \Rightarrow \quad \mathbf{t}_{0,\varepsilon} = \mathbf{P}_\varepsilon \mathbf{n}_0 = \frac{d\mathbf{f}_\varepsilon}{d\Gamma_0} \tag{3.46}$$

3.5. Second Piola-Kirchhoff stress tensor

The product between the inverse of the deformation gradient tensor and the differential force defined in (3.43) leads to the definition of the second Piola-Kirchhoff stress tensor.

$$\begin{aligned}
 d\mathbf{f}'_{\mathcal{L}} &= \mathbf{F}_{\mathcal{L}}^{-1} d\mathbf{f}_{\mathcal{L}} \\
 &= \mathbf{F}_{\mathcal{L}}^{-1} \left[\boldsymbol{\sigma}_{\mathcal{L}} \left(\mathbf{F}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} d\Gamma_0 \right) \right] \\
 &= \mathbf{F}_{\mathcal{L}} \left(\mathbf{F}_{\mathcal{L}}^{-1} \boldsymbol{\sigma}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} \right) d\Gamma_0 \\
 &= \mathbf{S}_{\mathcal{L}} d\Gamma_0
 \end{aligned} \tag{3.47}$$

Thus, the second Piola-Kirchhoff stress tensor is defined as:

$$\mathbf{S}_{\mathcal{L}} = \mathbf{F}_{\mathcal{L}} \left(\mathbf{F}_{\mathcal{L}}^{-1} \boldsymbol{\sigma}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} \right) \tag{3.48}$$

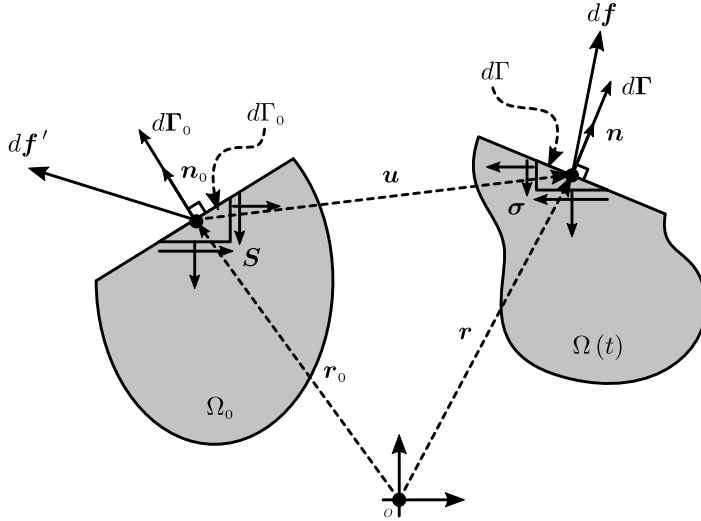


Figure 3.6. Cauchy stress tensor and second Piola-Kirchhoff stress tensor.

This tensor allows to calculate the modified differential force defined in (3.47) by means of the initial differential area. This modified differential force has no physical interpretation. Nevertheless, it allows to obtain an alternative stress tensor, which turns out to be work conjugate with the Green-Lagrange strain tensor (section 7.6.2). Two tensor magnitudes are work-conjugate, if the computation of their double dot product leads to the work per unit volume developed by the internal forces during the deformation process.

This stress tensor is symmetric. This property leads to symmetric stiffness matrices, as it will be proved later on. To prove this symmetry, the symmetry of the Cauchy stress tensor, which will be demonstrated later on in section 8.4, is also taken into

account.

$$\mathbf{S}_\mathcal{L} = F_\mathcal{L} \left(\mathbf{F}_\mathcal{L}^{-1} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T} \right) \quad \Longrightarrow \quad \begin{aligned} \mathbf{S}_\mathcal{L}^T &= F_\mathcal{L} \left(\mathbf{F}_\mathcal{L}^{-1} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T} \right)^T \\ &= F_\mathcal{L} \left(\mathbf{F}_\mathcal{L}^{-1} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T} \right) = \mathbf{S}_\mathcal{L} \end{aligned} \quad (3.49)$$

Moreover, it can be equivalently defined by means of the first Piola-Kirchhoff stress tensor (3.45).

$$\begin{aligned} \mathbf{S}_\mathcal{L} &= F_\mathcal{L} \left(\mathbf{F}_\mathcal{L}^{-1} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T} \right) \\ &= \mathbf{F}_\mathcal{L}^{-1} \left(F_\mathcal{L} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T} \right) \\ &= \mathbf{F}_\mathcal{L}^{-1} \mathbf{P}_\mathcal{L} \end{aligned} \quad (3.50)$$

The second Piola-Kirchhoff stress tensor allows to define another alternative stress vector ($\mathbf{t}'_{0,\mathcal{L}}$), that represents the modified differential force acting on the deformed configuration ($d\mathbf{f}'_\mathcal{L}$) per unit initial differential area ($d\Gamma_0$).

$$\left. \begin{aligned} d\mathbf{f}'_\mathcal{L} &= \mathbf{F}_\mathcal{L}^{-1} d\mathbf{f}_\mathcal{L} = \mathbf{S}_\mathcal{L} d\Gamma_0 \\ &= \mathbf{S}_\mathcal{L} (d\Gamma_0 \mathbf{n}_0) \\ &= (\mathbf{S}_\mathcal{L} \mathbf{n}_0) d\Gamma_0 \\ &= \mathbf{t}'_{0,\mathcal{L}} d\Gamma_0 \end{aligned} \right\} \quad \Longrightarrow \quad \mathbf{t}'_{0,\mathcal{L}} = \mathbf{S}_\mathcal{L} \mathbf{n}_0 = \frac{d\mathbf{f}'_\mathcal{L}}{d\Gamma_0} \quad (3.51)$$

3.5.1. Effect of a rotation

Let's consider that the structural behaviour of a given solid media subjected to a set of external loads at a specific load step t is known. As stated in section 1.10, the geometric transformation of a material vector $\delta\mathbf{r}_0$ that belongs to the reference material domain is ruled by the deformation gradient tensor.

$$\delta\mathbf{r}_t = \mathbf{F}_\mathcal{L}^t \delta\mathbf{r}_0 \quad (3.52)$$

And the second Piola-Kirchhoff stress tensor corresponding to this deformation gradient tensor is:

$$\mathbf{S}_\mathcal{L}^t = F_\mathcal{L}^t \left((\mathbf{F}_\mathcal{L}^t)^{-1} \boldsymbol{\sigma}_\mathcal{L}^t (\mathbf{F}_\mathcal{L}^t)^{-T} \right) \quad (3.53)$$

Let's also consider that the next load step $t + \Delta t$ consists in the application of a rotation to the solid. The material vector defined in (3.52) is modified by a rotation tensor as follows:

$$\begin{aligned} \delta\mathbf{r}_{t+\Delta t} &= \mathbf{R}_\mathcal{L} \delta\mathbf{r}_t \\ &= \mathbf{R}_\mathcal{L} (\mathbf{F}_\mathcal{L}^t \delta\mathbf{r}_0) \\ &= \underbrace{(\mathbf{R}_\mathcal{L} \mathbf{F}_\mathcal{L}^t)}_{\mathbf{F}_\mathcal{L}^{t+\Delta t}} \delta\mathbf{r}_0 \end{aligned} \quad (3.54)$$

The above rotation tensor turns out to be an orthogonal tensor and represents a proper rotation.

$$\mathbf{R}_\mathcal{L}^{-1} = \mathbf{R}_\mathcal{L}^T \quad \det(\mathbf{R}_\mathcal{L}) = 1 \quad (3.55)$$

After the application of the rotation, the deformation gradient tensor becomes:

$$\mathbf{F}_{\mathcal{L}}^{t+\Delta t} = \mathbf{R}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^t \quad (3.56)$$

And its corresponding second Piola-Kirchhoff stress tensor turns out to be:

$$\mathbf{S}_{\mathcal{L}}^{t+\Delta t} = F_{\mathcal{L}}^{t+\Delta t} \left((\mathbf{F}_{\mathcal{L}}^{t+\Delta t})^{-1} \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} (\mathbf{F}_{\mathcal{L}}^{t+\Delta t})^{-T} \right) \quad (3.57)$$

Since the rotation does not produce volume variation, the determinant of the current deformation gradient tensor turns out to be equivalent to the determinant of the previous load step.

$$\begin{aligned} F_{\mathcal{L}}^{t+\Delta t} &= \det(\mathbf{F}_{\mathcal{L}}^{t+\Delta t}) \\ &= \det(\mathbf{R}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^t) \\ &= \det(\mathbf{R}_{\mathcal{L}}) \det(\mathbf{F}_{\mathcal{L}}^t) \\ &= \det(\mathbf{F}_{\mathcal{L}}^t) \\ &= F_{\mathcal{L}}^t \end{aligned} \quad (3.58)$$

If the deformation gradient tensor (3.56) and its corresponding determinant (3.58) are substituted into (3.57), the second Piola-Kirchhoff stress tensor becomes:

$$\begin{aligned} \mathbf{S}_{\mathcal{L}}^{t+\Delta t} &= F_{\mathcal{L}}^t \left((\mathbf{R}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^t)^{-1} \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} (\mathbf{R}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^t)^{-T} \right) \\ &= F_{\mathcal{L}}^t \left((\mathbf{F}_{\mathcal{L}}^t)^{-1} \mathbf{R}_{\mathcal{L}}^T \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} \mathbf{R}_{\mathcal{L}} (\mathbf{F}_{\mathcal{L}}^t)^{-T} \right) \end{aligned} \quad (3.59)$$

According to the equation (3.41), the Cauchy stress tensor corresponding to the load step $t + \Delta t$ can be defined by means of its definition at the previous load step. From this equation, the inverse relation can be obtained.

$$\boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} = \mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}}^t \mathbf{R}_{\mathcal{L}}^T \quad \iff \quad \boldsymbol{\sigma}_{\mathcal{L}}^t = \mathbf{R}_{\mathcal{L}}^T \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} \mathbf{R}_{\mathcal{L}} \quad (3.60)$$

If the above stress tensor is identified in (3.59), and the definition of the second Piola-Kirchhoff stress tensor corresponding to the load step t (3.53) is recalled, it can be proven that the second Piola-Kirchhoff stress tensors corresponding to both load steps are equivalent.

$$\begin{aligned} \mathbf{S}_{\mathcal{L}}^{t+\Delta t} &= F_{\mathcal{L}}^t \left((\mathbf{F}_{\mathcal{L}}^t)^{-1} (\mathbf{R}_{\mathcal{L}}^T \boldsymbol{\sigma}_{\mathcal{L}}^{t+\Delta t} \mathbf{R}_{\mathcal{L}}) (\mathbf{F}_{\mathcal{L}}^t)^{-T} \right) \\ &= F_{\mathcal{L}}^t \left((\mathbf{F}_{\mathcal{L}}^t)^{-1} \boldsymbol{\sigma}_{\mathcal{L}}^t (\mathbf{F}_{\mathcal{L}}^t)^{-T} \right) \\ &= \mathbf{S}_{\mathcal{L}}^t \end{aligned} \quad (3.61)$$

It can be concluded that the second Piola-Kirchhoff stress tensor does not vary if a rotation is applied to the solid.

3.6. Overview and conclusions

The stress vector is defined with respect to a given plane, thus, an infinite number of stress vectors can be defined at a given material point that belongs to the deformed configuration. It is defined as the differential traction force acting on the deformed configuration per unit differential area. Note that this differential traction force depends on the plane taken into account, and that the differential area belongs to this plane.

According to this definition, the stress vector turns out to be a force per unit area which depends on the material point and time. Moreover, it also depends on the normal vector that defines the plane, since the direction of the traction force depends on the plane taken into account.

The linear momentum balance applied to a tetrahedron that belongs to the deformed material domain leads to the definition of the Cauchy stress vector ($\mathbf{t}_\mathcal{L}$). This equation defines the stress vector as the product between the Cauchy stress tensor ($\boldsymbol{\sigma}_\mathcal{L}$) and the normal unit vector (\mathbf{n}) that defines the plane with respect to which the stress vector is defined.

At a given material point, the Cauchy stress tensor turns out to be composed by the components of the stress vectors which are defined with respect to the Cartesian planes. This tensor represents the stress field experimented by the solid, defined as internal forces per unit area. If a rotation is applied to the deformed domain, the components of the Cauchy stress tensor are modified.

Alternative stress tensors can be defined, if the reference configuration is modified. When defining the Cauchy stress vector, the deformed material domain is the reference configuration. That is, stresses are defined as forces per unit deformed area. However, if the initial material domain becomes the reference one, the first Piola-Kirchhoff stress tensor ($\mathbf{P}_\mathcal{L}$) arises. This stress tensor allows to define a stress vector referred to the initial material domain, which is well-known. This alternative stress vector is defined as the differential traction force acting on the deformed configuration per unit initial differential area. The main disadvantage of this tensor is its non-symmetry, and its use to represent the stress field is usually rejected since it leads to non-symmetric stiffness matrices.

If the differential traction force is multiplied by the inverse of the deformation gradient tensor, the second Piola-Kirchhoff stress tensor ($\mathbf{S}_\mathcal{L}$) arises. Although this tensor defines another type of stress vector that has no clear physical interpretation, it fulfils some interesting and useful properties when a large displacement structural behaviour is faced. It defines a stress vector that represents a modified differential traction force (the product between the inverse of the deformation gradient tensor and the differential traction force) per unit initial differential area. Note that the initial domain is again adopted as the reference one. The second Piola-Kirchhoff stress tensor turns out to be symmetric and does not vary if a rotation is applied to a given deformed domain.

Finite strain field

4.1. Introduction

In order to fully understand the deformation experimented by the solid in the vicinity of a given material particle, the geometric transformation of a material vector is extensively analysed throughout this chapter.

The equation that rules this geometric variation contains the information that defines the change of volume, orientation, and shape that the solid undergoes. This information is necessary to properly define the displacement field, as well as its corresponding strain and stress fields.

4.2. Geometric transformation of a material vector

A material vector is a vector that links two material points that belong to a material domain. This vector experiments a change in its modulus and direction when the external loads are applied (figure 4.1).

Let consider a material vector $\delta\vec{\mathbf{r}}_0$ that belongs to the initial configuration. The expression of this vector with respect to the canonical basis is:

$$\delta\vec{\mathbf{r}}_0 = \sum_{k=1}^n \mathbf{e}_k \delta r_{0,k} \iff \begin{cases} \delta\vec{\mathbf{r}}_0 = \tilde{\mathbf{E}} \delta\mathbf{r}_0 \\ \tilde{\mathbf{E}} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n] \\ \delta\mathbf{r}_0 = \begin{cases} \delta r_{0,1} \\ \vdots \\ \delta r_{0,n} \end{cases} \end{cases} \quad (4.1)$$

As the matrix $\tilde{\mathbf{E}}$ is composed by the orthonormal vectors that compose the canonical basis, this matrix turns out to be the identity matrix. Thus, the matrix expression of

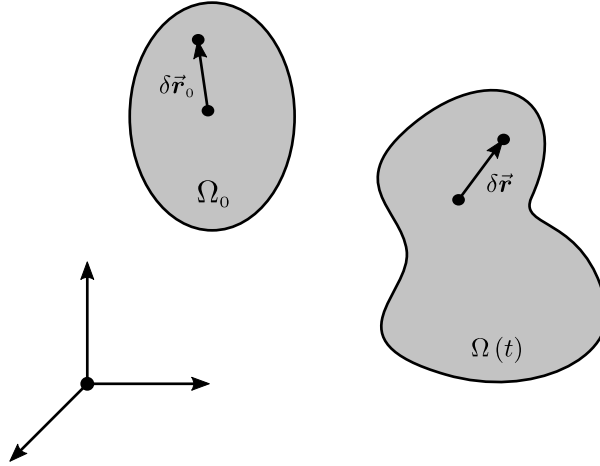


Figure 4.1. Material vector at a given instant of time.

the above material vector is reduced to:

$$\underset{\sim}{\mathbf{E}} = \mathbf{I} \implies \delta\vec{r}_0 = \delta\mathbf{r}_0 \quad (4.2)$$

After the application of the external forces, this vector changes its modulus and direction, and it becomes:

$$\delta\vec{r} = \delta\mathbf{r} \quad (4.3)$$

The above vector is the relative deformation vector defined in section 1.10. As proved in that section, it is defined by means of the deformation gradient tensor. Moreover, the deformation gradient tensor depends on the displacement gradient tensor. Thus,

$$\begin{aligned} \delta\mathbf{r} &= \mathbf{F}_{\mathcal{L}} \delta\mathbf{r}_0 + \mathcal{O}(\|\delta\mathbf{r}_0\|^2) \\ \mathbf{F}_{\mathcal{L}} &= \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{I} + \mathbf{J}_{\mathcal{L}} \\ \mathbf{J}_{\mathcal{L}} &= \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \end{aligned} \quad (4.4)$$

That is, the relative deformation vector defines the motion of a solid, in the vicinity of a given point (figure 4.2). As this vector depends on the deformation gradient tensor, this tensor is the one that contains the information related to the change of volume, orientation, and shape of the solid. This information is required to properly define the displacement, strain, and stress fields of a continuous solid media subjected to external forces.

4.2.1. Basis rotation

Let consider the geometric transformation of a generic material vector ruled by the deformation gradient tensor.

$$\delta\mathbf{r} = \mathbf{F}_{\mathcal{L}} \delta\mathbf{r}_0 \quad (4.5)$$

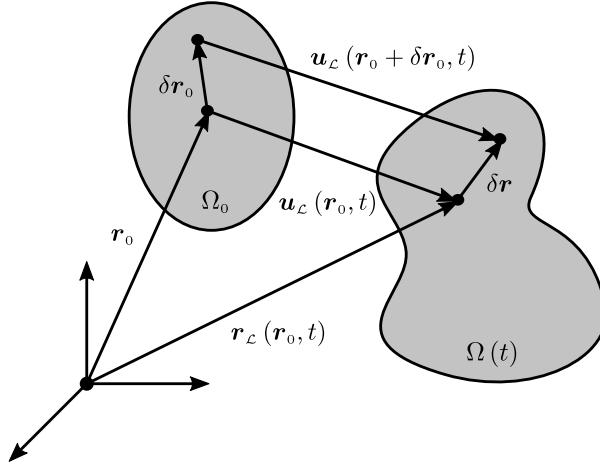


Figure 4.2. Geometric transformation of a material vector at a given instant of time.

This tensor is defined with respect to the canonical basis. If a rotation is now applied to each one of the unit vectors that compose this basis, a rotated one composed by orthonormal vectors is obtained. As stated in section A.13, the rotation tensor has to be orthogonal, since it represents a rotation. Then,

$$\mathbf{R}_c^{-1} = \mathbf{R}_c^T \quad (4.6)$$

And the rotated vectors that compose the new basis are:

$$\begin{aligned} \mathbf{e}'_i = \mathbf{R}_c \mathbf{e}_i &= \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} \mathbf{e}_i = \left\{ \begin{matrix} r_{1i} \\ \vdots \\ r_{ni} \end{matrix} \right\} \quad i = 1, \dots, n \quad \Leftrightarrow \\ \Leftrightarrow \underset{\sim}{\mathbf{E}'} &= [\mathbf{e}'_1 \quad \dots \quad \mathbf{e}'_n] = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} = \mathbf{R}_c \end{aligned} \quad (4.7)$$

If this rotated basis is adopted as a reference, the vector components, as well as the tensor components, change. In this section, the variation of these components is studied, and their calculation from the previous components expressed with respect to the canonical basis is obtained.

The reference material vector $\delta \vec{r}'_0$ and the deformed one $\delta \vec{r}'$ are defined with respect to the rotated basis as:

$$\begin{aligned} \delta \vec{r}'_0 &= \underset{\sim}{\mathbf{E}'} \delta \mathbf{r}'_0 = \mathbf{R}_c \delta \mathbf{r}'_0 \\ \delta \vec{r}' &= \underset{\sim}{\mathbf{E}'} \delta \mathbf{r}' = \mathbf{R}_c \delta \mathbf{r}' \end{aligned} \quad (4.8)$$

These equations have to be equal to the ones defined in (4.2) and (4.3), since they define the same vectors. Thus, if both pairs of equations are compared, the equations

that relate the vector components in both coordinate systems can be obtained.

$$\begin{aligned}\delta \mathbf{r}_0 &= \mathbf{R}_\mathcal{L} \delta \mathbf{r}'_0 \\ \delta \mathbf{r} &= \mathbf{R}_\mathcal{L} \delta \mathbf{r}'\end{aligned}\quad (4.9)$$

If the above equations are replaced into the expression that defines the geometric change of a generic material vector (4.5), the equation that defines the deformation gradient tensor expressed with respect to the rotated basis is obtained.

$$\underbrace{(\mathbf{R}_\mathcal{L} \delta \mathbf{r}')}_{\delta \mathbf{r}} = \mathbf{F}_\mathcal{L} \underbrace{(\mathbf{R}_\mathcal{L} \delta \mathbf{r}'_0)}_{\delta \mathbf{r}_0} \iff \delta \mathbf{r}' = \underbrace{(\mathbf{R}_\mathcal{L}^T \mathbf{F}_\mathcal{L} \mathbf{R}_\mathcal{L})}_{\mathbf{F}'_\mathcal{L}} \delta \mathbf{r}'_0 \quad (4.10)$$

So, the equation that defines the deformation gradient tensor expressed with respect to the rotated basis turns out to be:

$$\mathbf{F}'_\mathcal{L} = \mathbf{R}_\mathcal{L}^T \mathbf{F}_\mathcal{L} \mathbf{R}_\mathcal{L} \quad (4.11)$$

Where $\mathbf{F}_\mathcal{L}$ is the deformation gradient tensor expressed with respect to the canonical basis, and $\mathbf{R}_\mathcal{L}$ defines the rotation applied to each one of the canonical axes.

4.3. Polar decomposition of the deformation gradient tensor

According to the polar decomposition theorem [Halmos, 1958], any non-singular deformation gradient tensor can be decomposed as the product of the following tensors:

$$\forall \mathbf{F}_\mathcal{L} \mid \det(\mathbf{F}_\mathcal{L}) \neq 0 \implies \mathbf{F}_\mathcal{L} = \mathbf{R}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \quad (4.12)$$

Where $\mathbf{R}_\mathcal{L}$ is the finite rotation tensor, and $[\mathbf{I} + \mathbf{E}_\mathcal{L}]$ is the finite strain tensor. Moreover, the tensor $\mathbf{E}_\mathcal{L}$ is the so-called Biot strain tensor [Anand & Govindjee, 2020].

On the one hand, the rotation tensor has to be an orthogonal tensor, as it represents a rotation (A.13).

$$\mathbf{R}_\mathcal{L}^T = \mathbf{R}_\mathcal{L}^{-1} \iff \begin{cases} \mathbf{R}_\mathcal{L} \mathbf{R}_\mathcal{L}^T = \mathbf{I} \\ \mathbf{R}_\mathcal{L}^T \mathbf{R}_\mathcal{L} = \mathbf{I} \end{cases} \quad (4.13)$$

From the above property, the determinant of the finite rotation tensor can be deduced by applying the determinant operator at both sides of the equation.

$$\left. \begin{array}{l} \mathbf{R}_\mathcal{L} \mathbf{R}_\mathcal{L}^T = \mathbf{I} \\ \mathbf{R}_\mathcal{L}^T \mathbf{R}_\mathcal{L} = \mathbf{I} \end{array} \right\} \iff (\det(\mathbf{R}_\mathcal{L}))^2 = \det(\mathbf{I}) = 1 \iff \det(\mathbf{R}_\mathcal{L}) = \pm 1 \quad (4.14)$$

It can be concluded that the determinant of the rotation tensor has to be equal to one, as it has to represent a proper rotation. The determinant of an improper rotation has the opposite sign of a proper rotation [Salomon, 1999]. An improper rotation can be a reflection, or a rotation followed by a reflection. In this particular case, the

rotation of a material vector is studied. The improper rotation of a material vector has no physical meaning, so the rotation has to be a proper one.

$$\det(\mathbf{R}_\mathcal{L}) = 1 \quad (4.15)$$

On the other hand, the Biot strain tensor is symmetric.

$$\mathbf{E}_\mathcal{L}^T = \mathbf{E}_\mathcal{L} \quad (4.16)$$

And the finite strain tensor has to be positive definite.

$$\mathbf{x}^T [\mathbf{I} + \mathbf{E}_\mathcal{L}] \mathbf{x} > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (4.17)$$

In the following subsection, the computation of the Biot strain tensor and the finite rotation tensor is outlined.

4.3.1. Finite strain and finite rotation tensors

The right Cauchy-Green tensor and its properties are presented, in order to compute the finite strain and rotation tensors. This tensor depends on the deformation gradient tensor, and it is defined as stated below.

$$\mathbf{M}_\mathcal{L} = \mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} \quad (4.18)$$

According to this definition, it is straightforward to prove it turns out to be symmetric and positive definite.

$$\mathbf{M}_\mathcal{L}^T = (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L})^T = \mathbf{F}_\mathcal{L} \mathbf{F}_\mathcal{L}^T = \mathbf{M}_\mathcal{L} \quad (4.19)$$

$$\mathbf{x}^T \mathbf{M}_\mathcal{L} \mathbf{x} = \mathbf{x}^T (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L}) \mathbf{x} = (\mathbf{F}_\mathcal{L} \mathbf{x})^T (\mathbf{F}_\mathcal{L} \mathbf{x}) = \|\mathbf{F}_\mathcal{L} \mathbf{x}\|^2 > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (4.20)$$

Therefore, in a n -dimensional space, this matrix has n real positive eigenvalues associated to their corresponding eigenvectors. Furthermore, these eigenvectors are forced to compose an orthonormal basis.

$$\begin{aligned} \mathbf{M}_\mathcal{L} \mathbf{u}_i &= \lambda_i \mathbf{u}_i & i &= 1, \dots, n \\ \mathbf{u}_i^T \mathbf{u}_j &= \delta_{ij} \end{aligned} \quad (4.21)$$

The above eigenvalue problem can be rewritten as shown below.

$$\mathbf{M}_\mathcal{L} \mathbf{U} = \mathbf{U} \mathbf{\Lambda} \quad \left\{ \begin{array}{l} \mathbf{U} = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n], \quad \mathbf{U}^T = \mathbf{U}^{-1}, \quad \det(\mathbf{U}) = 1 \\ \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \quad \mathbf{\Lambda} \text{ positive definite} \end{array} \right. \quad (4.22)$$

The matrix \mathbf{U} is composed by the eigenvectors organized into columns. As the eigenvectors compose an orthonormal basis, this matrix turns out to be orthogonal. Thus, it can be interpreted as a rotation matrix.

$$\mathbf{U}^T \mathbf{U} = \mathbf{I} \quad \iff \quad \mathbf{U}^T = \mathbf{U}^{-1} \quad (4.23)$$

Moreover, the sense of the eigenvectors is defined so that they compose a proper rotation. Hence, its determinant has to be equal to one.

$$\det(\mathbf{U}) = 1 \quad (4.24)$$

And the matrix $\mathbf{\Lambda}$ is a diagonal matrix composed by the eigenvalues associated to the previous eigenvectors. Since the eigenvalues are positive, this diagonal matrix is also positive definite.

$$\mathbf{x}^T \mathbf{\Lambda} \mathbf{x} = \sum_{k=1}^n \underbrace{\lambda_k}_{>0} (x_k)^2 > 0 \quad \forall \mathbf{x} = \{x_k\}_{k=1, \dots, n} \neq \bar{\mathbf{0}} \quad (4.25)$$

On the one hand, if the polar decomposition of the deformation gradient tensor (4.12) is substituted into the definition of the right Cauchy-Green tensor (4.18), the effect of the rotation disappears as the rotation tensor is an orthogonal tensor.

$$\begin{aligned} \mathbf{F}_c^T \mathbf{F}_c &= \left(\mathbf{R}_c [\mathbf{I} + \mathbf{E}_c] \right)^T \left(\mathbf{R}_c [\mathbf{I} + \mathbf{E}_c] \right) \\ &= [\mathbf{I} + \mathbf{E}_c]^T \underbrace{(\mathbf{R}_c^T \mathbf{R}_c)}_{\mathbf{I}} [\mathbf{I} + \mathbf{E}_c] \\ &= [\mathbf{I} + \mathbf{E}_c]^T [\mathbf{I} + \mathbf{E}_c] \end{aligned} \quad (4.26)$$

On the other hand, from equation (4.22), an equivalent definition of the right Cauchy-Green tensor that depends on its eigenvalues and eigenvectors can be obtained.

$$\begin{aligned} \mathbf{F}_c^T \mathbf{F}_c &= \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T \\ &= \mathbf{U} [\mathbf{\Lambda}^{1/2} \mathbf{\Lambda}^{1/2}] \mathbf{U}^T \\ &= \mathbf{U} \mathbf{\Lambda}^{1/2} \underbrace{(\mathbf{U}^T \mathbf{U})}_{\mathbf{I}} \mathbf{\Lambda}^{1/2} \mathbf{U}^T \\ &= [\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T] [\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T] \\ &= [\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T]^T [\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T] \end{aligned} \quad (4.27)$$

If equations (4.26) and (4.27) are now compared, the finite strain tensor can be defined depending on the eigenvalues and the eigenvectors of the right Cauchy-Green tensor.

$$\mathbf{I} + \mathbf{E}_c = \mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T \quad (4.28)$$

And the Biot strain tensor can be computed as:

$$\begin{aligned} \mathbf{E}_c &= \mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T - \mathbf{I} \\ &= \mathbf{U} [\mathbf{\Lambda}^{1/2} - \mathbf{I}] \mathbf{U}^T \end{aligned} \quad (4.29)$$

Once the finite strain tensor (4.28) has been defined, the finite rotation tensor can be computed from the definition of the polar decomposition of the deformation gradient tensor (4.12).

$$\begin{aligned}\mathbf{R}_\mathcal{L} &= \mathbf{F}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}]^{-1} \\ &= \mathbf{F}_\mathcal{L} [\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T]^{-1} \\ &= \mathbf{F}_\mathcal{L} \mathbf{U} \mathbf{\Lambda}^{-1/2} \mathbf{U}^T\end{aligned}\tag{4.30}$$

According to the previous results, the polar decomposition of the deformation gradient tensor can be obtained by means of the eigenvalues and eigenvectors of the right Cauchy-Green tensor. Moreover, this decomposition allows to divide the geometric variation of a given material vector into two different phases, as stated below.

4.3.2. Physical interpretation

As it has already been proved in section 1.10, the deformation gradient tensor rules the geometric change of a material vector. According to its polar decomposition, this tensor can be decomposed into the product of a finite strain tensor and a finite rotation tensor (section 4.3).

$$\delta \mathbf{r} = \mathbf{F}_\mathcal{L} \delta \mathbf{r}_0 = \underbrace{\widehat{\mathbf{R}}_\mathcal{L}}_{\text{rotation}} \underbrace{\left([\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0 \right)}_{\text{strain}}\tag{4.31}$$

Thus, the geometric change of the material vector can be divided into two different phases. Firstly, the finite strain tensor modifies the modulus and the direction of the original material vector.

$$\delta \mathbf{r}_1 = [\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0\tag{4.32}$$

Then, the finite rotation tensor rotates the previous modified vector. As it is a rotation, its modulus does not change. It only modifies its direction.

$$\delta \mathbf{r} = \mathbf{R}_\mathcal{L} \delta \mathbf{r}_1\tag{4.33}$$

4.3.3. Biot strain tensor

According to the equation (4.29), the Biot strain tensor can be defined as:

$$\mathbf{E}_\mathcal{L} = \mathbf{U} [\mathbf{\Lambda}^{1/2} - \mathbf{I}] \mathbf{U}^T = \mathbf{U} \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{bmatrix} \mathbf{U}^T\tag{4.34}$$

The coefficients ε_i turn out to be the eigenvalues of the Biot strain tensor. They can be computed depending on the eigenvalues of the right Cauchy-Green tensor (λ_i) as shown below.

$$\left. \begin{array}{l} \varepsilon_i = \lambda_i^{1/2} - 1 \quad i = 1, \dots, n \\ \lambda_i \in (0, \infty) \end{array} \right\} \implies \varepsilon_i \in (-1, \infty)\tag{4.35}$$

If equations (4.22) and (4.34) are compared, it can be concluded that the Biot strain tensor and the right Cauchy-Green tensor have the same eigenvectors.

4.3.4. Determinant of the deformation gradient tensor

The polar decomposition of the deformation gradient tensor (4.12) also allows to calculate its determinant.

$$F_{\mathcal{L}} = \det(\mathbf{F}_{\mathcal{L}}) = \det\left(\mathbf{R}_{\mathcal{L}} [\mathbf{I} + \mathbf{E}_{\mathcal{L}}]\right) = \det(\mathbf{R}_{\mathcal{L}}) \det(\mathbf{I} + \mathbf{E}_{\mathcal{L}}) \quad (4.36)$$

As stated in (4.15), the determinant of the finite rotation tensor is equal to one, so the determinant of the deformation gradient tensor turns out to be equal to the determinant of the finite strain tensor.

$$F_{\mathcal{L}} = \underbrace{\det(\mathbf{R}_{\mathcal{L}})}_{=1} \det(\mathbf{I} + \mathbf{E}_{\mathcal{L}}) = \det(\mathbf{I} + \mathbf{E}_{\mathcal{L}}) \quad (4.37)$$

Furthermore, the finite strain tensor can be expressed depending on the eigenvalues and eigenvectors of the right Cauchy-Green tensor, as obtained in (4.28).

$$F_{\mathcal{L}} = \det(\mathbf{I} + \mathbf{E}_{\mathcal{L}}) = \det(\mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{U}^T) \quad (4.38)$$

As the matrix \mathbf{U} represents a proper rotation and its determinant is equal to one (4.24), the computation of the determinant is reduced to:

$$F_{\mathcal{L}} = \det(\mathbf{U}) \det(\mathbf{\Lambda}^{1/2}) \det(\mathbf{U}^T) = \underbrace{\det(\mathbf{U})}_{=1} \det(\mathbf{\Lambda}^{1/2}) \underbrace{\det(\mathbf{U}^T)}_{=1} = \det(\mathbf{\Lambda}^{1/2}) \quad (4.39)$$

The matrix $\mathbf{\Lambda}^{1/2}$ is diagonal, so its determinant can be calculated as the product of its diagonal components. Moreover, if equation (4.35) is taken into account, these components can be rewritten depending on the eigenvalues of the Biot strain tensor. So, the determinant of the deformation gradient tensor can be finally computed as follows:

$$F_{\mathcal{L}} = \det(\mathbf{\Lambda}^{1/2}) = \prod_{i=1}^n \lambda_i^{1/2} = \prod_{i=1}^n (1 + \varepsilon_i) \quad (4.40)$$

4.4. Finite strain

As the polar decomposition theorem states, the deformation gradient tensor can be decomposed into the product of a finite rotation and a finite strain tensor (section 4.3). On the one hand, the rotation tensor is a proper orthogonal tensor, since it represents a proper rotation (proved in section A.13). On the other hand, the finite strain tensor is symmetric and positive definite.

$$\mathbf{F}_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}} [\mathbf{I} + \mathbf{E}_{\mathcal{L}}] \quad (4.41)$$

Where:

$$\begin{aligned}\mathbf{R}_\mathcal{L} &\equiv \text{Finite rotation tensor} \\ \mathbf{I} + \mathbf{E}_\mathcal{L} &\equiv \text{Finite strain tensor} \\ \mathbf{E}_\mathcal{L} &\equiv \text{Biot strain tensor}\end{aligned}$$

Moreover, the geometric variation of a material vector is ruled by the deformation gradient tensor (1.23). If its polar decomposition is taken into account, the geometric transformation of a material vector can be expressed as follows.

$$\delta \mathbf{r} = \overbrace{\mathbf{R}_\mathcal{L}}^{\text{rotation}} \left(\underbrace{[\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0}_{\text{strain}} \right) + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (4.42)$$

This implies that, firstly, the material vector suffers the effect of the finite strain tensor. This tensor modifies its modulus and direction. Then, a finite rotation is applied to this vector. The rotation only changes its direction, it does not vary its norm.

4.4.1. Concept of finite strain

If the finite strain tensor is applied to a material vector, it modifies its direction and modulus. Thus, the finite strain is defined as the following geometric change of a given material vector.

$$\delta \mathbf{r}_E = [\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0 \quad (4.43)$$

The tensor $\mathbf{E}_\mathcal{L}$ is the so-called Biot strain tensor, which has to be symmetric.

$$\mathbf{E}_\mathcal{L}^T = \mathbf{E}_\mathcal{L} \quad (4.44)$$

Moreover, the finite strain tensor has to be positive definite. Hence, it fulfils the following property.

$$\mathbf{x}^T [\mathbf{I} + \mathbf{E}_\mathcal{L}] \mathbf{x} > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (4.45)$$

As stated in (4.34), the Biot strain tensor can be computed as:

$$\mathbf{E}_\mathcal{L} = \mathbf{U} \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{bmatrix} \mathbf{U}^T \quad (4.46)$$

Where the coefficients ε_i that compose the above diagonal matrix are the eigenvalues of the Biot strain tensor (4.35).

$$\varepsilon_i \in (-1, \infty) \quad (4.47)$$

And the matrix \mathbf{U} is composed by the eigenvectors associated to the previous eigenvalues. If the eigenvectors are forced to compose an orthonormal basis, this matrix

turns out to be orthogonal.

$$\left. \begin{array}{l} \mathbf{U}^T \mathbf{U} = \mathbf{I} \\ \mathbf{U} \mathbf{U}^T = \mathbf{I} \end{array} \right\} \iff \mathbf{U}^T = \mathbf{U}^{-1} \quad (4.48)$$

The previous matrix can also be forced to define a proper rotation, if the sense of the eigenvectors is properly defined. Thus, its determinant has to be equal to one.

$$\det(\mathbf{U}) = 1 \quad (4.49)$$

If the results obtained in (4.22) and (4.46) are compared, it can be concluded that the right Cauchy-Green tensor and the Biot strain tensor have the same eigenvectors. However, they do not have the same eigenvalues. The relation between both eigenvalues was stated in (4.35) as:

$$\left. \begin{array}{l} \varepsilon_i = \lambda_i^{1/2} - 1 \quad i = 1, \dots, n \\ \lambda_i \in (0, \infty) \end{array} \right\} \implies \varepsilon_i \in (-1, \infty) \quad (4.50)$$

Where λ_i are the eigenvalues of the right Cauchy-Green tensor, and ε_i are the ones corresponding to the Biot strain tensor.

4.4.2. Basis rotation

Let assume that the initial reference basis is the canonical one. If a rotation is applied to this basis, and the rotated basis is adopted as the new reference, the vector and tensor components vary. Therefore, this variation has to be defined.

$$\delta \mathbf{r}_E = [\mathbf{I} + \mathbf{E}_L] \delta \mathbf{r}_0 \quad \longrightarrow \quad \delta \mathbf{r}'_E = [\mathbf{I} + \mathbf{E}'_L] \delta \mathbf{r}'_0 \quad (4.51)$$

The equations that define the canonical vector components by means of the components expressed with respect to the rotated basis were already defined in (4.9). Therefore, the equations that relate both components are:

$$\begin{aligned} \delta \mathbf{r}_0 &= \mathbf{R}_L \delta \mathbf{r}'_0 \\ \delta \mathbf{r}_E &= \mathbf{R}_L \delta \mathbf{r}'_E \end{aligned} \quad (4.52)$$

The substitution of the above equations into the equation that defines the finite strain tensor applied to a material vector (4.43) leads to its definition with respect to the rotated basis.

$$\underbrace{(\mathbf{R}_L \delta \mathbf{r}'_E)}_{\delta \mathbf{r}_E} = [\mathbf{I} + \mathbf{E}_L] \underbrace{(\mathbf{R}_L \delta \mathbf{r}'_0)}_{\delta \mathbf{r}_0} \iff \delta \mathbf{r}'_E = \underbrace{(\mathbf{R}_L^T [\mathbf{I} + \mathbf{E}_L] \mathbf{R}_L)}_{\mathbf{I} + \mathbf{E}'_L} \delta \mathbf{r}'_0 \quad (4.53)$$

Hence, the finite strain tensor expressed with respect to the rotated basis turns out to be:

$$\mathbf{I} + \mathbf{E}'_L = \mathbf{I} + \mathbf{R}_L^T \mathbf{E}_L \mathbf{R}_L \quad (4.54)$$

And the Biot strain tensor:

$$\mathbf{E}'_L = \mathbf{R}_L^T \mathbf{E}_L \mathbf{R}_L \quad (4.55)$$

4.4.3. Biot strain tensor diagonalization

According to the definition of the Biot strain tensor stated in (4.34), the following equation that defines its diagonalization can be obtained.

$$\begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{bmatrix} = \mathbf{U}^T \mathbf{E}_\mathcal{L} \mathbf{U} \quad (4.56)$$

Moreover, the equation that defines the components of the Biot strain tensor with respect to a rotated basis was obtained in (4.55). If both equations are compared, it can be concluded that the rotation defined by $\mathbf{R}_\mathcal{L} = \mathbf{U}$ diagonalizes the Biot strain tensor.

$$\mathbf{E}'_\mathcal{L} = \mathbf{R}_\mathcal{L}^T \mathbf{E}_\mathcal{L} \mathbf{R}_\mathcal{L} = \mathbf{U}^T \mathbf{E}_\mathcal{L} \mathbf{U} = \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{bmatrix} \quad (4.57)$$

The coefficients ε_i are the eigenvalues of the Biot strain tensor, and the columns of the matrix \mathbf{U} are their corresponding eigenvectors.

Thus, if the finite strain that experiments a material vector is expressed with respect to this rotated basis, the finite strain tensor turns out to be diagonal.

$$\delta \mathbf{r}'_\mathcal{E} = [\mathbf{I} + \mathbf{E}'_\mathcal{L}] \delta \mathbf{r}'_0 = \begin{bmatrix} 1 + \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \varepsilon_n \end{bmatrix} \delta \mathbf{r}'_0 \quad (4.58)$$

The axes defined by this basis are also known as principal axes. And the coefficients that compose the diagonalized Biot strain tensor are the elongation factors that are applied to the material vector components along each one of them.

$$\varepsilon_i \in (-1, \infty) \quad \implies \quad (1 + \varepsilon_i) \in (0, \infty) \quad (4.59)$$

4.5. Alternative polar decomposition

The polar decomposition of the deformation gradient tensor can alternatively be rewritten as:

$$\mathbf{F}_\mathcal{L} = \mathbf{R}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}] = \mathbf{R}_\mathcal{L} \left(\frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \right) \left(\sqrt[n]{F_\mathcal{L}} [\mathbf{I}] \right) = \mathbf{R}_\mathcal{L} \mathbf{D}_\mathcal{L} \mathbf{H}_\mathcal{L} \quad (4.60)$$

Where n is the dimension of the space where the solid is defined.

This alternative polar decomposition divides the finite strain tensor into the product of two different tensors.

$$\begin{aligned} \mathbf{I} + \mathbf{E}_\mathcal{L} &= \mathbf{D}_\mathcal{L} \mathbf{H}_\mathcal{L} \\ \mathbf{D}_\mathcal{L} &= \frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \\ \mathbf{H}_\mathcal{L} &= \sqrt[n]{F_\mathcal{L}} [\mathbf{I}] \end{aligned} \quad (4.61)$$

These two tensors are extensively analysed in the following sections.

4.5.1. Inflation tensor (isotropic strain tensor)

On the one hand, the tensor $\mathbf{H}_\mathcal{L}$ is diagonal, and its determinant turns out to be equivalent to the determinant of the deformation gradient tensor.

$$\det(\mathbf{H}_\mathcal{L}) = \det\left(\sqrt[n]{F_\mathcal{L}}[\mathbf{I}]\right) = F_\mathcal{L} \quad (4.62)$$

This result indicates that this tensor is the one involved with the change of volume. It is the so-called inflation tensor or isotropic strain tensor. Since its determinant is equivalent to the determinant of the deformation gradient tensor, it can be concluded that the inflation tensor rules the volume variation in the vicinity of a given point.

4.5.2. Distortion tensor (isochoric strain tensor)

On the other hand, the tensor $\mathbf{D}_\mathcal{L}$ can be rewritten as shown below.

$$\begin{aligned} \mathbf{D}_\mathcal{L} &= \frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \\ &= \mathbf{I} + \frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] - \mathbf{I} \\ &= \mathbf{I} + \frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{E}_\mathcal{L} + (1 - \sqrt[n]{F_\mathcal{L}})\mathbf{I}] \end{aligned} \quad (4.63)$$

Its determinant can be derived from the computation of the determinant of the deformation gradient tensor.

$$F_\mathcal{L} = \det(\mathbf{F}_\mathcal{L}) = \det(\mathbf{R}_\mathcal{L} \mathbf{D}_\mathcal{L} \mathbf{H}_\mathcal{L}) = \det(\mathbf{R}_\mathcal{L}) \det(\mathbf{D}_\mathcal{L}) \det(\mathbf{H}_\mathcal{L}) \quad (4.64)$$

The determinant of the rotation tensor has to be equal to one, as it represents a proper rotation. If the result obtained in (4.62) is also taken into account, the determinant of the tensor $\mathbf{D}_\mathcal{L}$ can be derived.

$$\left. \begin{aligned} F_\mathcal{L} &= \det(\mathbf{R}_\mathcal{L}) \det(\mathbf{D}_\mathcal{L}) \det(\mathbf{H}_\mathcal{L}) \\ \det(\mathbf{R}_\mathcal{L}) &= 1 \\ \det(\mathbf{H}_\mathcal{L}) &= F_\mathcal{L} \end{aligned} \right\} \implies \det(\mathbf{D}_\mathcal{L}) = 1 \quad (4.65)$$

The above result indicates that this tensor does not produce volume variation. Hence, it has to be related to a distortion process. This tensor is the so-called distortion tensor or isochoric strain tensor, and it is defined as:

$$\begin{aligned} \mathbf{D}_\mathcal{L} &= \mathbf{I} + \mathbf{E}_{D,\mathcal{L}} \\ \mathbf{E}_{D,\mathcal{L}} &= \frac{1}{\sqrt[n]{F_\mathcal{L}}} [\mathbf{E}_\mathcal{L} + (1 - \sqrt[n]{F_\mathcal{L}})\mathbf{I}] \end{aligned} \quad (4.66)$$

Where $\mathbf{E}_{D,\mathcal{L}}$ is the deviatoric strain tensor.

4.5.3. Physical interpretation

If this alternative polar decomposition is taken into account, the geometric transformation of a given material vector (4.31) becomes:

$$\delta \mathbf{r} = \underbrace{\mathbf{R}_\mathcal{L}}_{\text{rotation}} \left(\underbrace{\mathbf{D}_\mathcal{L}}^{\text{distortion}} \left(\underbrace{\mathbf{H}_\mathcal{L}}_{\text{inflation}} \delta \mathbf{r}_0 \right) \right) + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (4.67)$$

Where:

$\mathbf{R}_\mathcal{L} \equiv$ Finite rotation tensor

$\mathbf{D}_\mathcal{L} \equiv$ Distortion tensor (isochoric strain tensor)

$\mathbf{H}_\mathcal{L} \equiv$ Inflation tensor (isotropic strain tensor)

Let's consider that an infinitesimal parallelepiped is defined at a given material point, whose reference position is defined by the position vector \mathbf{r}_0 . Its faces are parallel to the coordinate planes composed by the principal axes. Under these assumptions, the parallelepiped experiments the sequence of geometric changes stated below.

- The first geometric transformation that it experiments is the inflation. It is caused by the inflation tensor $\mathbf{H}_\mathcal{L}$, and it is the only one that implies a volume variation.
- The distortion is the second transformation that the parallelepiped experiments. The volume remains constant, and the tensor $\mathbf{D}_\mathcal{L}$ is the one that governs this process.
- Finally, the rotation produced by the finite rotation tensor $\mathbf{R}_\mathcal{L}$ is applied. This rotation orientates the parallelepiped according to the deformed configuration.

Figure 4.3 shows the initial position of an infinitesimal parallelepiped in the reference configuration, and its configuration at a given instant of time t .

Furthermore, figure 4.4 shows the whole geometric transformation that the parallelepiped experiments, decomposed in the following phases: inflation, distortion, and rotation.

If this entire process is known, the change of shape, volume, and orientation experimented by a solid after the application of the external loads can be entirely understood and defined. This information is necessary to properly define the displacement field, and its corresponding strain and stress fields.

4.6. Green-Lagrange strain tensor

The definition of the Green-Lagrange strain tensor arises when the difference between the square of the norm of a given material vector at a given instant of time t and the square of the norm of the undeformed material vector is computed.

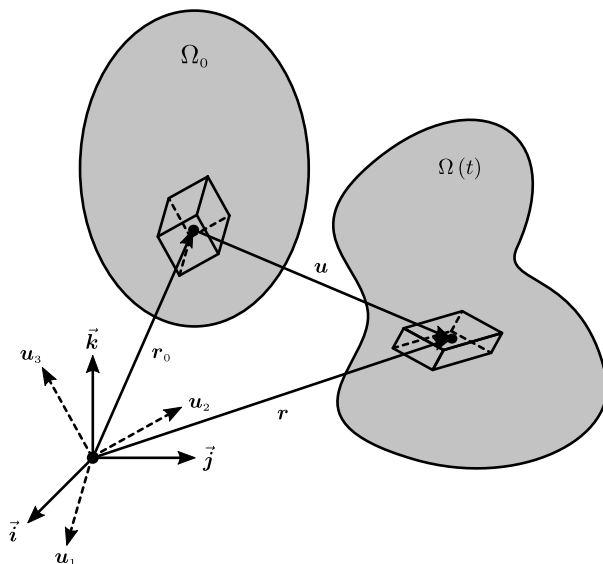


Figure 4.3. Parallelepiped geometric transformation at a given instant of time.

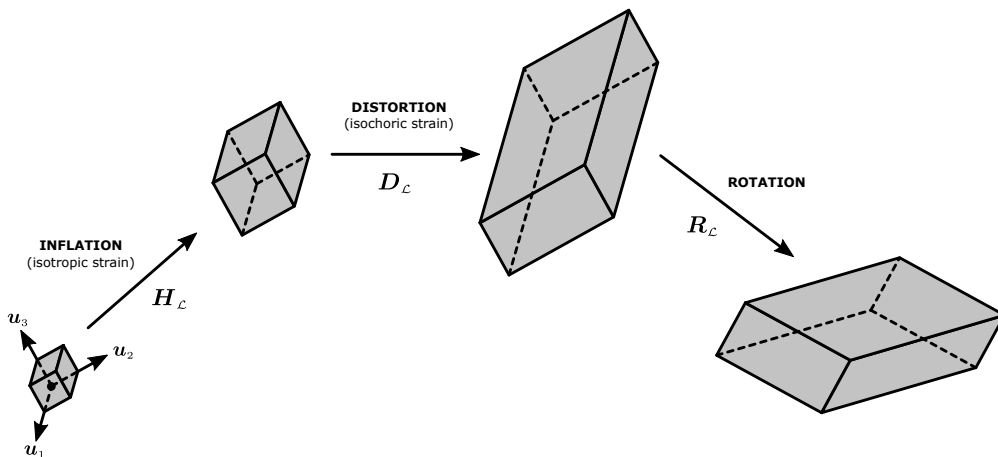


Figure 4.4. Alternative polar decomposition applied to the parallelepiped geometric transformation.

4.6.1. Definition in terms of the deformation gradient tensor

On the one hand, the square of the norm of the initial infinitesimal material vector is:

$$\|\delta \mathbf{r}_0\|^2 = \delta \mathbf{r}_0^T \delta \mathbf{r}_0 \tag{4.68}$$

On the other hand, as stated in (1.23), the deformation gradient tensor governs the geometric transformation of this material vector. Thus, the material vector at a given

instant of time is defined as:

$$\delta \mathbf{r} = \mathbf{F}_\mathcal{L} \delta \mathbf{r}_0 \quad (4.69)$$

And the square of its norm can be computed as shown below.

$$\begin{aligned} \|\delta \mathbf{r}\|^2 &= \delta \mathbf{r}^T \delta \mathbf{r} \\ &= (\mathbf{F}_\mathcal{L} \delta \mathbf{r}_0)^T (\mathbf{F}_\mathcal{L} \delta \mathbf{r}_0) \\ &= \delta \mathbf{r}_0^T (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L}) \delta \mathbf{r}_0 \end{aligned} \quad (4.70)$$

Therefore, the difference between (4.70) and (4.68) turns out to be:

$$\begin{aligned} \|\delta \mathbf{r}\|^2 - \|\delta \mathbf{r}_0\|^2 &= \delta \mathbf{r}_0^T (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L}) \delta \mathbf{r}_0 - \delta \mathbf{r}_0^T \delta \mathbf{r}_0 \\ &= \delta \mathbf{r}_0^T (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I}) \delta \mathbf{r}_0 \\ &= 2 \delta \mathbf{r}_0^T \left[\frac{1}{2} (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I}) \right] \delta \mathbf{r}_0 \\ &= 2 \delta \mathbf{r}_0^T \mathbf{E}_{G,\mathcal{L}} \delta \mathbf{r}_0 \end{aligned} \quad (4.71)$$

The definition of the Green-Lagrange strain tensor arises in the above equation. Its definition by means of the deformation gradient tensor is:

$$\mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I}) \quad (4.72)$$

4.6.2. Definition in terms of the displacement gradient tensor

If the definition of the deformation gradient tensor presented in (1.12) is recalled, the Green-Lagrange strain tensor can be defined in terms of the displacement gradient tensor.

$$\begin{aligned} \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I}) \\ &= \frac{1}{2} [(\mathbf{I} + \mathbf{J}_\mathcal{L})^T (\mathbf{I} + \mathbf{J}_\mathcal{L}) - \mathbf{I}] \\ &= \frac{1}{2} (\mathbf{J}_\mathcal{L} + \mathbf{J}_\mathcal{L}^T + \mathbf{J}_\mathcal{L}^T \mathbf{J}_\mathcal{L}) \\ &= \frac{1}{2} \left[\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right] \end{aligned} \quad (4.73)$$

The components of the above tensor are defined as:

$$\mathbf{E}_{G,\mathcal{L}} = [E_{ij}]_{\substack{j=1,\dots,n \\ i=1,\dots,n}} \quad E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial r_{0,j}} + \frac{\partial u_j}{\partial r_{0,i}} + \sum_{k=1}^n \frac{\partial u_k}{\partial r_{0,i}} \frac{\partial u_k}{\partial r_{0,j}} \right) \quad (4.74)$$

Moreover, the tensor defined in (4.73) can be rewritten in terms of the infinitesimal strain tensor (5.45), which is introduced in the following chapter.

$$\begin{aligned} \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\mathbf{J}_\mathcal{L} + \mathbf{J}_\mathcal{L}^T + \mathbf{J}_\mathcal{L}^T \mathbf{J}_\mathcal{L}) \\ &= \frac{1}{2} (\mathbf{J}_\mathcal{L} + \mathbf{J}_\mathcal{L}^T) + \frac{1}{2} \mathbf{J}_\mathcal{L}^T \mathbf{J}_\mathcal{L} \\ &= \boldsymbol{\mathcal{E}}_\mathcal{L} + \mathcal{O}(\|\mathbf{J}_\mathcal{L}\|^2) \end{aligned} \quad (4.75)$$

If the displacement gradients are small, the Green-Lagrange strain tensor turns out to be equivalent to the infinitesimal strain tensor.

$$\|\mathbf{J}_\mathcal{L}\| \ll \|\mathbf{I}\| \quad \Longrightarrow \quad \mathbf{E}_{G,\mathcal{L}} = \boldsymbol{\mathcal{E}}_\mathcal{L} + \mathcal{O}(\|\mathbf{J}_\mathcal{L}\|^2) \approx \boldsymbol{\mathcal{E}}_\mathcal{L} \quad (4.76)$$

4.6.3. Definition in terms of the Biot strain tensor

The polar decomposition of the deformation gradient tensor (4.12) can be substituted into the definition obtained in (4.72). It is recalled that the finite rotation tensor is an orthogonal tensor (4.13), and the Biot strain tensor is a symmetric one (4.16).

$$\begin{aligned} \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} \left(\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I} \right) \\ &= \frac{1}{2} \left[\left(\mathbf{R}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \right)^T \left(\mathbf{R}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \right) - \mathbf{I} \right] \\ &= \frac{1}{2} \left([\mathbf{I} + \mathbf{E}_\mathcal{L}]^T \underbrace{\mathbf{R}_\mathcal{L}^T \mathbf{R}_\mathcal{L}}_{\mathbf{I}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] - \mathbf{I} \right) \\ &= \mathbf{E}_\mathcal{L} + \frac{1}{2} \mathbf{E}_\mathcal{L}^2 \\ &= \mathbf{E}_\mathcal{L} + \mathcal{O}(\|\mathbf{E}_\mathcal{L}\|^2) \end{aligned} \quad (4.77)$$

This substitution allows to define the Green-Lagrange strain tensor by means of the Biot strain tensor. As shown in the above equation, the effect of the rotation is neglected, and the strain tensor only takes into account the effect of the Biot strain tensor. This tensor is the one that governs the variation of the norm of a given material vector, since the rotation does not modify its modulus.

It should be noted that, the Green-Lagrange strain tensor turns out to be equivalent to the Biot strain tensor if the strain field is infinitesimal.

$$\|\mathbf{E}_\mathcal{L}\| \ll \|\mathbf{I}\| \quad \Longrightarrow \quad \mathbf{E}_{G,\mathcal{L}} = \mathbf{E}_\mathcal{L} + \mathcal{O}(\|\mathbf{E}_\mathcal{L}\|^2) \approx \mathbf{E}_\mathcal{L} \quad (4.78)$$

4.6.4. Effect of a rotation

Let's consider a solid subjected to a set of external loads at a given load step t . The geometric transformation of a given material vector is ruled by the deformation gradient tensor, which defines the Green-Lagrange strain tensor corresponding to this load step.

$$\delta \mathbf{r}_t = \mathbf{F}_\mathcal{L}^t \delta \mathbf{r}_0 \quad \Longrightarrow \quad \mathbf{E}_{G,\mathcal{L}}^t = \frac{1}{2} \left[\left(\mathbf{F}_\mathcal{L}^t \right)^T \left(\mathbf{F}_\mathcal{L}^t \right) - \mathbf{I} \right] \quad (4.79)$$

If the next load step $t + \Delta t$ is a solid rotation, the material vector is rotated by a rotation tensor as shown below.

$$\begin{aligned} \delta \mathbf{r}_{t+\Delta t} &= \mathbf{R}_\mathcal{L} \delta \mathbf{r}_t \\ &= \mathbf{R}_\mathcal{L} \left(\mathbf{F}_\mathcal{L}^t \delta \mathbf{r}_0 \right) \\ &= \underbrace{\left(\mathbf{R}_\mathcal{L} \mathbf{F}_\mathcal{L}^t \right)}_{\mathbf{F}_\mathcal{L}^{t+\Delta t}} \delta \mathbf{r}_0 \end{aligned} \quad (4.80)$$

Thus, the deformation gradient tensor of the current load step is:

$$\mathbf{F}_{\mathcal{L}}^{t+\Delta t} = \mathbf{R}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^t \quad (4.81)$$

And its corresponding Green-Lagrange strain tensor turns out to be equivalent to the previous one.

$$\begin{aligned} \mathbf{E}_{G,\mathcal{L}}^{t+\Delta t} &= \frac{1}{2} \left[(\mathbf{F}_{\mathcal{L}}^{t+\Delta t})^T (\mathbf{F}_{\mathcal{L}}^{t+\Delta t}) - \mathbf{I} \right] \\ &= \frac{1}{2} \left[(\mathbf{F}_{\mathcal{L}}^t)^T \underbrace{\mathbf{R}_{\mathcal{L}}^T \mathbf{R}_{\mathcal{L}}}_{\mathbf{I}} (\mathbf{F}_{\mathcal{L}}^t) - \mathbf{I} \right] \\ &= \frac{1}{2} \left[(\mathbf{F}_{\mathcal{L}}^t)^T (\mathbf{F}_{\mathcal{L}}^t) - \mathbf{I} \right] \\ &= \mathbf{E}_{G,\mathcal{L}}^t \end{aligned} \quad (4.82)$$

Therefore, it can be concluded that the Green-Lagrange strain tensor does not vary if a rotation is applied to the solid. However, this tensor does vary if its corresponding displacement field is modified. This variation is analysed in the following subsections.

4.6.5. Green-Lagrange strain tensor variation

The Green-Lagrange strain tensor, corresponding to a given displacement field, is defined as shown below.

$$\begin{aligned} \mathbf{u}_{\mathcal{L}} &\implies \mathbf{J}_{\mathcal{L}} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \\ &\implies \mathbf{r}_{\mathcal{L}} = \mathbf{r}_0 + \mathbf{u}_{\mathcal{L}} \\ &\implies \mathbf{F}_{\mathcal{L}} = \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{I} + \mathbf{J}_{\mathcal{L}} \\ &\implies \mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left(\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} - \mathbf{I} \right) \end{aligned} \quad (4.83)$$

Let's consider that the above displacement field is modified by adding a compatible variation. Let's also assume that the gradients of the displacement field variation are small. Under these assumptions, the Green-Lagrange strain tensor corresponding to

the modified displacement field is computed as follows.

$$\begin{aligned}
 & \left. \begin{aligned} & \mathbf{u}'_{\mathcal{L}} = \mathbf{u}_{\mathcal{L}} + \delta\mathbf{u}_{\mathcal{L}} \\ & \left\| \frac{\partial\delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{aligned} \right\} \implies \\
 \implies & \quad \mathbf{J}'_{\mathcal{L}} = \frac{\partial\mathbf{u}'_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} + \frac{\partial\delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \delta\mathbf{J}_{\mathcal{L}} \\
 \implies & \quad \mathbf{r}'_{\mathcal{L}} = \mathbf{r}_0 + \mathbf{u}'_{\mathcal{L}} = \mathbf{r}_0 + (\mathbf{u}_{\mathcal{L}} + \delta\mathbf{u}_{\mathcal{L}}) = \underbrace{(\mathbf{r}_0 + \mathbf{u}_{\mathcal{L}})}_{\mathbf{r}_{\mathcal{L}}} + \delta\mathbf{u}_{\mathcal{L}} \\
 \implies & \quad \mathbf{F}'_{\mathcal{L}} = \frac{\partial\mathbf{r}'_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\mathbf{r}_{\mathcal{L}}}{\partial\mathbf{r}_0} + \frac{\partial\delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \mathbf{F}_{\mathcal{L}} + \underbrace{\delta\mathbf{F}_{\mathcal{L}}}_{\delta\mathbf{J}_{\mathcal{L}}} \tag{4.84} \\
 & \mathbf{E}'_{G,\mathcal{L}} = \frac{1}{2} \left[(\mathbf{F}'_{\mathcal{L}})^T (\mathbf{F}'_{\mathcal{L}}) - \mathbf{I} \right] \\
 & \quad = \frac{1}{2} \left[(\mathbf{F}_{\mathcal{L}} + \delta\mathbf{F}_{\mathcal{L}})^T (\mathbf{F}_{\mathcal{L}} + \delta\mathbf{F}_{\mathcal{L}}) - \mathbf{I} \right] \\
 \implies & \quad = \frac{1}{2} \left(\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \delta\mathbf{F}_{\mathcal{L}} + \delta\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \delta\mathbf{F}_{\mathcal{L}}^T \delta\mathbf{F}_{\mathcal{L}} - \mathbf{I} \right) \\
 & \quad = \frac{1}{2} \left(\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} - \mathbf{I} \right) + \frac{1}{2} \left(\delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \delta\mathbf{J}_{\mathcal{L}} + \underbrace{\delta\mathbf{J}_{\mathcal{L}}^T \delta\mathbf{J}_{\mathcal{L}}}_{\approx 0} \right) \\
 & \quad = \mathbf{E}_{G,\mathcal{L}} + \delta\mathbf{E}_{G,\mathcal{L}}
 \end{aligned}$$

Where $\mathbf{E}_{G,\mathcal{L}}$ is the Green-Lagrange strain tensor corresponding to the original displacement field, and $\delta\mathbf{E}_{G,\mathcal{L}}$ represents its variation. Hence, the Green-Lagrange strain tensor variation turns out to be:

$$\delta\mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left(\delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \delta\mathbf{J}_{\mathcal{L}} \right) \tag{4.85}$$

If the definition of the deformation gradient tensor presented in (1.12) is substituted into the above equation, the Green-Lagrange strain tensor variation can be rewritten as:

$$\begin{aligned}
 \delta\mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} \left(\delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \delta\mathbf{J}_{\mathcal{L}} \right) \\
 &= \frac{1}{2} \left[\delta\mathbf{J}_{\mathcal{L}}^T (\mathbf{I} + \mathbf{J}_{\mathcal{L}}) + (\mathbf{I} + \mathbf{J}_{\mathcal{L}})^T \delta\mathbf{J}_{\mathcal{L}} \right] \tag{4.86} \\
 &= \frac{1}{2} \left(\delta\mathbf{J}_{\mathcal{L}} + \delta\mathbf{J}_{\mathcal{L}}^T + \delta\mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta\mathbf{J}_{\mathcal{L}} \right)
 \end{aligned}$$

Furthermore, according to equation (1.20), the variation of the displacement gradient tensor can be expressed as:

$$\delta\mathbf{J}_{\mathcal{L}} = \frac{\partial\delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}} \bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \tag{4.87}$$

If the above definition is taken into account, the Green-Lagrange strain tensor variation obtained in (4.85) becomes:

$$\begin{aligned}
 \delta \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} \left(\delta \mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} \right) \\
 &= \frac{1}{2} \left[\left(\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \right)^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \left(\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \right) \right] \\
 &= \mathbf{F}_{\mathcal{L}}^T \frac{1}{2} \left[\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} + \left(\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}}
 \end{aligned} \tag{4.88}$$

Thus, it can be concluded that:

$$\begin{aligned}
 \delta \mathbf{E}_{G,\mathcal{L}} &= \mathbf{F}_{\mathcal{L}}^T \frac{1}{2} \left[\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} + \left(\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \iff \\
 \iff &\frac{1}{2} \left[\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} + \left(\frac{\partial \delta \mathbf{u}_{\varepsilon}}{\partial \mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} = \mathbf{F}_{\mathcal{L}}^{-T} \delta \mathbf{E}_{G,\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-1}
 \end{aligned} \tag{4.89}$$

The above equation is essential to derive the Total Lagrangian finite element formulation. It allows to transform integral equations that are defined over the unknown deformed domain, into equations that are defined with respect to the initial reference configuration, which is well-known.

4.6.6. Green-Lagrange strain tensor increment

In the previous section, a variation ($\delta \mathbf{u}_{\mathcal{L}}$) was applied to the displacement field. Let's now consider that the displacement field is again modified by adding a compatible increment ($\Delta \mathbf{u}_{\mathcal{L}}$), instead of a variation.

In nonlinear analysis, an iterative method has to be applied at each load step to obtain its corresponding displacement field. This increment represents the correction that the iterative method will apply at each step of the procedure, in order to reach convergence.

Let also assume that the gradients of the displacement field increment are small. Under these assumptions, the Green-Lagrange strain tensor corresponding to the mod-

ified displacement field becomes:

$$\begin{aligned}
 & \left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} &= \mathbf{u}_{\mathcal{L}} + \Delta\mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial\Delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} \right\| &\ll \|\mathbf{I}\| \end{aligned} \right\} \implies \\
 \implies & \mathbf{J}'_{\mathcal{L}} = \frac{\partial\mathbf{u}'_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} + \frac{\partial\Delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \Delta\mathbf{J}_{\mathcal{L}} \\
 \implies & \mathbf{r}'_{\mathcal{L}} = \mathbf{r}_0 + \mathbf{u}'_{\mathcal{L}} = \mathbf{r}_0 + (\mathbf{u}_{\mathcal{L}} + \Delta\mathbf{u}_{\mathcal{L}}) = \underbrace{(\mathbf{r}_0 + \mathbf{u}_{\mathcal{L}})}_{\mathbf{r}_{\mathcal{L}}} + \Delta\mathbf{u}_{\mathcal{L}} \\
 \implies & \mathbf{F}'_{\mathcal{L}} = \frac{\partial\mathbf{r}'_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\mathbf{r}_{\mathcal{L}}}{\partial\mathbf{r}_0} + \frac{\partial\Delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \mathbf{F}_{\mathcal{L}} + \underbrace{\Delta\mathbf{F}_{\mathcal{L}}}_{\Delta\mathbf{J}_{\mathcal{L}}} \tag{4.90} \\
 & \mathbf{E}'_{G,\mathcal{L}} = \frac{1}{2} [(\mathbf{F}'_{\mathcal{L}})^T (\mathbf{F}'_{\mathcal{L}}) - \mathbf{I}] \\
 & = \frac{1}{2} [(\mathbf{F}_{\mathcal{L}} + \Delta\mathbf{F}_{\mathcal{L}})^T (\mathbf{F}_{\mathcal{L}} + \Delta\mathbf{F}_{\mathcal{L}}) - \mathbf{I}] \\
 \implies & = \frac{1}{2} (\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \Delta\mathbf{F}_{\mathcal{L}} + \Delta\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \Delta\mathbf{F}_{\mathcal{L}}^T \Delta\mathbf{F}_{\mathcal{L}} - \mathbf{I}) \\
 & = \frac{1}{2} (\mathbf{F}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} - \mathbf{I}) + \frac{1}{2} (\Delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \Delta\mathbf{J}_{\mathcal{L}} + \underbrace{\Delta\mathbf{J}_{\mathcal{L}}^T \Delta\mathbf{J}_{\mathcal{L}}}_{\approx 0}) \\
 & = \mathbf{E}_{G,\mathcal{L}} + \Delta\mathbf{E}_{G,\mathcal{L}}
 \end{aligned}$$

Where $\mathbf{E}_{G,\mathcal{L}}$ is the Green-Lagrange strain tensor corresponding to the original displacement field, and $\Delta\mathbf{E}_{G,\mathcal{L}}$ represents its increment. Therefore, the Green-Lagrange strain tensor increment turns out to be:

$$\Delta\mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} (\Delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \Delta\mathbf{J}_{\mathcal{L}}) \tag{4.91}$$

Moreover, the above tensor can be rewritten if the definition of the deformation gradient tensor stated in (1.12) is taken into account, as:

$$\begin{aligned}
 \Delta\mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\Delta\mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \Delta\mathbf{J}_{\mathcal{L}}) \\
 &= \frac{1}{2} [\Delta\mathbf{J}_{\mathcal{L}}^T (\mathbf{I} + \mathbf{J}_{\mathcal{L}}) + (\mathbf{I} + \mathbf{J}_{\mathcal{L}})^T \Delta\mathbf{J}_{\mathcal{L}}] \tag{4.92} \\
 &= \frac{1}{2} (\Delta\mathbf{J}_{\mathcal{L}} + \Delta\mathbf{J}_{\mathcal{L}}^T + \Delta\mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \Delta\mathbf{J}_{\mathcal{L}})
 \end{aligned}$$

On the other hand, according to equation (1.20), the increment of the displacement gradient tensor can be expressed as:

$$\Delta\mathbf{J}_{\mathcal{L}} = \frac{\partial\Delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}_0} = \frac{\partial\Delta\mathbf{u}_{\mathcal{L}}}{\partial\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}} \tag{4.93}$$

The substitution of the above equation into the definition of the Green-Lagrange strain tensor increment (4.91) leads to the following equivalent definition.

$$\begin{aligned}\Delta \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} \left(\Delta \mathbf{J}_{\mathcal{L}}^T \mathbf{F}_{\mathcal{L}} + \mathbf{F}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) \\ &= \mathbf{F}_{\mathcal{L}}^T \frac{1}{2} \left[\frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}} + \left(\frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}} \right)^T \right] \Bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0,t)} \mathbf{F}_{\mathcal{L}}\end{aligned}\quad (4.94)$$

The increment applied in (4.90) can also be applied to a displacement field composed by the original one ($\mathbf{u}_{\mathcal{L}}$) plus a given variation ($\delta \mathbf{u}_{\mathcal{L}}$). The increment experimented by the Green-Lagrange strain tensor corresponding to this particular displacement field is analysed below.

4.6.7. Increment of the Green-Lagrange strain tensor variation

The Green-Lagrange strain tensor corresponding to a given displacement field can be computed by means of the displacement gradient tensor as:

$$\begin{aligned}\mathbf{u}_{\mathcal{L}} &\implies \mathbf{J}_{\mathcal{L}} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \\ &\implies \mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left(\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T + \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} \right)\end{aligned}\quad (4.95)$$

Let's consider that a compatible variation is applied to the above displacement field, whose gradients are small. The Green-Lagrange strain tensor corresponding to this modified displacement field becomes:

$$\begin{aligned}\left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} &= \mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \end{aligned} \right\} \implies \\ \implies \mathbf{J}'_{\mathcal{L}} &= \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}} \\ \mathbf{E}'_{G,\mathcal{L}} &= \frac{1}{2} \left[(\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) + (\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}})^T + (\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}})^T (\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) \right] \\ \implies &\approx \frac{1}{2} \left(\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T + \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} \right) + \frac{1}{2} \left(\delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T + \delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} \right) \\ &= \mathbf{E}_{G,\mathcal{L}} + \delta \mathbf{E}_{G,\mathcal{L}}\end{aligned}\quad (4.96)$$

Where $\mathbf{E}_{G,\mathcal{L}}$ is the Green-Lagrange strain tensor corresponding to the original displacement field, and $\delta \mathbf{E}_{G,\mathcal{L}}$ represents its variation. Therefore, the Green-Lagrange strain tensor variation is defined as:

$$\delta \mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left(\delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T + \delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} \right) \quad (4.97)$$

Let's now consider that an increment is applied to the previous displacement field, whose gradients are small, too. Under these assumptions, the computation of the

Green-Lagrange strain tensor is presented below.

$$\begin{aligned}
 & \left. \begin{aligned} \mathbf{u}_{\mathcal{L}}'' &= \mathbf{u}_{\mathcal{L}} + \Delta \mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \\ \left\| \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \end{aligned} \right\} \Rightarrow \\
 \Rightarrow \quad \mathbf{J}_{\mathcal{L}}'' &= \frac{\partial \mathbf{u}_{\mathcal{L}}''}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}} \\
 \mathbf{E}_{G,\mathcal{L}}'' &= \frac{1}{2} [\mathbf{J}_{\mathcal{L}}'' + (\mathbf{J}_{\mathcal{L}}'')^T + (\mathbf{J}_{\mathcal{L}}'')^T \mathbf{J}_{\mathcal{L}}''] \\
 &= \frac{1}{2} [(\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) + (\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}})^T + (\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}})^T (\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}})] \\
 &= \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T + \Delta \mathbf{J}_{\mathcal{L}}^T + \delta \mathbf{J}_{\mathcal{L}}^T + \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} + \\
 \Rightarrow \quad & + \Delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \underbrace{\Delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}}}_{\approx 0} + \Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \underbrace{\delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}}}_{\approx 0}) \\
 &\approx \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T + \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}}) + \frac{1}{2} (\delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T + \delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}}) + \\
 & + \frac{1}{2} (\Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T + \Delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}}) + \frac{1}{2} (\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}}) \\
 &= \mathbf{E}_{G,\mathcal{L}} + \delta \mathbf{E}_{G,\mathcal{L}} + \Delta \mathbf{E}_{G,\mathcal{L}} + \Delta (\delta \mathbf{E}_{G,\mathcal{L}})
 \end{aligned} \tag{4.98}$$

Where $\mathbf{E}_{G,\mathcal{L}}$ is the Green-Lagrange strain tensor corresponding to the original displacement field, $\delta \mathbf{E}_{G,\mathcal{L}}$ and $\Delta \mathbf{E}_{G,\mathcal{L}}$ symbolize its variation and increment, and $\Delta (\delta \mathbf{E}_{G,\mathcal{L}})$ represents the increment experimented by the Green-Lagrange strain tensor variation. Therefore, the Green-Lagrange strain tensor can be decomposed as the sum of the following tensors:

$$\mathbf{E}_{G,\mathcal{L}}'' = \mathbf{E}_{G,\mathcal{L}} + \delta \mathbf{E}_{G,\mathcal{L}} + \Delta \mathbf{E}_{G,\mathcal{L}} + \Delta (\delta \mathbf{E}_{G,\mathcal{L}}) \tag{4.99}$$

Where:

$$\begin{aligned}
 \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T + \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}}) \\
 \delta \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\delta \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}^T + \delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}}) \\
 \Delta \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T + \Delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}}) \\
 \Delta (\delta \mathbf{E}_{G,\mathcal{L}}) &= \frac{1}{2} (\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}})
 \end{aligned} \tag{4.100}$$

Therefore, the increment of the Green-Lagrange strain tensor variation is finally defined as:

$$\Delta (\delta \mathbf{E}_{G,\mathcal{L}}) = \frac{1}{2} (\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}}) \tag{4.101}$$

Once the Green-Lagrange strain tensor, as well as its variation and increment, are completely defined, it is convenient to propose their equivalent vectorial forms. When deriving a finite element formulation, manipulating their vectorial expressions is more convenient than operating with their tensor definitions.

4.6.8. Vectorial form of the Green-Lagrange strain tensor

If the Voigt notation (section A.14) is applied, an equivalent vectorial form of the Green-Lagrange strain tensor can be defined.

$$\mathbf{E}_{G,\mathcal{L}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \implies \bar{\mathbf{E}}_{G,\mathcal{L}} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{13} \\ 2E_{23} \end{Bmatrix} \quad (4.102)$$

The components of the Green-Lagrange strain tensor were defined in (4.74). Thus, its corresponding vectorial form is:

$$\bar{\mathbf{E}}_{G,\mathcal{L}} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{13} \\ 2E_{23} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial r_{0,1}} \right)^2 + \left(\frac{\partial u_2}{\partial r_{0,1}} \right)^2 + \left(\frac{\partial u_3}{\partial r_{0,1}} \right)^2 \right] \\ \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial r_{0,2}} \right)^2 + \left(\frac{\partial u_2}{\partial r_{0,2}} \right)^2 + \left(\frac{\partial u_3}{\partial r_{0,2}} \right)^2 \right] \\ \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial r_{0,3}} \right)^2 + \left(\frac{\partial u_2}{\partial r_{0,3}} \right)^2 + \left(\frac{\partial u_3}{\partial r_{0,3}} \right)^2 \right] \\ \frac{\partial u_1}{\partial r_{0,1}} \frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \frac{\partial u_2}{\partial r_{0,2}} + \frac{\partial u_3}{\partial r_{0,1}} \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,1}} \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_2}{\partial r_{0,1}} \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_2}{\partial r_{0,2}} \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \frac{\partial u_3}{\partial r_{0,3}} \end{Bmatrix} \quad (4.103)$$

The first term of the above vector turns out to be the vectorial form of the infinitesimal strain tensor (5.69). And the second one depends on the square of the norm of the displacement gradient tensor. Thus, the vectorial form of the Green-Lagrange strain tensor can be equivalently defined as follows.

$$\bar{\mathbf{E}}_{G,\mathcal{L}} = \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} + \mathcal{O}(\|\mathbf{J}_{\mathcal{L}}\|^2) \quad (4.104)$$

If the gradients of the displacement field are small, the second term can be neglected. Hence, it can be stated that the vectorial forms of the Green-Lagrange strain tensor and the infinitesimal strain tensor are equivalent.

$$\|\mathbf{J}_{\mathcal{L}}\| \ll \|\mathbf{I}\| \implies \bar{\mathbf{E}}_{G,\mathcal{L}} = \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} + \mathcal{O}(\|\mathbf{J}_{\mathcal{L}}\|^2) \approx \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} \quad (4.105)$$

In order to rewrite both terms of the vector defined in (4.103), a vectorial form of

the displacement gradient tensor is also defined.

$$\mathbf{J}_{\mathcal{L}} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_1}{\partial r_{0,2}} & \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,2}} & \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} & \frac{\partial u_3}{\partial r_{0,2}} & \frac{\partial u_3}{\partial r_{0,3}} \end{bmatrix} \implies \bar{\mathbf{J}}_{\mathcal{L}} = \left\{ \begin{array}{c} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \end{array} \right\} \quad (4.106)$$

The vectorial form of the infinitesimal strain tensor can be now defined in terms of the above vector as:

$$\bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} = \left\{ \begin{array}{c} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \end{array} \right\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \end{array} \right\} = \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} \quad (4.107)$$

The second term can also be expressed by means of the vectorial form of the dis-

placement gradient tensor.

$$\frac{1}{2} \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & 0 & 0 & \frac{\partial u_2}{\partial r_{0,1}} & 0 & 0 & \frac{\partial u_3}{\partial r_{0,1}} & 0 & 0 \\ 0 & \frac{\partial u_1}{\partial r_{0,2}} & 0 & 0 & \frac{\partial u_2}{\partial r_{0,2}} & 0 & 0 & \frac{\partial u_3}{\partial r_{0,2}} & 0 \\ 0 & 0 & \frac{\partial u_1}{\partial r_{0,3}} & 0 & 0 & \frac{\partial u_2}{\partial r_{0,3}} & 0 & 0 & \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} & \frac{\partial u_1}{\partial r_{0,1}} & 0 & \frac{\partial u_2}{\partial r_{0,2}} & \frac{\partial u_2}{\partial r_{0,1}} & 0 & \frac{\partial u_3}{\partial r_{0,2}} & \frac{\partial u_3}{\partial r_{0,1}} & 0 \\ \frac{\partial u_1}{\partial r_{0,3}} & 0 & \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,3}} & 0 & \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_3}{\partial r_{0,3}} & 0 & \frac{\partial u_3}{\partial r_{0,1}} \\ 0 & \frac{\partial u_1}{\partial r_{0,3}} & \frac{\partial u_1}{\partial r_{0,2}} & 0 & \frac{\partial u_2}{\partial r_{0,3}} & \frac{\partial u_2}{\partial r_{0,2}} & 0 & \frac{\partial u_3}{\partial r_{0,3}} & \frac{\partial u_3}{\partial r_{0,2}} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \end{Bmatrix} = \frac{1}{2} \mathbf{A}(\mathbf{J}_\mathcal{L}) \bar{\mathbf{J}}_\mathcal{L} \quad (4.108)$$

Therefore, if the previous terms are taken into account, the vectorial form of the Green-Lagrange strain tensor becomes:

$$\bar{\mathbf{E}}_{G,\mathcal{L}} = \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_\mathcal{L}) \right) \bar{\mathbf{J}}_\mathcal{L} \quad (4.109)$$

Where the matrices \mathbf{A}_C and \mathbf{A} are defined in (4.107) and (4.108) respectively. The matrix \mathbf{A}_C is constant, while \mathbf{A} turns out to depend on the displacement gradient tensor, that is, it depends on the displacement field.

4.6.9. Vectorial form of the Green-Lagrange strain tensor variation

As proved in the previous section, the vectorial form of the Green-Lagrange strain tensor corresponding to a given displacement field is computed as follows.

$$\begin{aligned} \mathbf{u}_\mathcal{L} &\implies \mathbf{J}_\mathcal{L} = \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \\ &\implies \bar{\mathbf{E}}_{G,\mathcal{L}} = \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_\mathcal{L}) \right) \bar{\mathbf{J}}_\mathcal{L} \end{aligned} \quad (4.110)$$

If the displacement field is now modified by adding a compatible variation, the computation of its corresponding Green-Lagrange strain tensor, expressed according to its vectorial form, is shown below. The gradients of the displacement field variation

are again assumed to be small.

$$\begin{aligned}
 & \left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} &= \mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \end{aligned} \right\} \implies \\
 \implies \quad \mathbf{J}'_{\mathcal{L}} &= \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}} \\
 \bar{\mathbf{E}}'_{G,\mathcal{L}} &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}'_{\mathcal{L}}) \right) \bar{\mathbf{J}}'_{\mathcal{L}} \\
 &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) \right) (\bar{\mathbf{J}}_{\mathcal{L}} + \delta \bar{\mathbf{J}}_{\mathcal{L}}) \\
 &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}}) \delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 \implies &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \left(\mathbf{A}(\mathbf{J}_{\mathcal{L}}) + \mathbf{A}(\delta \mathbf{J}_{\mathcal{L}}) \right) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \left(\mathbf{A}(\mathbf{J}_{\mathcal{L}}) + \mathbf{A}(\delta \mathbf{J}_{\mathcal{L}}) \right) \delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \underbrace{\mathbf{A}(\delta \mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}}}_{= \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \delta \bar{\mathbf{J}}_{\mathcal{L}}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \underbrace{\mathbf{A}(\delta \mathbf{J}_{\mathcal{L}}) \delta \bar{\mathbf{J}}_{\mathcal{L}}}_{\approx \mathbf{0}} \\
 &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \bar{\mathbf{J}}_{\mathcal{L}} + \left(\mathbf{A}_C + \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 &= \bar{\mathbf{E}}_{G,\mathcal{L}} + \delta \bar{\mathbf{E}}_{G,\mathcal{L}}
 \end{aligned} \tag{4.111}$$

Where $\bar{\mathbf{E}}_{G,\mathcal{L}}$ is the vectorial form of the Green-Lagrange strain tensor corresponding to the original displacement field, and $\delta \bar{\mathbf{E}}_{G,\mathcal{L}}$ represents its variation. Therefore, the vectorial form of the Green-Lagrange strain tensor variation turns out to be:

$$\delta \bar{\mathbf{E}}_{G,\mathcal{L}} = \left(\mathbf{A}_C + \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \delta \bar{\mathbf{J}}_{\mathcal{L}} \tag{4.112}$$

4.6.10. Vectorial form of the Green-Lagrange strain tensor increment

Let's consider that the displacement field is again modified by adding a compatible increment ($\Delta \mathbf{u}_{\mathcal{L}}$), instead of a variation ($\delta \mathbf{u}_{\mathcal{L}}$). It is also assumed that the gradients of the displacement field increment are small.

In nonlinear analysis, an iterative method has to be applied at each load step to obtain its corresponding displacement field. This increment represents the correction that the iterative method will apply at each step of the procedure, in order to reach convergence.

Under these assumptions, the vectorial form of the Green-Lagrange strain tensor

corresponding to the modified displacement field becomes:

$$\begin{aligned}
 & \left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} &= \mathbf{u}_{\mathcal{L}} + \Delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \end{aligned} \right\} \implies \\
 \implies \quad \mathbf{J}'_{\mathcal{L}} &= \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} \\
 \bar{\mathbf{E}}'_{G,\mathcal{L}} &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}'_{\mathcal{L}}) \right) \bar{\mathbf{J}}'_{\mathcal{L}} \\
 &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}) \right) (\bar{\mathbf{J}}_{\mathcal{L}} + \Delta \bar{\mathbf{J}}_{\mathcal{L}}) \\
 &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \Delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}) \Delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 \implies &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \Delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \left(\mathbf{A}(\mathbf{J}_{\mathcal{L}}) + \mathbf{A}(\Delta \mathbf{J}_{\mathcal{L}}) \right) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \left(\mathbf{A}(\mathbf{J}_{\mathcal{L}}) + \mathbf{A}(\Delta \mathbf{J}_{\mathcal{L}}) \right) \Delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 &= \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \Delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \underbrace{\mathbf{A}(\Delta \mathbf{J}_{\mathcal{L}}) \bar{\mathbf{J}}_{\mathcal{L}}}_{= \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \Delta \bar{\mathbf{J}}_{\mathcal{L}}} + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \Delta \bar{\mathbf{J}}_{\mathcal{L}} + \frac{1}{2} \underbrace{\mathbf{A}(\Delta \mathbf{J}_{\mathcal{L}}) \Delta \bar{\mathbf{J}}_{\mathcal{L}}}_{\approx \mathbf{0}} \\
 &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \bar{\mathbf{J}}_{\mathcal{L}} + \left(\mathbf{A}_C + \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \Delta \bar{\mathbf{J}}_{\mathcal{L}} \\
 &= \bar{\mathbf{E}}_{G,\mathcal{L}} + \Delta \bar{\mathbf{E}}_{G,\mathcal{L}}
 \end{aligned} \tag{4.113}$$

Where $\bar{\mathbf{E}}_{G,\mathcal{L}}$ is the vectorial form of the Green-Lagrange strain tensor corresponding to the original displacement field, and $\Delta \bar{\mathbf{E}}_{G,\mathcal{L}}$ represents its increment. Consequently, the vectorial form of the Green-Lagrange strain tensor increment turns out to be:

$$\Delta \bar{\mathbf{E}}_{G,\mathcal{L}} = \left(\mathbf{A}_C + \mathbf{A}(\mathbf{J}_{\mathcal{L}}) \right) \Delta \bar{\mathbf{J}}_{\mathcal{L}} \tag{4.114}$$

4.7. Incremental loading process

When dealing with a nonlinear analysis, the total load can not be applied in only one step, since a given load state has multiple possible solutions. This implies that the order in which the external loads are applied has to be taken into account. Otherwise, a structural behaviour that does not correspond to the real one may be obtained.

Furthermore, the equations to be solved compose a nonlinear system of equations. So, an iterative procedure is usually applied to obtain the structural response. This procedure is iterative and needs to start from a close approximation to the solution. To circumvent this inconvenience, an incremental loading process is usually adopted. That is, the total load is applied in a series of small load steps. In figure 4.5, the material domains that corresponds to two consecutive load steps are represented.

4.7.1. Total approach

On the one hand, the total approach is adopted if all magnitudes are referred to the initial reference configuration Ω_0 . That is, the magnitudes are defined according to their Lagrangian description.

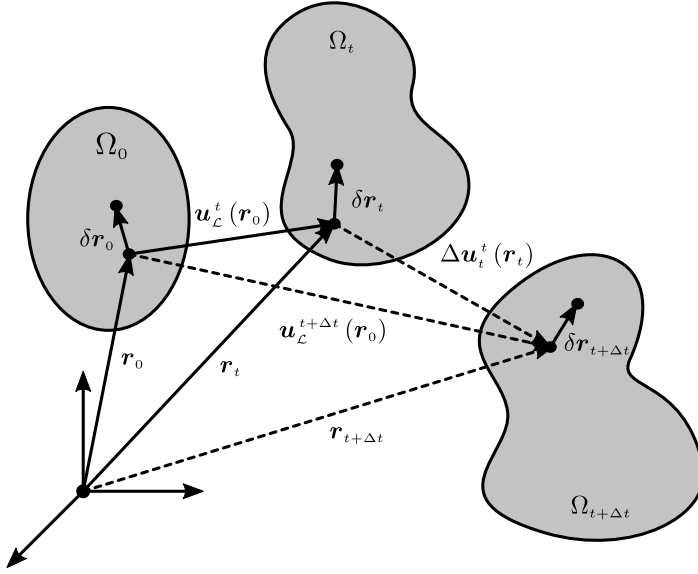


Figure 4.5. Deformation, displacement, and material vectors of a given particle, corresponding to two consecutive load steps.

Let's consider two consecutive load steps. The load step t has already been solved, thus, the main aim is to compute the displacement field that defines the unknown material domain $\Omega_{t+\Delta t}$.

The deformation vectors that define the position of a given material particle, whose initial position is defined by the vector \mathbf{r}_0 , are defined below.

$$\begin{aligned}\mathbf{r}_t &= \mathbf{r}_L^t(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_L^t(\mathbf{r}_0) \\ \mathbf{r}_{t+\Delta t} &= \mathbf{r}_L^{t+\Delta t}(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_L^{t+\Delta t}(\mathbf{r}_0)\end{aligned}\quad (4.115)$$

Their corresponding deformation gradient tensors are defined as follows.

$$\begin{aligned}\mathbf{F}_L^t &= \frac{d\mathbf{r}_L^t}{d\mathbf{r}_0} = \mathbf{I} + \frac{d\mathbf{u}_L^t}{d\mathbf{r}_0} = \mathbf{I} + \mathbf{J}_L^t \\ \mathbf{F}_L^{t+\Delta t} &= \frac{d\mathbf{r}_L^{t+\Delta t}}{d\mathbf{r}_0} = \mathbf{I} + \frac{d\mathbf{u}_L^{t+\Delta t}}{d\mathbf{r}_0} = \mathbf{I} + \mathbf{J}_L^{t+\Delta t}\end{aligned}\quad (4.116)$$

And the displacement gradient tensors are:

$$\begin{aligned}\mathbf{J}_L^t &= \frac{d\mathbf{u}_L^t}{d\mathbf{r}_0} \\ \mathbf{J}_L^{t+\Delta t} &= \frac{d\mathbf{u}_L^{t+\Delta t}}{d\mathbf{r}_0}\end{aligned}\quad (4.117)$$

Furthermore, the displacement field corresponding to the current load step $t + \Delta t$ can be defined by means of the displacement field of the previous load step, by adding

a given increment (figure 4.5).

$$\begin{aligned}\mathbf{u}_{\mathcal{L}}^{t+\Delta t}(\mathbf{r}_0) &= \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) + \Delta \mathbf{u}_t^t(\mathbf{r}_t) \Big|_{\mathbf{r}_t=\mathbf{r}_{\mathcal{L}}^t(\mathbf{r}_0)} \\ &= \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) + \Delta \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0)\end{aligned}\quad (4.118)$$

If the above definition is taken into account, the deformation vector of the current load step can be equivalently defined by means of the deformation vector of the previous one.

$$\begin{aligned}\mathbf{r}_{\mathcal{L}}^{t+\Delta t}(\mathbf{r}_0) &= \mathbf{r}_0 + \mathbf{u}_{\mathcal{L}}^{t+\Delta t}(\mathbf{r}_0) \\ &= \mathbf{r}_0 + \left[\mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) + \Delta \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) \right] \\ &= \left[\mathbf{r}_0 + \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) \right] + \Delta \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) \\ &= \mathbf{r}_{\mathcal{L}}^t(\mathbf{r}_0) + \Delta \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0)\end{aligned}\quad (4.119)$$

According to the above definition, its corresponding deformation gradient tensor becomes:

$$\mathbf{F}_{\mathcal{L}}^{t+\Delta t} = \frac{d\mathbf{r}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} = \frac{d\mathbf{r}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} = \mathbf{F}_{\mathcal{L}}^t + \Delta \mathbf{F}_{\mathcal{L}}^t \quad (4.120)$$

And the displacement gradient tensor turns out to be:

$$\mathbf{J}_{\mathcal{L}}^{t+\Delta t} = \frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} = \frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} = \mathbf{J}_{\mathcal{L}}^t + \Delta \mathbf{J}_{\mathcal{L}}^t \quad (4.121)$$

If equations (4.120) and (4.121) are compared, it can be concluded that the increment of the deformation gradient tensor and the increment of the displacement gradient tensor are equivalent.

$$\Delta \mathbf{F}_{\mathcal{L}}^t = \Delta \mathbf{J}_{\mathcal{L}}^t = \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \quad (4.122)$$

Once the deformation vector and the displacement vectors of the current load step are defined, as well as their corresponding gradient tensors, the increment experimented by the Green-Lagrange strain tensor between consecutive load steps can be analysed.

4.7.2. Total incremental Green-Lagrange strain tensor

The incremental Green-Lagrange strain tensor is defined as the difference between the strain tensors corresponding to two consecutive load steps.

$$\Delta \mathbf{E}_{G,\mathcal{L}}^t = \mathbf{E}_{G,\mathcal{L}}^{t+\Delta t} - \mathbf{E}_{G,\mathcal{L}}^t \quad (4.123)$$

The Green-Lagrange strain tensor corresponding to the load step t is defined as:

$$\mathbf{E}_{G,\mathcal{L}}^t = \frac{1}{2} \left[\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T + \left(\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T \frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right] \quad (4.124)$$

And the one corresponding to the following load step is:

$$\mathbf{E}_{G,\mathcal{L}}^{t+\Delta t} = \frac{1}{2} \left[\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right)^T + \left(\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right)^T \frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right] \quad (4.125)$$

Therefore, the difference between the last two tensors defines the increment of the Green-Lagrange strain tensor.

$$\begin{aligned} \Delta \mathbf{E}_{G,\mathcal{L}}^t &= \frac{1}{2} \left[\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right)^T + \left(\frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right)^T \frac{d\mathbf{u}_{\mathcal{L}}^{t+\Delta t}}{d\mathbf{r}_0} \right] \\ &\quad - \frac{1}{2} \left[\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T + \left(\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T \frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right] \end{aligned} \quad (4.126)$$

In addition, the displacement field is defined as stated in (4.118).

$$\mathbf{u}_{\mathcal{L}}^{t+\Delta t}(\mathbf{r}_0) = \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) + \Delta \mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0) \quad (4.127)$$

If the above definition is taken into account, the increment of the Green-Lagrange strain tensor is reduced to:

$$\Delta \mathbf{E}_{G,\mathcal{L}}^t = \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \left(\frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T + \left(\frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \left(\frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T \frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} + \left(\frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right)^T \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right] \quad (4.128)$$

And the components of the above tensor are defined as follows:

$$\begin{aligned} \Delta \mathbf{E}_{G,\mathcal{L}}^t &= [\Delta E_{ij}^t]_{\substack{j=1,\dots,n \\ i=1,\dots,n}} \\ \Delta E_{ij}^t &= \frac{1}{2} \left(\frac{d\Delta u_i}{dr_{0,j}} + \frac{d\Delta u_j}{dr_{0,i}} + \sum_{k=1}^n \frac{du_k}{dr_{0,i}} \frac{d\Delta u_k}{dr_{0,j}} + \sum_{k=1}^n \frac{d\Delta u_k}{dr_{0,i}} \frac{du_k}{dr_{0,j}} + \sum_{k=1}^n \frac{d\Delta u_k}{dr_{0,i}} \frac{d\Delta u_k}{dr_{0,j}} \right) \end{aligned} \quad (4.129)$$

Moreover, they can be separated into two different components:

$$\Delta E_{ij}^t = \Delta E_{ij}^L + \Delta E_{ij}^N \quad (4.130)$$

Where ΔE_{ij}^L is composed by the terms that are linear with respect to the increment of the displacement field. The first two addends define the increment of the infinitesimal strain tensor $\Delta \mathcal{E}_{ij}$ (5.66), and the last ones depend on the displacement field of the previous load step.

$$\Delta E_{ij}^L = \underbrace{\frac{1}{2} \left(\frac{d\Delta u_i}{dr_{0,j}} + \frac{d\Delta u_j}{dr_{0,i}} \right)}_{\Delta \mathcal{E}_{ij}} + \frac{1}{2} \sum_{k=1}^n \left(\frac{du_k}{dr_{0,i}} \frac{d\Delta u_k}{dr_{0,j}} + \frac{d\Delta u_k}{dr_{0,i}} \frac{du_k}{dr_{0,j}} \right) \quad (4.131)$$

And ΔE_{ij}^N contains the terms that are nonlinear with respect to the increment of the displacement field.

$$\Delta E_{ij}^N = \frac{1}{2} \sum_{k=1}^n \frac{d\Delta u_k}{dr_{0,i}} \frac{d\Delta u_k}{dr_{0,j}} \quad (4.132)$$

Note that the definition of this incremental strain tensor can also be obtained by computing the difference between the square of the norm of a given material vector at

two consecutive load steps.

$$\begin{aligned}
 \|\delta\mathbf{r}_{t+\Delta t}\|^2 - \|\delta\mathbf{r}_t\|^2 &= \left(\|\delta\mathbf{r}_{t+\Delta t}\|^2 - \|\delta\mathbf{r}_0\|^2\right) - \left(\|\delta\mathbf{r}_t\|^2 - \|\delta\mathbf{r}_0\|^2\right) \\
 &= 2\delta\mathbf{r}_0^T \mathbf{E}_{G,\mathcal{L}}^{t+\Delta t} \delta\mathbf{r}_0 - 2\delta\mathbf{r}_0^T \mathbf{E}_{G,\mathcal{L}}^t \delta\mathbf{r}_0 \\
 &= 2\delta\mathbf{r}_0^T \left(\mathbf{E}_{G,\mathcal{L}}^{t+\Delta t} - \mathbf{E}_{G,\mathcal{L}}^t\right) \delta\mathbf{r}_0 \\
 &= 2\delta\mathbf{r}_0^T \Delta \mathbf{E}_{G,\mathcal{L}}^t \delta\mathbf{r}_0
 \end{aligned} \tag{4.133}$$

4.7.3. Updated approach

The second option is to adopt the updated approach instead of the total one. According to this updated point of view, the last computed material domain becomes the new reference configuration.

Let's consider dealing with the load step $t + \Delta t$, and the configuration Ω_t becomes the reference domain. Under these assumptions, the deformation vector that defines the position of a given material particle that belongs to the unknown material domain turns out to be:

$$\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) = \mathbf{r}_t + \Delta \mathbf{u}_t^t(\mathbf{r}_t) \tag{4.134}$$

The material vector that corresponds to the current load step can be obtained by means of the one that belongs to the reference configuration, as stated below.

$$\begin{aligned}
 \delta\mathbf{r}_{t+\Delta t} &= \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t + \delta\mathbf{r}_t) - \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) \\
 &= \left[\mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) + \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t}(\mathbf{r}_t) \delta\mathbf{r}_t + \mathcal{O}(\|\delta\mathbf{r}_t\|^2) \right] - \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) \\
 &= \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \delta\mathbf{r}_t + \mathcal{O}(\|\delta\mathbf{r}_t\|^2)
 \end{aligned} \tag{4.135}$$

Therefore, the tensor that rules this geometric transformation turns out to be:

$$\frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} = \frac{d}{d\mathbf{r}_t}(\mathbf{r}_t + \Delta \mathbf{u}_t^t) = \mathbf{I} + \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \tag{4.136}$$

4.7.4. Updated incremental Green-Lagrange strain tensor

To obtain the updated incremental Green-Lagrange strain tensor, the difference between the square of the norm of a given infinitesimal material vector that corresponds to two consecutive load steps is computed, as exposed in (4.133).

$$\|\delta\mathbf{r}_{t+\Delta t}\|^2 - \|\delta\mathbf{r}_t\|^2 = \delta\mathbf{r}_{t+\Delta t}^T \delta\mathbf{r}_{t+\Delta t} - \delta\mathbf{r}_t^T \delta\mathbf{r}_t \tag{4.137}$$

However, the updated approach is now adopted instead of the total one. That is, the last computed material domain becomes the reference configuration, instead of the initial one.

The definition of the current material vector by means of the previous one was defined in (4.135) and (4.136).

$$\delta \mathbf{r}_{t+\Delta t} = \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \delta \mathbf{r}_t = \left(\mathbf{I} + \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right) \delta \mathbf{r}_t \quad (4.138)$$

Thus, equation (4.137) becomes:

$$\begin{aligned} \|\delta \mathbf{r}_{t+\Delta t}\|^2 - \|\delta \mathbf{r}_t\|^2 &= \delta \mathbf{r}_{t+\Delta t}^T \delta \mathbf{r}_{t+\Delta t} - \delta \mathbf{r}_t^T \delta \mathbf{r}_t \\ &= \delta \mathbf{r}_t^T \left(\frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \right)^T \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \delta \mathbf{r}_t - \delta \mathbf{r}_t^T \delta \mathbf{r}_t \\ &= \delta \mathbf{r}_t^T \left[\left(\frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \right)^T \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} - \mathbf{I} \right] \delta \mathbf{r}_t \\ &= \delta \mathbf{r}_t^T \left[\left(\mathbf{I} + \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right)^T \left(\mathbf{I} + \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right) - \mathbf{I} \right] \delta \mathbf{r}_t \\ &= 2 \delta \mathbf{r}_t^T \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} + \left(\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right)^T + \left(\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right)^T \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right] \delta \mathbf{r}_t \\ &= 2 \delta \mathbf{r}_t^T \Delta \mathbf{E}_{G,t}^t \delta \mathbf{r}_t \end{aligned} \quad (4.139)$$

And the updated incremental Green-Lagrange strain tensor turns out to be:

$$\Delta \mathbf{E}_{G,t}^t = \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} + \left(\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right)^T + \left(\frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right)^T \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right] \quad (4.140)$$

And the components of the above tensor can be defined as:

$$\begin{aligned} \Delta \mathbf{E}_{G,t}^t &= [\Delta E_{ij}]_{\substack{j=1,\dots,n \\ i=1,\dots,n}} \\ \Delta E_{ij} &= \frac{1}{2} \left(\frac{d\Delta u_i}{dr_{t,j}} + \frac{d\Delta u_j}{dr_{t,i}} + \sum_{k=1}^n \frac{d\Delta u_k}{dr_{t,i}} \frac{d\Delta u_k}{dr_{t,j}} \right) \end{aligned} \quad (4.141)$$

Moreover, they can be separated into two different components.

$$\begin{aligned} \Delta E_{ij} &= \Delta E_{ij}^L + \Delta E_{ij}^N \\ \Delta E_{ij}^L &= \frac{1}{2} \left(\frac{d\Delta u_i}{dr_{t,j}} + \frac{d\Delta u_j}{dr_{t,i}} \right) \\ \Delta E_{ij}^N &= \frac{1}{2} \sum_{k=1}^n \frac{d\Delta u_k}{dr_{t,i}} \frac{d\Delta u_k}{dr_{t,j}} \end{aligned} \quad (4.142)$$

The first one is composed by the terms that are linear with respect to the increment of the displacement field. And the second one contains the term which is nonlinear with respect to the increment of the displacement field.

4.8. Overview and conclusions

In order to understand the deformation that a solid experiments in the vicinity of a given material particle, the geometric transformation of a material vector is analysed in detail throughout this chapter. This transformation turns out to be governed by the deformation gradient tensor ($\mathbf{F}_\mathcal{L}$), which, in turn, depends on the displacement gradient tensor ($\mathbf{J}_\mathcal{L}$). Therefore, it can be concluded that the displacement gradient tensor contains the necessary information to define the change of volume, orientation, and shape that the solid undergoes. This information is required to properly define the displacement, strain, and stress fields.

The polar decomposition of the deformation gradient tensor leads to a clearer physical interpretation of the geometric transformation undergone by the material vector. It allows to decompose the deformation gradient tensor as the product between the finite rotation tensor ($\mathbf{R}_\mathcal{L}$) and the finite strain tensor ($[\mathbf{I} + \mathbf{E}_\mathcal{L}]$), where $\mathbf{E}_\mathcal{L}$ is the Biot strain tensor. Firstly, the finite strain tensor modifies the modulus and direction of the material vector, by means of a pure stretch transformation. Then, the finite rotation tensor rotates the previous modified material vector to orientate it according to the deformed geometry.

An alternative polar decomposition can also be considered. This decomposition divides the finite strain tensor as the product between the inflation tensor ($\mathbf{H}_\mathcal{L}$) and the distortion tensor ($\mathbf{D}_\mathcal{L}$). The inflation tensor rules the change of volume in the vicinity of a given material particle, whereas the distortion tensor is the one that governs the distortion process without volume variation.

Another tensor that fulfils interesting properties is the Green-Lagrange strain tensor ($\mathbf{E}_{G,\mathcal{L}}$). This tensor arises when the difference between the square of the norm of a given material vector at a given instant of time and the square of the norm of its initial geometry is computed. The Green-Lagrange strain tensor does not vary if a rotation is applied to the solid. This is an important property when dealing with a solid that behaves with large displacements and/or large displacement gradients, since a rigid rotation does not produce a variation of the strain field. In addition, this tensor turns out to be work conjugate with the second Piola-Kirchhoff stress tensor, which is also invariant when rigid rotations are applied. Thus, their double dot product leads to the work per unit volume developed by the internal forces during the deformation process. These tensors are a suitable pair of magnitudes to represent the strain and stress fields when carrying out a nonlinear analysis. Consequently, a constitutive equation that relate both magnitudes has to be defined.

On the other hand, the incremental loading process usually adopted in nonlinear analysis is presented. Since a given load state has multiple possible solutions, the total load can not be applied in only one step, and the load history has to be taken into account to reach the correct structural response. Furthermore, the nonlinear equations that govern the structural behaviour are usually solved by an iterative procedure, which needs to start from a close approximation to the solution. The incremental loading

process allows starting from a close approximation, as well as to take into account the load history.

Finally, two analysis approaches can be adopted, depending on the material domain taken as the reference configuration. The total approach references all magnitudes to the initial configuration during the entire incremental loading process, whereas the updated one updates the reference domain at each load step. That is, the material domain computed at the previous load step becomes the reference one at each incremental step.

Infinitesimal strain field

5.1. Introduction

In the previous chapters, the geometric transformation suffered by a material vector was not considered to be infinitesimal. Along this one, infinitesimal geometric transformations are considered. It should be noted that this consideration does not imply that the solid behaves with small displacements. Although the material vectors experiment an infinitesimal geometric transformation, the solid may experience large displacements.

The mathematical condition that allows to state that a geometric transformation is infinitesimal is derived, as well as the particular polar decomposition that can be applied in this specific case. The polar decomposition of the deformation gradient tensor is simpler to obtain, since solving an eigenvalue problem is not yet required. Therefore, the structural analysis is simplified and computational advantages arise.

5.2. Geometric transformation of a material vector

As proved in section 1.10, the definition of a material vector at a given instant of time t can be obtained by calculating the difference between the deformation vector corresponding to two close material particles (figure 5.1).

$$\begin{aligned}
 \delta \mathbf{r} &= \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0 + \delta \mathbf{r}_0, t) - \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) \\
 &= \left[\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) + \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0}(\mathbf{r}_0, t) \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \right] - \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0, t) \\
 &= \frac{\partial \mathbf{r}_{\mathcal{L}}}{\partial \mathbf{r}_0}(\mathbf{r}_0, t) \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2)
 \end{aligned} \tag{5.1}$$

Therefore, it can be concluded that the deformation gradient tensor rules the geo-

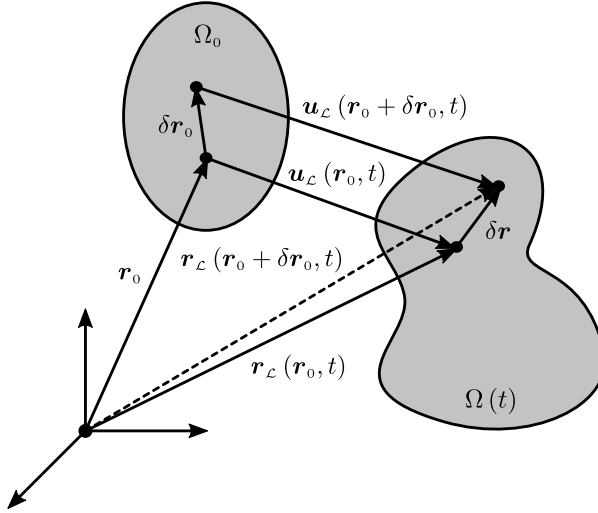


Figure 5.1. Geometric transformation of a material vector at a given instant of time.

metric transformation of a given material vector over time.

$$\begin{aligned}
 \delta \mathbf{r} &= \mathbf{F}_L \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \\
 \mathbf{F}_L &= \frac{\partial \mathbf{r}_L}{\partial \mathbf{r}_0} = \mathbf{I} + \mathbf{J}_L \\
 \mathbf{J}_L &= \frac{\partial \mathbf{u}_L}{\partial \mathbf{r}_0}
 \end{aligned} \tag{5.2}$$

If the norm of the undeformed material vector is much lower than one, the second term can be neglected, and the above equation is reduced to:

$$\begin{aligned}
 \delta \mathbf{r} &\approx \mathbf{F}_L \delta \mathbf{r}_0 \\
 \|\delta \mathbf{r}_0\| \ll 1 \quad \implies \quad &= [\mathbf{I} + \mathbf{J}_L] \delta \mathbf{r}_0 \\
 &= \delta \mathbf{r}_0 + \mathbf{J}_L \delta \mathbf{r}_0
 \end{aligned} \tag{5.3}$$

5.2.1. Infinitesimal geometric transformation

According to the result obtained in (5.3), if the norm of the initial material vector is much lower than one, its geometric transformation is defined by adding another vector that depends on the displacement gradient tensor to the original one.

$$\delta \mathbf{r} \approx \delta \mathbf{r}_0 + \mathbf{J}_L \delta \mathbf{r}_0 \tag{5.4}$$

This geometric transformation is considered to be infinitesimal when the norm of the vector added to the original material vector is much lower than the norm of the original one. If this condition holds, it can be assumed that the norm of the material vector does not vary.

$$\|\mathbf{J}_L \delta \mathbf{r}_0\| \ll \|\delta \mathbf{r}_0\| \quad \forall \delta \mathbf{r}_0 \neq \bar{\mathbf{0}} \quad \implies \quad \|\delta \mathbf{r}\| \approx \|\delta \mathbf{r}_0\| \tag{5.5}$$

The above inequality can be equivalently rewritten as:

$$\frac{\|\mathbf{J}_\varepsilon \delta \mathbf{r}_0\|}{\|\delta \mathbf{r}_0\|} \ll 1 \quad \forall \delta \mathbf{r}_0 \neq \bar{\mathbf{0}} \quad (5.6)$$

As condition (5.6) has to be fulfilled for all non-null vectors, it can also be stated that:

$$\max_{\delta \mathbf{r}_0 \neq \bar{\mathbf{0}}} \frac{\|\mathbf{J}_\varepsilon \delta \mathbf{r}_0\|}{\|\delta \mathbf{r}_0\|} \ll 1 \quad (5.7)$$

If the definition of the norm of a second order tensor exposed in section (A.2.2) is taken into account, the left-hand side of the previous inequality turns out to be the norm of the displacement gradient tensor. Moreover, the right-hand side can be interpreted as the norm of the second order unit tensor.

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \quad (5.8)$$

Therefore, if the norm of the displacement gradient tensor is much lower than the norm of the unit tensor, the geometric transformation can be considered as an infinitesimal one. If this condition is fulfilled, it usually said that the displacements gradients are small.

Nevertheless, this condition does not imply that the displacement experimented by each material particle is small. In real practice, it is common dealing with structures that experiment large displacements and small displacement gradients. This structural behaviour is illustrated in the next section with a simple example.

5.2.2. Large displacements and small displacement gradients

In figure 5.2, a structure that experiments large displacements and small displacement gradients is presented. The structure is composed by an articulated rod, whose vertical displacement is allowed. The displacements applied to its nodes fulfil the following conditions:

$$\begin{aligned} |a| \ll L_0 &\iff \left| \frac{a}{L_0} \right| \ll 1 \\ |\alpha L_0| \ll L_0 &\iff |\alpha| \ll 1 \end{aligned} \quad (5.9)$$

If the coordinate axes shown in figure 5.2 are adopted, the position vector of a material particle that belongs to the reference configuration is:

$$\mathbf{r}_0 = \begin{Bmatrix} r_{0,1} \\ 0 \end{Bmatrix} \quad (5.10)$$

And the displacement field is defined as:

$$\mathbf{u}_\varepsilon(\mathbf{r}_0) = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a + \alpha r_{0,1} \end{Bmatrix} \quad (5.11)$$

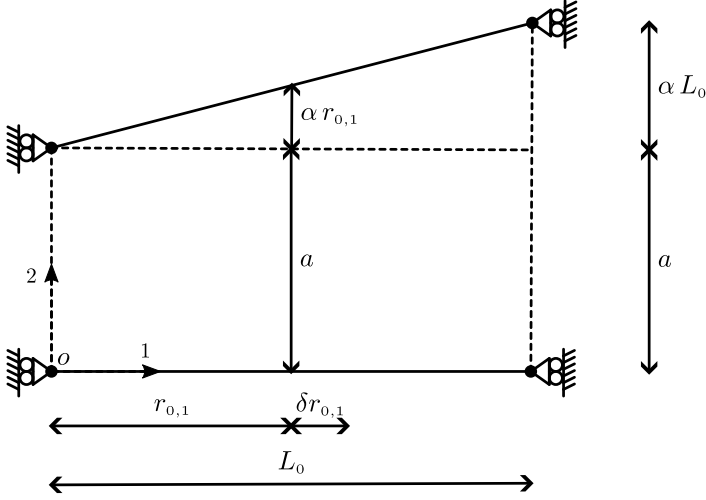


Figure 5.2. Single articulated rod with prescribed vertical displacements at its nodes.

Its corresponding displacement gradient tensor turns out to be:

$$\mathbf{J}_{\mathcal{L}} = \frac{d\mathbf{u}_{\mathcal{L}}}{d\mathbf{r}_0} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \quad (5.12)$$

On the other hand, the components of a given material vector defined in the initial domain are:

$$\delta\mathbf{r}_0 = \begin{Bmatrix} \delta r_{0,1} \\ 0 \end{Bmatrix} \quad (5.13)$$

And the norm of the displacement gradient tensor defined in (5.12) is:

$$\|\mathbf{J}_{\mathcal{L}}\| = \frac{\|\mathbf{J}_{\mathcal{L}} \delta\mathbf{r}_0\|}{\|\delta\mathbf{r}_0\|} = \frac{|\alpha \delta r_{0,1}|}{|\delta r_{0,1}|} = |\alpha| \ll 1 \quad \Leftrightarrow \quad \|\mathbf{J}_{\mathcal{L}}\| \ll \|\mathbf{I}\| \quad (5.14)$$

Therefore, under the assumptions presented in (5.9), the displacement gradients are small, and the displacements are large if they are compared to the initial length of the rod.

$$\frac{\|\mathbf{u}_{\mathcal{L}}(\mathbf{r}_0)\|}{L_0} = \frac{|a + \alpha r_{0,1}|}{L_0} = \underbrace{\left| \frac{a}{L_0} \right|}_{\ll 1} + \underbrace{\left| \frac{\alpha r_{0,1}}{L_0} \right|}_{\ll 1} \ll 1 \quad (5.15)$$

Furthermore, if it assumed that $a = 0$, the displacements become small. In this particular case, both the displacements and the displacement gradients turn out to be small.

5.3. Approximate polar decomposition of the deformation gradient tensor

As stated in (5.3), the geometric transformation of an infinitesimal material vector is ruled by the deformation gradient tensor. Moreover, this tensor can be expressed in terms of the displacement gradient tensor.

$$\delta \mathbf{r} = \mathbf{F}_\varepsilon \delta \mathbf{r}_0 = [\mathbf{I} + \mathbf{J}_\varepsilon] \delta \mathbf{r}_0 \quad (5.16)$$

Let's consider that the above geometric transformation is infinitesimal. That is, the norm of the displacement gradient tensor fulfils condition (5.8):

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \quad \text{where} \quad \begin{cases} \|\mathbf{J}_\varepsilon\| = \max_{\delta \mathbf{r}_0 \neq \mathbf{0}} \frac{\|\mathbf{J}_\varepsilon \delta \mathbf{r}_0\|}{\|\delta \mathbf{r}_0\|} \\ \|\mathbf{I}\| = 1 \end{cases} \quad (5.17)$$

If the above condition is fulfilled, the deformation gradient tensor can be accurately decomposed according to the following approximate polar decomposition.

$$\mathbf{F}_\varepsilon \approx \mathcal{F}_\varepsilon = \mathcal{R}_\varepsilon [\mathbf{I} + \mathcal{E}_\varepsilon] \quad (5.18)$$

The above tensors are defined by means of the displacement gradient tensor, as:

$$\begin{aligned} \mathbf{R}_\varepsilon &\approx \mathcal{R}_\varepsilon = \mathbf{I} + \mathcal{W}_\varepsilon \\ \mathcal{W}_\varepsilon &= \frac{1}{2} (\mathbf{J}_\varepsilon - \mathbf{J}_\varepsilon^T) \\ \mathbf{E}_\varepsilon &\approx \mathcal{E}_\varepsilon = \frac{1}{2} (\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T) \end{aligned} \quad (5.19)$$

The tensor \mathcal{R}_ε is the infinitesimal rotation tensor, and \mathcal{E}_ε is the so-called infinitesimal strain tensor. Both tensors depend on the displacement gradient tensor, which verifies condition (5.17). Hence, they also verify that their norm is much lower than the norm of the unit tensor. This implies that these tensors represent infinitesimal geometric transformations.

$$\begin{aligned} \|\mathcal{W}_\varepsilon\| &= \mathcal{O}(\|\mathbf{J}_\varepsilon\|) \ll \|\mathbf{I}\| \\ \|\mathcal{E}_\varepsilon\| &= \mathcal{O}(\|\mathbf{J}_\varepsilon\|) \ll \|\mathbf{I}\| \end{aligned} \quad (5.20)$$

5.3.1. Demonstration

On the one hand, the computation of the approximate polar decomposition presented in the previous section leads to:

$$\begin{aligned} \mathcal{R}_\varepsilon [\mathbf{I} + \mathcal{E}_\varepsilon] &= \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon - \mathbf{J}_\varepsilon^T) \right] \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T) \right] \\ &= \mathbf{I} + \mathbf{J}_\varepsilon + \mathcal{O}(\|\mathbf{J}_\varepsilon\|^2) \\ &= \mathbf{F}_\varepsilon + \mathcal{O}(\|\mathbf{J}_\varepsilon\|^2) \end{aligned} \quad (5.21)$$

Thus, if the displacement gradients are small, the second term can be neglected, and the deformation gradient tensor is accurately approximated.

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \quad \Longrightarrow \quad \mathcal{R}_\varepsilon [\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] \approx \mathbf{F}_\varepsilon \quad (5.22)$$

On the other hand, the infinitesimal rotation tensor orthogonality has to be checked.

$$\begin{aligned} \mathcal{R}_\varepsilon \mathcal{R}_\varepsilon^T &= \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon - \mathbf{J}_\varepsilon^T) \right] \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon^T - \mathbf{J}_\varepsilon) \right] = \mathbf{I} + \mathcal{O}(\|\mathbf{J}_\varepsilon\|^2) \\ \mathcal{R}_\varepsilon^T \mathcal{R}_\varepsilon &= \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon^T - \mathbf{J}_\varepsilon) \right] \left[\mathbf{I} + \frac{1}{2} (\mathbf{J}_\varepsilon - \mathbf{J}_\varepsilon^T) \right] = \mathbf{I} + \mathcal{O}(\|\mathbf{J}_\varepsilon\|^2) \end{aligned} \quad (5.23)$$

If the displacement gradients are small, the infinitesimal rotation tensor does represent a rotation, since it approximately fulfils the condition obtained in section (A.13).

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \quad \Longrightarrow \quad \left\{ \begin{array}{l} \mathcal{R}_\varepsilon \mathcal{R}_\varepsilon^T \approx \mathbf{1} \\ \mathcal{R}_\varepsilon^T \mathcal{R}_\varepsilon \approx \mathbf{1} \end{array} \right\} \quad \Longleftrightarrow \quad \mathcal{R}_\varepsilon^{-1} \approx \mathcal{R}_\varepsilon^T \quad (5.24)$$

Furthermore, the infinitesimal strain tensor $\boldsymbol{\varepsilon}_\varepsilon$ is symmetric.

$$\boldsymbol{\varepsilon}_\varepsilon^T = \frac{1}{2} (\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T)^T = \boldsymbol{\varepsilon}_\varepsilon \quad (5.25)$$

And the tensor $[\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon]$ is positive definite. These conditions are required according to the polar decomposition stated in section 4.3.

$$\mathbf{x}^T [\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] \mathbf{x} = \mathbf{x}^T \mathbf{x} + \mathbf{x}^T \boldsymbol{\varepsilon}_\varepsilon \mathbf{x} = \underbrace{\|\mathbf{x}\|^2 (1 + \mathcal{O}(\|\boldsymbol{\varepsilon}_\varepsilon\|))}_{>0} > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (5.26)$$

Therefore, it can be concluded that the approximate polar decomposition can be applied with accurate results, if the displacement gradient tensor fulfils the condition defined in (5.17). That is, if the displacement gradients are small, the polar decomposition can be defined by means of the displacement gradient tensor. This implies that solving an eigenvalue problem is not required to obtain the polar decomposition, as stated in section (4.3).

5.3.2. Physical interpretation

The polar decomposition decomposes the deformation gradient tensor as the product of a rotation and a strain tensor.

Moreover, if the geometrical transformation of a material vector is infinitesimal, the polar decomposition can be obtained with less computational effort by applying the approximate polar decomposition. Thus, the infinitesimal geometric transformation of a given material vector turns out to be:

$$\delta \mathbf{r} \approx \mathcal{F}_\varepsilon \delta \mathbf{r}_0 = \underbrace{\mathcal{R}_\varepsilon}_{\text{inf. rotation}} \left(\underbrace{[\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] \delta \mathbf{r}_0}_{\text{inf. strain}} \right) \quad (5.27)$$

Where:

$$\begin{aligned}\mathcal{R}_\mathcal{L} &\equiv \text{Infinitesimal rotation tensor} \\ \mathcal{E}_\mathcal{L} &\equiv \text{Infinitesimal strain tensor}\end{aligned}$$

Firstly, the infinitesimal strain tensor modifies the norm and the direction of the material vector. Then, the infinitesimal rotation tensor applies an infinitesimal rotation to the previous modified vector. Both tensors apply an infinitesimal change to the material vectors, as proved in (5.20).

5.3.3. Infinitesimal strain tensor

The eigenvalues of the infinitesimal strain tensor and their corresponding eigenvectors verify the following equations:

$$\begin{aligned}\mathcal{E}_\mathcal{L} \mathbf{v}_i &= \epsilon_i \mathbf{v}_i \quad i = 1, \dots, n \\ \mathbf{v}_i^T \mathbf{v}_i &= \delta_{ij}\end{aligned}\tag{5.28}$$

In the above equation, the eigenvectors are considered to compose an orthonormal basis. These equations can be equivalently rewritten in matrix notation, as shown below.

$$\begin{aligned}\mathcal{E}_\mathcal{L} \mathbf{U} &= \mathbf{U} \mathcal{E}^* \\ \mathbf{U}^T \mathbf{U} &= \mathbf{I}\end{aligned}\tag{5.29}$$

Where \mathcal{E}^* is a diagonal matrix composed by the eigenvalues of the infinitesimal strain tensor.

$$\mathcal{E}^* = \begin{bmatrix} \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_n \end{bmatrix}\tag{5.30}$$

Since the infinitesimal strain tensor represents an infinitesimal geometric transformation, the absolute value of its eigenvalues verify the following condition.

$$\|\mathcal{E}_\mathcal{L}\| = \mathcal{O}(\|\mathcal{J}_\mathcal{L}\|) \ll \|\mathbf{I}\| \quad \implies \quad |\epsilon_i| \ll 1 \quad i = 1, \dots, n\tag{5.31}$$

And \mathbf{U} is composed by its corresponding eigenvectors arranged in columns. As it is an orthogonal matrix, it represents a rotation. Moreover, the sense of the eigenvectors are properly defined to define a proper rotation.

$$\mathbf{U} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n] \quad \implies \quad \begin{cases} \mathbf{U}^T = \mathbf{U}^{-1} \\ \det(\mathbf{U}) = 1 \end{cases}\tag{5.32}$$

According to (5.29), it can be concluded that the infinitesimal strain tensor can be defined as:

$$\mathcal{E}_\mathcal{L} = \mathbf{U} \mathcal{E}^* \mathbf{U}^T = \mathbf{U} \begin{bmatrix} \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_n \end{bmatrix} \mathbf{U}^T \quad \text{where} \quad \begin{cases} \mathbf{U}^T = \mathbf{U}^{-1}, & \det(\mathbf{U}) = 1 \\ |\epsilon_i| \ll 1 & i = 1, \dots, n \end{cases}\tag{5.33}$$

5.3.4. Determinant of the deformation gradient tensor

The approximate polar decomposition (5.18) can be applied to obtain the determinant of the deformation gradient tensor that rules the infinitesimal geometric transformation of a given material vector.

$$F_{\mathcal{L}} \approx \mathcal{F}_{\mathcal{L}} = \det(\mathcal{F}_{\mathcal{L}}) = \det(\mathcal{R}_{\mathcal{L}} [\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}]) = \det(\mathcal{R}_{\mathcal{L}}) \det(\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}) \quad (5.34)$$

On the one hand, the determinant of the infinitesimal rotation tensor can be obtained by applying the determinant operator to both sides of the equation (5.24).

$$\left. \begin{array}{l} \mathcal{R}_{\mathcal{L}} \mathcal{R}_{\mathcal{L}}^T \approx \mathbf{I} \\ \mathcal{R}_{\mathcal{L}}^T \mathcal{R}_{\mathcal{L}} \approx \mathbf{I} \end{array} \right\} \iff \left(\det(\mathcal{R}_{\mathcal{L}}) \right)^2 \approx \det(\mathbf{I}) = 1 \iff \det(\mathcal{R}_{\mathcal{L}}) \approx \pm 1 \quad (5.35)$$

If the geometric transformation of the material vector has physical sense, the infinitesimal rotation tensor has to represent a proper rotation. Thus, it can be concluded that the determinant of the infinitesimal rotation tensor has to be equal to one.

$$\det(\mathcal{R}_{\mathcal{L}}) \approx 1 \quad (5.36)$$

On the other hand, the definition of the infinitesimal strain tensor (5.33) can be applied to calculate the determinant of the infinitesimal strain.

$$\begin{aligned} \det(\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}) &= \det(\mathbf{I} + \mathbf{u} \boldsymbol{\varepsilon}^* \mathbf{u}^T) \\ &= \det(\mathbf{u} [\mathbf{I} + \boldsymbol{\varepsilon}^*] \mathbf{u}^T) \\ &= \det(\mathbf{u}) \det(\mathbf{I} + \boldsymbol{\varepsilon}^*) \det(\mathbf{u}^T) \\ &= \det(\mathbf{u}) \det(\mathbf{I} + \boldsymbol{\varepsilon}^*) \det(\mathbf{u}) \end{aligned} \quad (5.37)$$

Moreover, the determinant of the tensor \mathbf{u} was defined in (5.32) by taking into account that it represents a proper rotation. Therefore, the above equation is reduced to:

$$\det(\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}) = \det(\mathbf{I} + \boldsymbol{\varepsilon}^*) = \prod_{i=1}^n (1 + \epsilon_i) \quad (5.38)$$

As stated in (5.33), the absolute value of the infinitesimal strain tensor eigenvalues is much lower than one. Thus, the above product can be approximated as:

$$\det(\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}) = \prod_{i=1}^n (1 + \epsilon_i) \approx 1 + \sum_{i=1}^n \epsilon_i = 1 + \text{Tr}(\boldsymbol{\varepsilon}^*) \quad (5.39)$$

If equation (5.33) is recalled, the tensor $\boldsymbol{\varepsilon}^*$ can be equivalently rewritten by means of the infinitesimal strain tensor. Hence, the above equation becomes:

$$\begin{aligned} \det(\mathbf{I} + \boldsymbol{\varepsilon}_{\mathcal{L}}) &= 1 + \text{Tr}(\boldsymbol{\varepsilon}^*) \\ &= 1 + \text{Tr}(\mathbf{u}^T \boldsymbol{\varepsilon}_{\mathcal{L}} \mathbf{u}) \\ &= 1 + \text{Tr}(\boldsymbol{\varepsilon}_{\mathcal{L}} \underbrace{\mathbf{u} \mathbf{u}^T}_{=\mathbf{I}}) \\ &= 1 + \text{Tr}(\boldsymbol{\varepsilon}_{\mathcal{L}}) \end{aligned} \quad (5.40)$$

Thus, if the results obtained in (5.36) and (5.40) are substituted into (5.34), the determinant of the deformation gradient tensor can be approximately computed as:

$$F_{\mathcal{L}} \approx 1 + \text{Tr}(\boldsymbol{\mathcal{E}}_{\mathcal{L}}) \quad (5.41)$$

Where:

$$|\text{Tr}(\boldsymbol{\mathcal{E}}_{\mathcal{L}})| = \left| \sum_{i=1}^n \epsilon_i \right| \ll 1 \quad (5.42)$$

5.4. Infinitesimal strain

As stated in section 5.2.1, the geometric transformation of a material vector is considered infinitesimal if the norm of the displacement gradient tensor verifies the following condition.

$$\|\mathbf{J}_{\mathcal{L}}\| \ll \|\mathbf{I}\| \quad (5.43)$$

If the above condition is fulfilled, the approximate polar decomposition can be accurately applied to the deformation gradient tensor (section 5.3). That is, the deformation gradient tensor is decomposed as the product between the infinitesimal rotation tensor and the infinitesimal strain.

$$\mathbf{F}_{\mathcal{L}} \approx \mathcal{F}_{\mathcal{L}} = \mathcal{R}_{\mathcal{L}} [\mathbf{I} + \boldsymbol{\mathcal{E}}_{\mathcal{L}}] \quad (5.44)$$

Both the infinitesimal rotation tensor $\mathcal{R}_{\mathcal{L}}$ and the infinitesimal strain tensor $\boldsymbol{\mathcal{E}}_{\mathcal{L}}$ are defined by means of the displacement gradient tensor, as shown below.

$$\begin{aligned} \mathbf{R}_{\mathcal{L}} &\approx \mathcal{R}_{\mathcal{L}} = \mathbf{I} + \boldsymbol{\mathcal{W}}_{\mathcal{L}} \\ \boldsymbol{\mathcal{W}}_{\mathcal{L}} &= \frac{1}{2}(\mathbf{J}_{\mathcal{L}} - \mathbf{J}_{\mathcal{L}}^T) \\ \boldsymbol{\mathcal{E}}_{\mathcal{L}} &\approx \boldsymbol{\mathcal{E}}_{\mathcal{L}} = \frac{1}{2}(\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T) \end{aligned} \quad (5.45)$$

Since both tensors depend on the displacement gradient tensor, and this tensor verifies condition (5.43), they also represent an infinitesimal geometric transformation. That is, they also verify that their norm is much lower than the norm of the unit tensor. Thus,

$$\begin{aligned} \|\boldsymbol{\mathcal{W}}_{\mathcal{L}}\| &= \mathcal{O}(\|\mathbf{J}_{\mathcal{L}}\|) \ll \|\mathbf{I}\| \\ \|\boldsymbol{\mathcal{E}}_{\mathcal{L}}\| &= \mathcal{O}(\|\mathbf{J}_{\mathcal{L}}\|) \ll \|\mathbf{I}\| \end{aligned} \quad (5.46)$$

Under these assumptions, the infinitesimal rotation tensor produces a proper infinitesimal rotation, as stated in (5.24), and the infinitesimal strain tensor is a symmetric tensor. Moreover, the tensor $[\mathbf{I} + \boldsymbol{\mathcal{E}}_{\mathcal{L}}]$ that rules the infinitesimal strain turns out to be positive definite, as proved in (5.26). That is:

$$\begin{aligned} \mathcal{R}_{\mathcal{L}}^{-1} &\approx \mathcal{R}_{\mathcal{L}}^T & \det(\mathcal{R}_{\mathcal{L}}) &\approx 1 \\ \boldsymbol{\mathcal{E}}_{\mathcal{L}}^T &= \boldsymbol{\mathcal{E}}_{\mathcal{L}} & \mathbf{x}^T [\mathbf{I} + \boldsymbol{\mathcal{E}}_{\mathcal{L}}] \mathbf{x} &> 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \end{aligned} \quad (5.47)$$

To sum up, if the norm of the displacement tensor is much lower than the norm of the unit tensor, the approximate polar decomposition can be accurately applied to the deformation gradient tensor. And the geometric transformation of a given material vector becomes:

$$\delta \mathbf{r} \approx \underbrace{\widehat{\mathcal{R}}_{\mathcal{L}}}_{\text{inf. rotation}} \left(\underbrace{[\mathbf{I} + \mathcal{E}_{\mathcal{L}}]}_{\text{inf. strain}} \delta \mathbf{r}_0 \right) + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (5.48)$$

According to this approximate decomposition, the material vector firstly suffers an infinitesimal strain transformation. Then, an infinitesimal rotation is applied to the previous deformed material vector.

5.4.1. Concept of infinitesimal strain

Let's consider that an infinitesimal strain is applied to a given material vector. This infinitesimal geometric change is defined as:

$$\delta \mathbf{r}_{\mathcal{E}} = [\mathbf{I} + \mathcal{E}_{\mathcal{L}}] \delta \mathbf{r}_0 \quad (5.49)$$

Where $\mathcal{E}_{\mathcal{L}}$ is the so-called infinitesimal strain tensor (5.19), which is defined in terms of the displacement gradient tensor as:

$$\mathcal{E}_{\mathcal{L}} = \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T) \quad (5.50)$$

According to the above definition, the infinitesimal strain tensor turns out to be symmetric. Thus,

$$\mathcal{E}_{\mathcal{L}}^T = \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T)^T = \mathcal{E}_{\mathcal{L}} \quad (5.51)$$

Since the infinitesimal strain tensor depends on the displacement gradient tensor, this tensor represents an infinitesimal geometric change.

$$\|\mathbf{J}_{\mathcal{L}}\| \ll \|\mathbf{I}\| \quad \Longrightarrow \quad \|\mathcal{E}_{\mathcal{L}}\| = \mathcal{O}(\|\mathbf{J}_{\mathcal{L}}\|) \ll \|\mathbf{I}\| \quad (5.52)$$

In addition, the tensor $[\mathbf{I} + \mathcal{E}_{\mathcal{L}}]$ has to be positive definite. Thus, it verifies that:

$$\mathbf{x}^T [\mathbf{I} + \mathcal{E}_{\mathcal{L}}] \mathbf{x} = \mathbf{x}^T \mathbf{x} + \mathbf{x}^T \mathcal{E}_{\mathcal{L}} \mathbf{x} = \underbrace{\|\mathbf{x}\|^2}_{>0} \left(1 + \mathcal{O}(\|\mathcal{E}_{\mathcal{L}}\|) \right) > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (5.53)$$

On the other hand, the infinitesimal strain tensor was defined in (5.33), as:

$$\mathcal{E}_{\mathcal{L}} = \mathbf{U} \begin{bmatrix} \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_n \end{bmatrix} \mathbf{U}^T \quad (5.54)$$

Where the coefficients ϵ_i , that compose the diagonal tensor, are the infinitesimal strain tensor eigenvalues. As the geometric change is infinitesimal, these coefficients verify that:

$$|\epsilon_i| \ll 1 \quad i = 1, \dots, n \quad (5.55)$$

And the tensor \mathbf{U} is composed by the eigenvectors corresponding to the previous eigenvalues, arranged in columns. These eigenvectors can be forced to compose an orthogonal basis, and their sense can be adequately defined, so that they represent a proper rotation. Thus, it verifies that:

$$\mathbf{U} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n] \quad \Longrightarrow \quad \begin{cases} \mathbf{U}^T = \mathbf{U}^{-1} \\ \det(\mathbf{U}) = 1 \end{cases} \quad (5.56)$$

5.4.2. Infinitesimal strain tensor diagonalization

According to equation (5.33), the equation that defines the infinitesimal strain tensor diagonalization turns out to be:

$$\begin{bmatrix} \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_n \end{bmatrix} = \mathbf{U}^T \boldsymbol{\mathcal{E}}_{\mathcal{L}} \mathbf{U} \quad (5.57)$$

To obtain the components of the infinitesimal strain tensor with respect to a rotated basis, the result obtained in (4.55) can be recalled. So, the components of the infinitesimal strain tensor with respect to the rotated basis are:

$$\boldsymbol{\mathcal{E}}'_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}}^T \boldsymbol{\mathcal{E}}_{\mathcal{L}} \mathbf{R}_{\mathcal{L}} \quad (5.58)$$

The comparison of equations (5.57) and (5.58) allows to conclude that the rotation defined by $\mathbf{R}_{\mathcal{L}} = \mathbf{U}$ diagonalizes the infinitesimal strain tensor.

$$\boldsymbol{\mathcal{E}}'_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}}^T \boldsymbol{\mathcal{E}}_{\mathcal{L}} \mathbf{R}_{\mathcal{L}} = \mathbf{U}^T \boldsymbol{\mathcal{E}}_{\mathcal{L}} \mathbf{U} = \begin{bmatrix} \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_n \end{bmatrix} \quad (5.59)$$

The coefficients ϵ_i are the eigenvalues of the infinitesimal strain tensor, and \mathbf{U} is composed by their corresponding eigenvectors, arranged in columns. These eigenvectors are defined, so they compose an orthonormal basis and a proper rotation.

Therefore, the infinitesimal strain experimented by a material vector, expressed with respect to this rotated basis, turns out to be diagonal.

$$\delta \mathbf{r}' = [\mathbf{I} + \boldsymbol{\mathcal{E}}'_{\mathcal{L}}] \delta \mathbf{r}'_0 = \begin{bmatrix} 1 + \epsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \epsilon_n \end{bmatrix} \delta \mathbf{r}'_0 \quad (5.60)$$

The axes defined by the eigenvectors are also known as principal axes. And the coefficients that compose the diagonalized infinitesimal strain tensor are the elongation factors that are applied to the material vector components along each one of them. These factors verify that:

$$|\epsilon_i| \ll 1 \quad \Longrightarrow \quad 1 + \epsilon_i \approx 1 \quad i = 1, \dots, n \quad (5.61)$$

5.5. Infinitesimal strain tensor

As stated in (5.19), the infinitesimal strain tensor of a given displacement field is defined by means of its corresponding displacement gradient tensor, as:

$$\begin{aligned} \mathbf{u}_\varepsilon &\implies \mathbf{J}_\varepsilon = \frac{\partial \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} \\ &\implies \boldsymbol{\mathcal{E}}_\varepsilon = \frac{1}{2} (\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T) \end{aligned} \quad (5.62)$$

5.5.1. Infinitesimal strain tensor variation

Let's consider the modification of the displacement field defined in the previous section by adding a compatible variation. Let's also assume that the gradients of the displacement field variation are small. Under these assumptions, the infinitesimal strain tensor corresponding to the modified displacement field turns out to be:

$$\begin{aligned} \left. \begin{array}{l} \mathbf{u}'_\varepsilon = \mathbf{u}_\varepsilon + \delta \mathbf{u}_\varepsilon \\ \left\| \frac{\partial \delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{array} \right\} &\implies \mathbf{J}'_\varepsilon = \frac{\partial \mathbf{u}'_\varepsilon}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} + \frac{\partial \delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} = \mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon \\ &\implies \boldsymbol{\mathcal{E}}'_\varepsilon = \frac{1}{2} [\mathbf{J}'_\varepsilon + (\mathbf{J}'_\varepsilon)^T] \\ &= \frac{1}{2} [(\mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon) + (\mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon)^T] \\ &= \frac{1}{2} (\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T) + \frac{1}{2} (\delta \mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon^T) \\ &= \boldsymbol{\mathcal{E}}_\varepsilon + \delta \boldsymbol{\mathcal{E}}_\varepsilon \end{aligned} \quad (5.63)$$

Where $\boldsymbol{\mathcal{E}}_\varepsilon$ is the infinitesimal strain tensor corresponding to the original displacement field, and $\delta \boldsymbol{\mathcal{E}}_\varepsilon$ represents its variation. Thus, the infinitesimal strain tensor variation turns out to be defined in terms of the displacement gradient tensor variation as follows.

$$\delta \boldsymbol{\mathcal{E}}_\varepsilon = \frac{1}{2} (\delta \mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon^T) \quad (5.64)$$

5.5.2. Infinitesimal strain tensor increment

In the previous section, a compatible variation ($\delta \mathbf{u}$) was applied to the displacement field. Let's consider that the displacement field is again modified by adding a compatible increment ($\Delta \mathbf{u}$), instead of a variation. It is also assumed that the gradients of the displacement field increment are small. Under these assumptions, the infinitesimal

strain tensor corresponding to the modified displacement field becomes:

$$\left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} = \mathbf{u}_{\mathcal{L}} + \Delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{aligned} \right\} \Rightarrow \mathbf{J}'_{\mathcal{L}} = \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}$$

$$\begin{aligned} \mathbf{E}'_{\mathcal{L}} &= \frac{1}{2} \left[\mathbf{J}'_{\mathcal{L}} + (\mathbf{J}'_{\mathcal{L}})^T \right] \\ &= \frac{1}{2} \left[(\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}) + (\mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}})^T \right] \\ &= \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T) + \frac{1}{2} (\Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T) \\ &= \mathbf{E}_{\mathcal{L}} + \Delta \mathbf{E}_{\mathcal{L}} \end{aligned} \tag{5.65}$$

Where $\mathbf{E}_{\mathcal{L}}$ is the infinitesimal strain tensor corresponding to the original displacement field, and $\Delta \mathbf{E}_{\mathcal{L}}$ represents its increment. Hence, the infinitesimal strain tensor increment turns out to be defined as:

$$\Delta \mathbf{E}_{\mathcal{L}} = \frac{1}{2} (\Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T) \tag{5.66}$$

5.6. Vectorial form of the infinitesimal strain tensor

The infinitesimal strain tensor was defined by means of the displacement gradient tensor in (5.19) as shown below.

$$\mathbf{E}_{\mathcal{L}} = \frac{1}{2} (\mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T) \quad \mathbf{J}_{\mathcal{L}} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_1}{\partial r_{0,2}} & \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,2}} & \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} & \frac{\partial u_3}{\partial r_{0,2}} & \frac{\partial u_3}{\partial r_{0,3}} \end{bmatrix} \tag{5.67}$$

And its components turn out to be:

$$\mathbf{E}_{\mathcal{L}} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \right) & \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \right) & \frac{\partial u_2}{\partial r_{0,2}} & \frac{1}{2} \left(\frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \right) & \frac{1}{2} \left(\frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \right) & \frac{\partial u_3}{\partial r_{0,3}} \end{bmatrix} \tag{5.68}$$

If the Voigt notation (section A.14) is applied, an equivalent vectorial form can be

adopted to handle the infinitesimal strain tensor as:

$$\boldsymbol{\varepsilon}_{\mathcal{L}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \implies \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \end{Bmatrix} \quad (5.69)$$

The above vector can be rewritten in terms of the vectorial form of the displacement gradient tensor defined in (4.106).

$$\bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} = \begin{Bmatrix} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \\ \frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \end{Bmatrix} = \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} \quad (5.70)$$

Furthermore, the vectorial form of the displacement gradient tensor can be ex-

pressed by means of a differential operator as follows.

$$\bar{\mathbf{J}}_{\mathcal{L}} = \left\{ \begin{array}{c} \frac{\partial u_1}{\partial r_{0,1}} \\ \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} \\ \frac{\partial u_2}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} \\ \frac{\partial u_3}{\partial r_{0,2}} \\ \frac{\partial u_3}{\partial r_{0,3}} \end{array} \right\} = \left[\begin{array}{ccc} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} = \boldsymbol{\partial}_0 \mathbf{u}_{\mathcal{L}} \quad (5.71)$$

Therefore, the vectorial form of the infinitesimal strain tensor (5.70) can be equivalently defined as:

$$\left. \begin{array}{l} \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} = \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} \\ \bar{\mathbf{J}}_{\mathcal{L}} = \boldsymbol{\partial}_0 \mathbf{u}_{\mathcal{L}} \end{array} \right\} \implies \begin{array}{l} \bar{\boldsymbol{\varepsilon}}_{\mathcal{L}} = \mathbf{A}_C (\boldsymbol{\partial}_0 \mathbf{u}_{\mathcal{L}}) \\ = (\mathbf{A}_C \boldsymbol{\partial}_0) \mathbf{u}_{\mathcal{L}} \\ = \mathbf{L}_0 \mathbf{u}_{\mathcal{L}} \end{array} \quad (5.72)$$

Where:

$$\mathbf{L}_0 = \mathbf{A}_C \boldsymbol{\partial}_0 = \left[\begin{array}{ccc} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \\ \frac{\partial}{\partial r_{0,2}} & \frac{\partial}{\partial r_{0,1}} & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & \frac{\partial}{\partial r_{0,3}} & \frac{\partial}{\partial r_{0,2}} \end{array} \right] \quad (5.73)$$

5.6.1. Vectorial form of the infinitesimal strain tensor variation

Definition in terms of the displacement field variation

The vectorial form of the infinitesimal strain tensor corresponding to a given displacement field was defined in (5.72) as:

$$\bar{\mathcal{E}}_{\mathcal{L}} = \mathbf{L}_0 \mathbf{u}_{\mathcal{L}} \quad (5.74)$$

Let's consider that the displacement field is modified, by adding a compatible variation. Let's also assume that the gradients of the displacement field variation are small. Under these assumptions, the vectorial form corresponding to the modified displacement field turns out to be:

$$\left. \begin{array}{l} \mathbf{u}'_{\mathcal{L}} = \mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{array} \right\} \implies \begin{array}{l} \bar{\mathcal{E}}'_{\mathcal{L}} = \mathbf{L}_0 \mathbf{u}'_{\mathcal{L}} \\ = \mathbf{L}_0 (\mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}}) \\ = \mathbf{L}_0 \mathbf{u}_{\mathcal{L}} + \mathbf{L}_0 \delta \mathbf{u}_{\mathcal{L}} \\ = \bar{\mathcal{E}}_{\mathcal{L}} + \delta \bar{\mathcal{E}}_{\mathcal{L}} \end{array} \quad (5.75)$$

Where $\bar{\mathcal{E}}_{\mathcal{L}}$ is the vectorial form of the infinitesimal strain tensor corresponding to the original displacement field, and $\delta \bar{\mathcal{E}}_{\mathcal{L}}$ represents its variation. Thus, the vectorial form of the infinitesimal strain tensor variation is:

$$\delta \bar{\mathcal{E}}_{\mathcal{L}} = \mathbf{L}_0 \delta \mathbf{u}_{\mathcal{L}} \quad (5.76)$$

Definition in terms of the displacement gradient tensor variation

The vectorial form of the infinitesimal strain tensor was equivalently defined in (5.70) by means of the vectorial form of the displacement gradient tensor variation, as:

$$\bar{\mathcal{E}}_{\mathcal{L}} = \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} \quad (5.77)$$

If the displacement field variation defined in the previous section is applied, the vectorial form of the infinitesimal strain tensor corresponding to the modified displacement field becomes:

$$\begin{aligned} \left. \begin{array}{l} \mathbf{u}'_{\mathcal{L}} = \mathbf{u}_{\mathcal{L}} + \delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{array} \right\} &\implies \mathbf{J}'_{\mathcal{L}} = \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \delta \mathbf{J}_{\mathcal{L}} \\ &\implies \bar{\mathbf{J}}'_{\mathcal{L}} = \bar{\mathbf{J}}_{\mathcal{L}} + \delta \bar{\mathbf{J}}_{\mathcal{L}} \\ &\implies \bar{\mathcal{E}}'_{\mathcal{L}} = \mathbf{A}_C \bar{\mathbf{J}}'_{\mathcal{L}} \\ &\implies \begin{array}{l} = \mathbf{A}_C (\bar{\mathbf{J}}_{\mathcal{L}} + \delta \bar{\mathbf{J}}_{\mathcal{L}}) \\ = \mathbf{A}_C \bar{\mathbf{J}}_{\mathcal{L}} + \mathbf{A}_C \delta \bar{\mathbf{J}}_{\mathcal{L}} \\ = \bar{\mathcal{E}}_{\mathcal{L}} + \delta \bar{\mathcal{E}}_{\mathcal{L}} \end{array} \end{aligned} \quad (5.78)$$

Where $\bar{\mathcal{E}}_\epsilon$ is the vectorial form of the infinitesimal strain tensor corresponding to the original displacement field, and $\delta\bar{\mathcal{E}}_\epsilon$ represents its variation. Therefore, the vectorial form of the infinitesimal strain tensor variation can be equivalently defined by means of the displacement gradient tensor variation as:

$$\delta\bar{\mathcal{E}}_\epsilon = \mathbf{A}_C \delta\bar{\mathbf{J}}_\epsilon \quad (5.79)$$

5.6.2. Vectorial form of the infinitesimal strain tensor increment

Definition in terms of the displacement field increment

Let's consider that the displacement field is modified by adding a compatible increment ($\Delta\mathbf{u}_\epsilon$), instead of a variation ($\delta\mathbf{u}_\epsilon$). Let's also assume that the gradients of the displacement field increment are small. Under these assumptions, the vectorial form of the infinitesimal strain tensor (5.72) are:

$$\left. \begin{array}{l} \mathbf{u}'_\epsilon = \mathbf{u}_\epsilon + \Delta\mathbf{u}_\epsilon \\ \left\| \frac{\partial\Delta\mathbf{u}_\epsilon}{\partial\mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{array} \right\} \implies \begin{array}{l} \bar{\mathcal{E}}'_\epsilon = \mathbf{L}_0 \mathbf{u}'_\epsilon \\ = \mathbf{L}_0 (\mathbf{u}_\epsilon + \Delta\mathbf{u}_\epsilon) \\ = \mathbf{L}_0 \mathbf{u}_\epsilon + \mathbf{L}_0 \Delta\mathbf{u}_\epsilon \\ = \bar{\mathcal{E}}_\epsilon + \Delta\bar{\mathcal{E}}_\epsilon \end{array} \quad (5.80)$$

Where $\bar{\mathcal{E}}_\epsilon$ is the vectorial form of the infinitesimal strain tensor corresponding to the original displacement field, and $\Delta\bar{\mathcal{E}}_\epsilon$ represents its increment. Hence, the vectorial form of the infinitesimal strain tensor increment turns out to be:

$$\Delta\bar{\mathcal{E}}_\epsilon = \mathbf{L}_0 \Delta\mathbf{u}_\epsilon \quad (5.81)$$

Definition in terms of the displacement gradient tensor increment

If the vector definition of the infinitesimal strain tensor presented in (5.70) is recalled, the above vector can be equivalently defined in terms of the vectorial form of the displacement gradient tensor increment. According to this definition, the vectorial form of the infinitesimal strain tensor corresponding to the modified displacement field becomes:

$$\begin{aligned} \left. \begin{array}{l} \mathbf{u}'_\epsilon = \mathbf{u}_\epsilon + \Delta\mathbf{u}_\epsilon \\ \left\| \frac{\partial\Delta\mathbf{u}_\epsilon}{\partial\mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{array} \right\} &\implies \mathbf{J}'_\epsilon = \frac{\partial\mathbf{u}'_\epsilon}{\partial\mathbf{r}_0} = \frac{\partial\mathbf{u}_\epsilon}{\partial\mathbf{r}_0} + \frac{\partial\Delta\mathbf{u}_\epsilon}{\partial\mathbf{r}_0} = \mathbf{J}_\epsilon + \Delta\mathbf{J}_\epsilon \\ &\implies \bar{\mathbf{J}}'_\epsilon = \bar{\mathbf{J}}_\epsilon + \Delta\bar{\mathbf{J}}_\epsilon \\ &\bar{\mathcal{E}}'_\epsilon = \mathbf{A}_C \bar{\mathbf{J}}'_\epsilon \\ &\implies \begin{array}{l} = \mathbf{A}_C (\bar{\mathbf{J}}_\epsilon + \Delta\bar{\mathbf{J}}_\epsilon) \\ = \mathbf{A}_C \bar{\mathbf{J}}_\epsilon + \mathbf{A}_C \Delta\bar{\mathbf{J}}_\epsilon \\ = \bar{\mathcal{E}}_\epsilon + \Delta\bar{\mathcal{E}}_\epsilon \end{array} \end{aligned} \quad (5.82)$$

Where $\bar{\boldsymbol{\varepsilon}}_\varepsilon$ is the vectorial form of the infinitesimal strain tensor corresponding to the original displacement field, and $\Delta\bar{\boldsymbol{\varepsilon}}_\varepsilon$ represents its increment. Consequently, the vectorial form of the infinitesimal strain tensor increment can be equivalently defined as:

$$\Delta\bar{\boldsymbol{\varepsilon}}_\varepsilon = \mathbf{A}_C \Delta\bar{\mathbf{J}}_\varepsilon \quad (5.83)$$

5.7. Alternative approximate polar decomposition

The approximate polar decomposition of the deformation gradient tensor stated in section (5.3) can alternatively be rewritten as:

$$\mathbf{F}_\varepsilon \approx \mathcal{F}_\varepsilon = \mathcal{R}_\varepsilon [\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] = \mathcal{R}_\varepsilon \left(\frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} [\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] \right) \left(\sqrt[n]{\mathcal{F}_\varepsilon} [\mathbf{I}] \right) = \mathcal{R}_\varepsilon \mathcal{D}_\varepsilon \mathcal{H}_\varepsilon \quad (5.84)$$

This alternative polar decomposition divides the infinitesimal strain into the product of two different tensors:

$$\begin{aligned} \mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon &= \mathcal{D}_\varepsilon \mathcal{H}_\varepsilon \\ \mathcal{D}_\varepsilon &= \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} [\mathbf{I} + \boldsymbol{\varepsilon}_\varepsilon] \\ \mathcal{H}_\varepsilon &= \sqrt[n]{\mathcal{F}_\varepsilon} [\mathbf{I}] \end{aligned} \quad (5.85)$$

Where \mathcal{H}_ε is the infinitesimal inflation tensor, and \mathcal{D}_ε is the infinitesimal distortion tensor. These two tensors are extensively analysed in the following sections.

5.7.1. Infinitesimal inflation tensor (infinitesimal isotropic strain tensor)

On the one hand, the tensor \mathcal{H}_ε is diagonal, and its determinant turns out to be approximately equal to the determinant of the deformation gradient tensor.

$$\det(\mathcal{H}_\varepsilon) = \det\left(\sqrt[n]{\mathcal{F}_\varepsilon} [\mathbf{I}]\right) = \mathcal{F}_\varepsilon \approx F_\varepsilon \quad (5.86)$$

The value of its determinant indicates that this tensor is the one involved with the change of volume. It is the so-called infinitesimal inflation tensor or infinitesimal isotropic strain tensor. Since its determinant is approximately equal to the determinant of the deformation gradient tensor, this tensor rules the volume change in the vicinity of a given point.

According to the result obtained in (5.41), the determinant of the approximate deformation gradient tensor is:

$$\mathcal{F}_\varepsilon \approx 1 + \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \quad (5.87)$$

Thus, the infinitesimal inflation tensor defined in (5.85) can be rewritten as:

$$\mathcal{H}_\varepsilon \approx \left(1 + \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon)\right)^{1/n} [\mathbf{I}] \quad (5.88)$$

Moreover, the coefficients that compose the diagonal of the above tensor can be expressed by its Taylor series expansion, as shown below.

$$\begin{aligned} f\left(\text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)\right) &= \left(1 + \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)\right)^{1/n} \\ &= f(0) + f'(0) \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon) + \mathcal{O}\left(|\text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)|^2\right) \end{aligned} \quad (5.89)$$

In (5.42) it has been proved that the absolute value of the infinitesimal strain tensor trace is much lower than one. Therefore, the above series expansion is reduced to:

$$|\text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)| \ll 1 \quad \implies \quad \left(1 + \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)\right)^{1/n} \approx f(0) + f'(0) \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon) \quad (5.90)$$

So, the diagonal components can be approximately computed as:

$$\left(1 + \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)\right)^{1/n} \approx 1 + \frac{1}{n} \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon) \quad (5.91)$$

And the infinitesimal inflation tensor (5.88) can be finally defined as:

$$\boldsymbol{\mathcal{H}}_\varepsilon \approx \left(1 + \frac{1}{n} \text{Tr}(\boldsymbol{\mathcal{E}}_\varepsilon)\right) [\mathbf{I}] \quad (5.92)$$

This tensor governs the volume variation in the vicinity of a given material particle, whereas the one that rules the distortion process without volume variation is presented below.

5.7.2. Infinitesimal distortion tensor (infinitesimal isochoric strain tensor)

On the other hand, the tensor $\boldsymbol{\mathcal{D}}_\varepsilon$ can be rewritten as follows:

$$\begin{aligned} \boldsymbol{\mathcal{D}}_\varepsilon &= \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} [\mathbf{I} + \boldsymbol{\mathcal{E}}_\varepsilon] \\ &= \mathbf{I} + \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} [\mathbf{I} + \boldsymbol{\mathcal{E}}_\varepsilon] - \mathbf{I} \\ &= \mathbf{I} + \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} \left[\boldsymbol{\mathcal{E}}_\varepsilon + \left(1 - \sqrt[n]{\mathcal{F}_\varepsilon}\right) \mathbf{I} \right] \end{aligned} \quad (5.93)$$

If the alternative polar decomposition stated in (5.84) is applied, the determinant of the deformation gradient tensor becomes:

$$F_\varepsilon \approx \mathcal{F}_\varepsilon = \det(\boldsymbol{\mathcal{F}}_\varepsilon) = \det(\boldsymbol{\mathcal{R}}_\varepsilon \boldsymbol{\mathcal{D}}_\varepsilon \boldsymbol{\mathcal{H}}_\varepsilon) = \det(\boldsymbol{\mathcal{R}}_\varepsilon) \det(\boldsymbol{\mathcal{D}}_\varepsilon) \det(\boldsymbol{\mathcal{H}}_\varepsilon) \quad (5.94)$$

The determinant of the infinitesimal rotation tensor has to be approximately equal to one, as it represents a proper rotation (5.36). Furthermore, the determinant of

the tensor \mathcal{H}_ε has already been obtained in (5.86). So, it can be concluded that the determinant of the tensor \mathcal{D}_ε has to be equal to one.

$$\left. \begin{aligned} \mathcal{F}_\varepsilon &= \det(\mathcal{R}_\varepsilon) \det(\mathcal{D}_\varepsilon) \det(\mathcal{H}_\varepsilon) \\ \det(\mathcal{R}_\varepsilon) &\approx 1 \\ \det(\mathcal{H}_\varepsilon) &\approx \mathcal{F}_\varepsilon \end{aligned} \right\} \implies \det(\mathcal{D}_\varepsilon) \approx 1 \quad (5.95)$$

The value of its determinant indicates that this tensor rules a geometric change without volume variation, hence, it is related with a distortion process. It is the so-called infinitesimal distortion tensor or infinitesimal isochoric strain tensor, and it is defined as shown below.

$$\mathcal{D}_\varepsilon = \mathbf{I} + \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} \left[\boldsymbol{\varepsilon}_\varepsilon + \left(1 - \sqrt[n]{\mathcal{F}_\varepsilon} \right) \mathbf{I} \right] \quad (5.96)$$

This definition of the infinitesimal distortion tensor can be simplified. To do so, the result obtained in (5.91) is recalled:

$$\mathcal{F}_\varepsilon^{1/n} = \left(1 + \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \right)^{1/n} \approx 1 + \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \quad \text{since} \quad |\text{Tr}(\boldsymbol{\varepsilon}_\varepsilon)| \ll 1 \quad (5.97)$$

And its inverse turns out to be:

$$\mathcal{F}_\varepsilon^{-1/n} = \frac{1}{\mathcal{F}_\varepsilon^{1/n}} \approx \frac{1}{1 + \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon)} = \frac{1 - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon)}{\underbrace{1 - \frac{1}{n^2} \text{Tr}^2(\boldsymbol{\varepsilon}_\varepsilon)}_{\approx 0}} \approx 1 - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \quad (5.98)$$

Thus, if the coefficients stated in (5.97) and (5.98) are substituted into the equation that defines the infinitesimal distortion tensor (5.96), an equivalent definition can be obtained:

$$\begin{aligned} \mathcal{D}_\varepsilon &= \mathbf{I} + \frac{1}{\sqrt[n]{\mathcal{F}_\varepsilon}} \left[\boldsymbol{\varepsilon}_\varepsilon + \left(1 - \sqrt[n]{\mathcal{F}_\varepsilon} \right) \mathbf{I} \right] \\ &\approx \mathbf{I} + \left(1 - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \right) \left[\boldsymbol{\varepsilon}_\varepsilon - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} \right] \\ &\approx \mathbf{I} + \left(\boldsymbol{\varepsilon}_\varepsilon - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} \right) \end{aligned} \quad (5.99)$$

In addition, it can be equivalently rewritten in terms of the infinitesimal deviatoric strain tensor as:

$$\begin{aligned} \mathcal{D}_\varepsilon &= \mathbf{I} + \boldsymbol{\varepsilon}_{D,\varepsilon} \\ \boldsymbol{\varepsilon}_{D,\varepsilon} &= \boldsymbol{\varepsilon}_\varepsilon - \frac{1}{n} \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} \end{aligned} \quad (5.100)$$

Where $\boldsymbol{\varepsilon}_{D,\varepsilon}$ is the infinitesimal deviatoric strain tensor.

The alternative polar decomposition presented in the previous sections allows to obtain a clearer physical interpretation of the geometric transformation that occurs in the vicinity of each material particle.

5.7.3. Physical interpretation

If this alternative infinitesimal polar decomposition is applied to the deformation gradient tensor, the infinitesimal geometric change of a material vector can be expressed as follows:

$$\delta \mathbf{r} \approx \underbrace{\mathcal{R}_c}_{\text{inf. rotation}} \left(\underbrace{\mathcal{D}_c}_{\text{inf. distortion}} \left(\underbrace{\mathcal{H}_c}_{\text{inf. inflation}} \delta \mathbf{r}_0 \right) \right) + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (5.101)$$

Where:

$\mathcal{R}_c \equiv$ Infinitesimal rotation tensor

$\mathcal{D}_c \equiv$ Infinitesimal distortion tensor (infinitesimal isochoric strain tensor)

$\mathcal{H}_c \equiv$ Infinitesimal inflation tensor (infinitesimal isotropic strain tensor)

Let's consider the definition of an infinitesimal parallelepiped at a given material point, whose reference position is defined by the position vector \mathbf{r}_0 . Let's also consider that its faces are parallel to the coordinate planes composed by the principal axes. The geometric changes that the parallelepiped experiments are:

- The infinitesimal inflation caused by the infinitesimal inflation tensor \mathcal{H}_c . This is the only transformation that involves a volume variation, and it is the first geometric change.
- The infinitesimal distortion, produced by the distortion tensor \mathcal{D}_c . As it is a distortion process, the volume of the parallelepiped remains constant.
- And finally, the infinitesimal rotation produced by the infinitesimal rotation tensor \mathcal{R}_c . This rotation orientates the parallelepiped according to the deformed configuration.

Figure 5.3 shows the initial location of an infinitesimal parallelepiped on the reference configuration, and its configuration at a given instant of time t .

Furthermore, figure 5.4 shows the whole geometric transformation that experiments the infinitesimal parallelepiped, decomposed in an inflation, followed by a distortion and a rotation.

If this entire process is known, the change of shape, volume, and orientation experimented by a solid after the application of the external loads can be entirely understood and defined. This information is necessary to properly define the displacement field, and its corresponding strain and stress fields. So, the polar decomposition of the deformation gradient tensor is a powerful tool to analyse solids, when they are assumed to experiment large displacements.

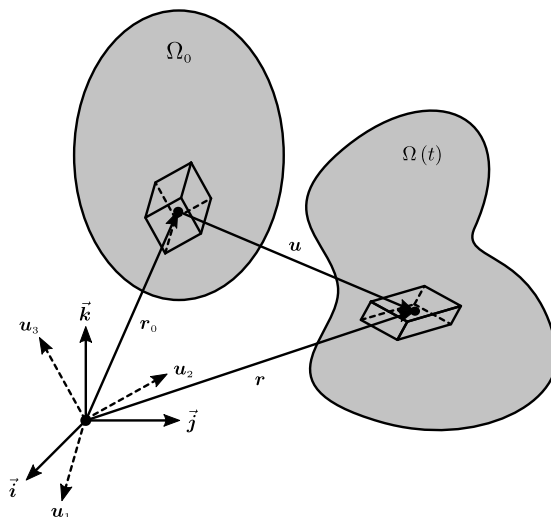


Figure 5.3. Parallelepiped infinitesimal geometric transformation at a given instant of time.

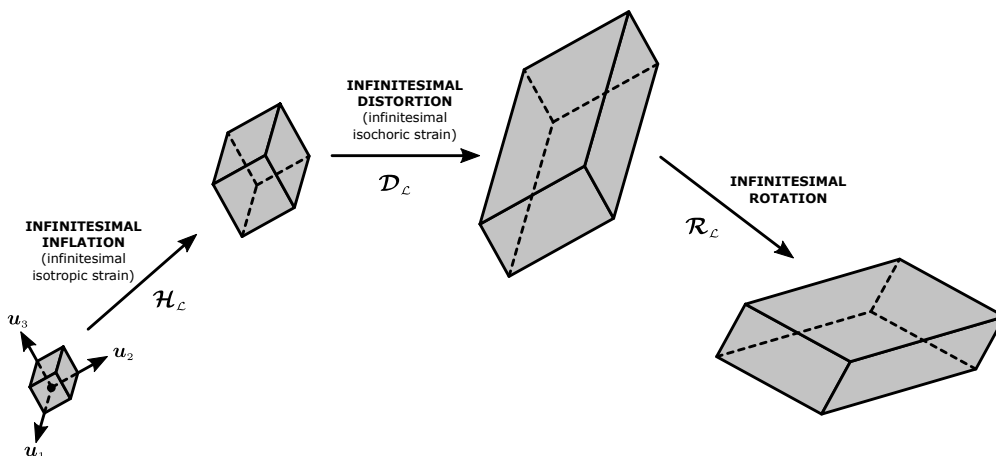


Figure 5.4. Alternative approximate polar decomposition applied to the parallelepiped infinitesimal geometric transformation.

5.8. Overview and conclusions

This chapter is focused on the infinitesimal geometric transformation of a material vector. Its geometric variation turns out to be infinitesimal if the displacement gradients are small. Note that a structure that behaves with small displacement gradients may undergo large displacements. Thus, it can be concluded that a structural response with small displacement gradients does not imply a small displacement behaviour.

If the displacement gradients are small, the approximate polar decomposition can be accurately applied to decompose the deformation gradient tensor. This decomposi-

tion defines the deformation gradient tensor as the product between the infinitesimal rotation tensor ($\mathcal{R}_\mathcal{L}$) and the infinitesimal strain ($[\mathbf{I} + \mathcal{E}_\mathcal{L}]$), where $\mathcal{E}_\mathcal{L}$ is the infinitesimal strain tensor. Both tensors turn out to be defined by means of the displacement gradient tensor. Therefore, if the displacement gradients are small, there is no need to solve an eigenvalue problem to obtain the polar decomposition of the deformation gradient tensor. This is a major advantage, since less computational effort is required.

The polar decomposition clarifies the physical interpretation of the geometric transformation suffered by the material vector. Firstly, the infinitesimal strain modifies the modulus and direction of the material vector by means of a pure infinitesimal stretch transformation. Then, the infinitesimal rotation tensor rotates the previous modified material vector to orientate it according to the deformed geometry. Both effects turn out to be infinitesimal geometric transformations.

Furthermore, an alternative polar decomposition can equivalently be stated. This decomposition divides the infinitesimal strain effect into an infinitesimal inflation, followed by an infinitesimal distortion. The inflation is ruled by the infinitesimal inflation tensor ($\mathcal{H}_\mathcal{L}$), which causes a volume variation in the vicinity of a given material particle. Whereas, the distortion effect is governed by the infinitesimal distortion tensor ($\mathcal{D}_\mathcal{L}$), which produces a distortion without volume variation.

The polar decomposition of the deformation gradient tensor is a powerful tool to analyse solids that experiment large displacements and/or large displacement gradients. If the displacement gradients are small, the structural analysis is simplified and the polar decomposition can be directly obtained by means of the displacement field gradients.

Incremental approach

6.1. Introduction

The geometric transformation of a material vector between $t = 0$ and a given instant of time t was extensively analysed in the previous chapters. Whether this geometric variation is finite or infinitesimal, the deformation gradient tensor is the one that governs the time evolution of a given material vector.

However, this chapter focuses on the incremental geometric transformation experimented by a material vector between two consecutive infinitesimal time steps. The equation that defines this incremental geometric transformation, as well as the polar decomposition of the tensor that governs this geometric transformation, are derived.

6.2. Incremental approach

As proved in section 1.10, the deformation gradient tensor rules the geometric transformation experimented by a material vector between $t = 0$ and a given instant of time t .

$$\underbrace{\delta \mathbf{r}_0}_{(t=0)} \longrightarrow \underbrace{\delta \mathbf{r}_t}_{(\text{instant } t)} = \mathbf{F}_C \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (6.1)$$

An incremental approach is now adopted (figure 6.1). The equation that defines the geometric transformation of a given material vector between two consecutive infinitesimal time steps turns out to be:

$$\begin{aligned} \underbrace{\delta \mathbf{r}_t}_{(\text{instant } t)} &\longrightarrow \underbrace{\delta \mathbf{r}_{t+dt}}_{(\text{instant } t+dt)} = \delta \mathbf{r}_t + \frac{\partial \delta \mathbf{r}}{\partial t} dt + \mathcal{O}(dt^2) \\ &\approx \delta \mathbf{r}_t + \frac{\partial \delta \mathbf{r}}{\partial t} dt \end{aligned} \quad (6.2)$$

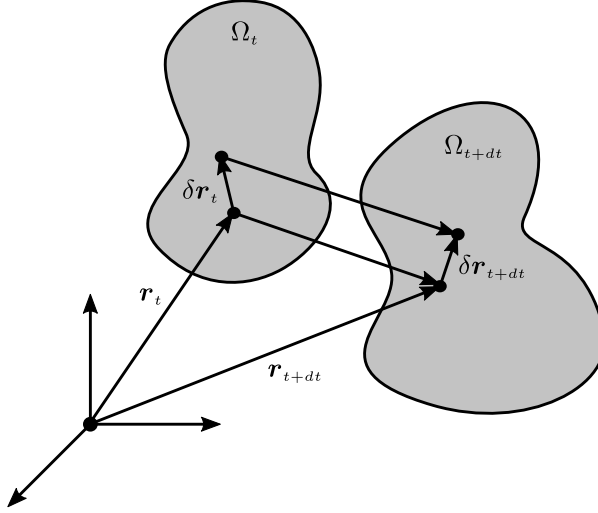


Figure 6.1. Geometric transformation of a material vector between two consecutive infinitesimal time steps.

According to the result obtained in (6.2), the time variation of the material vector has to be defined in order to obtain the equation that defines its incremental geometric change. The derivation of equation (6.1) with respect to time leads to:

$$\frac{\partial \delta \mathbf{r}}{\partial t} = \frac{\partial \mathbf{F}_{\mathcal{L}}}{\partial t} \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (6.3)$$

As stated in (1.26), the deformation gradient tensor is invertible. Consequently, the inverse geometric transformation can be defined as follows.

$$\delta \mathbf{r}_0 = \mathbf{F}_{\mathcal{L}}^{-1} \delta \mathbf{r} + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (6.4)$$

On the other hand, the deformed material vector depends on the initial one. Thus, it can be easily seen that:

$$\delta \mathbf{r} = \mathbf{F}_{\mathcal{L}} \delta \mathbf{r}_0 + \mathcal{O}(\|\mathbf{r}_0\|^2) \quad \implies \quad \mathcal{O}(\|\delta \mathbf{r}\|^2) = \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \quad (6.5)$$

Therefore, if equation (6.4) is now substituted into (6.3), and the property stated in (6.5) is taken into account, the time variation of the material vector becomes:

$$\frac{\partial \delta \mathbf{r}}{\partial t} = \left[\frac{\partial \mathbf{F}_{\mathcal{L}}}{\partial t} \mathbf{F}_{\mathcal{L}}^{-1} \right] \delta \mathbf{r} + \mathcal{O}(\|\delta \mathbf{r}\|^2) \quad (6.6)$$

And the incremental geometric transformation of the material vector (6.2) can be finally rewritten by means of the velocity gradient tensor $\mathbf{l}_{\mathcal{L}}$ as:

$$\delta \mathbf{r}_{t+dt} = [\mathbf{I} + \mathbf{l}_{\mathcal{L}} dt] \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \quad \text{with} \quad \mathbf{l}_{\mathcal{L}} = \frac{\partial \mathbf{F}_{\mathcal{L}}}{\partial t} \mathbf{F}_{\mathcal{L}}^{-1} \quad (6.7)$$

6.2.1. Velocity gradient tensor

As stated in the previous section, the tensor that defines the incremental geometric transformation of a material vector is the velocity gradient tensor. The substitution of the time derivative of the deformation gradient tensor (1.14) into the definition of the velocity gradient tensor (6.7) leads to:

$$\begin{aligned} \mathbf{l}_\mathcal{L}(\mathbf{r}_0, t) &= \frac{\partial \mathbf{F}_\mathcal{L}}{\partial t} \mathbf{F}_\mathcal{L}^{-1} \\ &= \frac{\partial \mathbf{a}_\mathcal{L}}{\partial \mathbf{r}_0} \mathbf{F}_\mathcal{L}^{-1} \end{aligned} \quad (6.8)$$

According to (1.20), the above equation can be equivalently rewritten as:

$$\mathbf{l}_\mathcal{L}(\mathbf{r}_0, t) = \frac{\partial \mathbf{a}_\mathcal{L}}{\partial \mathbf{r}_0} \mathbf{F}_\mathcal{L}^{-1} = \frac{\partial \mathbf{a}_\mathcal{E}}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0, t)} = \mathbf{l}_\mathcal{E}(\mathbf{r}, t) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0, t)} \quad (6.9)$$

The previous result proves that the tensor that defines the incremental geometric transformation turns out to be the gradient tensor of the Eulerian velocity vector. Consequently, this tensor is usually called as the velocity gradient tensor.

6.2.2. Infinitesimal geometric transformation

As equation (6.7) states, the incremental geometric transformation of a material vector depends on the velocity gradient tensor.

$$\begin{aligned} \delta \mathbf{r}_{t+dt} &= [\mathbf{I} + \mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \\ &= \delta \mathbf{r}_t + [\mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \end{aligned} \quad (6.10)$$

According to the above result, it can be concluded that the incremental transformation is an infinitesimal one, since the norm of the vector added to the original material vector is much smaller than the norm of the original one.

$$\|[\mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}\| \ll \|\delta \mathbf{r}\| \quad \Longleftrightarrow \quad \frac{\|[\mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}\|}{\|\delta \mathbf{r}\|} \ll 1 \quad \forall \delta \mathbf{r} \neq \bar{\mathbf{0}} \quad (6.11)$$

Since the above inequality has to be fulfilled for all non-null material vectors, it can also be stated that:

$$\max_{\delta \mathbf{r} \neq \bar{\mathbf{0}}} \frac{\|[\mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}\|}{\|\delta \mathbf{r}\|} \ll 1 \quad (6.12)$$

According to the norm of a second order tensor stated in (A.49), the left-hand side of the previous inequality turns out to be the norm of the tensor that rules the incremental geometric transformation. On the other hand, the right-hand side is the norm of the second order unit tensor (A.50). Consequently, the condition that indicates that the incremental transformation is an infinitesimal one can be alternatively rewritten as:

$$\|\mathbf{l}_\mathcal{L} dt\| \ll \|\mathbf{I}\| \quad \text{where} \quad \begin{cases} \|\mathbf{l}_\mathcal{L} dt\| = \max_{\delta \mathbf{r} \neq \bar{\mathbf{0}}} \frac{\|[\mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}\|}{\|\delta \mathbf{r}\|} \\ \|\mathbf{I}\| = 1 \end{cases} \quad (6.13)$$

In the incremental approach, the velocity gradient tensor is the one that contains the information related to the change of volume, orientation, and shape between two consecutive infinitesimal time steps. This information is required to properly define the structural behaviour.

6.3. Polar decomposition of the tensor that rules the incremental approach

As the incremental approach defines an infinitesimal geometric transformation, the approximate polar decomposition stated in section (5.3) can be applied. According to this decomposition, the tensor can be decomposed as:

$$[\mathbf{I} + \mathbf{l}_\varepsilon dt] = [\mathbf{I} + \mathbf{w}_\varepsilon dt] [\mathbf{I} + \mathbf{e}_\varepsilon dt] , \quad (6.14)$$

where:

$$\begin{aligned} \mathbf{w}_\varepsilon &= \frac{1}{2}(\mathbf{l}_\varepsilon - \mathbf{l}_\varepsilon^T) \equiv \text{spin tensor}, \\ \mathbf{e}_\varepsilon &= \frac{1}{2}(\mathbf{l}_\varepsilon + \mathbf{l}_\varepsilon^T) \equiv \text{strain-rate tensor}. \end{aligned} \quad (6.15)$$

If the second order infinitesimals are neglected, the above decomposition can be verified.

$$\begin{aligned} [\mathbf{I} + \mathbf{w}_\varepsilon dt] [\mathbf{I} + \mathbf{e}_\varepsilon dt] &= \left[\mathbf{I} + \frac{1}{2}(\mathbf{l}_\varepsilon - \mathbf{l}_\varepsilon^T) dt \right] \left[\mathbf{I} + \frac{1}{2}(\mathbf{l}_\varepsilon + \mathbf{l}_\varepsilon^T) dt \right] \\ &\approx \mathbf{I} + \frac{1}{2}(\mathbf{l}_\varepsilon + \mathbf{l}_\varepsilon^T) dt + \frac{1}{2}(\mathbf{l}_\varepsilon - \mathbf{l}_\varepsilon^T) dt \\ &= \mathbf{I} + \mathbf{l}_\varepsilon dt \end{aligned} \quad (6.16)$$

On the one hand, the tensor that defines the incremental rotation is the spin tensor. This tensor fulfils the condition to define an infinitesimal geometric transformation, since it is multiplied by dt .

$$\|\mathbf{w}_\varepsilon dt\| \ll \|\mathbf{I}\| \quad (6.17)$$

According to its definition, the incremental rotation tensor turns out to be skew-symmetric.

$$(\mathbf{w}_\varepsilon dt)^T = \frac{1}{2}(\mathbf{l}_\varepsilon - \mathbf{l}_\varepsilon^T)^T dt = -\mathbf{w}_\varepsilon dt \quad (6.18)$$

Since it is a skew-symmetric tensor, and its components are infinitesimal, this tensor turns out to represent an infinitesimal rotation [Oliver & Agelet, 2002]. Furthermore, it has to be orthogonal to properly represent a rotation (section A.13). To prove it, the second order infinitesimal are again neglected.

$$\left. \begin{aligned} [\mathbf{I} + \mathbf{w}_\varepsilon dt]^T [\mathbf{I} + \mathbf{w}_\varepsilon dt] &\approx \mathbf{I} \\ [\mathbf{I} + \mathbf{w}_\varepsilon dt] [\mathbf{I} + \mathbf{w}_\varepsilon dt]^T &\approx \mathbf{I} \end{aligned} \right\} \iff [\mathbf{I} + \mathbf{w}_\varepsilon dt]^{-1} \approx [\mathbf{I} + \mathbf{w}_\varepsilon dt]^T \quad (6.19)$$

On the other hand, the strain-rate tensor has to be symmetric.

$$(\mathbf{e}_\mathcal{L} dt)^T = \frac{1}{2}(\mathbf{l}_\mathcal{L} + \mathbf{l}_\mathcal{L}^T)^T dt = \mathbf{e}_\mathcal{L} dt \quad (6.20)$$

And the tensor $[\mathbf{I} + \mathbf{e}_\mathcal{L} dt]$ has to be positive definite, since this tensor represents the incremental strain that the material vector experiments.

$$\mathbf{x}^T [\mathbf{I} + \mathbf{e}_\mathcal{L} dt] \mathbf{x} = \mathbf{x}^T \mathbf{x} + \mathbf{x}^T (\mathbf{e}_\mathcal{L} dt) \mathbf{x} = \|\mathbf{x}\|^2 + (\mathbf{x}^T \mathbf{e}_\mathcal{L} \mathbf{x}) dt > 0 \quad \forall \mathbf{x} \neq \bar{\mathbf{0}} \quad (6.21)$$

To sum up, if the infinitesimal polar decomposition is taken into account, the incremental geometric transformation of a material vector can be expressed as shown below.

$$\delta \mathbf{r}_{t+dt} = \underbrace{[\mathbf{I} + \mathbf{w}_\mathcal{L} dt]}_{\text{incr. rotation}} \left(\overbrace{[\mathbf{I} + \mathbf{e}_\mathcal{L} dt]}^{\text{incr. strain}} \delta \mathbf{r}_t \right) + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \quad (6.22)$$

Firstly, the incremental strain tensor modifies the modulus and direction of the material vector corresponding to a given instant of time t . Then, the incremental rotation tensor rotates the previous modified vector, to properly orientate it according to the deformed geometry corresponding to the instant $t + dt$. Both geometric transformation turn out to be infinitesimal.

6.3.1. Determinant of the tensor that rules the incremental approach

As obtained in (6.7), the geometric transformation of a material vector between two consecutive infinitesimal time steps is defined by the velocity gradient tensor.

$$\delta \mathbf{r}_{t+dt} = [\mathbf{I} + \mathbf{l}_\mathcal{L} dt] \delta \mathbf{r}_t \quad (6.23)$$

The computation of the determinant of the tensor that rules the above geometric transformation is analogous to the one calculated in the infinitesimal case (5.41), writing $(\mathbf{l}_\mathcal{L} dt)$ instead of $\mathcal{E}_\mathcal{L}$.

$$\begin{aligned} \det(\mathbf{I} + \mathbf{l}_\mathcal{L} dt) &\approx 1 + \text{Tr}(\mathbf{l}_\mathcal{L} dt) \\ &= 1 + \text{Tr}(\mathbf{l}_\mathcal{L}) dt \end{aligned} \quad (6.24)$$

In addition, if the definition of the strain-rate tensor in terms of the velocity gradient tensor (6.15) is taken into account, it can be concluded that the trace of the strain-rate tensor is equivalent to the trace of the velocity gradient tensor.

$$\mathbf{e}_\mathcal{L} = \frac{1}{2}(\mathbf{l}_\mathcal{L} + \mathbf{l}_\mathcal{L}^T) \quad \implies \quad \text{Tr}(\mathbf{e}_\mathcal{L}) = \text{Tr}(\mathbf{l}_\mathcal{L}) \quad (6.25)$$

Therefore, the determinant (6.24) can be computed by means of the strain-rate tensor as:

$$\begin{aligned} \det(\mathbf{I} + \mathbf{l}_\mathcal{L} dt) &\approx 1 + \text{Tr}(\mathbf{l}_\mathcal{L}) dt \\ &= 1 + \text{Tr}(\mathbf{e}_\mathcal{L}) dt \end{aligned} \quad (6.26)$$

6.3.2. Alternative polar decomposition

An alternative polar decomposition of the tensor that defines the incremental geometric transformation can be stated, by separating the incremental strain into the product of an incremental distortion and an incremental inflation.

The equations that define these tensors can be obtained from (5.92) and (5.99). These equations define the infinitesimal distortion tensor $\mathcal{D}_\mathcal{L}$ and the infinitesimal inflation tensor $\mathcal{H}_\mathcal{L}$, by means of the infinitesimal strain tensor $\mathcal{E}_\mathcal{L}$. The incremental version of these tensors is obtained by writing $(\mathbf{e}_\mathcal{L} dt)$ instead of $\mathcal{E}_\mathcal{L}$.

$$\begin{aligned} \mathbf{I} + \mathbf{l}_\mathcal{L} dt &= [\mathbf{I} + \mathbf{w}_\mathcal{L} dt] [\mathbf{I} + \mathbf{e}_\mathcal{L} dt] \\ &= [\mathbf{I} + \mathbf{w}_\mathcal{L} dt] \left[\mathbf{I} + \left(\mathbf{e}_\mathcal{L} - \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) \mathbf{I} \right) dt \right] \left[\left(1 + \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) dt \right) \mathbf{I} \right], \end{aligned} \quad (6.27)$$

where:

$$\begin{aligned} \mathbf{I} + \mathbf{w}_\mathcal{L} dt &\equiv \text{incremental rotation tensor,} \\ \mathbf{I} + \left(\mathbf{e}_\mathcal{L} - \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) \mathbf{I} \right) dt &\equiv \text{incremental distortion tensor,} \\ \left(1 + \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) dt \right) \mathbf{I} &\equiv \text{incremental inflation tensor.} \end{aligned} \quad (6.28)$$

Moreover, the incremental distortion tensor can be rewritten by means of the incremental deviatoric strain tensor $(\mathbf{e}_{D,\mathcal{L}})$ as:

$$\mathbf{I} + \mathbf{e}_{D,\mathcal{L}} dt \quad \text{with} \quad \mathbf{e}_{D,\mathcal{L}} = \mathbf{e}_\mathcal{L} - \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) \mathbf{I} \quad (6.29)$$

Thus, the relative motion of the media in the vicinity of a given point can be expressed as follows:

$$\delta \mathbf{r}_{t+dt} = \underbrace{[\mathbf{I} + \mathbf{w}_\mathcal{L} dt]}_{\text{incr. rotation}} \left(\underbrace{[\mathbf{I} + \mathbf{e}_{D,\mathcal{L}} dt]}_{\text{incr. distortion}} \left(\underbrace{\left[\left(1 + \frac{1}{n} \text{Tr}(\mathbf{e}_\mathcal{L}) dt \right) \mathbf{I} \right]}_{\text{incr. inflation}} \delta \mathbf{r}_t \right) \right) + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \quad (6.30)$$

Let's consider that a parallelepiped is defined at a specific material point, whose position at a given instant of time t is defined by the position vector \mathbf{r}_t . Its faces are oriented according to the directions defined by the main axes (figure 6.2).

- The first geometric transformation that the parallelepiped experiments is the incremental inflation, ruled by the incremental inflation tensor. This is the only geometric transformation that involves a volume variation.
- The second one is the incremental distortion, governed by the incremental distortion tensor. As it is a distortion process, the volume of the parallelepiped remains constant.

- Finally, an incremental rotation produced by the spin tensor is applied. This rotation orientates the parallelepiped according to the deformed configuration corresponding to time $t + dt$.

Figure 6.2 illustrates the definition of a parallelepiped at a given instant of time t , and the incremental geometric transformation that it suffers after an infinitesimal time step.

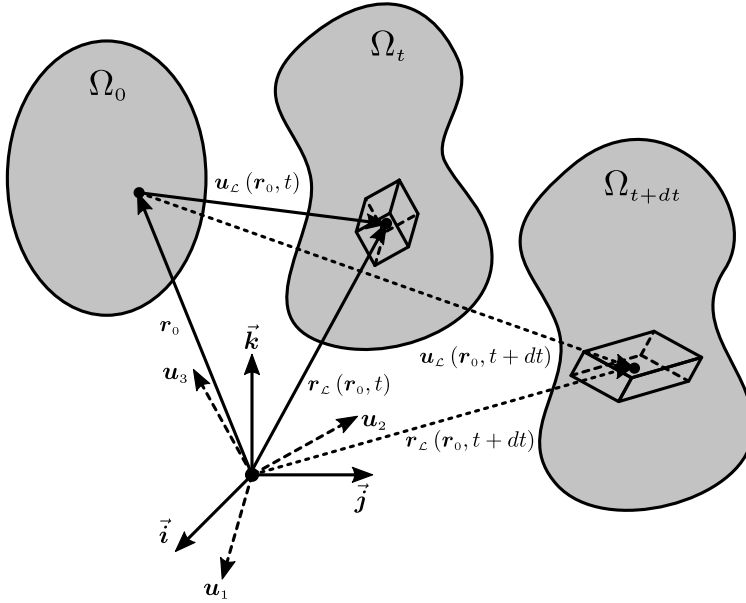


Figure 6.2. Parallelepiped incremental geometric transformation experimented between two consecutive infinitesimal time steps.

And figure 6.3 shows the decomposition of the whole incremental geometric transformation. That is, the inflation, distortion, and rotation that the initial volume experiments.

If this entire process is known, the change of shape, volume, and orientation that a solid experiments between two consecutive infinitesimal time steps can be entirely understood and defined. This information is necessary to properly define the displacement field, and its corresponding strain and stress fields.

6.4. Overview and conclusions

Up to now, the geometric transformation experimented by a given material vector between $t = 0$ and t was extensively analysed. Whether the variation is finite or infinitesimal, both geometric changes turn out to be ruled by the deformation gradient tensor. The polar decomposition allows to define the deformation gradient tensor as the product between a rotation tensor and a strain tensor.

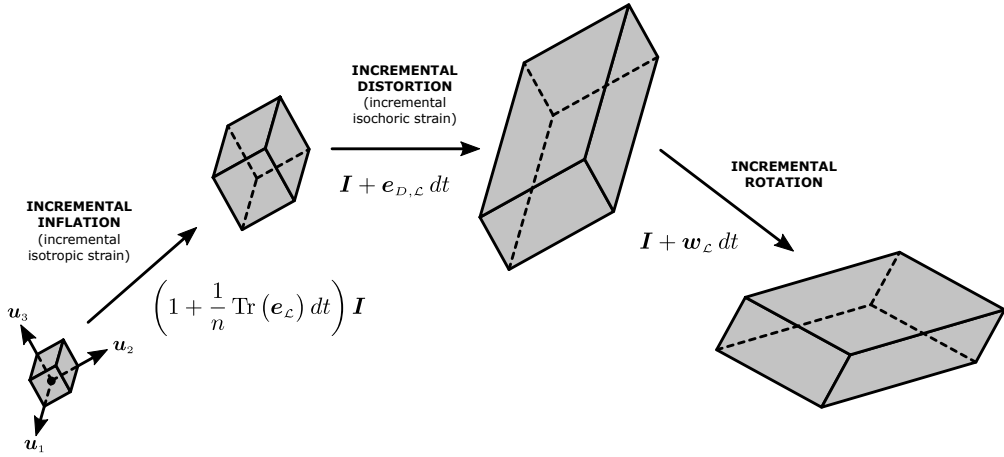


Figure 6.3. Alternative polar decomposition applied to the parallelepiped incremental geometric transformation.

Nevertheless, this chapter adopts an incremental approach, focused on the incremental geometric transformation between two consecutive infinitesimal time steps. The geometric transformation experimented by a given material vector between t and $t + dt$ turns out to be ruled by the velocity gradient tensor. As this geometric variation is infinitesimal, the approximate polar decomposition can be accurately applied to decompose the tensor that governs this incremental geometric transformation. According to the approximate decomposition, this tensor can be defined as the product between an incremental rotation and an incremental strain.

Furthermore, the incremental strain can be equivalently defined as an incremental inflation followed by an incremental distortion. The incremental inflation rules the incremental volume variation in the vicinity of a given material point, whereas the incremental distortion does not imply volume variation.

This incremental approach can be extended to describe the incremental loading process, usually carried out in nonlinear analysis. When dealing with an incremental loading procedure, the time variable t becomes a variable which indicates the current load step. If the incremental load is small enough at each step of the loading process, the incremental geometric transformation of a given material vector is governed by the formulation presented in this chapter.

Constitutive equations

7.1. Introduction

In structural analysis, the equation that states the relation between the strain field and its corresponding stress field is the so-called constitutive equation. As this work analyses the structural behaviour of elastic solids, this chapter focuses on the definition of the constitutive equation both in linear and nonlinear elasticity theory.

7.2. Elastic solids

As proved in section 1.10, the geometric variation of a material vector over time is ruled by the deformation gradient tensor. Moreover, the deformation gradient tensor can be expressed by means of the displacement gradient tensor as:

$$\begin{aligned}\delta \mathbf{r} &= \mathbf{F}_\mathcal{L} \delta \mathbf{r}_0 + \mathcal{O}(\|\delta \mathbf{r}_0\|^2) \\ \mathbf{F}_\mathcal{L} &= \frac{\partial \mathbf{r}_\mathcal{L}}{\partial \mathbf{r}_0} = \mathbf{I} + \mathbf{J}_\mathcal{L} \\ \mathbf{J}_\mathcal{L} &= \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0}\end{aligned}\tag{7.1}$$

This implies that the deformation gradient tensor governs the motion of a solid in the vicinity of a given point. This tensor is the one that contains the information related to the change of volume, orientation, and shape experimented by the solid. This information is required to properly define the displacement, strain, and stress fields of a continuous solid media subjected to external forces.

Furthermore, as stated in section 4.3, the polar decomposition theorem allows to decompose the deformation gradient tensor as the product between a finite rotation tensor and a finite strain tensor.

$$\delta \mathbf{r} = \mathbf{R}_\mathcal{L} \left(\underbrace{[\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0}_{\delta \mathbf{r}_1} \right) + \mathcal{O}(\|\delta \mathbf{r}_0\|^2)\tag{7.2}$$

Consequently, the geometric change of a material vector can be divided into two phases:

- Firstly, the finite strain tensor modifies the direction and modulus of the material vector.

$$\delta \mathbf{r}_1 = [\mathbf{I} + \mathbf{E}_\varepsilon] \delta \mathbf{r}_0 \quad (7.3)$$

- Finally, the finite rotation tensor applies a finite rotation to the material vector obtained in the previous step. As it is a rotation, its modulus does not vary. The material vector only experiments a variation in its direction.

$$\delta \mathbf{r} = \mathbf{R}_\varepsilon \delta \mathbf{r}_1 \quad \|\delta \mathbf{r}\| = \|\delta \mathbf{r}_1\| \quad (7.4)$$

The first step is the only one that generates internal stresses, since the second one produces a finite rotation. The finite rotation tensor is the one in charge of properly orientating the previous modified material vector, according to the deformed configuration.

Therefore, the polar decomposition allows to state that, both the finite strain tensor and the finite rotation tensor are the ones that define the stress field.

$$\left. \begin{array}{l} \sigma_\varepsilon = \sigma_\varepsilon(\mathbf{F}_\varepsilon) \\ \mathbf{F}_\varepsilon = \mathbf{R}_\varepsilon [\mathbf{I} + \mathbf{E}_\varepsilon] \end{array} \right\} \implies \sigma_\varepsilon = \sigma_\varepsilon(\mathbf{R}_\varepsilon, \mathbf{E}_\varepsilon) \quad (7.5)$$

In elasticity theory, the stress tensor is considered to depend only on the value of the deformation gradient tensor at a given instant of time. That is, it does not depend on its previous values. Thus, if a set of external loads are applied, and then the solid is unloaded, it will recover its initial configuration.

7.3. Nonlinear elasticity

To define the stress tensor in nonlinear elasticity theory, the polar decomposition of the deformation gradient tensor (section 4.3) is recalled.

$$\mathbf{F}_\varepsilon = \mathbf{R}_\varepsilon [\mathbf{I} + \mathbf{E}_\varepsilon] \quad (7.6)$$

In equation (7.3) it has been proved that the finite strain tensor is the one that modifies the modulus and direction of the material vector. Thus, firstly, the stress field is defined depending on the Biot strain tensor.

$$\sigma'_\varepsilon = \Psi(\mathbf{E}_\varepsilon) \quad (7.7)$$

The stress tensor has to be symmetric, if the conservation of angular momentum is fulfilled (proved in section 8.4). Thus, the above tensor verifies that:

$$\Psi^T = \Psi \quad (7.8)$$

Once the preliminary stress tensor (7.7) is defined, its corresponding stress vector can be computed. This vector is associated with a specific plane, defined by its normal vector \mathbf{n}' . Hence, according to the stress vector definition stated in (3.36), the stress vector turns out to be:

$$\left. \begin{array}{l} \mathbf{t}'_{\mathcal{L}} = \boldsymbol{\sigma}'_{\mathcal{L}} \mathbf{n}' \\ \boldsymbol{\sigma}'_{\mathcal{L}} = \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \end{array} \right\} \implies \mathbf{t}'_{\mathcal{L}} = \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \mathbf{n}' \quad (7.9)$$

If the rotation defined by the finite rotation tensor is now applied to the above vector, the stress vector orientated according to the deformed configuration is obtained.

$$\mathbf{t}_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}} \mathbf{t}'_{\mathcal{L}} \quad (7.10)$$

In case a finite rotation is applied to the stress vector, the normal vector that defines its corresponding plane also suffers a rotation. The following rotated normal vector defines the plane associated with the rotated stress vector.

$$\mathbf{n} = \mathbf{R}_{\mathcal{L}} \mathbf{n}' \quad (7.11)$$

From equations (7.10) and (7.11), the preliminaries stress vector ($\mathbf{t}'_{\mathcal{L}}$) and normal vector (\mathbf{n}') can be defined by means of their rotated definitions. The orthogonality of the finite rotation tensor (4.13) is recalled, as it represents a rotation. Thus, the substitution of these vectors into (7.9) leads to definition of the rotated stress tensor.

$$\left. \begin{array}{l} \mathbf{t}'_{\mathcal{L}} = \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \mathbf{n}' \\ \mathbf{t}'_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}}^T \mathbf{t}_{\mathcal{L}} \\ \mathbf{n}' = \mathbf{R}_{\mathcal{L}}^T \mathbf{n} \end{array} \right\} \implies \underbrace{\mathbf{R}_{\mathcal{L}}^T \mathbf{t}_{\mathcal{L}}}_{\mathbf{t}'_{\mathcal{L}}} = \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \underbrace{\mathbf{R}_{\mathcal{L}}^T \mathbf{n}}_{\mathbf{n}'} \iff \mathbf{t}_{\mathcal{L}} = \underbrace{\left(\mathbf{R}_{\mathcal{L}} \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \mathbf{R}_{\mathcal{L}}^T \right)}_{\boldsymbol{\sigma}_{\mathcal{L}}} \mathbf{n} \quad (7.12)$$

In the above equation, the stress tensor ($\boldsymbol{\sigma}_{\mathcal{L}}$) oriented according to the deformed configuration is defined as:

$$\boldsymbol{\sigma}_{\mathcal{L}} = \mathbf{R}_{\mathcal{L}} \boldsymbol{\Psi}(\mathbf{E}_{\mathcal{L}}) \mathbf{R}_{\mathcal{L}}^T \quad (7.13)$$

According to the above result, it can be concluded that the stress field does depend on the Biot strain tensor and the finite rotation tensor, as stated before in (7.5). In structural analysis, the equation that describes the relation between the stress field and the strain field is usually known as the constitutive equation. Therefore, the above equation defines the constitutive equation.

If the strain field turns out to be infinitesimal, the definition of the tensor $\boldsymbol{\Psi}$ that defines the above constitutive equation is simplified. This particular case is extensively analysed in the following section.

7.4. Linear elasticity: infinitesimal strain field

To consider that the strain field is infinitesimal is the main assumption adopted in linear elasticity. As stated in section 5.4.1, the strain tensor is considered to produce an

infinitesimal geometric transformation to the material vector, if the Biot strain tensor fulfils the following condition:

$$\|\mathbf{E}_\mathcal{L}\| \ll \|\mathbf{I}\| \quad (7.14)$$

If the above condition holds, the first geometric change of the material vector described in (7.3) is reduced to:

$$\left. \begin{aligned} \delta \mathbf{r}_1 &= [\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta \mathbf{r}_0 = \delta \mathbf{r}_0 + \mathbf{E}_\mathcal{L} \delta \mathbf{r}_0 \\ \|\mathbf{E}_\mathcal{L} \delta \mathbf{r}_0\| &\ll \|\delta \mathbf{r}_0\| \end{aligned} \right\} \implies \delta \mathbf{r}_1 \approx \delta \mathbf{r}_0 \quad (7.15)$$

This particular case simplifies the definition of the tensor Ψ that defines the constitutive equation presented in (7.13), since this tensor can be linearized in terms of the Biot strain tensor. This will be proved later on in this chapter.

Moreover, it should be noted that, the fulfilment of the condition (7.14) does not imply neither small displacements nor small displacement gradients. That is, some structures may experiment large displacements and/or large displacement gradients, but their corresponding strain field is infinitesimal. These structural responses are exemplified in the following subsections.

7.4.1. Large displacements and infinitesimal strain field

The single articulated rod with prescribed displacements at its nodes (section 5.2.2) is again analysed (figure 7.1).

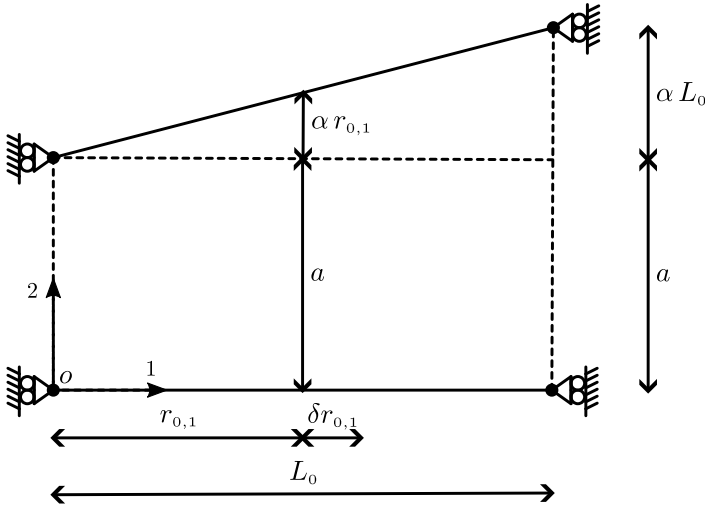


Figure 7.1. Single articulated rod with prescribed displacements at its nodes.

If the prescribed node displacements fulfil the conditions exposed in (5.9), the displacements that experiments the structure are large, and its displacement gradients

are small. These conditions are:

$$\begin{aligned} |a| \ll L_0 &\iff \left| \frac{a}{L_0} \right| \ll 1 \\ |\alpha L_0| \ll L_0 &\iff |\alpha| \ll 1 \end{aligned} \quad (7.16)$$

Since the displacement gradients are small, the infinitesimal strain tensor presented in (5.19) accurately approximates the strain field. Its definition depends on the displacement gradient tensor. So, if the definition of the displacement gradient tensor presented in (5.12) is taken into account, the infinitesimal strain tensor turns out to be:

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \implies \mathbf{E}_\varepsilon \approx \frac{1}{2}(\mathbf{J}_\varepsilon + \mathbf{J}_\varepsilon^T) = \frac{1}{2} \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \quad \text{where } |\alpha| \ll 1 \quad (7.17)$$

And the norm of the above tensor is:

$$\|\mathbf{E}_\varepsilon\| = \frac{\|\mathbf{E}_\varepsilon \delta \mathbf{r}_0\|}{\|\delta \mathbf{r}_0\|} = \frac{\frac{1}{2} |\alpha \delta r_{0,1}|}{\frac{|\delta r_{0,1}|}{2}} = \frac{|\alpha|}{2} \ll 1 \quad (7.18)$$

According to the above result, it can be concluded that the norm of the strain tensor is much lower than one. Therefore, the strain tensor fulfils the condition to produce an infinitesimal geometric transformation (presented in section 5.2.1) when it is applied to a given material vector.

$$\|\mathbf{E}_\varepsilon\| \ll \|\mathbf{I}\| \quad (7.19)$$

This example illustrates that an infinitesimal strain field does not imply an infinitesimal displacement field. Furthermore, it does not necessarily imply small displacement gradients, as exemplified in the following subsection.

7.4.2. Large displacement gradients and infinitesimal strain field

Let's consider a simply supported beam subjected to an external point load applied at the centre of the beam (figure 7.2).

After the application of the external load, a rotation is applied around one of the beam supports. Let's consider that the direction of the load is also modified by the rotation, and the angle of rotation is large enough, so it can be stated that:

$$|\alpha| \ll 1 \quad (7.20)$$

The case without the rigid rotation is firstly studied. If the behaviour of the beam verifies the linear analysis assumptions, its displacements and displacement gradients are small. Since the displacements gradients are small, the strain field can be approximated by the infinitesimal strain tensor. Thus, the strain field turns out to be infinitesimal.

Then, a rigid rotation is applied to this deformed configuration. The displacements and the displacement gradients become large, but the strain field remains infinitesimal.

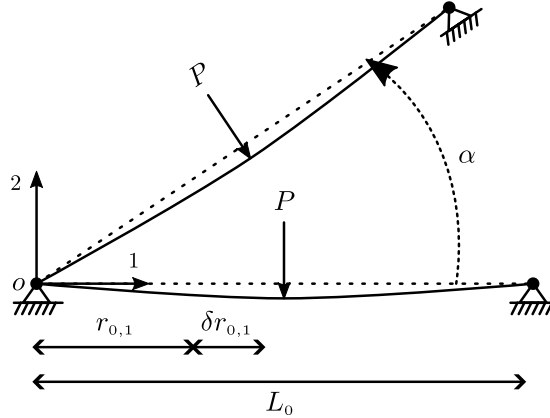


Figure 7.2. Rotation applied to a simply supported beam subjected to an external vertical point load.

This case illustrates that an infinitesimal strain field does not imply necessarily small displacement gradients.

To prove that the displacements and the displacement gradients become large when the rotation is applied, the modulus of the displacement vector and the norm of its corresponding gradient tensor are computed.

On the one hand, the position vector of a material point that belongs to the reference configuration is defined as:

$$\mathbf{r}_0 = \begin{Bmatrix} r_{0,1} \\ 0 \end{Bmatrix} \quad (7.21)$$

The displacement that this particle experiments can be calculated depending on the rotation angle as shown below.

$$\mathbf{u}_L(\mathbf{r}_0) = \begin{Bmatrix} -(r_{0,1} - r_{0,1} \cos \alpha) \\ r_{0,1} \sin \alpha \end{Bmatrix} = \begin{Bmatrix} r_{0,1} (\cos \alpha - 1) \\ r_{0,1} \sin \alpha \end{Bmatrix} \quad (7.22)$$

And the norm of the displacement vector is:

$$\|\mathbf{u}_L\| = \sqrt{r_{0,1}^2 (\cos \alpha - 1)^2 + r_{0,1}^2 \sin^2 \alpha} \quad (7.23)$$

On the other hand, the gradient tensor corresponding to the previous displacement field turns out to be:

$$\mathbf{J}_L = \frac{d\mathbf{u}_L}{d\mathbf{r}_0} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_1}{\partial r_{0,2}} \\ \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,2}} \end{bmatrix} = \begin{bmatrix} \cos \alpha - 1 & 0 \\ \sin \alpha & 0 \end{bmatrix} \quad (7.24)$$

The components of a generic relative deformation vector defined in the reference configuration are:

$$\delta \mathbf{r}_0 = \begin{Bmatrix} \delta r_{0,1} \\ 0 \end{Bmatrix} \quad (7.25)$$

Hence, the norm of the displacement gradient tensor is computed by means of the rotation angle as follows.

$$\|\mathbf{J}_\mathcal{L}\| = \frac{\|\mathbf{J}_\mathcal{L} \delta \mathbf{r}_0\|}{\|\delta \mathbf{r}_0\|} = \sqrt{\frac{(\cos \alpha - 1)^2 \delta r_{0,1}^2 + \sin^2 \alpha \delta r_{0,1}^2}{\delta r_{0,1}^2}} = \sqrt{2(1 - \cos \alpha)} \quad (7.26)$$

According to the results obtained in (7.23) and (7.26), it can be concluded that both the displacements and the displacement gradients are large, if the rotation angle fulfils the condition (7.20). As a conclusion:

$$\left. \begin{aligned} \|\mathbf{u}_\mathcal{L}\| &= \sqrt{r_{0,1}^2 (\cos \alpha - 1)^2 + r_{0,1}^2 \sin^2 \alpha} \ll 1 \\ \|\mathbf{J}_\mathcal{L}\| &= \sqrt{2(1 - \cos \alpha)} \ll 1 \end{aligned} \right\} \quad \text{if} \quad |\alpha| \ll 1 \quad (7.27)$$

This structural behaviour illustrates that an infinitesimal strain field does not imply small displacement gradients.

7.4.3. Isotropic medium

Let's focus now on isotropic mediums, whose mechanical properties do not depend on the direction taken into account.

To express the components of the tensor Ψ , the usual engineering nomenclature is adopted. The diagonal components are the normal stresses σ , and the non-diagonal components are the shear stresses τ .

$$\Psi(\mathbf{E}_\mathcal{L}) = \boldsymbol{\sigma}'_\mathcal{L} = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{12} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{13} & \sigma'_{23} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \sigma'_1 & \tau'_{12} & \tau'_{13} \\ \tau'_{12} & \sigma'_2 & \tau'_{23} \\ \tau'_{13} & \tau'_{23} & \sigma'_3 \end{bmatrix} \quad (7.28)$$

The equation that defines the relationship between the above tensor and the Biot strain tensor is required. The main assumption of the linear elasticity theory is to consider that the strain field is infinitesimal (condition 7.14). Thus, the strain tensor can be approximated by the infinitesimal strain tensor that arises from the approximate polar decomposition exposed in section 5.3 as:

$$\mathbf{E}_\mathcal{L} \approx \boldsymbol{\varepsilon}_\mathcal{L} = \frac{1}{2} (\mathbf{J}_\mathcal{L} + \mathbf{J}_\mathcal{L}^T) \quad (7.29)$$

In a three-dimensional space, the displacement gradient tensor turns out to be:

$$\mathbf{J}_\mathcal{L} = \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{\partial u_1}{\partial r_{0,2}} & \frac{\partial u_1}{\partial r_{0,3}} \\ \frac{\partial u_2}{\partial r_{0,1}} & \frac{\partial u_2}{\partial r_{0,2}} & \frac{\partial u_2}{\partial r_{0,3}} \\ \frac{\partial u_3}{\partial r_{0,1}} & \frac{\partial u_3}{\partial r_{0,2}} & \frac{\partial u_3}{\partial r_{0,3}} \end{bmatrix} \quad (7.30)$$

Consequently, the Biot strain tensor approximation turns out to be:

$$\mathbf{E}_\varepsilon \approx \begin{bmatrix} \frac{\partial u_1}{\partial r_{0,1}} & \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \right) & \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,2}} + \frac{\partial u_2}{\partial r_{0,1}} \right) & \frac{\partial u_2}{\partial r_{0,2}} & \frac{1}{2} \left(\frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,1}} \right) & \frac{1}{2} \left(\frac{\partial u_2}{\partial r_{0,3}} + \frac{\partial u_3}{\partial r_{0,2}} \right) & \frac{\partial u_3}{\partial r_{0,3}} \end{bmatrix} \quad (7.31)$$

For convenience, the components of the above strain tensor are renamed according to the usual engineering nomenclature. The diagonal components are the normal strains ε , and the non-diagonal ones are the shear strains γ . Thus, the Biot strain tensor can be rewritten as:

$$\mathbf{E}_\varepsilon = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_3 \end{bmatrix} \quad (7.32)$$

The components of the above strain tensor are defined below.

Shear stress vs. shear strain

The equation that states the relation between the shear stress (τ) and the shear strain (γ) is now defined. This equation can be expressed by the Taylor series expansion about the point of null shear strain as:

$$\tau(\gamma) = \tau(0) + \frac{d\tau}{d\gamma}(0) \gamma + \mathcal{O}(|\gamma|^2) \quad (7.33)$$

If there are no residual stresses, the first term becomes null. Moreover, the last term can be neglected, if the strain field is infinitesimal. Thus, the series expansion is reduced to:

$$\left. \begin{array}{l} \tau(\gamma) = \tau(0) + \frac{d\tau}{d\gamma}(0) \gamma + \mathcal{O}(|\gamma|^2) \\ \tau(0) = 0 \\ \mathcal{O}(|\gamma|^2) \approx 0 \quad \text{if} \quad |\gamma| \ll 1 \end{array} \right\} \implies \tau(\gamma) \approx \frac{d\tau}{d\gamma}(0) \gamma \quad (7.34)$$

Therefore, the shear stress can be defined as a linear function of the shear strain.

$$\tau(\gamma) \approx G \gamma \quad \text{with} \quad G = \frac{d\tau}{d\gamma}(0) \quad (7.35)$$

Where the constant G defined in the above equation is the so-called shear modulus in the literature. As the medium is isotropic, this constant does not depend on the direction. Therefore, if the strain field is infinitesimal, the equation that defines the shear stress by means of the shear strain is linear, and the constant that defines this

linear dependence turns out to be the shear modulus.

$$\begin{aligned}\tau_{12} &= G \gamma_{12} \\ \tau_{13} &= G \gamma_{13} \\ \tau_{23} &= G \gamma_{23}\end{aligned}\tag{7.36}$$

Normal stress vs. normal strain

The second equation to define is the relation between the normal stress (σ) and the normal strain (ε). The Taylor series expansion about the point of null normal strain that defines this equation is:

$$\sigma(\varepsilon) = \sigma(0) + \frac{d\sigma}{d\varepsilon}(0) \varepsilon + \mathcal{O}(|\varepsilon|^2)\tag{7.37}$$

It is again considered that there are no residual stresses, so the first term of the series expansion is zero. Furthermore, as the strain field is infinitesimal, the last term can be neglected. Thus, the series expansion is reduced to:

$$\left. \begin{aligned}\sigma(\varepsilon) &= \sigma(0) + \frac{d\sigma}{d\varepsilon}(0) \varepsilon + \mathcal{O}(|\varepsilon|^2) \\ \sigma(0) &= 0 \\ \mathcal{O}(|\varepsilon|^2) &\approx 0 \quad \text{if} \quad |\varepsilon| \ll 1\end{aligned}\right\} \implies \sigma(\varepsilon) = \frac{d\sigma}{d\varepsilon}(0) \varepsilon\tag{7.38}$$

Hence, the normal stress can be defined as a linear function of the normal strain.

$$\sigma(\varepsilon) \approx E \varepsilon \quad \text{with} \quad E = \frac{d\sigma}{d\varepsilon}(0)\tag{7.39}$$

The relation between both variables is defined by a constant, which is the so-called Young's modulus or elastic modulus of the material. It can be concluded that, if the strain field is infinitesimal, the equation that defines the normal stress by means of the normal strain can be accurately computed as a linear function, and the constant that defines this linear dependence turns out to be the Young's modulus's of the material.

From equation (7.39), the equation that defines the normal strain by means of the normal stress turns out to be:

$$\varepsilon \approx \frac{\sigma}{E}\tag{7.40}$$

So, in a three-dimensional problem, the normal strain corresponding to the normal stress σ_1 turns out to be:

$$\varepsilon_1^{\sigma_1} = \frac{\sigma_1}{E}\tag{7.41}$$

However, the normal stress σ_1 also originates a normal strain in the other perpendicular directions (directions 2 and 3). These strains are proportional to the above normal strain, and have opposite sign. The proportionality constant is the so-called Poisson's ratio, which is represented by the symbol ν .

$$\varepsilon_2^{\sigma_1} = \varepsilon_3^{\sigma_1} = -\nu \varepsilon_1^{\sigma_1} = -\nu \frac{\sigma_1}{E}\tag{7.42}$$

The same deduction can be carried out with the remaining normal stresses (σ_2 and σ_3). The normal strain they cause along each one of the independent directions are:

$$\begin{aligned} \sigma_2 &\longrightarrow \begin{cases} \varepsilon_2 \sigma_2 = \frac{\sigma_2}{E} \\ \varepsilon_1 \sigma_2 = \varepsilon_3 \sigma_2 = -\nu \varepsilon_2 \sigma_2 = -\nu \frac{\sigma_2}{E} \end{cases} \\ \sigma_3 &\longrightarrow \begin{cases} \varepsilon_3 \sigma_3 = \frac{\sigma_3}{E} \\ \varepsilon_1 \sigma_3 = \varepsilon_2 \sigma_3 = -\nu \varepsilon_3 \sigma_3 = -\nu \frac{\sigma_3}{E} \end{cases} \end{aligned} \quad (7.43)$$

If the contributions of the three normal stresses are taken into account, the total strain corresponding to each one of the independent directions can be computed. As the strain field is infinitesimal, the effect of each normal stress can be superposed:

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1 \sigma_1 + \varepsilon_1 \sigma_2 + \varepsilon_1 \sigma_3 = \frac{1}{E} \left[\sigma_1 - \nu (\sigma_2 + \sigma_3) \right] \\ \varepsilon_2 &= \varepsilon_2 \sigma_1 + \varepsilon_2 \sigma_2 + \varepsilon_2 \sigma_3 = \frac{1}{E} \left[\sigma_2 - \nu (\sigma_1 + \sigma_3) \right] \\ \varepsilon_3 &= \varepsilon_3 \sigma_1 + \varepsilon_3 \sigma_2 + \varepsilon_3 \sigma_3 = \frac{1}{E} \left[\sigma_3 - \nu (\sigma_1 + \sigma_2) \right] \end{aligned} \quad (7.44)$$

Moreover, equation (7.44) can be rewritten by means of the trace of the stress tensor as:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E} \left[\sigma_1 (1 + \nu) - \nu \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) \right] \\ \varepsilon_2 &= \frac{1}{E} \left[\sigma_2 (1 + \nu) - \nu \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) \right] \\ \varepsilon_3 &= \frac{1}{E} \left[\sigma_3 (1 + \nu) - \nu \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) \right] \end{aligned} \quad (7.45)$$

On the other hand, the sum of the above equations leads to the trace of the strain tensor.

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \text{Tr}(\mathbf{E}_\mathcal{L}) = \frac{1 - 2\nu}{E} \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) \quad (7.46)$$

From the above equation, the trace of the stress tensor can be obtained by means of the trace of the strain tensor.

$$\text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) = \frac{E}{1 - 2\nu} \text{Tr}(\mathbf{E}_\mathcal{L}) \quad (7.47)$$

Equations (7.45) define the normal strains in terms of the normal stresses. From these equations, the ones that define the inverse relation between both variables can be obtained as:

$$\begin{aligned} \sigma_1 &= \frac{\nu}{1 + \nu} \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) + \frac{E}{1 + \nu} \varepsilon_1 \\ \sigma_2 &= \frac{\nu}{1 + \nu} \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) + \frac{E}{1 + \nu} \varepsilon_2 \\ \sigma_3 &= \frac{\nu}{1 + \nu} \text{Tr}(\boldsymbol{\sigma}_\mathcal{L}) + \frac{E}{1 + \nu} \varepsilon_3 \end{aligned} \quad (7.48)$$

And equation (7.47) can be replaced into the previous equations.

$$\begin{aligned}\sigma_1 &= \frac{\nu E}{(1+\nu)(1-2\nu)} \text{Tr}(\mathbf{E}_\mathcal{L}) + \frac{E}{1+\nu} \varepsilon_1 \\ \sigma_2 &= \frac{\nu E}{(1+\nu)(1-2\nu)} \text{Tr}(\mathbf{E}_\mathcal{L}) + \frac{E}{1+\nu} \varepsilon_2 \\ \sigma_3 &= \frac{\nu E}{(1+\nu)(1-2\nu)} \text{Tr}(\mathbf{E}_\mathcal{L}) + \frac{E}{1+\nu} \varepsilon_3\end{aligned}\quad (7.49)$$

These are the so-called Lamé's equations, which are usually written as:

$$\begin{aligned}\sigma_1 &= \lambda \text{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_1 \\ \sigma_2 &= \lambda \text{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_2 \\ \sigma_3 &= \lambda \text{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_3\end{aligned}\quad (7.50)$$

where λ and μ are the Lamé's parameters.

$$\begin{aligned}\lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)}\end{aligned}\quad (7.51)$$

It can be proved that the shear modulus G can be defined in terms of the Young's modulus E and the Poisson's ratio ν , as follows. This demonstration can be checked in [Hernández, 2000], among other reference textbooks.

$$G = \frac{E}{2(1+\nu)}\quad (7.52)$$

Consequently, the Lamé's parameter μ turns out to be equivalent to the shear modulus G .

$$\mu = G = \frac{E}{2(1+\nu)}\quad (7.53)$$

Once the equations (7.50) that define the normal stresses in terms of the normal strains are completely defined, the tensor Ψ (7.28) that is involved in the definition of the stress tensor (7.13) can be obtained by means of the Biot strain tensor. This mathematical construction leads to the so-called Lamé's equation, which is presented below.

Lamé's equation

The tensor Ψ was defined in (7.28) as:

$$\Psi(\mathbf{E}_\mathcal{L}) = \boldsymbol{\sigma}'_\mathcal{L} = \begin{bmatrix} \sigma'_1 & \tau'_{12} & \tau'_{13} \\ \tau'_{12} & \sigma'_2 & \tau'_{23} \\ \tau'_{13} & \tau'_{23} & \sigma'_3 \end{bmatrix}\quad (7.54)$$

The components of the above tensor can be defined according to the equations exposed in (7.36) and (7.50). It is also recalled that the shear modulus G is equivalent to the Lamé's parameter μ , as stated in (7.53). Thus, the components are defined as follows.

$$\begin{aligned}
 \sigma_1' &\approx \lambda \operatorname{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_1 & \tau_{12}' &\approx \mu \gamma_{12} \\
 \sigma_2' &\approx \lambda \operatorname{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_2 & \tau_{13}' &\approx \mu \gamma_{13} \\
 \sigma_3' &\approx \lambda \operatorname{Tr}(\mathbf{E}_\mathcal{L}) + 2\mu \varepsilon_3 & \tau_{23}' &\approx \mu \gamma_{23}
 \end{aligned} \tag{7.55}$$

The above equations can be gathered in tensor notation as:

$$\boldsymbol{\Psi}(\mathbf{E}_\mathcal{L}) \approx \lambda \operatorname{Tr}(\mathbf{E}_\mathcal{L}) \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}} + 2\mu \underbrace{\begin{bmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_3 \end{bmatrix}}_{\mathbf{E}_\mathcal{L}} \tag{7.56}$$

Therefore, if the strain field is infinitesimal, that is, if the Biot strain tensor fulfils condition (7.14), the tensor $\boldsymbol{\Psi}$ that defines the stress tensor (7.13) can be linearized by means of the Biot strain tensor.

$$\boldsymbol{\Psi}(\mathbf{E}_\mathcal{L}) \approx \lambda \operatorname{Tr}(\mathbf{E}_\mathcal{L}) \mathbf{I} + 2\mu \mathbf{E}_\mathcal{L} \tag{7.57}$$

Besides an infinitesimal strain field, the displacement gradients can be small. If both conditions are fulfilled, the effect of the rotation can be neglected and the definition of the stress tensor is simplified, as proven in the following section.

7.4.4. Small displacement gradients and infinitesimal strain field

As stated in (5.8), the gradients of the displacements that the solid experiments are small, if the norm of the displacement gradient tensor verifies the condition:

$$\|\mathbf{J}_\mathcal{L}\| \ll \|\mathbf{I}\| \tag{7.58}$$

In addition, the approximate polar decomposition presented in section 5.3 can be applied to decompose the deformation gradient tensor

$$\mathbf{F}_\mathcal{L} \approx \mathcal{R}_\mathcal{L} [\mathbf{I} + \boldsymbol{\mathcal{E}}_\mathcal{L}] , \tag{7.59}$$

where $\mathcal{R}_\mathcal{L}$ is the infinitesimal rotation tensor, and $\boldsymbol{\mathcal{E}}_\mathcal{L}$ is the infinitesimal strain tensor. Both of them are defined by means of the displacement gradient tensor.

$$\begin{aligned}
 \mathcal{R}_\mathcal{L} &= \mathbf{I} + \underbrace{\frac{1}{2}(\mathbf{J}_\mathcal{L} - \mathbf{J}_\mathcal{L}^T)}_{\boldsymbol{\mathcal{W}}_\mathcal{L}} \\
 \boldsymbol{\mathcal{E}}_\mathcal{L} &= \frac{1}{2}(\mathbf{J}_\mathcal{L} + \mathbf{J}_\mathcal{L}^T)
 \end{aligned} \tag{7.60}$$

Since the displacement gradient tensor verifies condition (7.58), the norm of the above tensors also verify that:

$$\begin{aligned}\|\mathcal{W}_\varepsilon\| &= \mathcal{O}(\|\mathbf{J}_\varepsilon\|) \ll \|\mathbf{I}\| \\ \|\boldsymbol{\varepsilon}_\varepsilon\| &= \mathcal{O}(\|\mathbf{J}_\varepsilon\|) \ll \|\mathbf{I}\|\end{aligned}\tag{7.61}$$

If the displacement gradients are small, the tensors involved in the definition of the stress tensor exposed in (7.13) can be accurately substituted by the infinitesimal rotation tensor and the infinitesimal strain tensor.

$$\begin{aligned}\boldsymbol{\sigma}_\varepsilon &= \mathbf{R}_\varepsilon \boldsymbol{\Psi}(\mathbf{E}_\varepsilon) \mathbf{R}_\varepsilon^T \\ &\approx \mathbf{R}_\varepsilon \boldsymbol{\Psi}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{R}_\varepsilon^T\end{aligned}\tag{7.62}$$

Furthermore, as the strain field is infinitesimal (proved in (7.61)), the main linear elasticity assumption (7.14) is fulfilled. This implies that, if the medium is isotropic, the tensor $\boldsymbol{\Psi}$ can be approximated by the Lamé's equation (7.57).

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}_\varepsilon) \approx \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}_\varepsilon\tag{7.63}$$

The above tensor and the infinitesimal rotation tensor exposed in (7.60) can now be substituted into the definition of the stress tensor (7.62):

$$\begin{aligned}\boldsymbol{\sigma}_\varepsilon &\approx (\mathbf{I} + \mathcal{W}_\varepsilon) (\lambda \operatorname{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}_\varepsilon) (\mathbf{I} - \mathcal{W}_\varepsilon) \\ &= (\lambda \operatorname{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}_\varepsilon) + \mathcal{O}(\|\mathbf{J}_\varepsilon\|^2)\end{aligned}\tag{7.64}$$

Since the displacement gradients are small, the second term can be neglected, and the above equation is reduced to:

$$\boldsymbol{\sigma}_\varepsilon \approx \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}_\varepsilon\tag{7.65}$$

Therefore, the effect of the infinitesimal rotation can be neglected, and the stress tensor turns out to be defined by means of the infinitesimal strain tensor. This is the so-called linear elastic constitutive equation.

In the following section, an equivalent definition of the above constitutive equation is derived, where the relation between the Cauchy stress tensor and the infinitesimal strain tensor is stated by means of the constitutive tensor.

7.5. Cauchy stress tensor vs. infinitesimal strain tensor

From the result obtained in (7.65), it can be concluded that if the strain field is infinitesimal (condition 7.14), and the displacement gradients are small (condition 7.58), the Cauchy stress tensor can be defined by means of the infinitesimal strain tensor.

$$\left. \begin{aligned}\|\mathbf{E}_\varepsilon\| &\ll \|\mathbf{I}\| \\ \|\mathbf{J}_\varepsilon\| &\ll \|\mathbf{I}\|\end{aligned} \right\} \implies \boldsymbol{\sigma}_\varepsilon = \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon)\tag{7.66}$$

7.5.1. Tensor constitutive equation

The above tensor function can be equivalently defined according to the Taylor series expansion about the point of null infinitesimal strain, as:

$$\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon) = \boldsymbol{\sigma}_\varepsilon(\mathbf{0}) + \frac{d\boldsymbol{\sigma}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon}(\mathbf{0}) : \boldsymbol{\varepsilon}_\varepsilon + \mathcal{O}(\|\boldsymbol{\varepsilon}_\varepsilon\|^2) \quad (7.67)$$

The tensor operation represented by the symbol $:$ turns out to be the double dot product between a fourth order tensor and the infinitesimal strain tensor. The first term of the above series is null, if there are no residual stresses, and the last one can be neglected, as the strain field is infinitesimal. So, the series expansion is reduced to:

$$\left. \begin{aligned} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon) &= \boldsymbol{\sigma}_\varepsilon(\mathbf{0}) + \frac{d\boldsymbol{\sigma}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon}(\mathbf{0}) : \boldsymbol{\varepsilon}_\varepsilon + \mathcal{O}(\|\boldsymbol{\varepsilon}_\varepsilon\|^2) \\ \boldsymbol{\sigma}_\varepsilon(\mathbf{0}) &= \mathbf{0} \\ \|\boldsymbol{\varepsilon}_\varepsilon\| \ll \|\mathbf{I}\| &\implies \mathcal{O}(\|\boldsymbol{\varepsilon}_\varepsilon\|^2) \approx \mathbf{0} \end{aligned} \right\} \implies \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon) \approx \frac{d\boldsymbol{\sigma}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon}(\mathbf{0}) : \boldsymbol{\varepsilon}_\varepsilon \quad (7.68)$$

Therefore, if the assumptions presented in this section are fulfilled, the constitutive equation can be defined as:

$$\boldsymbol{\sigma}_\varepsilon = \mathbf{D}_4(\mathbf{0}) : \boldsymbol{\varepsilon}_\varepsilon \quad \text{with} \quad \mathbf{D}_4(\mathbf{0}) = \frac{d\boldsymbol{\sigma}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon}(\mathbf{0}), \quad (7.69)$$

where \mathbf{D}_4 is a fourth order tensor and is the so-called constitutive tensor. Its components are defined as follows.

$$\mathbf{D}_4(\mathbf{0}) = [D_{ijkl}(\mathbf{0})]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \quad D_{ijkl}(\mathbf{0}) = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}(\mathbf{0}) \quad (7.70)$$

7.5.2. Fourth order constitutive tensor

According to equation (7.69), the fourth order constitutive tensor turns out to be the derivative of the Cauchy stress tensor with respect to the infinitesimal strain tensor.

The definition of the Cauchy stress tensor, if the displacement gradients are small and the strain field is infinitesimal, was obtained in (7.65). Its derivation with respect to the infinitesimal strain tensor leads to an equivalent definition of the constitutive tensor.

$$\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon) = \lambda \text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I}_2 + 2\mu \boldsymbol{\varepsilon}_\varepsilon \implies \frac{d\boldsymbol{\sigma}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon} = \lambda \frac{d}{d\boldsymbol{\varepsilon}_\varepsilon} \left(\text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I}_2 \right) + 2\mu \frac{d\boldsymbol{\varepsilon}_\varepsilon}{d\boldsymbol{\varepsilon}_\varepsilon} \quad (7.71)$$

The tensor whose derivative is computed in the first term is defined below.

$$\text{Tr}(\boldsymbol{\varepsilon}_\varepsilon) \mathbf{I}_2 = [A_{ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \quad A_{ij} = \delta_{ij} \sum_{m=1}^3 \varepsilon_{mm} \quad (7.72)$$

In this section, the subscript $_2$ is added to the second order unit tensor to clarify its order. Later on in this chapter, the fourth order unit tensor appears, so the order of the unit tensors are indicated to avoid confusion. To compute the derivative of the above tensor, with respect to the infinitesimal strain tensor, the definition of the tensor $\mathbf{I}_2 \otimes \mathbf{I}_2$ exposed in (A.92) is recalled.

$$\left. \begin{aligned} \frac{d}{d\boldsymbol{\mathcal{E}}_\mathcal{L}} \left(\text{Tr}(\boldsymbol{\mathcal{E}}_\mathcal{L}) \mathbf{I}_2 \right) &= [B_{ijkl}]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \\ B_{ijkl} &= \frac{\partial A_{ij}}{\partial \mathcal{E}_{kl}} = \frac{\partial}{\partial \mathcal{E}_{kl}} \left(\delta_{ij} \sum_{m=1}^3 \mathcal{E}_{mm} \right) \\ &= \delta_{ij} \frac{\partial}{\partial \mathcal{E}_{kl}} \left(\sum_{m=1}^3 \mathcal{E}_{mm} \right) \\ &= \delta_{ij} \delta_{kl} \end{aligned} \right\} \implies \frac{d}{d\boldsymbol{\mathcal{E}}_\mathcal{L}} \left(\text{Tr}(\boldsymbol{\mathcal{E}}_\mathcal{L}) \mathbf{I}_2 \right) = \mathbf{I}_2 \otimes \mathbf{I}_2 \quad (7.73)$$

On the other hand, the derivative involved in the second term turns out to be equivalent to the fourth order unit tensor \mathbf{I}_4 , defined in (A.88).

$$\left. \begin{aligned} \frac{d\boldsymbol{\mathcal{E}}_\mathcal{L}}{d\boldsymbol{\mathcal{E}}_\mathcal{L}} &= [C_{ijkl}]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \\ C_{ijkl} &= \frac{\partial E_{ij}}{\partial E_{kl}} = \delta_{ik} \delta_{jl} \end{aligned} \right\} \implies \frac{d\boldsymbol{\mathcal{E}}_\mathcal{L}}{d\boldsymbol{\mathcal{E}}_\mathcal{L}} = \mathbf{I}_4 \quad (7.74)$$

So, the substitution of (7.73) and (7.74) into the derivative of the Cauchy stress tensor (7.71) leads to:

$$\mathbf{D}_4 = \frac{d\boldsymbol{\sigma}_\mathcal{L}}{d\boldsymbol{\mathcal{E}}_\mathcal{L}} = \lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4 \quad (7.75)$$

Since the above tensor is constant, it can be concluded that the fourth order tensor (7.69) turns out to be:

$$\mathbf{D}_4(\mathbf{0}) = \lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4 \quad (7.76)$$

Therefore, the constitutive equation presented in (7.69) becomes:

$$\boldsymbol{\sigma}_\mathcal{L} = \left[\lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4 \right] : \boldsymbol{\mathcal{E}}_\mathcal{L} \quad (7.77)$$

Furthermore, since the infinitesimal strain tensor is symmetric, the fourth order unit tensor $\bar{\mathbf{I}}_4$ defined in (A.88) leads to the same result as \mathbf{I}_4 when it is applied to the infinitesimal strain tensor.

$$\begin{aligned} \mathbf{I}_4 : \boldsymbol{\mathcal{E}}_\mathcal{L} &= \boldsymbol{\mathcal{E}}_\mathcal{L} \\ \bar{\mathbf{I}}_4 : \boldsymbol{\mathcal{E}}_\mathcal{L} &= \boldsymbol{\mathcal{E}}_\mathcal{L}^T = \boldsymbol{\mathcal{E}}_\mathcal{L} \end{aligned} \quad (7.78)$$

This implies that the fourth order unit tensor \mathbf{I}_4 can be replaced by its symmetric term $\mathbf{I}_4^{\text{sym}}$, defined in (A.94).

$$\begin{aligned}
 \mathbf{I}_4 : \boldsymbol{\varepsilon}_\varepsilon &= \boldsymbol{\varepsilon}_\varepsilon = \frac{1}{2} (\boldsymbol{\varepsilon}_\varepsilon + \boldsymbol{\varepsilon}_\varepsilon) \\
 &= \frac{1}{2} (\mathbf{I}_4 : \boldsymbol{\varepsilon}_\varepsilon + \bar{\mathbf{I}}_4 : \boldsymbol{\varepsilon}_\varepsilon) \\
 &= \frac{1}{2} (\mathbf{I}_4 + \bar{\mathbf{I}}_4) : \boldsymbol{\varepsilon}_\varepsilon \\
 &= \mathbf{I}_4^{\text{sym}} : \boldsymbol{\varepsilon}_\varepsilon
 \end{aligned} \tag{7.79}$$

Thus, if the above result is taken into account, the Cauchy stress tensor (7.77) becomes:

$$\boldsymbol{\sigma}_\varepsilon = \left[\lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4^{\text{sym}} \right] : \boldsymbol{\varepsilon}_\varepsilon \tag{7.80}$$

And the constitutive equation can be finally defined as:

$$\boldsymbol{\sigma}_\varepsilon = \mathbf{D}_4(\mathbf{0}) : \boldsymbol{\varepsilon}_\varepsilon \quad \text{with} \quad \mathbf{D}_4(\mathbf{0}) = \lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4^{\text{sym}}, \tag{7.81}$$

where $\mathbf{D}_4(\mathbf{0})$ is the constitutive tensor. The components of the tensors $\mathbf{I}_2 \otimes \mathbf{I}_2$ and $\mathbf{I}_4^{\text{sym}}$ were defined in (A.92) and (A.95), respectively. If these definitions are taken into account, the components of the above constitutive tensor can be finally defined as:

$$\mathbf{D}_4(\mathbf{0}) = \left[D_{ijkl}(\mathbf{0}) \right]_{\substack{i=1,2,3 \\ j=1,2,3 \\ i=1,3,3 \\ j=1,3,3}} \quad D_{ijkl}(\mathbf{0}) = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{7.82}$$

7.5.3. Vector constitutive equation

The Voigt notation exposed in section A.14 allows to propose the equivalent vector equation that defines the constitutive equation presented in the previous section.

$$\boldsymbol{\sigma}_\varepsilon = \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varepsilon}_\varepsilon) \quad \implies \quad \bar{\boldsymbol{\sigma}}_\varepsilon = \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\boldsymbol{\varepsilon}}_\varepsilon) \tag{7.83}$$

And the Taylor series expansion of the above vector function, defined about the point of null infinitesimal strain, turns out to be:

$$\bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\boldsymbol{\varepsilon}}_\varepsilon) = \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\mathbf{0}}) + \frac{d\bar{\boldsymbol{\sigma}}_\varepsilon}{d\bar{\boldsymbol{\varepsilon}}_\varepsilon}(\bar{\mathbf{0}}) \bar{\boldsymbol{\varepsilon}}_\varepsilon + \mathcal{O}(\|\bar{\boldsymbol{\varepsilon}}_\varepsilon\|^2) \tag{7.84}$$

The above series can be simplified, since the first term is null if there are no residual stresses, and the last one can be neglected as the strain field is infinitesimal. Thus, the series expansion is reduced to:

$$\left. \begin{aligned}
 \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\boldsymbol{\varepsilon}}_\varepsilon) &= \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\mathbf{0}}) + \frac{d\bar{\boldsymbol{\sigma}}_\varepsilon}{d\bar{\boldsymbol{\varepsilon}}_\varepsilon}(\bar{\mathbf{0}}) \bar{\boldsymbol{\varepsilon}}_\varepsilon + \mathcal{O}(\|\bar{\boldsymbol{\varepsilon}}_\varepsilon\|^2) \\
 \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\mathbf{0}}) &= \bar{\mathbf{0}} \\
 \|\bar{\boldsymbol{\varepsilon}}_\varepsilon\| \ll 1 &\implies \mathcal{O}(\|\bar{\boldsymbol{\varepsilon}}_\varepsilon\|^2) \approx \bar{\mathbf{0}}
 \end{aligned} \right\} \implies \bar{\boldsymbol{\sigma}}_\varepsilon(\bar{\boldsymbol{\varepsilon}}_\varepsilon) \approx \frac{d\bar{\boldsymbol{\sigma}}_\varepsilon}{d\bar{\boldsymbol{\varepsilon}}_\varepsilon}(\bar{\mathbf{0}}) \bar{\boldsymbol{\varepsilon}}_\varepsilon \tag{7.85}$$

Therefore, if the displacement gradients are small, and the infinitesimal strain is infinitesimal, the constitutive equation can be defined as a linear function of the infinitesimal strain field as:

$$\bar{\boldsymbol{\sigma}}_{\mathcal{L}} = \mathbf{D}_2(\bar{\mathbf{0}}) \bar{\boldsymbol{\mathcal{E}}}_{\mathcal{L}} \quad \text{with} \quad \mathbf{D}_2(\bar{\mathbf{0}}) = \frac{d\bar{\boldsymbol{\sigma}}_{\mathcal{L}}}{d\bar{\boldsymbol{\mathcal{E}}}_{\mathcal{L}}}(\bar{\mathbf{0}}), \quad (7.86)$$

where \mathbf{D}_2 is the constitutive tensor. This tensor turns out to be a second order tensor, if the vector notation is adopted instead of the tensor one. And its components turn out to be:

$$\mathbf{D}_2(\bar{\mathbf{0}}) = [D_{ij}(\bar{\mathbf{0}})]_{\substack{i=1,\dots,6 \\ j=1,\dots,6}} \quad D_{ij}(\bar{\mathbf{0}}) = \frac{\partial \sigma_{ij}}{\partial \mathcal{E}_{kl}}(\bar{\mathbf{0}}) \quad (7.87)$$

7.5.4. Second order constitutive tensor

From the stress tensor defined in (7.65), the components of the second order constitutive tensor can be obtained.

The Voigt notation presented in section A.14 is the one that defines the equivalent vector expression of the Cauchy stress tensor.

$$\boldsymbol{\sigma}_{\mathcal{L}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \implies \bar{\boldsymbol{\sigma}}_{\mathcal{L}} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} \quad (7.88)$$

This notation can also be applied to the strain tensor, to obtain its equivalent vector definition.

$$\boldsymbol{\mathcal{E}}_{\mathcal{L}} = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} \\ \mathcal{E}_{12} & \mathcal{E}_{22} & \mathcal{E}_{23} \\ \mathcal{E}_{13} & \mathcal{E}_{23} & \mathcal{E}_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_3 \end{bmatrix} \implies \bar{\boldsymbol{\mathcal{E}}}_{\mathcal{L}} = \begin{Bmatrix} \mathcal{E}_{11} \\ \mathcal{E}_{22} \\ \mathcal{E}_{33} \\ 2\mathcal{E}_{12} \\ 2\mathcal{E}_{13} \\ 2\mathcal{E}_{23} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \quad (7.89)$$

According to the result obtained in (7.55), the components of the vector expression of the stress tensor are defined by means of the Lamé's parameters, as shown below.

$$\begin{aligned} \sigma_1 &= \lambda \text{Tr}(\boldsymbol{\mathcal{E}}_{\mathcal{L}}) + 2\mu \varepsilon_1 & \tau_{12} &= \mu \gamma_{12} \\ \sigma_2 &= \lambda \text{Tr}(\boldsymbol{\mathcal{E}}_{\mathcal{L}}) + 2\mu \varepsilon_2 & \tau_{13} &= \mu \gamma_{13} \\ \sigma_3 &= \lambda \text{Tr}(\boldsymbol{\mathcal{E}}_{\mathcal{L}}) + 2\mu \varepsilon_3 & \tau_{23} &= \mu \gamma_{23} \end{aligned} \quad (7.90)$$

Therefore, if the above equations are gathered, the vector expression of the Cauchy stress tensor can be defined as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \quad (7.91)$$

The above equation turns out to be equivalent to the one obtained in (7.86). Thus, the second order tensor that defines the previous linear relation between the vector expressions of the stress tensor and the strain tensor is the constitutive tensor $\mathbf{D}_2(\bar{\mathbf{0}})$:

$$\mathbf{D}_2(\bar{\mathbf{0}}) = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \quad (7.92)$$

The substitution of the Lamé's parameters (7.51) into the above tensor, allows to rewrite its components in terms of the Young's modulus and the Poisson's ratio.

$$\begin{aligned} 2\mu + \lambda &= \frac{E}{1 + \nu} + \frac{E\nu}{(1 - 2\nu)(1 + \nu)} = \frac{E(1 - \nu)}{(1 - 2\nu)(1 + \nu)} \\ \mu &= \frac{E}{2(1 + \nu)} = \frac{E(1 - 2\nu)}{2(1 + \nu)(1 - 2\nu)} = \frac{E(0.5 - \nu)}{(1 + \nu)(1 - 2\nu)} \\ \lambda &= \frac{E\nu}{(1 - 2\nu)(1 + \nu)} \end{aligned} \quad (7.93)$$

Consequently, the constitutive tensor is reduced to:

$$\mathbf{D}_2(\bar{\mathbf{0}}) = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 - \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 - \nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 - \nu \end{bmatrix} \quad (7.94)$$

To sum up, when dealing with an infinitesimal strain field, and the displacement gradients are small, the constitutive equation can be directly defined by means of the infinitesimal strain tensor. And a linear relation between the Cauchy stress tensor and the infinitesimal strain tensor can be stated by means of the constitutive tensor.

If these assumptions are not fulfilled, this constitutive equation does not hold any more, and the Cauchy stress tensor has to be defined by means of the rotation tensor and the Biot strain tensor, as stated in (7.13). Nevertheless, more suitable tensor magnitudes can be adopted to define the strain and stress fields, if the solid behaviour does not fulfil these assumptions. This topic is extensively discussed in the following section.

7.6. Second Piola-Kirchhoff stress tensor vs. Green-Lagrange strain tensor

On the one hand, the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor remain invariant if a rotation is applied to a solid subjected to external loads. This is demonstrated in sections 3.5.1 and 4.6.4, respectively.

On the other hand, both tensors depend on the deformation gradient tensor, which does not vary if a translation is applied to the solid. Therefore, it can be concluded that both tensors also remain constant if a translation is applied to the solid.

Consequently, both the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor remain invariant if a rigid motion (rotation and/or translation) is applied to a solid. This is an important property to take into account when a large displacement analysis is carried out. If the solid experiments large displacements, these tensors can be applied to properly describe the strain field and its corresponding stress field.

Therefore, a constitutive equation that defines the mathematical relation between them can be defined.

$$\mathbf{S}_\mathcal{L} = \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}) \quad (7.95)$$

7.6.1. Incremental tensor constitutive equation

In this section, the equation that defines the second Piola-Kirchhoff stress tensor increment by means of the Green-Lagrange strain tensor increment is derived. The increment of the Green-Lagrange strain tensor is originated by an increment of the displacement field. This equation is essential to develop a nonlinear finite element analysis, since it will be carried out by an incremental loading procedure.

In figure (7.3), the relation between the second Piola-Kirchhoff stress and the Green-Lagrange strain, corresponding to a simple one-dimensional case, is represented.

The second Piola-Kirchhoff stress tensor increment, corresponding to a given displacement field increment $\Delta \mathbf{u}_\mathcal{L}$, can be defined as the difference between the second Piola-Kirchhoff stress tensor corresponding to each one of the displacement fields:

$$\Delta \mathbf{S}_\mathcal{L} = \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}(\mathbf{u}_\mathcal{L} + \Delta \mathbf{u}_\mathcal{L})) - \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}(\mathbf{u}_\mathcal{L})) \quad (7.96)$$

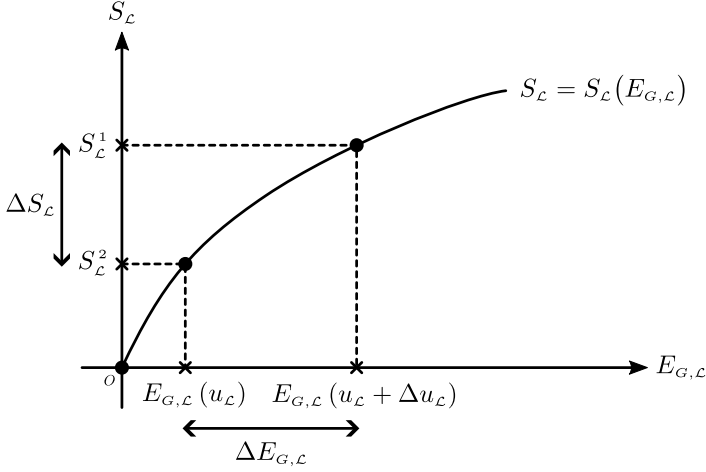


Figure 7.3. Second Piola-Kirchhoff stress increment vs. Green-Lagrange strain increment (one-dimensional case).

The first term of the above difference can be expressed according to the Taylor series expansion as:

$$\mathbf{S}_L(\mathbf{E}_{G,L}(\mathbf{u}_L + \Delta \mathbf{u}_L)) = \mathbf{S}_L(\mathbf{E}_{G,L}(\mathbf{u}_L)) + \frac{d\mathbf{S}_L}{d\mathbf{E}_{G,L}}(\mathbf{E}_{G,L}(\mathbf{u}_L)) : \Delta \mathbf{E}_{G,L} + \mathcal{O}(\|\Delta \mathbf{E}_{G,L}\|^2) \quad (7.97)$$

The substitution of the above series expansion into the definition of the second Piola-Kirchhoff stress tensor increment (7.96) leads to:

$$\Delta \mathbf{S}_L = \frac{d\mathbf{S}_L}{d\mathbf{E}_{G,L}}(\mathbf{E}_{G,L}(\mathbf{u}_L)) : \Delta \mathbf{E}_{G,L} + \mathcal{O}(\|\Delta \mathbf{E}_{G,L}\|^2) \quad (7.98)$$

And if the Green-Lagrange strain increments are considered to be small, it can be concluded that:

$$\Delta \mathbf{S}_L \approx \frac{d\mathbf{S}_L}{d\mathbf{E}_{G,L}}(\mathbf{E}_{G,L}(\mathbf{u}_L)) : \Delta \mathbf{E}_{G,L} \quad (7.99)$$

The Green-Lagrange strain tensor increment was defined in (4.92), and it is the effect of considering that the displacement field is modified by a displacement field increment. The gradients of the displacement field increment are assumed to be small, so the assumption made in (7.99) is fulfilled, and the Green-Lagrange strain increments

are small.

$$\left. \begin{aligned} \mathbf{u}'_{\mathcal{L}} &= \mathbf{u}_{\mathcal{L}} + \Delta \mathbf{u}_{\mathcal{L}} \\ \left\| \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} \right\| &\ll \| \mathbf{I} \| \end{aligned} \right\} \Rightarrow \mathbf{J}'_{\mathcal{L}} = \frac{\partial \mathbf{u}'_{\mathcal{L}}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} + \frac{\partial \Delta \mathbf{u}_{\mathcal{L}}}{\partial \mathbf{r}_0} = \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}} \\ \Rightarrow \| \Delta \mathbf{J}_{\mathcal{L}} \| &\ll \| \mathbf{I} \| \\ \Rightarrow \Delta \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} \left(\Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T + \Delta \mathbf{J}_{\mathcal{L}}^T \mathbf{J}_{\mathcal{L}} + \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) \\ \Rightarrow \| \Delta \mathbf{E}_{G,\mathcal{L}} \| &\ll \| \mathbf{I} \| \end{aligned} \tag{7.100}$$

It can be concluded that, under the above assumption, the second Piola-Kirchhoff stress tensor increment defined by means of the Green-Lagrange strain tensor increment turns out to be:

$$\Delta \mathbf{S}_{\mathcal{L}} \approx \mathbf{C}_4(\mathbf{E}_{G,\mathcal{L}}) : \Delta \mathbf{E}_{G,\mathcal{L}} \quad \text{with} \quad \mathbf{C}_4(\mathbf{E}_{G,\mathcal{L}}) = \frac{d\mathbf{S}_{\mathcal{L}}}{d\mathbf{E}_{G,\mathcal{L}}}(\mathbf{E}_{G,\mathcal{L}}), \tag{7.101}$$

where \mathbf{C}_4 is a fourth order tensor, and is known as the constitutive tensor. It is composed by 81 components, which are defined as shown below.

$$\mathbf{C}_4 = [C_{ijkl}]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \quad C_{ijkl} = \frac{\partial S_{ij}}{\partial E_{kl}} \tag{7.102}$$

7.6.2. Fourth order constitutive tensor

It is usually established that two tensor magnitudes are conjugate if their double dot product gives the work per unit volume developed by the internal forces during the deformation process.

It can be proven that the first Piola-Kirchhoff stress tensor is work conjugate with the deformation gradient tensor, and the second Piola-Kirchhoff stress tensor turns out to be work conjugate with the Green-Lagrange strain tensor [De Borst et al., 2012; Capaldi, 2012; Bonet et al., 2016].

Work conjugacy	
Stress tensor	Strain tensor
• First P-K stress tensor ($\mathbf{P}_{\mathcal{L}}$)	• Deformation gradient tensor ($\mathbf{F}_{\mathcal{L}}$)
• Second P-K stress tensor ($\mathbf{S}_{\mathcal{L}}$)	• G-L strain tensor ($\mathbf{E}_{G,\mathcal{L}}$)

Table 7.1. Conjugate tensor magnitudes.

As the first Piola-Kirchhoff stress tensor and the deformation gradient tensor are work conjugate magnitudes (table 7.1), it can be stated that the work per unit volume developed by the internal stresses during a differential part of the deformation process

is:

$$d\Psi = \mathbf{P}_\mathcal{L} : d\mathbf{F}_\mathcal{L} \quad (7.103)$$

If the above magnitude is integrated over the whole deformation process, the work per unit volume developed by the internal forces during the entire deformation process is obtained.

$$\Psi(\mathbf{F}_\mathcal{L}) = \int_{\mathbf{0}}^{\mathbf{F}_\mathcal{L}} d\Psi = \int_{\mathbf{0}}^{\mathbf{F}_\mathcal{L}} \mathbf{P}_\mathcal{L} : d\mathbf{F}_\mathcal{L} \quad (7.104)$$

According to the above result, it can be concluded that the work per unit volume depends on the deformation gradient tensor. If its polar decomposition is taken into account (section 4.3), the deformation gradient tensor can be decomposed as the product of the finite rotation tensor and the finite strain tensor. The finite strain tensor is the tensor that modifies the modulus and direction of a given material vector. And the finite rotation tensor is the one that rotates the previous material vector.

$$\begin{aligned} \delta\mathbf{r} &= \mathbf{F}_\mathcal{L} \delta\mathbf{r}_0 \\ &= \mathbf{R}_\mathcal{L} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \delta\mathbf{r}_0 \end{aligned} \quad (7.105)$$

The material vector rotation does not produce work per unit volume, so, the finite strain tensor is the one that produces it. Therefore, the work per unit volume that the internal stresses develop over the deformation process only depends on the finite strain tensor.

$$\Psi = \Psi(\mathbf{I} + \mathbf{E}_\mathcal{L}) \quad (7.106)$$

If the polar decomposition of the deformation gradient tensor is substituted into the definition of the right Cauchy-Green tensor (4.18), the orthogonality of the finite rotation tensor (4.13) allows to get rid of the effect of the rotation.

$$\begin{aligned} \mathbf{M}_\mathcal{L} &= \mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} \\ &= [\mathbf{I} + \mathbf{E}_\mathcal{L}]^T \underbrace{\mathbf{R}_\mathcal{L}^T \mathbf{R}_\mathcal{L}}_{\mathbf{I}} [\mathbf{I} + \mathbf{E}_\mathcal{L}] \\ &= [\mathbf{I} + \mathbf{E}_\mathcal{L}]^2 \end{aligned} \quad (7.107)$$

Hence, the finite strain tensor can be defined as the square root of the right Cauchy-Green tensor.

$$\mathbf{I} + \mathbf{E}_\mathcal{L} = \mathbf{M}_\mathcal{L}^{1/2} \quad (7.108)$$

On the other hand, the Green-Lagrange strain tensor (4.72) can be expressed in terms of the right Cauchy-Green tensor as:

$$\left. \begin{aligned} \mathbf{E}_{G,\mathcal{L}} &= \frac{1}{2} (\mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} - \mathbf{I}) \\ \mathbf{M}_\mathcal{L} &= \mathbf{F}_\mathcal{L}^T \mathbf{F}_\mathcal{L} \end{aligned} \right\} \implies \mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} (\mathbf{M}_\mathcal{L} - \mathbf{I}) \quad (7.109)$$

From the above equation, the right Cauchy-Green tensor can be obtained in terms of the Green-Lagrange strain tensor according to:

$$\mathbf{M}_\mathcal{L} = 2\mathbf{E}_{G,\mathcal{L}} + \mathbf{I} \quad (7.110)$$

Therefore, the finite strain tensor defined in (7.108) finally becomes:

$$\left. \begin{aligned} \mathbf{I} + \mathbf{E}_{\mathcal{L}} &= \mathbf{M}_{\mathcal{L}}^{1/2} \\ \mathbf{M}_{\mathcal{L}} &= 2 \mathbf{E}_{G,\mathcal{L}} + \mathbf{I} \end{aligned} \right\} \implies \mathbf{I} + \mathbf{E}_{\mathcal{L}} = \left(2 \mathbf{E}_{G,\mathcal{L}} + \mathbf{I} \right)^{1/2} \quad (7.111)$$

In the above equation, the finite strain tensor is defined by means of the Green-Lagrange strain tensor. In (7.106) it has been stated that the work per unit volume developed by the internal forces depends on the finite strain tensor. According to that, it can be concluded that the work per unit volume finally depends on the Green-Lagrange strain tensor.

$$\Psi = \Psi(\mathbf{E}_{G,\mathcal{L}}) \implies d\Psi = \frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} : d\mathbf{E}_{G,\mathcal{L}} \quad (7.112)$$

Another pair of conjugate magnitudes are the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor (table 7.1). Therefore, it can be also stated that:

$$d\Psi = \mathbf{S}_{\mathcal{L}} : d\mathbf{E}_{G,\mathcal{L}} \quad (7.113)$$

If equations (7.112) and (7.113) are compared, an alternative definition of the second Piola-Kirchhoff stress tensor is obtained as:

$$\left. \begin{aligned} d\Psi &= \mathbf{S}_{\mathcal{L}} : d\mathbf{E}_{G,\mathcal{L}} \\ d\Psi &= \frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} : d\mathbf{E}_{G,\mathcal{L}} \end{aligned} \right\} \implies \mathbf{S}_{\mathcal{L}} = \frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} \quad (7.114)$$

The substitution of the above second Piola-Kirchhoff stress tensor definition into the fourth order constitutive tensor obtained in (7.101), leads to:

$$\left. \begin{aligned} \mathbf{C}_4 &= \frac{d\mathbf{S}_{\mathcal{L}}}{d\mathbf{E}_{G,\mathcal{L}}} \\ \mathbf{S}_{\mathcal{L}} &= \frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} \end{aligned} \right\} \implies \mathbf{C}_4 = \frac{d\mathbf{S}_{\mathcal{L}}}{d\mathbf{E}_{G,\mathcal{L}}} = \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} \right) \quad (7.115)$$

Consequently, it can be stated that the constitutive tensor turns out to be the second derivative of the internal work per unit volume, with respect to the Green-Lagrange strain tensor.

The first derivative of the internal work per unit volume is the following second order tensor:

$$\frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} = \begin{bmatrix} \frac{\partial \Psi}{\partial E_{11}} & \frac{\partial \Psi}{\partial E_{12}} & \frac{\partial \Psi}{\partial E_{13}} \\ \frac{\partial \Psi}{\partial E_{12}} & \frac{\partial \Psi}{\partial E_{22}} & \frac{\partial \Psi}{\partial E_{23}} \\ \frac{\partial \Psi}{\partial E_{13}} & \frac{\partial \Psi}{\partial E_{23}} & \frac{\partial \Psi}{\partial E_{33}} \end{bmatrix} = \left[\frac{\partial \Psi}{\partial E_{kl}} \right]_{\substack{k=1,2,3 \\ l=1,2,3}} \quad (7.116)$$

And its second derivative, that defines the constitutive tensor (7.115), is the following fourth order tensor.

$$\mathbf{C}_4 = \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} \right) = [C_{ijkl}]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \quad C_{ijkl} = \frac{\partial}{\partial E_{ij}} \left(\frac{\partial \Psi}{\partial E_{kl}} \right) \quad (7.117)$$

The fourth order constitutive tensor symmetries are analysed in the following subsection. Its symmetries allow to reduce the number of different components, so the definition of the constitutive tensor is simplified.

7.6.3. Fourth order constitutive tensor symmetries

The components of the constitutive tensor are defined in (7.117). If this definition is taken into account, it can be proven that the constitutive tensor has major symmetry.

$$C_{ijkl} = \frac{\partial}{\partial E_{ij}} \left(\frac{\partial \Psi}{\partial E_{jk}} \right) = \frac{\partial}{\partial E_{kl}} \left(\frac{\partial \Psi}{\partial E_{ij}} \right) = C_{klij} \quad (7.118)$$

This constitutive tensor defines the relation between the second Piola-Kirchhoff stress tensor increment and the Green-Lagrange strain tensor increment.

$$\Delta \mathbf{S}_{\mathcal{L}} = \mathbf{C}_4 : \Delta \mathbf{E}_{G,\mathcal{L}} \quad \Longleftrightarrow \quad \Delta S_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \Delta E_{kl} \quad (7.119)$$

Both the Green-Lagrange strain tensor increment and the second Piola-Kirchhoff stress tensor increment are symmetric. Thus, their components verify the following conditions:

$$\begin{aligned} (\Delta \mathbf{E}_{G,\mathcal{L}})^T = \Delta \mathbf{E}_{G,\mathcal{L}} &\Longleftrightarrow \Delta E_{ji} = \Delta E_{ij} \\ \Delta \mathbf{S}_{\mathcal{L}}^T = \Delta \mathbf{S}_{\mathcal{L}} &\Longleftrightarrow \Delta S_{ji} = \Delta S_{ij} \end{aligned} \quad (7.120)$$

The above conditions allow to prove that the constitutive tensor has minor symmetries too.

$$\left. \begin{aligned} \Delta S_{ji} &= \sum_{k=1}^3 \sum_{l=1}^3 C_{jikl} \Delta E_{kl} \\ \Delta S_{ij} &= \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \Delta E_{kl} \\ \Delta S_{ij} &= \sum_{k=1}^3 \sum_{l=1}^3 C_{ijlk} \Delta E_{lk} \end{aligned} \right\} \Longrightarrow C_{ijkl} = C_{ijlk} = C_{jikl} \quad (7.121)$$

The fourth order constitutive tensor symmetries turn out to be a major advantage, since the number of different tensor components is significantly reduced.

In the following section, the incremental constitutive equation is defined according to its equivalent vector notation. In this particular case, the constitutive tensor becomes a second order one, and the symmetries proven in this section will be recalled to define its components.

7.6.4. Incremental vector constitutive equation

The equation that establishes the relation between their equivalent vector expressions is now derived. That is, the equation that defines the vector expression of the second Piola-Kirchhoff stress tensor increment by means of the vector expression of the Green-Lagrange strain tensor increment is defined.

The vector expression of the second Piola-Kirchhoff stress tensor increment, corresponding to a given displacement field increment $\Delta \mathbf{u}_\varepsilon$, can be defined as the following difference.

$$\Delta \bar{\mathbf{S}}_\varepsilon = \bar{\mathbf{S}}_\varepsilon \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon + \Delta \mathbf{u}_\varepsilon) \right) - \bar{\mathbf{S}}_\varepsilon \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon) \right) \quad (7.122)$$

Where the first term of this difference can be expressed according to the Taylor series expansion as:

$$\bar{\mathbf{S}}_\varepsilon \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon + \Delta \mathbf{u}_\varepsilon) \right) = \bar{\mathbf{S}}_\varepsilon \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon) \right) + \frac{d\bar{\mathbf{S}}_\varepsilon}{d\bar{\mathbf{E}}_{G,\varepsilon}} \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon) \right) \Delta \bar{\mathbf{E}}_{G,\varepsilon} + \mathcal{O}(\|\Delta \bar{\mathbf{E}}_{G,\varepsilon}\|^2) \quad (7.123)$$

The substitution of the above series expansion into the definition of the second Piola-Kirchhoff stress tensor increment (7.122) leads to:

$$\Delta \bar{\mathbf{S}}_\varepsilon = \frac{d\bar{\mathbf{S}}_\varepsilon}{d\bar{\mathbf{E}}_{G,\varepsilon}} \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon) \right) \Delta \bar{\mathbf{E}}_{G,\varepsilon} + \mathcal{O}(\|\Delta \bar{\mathbf{E}}_{G,\varepsilon}\|^2) \quad (7.124)$$

And if the Green-Lagrange strain increments are considered to be small, it can be concluded that:

$$\Delta \bar{\mathbf{S}}_\varepsilon \approx \frac{d\bar{\mathbf{S}}_\varepsilon}{d\bar{\mathbf{E}}_{G,\varepsilon}} \left(\bar{\mathbf{E}}_{G,\varepsilon}(\mathbf{u}_\varepsilon) \right) \Delta \bar{\mathbf{E}}_{G,\varepsilon} \quad (7.125)$$

The vector expression of the Green-Lagrange strain tensor increment was defined in (4.114), and it is the effect of considering that the displacement field is modified by a displacement field increment. The gradients of the displacement field increment are assumed to be small, so the assumption made in (7.125) is fulfilled, and the Green-Lagrange strain increments are small.

$$\left. \begin{aligned} & \mathbf{u}'_\varepsilon = \mathbf{u}_\varepsilon + \Delta \mathbf{u}_\varepsilon \\ & \left\| \frac{\partial \Delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \end{aligned} \right\} \implies \begin{aligned} & \mathbf{J}'_\varepsilon = \frac{\partial \mathbf{u}'_\varepsilon}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} + \frac{\partial \Delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}_0} = \mathbf{J}_\varepsilon + \Delta \mathbf{J}_\varepsilon \\ & \implies \bar{\mathbf{J}}'_\varepsilon = \bar{\mathbf{J}}_\varepsilon + \Delta \bar{\mathbf{J}}_\varepsilon \\ & \implies \|\Delta \bar{\mathbf{J}}_\varepsilon\| \ll 1 \\ & \implies \Delta \bar{\mathbf{E}}_{G,\varepsilon} = \left(\mathbf{A}_C + \mathbf{A}_\varepsilon(\mathbf{J}_\varepsilon) \right) \Delta \bar{\mathbf{J}}_\varepsilon \\ & \implies \|\Delta \bar{\mathbf{E}}_{G,\varepsilon}\| \ll 1 \end{aligned} \quad (7.126)$$

It can be concluded that, under the above assumption, the vector expression of the second Piola-Kirchhoff stress tensor increment defined by means of the vector

expression of the Green-Lagrange strain tensor increment turns out to be:

$$\Delta \bar{\mathbf{S}}_{\mathcal{L}} \approx \mathbf{C}_2(\bar{\mathbf{E}}_{G,\mathcal{L}}) \Delta \bar{\mathbf{E}}_{G,\mathcal{L}} \quad \text{with} \quad \mathbf{C}_2(\bar{\mathbf{E}}_{G,\mathcal{L}}) = \frac{d\bar{\mathbf{S}}_{\mathcal{L}}}{d\bar{\mathbf{E}}_{G,\mathcal{L}}}(\bar{\mathbf{E}}_{G,\mathcal{L}}), \quad (7.127)$$

where \mathbf{C}_2 is the constitutive tensor. The subscript indicates that, in this particular case, the constitutive tensor is a second order constitutive tensor. It is composed by 36 components, which are defined as shown below.

$$\mathbf{C}_2 = [C_{ij}]_{\substack{i=1,\dots,6 \\ j=1,\dots,6}} \quad C_{ij} = \frac{\partial S_i}{\partial E_j} \quad (7.128)$$

7.6.5. Second order constitutive tensor

The Voigt notation (section A.14) is applied to define the vector expression of the second Piola-Kirchhoff stress tensor.

$$\Delta \mathbf{S}_{\mathcal{L}} = \begin{bmatrix} \Delta S_{11} & \Delta S_{12} & \Delta S_{13} \\ \Delta S_{12} & \Delta S_{22} & \Delta S_{23} \\ \Delta S_{13} & \Delta S_{23} & \Delta S_{33} \end{bmatrix} \implies \Delta \bar{\mathbf{S}}_{\mathcal{L}} = \begin{Bmatrix} \Delta S_{11} \\ \Delta S_{22} \\ \Delta S_{33} \\ \Delta S_{12} \\ \Delta S_{13} \\ \Delta S_{23} \end{Bmatrix} \quad (7.129)$$

The same criterion is applied to define the vector expression of the Green-Lagrange strain tensor.

$$\Delta \mathbf{E}_{G,\mathcal{L}} = \begin{bmatrix} \Delta E_{11} & \Delta E_{12} & \Delta E_{13} \\ \Delta E_{12} & \Delta E_{22} & \Delta E_{23} \\ \Delta E_{13} & \Delta E_{23} & \Delta E_{33} \end{bmatrix} \implies \Delta \bar{\mathbf{E}}_{G,\mathcal{L}} = \begin{Bmatrix} \Delta E_{11} \\ \Delta E_{22} \\ \Delta E_{33} \\ 2 \Delta E_{12} \\ 2 \Delta E_{13} \\ 2 \Delta E_{23} \end{Bmatrix} \quad (7.130)$$

The relations between both vectors is defined by means of the constitutive tensor \mathbf{C}_2 , which is a second order tensor.

$$\begin{Bmatrix} \Delta S_{11} \\ \Delta S_{22} \\ \Delta S_{33} \\ \Delta S_{12} \\ \Delta S_{13} \\ \Delta S_{23} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \Delta E_{11} \\ \Delta E_{22} \\ \Delta E_{33} \\ 2 \Delta E_{12} \\ 2 \Delta E_{13} \\ 2 \Delta E_{23} \end{Bmatrix} \iff \Delta \bar{\mathbf{S}}_{\mathcal{L}} = \mathbf{C}_2 \Delta \bar{\mathbf{E}}_{G,\mathcal{L}} \quad (7.131)$$

The components that compose the vector expression of the second Piola-Kirchhoff stress tensor can be obtained in its equivalent tensor equation defined in (7.101).

$$\Delta \mathbf{S}_{\mathcal{L}} = \mathbf{C}_4 : \Delta \mathbf{E}_{G,\mathcal{L}} \iff \Delta S_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \Delta E_{kl} \quad (7.132)$$

If the symmetry of the Green-Lagrange strain tensor increment (7.120) and the minor symmetry of the fourth order constitutive tensor (7.121) are taken into account, the above components become:

$$\begin{aligned}
 \Delta S_{ij} &= \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \Delta E_{kl} \\
 &= C_{ij11} \Delta E_{11} + C_{ij12} \Delta E_{12} + C_{ij13} \Delta E_{13} + \\
 &\quad + C_{ij21} \Delta E_{21} + C_{ij22} \Delta E_{22} + C_{ij23} \Delta E_{23} + \\
 &\quad + C_{ij31} \Delta E_{31} + C_{ij32} \Delta E_{32} + C_{ij33} \Delta E_{33} \\
 &= C_{ij11} \Delta E_{11} + C_{ij22} \Delta E_{22} + C_{ij33} \Delta E_{33} + \\
 &\quad + C_{ij12} (2 \Delta E_{12}) + C_{ij13} (2 \Delta E_{13}) + C_{ij23} (2 \Delta E_{23})
 \end{aligned} \tag{7.133}$$

That is, the 81 different components of the fourth order constitutive tensor are reduced to 36 independent components.

$$\begin{pmatrix} \Delta S_{11} \\ \Delta S_{22} \\ \Delta S_{33} \\ \Delta S_{12} \\ \Delta S_{13} \\ \Delta S_{23} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{bmatrix} \begin{pmatrix} \Delta E_{11} \\ \Delta E_{22} \\ \Delta E_{33} \\ 2 \Delta E_{12} \\ 2 \Delta E_{13} \\ 2 \Delta E_{23} \end{pmatrix} \tag{7.134}$$

If the major symmetry of the fourth order constitutive tensor (7.118) is now taken into account, the number of components is finally reduced to 21.

$$\begin{pmatrix} \Delta S_{11} \\ \Delta S_{22} \\ \Delta S_{33} \\ \Delta S_{12} \\ \Delta S_{13} \\ \Delta S_{23} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ & & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ & & & C_{1212} & C_{1213} & C_{1223} \\ & \text{sym} & & & C_{1313} & C_{1323} \\ & & & & & C_{2323} \end{bmatrix} \begin{pmatrix} \Delta E_{11} \\ \Delta E_{22} \\ \Delta E_{33} \\ 2 \Delta E_{12} \\ 2 \Delta E_{13} \\ 2 \Delta E_{23} \end{pmatrix} \tag{7.135}$$

Therefore, the second order constitutive tensor turns out to be symmetric.

$$\mathbf{C}_2^T = \mathbf{C}_2 \iff C_{ji} = C_{ij} \tag{7.136}$$

And its components defined by means of the fourth order constitutive tensor components are:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ & & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ & & & C_{1212} & C_{1213} & C_{1223} \\ & \text{sym} & & & C_{1313} & C_{1323} \\ & & & & & C_{2323} \end{bmatrix} \tag{7.137}$$

In some particular cases, a linear relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor can be defined by means of the linear constitutive tensor. This is a major advantage since the definition of the constitutive tensor is simplified. Moreover, as the relation between both magnitudes is linear, its incremental form is straightforward to obtain by means of the linear constitutive tensor.

In the following section, this simple constitutive equation is presented, as well as the conditions that the structural behaviour has to fulfil to accurately apply this model.

7.7. St. Venant-Kirchhoff constitutive model

One of the simplest models to deal with an elastic material is the St. Venant-Kirchhoff model [Bathe, 1996; Holzapfel, 2000; Belytschko et al., 2014; Kim, 2014]. For an isotropic material, this model defines the work per unit volume as:

$$\Psi(\mathbf{E}_{G,\mathcal{L}}) = \frac{1}{2}\lambda \text{Tr}^2(\mathbf{E}_{G,\mathcal{L}}) + \mu \text{Tr}\left((\mathbf{E}_{G,\mathcal{L}})^2\right) \quad (7.138)$$

Where λ and μ are the Lamé's parameters (7.51), which depend on the Young's modulus and the Poisson's ratio.

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad \mu = \frac{E}{2(1 + \nu)} \quad (7.139)$$

7.7.1. St. Venant-Kirchhoff constitutive equation

According to the result obtained in (7.114), the derivation of this function with respect to the Green-Lagrange strain tensor leads to the definition of the second Piola-Kirchhoff stress tensor ($\mathbf{S}_{\mathcal{L}}$) as:

$$\begin{aligned} \mathbf{S}_{\mathcal{L}} &= \frac{d\Psi(\mathbf{E}_{G,\mathcal{L}})}{d\mathbf{E}_{G,\mathcal{L}}} \\ &= \lambda \text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \right) + \mu \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}\left((\mathbf{E}_{G,\mathcal{L}})^2\right) \right) \end{aligned} \quad (7.140)$$

Let's focus now on the first term of the above equation. On the one hand, the trace of the Green-Lagrange strain tensor is defined as:

$$\mathbf{E}_{G,\mathcal{L}} = [E_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \Longrightarrow \quad \text{Tr}(\mathbf{E}_{G,\mathcal{L}}) = \sum_{k=1}^n E_{kk} \quad (7.141)$$

On the other hand, the derivative of the above trace with respect to the Green-

Lagrange strain tensor turns out to be equivalent to the second order unit tensor.

$$\left. \begin{aligned} \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \right) &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ A_{ij} &= \frac{\partial}{\partial E_{ij}} \left(\text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \right) \\ &= \frac{\partial}{\partial E_{ij}} \left(\sum_{k=1}^n E_{kk} \right) \\ &= \delta_{ij} \end{aligned} \right\} \implies \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \right) = \mathbf{I}_2 \quad (7.142)$$

In the above equation, the subscript $_2$ is added to the unit tensor, to indicate that it is the second order unit tensor. Later on in this section, the fourth order unit tensor will also show up. Thus, subscripts are added to indicate the order and avoid confusion.

To obtain the second term, the square of the Green-Lagrange strain tensor is defined, and its trace is computed.

$$\left. \begin{aligned} (\mathbf{E}_{G,\mathcal{L}})^2 = \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ B_{ij} &= \sum_{l=1}^n E_{il} E_{lj} \end{aligned} \right\} \implies \text{Tr}(\mathbf{B}) = \sum_{k=1}^n B_{kk} = \sum_{k=1}^n \sum_{l=1}^n E_{kl} E_{lk} \quad (7.143)$$

And the derivative of the above trace with respect to the Green-Lagrange strain tensor can be calculated, considering the symmetry of the Green-Lagrange strain tensor, as:

$$\left. \begin{aligned} \frac{d}{d\mathbf{E}_G} \left(\text{Tr}(\mathbf{B}) \right) &= [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ C_{ij} &= \frac{\partial}{\partial E_{ij}} \left(\sum_{k,l} E_{kl} E_{lk} \right) \\ &= \sum_{k,l} \left(\frac{\partial E_{kl}}{\partial E_{ij}} E_{lk} + E_{kl} \frac{\partial E_{lk}}{\partial E_{ij}} \right) \\ &= \delta_{ki} \delta_{lj} E_{lk} + E_{kl} \delta_{li} \delta_{kj} \\ &= 2 E_{ji} \\ &= 2 E_{ij} \end{aligned} \right\} \implies \frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}(\mathbf{B}) \right) = 2 \mathbf{E}_{G,\mathcal{L}} \quad (7.144)$$

Moreover, the property of the fourth order unit tensor \mathbf{I}_4 exposed in (A.90) allows to rewrite the above equation as:

$$\frac{d}{d\mathbf{E}_{G,\mathcal{L}}} \left(\text{Tr}(\mathbf{B}) \right) = 2 \mathbf{E}_{G,\mathcal{L}} = 2 \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} \quad (7.145)$$

The substitution of the results obtained in (7.142) and (7.145) into the definition of the second Piola-Kirchhoff stress tensor (7.140) leads to:

$$\mathbf{S}_{\mathcal{L}} = \lambda \text{Tr}(\mathbf{E}_{G,\mathcal{L}}) \mathbf{I}_2 + 2\mu \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} \quad (7.146)$$

The tensor $\mathbf{I}_2 \otimes \mathbf{I}_2$ presented in (A.92) is another fourth order tensor that fulfils an interesting property that can be applied to the above equation. If the property exposed in (A.93) is taken into account, the definition of the second Piola-Kirchhoff stress tensor becomes:

$$\begin{aligned} \mathbf{S}_\mathcal{L} &= \lambda \operatorname{Tr}(\mathbf{E}_{G,\mathcal{L}}) \mathbf{I}_2 + 2\mu \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} \\ &= \lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) : \mathbf{E}_{G,\mathcal{L}} + 2\mu \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} \\ &= \left[\lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4 \right] : \mathbf{E}_{G,\mathcal{L}} \end{aligned} \quad (7.147)$$

Furthermore, since the Green-Lagrange strain tensor is symmetric, the other fourth order unit tensor $\bar{\mathbf{I}}_4$ defined in (A.88) can be equivalently used, and the same result is obtained.

$$\begin{aligned} \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} &= \mathbf{E}_{G,\mathcal{L}} \\ \bar{\mathbf{I}}_4 : \mathbf{E}_{G,\mathcal{L}} &= (\mathbf{E}_{G,\mathcal{L}})^T = \mathbf{E}_{G,\mathcal{L}} \end{aligned} \quad (7.148)$$

This implies that the fourth order unit tensor \mathbf{I}_4 can be replaced by its symmetric term $\mathbf{I}_4^{\text{sym}}$, defined in (A.94).

$$\begin{aligned} \mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} &= \mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} (\mathbf{E}_{G,\mathcal{L}} + \mathbf{E}_{G,\mathcal{L}}) \\ &= \frac{1}{2} (\mathbf{I}_4 : \mathbf{E}_{G,\mathcal{L}} + \bar{\mathbf{I}}_4 : \mathbf{E}_{G,\mathcal{L}}) \\ &= \frac{1}{2} (\mathbf{I}_4 + \bar{\mathbf{I}}_4) : \mathbf{E}_{G,\mathcal{L}} \\ &= \mathbf{I}_4^{\text{sym}} : \mathbf{E}_{G,\mathcal{L}} \end{aligned} \quad (7.149)$$

Therefore, the second Piola-Kirchhoff stress tensor (7.147) becomes:

$$\mathbf{S}_\mathcal{L} = \left[\lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4^{\text{sym}} \right] : \mathbf{E}_{G,\mathcal{L}} \quad (7.150)$$

7.7.2. St. Venant-Kirchhoff constitutive tensor

As proved in the previous section, the constitutive equation can be finally defined as:

$$\mathbf{S}_\mathcal{L} = \mathbf{C}_4 : \mathbf{E}_{G,\mathcal{L}} \quad \text{with} \quad \mathbf{C}_4 = \lambda (\mathbf{I}_2 \otimes \mathbf{I}_2) + 2\mu \mathbf{I}_4^{\text{sym}}, \quad (7.151)$$

where \mathbf{C}_4 is the constitutive tensor. The components of the tensors $\mathbf{I}_2 \otimes \mathbf{I}_2$ and $\mathbf{I}_4^{\text{sym}}$ were defined in (A.92) and (A.95), respectively. If these definitions are taken into account, the components of the above constitutive tensor can be finally defined as:

$$\mathbf{C}_4 = \left[C_{ijkl} \right]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,3,3 \\ l=1,3,3}} \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (7.152)$$

This tensor components are constant, and they turn out to be equivalent to the ones that define the relation between the Cauchy stress tensor and the infinitesimal strain tensor, exposed in (7.81) and (7.82).

7.7.3. St. Venant-Kirchhoff model applications

The constitutive equation that defines the second Piola-Kirchhoff stress tensor by means of the Green-Lagrange strain tensor can be expressed according to the Taylor series expansion about the point of null Green-Lagrange strains.

$$\mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}) = \mathbf{S}_\mathcal{L}(\mathbf{0}) + \frac{d\mathbf{S}_\mathcal{L}}{d\mathbf{E}_{G,\mathcal{L}}}(\mathbf{0}) : \mathbf{E}_{G,\mathcal{L}} + \mathcal{O}(\|\mathbf{E}_{G,\mathcal{L}}\|^2) \quad (7.153)$$

The first term of the above series is null if there are no residual stresses, and the last one can be neglected if the Green-Lagrange strains are small. Thus, the series expansion is reduced to:

$$\left. \begin{aligned} \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}) &= \mathbf{S}_\mathcal{L}(\mathbf{0}) + \frac{d\mathbf{S}_\mathcal{L}}{d\mathbf{E}_{G,\mathcal{L}}}(\mathbf{0}) : \mathbf{E}_{G,\mathcal{L}} + \mathcal{O}(\|\mathbf{E}_{G,\mathcal{L}}\|^2) \\ \mathbf{S}_\mathcal{L}(\mathbf{0}) &= \mathbf{0} \\ \mathcal{O}(\|\mathbf{E}_{G,\mathcal{L}}\|^2) &\approx \mathbf{0} \quad \text{if} \quad \|\mathbf{E}_{G,\mathcal{L}}\| \ll 1 \end{aligned} \right\} \implies \mathbf{S}_\mathcal{L} \approx \frac{d\mathbf{S}_\mathcal{L}}{d\mathbf{E}_{G,\mathcal{L}}}(\mathbf{0}) : \mathbf{E}_{G,\mathcal{L}} \quad (7.154)$$

If the above assumptions are fulfilled, the constitutive equation can be finally defined as:

$$\mathbf{S}_\mathcal{L} = \mathbf{C}_4(\mathbf{0}) : \mathbf{E}_{G,\mathcal{L}} \quad \text{with} \quad \mathbf{C}_4(\mathbf{0}) = \frac{d\mathbf{S}_\mathcal{L}}{d\mathbf{E}_{G,\mathcal{L}}}(\mathbf{0}), \quad (7.155)$$

where $\mathbf{C}_4(\mathbf{0})$ is the constitutive tensor. When dealing with a one-dimensional problem, this tensor turns out to be the tangent of the curve represented in figure 7.4, evaluated at the point of null Green-Lagrange strain. Equation (7.155) represents the linear approach to a relation which, in general, is nonlinear. This approach is only valid when the Green-Lagrange strains are small. Nevertheless, the displacements do not have to be necessarily small, since small Green-Lagrange strains may correspond to large displacements.

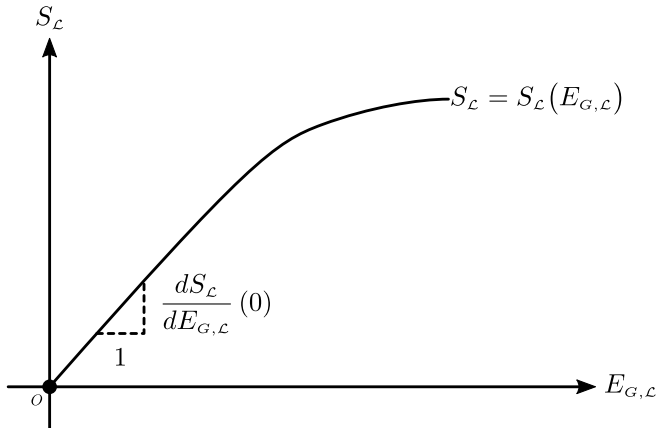


Figure 7.4. Second Piola-Kirchhoff stress vs. Green-Lagrange strain (one-dimensional case).

The structure of the St. Venant-Kirchhoff constitutive equation (7.150) is analogous to the one exposed in (7.155). Hence, it also represents a linear approach to the constitutive relation. Thus, this model can be properly applied when the Green-Lagrange strains are small.

Furthermore, the St. Venant-Kirchhoff constitutive tensor turns out to be equivalent to the linear elastic constitutive tensor (7.81). As it will be demonstrated below, the constitutive tensor can be approximated by the linear elastic one, if the strain field is infinitesimal.

Let's consider a given deformation gradient tensor composed by a finite strain tensor. If the strain field is infinitesimal, it can be proven that the second Piola-Kirchhoff stress tensor (3.48) is equivalent to the Cauchy stress tensor; and the Green-Lagrange strain tensor (4.77) turns out to be equivalent to the Biot strain tensor.

$$\left. \begin{array}{l} \mathbf{F}_\mathcal{L} = \mathbf{I} + \underbrace{\mathbf{E}_\mathcal{L}}_{J_\mathcal{L}} \\ \|\underbrace{\mathbf{E}_\mathcal{L}}_{J_\mathcal{L}}\| \ll \|\mathbf{I}\| \end{array} \right\} \implies \mathbf{F}_\mathcal{L} \approx \mathbf{I} \tag{7.156}$$

$$\begin{aligned}
 &\implies F_\mathcal{L} = \det(\mathbf{F}_\mathcal{L}) \approx \det(\mathbf{I}) = 1 \\
 &\implies \mathbf{E}_{G,\mathcal{L}} = \mathbf{E}_\mathcal{L} + \mathcal{O}(\|\mathbf{E}_\mathcal{L}\|^2) \approx \mathbf{E}_\mathcal{L} \\
 &\implies \mathbf{S}_\mathcal{L} = F_\mathcal{L}(\mathbf{F}_\mathcal{L}^{-1} \boldsymbol{\sigma}_\mathcal{L} \mathbf{F}_\mathcal{L}^{-T}) \approx \boldsymbol{\sigma}_\mathcal{L}
 \end{aligned}$$

Thus, the St. Venant-Kirchhoff constitutive equation (7.155) is reduced to:

$$\underbrace{\mathbf{S}_\mathcal{L}}_{\approx \boldsymbol{\sigma}_\mathcal{L}} = \mathbf{C}_4(\mathbf{0}) : \underbrace{\mathbf{E}_{G,\mathcal{L}}}_{\approx \mathbf{E}_\mathcal{L}} \iff \boldsymbol{\sigma}_\mathcal{L} \approx \mathbf{C}_4(\mathbf{0}) : \mathbf{E}_\mathcal{L} \tag{7.157}$$

If the above equation is compared to the linear elastic constitutive equation (7.69), it can be concluded that the St. Venant-Kirchhoff constitutive tensor is equivalent to the linear elastic one.

$$\left. \begin{array}{l} \boldsymbol{\sigma}_\mathcal{L} \approx \mathbf{C}_4(\mathbf{0}) : \mathbf{E}_\mathcal{L} \\ \boldsymbol{\sigma}_\mathcal{L} \approx \mathbf{D}_4(\mathbf{0}) : \mathbf{E}_\mathcal{L} \end{array} \right\} \implies \mathbf{C}_4(\mathbf{0}) \approx \mathbf{D}_4(\mathbf{0}) \tag{7.158}$$

Let's now apply a rigid rotation to the solid. As proved in section 3.5.1, the second Piola-Kirchhoff stress tensor does not vary. That is, it remains equivalent to the Cauchy stress tensor corresponding to the previous deformed configuration. Furthermore, the Green-Lagrange strain tensor also remains constant if a rotation is applied, as demonstrated in section 4.6.4. And the Cauchy stress tensor is obtained by applying

a rotation to the previous one, as shown in section 7.3.

$$\begin{aligned}
 \left. \begin{aligned}
 \mathbf{F}'_{\mathcal{L}} &= \mathbf{R}_{\mathcal{L}} [\mathbf{I} + \mathbf{E}_{\mathcal{L}}] \\
 \mathbf{R}_{\mathcal{L}}^{-1} &= \mathbf{R}_{\mathcal{L}}^T \\
 \det(\mathbf{R}_{\mathcal{L}}) &= 1
 \end{aligned} \right\} \implies \mathbf{F}'_{\mathcal{L}} \approx \mathbf{R}_{\mathcal{L}} \\
 \implies F'_{\mathcal{L}} &= \det(\mathbf{F}'_{\mathcal{L}}) \approx \det(\mathbf{R}_{\mathcal{L}}) = \det(\mathbf{I}) = 1 \\
 \implies \boldsymbol{\sigma}'_{\mathcal{L}} &= \mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}} \mathbf{R}_{\mathcal{L}}^T \\
 \implies \mathbf{E}'_{G,\mathcal{L}} &= \mathbf{E}_{G,\mathcal{L}} \approx \mathbf{E}_{\mathcal{L}} \\
 \implies \mathbf{S}'_{\mathcal{L}} &= F'_{\mathcal{L}} \left((\mathbf{F}'_{\mathcal{L}})^{-1} \boldsymbol{\sigma}'_{\mathcal{L}} (\mathbf{F}'_{\mathcal{L}})^{-T} \right) \approx \mathbf{R}_{\mathcal{L}}^T \mathbf{R}_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}} \mathbf{R}_{\mathcal{L}}^T \mathbf{R}_{\mathcal{L}} = \boldsymbol{\sigma}_{\mathcal{L}}
 \end{aligned} \tag{7.159}$$

Hence, the St. Venant-Kirchhoff constitutive equation corresponding to this rotated configuration becomes:

$$\underbrace{\mathbf{S}'_{\mathcal{L}}}_{\approx \boldsymbol{\sigma}_{\mathcal{L}}} = \mathbf{C}'_4(\mathbf{0}) : \underbrace{\mathbf{E}'_{G,\mathcal{L}}}_{\approx \mathbf{E}_{\mathcal{L}}} \iff \boldsymbol{\sigma}_{\mathcal{L}} \approx \mathbf{C}'_4(\mathbf{0}) : \mathbf{E}_{\mathcal{L}} \tag{7.160}$$

The Cauchy stress tensor was equivalently defined according to the linear elastic constitutive equation in (7.158). So, if the above equation is compared to the linear elastic one, it can be concluded that the St. Venant-Kirchhoff constitutive tensor is still equivalent to the linear elastic constitutive tensor.

$$\left. \begin{aligned}
 \boldsymbol{\sigma}_{\mathcal{L}} &\approx \mathbf{C}'_4(\mathbf{0}) : \mathbf{E}_{\mathcal{L}} \\
 \boldsymbol{\sigma}_{\mathcal{L}} &\approx \mathbf{D}_4(\mathbf{0}) : \mathbf{E}_{\mathcal{L}}
 \end{aligned} \right\} \implies \mathbf{C}'_4(\mathbf{0}) \approx \mathbf{D}_4(\mathbf{0}) \tag{7.161}$$

According to the previous results, it can be concluded that the constitutive tensor can be approximated by the linear elastic one if the strain field is infinitesimal. Even if the displacements and/or the displacement gradients are large, this conclusion holds.

7.8. Overview and conclusions

The structural behaviour of elastic solids is analysed in this work. The value of the stress field of an elastic solid only depends on the value of the deformation gradient tensor. If the polar decomposition of the deformation gradient tensor is taken into account, it can be equivalently stated that the stress tensor depends on both the finite rotation tensor and the Biot strain tensor. Moreover, the stress tensor does not depend on the previous values of the deformation gradient tensor. That is, it only depends on the value of the deformation gradient tensor at a given instant of time. Therefore, if the external applied loads are removed, the initial configuration is completely recovered.

The definition of the stress tensor is simplified in linear elasticity theory. The main assumption is to consider an infinitesimal strain field, which does not necessarily imply neither small displacements nor small displacement gradients. This hypothesis allows to simplify the definition of the Cauchy stress tensor, since the term that depends on the

strain field can be linearized by means of the infinitesimal strain tensor. Furthermore, if the displacement gradients are small, the effect of the infinitesimal rotation can be neglected, and the Cauchy stress tensor can be finally defined only by means of the infinitesimal strain tensor. Therefore, the equation that relates both magnitudes can be linearized, and the tensor that defines this linear relation is the so-called linear constitutive tensor.

If the assumptions of the linear elasticity are not fulfilled, a proper definition of the Cauchy stress tensor by means of the rotation tensor and the Biot strain tensor has to be stated. Nevertheless, more suitable tensor magnitudes can be adopted to define the strain and stress fields.

Both the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor remain invariant if a rigid motion (rotation and/or translation) is applied to the solid. This is an important property to take into account when a large displacement analysis is carried out. If the solid experiments large displacements, with large or small displacement gradients, these tensors can be adopted to properly describe the strain field and its corresponding stress field. Therefore, a constitutive equation that defines their mathematical relation has to be stated.

In order to develop a nonlinear finite element analysis, it is essential to define the incremental constitutive equation, since the analysis will be carried out by an incremental loading procedure. Hence, the incremental equation that defines the second Piola-Kirchhoff stress tensor increment by means of the Green-Lagrange strain tensor increment is derived in this chapter. Since the gradients of the displacement field increment between consecutive load steps are assumed to be small, its corresponding Green-Lagrange strain increment turns out to be small. Thus, the relation between both tensor magnitudes can be defined as a linear relation by means of the constitutive tensor.

If the strain field is infinitesimal, even if the displacements and/or the displacement gradients are large, the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor can be accurately approximated as a linear relation defined by means of the linear constitutive tensor. That is, the same constitutive tensor that states the linear relation between the Cauchy stress tensor and the infinitesimal strain tensor defines the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor. This is the so-called St. Venant-Kirchhoff constitutive equation, which is one of the simplest constitutive models.

Static equilibrium equations

8.1. Introduction

The equations that define the balance of mass, linear momentum and angular momentum were derived in chapter 2. In this chapter, these magnitudes are considered to not vary. This assumption leads to the equation that defines the value of the density field over time, as well as the equations that govern the static equilibrium of forces and torques.

In Solid Mechanics, it is not common to deal with mass sources. Hence, it is usually considered that the mass remains constant. The mass conservation allows to obtain the equation that defines the value of the density field over time. Furthermore, the imposition of the linear momentum conservation allows to derive the equation that rules the static equilibrium of forces, whereas the angular momentum conservation leads to the equation that governs the equilibrium of torques.

8.2. Mass conservation

In Solid Mechanics, it is not usual to deal with mass sources. That is, it is usually considered that the mass is conserved and there is no mass variation. If there are no mass sources, the variable $\varphi_{\mathcal{L}}$ that represent the mass source per unit time per unit volume becomes null.

$$\varphi_{\mathcal{L}} = 0 \tag{8.1}$$

Under this assumption, the differential equation that defines the mass balance presented in (2.9) is reduced to:

$$\frac{\partial}{\partial t}(\rho_{\mathcal{L}} F_{\mathcal{L}}) = 0 \quad \forall \mathbf{r}_0 \in \mathring{\Omega}_0, \forall t. \tag{8.2}$$

Where $\rho_{\mathcal{L}}$ is the Lagrangian description of the density, and $F_{\mathcal{L}}$ is the determinant of the deformation gradient tensor.

The above differential equation has analytical solution. As the time variation is zero, the product of the density field and the determinant of the deformation gradient tensor has to be constant over time. Thus, it can be stated that:

$$\rho_{\mathcal{L}}(\mathbf{r}_0, t) F_{\mathcal{L}}(\mathbf{r}_0, t) = \mathbb{C}(\mathbf{r}_0) \quad (8.3)$$

From the above equation, the equation that defines the density field turns out to be:

$$\rho_{\mathcal{L}}(\mathbf{r}_0, t) = \frac{\mathbb{C}(\mathbf{r}_0)}{F_{\mathcal{L}}(\mathbf{r}_0, t)} \quad (8.4)$$

And the value of the function \mathbb{C} can be obtained if the equation is evaluated at instant $t = 0$.

$$\mathbb{C}(\mathbf{r}_0) = \rho_{\mathcal{L}}(\mathbf{r}_0, 0) F_{\mathcal{L}}(\mathbf{r}_0, 0) \quad (8.5)$$

On the one hand, the value of the initial density field is known. On the other hand, the value of the determinant of the deformation gradient tensor at $t = 0$ is also known. As stated in (1.29), the determinant of the deformation gradient tensor can be interpreted as the volume ratio with respect to the initial volume. Since there is no change in volume before the application of the external loads, the determinant of the deformation gradient tensor has to be equal to one.

$$F_{\mathcal{L}}(\mathbf{r}_0, 0) = 1 \quad (8.6)$$

Therefore, the function \mathbb{C} turns out to be equivalent to the initial density field.

$$\mathbb{C}(\mathbf{r}_0) = \rho_{\mathcal{L}}(\mathbf{r}_0, 0) \quad (8.7)$$

And the equation that defines the density field over time (8.4) becomes:

$$\rho_{\mathcal{L}}(\mathbf{r}_0, t) = \frac{\rho_{\mathcal{L}}(\mathbf{r}_0, 0)}{F_{\mathcal{L}}(\mathbf{r}_0, t)} \quad \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0, \forall t \quad (8.8)$$

8.3. Linear momentum conservation

Let's consider a static problem without mass sources. This implies that the velocity field \mathbf{a}_{ε} and the mass source φ_{ε} are null over the whole material domain. Moreover, the variables no longer depend on time.

$$\begin{aligned} \mathbf{a}_{\varepsilon}(\mathbf{r}) &= \bar{\mathbf{0}} & \forall \mathbf{r} \in \Omega \\ \varphi_{\varepsilon}(\mathbf{r}) &= 0 & \forall \mathbf{r} \in \Omega \end{aligned} \quad (8.9)$$

Under these assumptions, the Eulerian differential equation that defines the linear momentum balance equation (2.27) is reduced to:

$$\mathbf{div}(-\boldsymbol{\sigma}_{\varepsilon}^T) = \mathbf{b}_{\varepsilon} \rho_{\varepsilon} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (8.10)$$

Therefore, the equation that rules the static equilibrium of forces of a given solid subjected to external loads turns out to be:

$$\mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}^T) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon} = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (8.11)$$

8.3.1. Alternative derivation

The equation that governs the conservation of linear momentum can be alternatively derived if the Eulerian integral form of the linear momentum balance equation (2.30) is recalled. If there are no mass sources, this integral form becomes:

$$\frac{\partial}{\partial t} \left(\iiint_{\Omega} \mathbf{a}_{\varepsilon} \rho_{\varepsilon} d\Omega \right) = \iiint_{\Omega} \left[\mathbf{b}_{\varepsilon} \rho_{\varepsilon} + \mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}^T) \right] d\Omega \quad (8.12)$$

If the system is in static equilibrium, the time variation of the linear momentum has to be equal to zero. Hence, the above equation is reduced to:

$$\iiint_{\Omega} \left[\mathbf{b}_{\varepsilon} \rho_{\varepsilon} + \mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}^T) \right] d\Omega = \bar{\mathbf{0}} \quad (8.13)$$

Moreover, according to the localization theorem exposed in section C.2, it can be stated that the kernel of the above integral has to be null.

$$\mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}^T) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon} = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega}, \forall t \quad (8.14)$$

This equation is the one that rules the static equilibrium of a solid subjected to external forces, and it matches the one obtained before in (8.11).

8.4. Angular momentum conservation

As obtained in section 2.4.2, the Eulerian integral form of the angular momentum balance equation, considering that there are no mass sources, turns out to be:

$$\frac{\partial}{\partial t} \left[\iiint_{\Omega} (\mathbf{r} \wedge \mathbf{a}_{\varepsilon} \rho_{\varepsilon}) d\Omega \right] = \iiint_{\Omega} (\mathbf{r} \wedge \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) d\Omega + \iint_{\Gamma} (\mathbf{r} \wedge \mathbf{t}_{\varepsilon}) d\Gamma \quad (8.15)$$

It is considered that the solid is in static equilibrium, and the material is not capable of absorbing angular momentum per unit volume. Under these assumptions, the time variation of the angular momentum has to be zero. Thus, the above equation is reduced to:

$$\iiint_{\Omega} (\mathbf{r} \wedge \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) d\Omega + \iint_{\Gamma} (\mathbf{r} \wedge \mathbf{t}_{\varepsilon}) d\Gamma = \bar{\mathbf{0}} \quad (8.16)$$

The stress vector was defined in (3.34) as:

$$\mathbf{t}_{\varepsilon} = \boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n} \quad (8.17)$$

Hence, the angular momentum conservation equation becomes:

$$\iiint_{\Omega} (\mathbf{r} \wedge \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) d\Omega + \iint_{\Gamma} \left[\mathbf{r} \wedge (\boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n}) \right] d\Gamma = \bar{\mathbf{0}} \quad (8.18)$$

The above surface integral has to be transformed into a volume integral defined over the deformed material domain.

On the one hand, the definition of the components of a vector product (A.20) is recalled, as well as the definition of the components of the resulting vector of the product of a second order tensor and a first order one (A.67).

$$\begin{aligned} \mathbf{r} \wedge \mathbf{t}_\varepsilon = \{x_i\}_{i=1,2,3} &\implies x_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j t_k \\ \mathbf{t}_\varepsilon = \boldsymbol{\sigma}_\varepsilon^T \mathbf{n} = \{t_k\}_{k=1,2,3} &\implies t_k = \sum_{l=1}^3 \sigma_{lk} n_l \end{aligned} \quad (8.19)$$

Therefore, the second term of the angular momentum conservation (8.18) can be equivalently expressed as:

$$\begin{aligned} \iint_\Gamma [\mathbf{r} \wedge (\boldsymbol{\sigma}_\varepsilon^T \mathbf{n})] d\Gamma &= \left\{ \iint_\Gamma \left[\sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j \left(\sum_{l=1}^3 \sigma_{lk} n_l \right) \right] d\Gamma \right\}_{i=1,2,3} \\ &= \left\{ \iint_\Gamma \left[\sum_{l=1}^3 \left(\sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j \sigma_{lk} \right) n_l \right] d\Gamma \right\}_{i=1,2,3} \end{aligned} \quad (8.20)$$

The above equation can be rewritten in a more compact form as:

$$\iint_\Gamma [\mathbf{r} \wedge (\boldsymbol{\sigma}_\varepsilon^T \mathbf{n})] d\Gamma = \left\{ \iint_\Gamma \left(\sum_{l=1}^3 y_{il} n_l \right) d\Gamma \right\}_{i=1,2,3} \quad (8.21)$$

Where:

$$\mathbf{Y} = [y_{il}]_{\substack{i=1,2,3 \\ l=1,2,3}} \quad y_{il} = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j \sigma_{lk} \quad (8.22)$$

On the other hand, the divergence theorem applied to a tensor field (B.44) is recalled. Its application to the above tensor field leads to:

$$\begin{aligned} \iint_\Gamma \mathbf{Y} \mathbf{n} d\Gamma &= \iiint_\Omega \mathbf{div}(\mathbf{Y}) d\Omega \iff \\ \iff \left\{ \iint_\Gamma \left(\sum_{l=1}^3 y_{il} n_l \right) d\Gamma \right\}_{i=1,2,3} &= \left\{ \sum_{l=1}^3 \frac{\partial y_{il}}{\partial r_l} \right\}_{i=1,2,3} \end{aligned} \quad (8.23)$$

Thus, if the statement of the divergence theorem is taken into account, the equation (8.21) can be rewritten as a volume integral.

$$\iint_\Gamma [\mathbf{r} \wedge (\boldsymbol{\sigma}_\varepsilon^T \mathbf{n})] d\Gamma = \left\{ \iiint_\Omega \left(\sum_{l=1}^3 \frac{\partial y_{il}}{\partial r_l} \right) d\Omega \right\}_{i=1,2,3} \quad (8.24)$$

Equation (8.22) can now be substituted into the above equation, and the derivatives can be calculated.

$$\begin{aligned}
 & \iint_{\Gamma} [\mathbf{r} \wedge (\boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n})] d\Gamma = \\
 & = \left\{ \iiint_{\Omega} \left[\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \frac{\partial}{\partial r_l} (\varepsilon_{ijk} r_j \sigma_{lk}) \right] d\Omega \right\}_{i=1,2,3} \\
 & = \left\{ \iiint_{\Omega} \left[\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \left(\frac{\partial}{\partial r_l} (\varepsilon_{ijk} r_j) \sigma_{lk} + (\varepsilon_{ijk} r_j) \frac{\partial \sigma_{lk}}{\partial r_l} \right) \right] d\Omega \right\}_{i=1,2,3} \\
 & = \left\{ \iiint_{\Omega} \left[\sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \sigma_{jk} + \sum_{j=1}^3 \sum_{k=1}^3 \left(\varepsilon_{ijk} r_j \sum_{l=1}^3 \frac{\partial \sigma_{lk}}{\partial r_l} \right) \right] d\Omega \right\}_{i=1,2,3}
 \end{aligned} \tag{8.25}$$

To simplify the previous equation, the definition of the double dot product between a third order tensor and a second order tensor (section A.7) is recalled, as well as the definition of the divergence of a tensor field (section B.7.2), and the definition of the vector product between two first order tensors (section A.1.5).

$$\begin{aligned}
 \boldsymbol{\alpha} &= \boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} = \{\alpha_i\}_{i=1,2,3} & \text{where} & \quad \alpha_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \sigma_{jk} \\
 \boldsymbol{\beta} &= \text{div}(\boldsymbol{\sigma}_{\varepsilon}^T) = \{\beta_k\}_{k=1,2,3} & \text{where} & \quad \beta_k = \sum_{l=1}^3 \frac{\partial \sigma_{lk}}{\partial r_l} \\
 \boldsymbol{\gamma} &= \mathbf{r} \wedge \boldsymbol{\beta} = \{\gamma_i\}_{i=1,2,3} & \text{where} & \quad \gamma_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j \beta_k
 \end{aligned} \tag{8.26}$$

The introduction of the above definitions into (8.25) leads to:

$$\iint_{\Gamma} [\mathbf{r} \wedge (\boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n})] d\Gamma = \left\{ \iiint_{\Omega} \left(\alpha_i + \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} r_j \beta_k \right) d\Omega \right\}_{i=1,2,3} \tag{8.27}$$

Or, if the tensor notation is adopted, instead of the index one, the above equation can be equivalently expressed as:

$$\iint_{\Gamma} [\mathbf{r} \wedge (\boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n})] d\Gamma = \iiint_{\Omega} [\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} + \mathbf{r} \wedge \text{div}(\boldsymbol{\sigma}_{\varepsilon}^T)] d\Omega \tag{8.28}$$

The substitution of this result into the equation that defines the conservation of

angular momentum (8.18), leads to:

$$\begin{aligned}
 & \iint\int_{\Omega} (\mathbf{r} \wedge \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) d\Omega + \iint_{\Gamma} [\mathbf{r} \wedge (\boldsymbol{\sigma}_{\varepsilon}^T \mathbf{n})] d\Gamma = \\
 & = \iint\int_{\Omega} (\mathbf{r} \wedge \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) d\Omega + \iint\int_{\Omega} [\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} + \mathbf{r} \wedge \operatorname{div}(\boldsymbol{\sigma}_{\varepsilon}^T)] d\Omega \quad (8.29) \\
 & = \iint\int_{\Omega} \left[\mathbf{r} \wedge \underbrace{(\operatorname{div}(\boldsymbol{\sigma}_{\varepsilon}^T) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon})}_{=\bar{\mathbf{0}}} + \boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} \right] d\Omega = \bar{\mathbf{0}}
 \end{aligned}$$

The first term of the above equation is equal to zero due to the equation of conservation of linear momentum (8.14). The integral equation is then reduced to:

$$\iint\int_{\Omega} \left[\mathbf{r} \wedge \underbrace{(\operatorname{div}(\boldsymbol{\sigma}_{\varepsilon}^T) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon})}_{=\bar{\mathbf{0}}} + \boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} \right] d\Omega = \iint\int_{\Omega} (\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon}) d\Omega = \bar{\mathbf{0}} \quad (8.30)$$

And the localization theorem (section C.2.2) allows to conclude that the integrand of the above integral equation has to be equal to the null vector.

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} = \bar{\mathbf{0}} \quad (8.31)$$

Furthermore, this equation can be rewritten in index notation, as:

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_{\varepsilon} = \left\{ \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \sigma_{jk} \right\}_{i=1,2,3} = \left\{ \begin{array}{l} \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{1jk} \sigma_{jk} \\ \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{2jk} \sigma_{jk} \\ \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{3jk} \sigma_{jk} \end{array} \right\} = \left\{ \begin{array}{l} \boldsymbol{\varepsilon}_1 : \boldsymbol{\sigma}_{\varepsilon} \\ \boldsymbol{\varepsilon}_2 : \boldsymbol{\sigma}_{\varepsilon} \\ \boldsymbol{\varepsilon}_3 : \boldsymbol{\sigma}_{\varepsilon} \end{array} \right\} = \bar{\mathbf{0}} \quad (8.32)$$

Where:

$$\begin{aligned}
 \boldsymbol{\varepsilon}_1 &= [\boldsymbol{\varepsilon}_{1ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \\
 \boldsymbol{\varepsilon}_2 &= [\boldsymbol{\varepsilon}_{2ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \\
 \boldsymbol{\varepsilon}_3 &= [\boldsymbol{\varepsilon}_{3ij}]_{\substack{i=1,2,3 \\ j=1,2,3}}
 \end{aligned} \quad (8.33)$$

If the definition of the Levi-Civita symbol (A.21) is taken into account, the previous double dot products can be computed. The condition that the components of the stress tensor have to verify if the angular momentum conservation is fulfilled are then

obtained.

$$\varepsilon : \boldsymbol{\sigma}_\varepsilon = \left\{ \begin{array}{l} \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array} \right] : \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] \\ \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] : \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] : \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] \end{array} \right\} = \left\{ \begin{array}{l} \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{array} \right\} = \bar{\mathbf{0}} \implies \left\{ \begin{array}{l} \sigma_{23} = \sigma_{32} \\ \sigma_{13} = \sigma_{31} \\ \sigma_{12} = \sigma_{21} \end{array} \right. \quad (8.34)$$

Therefore, it can be concluded that, the Cauchy stress tensor has to be symmetric if the angular momentum conservation is fulfilled.

$$\boldsymbol{\sigma}_\varepsilon = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \implies \boldsymbol{\sigma}_\varepsilon^T = \boldsymbol{\sigma}_\varepsilon \quad (8.35)$$

8.5. Overview and conclusions

In Solid Mechanics, it is not usual to deal with mass sources, so the mass is usually considered to not vary. If the assumption of mass conservation is taken into account, the equation that defines the value of the density field over time can be obtained from the Lagrangian differential form of the mass balance equation (2.9). The density of the media turns out to depend on the initial density field and the determinant of the deformation gradient tensor.

Moreover, if the static equilibrium is analysed, the equilibrium of forces and torques has to be verified.

Under the assumptions of static equilibrium and mass conservation, the Eulerian differential form of the linear momentum balance equation (2.27) leads to the equation that rules the static equilibrium of forces. This equation can be equivalently obtained if the linear momentum conservation is taken into account. That is, the time derivative of the linear momentum has to be zero. The substitution of this time derivative into the Eulerian integral form of the linear momentum balance equation (2.30) allows to alternatively obtain the equation that governs the static equilibrium of forces.

In addition, it is considered that the material is not capable of absorbing angular momentum per unit volume. Therefore, the angular momentum conservation is verified, and the angular momentum time variation has to be null. The substitution of this time derivative into the Eulerian integral form of the angular momentum balance equation (2.34) leads to the symmetry of the Cauchy stress tensor. Consequently, it can be concluded that the Cauchy stress tensor has to be symmetric if the angular momentum conservation is fulfilled.

Linear finite element analysis

9.1. Introduction

The equations that govern the structural response of a given solid media subjected to external loads are considerably simplified if the assumptions made in linear theory are applied. In this chapter, the principles of the linear solid mechanics that allow to address a structural analysis by means of the linear finite element method are presented.

A wide historical overview of the finite element analysis can be checked in [Madier \[2020\]](#); [Liu et al. \[2022\]](#), among other references.

The main aim of this chapter is to state the hypotheses assumed in linear analysis, and to clearly identify their implications. The equations that compose the mathematical model that allows to simulate the structural response of a given solid media subjected to external loads are derived. Then, the finite element method is applied to obtain its structural behaviour.

Once this linear analysis is completely posed, the analogous nonlinear analysis is carried out in the following chapters. The main differences between both analysis are the assumptions made about the structural response. The nonlinear hypotheses considerably difficult the structural analysis.

9.2. Conceptual problem

Let's consider that the structural behaviour of a given solid subjected to a set of external forces is analysed. The main aim is to obtain the displacement field experimented by the solid after the application of the external forces, as well as the strain and stress fields corresponding to this displacement field.

9.3. Reference material domain

The reference material domain is Ω_0 , and its surface is divided into two subdomains (figure 9.1). The first one is the surface where the displacements are prescribed, and the second one is where the external surface loads are applied. The union of both subdomains generates the whole solid surface.

$$\partial\Omega_0 = \Gamma_0 = \Gamma_0^\sigma \cup \Gamma_0^u \quad (9.1)$$

The external surface forces may not be applied over the entire subdomain. In this particular case, this subdomain is, in turn, divided into two areas. A null surface load is considered where there are no surface forces.

Besides, the following position vector defines the initial position of a given material particle.

$$\mathbf{r}_0 \in \Omega_0 \quad (9.2)$$

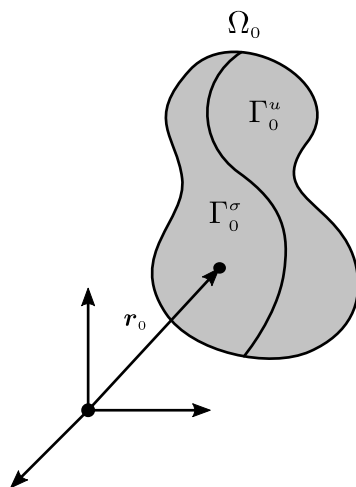


Figure 9.1. Reference material domain.

9.4. Linear analysis hypotheses

If the analysis of the structural behaviour is carried out in linear theory, two main hypotheses that considerably simplify the analysis are adopted. Nevertheless, if the real structural behaviour does not verify these assumptions, the results obtained with this theory do not correspond to the real structural response.

9.4.1. Small displacements

On the one hand, the displacements that the solid experiments are considered small.

$$\|\mathbf{u}_\varepsilon\| \ll 1 \quad (9.3)$$

This assumption allows to consider that the initial and final position of a given material particle are approximately equivalent, that is, the reference configuration and the deformed material domain can be considered coincident.

$$\left. \begin{array}{l} \mathbf{r}_{\mathcal{L}}(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_{\mathcal{L}}(\mathbf{r}_0) \\ \|\mathbf{u}_{\mathcal{L}}\| \ll 1 \end{array} \right\} \implies \mathbf{r}_{\mathcal{L}} \approx \mathbf{r}_0 \iff \Omega \approx \Omega_0 \quad (9.4)$$

This hypothesis simplifies the analysis, since it allows to apply the equilibrium equation over the reference material domain, which is well-known. Without this assumption, the equilibrium conditions have to be imposed over the deformed material domain, which is the unknown to solve. And the determinant of the deformation gradient tensor turns out to be equal to one, since the final and the initial material domains are considered to be equivalent.

$$F_{\mathcal{L}}(\mathbf{r}_0) = \frac{d\Omega}{d\Omega_0} = 1 \quad (9.5)$$

Furthermore, the Lagrangian and the Eulerian description of a given magnitude are equivalent. Hence, it is not necessary to add the subscripts $_{\mathcal{L}}$ and $_{\mathcal{E}}$ that specify the description adopted.

$$\psi_{\mathcal{E}}(\mathbf{r}) \approx \psi_{\mathcal{L}}(\mathbf{r}_0) = \psi(\mathbf{r}_0) \quad (9.6)$$

9.4.2. Small displacement gradients

On the other hand, the displacement gradients are also considered to be small.

$$\|\mathbf{J}\| \ll \|\mathbf{I}\| \quad (9.7)$$

This assumption allows to apply the approximate polar decomposition, presented in section 5.3, to decompose the deformation gradient tensor. If the displacement gradients are small, the deformation gradient tensor can be accurately decomposed by means of the displacement gradient tensor. Without this assumption, an eigenvalue problem has to be solved in order to obtain its polar decomposition, as exposed in section 4.3.1.

9.5. Density field

If the mass does not vary over time, the mass conservation equation has to be fulfilled, and the density variation over time was defined in (8.8) as shown below.

$$\rho_{\mathcal{L}}(\mathbf{r}_0, t) = \frac{\rho_{\mathcal{L}}(\mathbf{r}_0, 0)}{F_{\mathcal{L}}(\mathbf{r}_0, t)} \quad \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0, \forall t \quad (9.8)$$

Under the linear analysis assumptions, the determinant of the deformation gradient tensor turns out to be equal to one (9.5). Therefore, if a static linear analysis is carried out, the density field is considered to remain constant.

$$\rho(\mathbf{r}_0) = \frac{\rho^0(\mathbf{r}_0)}{F(\mathbf{r}_0)} = \rho^0(\mathbf{r}_0) \quad (9.9)$$

As the initial density field ρ^0 is known, the density field is not an unknown to calculate.

9.6. Static equilibrium equations

9.6.1. Equilibrium of forces

The equation that rules the static equilibrium of forces of a given solid subjected to external forces was defined in (8.11) as:

$$\mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}^T) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon} = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \mathring{\Omega} \quad (9.10)$$

If the linear analysis hypotheses are adopted, the initial and deformed material domain turn out to be equivalent (9.4), the Lagrangian and Eulerian descriptions of a given magnitude (9.6) become equivalent, and the density remains constant (9.9). Thus, the above equation is reduced to:

$$\mathbf{div}(\boldsymbol{\sigma}^T) + \mathbf{b} \rho^0 = \bar{\mathbf{0}} \quad \forall \mathbf{r}_0 \in \mathring{\Omega}_0 \quad (9.11)$$

9.6.2. Equilibrium of moments

As proved in section 8.4, the Cauchy stress tensor is symmetric if the angular momentum conservation is verified. That is, the sum of all torques is zero, and the equilibrium of moments is fulfilled.

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma} \quad (9.12)$$

9.7. Compatibility equation

The compatibility equation is the one that defines the strain field by means of the displacement field.

According to the approximate polar decomposition (section 5.3), the strain field can be approximated by the infinitesimal strain tensor (5.45), which is completely defined by means of the displacement gradient tensor as:

$$\mathbf{E} \approx \boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T) \quad (9.13)$$

Since the displacement gradient tensor fulfils condition (9.7), the strain field turns out to be infinitesimal.

$$\|\mathbf{E}\| = \mathcal{O}(\|\mathbf{J}\|) \ll \|\mathbf{I}\| \quad (9.14)$$

9.8. Constitutive equation

The equation that defines the relation between the stress and strain fields is the so-called constitutive equation.

Since the strain field is infinitesimal and the displacement gradients are small, the effect of the rotation on the definition of the Cauchy stress tensor can be neglected. And the stress tensor can be defined only by means of the strain tensor, as proved in section 7.4.4.

$$\boldsymbol{\sigma}(\boldsymbol{\mathcal{E}}) \approx \lambda \text{Tr}(\boldsymbol{\mathcal{E}})\mathbf{I} + 2\mu \boldsymbol{\mathcal{E}} \quad (9.15)$$

9.9. Boundary conditions

In this study, the displacement field and the value of the stress vector can be prescribed on a given portion of the solid surface. Therefore, there are two types of boundary conditions, which are described in the following sections.

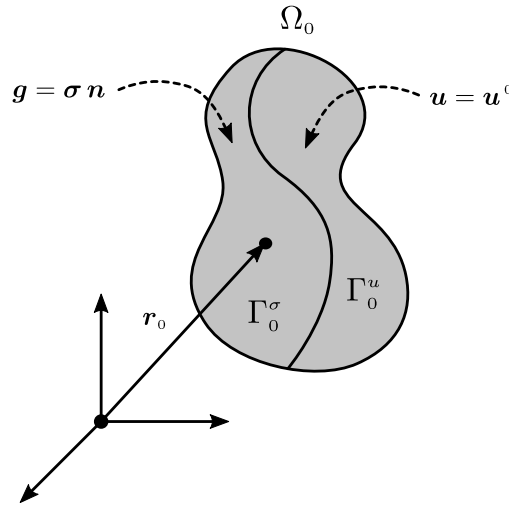


Figure 9.2. Reference material domain and boundary conditions.

9.9.1. Essential boundary conditions

The first ones are usually known as essential boundary conditions, which are based on the definition of the displacement field on a specific solid surface.

$$\mathbf{u}(\mathbf{r}_0) = \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.16)$$

9.9.2. Natural boundary conditions

Moreover, surface external loads are applied on the other portion of the solid surface. These second type are the so-called natural boundary conditions, and they are based

on the definition of the stress vector on the solid surface where the external loads are applied.

$$\mathbf{g}(\mathbf{r}_0) = \boldsymbol{\sigma}(\mathbf{r}_0) \mathbf{n}_0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^\sigma \quad (9.17)$$

It is shown later on in this chapter why they are usually known as natural boundary conditions.

9.10. Unknowns

The main unknown to solve is the displacement field that the solid experiments. Once this field is known, the compatibility equation (9.13) allows to define the strain field by means of the displacement field. And finally, the constitutive equation (9.15) defines the stress field corresponding to the previous strain field.

$$\mathbf{u}(\mathbf{r}_0) \implies \boldsymbol{\mathcal{E}}(\mathbf{u}(\mathbf{r}_0)) \implies \boldsymbol{\sigma}(\boldsymbol{\mathcal{E}}(\mathbf{u}(\mathbf{r}_0))) \quad (9.18)$$

In addition, once the stress field is defined, the reaction that appears on the surface where the essential boundary condition is applied can be computed, as shown below.

$$\mathbf{g}_R(\mathbf{r}_0) = \boldsymbol{\sigma}(\mathbf{r}_0) \mathbf{n}_0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.19)$$

9.11. Mathematical model

The solid medium is assumed to be homogenous and isotropic. That is, the mechanical properties are constant over the whole material domain, and they do not depend on the direction taken into account.

The solid is subjected to volumetric \mathbf{b} , and superficial \mathbf{g} forces. These forces are forces per unit mass and forces per unit area, respectively.

And the material mechanical properties are known. The Young's modulus is represented by E and the Poisson's ratio by ν . The Lamé's parameters (7.51) are defined in terms of the Young's modulus and the Poisson's ratio, as:

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)} \quad \mu = \frac{E}{2(1+\nu)} \quad (9.20)$$

The main aim is to obtain the displacement field that the solid experiments due to the application of the external loads, and its corresponding stress field.

$$\mathbf{u}(\mathbf{r}_0), \boldsymbol{\sigma}(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Omega_0 \quad (9.21)$$

That verify the following equations.

$$\begin{aligned}
 \operatorname{div}(\boldsymbol{\sigma}) + \mathbf{b} \rho^0 &= \bar{\mathbf{0}} & \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0 & \quad (\text{Equilibrium of forces (9.11) and moments (9.12)}) \\
 \boldsymbol{\sigma} &= 2\mu \boldsymbol{\mathcal{E}} + \lambda \operatorname{Tr}(\boldsymbol{\mathcal{E}}) \mathbf{I} & \mathbf{r}_0 \in \Omega_0 & \quad (\text{Constitutive equation (9.15)}) \\
 \boldsymbol{\mathcal{E}} &= \frac{1}{2} \left[\frac{d\mathbf{u}}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}}{d\mathbf{r}_0} \right)^T \right] & \mathbf{r}_0 \in \Omega_0 & \quad (\text{Compatibility equation (9.13)}) \\
 -\boldsymbol{\sigma} \mathbf{n}_0 + \mathbf{g} &= \bar{\mathbf{0}} & \forall \mathbf{r}_0 \in \Gamma_0^\sigma & \quad (\text{Natural boundary condition (9.17)}) \\
 \mathbf{u} &= \mathbf{u}^0 & \forall \mathbf{r}_0 \in \Gamma_0^u & \quad (\text{Essential boundary condition (9.16)})
 \end{aligned} \tag{9.22}$$

9.12. Strong form

In this section, the original form of the problem is summed up, which is usually known as the strong form.

The main goal of this analysis is to obtain the displacement field experimented by the solid, after the application of the external loads.

$$\mathbf{u}(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Omega_0 \tag{9.23}$$

The above displacement field verifies the following equations, expressed in terms of the equilibrium equation residual \mathbf{R}_σ and the natural boundary condition residual \mathbf{R}_Γ .

$$\begin{aligned}
 \mathbf{R}_\sigma &= \bar{\mathbf{0}} & \forall \mathbf{r}_0 \in \overset{\circ}{\Omega}_0 & \\
 \mathbf{R}_\Gamma &= \bar{\mathbf{0}} & \forall \mathbf{r}_0 \in \Gamma_0^\sigma &
 \end{aligned} \tag{9.24}$$

Where:

$$\begin{aligned}
 \mathbf{R}_\sigma &= \operatorname{div}(\boldsymbol{\sigma}) + \mathbf{b} \rho^0 & \mathbf{r}_0 \in \overset{\circ}{\Omega}_0 & \\
 \boldsymbol{\sigma} &= 2\mu \boldsymbol{\mathcal{E}} + \lambda \operatorname{Tr}(\boldsymbol{\mathcal{E}}) \mathbf{I} & \mathbf{r}_0 \in \Omega_0 & \\
 \boldsymbol{\mathcal{E}} &= \frac{1}{2} \left[\frac{d\mathbf{u}}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}}{d\mathbf{r}_0} \right)^T \right] & \mathbf{r}_0 \in \Omega_0 & \\
 \mathbf{R}_\Gamma &= -\boldsymbol{\sigma} \mathbf{n}_0 + \mathbf{g} & \mathbf{r}_0 \in \Gamma_0^\sigma & \\
 \mathbf{u} &= \mathbf{u}^0 & \forall \mathbf{r}_0 \in \Gamma_0^u &
 \end{aligned} \tag{9.25}$$

9.13. Weak form

In order to obtain the weak form of the previous strong form, the weighted residual method exposed in section C.3 is applied. The displacement field (9.23) fulfils now the following equation:

$$\iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{R}_\sigma d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{R}_\Gamma d\Gamma_0 = 0 \quad \forall \boldsymbol{\omega} \in H_w \tag{9.26}$$

Where $\boldsymbol{\omega}$ are suitable test functions, and the residuals \mathbf{R}_σ and \mathbf{R}_Γ were defined in (9.25).

9.13.1. Equivalent weak form derivation

If the property of the divergence proved in (B.30) is taken into account, it can be stated that:

$$\operatorname{div}(\boldsymbol{\sigma}^T \boldsymbol{\omega}) = \boldsymbol{\omega}^T \operatorname{div}(\boldsymbol{\sigma}) + \operatorname{Tr}\left(\boldsymbol{\sigma}^T \frac{d\boldsymbol{\omega}}{d\mathbf{r}_0}\right) \quad (9.27)$$

In the above equation, $\boldsymbol{\omega}$ is a vector test function and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. But the stress tensor is symmetric, as demonstrated in (8.35). Hence, the above equation becomes:

$$\operatorname{div}(\boldsymbol{\sigma} \boldsymbol{\omega}) = \boldsymbol{\omega}^T \operatorname{div}(\boldsymbol{\sigma}) + \operatorname{Tr}\left(\boldsymbol{\sigma} \frac{d\boldsymbol{\omega}}{d\mathbf{r}_0}\right) \quad (9.28)$$

The integration of the above equation over the deformed material domain leads to:

$$\iiint_{\Omega_0} \operatorname{div}(\boldsymbol{\sigma} \boldsymbol{\omega}) d\Omega_0 = \iiint_{\Omega_0} \boldsymbol{\omega}^T \operatorname{div}(\boldsymbol{\sigma}) d\Omega_0 + \iiint_{\Omega_0} \operatorname{Tr}\left(\boldsymbol{\sigma} \frac{d\boldsymbol{\omega}}{d\mathbf{r}_0}\right) d\Omega_0 \quad (9.29)$$

The left-hand side of this integral equation can be equivalently expressed as a surface integral, if the divergence theorem (section B.7.1) is applied.

$$\begin{aligned} \iiint_{\Omega_0} \operatorname{div}(\boldsymbol{\sigma} \boldsymbol{\omega}) d\Omega_0 &= \iint_{\Gamma_0} (\boldsymbol{\sigma} \boldsymbol{\omega})^T \mathbf{n}_0 d\Gamma_0 \\ &= \iint_{\Gamma_0} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 \end{aligned} \quad (9.30)$$

If the above equation is substituted into (9.29), and the property (A.79) is taken into account, equation (9.29) can be rewritten as:

$$\iiint_{\Omega_0} \boldsymbol{\omega}^T \operatorname{div}(\boldsymbol{\sigma}) d\Omega_0 = \iint_{\Gamma_0} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 - \iiint_{\Omega_0} \operatorname{Tr}\left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma}\right) d\Omega_0 \quad (9.31)$$

The substitution of the above equation into the first term of the weak form (9.26) leads to:

$$\begin{aligned} \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{R}_\sigma d\Omega_0 &= \iiint_{\Omega_0} \boldsymbol{\omega}^T [\operatorname{div}(\boldsymbol{\sigma}) + \mathbf{b} \rho^0] d\Omega_0 \\ &= \iiint_{\Omega_0} \boldsymbol{\omega}^T \operatorname{div}(\boldsymbol{\sigma}) d\Omega_0 + \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 \\ &= \left[\iint_{\Gamma_0} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 - \iiint_{\Omega_0} \operatorname{Tr}\left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma}\right) d\Omega_0 \right] + \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 \\ &= \left[\iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 \right] - \\ &\quad - \iiint_{\Omega_0} \operatorname{Tr}\left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma}\right) d\Omega_0 + \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 \end{aligned} \quad (9.32)$$

If the above result is now substituted into (9.26), the weak form becomes:

$$\begin{aligned}
 \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{R}_\sigma d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{R}_\Gamma d\Gamma_0 &= \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}^T (\boldsymbol{\sigma} \mathbf{n}_0) d\Gamma_0 - \\
 &\quad - \iiint_{\Omega_0} \text{Tr} \left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 + \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 + \\
 &\quad + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T (-\boldsymbol{\sigma} \mathbf{n}_0 + \mathbf{g}) d\Gamma_0 = \\
 &= - \iiint_{\Omega_0} \text{Tr} \left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 + \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 + \\
 &\quad + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}^T \underbrace{(\boldsymbol{\sigma} \mathbf{n}_0)}_{\mathbf{g}_R} d\Gamma_0 = 0
 \end{aligned} \tag{9.33}$$

Where \mathbf{g}_R is the reaction that appears on the surface where the essential boundary condition is applied.

$$\mathbf{g}_R = \boldsymbol{\sigma} \mathbf{n}_0 \quad \forall \mathbf{r}_0 \in \Gamma_0^u \tag{9.34}$$

Therefore, the equation that defines the equivalent weak form can be obtained from (9.33) as:

$$\iiint_{\Omega_0} \text{Tr} \left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 = \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}^T \mathbf{g}_R d\Gamma_0 \tag{9.35}$$

And an equivalent weak form can be stated, as shown in the following subsection.

9.13.2. Equivalent weak form statement

The main aim of the linear analysis is to obtain the displacement field that verifies the essential boundary condition. In addition, the value of the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}(\mathbf{r}_0) = \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \tag{9.36}$$

$$\mathbf{g}_R(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \tag{9.37}$$

The above unknowns verify that:

$$\begin{aligned}
 &\iiint_{\Omega_0} \text{Tr} \left(\frac{d\boldsymbol{\omega}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 = \\
 &= \iiint_{\Omega_0} \boldsymbol{\omega}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}^T \mathbf{g}_R d\Gamma_0 \quad \forall \boldsymbol{\omega} \in H_\omega
 \end{aligned} \tag{9.38}$$

where:

$$\begin{aligned}
 \boldsymbol{\sigma} &= 2\mu \boldsymbol{\mathcal{E}} + \lambda \text{Tr}(\boldsymbol{\mathcal{E}}) \mathbf{I} \quad \mathbf{r}_0 \in \Omega_0 \\
 \boldsymbol{\mathcal{E}} &= \frac{1}{2} \left[\frac{d\mathbf{u}}{d\mathbf{r}_0} + \left(\frac{d\mathbf{u}}{d\mathbf{r}_0} \right)^T \right] \quad \mathbf{r}_0 \in \Omega_0
 \end{aligned} \tag{9.39}$$

The role of the natural boundary condition in the previous weak form is discussed below.

9.13.3. Effect of the natural boundary condition

The natural boundary condition is introduced into the weak form (9.38) by the following term:

$$\iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{g} \, d\Gamma_0 \quad (9.40)$$

If there is no surface load, the boundary condition and its corresponding term become:

$$\mathbf{g} = \boldsymbol{\sigma} \mathbf{n}_0 = \bar{\mathbf{0}} \quad \forall \mathbf{r}_0 \in \Gamma_0^\sigma \quad \implies \quad \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}^T \mathbf{g} \, d\Gamma_0 = 0 \quad (9.41)$$

Thus, its effect on the weak form automatically disappears, and the boundary condition seems to be satisfied naturally. This is why they are usually known as natural boundary conditions.

9.13.4. Alternative left-hand side

Let's consider the test functions as a variation of the displacement field. This variation has to be compatible with the essential boundary condition.

$$\mathbf{w} = \delta \mathbf{u} \quad (9.42)$$

The new displacement field turns out to be:

$$\mathbf{u}' = \mathbf{u} + \delta \mathbf{u} \quad (9.43)$$

And its corresponding displacement gradient tensor is:

$$\mathbf{J}' = \frac{d\mathbf{u}'}{d\mathbf{r}_0} = \frac{d}{d\mathbf{r}_0}(\mathbf{u} + \delta \mathbf{u}) = \frac{d\mathbf{u}}{d\mathbf{r}_0} + \frac{d\delta \mathbf{u}}{d\mathbf{r}_0} = \mathbf{J} + \delta \mathbf{J} \quad (9.44)$$

Thus, the displacement gradient tensor variation is:

$$\delta \mathbf{J} = \frac{d\delta \mathbf{u}}{d\mathbf{r}_0} \quad (9.45)$$

The gradients of the displacement field variation are also assumed to be small.

$$\|\delta \mathbf{J}\| \ll \|\mathbf{I}\| \quad (9.46)$$

On the other hand, the infinitesimal strain tensor corresponding to the displacement field (9.43) is defined as:

$$\begin{aligned} \boldsymbol{\varepsilon}' &= \frac{1}{2} [\mathbf{J}' + (\mathbf{J}')^T] \\ &= \frac{1}{2} [(\mathbf{J} + \delta \mathbf{J}) + (\mathbf{J} + \delta \mathbf{J})^T] \\ &= \frac{1}{2} (\mathbf{J} + \mathbf{J}^T) + \frac{1}{2} (\delta \mathbf{J} + \delta \mathbf{J}^T) \\ &= \boldsymbol{\varepsilon} + \delta \boldsymbol{\varepsilon} \end{aligned} \quad (9.47)$$

Therefore, the infinitesimal strain tensor variation is defined as:

$$\delta \boldsymbol{\mathcal{E}} = \frac{1}{2} (\delta \boldsymbol{\mathcal{J}} + \delta \boldsymbol{\mathcal{J}}^T) \quad (9.48)$$

In addition, the displacement gradient tensor variation can be decomposed as the sum of a symmetric and a skew-symmetric tensor.

$$\delta \boldsymbol{\mathcal{J}} = \delta \boldsymbol{\mathcal{E}} + \delta \boldsymbol{\mathcal{W}} \quad \text{with} \quad \begin{cases} \delta \boldsymbol{\mathcal{E}} = \frac{1}{2} (\delta \boldsymbol{\mathcal{J}} + \delta \boldsymbol{\mathcal{J}}^T) = [\delta \mathcal{E}_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \delta \boldsymbol{\mathcal{W}} = \frac{1}{2} (\delta \boldsymbol{\mathcal{J}} - \delta \boldsymbol{\mathcal{J}}^T) = [\delta \mathcal{W}_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{cases} \quad (9.49)$$

The symmetric term turns out to be equivalent to the infinitesimal strain tensor variation (9.48).

$$\delta \boldsymbol{\mathcal{E}}^T = \delta \boldsymbol{\mathcal{E}} \iff \delta \mathcal{E}_{ji} = \delta \mathcal{E}_{ij} \quad (9.50)$$

And the components of the skew-symmetric one verify that:

$$\delta \boldsymbol{\mathcal{W}}^T = -\delta \boldsymbol{\mathcal{W}} \iff \delta \mathcal{W}_{ji} = \begin{cases} -\delta \mathcal{W}_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \quad (9.51)$$

Let's focus now on the left-hand side of the weak form (9.38). If the test functions are considered as a compatible variation of the displacement field, the left-hand side becomes:

$$\iiint_{\Omega_0} \text{Tr} \left(\frac{d\delta \mathbf{u}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 \quad (9.52)$$

An equivalent expression of the above integrand can be obtained, if the property of the trace operator stated in (A.81) and the symmetry of the Cauchy stress tensor are applied, as well as the decomposition of the displacement gradient tensor variation obtained in (9.49). Furthermore, it is recalled that the double dot product between a skew-symmetric tensor and a symmetric one is equal to zero.

$$\text{Tr} \left(\frac{d\delta \mathbf{u}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) = \delta \boldsymbol{\mathcal{J}} : \boldsymbol{\sigma}^T = (\delta \boldsymbol{\mathcal{E}} + \delta \boldsymbol{\mathcal{W}}) : \boldsymbol{\sigma} = \delta \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma} + \underbrace{\delta \boldsymbol{\mathcal{W}} : \boldsymbol{\sigma}}_{=0} = \delta \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma} \quad (9.53)$$

The integration of the above result over the reference material domain leads to an equivalent left-hand side of the weak form.

$$\iiint_{\Omega_0} \text{Tr} \left(\frac{d\delta \mathbf{u}}{d\mathbf{r}_0} \boldsymbol{\sigma} \right) d\Omega_0 = \iiint_{\Omega_0} \delta \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma} d\Omega_0 \quad (9.54)$$

9.13.5. Switch to vector notation

In order to move from tensor to vector notation, the Voigt notation (section A.14) is now applied. The vectorial form of the infinitesimal strain tensor variation is:

$$\delta \mathcal{E} = \begin{bmatrix} \delta \mathcal{E}_{11} & \delta \mathcal{E}_{12} & \delta \mathcal{E}_{13} \\ \delta \mathcal{E}_{12} & \delta \mathcal{E}_{22} & \delta \mathcal{E}_{23} \\ \delta \mathcal{E}_{13} & \delta \mathcal{E}_{23} & \delta \mathcal{E}_{33} \end{bmatrix} \implies \delta \bar{\mathcal{E}} = \begin{Bmatrix} \delta \mathcal{E}_{11} \\ \delta \mathcal{E}_{22} \\ \delta \mathcal{E}_{33} \\ 2 \delta \mathcal{E}_{12} \\ 2 \delta \mathcal{E}_{13} \\ 2 \delta \mathcal{E}_{23} \end{Bmatrix} \quad (9.55)$$

And the vector definition of the Cauchy stress tensor is:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \implies \bar{\boldsymbol{\sigma}} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \quad (9.56)$$

According to the above definitions, the double dot product between the infinitesimal strain tensor variation and the Cauchy stress tensor is equivalent to the scalar product of their corresponding vector definitions.

$$\delta \mathcal{E} : \boldsymbol{\sigma} = \delta \bar{\mathcal{E}}^T \bar{\boldsymbol{\sigma}} \quad (9.57)$$

Therefore, the left-hand side of the weak form (9.54) can be reduced to:

$$\iiint_{\Omega_0} \delta \mathcal{E} : \boldsymbol{\sigma} \, d\Omega_0 = \iiint_{\Omega_0} \delta \bar{\mathcal{E}}^T \bar{\boldsymbol{\sigma}} \, d\Omega_0 \quad (9.58)$$

Furthermore, the vectorial form of the infinitesimal strain tensor variation was defined in (5.76), as:

$$\delta \bar{\mathcal{E}} = \mathbf{L}_0 \delta \mathbf{u} \quad (9.59)$$

So, the substitution of the above definition into (9.58) leads to:

$$\iiint_{\Omega_0} \delta \bar{\mathcal{E}}^T \bar{\boldsymbol{\sigma}} \, d\Omega_0 = \iiint_{\Omega_0} (\mathbf{L}_0 \delta \mathbf{u})^T \bar{\boldsymbol{\sigma}} \, d\Omega_0 \quad (9.60)$$

On the other hand, the equivalent vectorial forms of the infinitesimal strain tensor and the Cauchy stress tensor were defined in (5.72) and (7.86), respectively, as:

$$\begin{aligned} \bar{\boldsymbol{\sigma}} &= \mathbf{D}_2 \bar{\mathcal{E}} \\ \bar{\mathcal{E}} &= \mathbf{L}_0 \mathbf{u} \end{aligned} \quad (9.61)$$

9.13.6. Alternative weak form in vector notation

The main goal is to obtain the displacement field that verifies the essential boundary condition. Moreover, the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}(\mathbf{r}_0) = \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.62)$$

$$\mathbf{g}_R(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.63)$$

The above unknowns verify the following weak form:

$$\begin{aligned} & \iiint_{\Omega_0} (\mathbf{L}_0 \delta \mathbf{u})^T \bar{\boldsymbol{\sigma}} d\Omega_0 = \\ & = \iiint_{\Omega_0} \delta \mathbf{u}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}^T \mathbf{g}_R d\Gamma_0 \quad \forall \delta \mathbf{u} \in H_{\delta u} \end{aligned} \quad (9.64)$$

Where the vectorial form of the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ written by means of the vectorial form of the infinitesimal strain tensor $\bar{\boldsymbol{\mathcal{E}}}$ (constitutive equation), and the vectorial form of the infinitesimal strain tensor $\bar{\boldsymbol{\mathcal{E}}}$ expressed in terms of the displacement field \mathbf{u} (compatibility equation), were defined in (9.61).

9.14. Trial functions basis

The trial functions basis is defined as the set composed by the following functions.

$$\{\phi_i(\mathbf{r}_0)\}_{i=1,\dots,\eta} \quad (9.65)$$

These functions allow to state an approximation to the unknown displacement field. This approach is defined as the sum of an initial approximation, plus a correction term that modifies this initial value.

$$\mathbf{u}(\mathbf{r}_0) \approx \mathbf{u}^h(\mathbf{r}_0) = \underbrace{\mathbf{u}_p(\mathbf{r}_0)}_{\text{approximation}} + \underbrace{\Delta \mathbf{u}_p(\mathbf{r}_0)}_{\text{correction}} \quad \mathbf{u}^h \in H_u^h \subset H_u \quad (9.66)$$

The above decomposition allows taking into account an initial approximation to the displacement field. In case there is no initial approximation, this term becomes null, and the correction term will completely approximate the displacement field.

Both terms are defined as a linear combination of the trial functions, as stated in the following sections.

9.14.1. Initial displacement field approximation

On the one hand, the initial approximation is user defined, so it is completely known. This term is defined as a linear combination of the functions that compose the trial

functions basis. The coefficients that define this linear combination are known.

$$\begin{aligned}
 \mathbf{u}_p(\mathbf{r}_0) &= \sum_{i=1}^{\eta} \underbrace{[\phi_i(\mathbf{r}_0) \mathbf{I}]}_{\phi_i(\mathbf{r}_0)} \boldsymbol{\alpha}_i \\
 &= \sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) \boldsymbol{\alpha}_i \\
 &= [\phi_1(\mathbf{r}_0) \quad \cdots \quad \phi_{\eta}(\mathbf{r}_0)] \begin{Bmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_{\eta} \end{Bmatrix} \\
 &= \boldsymbol{\phi}(\mathbf{r}_0) \boldsymbol{\alpha}
 \end{aligned} \tag{9.67}$$

Once this approximation is defined, its corresponding strain and stress fields can be computed. The definitions of both fields were exposed in (9.61). The strain field corresponding to the initial displacement field approximation is:

$$\bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) = \mathbf{L}_0 \mathbf{u}_p(\mathbf{r}_0) = \mathbf{L}_0 \boldsymbol{\phi}(\mathbf{r}_0) \boldsymbol{\alpha} \tag{9.68}$$

And its corresponding stress field turns out to be:

$$\bar{\boldsymbol{\sigma}}_p(\mathbf{r}_0) = \mathbf{D}_2 \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) = \mathbf{D}_2 \mathbf{L}_0 \boldsymbol{\phi}(\mathbf{r}_0) \boldsymbol{\alpha} \tag{9.69}$$

9.14.2. Displacement field correction

On the other hand, the correction term is also defined as a linear combination of trial functions. However, the coefficients which go with each trial function are unknown. The main goal is to obtain these coefficients.

$$\begin{aligned}
 \Delta \mathbf{u}_p(\mathbf{r}_0) &= \sum_{i=1}^{\eta} \underbrace{[\phi_i(\mathbf{r}_0) \mathbf{I}]}_{\phi_i(\mathbf{r}_0)} \Delta \boldsymbol{\alpha}_i \\
 &= \sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) \Delta \boldsymbol{\alpha}_i \\
 &= [\phi_1(\mathbf{r}_0) \quad \cdots \quad \phi_{\eta}(\mathbf{r}_0)] \begin{Bmatrix} \Delta \boldsymbol{\alpha}_1 \\ \vdots \\ \Delta \boldsymbol{\alpha}_{\eta} \end{Bmatrix} \\
 &= \boldsymbol{\phi}(\mathbf{r}_0) \Delta \boldsymbol{\alpha}
 \end{aligned} \tag{9.70}$$

Its corresponding strain field is:

$$\Delta \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) = \mathbf{L}_0 \Delta \mathbf{u}_p(\mathbf{r}_0) = \mathbf{L}_0 \boldsymbol{\phi}(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \tag{9.71}$$

And the stress field, in terms of the above strain field, is:

$$\Delta \bar{\boldsymbol{\sigma}}_p(\mathbf{r}_0) = \mathbf{D}_2 \Delta \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) = \mathbf{D}_2 \mathbf{L}_0 \boldsymbol{\phi}(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \tag{9.72}$$

9.14.3. Displacement, strain, and stress field approximations

Once the initial approximation (9.67) is completely defined, the correction term (9.70) is added to complete the displacement field approximation.

$$\mathbf{u}(\mathbf{r}_0) \approx \mathbf{u}^h(\mathbf{r}_0) = \mathbf{u}_p(\mathbf{r}_0) + \Delta \mathbf{u}_p(\mathbf{r}_0) = \mathbf{u}_p(\mathbf{r}_0) + \phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \quad (9.73)$$

Both the initial approximation and the corrector term have their corresponding strain fields, defined in (9.68) and (9.71), respectively. Thus, the strain field approximation is defined as the sum of both contributions.

$$\bar{\boldsymbol{\epsilon}}(\mathbf{r}_0) \approx \bar{\boldsymbol{\epsilon}}^h(\mathbf{r}_0) = \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) + \Delta \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) = \bar{\boldsymbol{\epsilon}}_p(\mathbf{r}_0) + \mathbf{L}_0 \phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \quad (9.74)$$

The stress field approximation is also composed by the contributions of both terms. The stress field corresponding to the initial approximation is obtained in (9.69), and the one corresponding to the correction term in (9.72).

$$\bar{\boldsymbol{\sigma}}(\mathbf{r}_0) \approx \bar{\boldsymbol{\sigma}}^h(\mathbf{r}_0) = \bar{\boldsymbol{\sigma}}_p(\mathbf{r}_0) + \Delta \bar{\boldsymbol{\sigma}}_p(\mathbf{r}_0) = \bar{\boldsymbol{\sigma}}_p(\mathbf{r}_0) + \mathbf{D}_2 \mathbf{L}_0 \phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \quad (9.75)$$

Substitution in the weak form

The difference between the real displacement field and the initial approximation ($\mathbf{u} - \mathbf{u}_p$) represents the term that the corrector has to achieve to obtain the exact solution.

In general, the trial functions are not able to generate this difference. Thus, the displacement field approximation (9.73) does not turn out to be equivalent to the real displacement field.

$$\mathbf{u}^h(\mathbf{r}_0) \neq \mathbf{u}(\mathbf{r}_0) \quad (9.76)$$

Consequently, as a general rule, the weak form (9.64) is not verified.

$$\iiint_{\Omega_0} (\mathbf{L}_0 \delta \mathbf{u})^T \bar{\boldsymbol{\sigma}}^h d\Omega_0 \neq \iiint_{\Omega_0} \delta \mathbf{u}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}^T \mathbf{g}_R^h d\Gamma_0 \quad (9.77)$$

Nevertheless, if ($\mathbf{u} - \mathbf{u}_p$) is contained in the subspace generated by the trial functions, the weak form is automatically fulfilled. But, in general, the weak form is not strictly verified.

From now on, a displacement field that does verify the weak form should be considered. To circumvent this inconvenience, a discretized subspace of test functions is adopted, that allows to approximate the test functions as a linear combination of the functions that compose this subspace. This methodology is presented in the following section.

9.15. Test functions approximation

The following subspace is the one taken into account to build the test function approximation.

$$\{\delta u_j(\mathbf{r}_0)\}_{j=1, \dots, \eta} \quad (9.78)$$

And the test functions are defined as the linear combination of the functions that compose the above subspace.

$$\begin{aligned}
 \delta \mathbf{u}(\mathbf{r}_0) &\approx \delta \mathbf{u}^h(\mathbf{r}_0) = \sum_{j=1}^{\eta} \underbrace{[\delta u_j(\mathbf{r}_0) \mathbf{I}]}_{\Omega_j(\mathbf{r}_0)} \beta_j \\
 &= \sum_{j=1}^{\eta} \Omega_j(\mathbf{r}_0) \beta_j \\
 &= [\Omega_1(\mathbf{r}_0) \quad \cdots \quad \Omega_{\eta}(\mathbf{r}_0)] \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{\eta} \end{Bmatrix} \\
 &= \mathbf{\Omega}(\mathbf{r}_0) \boldsymbol{\beta} \quad \delta \mathbf{u}^h \in H_{\delta u}^h \subset H_{\delta u}
 \end{aligned} \tag{9.79}$$

9.15.1. Weak form approximation

The objective of the linear solid analysis is to obtain an approximation to the displacement field that verifies the essential boundary condition. Moreover, an approximation to the reaction that arises on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}^h(\mathbf{r}_0) \in H_u^h \quad | \quad \mathbf{u}^h(\mathbf{r}_0) = \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \tag{9.80}$$

$$\mathbf{g}_R^h(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Gamma_0^u \tag{9.81}$$

The above unknowns verify the following weak form:

$$\begin{aligned}
 &\iiint_{\Omega_0} (\mathbf{L}_0 \delta \mathbf{u}^h)^T \bar{\boldsymbol{\sigma}}^h d\Omega_0 = \\
 &= \iiint_{\Omega_0} (\delta \mathbf{u}^h)^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^{\sigma}} (\delta \mathbf{u}^h)^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} (\delta \mathbf{u}^h)^T \mathbf{g}_R^h d\Gamma_0 \quad \forall \delta \mathbf{u}^h \in H_{\delta u}^h
 \end{aligned} \tag{9.82}$$

9.15.2. Resulting system of linear equations

The test functions were approximated in the previous section as:

$$\delta \mathbf{u}^h = \mathbf{\Omega} \boldsymbol{\beta} \tag{9.83}$$

Consequently, the weak form (9.82) becomes:

$$\boldsymbol{\beta}^T \left[\iiint_{\Omega_0} (\mathbf{L}_0 \mathbf{\Omega})^T \bar{\boldsymbol{\sigma}}^h d\Omega_0 - \iiint_{\Omega_0} \mathbf{\Omega}^T \mathbf{b} \rho^0 d\Omega_0 - \iint_{\Gamma_0^{\sigma}} \mathbf{\Omega}^T \mathbf{g} d\Gamma_0 - \iint_{\Gamma_0^u} \mathbf{\Omega}^T \mathbf{g}_R^h d\Gamma_0 \right] = 0 \quad \forall \boldsymbol{\beta} \tag{9.84}$$

Or equivalently:

$$\iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega})^T \bar{\boldsymbol{\sigma}}^h d\Omega_0 = \iiint_{\Omega_0} \boldsymbol{\Omega}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\Omega}^T \mathbf{g}_R^h d\Gamma_0 \quad (9.85)$$

Furthermore, the stress field approximation was defined in (9.75) as:

$$\bar{\boldsymbol{\sigma}}^h = \bar{\boldsymbol{\sigma}}_p + \mathbf{D}_2 \mathbf{L}_0 \phi \Delta \boldsymbol{\alpha} \quad (9.86)$$

The substitution of the above definition into the weak form (9.85) leads to:

$$\begin{aligned} & \left[\iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega})^T \mathbf{D}_2 (\mathbf{L}_0 \phi) d\Omega_0 \right] \Delta \boldsymbol{\alpha} = \\ & = \iiint_{\Omega_0} \boldsymbol{\Omega}^T \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}^T \mathbf{g} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\Omega}^T \mathbf{g}_R^h d\Gamma_0 - \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega})^T \bar{\boldsymbol{\sigma}}_p d\Omega_0 \end{aligned} \quad (9.87)$$

The above equation can be equivalently written as the following system of linear equations.

$$\begin{aligned} \mathbf{K} \Delta \boldsymbol{\alpha} &= \mathbf{f} + \iint_{\Gamma_0^u} \boldsymbol{\Omega}^T \mathbf{g}_R^h d\Gamma_0 \\ \mathbf{K} &= \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega})^T \mathbf{D}_2 (\mathbf{L}_0 \phi) d\Omega_0 \\ \mathbf{f} &= \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}^T \mathbf{g} d\Gamma_0 + \iiint_{\Omega_0} \left[\boldsymbol{\Omega}^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega})^T \bar{\boldsymbol{\sigma}}_p \right] d\Omega_0 \end{aligned} \quad (9.88)$$

Therefore, the original equilibrium problem has been substituted by the above system of linear equations. The order of the linear system is η , as there are η unknowns.

$$\Delta \boldsymbol{\alpha} = \{ \Delta \boldsymbol{\alpha}_i \}_{i=1, \dots, \eta} \quad (9.89)$$

But the reaction \mathbf{g}_R^h is also an unknown. Hence, there are actually $\eta + 1$ unknowns. Fortunately, this inconvenience can be overcome if the essential boundary condition is considered as an additional equation.

The application of the essential boundary condition leads to the following equation.

$$\left. \begin{aligned} \mathbf{u}(\mathbf{r}_0) &\approx \mathbf{u}_p(\mathbf{r}_0) + \phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \\ \mathbf{u}(\mathbf{r}_0) &= \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \end{aligned} \right\} \implies \mathbf{u}^0(\mathbf{r}_0) = \mathbf{u}_p(\mathbf{r}_0) + \phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.90)$$

The additional equation can be equivalently written as shown below.

$$\phi(\mathbf{r}_0) \Delta \boldsymbol{\alpha} = \mathbf{u}^0(\mathbf{r}_0) - \mathbf{u}_p(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.91)$$

9.15.3. Simplification of the system of linear equations

If the initial approximation fulfils the essential boundary condition, and the trial functions are null over the surface where this boundary condition is applied, the displacement field approximation automatically verifies the essential boundary condition.

$$\left. \begin{aligned} \mathbf{u}^h(\mathbf{r}_0) &= \mathbf{u}_p(\mathbf{r}_0) + \boldsymbol{\phi}(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \\ \mathbf{u}_p(\mathbf{r}_0) &= \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \\ \boldsymbol{\phi}(\mathbf{r}_0) &= \mathbf{0} \quad \forall \mathbf{r}_0 \in \Gamma_0^u \end{aligned} \right\} \implies \mathbf{u}^h(\mathbf{r}_0) = \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (9.92)$$

Moreover, if the test functions are null over the surface where the essential boundary condition is applied, the reaction is not involved in the formulation.

$$\boldsymbol{\Omega}(\mathbf{r}_0) = \mathbf{0} \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad \implies \iint_{\Gamma_0^u} \boldsymbol{\Omega}^T \mathbf{g}_R^h d\Gamma_0 = \bar{\mathbf{0}} \quad (9.93)$$

Thus, the linear system (9.88) to be solved is reduced to:

$$\mathbf{K} \Delta \boldsymbol{\alpha} = \mathbf{f} \quad (9.94)$$

Where the vector composed by the unknowns is:

$$\Delta \boldsymbol{\alpha} = \{\Delta \alpha_i\}_{i=1, \dots, \eta} \quad (9.95)$$

And the matrix and the independent vector are:

$$\begin{aligned} \mathbf{K} &= \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega})^T \mathbf{D}_2 (\mathbf{L}_0 \boldsymbol{\phi}) d\Omega_0 \\ \mathbf{f} &= \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}^T \mathbf{g} d\Gamma_0 + \iiint_{\Omega_0} [\boldsymbol{\Omega}^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega})^T \bar{\boldsymbol{\sigma}}_p] d\Omega_0 \end{aligned} \quad (9.96)$$

The definition of the tensors $\boldsymbol{\phi}$ and $\boldsymbol{\Omega}$ were stated in (9.70) and (9.79), respectively.

$$\begin{aligned} \boldsymbol{\Omega} &= [\boldsymbol{\Omega}_1 \quad \dots \quad \boldsymbol{\Omega}_\eta] \\ \boldsymbol{\phi} &= [\boldsymbol{\phi}_1 \quad \dots \quad \boldsymbol{\phi}_\eta] \end{aligned} \quad (9.97)$$

The substitution of the above tensors into the definition of the matrix and independent vector (9.96) leads to:

$$\begin{aligned} \mathbf{K} &= \iiint_{\Omega_0} \begin{bmatrix} (\mathbf{L}_0 \boldsymbol{\Omega}_1)^T \\ \vdots \\ (\mathbf{L}_0 \boldsymbol{\Omega}_\eta)^T \end{bmatrix} \mathbf{D}_2 [\mathbf{L}_0 \boldsymbol{\phi}_1 \quad \dots \quad \mathbf{L}_0 \boldsymbol{\phi}_\eta] d\Omega_0 \\ \mathbf{f} &= \iiint_{\Omega_0} \begin{bmatrix} \boldsymbol{\Omega}_1^T \\ \vdots \\ \boldsymbol{\Omega}_\eta^T \end{bmatrix} \mathbf{b} \rho^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \begin{bmatrix} \boldsymbol{\Omega}_1^T \\ \vdots \\ \boldsymbol{\Omega}_\eta^T \end{bmatrix} \mathbf{g} d\Gamma_0 - \iiint_{\Omega_0} \begin{bmatrix} (\mathbf{L}_0 \boldsymbol{\Omega}_1)^T \\ \vdots \\ (\mathbf{L}_0 \boldsymbol{\Omega}_\eta)^T \end{bmatrix} \bar{\boldsymbol{\sigma}}_p d\Omega_0 \end{aligned} \quad (9.98)$$

Therefore, their components can be defined as:

$$\begin{aligned}
 \mathbf{K} &= [\mathbf{K}_{ij}]_{\substack{j=1,\dots,\eta \\ i=1,\dots,\eta}} & \mathbf{K}_{ij} &= \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \\
 \mathbf{f} &= \{\mathbf{f}_i\}_{j=1,\dots,\eta} & \mathbf{f}_i &= \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}_i^T \mathbf{g} d\Gamma_0 + \iiint_{\Omega_0} [\boldsymbol{\Omega}_i^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \bar{\boldsymbol{\sigma}}_p] d\Omega_0
 \end{aligned} \tag{9.99}$$

Where \mathbf{K} is the so-called stiffness matrix, and \mathbf{f} is usually known as the external forces vector.

If the test functions are defined equivalent to the trial ones, the stiffness matrix becomes symmetric. This election is the basis of the Bubnov-Galerkin method, which is presented in the following section.

9.16. Bubnov-Galerkin method

The Bubnov-Galerkin method assumes that the test and trial functions are equivalent [Mikhlin & Chambers, 1964; Navarrina et al., 2009].

$$\boldsymbol{\Omega} = \boldsymbol{\phi} \iff H_{\delta u}^h = H_u^h \tag{9.100}$$

Thus, the linear system of equations (9.88) becomes:

$$\begin{aligned}
 \mathbf{K} \Delta \boldsymbol{\alpha} &= \mathbf{f} + \iint_{\Gamma_0^u} \boldsymbol{\phi}^T \mathbf{g}_R^h d\Gamma_0 \\
 \mathbf{K} &= \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\phi})^T \mathbf{D}_2 (\mathbf{L}_0 \boldsymbol{\phi}) d\Omega_0 \\
 \mathbf{f} &= \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}^T \mathbf{g} d\Gamma_0 + \iiint_{\Omega_0} [\boldsymbol{\phi}^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\phi})^T \bar{\boldsymbol{\sigma}}_p] d\Omega_0
 \end{aligned} \tag{9.101}$$

And the components of both the stiffness matrix and the forces vector turn out to be:

$$\begin{aligned}
 \mathbf{K} &= [\mathbf{K}_{ij}]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} & \mathbf{K}_{ij} &= \iiint_{\Omega_0} (\mathbf{L}_0 \phi_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \\
 \mathbf{f} &= \{\mathbf{f}_i\}_{i=1,\dots,\eta} & \mathbf{f}_i &= \iint_{\Gamma_0^\sigma} \phi_i^T \mathbf{g} d\Gamma_0 + \iiint_{\Omega_0} [\phi_i^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \phi_i)^T \bar{\boldsymbol{\sigma}}_p] d\Omega_0
 \end{aligned} \tag{9.102}$$

Furthermore, if the initial approximation fulfils the essential boundary condition, and the trial functions are null over the surface where this condition is applied, the essential boundary condition is automatically fulfilled. Moreover, the reaction vector does not appear in the formulation.

$$\left. \begin{aligned}
 \mathbf{u}^h(\mathbf{r}_0) &= \mathbf{u}_p(\mathbf{r}_0) + \boldsymbol{\phi}(\mathbf{r}_0) \Delta \boldsymbol{\alpha} \\
 \mathbf{u}_p(\mathbf{r}_0) &= \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \\
 \boldsymbol{\phi}(\mathbf{r}_0) &= \mathbf{0} \quad \forall \mathbf{r}_0 \in \Gamma_0^u
 \end{aligned} \right\} \implies \left\{ \begin{aligned}
 \mathbf{u}^h(\mathbf{r}_0) &= \mathbf{u}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \\
 \iint_{\Gamma_0^u} \boldsymbol{\phi}^T \mathbf{g}_R^h d\Gamma_0 &= \mathbf{0}
 \end{aligned} \right. \tag{9.103}$$

Consequently, the system of linear equations to be solved is reduced to:

$$\mathbf{K} \Delta \boldsymbol{\alpha} = \mathbf{f} \quad (9.104)$$

9.16.1. Stiffness matrix

The stiffness matrix and its components were defined in (9.101) and (9.102) as:

$$\begin{aligned} \mathbf{K} &= \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\phi})^T \mathbf{D}_2 (\mathbf{L}_0 \boldsymbol{\phi}) d\Omega_0 \\ \mathbf{K} &= [\mathbf{K}_{ij}]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} \quad \mathbf{K}_{ij} = \iiint_{\Omega_0} (\mathbf{L}_0 \boldsymbol{\phi}_i)^T \mathbf{D}_2 (\mathbf{L}_0 \boldsymbol{\phi}_j) d\Omega_0 \end{aligned} \quad (9.105)$$

It can be rewritten in a more compact form in terms of the tensor \mathbf{B}_0 . This tensor is defined by means of the tensor $\boldsymbol{\phi}$, which was presented in equation (9.70).

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{L}_0 \boldsymbol{\phi} \\ &= \mathbf{L}_0 [\boldsymbol{\phi}_1 \quad \cdots \quad \boldsymbol{\phi}_\eta] \\ &= [\mathbf{L}_0 \boldsymbol{\phi}_1 \quad \cdots \quad \mathbf{L}_0 \boldsymbol{\phi}_\eta] \\ &= [\mathbf{B}_0^{(1)} \quad \cdots \quad \mathbf{B}_0^{(\eta)}] \end{aligned} \quad (9.106)$$

Where:

$$\mathbf{B}_0^{(k)} = \mathbf{L}_0 \boldsymbol{\phi}_k = \begin{bmatrix} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \\ \frac{\partial}{\partial r_{0,2}} & \frac{\partial}{\partial r_{0,1}} & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & \frac{\partial}{\partial r_{0,3}} & \frac{\partial}{\partial r_{0,2}} \end{bmatrix} \begin{bmatrix} \phi_k & 0 & 0 \\ 0 & \phi_k & 0 \\ 0 & 0 & \phi_k \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_k}{\partial r_{0,1}} & 0 & 0 \\ 0 & \frac{\partial \phi_k}{\partial r_{0,2}} & 0 \\ 0 & 0 & \frac{\partial \phi_k}{\partial r_{0,3}} \\ \frac{\partial \phi_k}{\partial r_{0,2}} & \frac{\partial \phi_k}{\partial r_{0,1}} & 0 \\ \frac{\partial \phi_k}{\partial r_{0,3}} & 0 & \frac{\partial \phi_k}{\partial r_{0,1}} \\ 0 & \frac{\partial \phi_k}{\partial r_{0,3}} & \frac{\partial \phi_k}{\partial r_{0,2}} \end{bmatrix} \quad (9.107)$$

Thus, the stiffness matrix becomes:

$$\begin{aligned} \mathbf{K} &= \iiint_{\Omega_0} \mathbf{B}_0^T \mathbf{D}_2 \mathbf{B}_0 d\Omega_0 \\ \mathbf{K} &= [\mathbf{K}_{ij}]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} \quad \mathbf{K}_{ij} = \iiint_{\Omega_0} (\mathbf{B}_0^{(i)})^T \mathbf{D}_2 \mathbf{B}_0^{(j)} d\Omega_0 \end{aligned} \quad (9.108)$$

According to this definition, the stiffness matrix turns out to be symmetric. To prove this property, the symmetry of the linear constitutive tensor defined in (7.94) is

recalled.

$$\begin{aligned} \mathbf{K}^T &= \left(\iiint_{\Omega_0} \mathbf{B}_0^T \mathbf{D}_2 \mathbf{B}_0 d\Omega_0 \right)^T = \iiint_{\Omega_0} \mathbf{B}_0^T \mathbf{D}_2 \mathbf{B}_0 d\Omega_0 = \mathbf{K} \\ \mathbf{K}_{ij}^T &= \left(\iiint_{\Omega_0} (\mathbf{B}_0^{(i)})^T \mathbf{D}_2 \mathbf{B}_0^{(j)} d\Omega_0 \right)^T = \iiint_{\Omega_0} (\mathbf{B}_0^{(j)})^T \mathbf{D}_2 \mathbf{B}_0^{(i)} d\Omega_0 = \mathbf{K}_{ji} \end{aligned} \quad (9.109)$$

9.17. Reference material domain discretization

Let's consider that the reference configuration Ω_0 is composed by the union of n finite elements.

$$\Omega_0 = \bigcup_{e=1}^n \Omega_0^e \quad (9.110)$$

Where Ω_0^e is the e -th finite element. The intersection of different finite elements is assumed to be equal to the empty set.

$$\overset{\circ}{\Omega}_0^e \cap \overset{\circ}{\Omega}_0^f = \emptyset \quad \forall e \neq f \quad (9.111)$$

This assumption allows to divide an integral defined over the whole reference domain, as the sum of the integrals defined over the domain corresponding to each finite element. For instance, the integral of a given function φ turns out to be divided as:

$$\iiint_{\Omega_0} \varphi(\mathbf{r}_0) d\Omega_0 = \sum_{e=1}^n \iiint_{\Omega_0^e} \varphi(\mathbf{r}_0) d\Omega_0 \quad (9.112)$$

9.17.1. Element stiffness matrix

The reference domain discretization allows to state the stiffness matrix (9.99) as the sum of the contribution of each finite element.

$$\begin{aligned} \mathbf{K} &= \sum_{e=1}^n \hat{\mathbf{K}}^e \\ \hat{\mathbf{K}}^e &= \left[\hat{\mathbf{K}}_{ij}^e \right]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} \quad \hat{\mathbf{K}}_{ij}^e = \iiint_{\Omega_0^e} (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \end{aligned} \quad (9.113)$$

The hat operator indicates that the subscripts of each component refer to a global numeration. In the next section, the nodes that compose the mesh will be defined. They can be numbered according to a global or a local criterion. If the whole set of nodes is taken into account, a global numeration is applied. However, if the nodes of a given finite element are numbered, a local numeration is adopted.

9.17.2. Element forces vector

The forces vector (9.99) can also be rewritten as the sum of each element contribution, as shown below.

$$\begin{aligned} \mathbf{f} &= \iint_{\Gamma_0^\sigma} \boldsymbol{\Omega}^T \mathbf{g} \, d\Gamma_0 + \sum_{e=1}^n \hat{\mathbf{f}}^e \\ \hat{\mathbf{f}}^e &= \left\{ \hat{\mathbf{f}}_i^e \right\}_{i=1, \dots, \eta} \quad \hat{\mathbf{f}}_i^e = \iiint_{\Omega_0^e} \left[\boldsymbol{\Omega}_i^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \bar{\boldsymbol{\sigma}}_p \right] d\Omega_0 \end{aligned} \quad (9.114)$$

9.18. Trial functions definition

To simplify the formulation, and reduce the computation time, the trial functions are defined according to the criterion exposed in the following subsections.

9.18.1. Nodal points

The following set of nodal points is defined:

$$\left\{ \mathbf{r}_{0,k} \right\}_{k=1, \dots, \eta} \quad (9.115)$$

On the one hand, the trial functions defined in section 9.14 are considered to fulfil the following condition.

$$\left\{ \phi_i(\mathbf{r}_0) \right\}_{i=1, \dots, \eta} \quad \Longrightarrow \quad \phi_i(\mathbf{r}_{0,k}) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \quad (9.116)$$

That is, each trial function is associated with a nodal point. The function adopts a unitary value at its corresponding node, whereas it is null at the remaining ones.

The displacement field approximation was defined in (9.73) as:

$$\begin{aligned} \mathbf{u}^h(\mathbf{r}_0) &= \mathbf{u}_p(\mathbf{r}_0) + \Delta \mathbf{u}_p(\mathbf{r}_0) \\ &= \sum_{i=1}^{\eta} [\phi_i(\mathbf{r}_0) \mathbf{I}] \boldsymbol{\alpha}_i + \sum_{i=1}^{\eta} [\phi_i(\mathbf{r}_0) \mathbf{I}] \Delta \boldsymbol{\alpha}_i \\ &= \sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) (\boldsymbol{\alpha}_i + \Delta \boldsymbol{\alpha}_i) \end{aligned} \quad (9.117)$$

If the trial functions verify condition (9.116), the evaluation of the above approximation at the nodal points leads to:

$$\begin{aligned} \mathbf{u}^h(\mathbf{r}_{0,k}) &= \sum_{i=1}^{\eta} \phi_i(\mathbf{r}_{0,k}) (\boldsymbol{\alpha}_i + \Delta \boldsymbol{\alpha}_i) \\ &= \sum_{i=1}^{\eta} \delta_{ik} (\boldsymbol{\alpha}_i + \Delta \boldsymbol{\alpha}_i) \\ &= \boldsymbol{\alpha}_k + \Delta \boldsymbol{\alpha}_k \end{aligned} \quad (9.118)$$

Therefore, the value α_k represents the initial displacement approximation experimented by the k -th node, and the unknown $\Delta\alpha_k$ represents its corresponding correction.

The imposition of the condition (9.116) simplifies the physical interpretation of the result obtained from the system of linear equations, since it turns out to be the displacement that the nodal points experiment.

9.18.2. Local support

On the other hand, the trial functions ϕ_j and the test ones δu_i are defined with local support. That is, they have to be null over a large amount of finite elements, to obtain some computational advantages.

If they fulfil this condition, many contributions $\hat{\mathbf{K}}_{ij}^e$ and $\hat{\mathbf{f}}_i^e$ will be zero. Therefore, many components of the stiffness matrix \mathbf{K}_{ij} might be zero. This implies a significant reduction of the computation time required to compute the integrals. Furthermore, the stiffness matrix will be sparse. Hence, the number of components is significantly reduced, and the memory storage needed is much lower.

The stiffness matrix and element forces vector corresponding to a given finite element were defined in (9.113) and (9.114), as:

$$\begin{aligned} \hat{\mathbf{K}}^e &= \left[\hat{\mathbf{K}}_{ij}^e \right]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} & \hat{\mathbf{K}}_{ij}^e &= \iiint_{\Omega_0^e} (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \\ \hat{\mathbf{f}}^e &= \left\{ \hat{\mathbf{f}}_i^e \right\}_{i=1,\dots,\eta} & \hat{\mathbf{f}}_i^e &= \iiint_{\Omega_0^e} \left[\boldsymbol{\Omega}_i^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \bar{\boldsymbol{\sigma}}_p \right] d\Omega_0 \end{aligned} \quad (9.119)$$

Where:

$$\begin{aligned} \boldsymbol{\Omega}_i &= \delta u_i(\mathbf{r}_0) \mathbf{I} \\ \phi_j &= \phi_j(\mathbf{r}_0) \mathbf{I} \end{aligned} \quad (9.120)$$

If the test functions δu_i and the trial ones ϕ_j are null at a huge part of the reference configuration, it is only required to compute the elemental contributions $\hat{\mathbf{K}}_{ij}^e$ and $\hat{\mathbf{f}}_i^e$ that satisfy both of the following conditions:

$$\delta u_i(\mathbf{r}_0) \neq 0 \quad \text{and} \quad \phi_j(\mathbf{r}_0) \neq 0 \quad \forall \mathbf{r}_0 \in \Omega_0^e \quad (9.121)$$

Furthermore, the stiffness matrix component \mathbf{K}_{ij} is null, if one of the following conditions is verified for the whole set of finite elements.

$$\delta u_i(\mathbf{r}_0) = 0 \quad \text{or} \quad \phi_j(\mathbf{r}_0) = 0 \quad \forall \mathbf{r}_0 \in \Omega_0^e, \quad \forall e \quad (9.122)$$

9.18.3. Partition of unity

If a translation is applied to the solid, the displacement field approximation has to be able to represent this rigid motion. Let's consider the translation represented by the following displacement field.

$$\mathbf{u}(\mathbf{r}_0) = \mathbf{a} \quad \mathbf{a} \in \mathbb{R}, \quad \forall \mathbf{r}_0 \in \Omega_0 \quad (9.123)$$

As stated before, the displacement field approximation evaluated at the nodal points represents the displacement experimented by the nodes (9.118). In order to be capable of representing the above translation, the nodal displacements have to be equivalent to the vector that defines this rigid motion.

$$\mathbf{u}^h(\mathbf{r}_{0,i}) = \boldsymbol{\alpha}_i + \Delta\boldsymbol{\alpha}_i = \mathbf{a} \quad i = 1, \dots, \eta \quad (9.124)$$

If the above result is taken into account, the displacement field approximation (9.117) becomes:

$$\mathbf{u}^h(\mathbf{r}_0) = \sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) \underbrace{(\boldsymbol{\alpha}_i + \Delta\boldsymbol{\alpha}_i)}_{=\mathbf{a}} = \left(\sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) \right) \mathbf{a} \quad (9.125)$$

Therefore, if the sum of the trial functions is equal to one, the displacement field approximation properly represents a translation.

$$\sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) = 1 \quad \Longrightarrow \quad \mathbf{u}^h(\mathbf{r}_0) = \underbrace{\left(\sum_{i=1}^{\eta} \phi_i(\mathbf{r}_0) \right)}_{=1} \mathbf{a} = \mathbf{a} \quad (9.126)$$

The above condition is usually known as the partition of unity [Melenk & Babuška, 1996; Babuška & Melenk, 1997]. It turns out to be an essential condition that the trial functions have to fulfil to properly address translations. In addition, if this condition is satisfied, the stain and stress fields corresponding to the displacement field approximation (9.126) are null.

$$\begin{aligned} \mathbf{u}^h(\mathbf{r}_0) = \mathbf{a} &\Longrightarrow \bar{\boldsymbol{\epsilon}}^h = \mathbf{L}_0 \mathbf{u}^h = \bar{\mathbf{0}} \\ &\Longrightarrow \bar{\boldsymbol{\sigma}}^h = \mathbf{D}_2 \bar{\boldsymbol{\epsilon}}^h = \bar{\mathbf{0}} \end{aligned} \quad (9.127)$$

Otherwise, a translation implies the existence of a stress field, which is incorrect and has no physical sense.

9.19. Three-dimensional finite element

A three-dimensional finite element composed by N nodal points is defined, in order to obtain its particular elemental stiffness matrix and elemental forces vector.

The elemental stiffness matrix and the elemental forces vector were defined in (9.113) and (9.114), respectively.

$$\begin{aligned} \hat{\mathbf{K}}^e &= \left[\hat{\mathbf{K}}_{ij}^e \right]_{\substack{i=1,\dots,\eta \\ j=1,\dots,\eta}} & \hat{\mathbf{K}}_{ij}^e &= \iiint_{\Omega_0^e} (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \\ \hat{\mathbf{f}}^e &= \left\{ \hat{\mathbf{f}}_i^e \right\}_{i=1,\dots,\eta} & \hat{\mathbf{f}}_i^e &= \iiint_{\Omega_0^e} \left[\boldsymbol{\Omega}_i^T \mathbf{b} \rho^0 - (\mathbf{L}_0 \boldsymbol{\Omega}_i)^T \bar{\boldsymbol{\sigma}}_p \right] d\Omega_0 \end{aligned} \quad (9.128)$$

Where the tensors $\boldsymbol{\Omega}_i$ and ϕ_j are composed by the trial and test functions, as:

$$\begin{aligned} \boldsymbol{\Omega}_i &= \delta u_i(\mathbf{r}_0) \mathbf{I} \\ \phi_j &= \phi_j(\mathbf{r}_0) \mathbf{I} \end{aligned} \quad (9.129)$$

As stated in section 9.18.1, the trial functions become unitary at their corresponding node, and zero at the remaining ones. Moreover, it can be imposed that the only non-zero functions over a given finite element are those associated with the nodal points of that element.

$$\phi_i(\mathbf{r}_0) = \underbrace{\phi_i(\mathbf{r}_0)}_{\neq 0} \mathbf{I} \neq \mathbf{0} \quad i = a, b, c, \dots \quad \forall \mathbf{r}_0 \in \Omega_0^e \quad \forall e \quad (9.130)$$

Where a, b, c, \dots are the nodes of a given finite element e . Therefore, the non-zero components of the elemental stiffness matrix and the elemental forces vector turn out to be:

$$\begin{aligned} \hat{\mathbf{K}}_{ij}^e &\neq \mathbf{0} & j = a, b, c, \dots \\ \hat{\mathbf{f}}_i^e &\neq \bar{\mathbf{0}} & i = a, b, c, \dots \end{aligned} \quad (9.131)$$

If the Bubnov-Galerkin method is applied, the test and trial functions become equivalent.

$$\delta u_i(\mathbf{r}_0) = \phi_i(\mathbf{r}_0) \quad \iff \quad \Omega_i = \phi_i \quad (9.132)$$

Thus, the components of the element stiffness matrix become:

$$\hat{\mathbf{K}}_{ij}^e = \iiint_{\Omega_0^e} (\mathbf{L}_0 \phi_i)^T \mathbf{D}_2 (\mathbf{L}_0 \phi_j) d\Omega_0 \quad (9.133)$$

And the number of non-zero components is reduced to:

$$\hat{\mathbf{K}}_{ij}^e \neq \mathbf{0} \quad i = a, b, c, \dots \quad j = a, b, c, \dots \quad (9.134)$$

Consequently, if the zero components are not taken into account, the elemental stiffness matrix and the elemental forces vector are reduced to:

$$\mathbf{K}^e = \begin{bmatrix} \hat{\mathbf{K}}_{aa}^e & \hat{\mathbf{K}}_{ab}^e & \hat{\mathbf{K}}_{ac}^e & \dots \\ \hat{\mathbf{K}}_{ba}^e & \hat{\mathbf{K}}_{bb}^e & \hat{\mathbf{K}}_{bc}^e & \dots \\ \hat{\mathbf{K}}_{ca}^e & \hat{\mathbf{K}}_{cb}^e & \hat{\mathbf{K}}_{cc}^e & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \mathbf{f}^e = \begin{Bmatrix} \hat{\mathbf{f}}_a^e \\ \hat{\mathbf{f}}_b^e \\ \hat{\mathbf{f}}_c^e \\ \vdots \end{Bmatrix} \quad (9.135)$$

9.19.1. Nodal numeration, master element and shape functions

In order to properly identify the nodal points of a given finite element, two different numerations are defined. On the one hand, the global numeration takes into account the whole set of nodal points that compose the entire mesh. All of them are numbered according to this global numeration. On the other hand, an alternative numeration that only takes into account the nodes of a given finite element is defined. This one is more suitable to work at a local level. And an equivalence between both numerations can be established.

Moreover, a master element in a space of normalized coordinates is usually adopted, where the so-called shape functions are defined. These functions are equivalent to the

trial ones, but defined in a normalized space. The geometric transformation that allows moving from the material domain to the normalized one has to be properly defined. Once, this geometric transformation is completely defined, the integration domain of each elemental contribution can be changed, and the integration can be now carried out in the normalized space.

This master element allows to define the displacement field at a given finite element in terms of the material coordinates or by means of the normalized ones. But the shape functions can also be applied to interpolate the reference geometry. The finite elements that employ the same functions to interpolate the displacement field and the geometry are usually known as isoparametric elements [Taig, 1962; Ergatoudis et al., 1968].

9.20. Overview and conclusions

9.20.1. Linear analysis strategy

In the current chapter, the structural behaviour of a given solid subjected to a set of external forces is analysed.

According to the definition of the reference domain, two types of boundary conditions are considered. The first ones are usually known as essential boundary conditions, which are based on the definition of the displacement field on a specific solid surface. The second type is the so-called natural boundary conditions, which define the stress vector on the surface where the external loads are applied.

If the analysis is carried out in linear theory, two main hypotheses that considerably simplify the analysis are adopted.

- On the one hand, the displacements that the solid experiments are considered to be small. This hypothesis allows to consider that the initial and the deformed domain are coincident. Thus, the equilibrium equations can be imposed over the reference configuration, which is well-known. This also implies that there is no volume variation between both configurations.
- On the other hand, the displacement gradients are assumed to be small. This assumption allows to apply the approximate polar decomposition to decompose the deformation gradient tensor by means of the displacement gradient tensor. Without this assumption, an eigenvalue problem has to be solved in order to obtain its polar decomposition. This simplification turns out to be a major advantage, as it reduces computational effort.

Nevertheless, if the real structural behaviour does not verify the linear analysis assumptions, the results obtained with this theory do not correspond to the real structural response.

As the mass and the volume are assumed to remain constant, the density field is also considered to not vary. Hence, the density field is not an unknown to calculate

if the linear approach is adopted, as the initial density field is known and it remains constant.

The equations that govern the static equilibrium were also derived in this chapter, both the equation that rules the equilibrium of forces and the one that states the equilibrium of moments. The Cauchy stress tensor turns out to be symmetric if the angular momentum conservation is verified. The conservation of angular momentum implies that the sum of all torques is zero. Therefore, the equation that rules the equilibrium of moments is reduced to the symmetry of the Cauchy stress tensor.

The compatibility equation is the one that defines the strain field by means of the displacement field. According to the approximate polar decomposition, the strain field can be approximated by the infinitesimal strain tensor, which is completely defined by means of the displacement gradient tensor. Since the displacement gradients are small and the strain field depends on the displacement gradient tensor, the strain field turns out to be infinitesimal in linear analysis.

And the equation that defines the relation between the stress and the strain fields is the constitutive equation. As the strain field is infinitesimal, and the displacement gradients are small, the effect of the rotation can be neglected to define the Cauchy stress tensor. Consequently, the stress tensor can be completely defined by means of the infinitesimal strain tensor. That is, the Lamé's equation can be applied to define the constitutive equation.

All the equations mentioned before compose the mathematical model that allows to obtain the structural response under the assumptions of the linear analysis. The main unknown to solve is the displacement field that the solid experiments. Once this field is known, the compatibility equation allows to compute the strain field by means of the displacement field. And finally, the constitutive equation states the stress field that corresponds to the previous strain field. In addition, once the stress field is defined, the reaction that appears on the surface where the essential boundary condition is applied can be computed if required.

9.20.2. Linear finite element analysis

The linear finite element analysis derived in this chapter allows to obtain the displacement field experimented by the solid after the application of the external forces, under the assumptions of small displacements and small displacement gradients.

Both the strong and its corresponding weak form are derived. The definition of the trial functions basis allows to build a displacement field approximation, which can be defined as an initial approximation plus an additional term that corrects the former one. In general, the difference between the exact solution and the initial displacement field approximation can not be generated by the subspace composed by the trial functions basis. Hence, the displacement field approximation does not usually verify the weak form. In order to obtain a solution which does verify the weak form, a test functions basis is proposed to generate an approximated test function which leads to an approximated weak form.

The original equilibrium problem is then substituted by a system of linear equations. Its corresponding matrix is usually known as the stiffness matrix, and the independent vector is the so-called forces vector. If the initial approximation fulfils the essential boundary condition, and the trial functions are null over the surface where this boundary condition is applied, the displacement field approximation automatically verifies the essential boundary condition. Moreover, if the test functions are null over the surface where the essential boundary condition is applied, the reaction is not involved in the formulation.

The Bubnov-Galerkin method is commonly applied. This method is based on the application of the same functions to define the trial and test functions, which leads to symmetric stiffness matrices.

Once the resulting system of linear equations is completely defined, the material domain is divided into a set of finite elements. This allows to rewrite integrals as the sum of the integrals defined over the domain corresponding to each finite element. Therefore, the components of the stiffness matrix and the forces vector become the sum of the contribution of each finite element.

Furthermore, each trial function can be defined associated with a given nodal point. The function adopts a unitary value at its corresponding node, whereas it becomes null at the remaining ones. This condition simplifies the physical interpretation of the result obtained from the system of linear equations, since it turns out to represent the displacement experimented by the nodal point. On the other hand, the trial functions and the test ones can be defined with local support. If they are defined null over a large amount of finite elements, some computational advantages are obtained, such as a significant reduction of computation time and storage memory required.

Finally, the trial functions have to fulfil an essential condition usually known as the partition of unity in order to properly address translations. If this condition is not satisfied, a translation implies the existence of a stress field, which is incorrect and has no physical sense.

Total Lagrangian finite element analysis

10.1. Introduction

The linear finite element formulation presented in the previous chapter was derived under the assumptions of small displacements and small displacement gradients. These hypotheses allow to simplify both the solid mechanics and the obtention of the structural behaviour by means of the finite element method.

However, these hypotheses are no longer adopted in nonlinear analysis, and the solid is assumed to experiment large displacements and large displacement gradients. These hypotheses difficult the analysis. Their implications are extensively analysed in the current chapter, and the Total Lagrangian finite element formulation is derived. This formulation allows to obtain the nonlinear behaviour of a given solid subjected to external loads.

An overview of this formulation can be checked in [Bathe et al. \[1975\]](#); [Bathe & Bolourchi \[1979\]](#); [Bathe \[1996\]](#), among other reference textbooks and papers. Some classical nonlinear finite element formulations are written entirely in index notation. Besides the indexes that correspond to the components of each tensor magnitude, the one that indicates the load level, as well as the one that clarify the reference configuration adopted, have to be added too. Index notation is convenient for coding and implementing algorithms into a computer software. However, it is not the best option to understand the underlying physics of a given problem.

In this work, the use of index notation is avoided whenever possible for facilitating the understanding of the concepts. If required, it is applied as an intermediate step to reach a final tensor notation. The main aim of this chapter is to fully comprehend and follow the complete derivation of this nonlinear finite element formulation.

10.2. Reference configuration and deformed material domain

The conceptual problem and the reference material domain were already presented in sections 9.2 and 9.3, respectively.

After the application of the external loads, the reference material domain becomes Ω , and its corresponding surface becomes:

$$\partial\Omega = \Gamma = \Gamma^\sigma \cup \Gamma^u \quad (10.1)$$

The deformed surface is again divided into two subdomains: the surface where the displacements are prescribed (Γ^u), and the surface where the surface forces are applied (Γ^σ).

The deformation vector is the one that defines the position of a given material particle, whose initial position is defined by the position vector \mathbf{r}_0 . This vector turns out to be equivalent to the sum of the initial position vector plus the displacement that the material particle experiments (figure 10.1).

$$\mathbf{r}_\mathcal{L}(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_\mathcal{L}(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Omega_0 \quad (10.2)$$

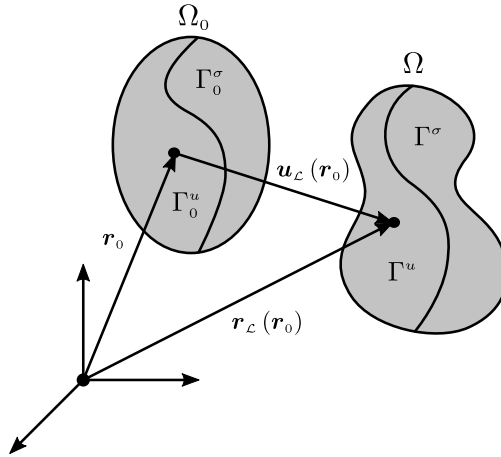


Figure 10.1. Reference configuration, deformed material domain, and vectors that define the initial and final position of a given material particle.

10.3. Nonlinear analysis hypotheses

In contrast to linear analysis, the structure is now supposed to experiment large displacements and large displacement gradients. These hypotheses allow to study structures that do not verify the linear assumptions, that is, structures that do not experiment small displacements nor small displacement gradients can be properly analysed.

Moreover, if the nonlinear theory is adopted, structures that behave according to the linear assumptions can also be properly simulated.

10.3.1. Large displacements

On the one hand, it is considered the displacements that the solid experiments can be large.

$$\|\mathbf{u}_\varepsilon\| \ll 1 \quad (10.3)$$

Consequently, the initial and final position of a given material particle can not be considered equivalent. Thus, the reference configuration and the deformed material domain can not be considered coincident.

$$\left. \begin{array}{l} \mathbf{r}_\varepsilon(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_\varepsilon(\mathbf{r}_0) \\ \|\mathbf{u}_\varepsilon\| \ll 1 \end{array} \right\} \implies \mathbf{r} \not\approx \mathbf{r}_0 \iff \Omega \not\approx \Omega_0 \quad (10.4)$$

The above hypothesis leads to impose the equilibrium conditions over the deformed material domain, which is the unknown to be solved. This is an important inconvenience which has to be overcome in the following sections.

As the initial and deformed material domains are no longer considered equivalent, the determinant of the deformation gradient tensor is no longer equal to one.

$$F_\varepsilon(\mathbf{r}_0) \neq 1 \quad (10.5)$$

In addition, the Lagrangian and Eulerian descriptions of a given magnitude are not equivalent. Both descriptions have to be differentiated, so the subscripts ε and ε are added to specify the description adopted.

$$\psi_\varepsilon(\mathbf{r}) \not\approx \psi_\varepsilon(\mathbf{r}_0) \quad (10.6)$$

The equivalence between both descriptions was defined in section 1.8 as shown below.

$$\psi_\varepsilon(\mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}_\varepsilon(\mathbf{r}_0)} = \psi_\varepsilon(\mathbf{r}_0) \quad (10.7)$$

10.3.2. Large displacement gradients

On the other hand, it is considered that the displacement gradients can be large.

$$\|\mathbf{J}_\varepsilon\| \ll \|\mathbf{I}\| \quad (10.8)$$

Thus, the approximate polar decomposition (section 5.3) of the deformation gradient tensor is no longer acceptable. To properly decompose the deformation gradient tensor into an expansion and a rotation, an eigenvalue problem must be solved, as exposed in section 4.3.1.

$$\mathbf{F}_\varepsilon = \mathbf{R}_\varepsilon [\mathbf{I} + \mathbf{E}_\varepsilon] \quad (10.9)$$

The tensor \mathbf{R}_ε is the finite rotation tensor, and \mathbf{E}_ε is the Biot strain tensor. As stated in equation (7.13), the Lagrangian description of the Cauchy stress tensor can be expressed, by means of these tensors, as:

$$\boldsymbol{\sigma}_\varepsilon = \mathbf{R}_\varepsilon \boldsymbol{\Psi}(\mathbf{E}_\varepsilon) \mathbf{R}_\varepsilon^T \quad (10.10)$$

Once the hypotheses of nonlinear analysis are stated, the equations that rule the static equilibrium have to be defined according to these assumptions.

10.4. Static equilibrium equations

Static equilibrium occurs if the equilibrium of forces and moments is fulfilled. These equations are presented in the following subsections.

10.4.1. Equilibrium of forces

On the one hand, the equilibrium of forces has to be verified. The equation that rules the static equilibrium of forces of a given solid subjected to external forces was defined in (8.11) as:

$$\mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T) + \mathbf{b}_\varepsilon \rho_\varepsilon = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (10.11)$$

10.4.2. Equilibrium of moments

On the other hand, the equilibrium of moments has to be fulfilled too. As proved in section 8.4, the Cauchy stress tensor is symmetric if the angular momentum conservation is verified.

$$\boldsymbol{\sigma}_\varepsilon^T = \boldsymbol{\sigma}_\varepsilon \quad \forall \mathbf{r} \in \Omega \quad (10.12)$$

If the above condition holds, it can be stated that the sum of all torques is zero, and the equilibrium of moments is fulfilled.

10.5. Density field

If the mass is considered to not vary over time, the mass conservation equation has to be verified, and the density can be defined, according to (8.8) as:

$$\rho_\varepsilon(\mathbf{r}_o, t) = \frac{\rho_\varepsilon(\mathbf{r}_o, 0)}{F_\varepsilon(\mathbf{r}_o, t)} \quad \forall \mathbf{r}_o \in \overset{\circ}{\Omega}_o, \forall t \quad (10.13)$$

If a static analysis is carried out, the final density field can be obtained from the above equation. The time variable t vanishes, and the density field is finally defined as:

$$\rho_\varepsilon(\mathbf{r}_o) = \frac{\rho_\varepsilon^0(\mathbf{r}_o)}{F_\varepsilon(\mathbf{r}_o)} \quad (10.14)$$

10.6. Boundary conditions

Two types of boundary conditions are again differentiated: the imposition of the displacement field and the definition of the stress vector on a given portion of the solid surface.

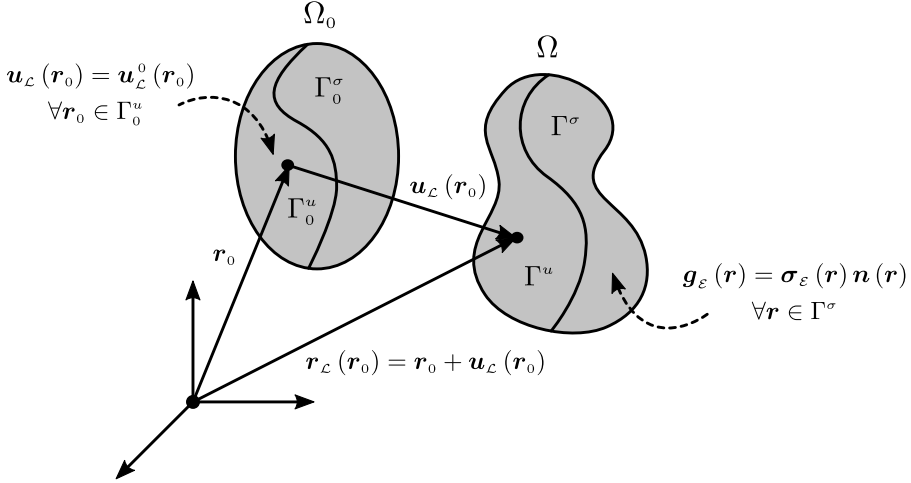


Figure 10.2. Reference configuration, deformed material domain, and boundary conditions.

10.6.1. Essential boundary conditions

The first ones are the essential boundary conditions, which are based on the definition of the displacement field on a specific solid surface. That is, they represent prescribed displacements on a given surface.

$$\mathbf{u}_\varepsilon(\mathbf{r}_0) = \mathbf{u}_\varepsilon^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.15)$$

10.6.2. Natural boundary conditions

Moreover, surface external loads are applied on the other portion of the solid surface. These are the so-called natural boundary conditions, and they are based on the definition of the stress vector on the surface where the external surface forces are applied.

$$\mathbf{g}_\varepsilon(\mathbf{r}) = \boldsymbol{\sigma}_\varepsilon(\mathbf{r}) \mathbf{n}(\mathbf{r}) \quad \forall \mathbf{r} \in \Gamma^\sigma \quad (10.16)$$

10.7. Mathematical model

A homogenous and isotropic solid medium is again considered. That is, the mechanical properties are constant over the whole material domain, and they do not depend on the direction.

The solid is subjected to volumetric \mathbf{b}_ε and surface \mathbf{g}_ε forces. These loads are forces per unit mass and forces per unit area, respectively.

And the material mechanical properties are known. The Young's modulus is represented by E and the Poisson's ratio by ν . The Lamé's parameters (7.51) are defined

in terms of the Young's modulus and the Poisson's ratio, as shown below.

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)} \quad \mu = \frac{E}{2(1+\nu)} \quad (10.17)$$

The main aim is to obtain the displacement field that the solid experiments due to the application of the external loads, and its corresponding stress field.

$$\mathbf{u}_\mathcal{L}(\mathbf{r}_0), \boldsymbol{\sigma}_\mathcal{L}(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Omega_0 \quad (10.18)$$

The above unknowns verify the following equations:

$$\begin{aligned} \operatorname{div}(\boldsymbol{\sigma}_\varepsilon) + \mathbf{b}_\varepsilon \rho_\varepsilon &= \bar{\mathbf{0}} & \forall \mathbf{r} \in \overset{\circ}{\Omega} & \quad (\text{Equilibrium of forces (10.11) and moments (10.12)}) \\ -\boldsymbol{\sigma}_\varepsilon \mathbf{n} + \mathbf{g}_\varepsilon &= \bar{\mathbf{0}} & \forall \mathbf{r} \in \Gamma^\sigma & \quad (\text{Natural boundary condition (10.16)}) \\ \mathbf{u}_\mathcal{L} &= \mathbf{u}_\mathcal{L}^0 & \forall \mathbf{r}_0 \in \Gamma_0^u & \quad (\text{Essential boundary condition (10.15)}) \end{aligned} \quad (10.19)$$

Note that the magnitudes involved in the definition of the equilibrium equation, as well as the ones that appear in the natural boundary condition, are expressed according to their Eulerian description. However, the other magnitudes are defined according to their Lagrangian description.

Later on in this chapter, a strategy to obtain all magnitudes described according to their Lagrangian description will be outlined. This conversion will drive to adopt the Green-Lagrange strain tensor (4.73), as the compatibility equation.

$$\mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left[\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right] \quad (10.20)$$

And the second Piola-Kirchhoff stress tensor defined by means of the above Green-Lagrange strain tensor (section 7.6), as the constitutive equation.

$$\mathbf{S}_\mathcal{L} = \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}) \quad (10.21)$$

10.8. Strong form

In this section, the strong form is stated. As the original form of the problem turns out to be mathematically exact, it is usually known as the strong form.

The main goal of the analysis is to obtain the displacement field experimented by the solid after the application of the external loads, that verifies the essential boundary condition.

$$\mathbf{u}_\mathcal{L}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_\mathcal{L}(\mathbf{r}_0) = \mathbf{u}_\mathcal{L}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.22)$$

The above displacement field verifies the following equations, expressed in terms of the equilibrium equation residual $\mathbf{R}_{\sigma,\varepsilon}$ and the natural boundary condition residual $\mathbf{R}_{\Gamma,\varepsilon}$.

$$\begin{aligned} \mathbf{R}_{\sigma,\varepsilon} &= \bar{\mathbf{0}} & \forall \mathbf{r} \in \overset{\circ}{\Omega} \\ \mathbf{R}_{\Gamma,\varepsilon} &= \bar{\mathbf{0}} & \forall \mathbf{r} \in \Gamma^\sigma \end{aligned} \quad (10.23)$$

Where:

$$\begin{aligned} \mathbf{R}_{\sigma,\varepsilon} &= \mathbf{div}(\boldsymbol{\sigma}_\varepsilon) + \mathbf{b}_\varepsilon \rho_\varepsilon & \mathbf{r} \in \overset{\circ}{\Omega} \\ \mathbf{R}_{\Gamma,\varepsilon} &= -\boldsymbol{\sigma}_\varepsilon \mathbf{n} + \mathbf{g}_\varepsilon & \mathbf{r} \in \Gamma^\sigma \end{aligned} \quad (10.24)$$

Once the strong form is completely defined, the weighted residual method allows to obtain its corresponding weak form, as stated in the following section.

10.9. Eulerian weak form

The weighted residual method exposed in section C.3 is now applied, in order to obtain the weak form of the previous strong form. The displacement field (10.22) verifies now the following equation:

$$\iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{R}_{\sigma,\varepsilon} d\Omega + \iint_{\Gamma^\sigma} \boldsymbol{\omega}_\varepsilon^T \mathbf{R}_{\Gamma,\varepsilon} d\Gamma = 0 \quad \forall \boldsymbol{\omega}_\varepsilon \in H_\omega \quad (10.25)$$

Where $\boldsymbol{\omega}_\varepsilon$ are suitable test functions, and the residuals $\mathbf{R}_{\sigma,\varepsilon}$ and $\mathbf{R}_{\Gamma,\varepsilon}$ were defined in (10.24).

10.9.1. Equivalent Eulerian weak form derivation

If the property of the divergence proved in (B.30) is taken into account, it can be stated that:

$$\mathbf{div}(\boldsymbol{\sigma}_\varepsilon^T \boldsymbol{\omega}_\varepsilon) = \boldsymbol{\omega}_\varepsilon^T \mathbf{div}(\boldsymbol{\sigma}_\varepsilon) + \text{Tr} \left(\boldsymbol{\sigma}_\varepsilon^T \frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \right) \quad (10.26)$$

But the stress tensor is symmetric, as demonstrated in (8.35). Hence, the above equation becomes:

$$\mathbf{div}(\boldsymbol{\sigma}_\varepsilon \boldsymbol{\omega}_\varepsilon) = \boldsymbol{\omega}_\varepsilon^T \mathbf{div}(\boldsymbol{\sigma}_\varepsilon) + \text{Tr} \left(\boldsymbol{\sigma}_\varepsilon \frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \right) \quad (10.27)$$

The integration of the above equation over the deformed material domain leads to:

$$\iiint_{\Omega} \mathbf{div}(\boldsymbol{\sigma}_\varepsilon \boldsymbol{\omega}_\varepsilon) d\Omega = \iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{div}(\boldsymbol{\sigma}_\varepsilon) d\Omega + \iiint_{\Omega} \text{Tr} \left(\boldsymbol{\sigma}_\varepsilon \frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \right) d\Omega \quad (10.28)$$

If the divergence theorem (section B.7.1) is applied, the left-hand side of the above integral equation can be equivalently expressed as a surface integral.

$$\begin{aligned} \iiint_{\Omega} \mathbf{div}(\boldsymbol{\sigma}_\varepsilon \boldsymbol{\omega}_\varepsilon) d\Omega &= \iint_{\Gamma} (\boldsymbol{\sigma}_\varepsilon \boldsymbol{\omega}_\varepsilon)^T \mathbf{n} d\Gamma \\ &= \iint_{\Gamma} \boldsymbol{\omega}_\varepsilon^T (\boldsymbol{\sigma}_\varepsilon \mathbf{n}) d\Gamma \end{aligned} \quad (10.29)$$

Therefore, equation (10.28) can be rewritten as:

$$\iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{div}(\boldsymbol{\sigma}_\varepsilon) d\Omega = \iint_{\Gamma} \boldsymbol{\omega}_\varepsilon^T (\boldsymbol{\sigma}_\varepsilon \mathbf{n}) d\Gamma - \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) d\Omega \quad (10.30)$$

The above equation can be substituted into the first addend of the weak form (10.25). So,

$$\begin{aligned}
 \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{R}_{\sigma, \varepsilon} d\Omega &= \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \left[\mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}) + \mathbf{b}_{\varepsilon} \rho_{\varepsilon} \right] d\Omega \\
 &= \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{div}(\boldsymbol{\sigma}_{\varepsilon}) d\Omega + \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega \\
 &= \left[\iint_{\Gamma} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma - \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega \right] + \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega \\
 &= \left[\iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma \right] - \\
 &\quad - \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega + \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega
 \end{aligned} \tag{10.31}$$

And the substitution of the above result into the weak form (10.25) leads to:

$$\begin{aligned}
 \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{R}_{\sigma, \varepsilon} d\Omega + \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{R}_{\Gamma, \varepsilon} d\Gamma &= \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma - \\
 &\quad - \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega + \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega + \\
 &\quad + \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T (-\boldsymbol{\sigma}_{\varepsilon} \mathbf{n} + \mathbf{g}_{\varepsilon}) d\Gamma = \\
 &= - \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega + \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega + \\
 &\quad + \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{\varepsilon} d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_{\varepsilon}^T \underbrace{(\boldsymbol{\sigma}_{\varepsilon} \mathbf{n})}_{\mathbf{g}_{R, \varepsilon}} d\Gamma = 0
 \end{aligned} \tag{10.32}$$

Where $\mathbf{g}_{R, \varepsilon}$ is the reaction that appears on the surface where the essential boundary condition is applied.

$$\mathbf{g}_{R, \varepsilon} = \boldsymbol{\sigma}_{\varepsilon} \mathbf{n} \quad \forall \mathbf{r} \in \Gamma^u \tag{10.33}$$

Therefore, the equation that defines the equivalent weak form can be obtained from (10.32) as:

$$\iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega = \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega + \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{\varepsilon} d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{R, \varepsilon} d\Gamma \tag{10.34}$$

10.9.2. Equivalent Eulerian weak form statement

The main aim of the structural analysis is to obtain the displacement field that verifies the essential boundary condition. Moreover, the value of the reaction that appears on the surface where the essential boundary condition is applied may also be

required.

$$\mathbf{u}_\mathcal{L}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_\mathcal{L}(\mathbf{r}_0) = \mathbf{u}_\mathcal{L}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.35)$$

$$\mathbf{g}_{R,\varepsilon}(\mathbf{r}) \quad \mathbf{r} \in \Gamma^u \quad (10.36)$$

The above unknowns verify the following equation.

$$\begin{aligned} & \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) d\Omega = \\ & = \iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{b}_\varepsilon \rho_\varepsilon d\Omega + \iint_{\Gamma^\sigma} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_\varepsilon d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_{R,\varepsilon} d\Gamma \quad \forall \boldsymbol{\omega}_\varepsilon \in H_\omega \end{aligned} \quad (10.37)$$

10.10. Lagrangian weak form

The weak form obtained in the previous section is an Eulerian one, since it is composed by integrals defined over the unknown deformed material domain, and the magnitudes involved in it are described according to their Eulerian description. This is an important issue, since a computation over an unknown material domain can not be performed.

To overcome this inconvenience, the main aim of this section is to obtain an equivalent weak form composed by integrals defined over the initial material domain, which is well-know. If this objective is accomplished, the integrands will be composed by magnitudes defined according to their Lagrangian description, and the entire weak form statement will be composed by Lagrangian magnitudes.

10.10.1. Lagrangian left-hand side

Let's focus now on the left-hand side of the Eulerian weak form (10.37). If the property of the trace operator stated in (A.81) is applied as, an equivalent expression of the integrand can be obtained as:

$$\text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) = \frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} : \boldsymbol{\sigma}_\varepsilon^T \quad (10.38)$$

Moreover, the test function gradient tensor can be decomposed as the sum of a symmetric and a skew-symmetric tensor.

$$\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} = \mathbf{A} + \mathbf{B} \quad \text{with} \quad \begin{cases} \mathbf{A} = \frac{1}{2} \left[\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} + \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \right)^T \right] = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} = \frac{1}{2} \left[\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} - \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \right)^T \right] = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{cases} \quad (10.39)$$

Where the symmetric and skew-symmetric terms verify that:

$$\begin{aligned} \mathbf{A}^T = \mathbf{A} & \iff A_{ji} = A_{ij} \\ \mathbf{B}^T = -\mathbf{B} & \iff B_{ji} = \begin{cases} -B_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \end{aligned} \quad (10.40)$$

If the above decomposition and the symmetry of the Cauchy stress tensor are taken into account, the integrand (10.38) can be simplified as shown below. It is also recalled that the double dot product between a skew-symmetric tensor and a symmetric one is equal to zero.

$$\text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{dr} \boldsymbol{\sigma}_\varepsilon \right) = \frac{d\boldsymbol{\omega}_\varepsilon}{dr} : \boldsymbol{\sigma}_\varepsilon^T = (\mathbf{A} + \mathbf{B}) : \boldsymbol{\sigma}_\varepsilon = \mathbf{A} : \boldsymbol{\sigma}_\varepsilon + \underbrace{\mathbf{B} : \boldsymbol{\sigma}_\varepsilon}_{=0} = \mathbf{A} : \boldsymbol{\sigma}_\varepsilon \quad (10.41)$$

The integration of the above result over the material domain leads to an equivalent expression of the left-hand side.

$$\iiint_\Omega \text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{dr} \boldsymbol{\sigma}_\varepsilon \right) d\Omega = \iiint_\Omega \frac{1}{2} \left[\frac{d\mathbf{w}_\varepsilon}{dr} + \left(\frac{d\mathbf{w}_\varepsilon}{dr} \right)^T \right] : \boldsymbol{\sigma}_\varepsilon d\Omega \quad (10.42)$$

Let's consider the test functions as a displacement field variation compatible with the essential boundary condition.

$$\boldsymbol{\omega}_\varepsilon = \delta \mathbf{u}_\varepsilon \quad (10.43)$$

And its corresponding gradient tensor is:

$$\frac{d\boldsymbol{\omega}_\varepsilon}{dr} = \frac{d\delta \mathbf{u}_\varepsilon}{dr} \quad (10.44)$$

The consideration of the test functions as a compatible displacement field variation, and the change of the integration domain according to the methodology presented in section 1.14, lead to the equivalent Lagrangian description of the left-hand side (10.42).

$$\begin{aligned} \iiint_\Omega \text{Tr} \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \boldsymbol{\sigma}_\varepsilon \right) d\Omega &= \iiint_\Omega \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\varepsilon}{dr} + \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \right)^T \right] : \boldsymbol{\sigma}_\varepsilon d\Omega \\ &= \iiint_{\Omega_0} \left\{ \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\varepsilon}{dr} + \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \right)^T \right] : \boldsymbol{\sigma}_\varepsilon \right\} \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0)} F_\mathcal{L} d\Omega_0 \\ &= \iiint_{\Omega_0} \boldsymbol{\sigma}_\mathcal{L} : \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\varepsilon}{dr} + \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0)} F_\mathcal{L} d\Omega_0 \end{aligned} \quad (10.45)$$

On the other hand, the result obtained in (4.89) is recalled.

$$\frac{1}{2} \left[\frac{\partial \delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}} + \left(\frac{\partial \delta \mathbf{u}_\varepsilon}{\partial \mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0,t)} = \mathbf{F}_\mathcal{L}^{-T} \delta \mathbf{E}_{G,\mathcal{L}} \mathbf{F}_\mathcal{L}^{-1} \quad (10.46)$$

The above equation is now substituted into the Lagrangian left-hand side (10.45). To operate, the trace operator properties mentioned in (A.79,A.81) are recalled, as well as the symmetry of the Green-Lagrange strain tensor, and the definition of the second

Piola-Kirchhoff stress tensor (3.48).

$$\begin{aligned}
 \iiint_{\Omega} \text{Tr} \left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \boldsymbol{\sigma}_{\varepsilon} \right) d\Omega &= \iiint_{\Omega_0} \boldsymbol{\sigma}_{\varepsilon} : \frac{1}{2} \left[\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}_{\varepsilon}(\mathbf{r}_0)} F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \boldsymbol{\sigma}_{\varepsilon} : \left(\mathbf{F}_{\varepsilon}^{-T} \delta \mathbf{E}_{G,\varepsilon} \mathbf{F}_{\varepsilon}^{-1} \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \text{Tr} \left(\boldsymbol{\sigma}_{\varepsilon} \left(\mathbf{F}_{\varepsilon}^{-T} \delta \mathbf{E}_{G,\varepsilon} \mathbf{F}_{\varepsilon}^{-1} \right) \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \text{Tr} \left(\left(\mathbf{F}_{\varepsilon}^{-T} \delta \mathbf{E}_{G,\varepsilon} \mathbf{F}_{\varepsilon}^{-1} \right) \boldsymbol{\sigma}_{\varepsilon} \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \text{Tr} \left(\left(\mathbf{F}_{\varepsilon}^{-T} \delta \mathbf{E}_{G,\varepsilon} \right) \left(\mathbf{F}_{\varepsilon}^{-1} \boldsymbol{\sigma}_{\varepsilon} \right) \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \text{Tr} \left(\left(\mathbf{F}_{\varepsilon}^{-1} \boldsymbol{\sigma}_{\varepsilon} \right) \left(\mathbf{F}_{\varepsilon}^{-T} \delta \mathbf{E}_{G,\varepsilon} \right) \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \text{Tr} \left(\left(\mathbf{F}_{\varepsilon}^{-1} \boldsymbol{\sigma}_{\varepsilon} \mathbf{F}_{\varepsilon}^{-T} \right) \delta \mathbf{E}_{G,\varepsilon} \right) F_{\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} F_{\varepsilon} \left(\mathbf{F}_{\varepsilon}^{-1} \boldsymbol{\sigma}_{\varepsilon} \mathbf{F}_{\varepsilon}^{-T} \right) : \delta \mathbf{E}_{G,\varepsilon} d\Omega_0 \\
 &= \iiint_{\Omega_0} \mathbf{S}_{\varepsilon} : \delta \mathbf{E}_{G,\varepsilon} d\Omega_0
 \end{aligned} \tag{10.47}$$

The above equivalent Lagrangian left-hand side turns out to depend on the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor variation.

This equivalence is the basis of the Total Lagrangian formulation. The integral, initially defined over the unknown material domain, is now referred to the well-known reference configuration. Moreover, this transformation allows to manipulate magnitudes which are defined according to their Lagrangian description, instead of dealing with their Eulerian one.

10.10.2. Lagrangian right-hand side

The right-hand side of the Eulerian weak form (10.37) is:

$$\iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega + \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{\varepsilon} d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{R,\varepsilon} d\Gamma \tag{10.48}$$

The external forces are involved in the above equation, which are supposed to not depend on the displacement field. These are the so-called conservative forces. This assumption can be made when dealing with usual load cases, such as usage structural overload or self-weight loads. If the external loads do depend on the displacement field (non-conservative forces), their treatment has to be addressed differently.

As stated in (10.14), the mass conservation allows to obtain the density field corresponding to the deformed material domain. From this equation, the determinant of

the deformation gradient tensor by means of the initial and final density fields can also be obtained.

$$\rho_{\mathcal{L}} = \frac{\rho_{\mathcal{L}}^0}{F_{\mathcal{L}}} \iff F_{\mathcal{L}} = \frac{\rho_{\mathcal{L}}^0}{\rho_{\mathcal{L}}} \quad (10.49)$$

Let's focus now on the first addend of the right-hand side (10.48). The integration domain can be changed according to the methodology proposed in section 1.14. If the above equation is also taken into account, the determinant of the deformation gradient tensor can be replaced by an equivalent definition in terms of the initial and final density fields.

$$\begin{aligned} \iiint_{\Omega} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega &= \iiint_{\Omega_0} (\boldsymbol{\omega}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon}) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} F_{\mathcal{L}} d\Omega_0 \\ &= \iiint_{\Omega_0} (\boldsymbol{\omega}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}) \left(\frac{\rho_{\mathcal{L}}^0}{\rho_{\mathcal{L}}} \right) d\Omega_0 \\ &= \iiint_{\Omega_0} \boldsymbol{\omega}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 \end{aligned} \quad (10.50)$$

The same strategy is applied to the second addend, in order to obtain an integral defined over the reference material domain. To achieve this goal, the methodology presented in section 1.14 is again applied, and the equation that defines the area variation (1.55) is recalled. Furthermore, the definitions of the stress vector (3.35) and the first Piola-Kirchhoff stress tensor (3.45) are also recalled.

$$\begin{aligned} \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \mathbf{g}_{\varepsilon} d\Gamma &= \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T (\boldsymbol{\sigma}_{\varepsilon} \mathbf{n}) d\Gamma \\ &= \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \boldsymbol{\sigma}_{\varepsilon} (d\Gamma \mathbf{n}) \\ &= \iint_{\Gamma^{\sigma}} \boldsymbol{\omega}_{\varepsilon}^T \boldsymbol{\sigma}_{\varepsilon} d\Gamma \\ &= \iint_{\Gamma_0^{\sigma}} (\boldsymbol{\omega}_{\varepsilon}^T \boldsymbol{\sigma}_{\varepsilon}) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} (F_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T} d\Gamma_0) \\ &= \iint_{\Gamma_0^{\sigma}} \boldsymbol{\omega}_{\mathcal{L}}^T (F_{\mathcal{L}} \boldsymbol{\sigma}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-T}) (d\Gamma_0 \mathbf{n}_0) \\ &= \iint_{\Gamma_0^{\sigma}} \boldsymbol{\omega}_{\mathcal{L}}^T \mathbf{P}_{\mathcal{L}} (d\Gamma_0 \mathbf{n}_0) \\ &= \iint_{\Gamma_0^{\sigma}} \boldsymbol{\omega}_{\mathcal{L}}^T (\mathbf{P}_{\mathcal{L}} \mathbf{n}_0) d\Gamma_0 \\ &= \iint_{\Gamma_0^{\sigma}} \boldsymbol{\omega}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 \end{aligned} \quad (10.51)$$

In the above result, the vector $\mathbf{g}_{0,\mathcal{L}}$ was introduced, which represents the differential force acting on the deformed solid per unit initial area. To clarify its physical interpretation, equation (3.44) is recalled, where the definition of the first Piola-Kirchhoff

stress vector arises.

$$\left. \begin{aligned} d\mathbf{f}_\mathcal{L} &= \mathbf{P}_\mathcal{L} d\Gamma_0 \\ &= \mathbf{P}_\mathcal{L} (d\Gamma_0 \mathbf{n}_0) \\ &= (\mathbf{P}_\mathcal{L} \mathbf{n}_0) d\Gamma_0 \\ &= \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 \end{aligned} \right\} \implies \mathbf{g}_{0,\mathcal{L}} = \frac{d\mathbf{f}_\mathcal{L}}{d\Gamma_0} \quad \forall \mathbf{r}_0 \in \Gamma_0^\sigma \quad (10.52)$$

The manipulation of the third addend is analogous to the one performed in (10.51). Hence, this term becomes:

$$\iint_{\Gamma^u} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_{R,\varepsilon} d\Gamma = \iint_{\Gamma_0^u} \boldsymbol{\omega}_\mathcal{L}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 \quad (10.53)$$

Where $\mathbf{g}_{R,0,\mathcal{L}}$ represents the differential force acting on the deformed solid surface where the essential boundary condition is applied, per unit undeformed area.

$$\mathbf{g}_{R,0,\mathcal{L}} = \frac{d\mathbf{f}_\mathcal{L}}{d\Gamma_0} \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.54)$$

The addition of equations (10.50), (10.51), and (10.53), defines the equivalent Lagrangian right-hand side of the Eulerian weak form.

$$\begin{aligned} &\iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{b}_\varepsilon \rho_\varepsilon d\Omega + \iint_{\Gamma^\sigma} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_\varepsilon d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_{R,\varepsilon} d\Gamma = \\ &= \iiint_{\Omega_0} \boldsymbol{\omega}_\mathcal{L}^T \mathbf{b}_\mathcal{L} \rho_\mathcal{L}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\omega}_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \iint_{\Gamma_0^u} \boldsymbol{\omega}_\mathcal{L}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 \end{aligned} \quad (10.55)$$

The above equation allows to move from an Eulerian to a Lagrangian description of the involved magnitudes. Moreover, the resulting integrals are defined over the initial undeformed domain, which is known, instead of dealing with integrals defined over the unknown material one.

The test functions are finally considered as a displacement field variation compatible with the essential boundary condition.

$$\boldsymbol{\omega}_\mathcal{L} = \delta \mathbf{u}_\mathcal{L} \quad (10.56)$$

Therefore, the Lagrangian right-hand side (10.55) becomes:

$$\iiint_{\Omega_0} \delta \mathbf{u}_\mathcal{L}^T \mathbf{b}_\mathcal{L} \rho_\mathcal{L}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}_\mathcal{L}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 \quad (10.57)$$

And the complete Lagrangian weak form remains as stated in the following subsection.

10.10.3. Lagrangian weak form statement

The Lagrangian left-hand side (10.47) and the right-hand one (10.57) lead to this equivalent Lagrangian weak form.

As stated before, the objective of the structural analysis is to obtain the displacement field that verifies the essential boundary condition. Moreover, the value of the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}_\mathcal{L}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_\mathcal{L}(\mathbf{r}_0) = \mathbf{u}_\mathcal{L}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.58)$$

$$\mathbf{g}_{R,0,\mathcal{L}}(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Gamma_0^u \quad (10.59)$$

The above unknowns verify the following equation.

$$\begin{aligned} & \iiint_{\Omega_0} \mathbf{S}_\mathcal{L} : \delta \mathbf{E}_{G,\mathcal{L}} d\Omega_0 = \\ & = \iiint_{\Omega_0} \delta \mathbf{u}_\mathcal{L}^T \mathbf{b}_\mathcal{L} \rho_\mathcal{L}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}_\mathcal{L}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 \quad \forall \delta \mathbf{u}_\mathcal{L} \in H_{\delta u} \end{aligned} \quad (10.60)$$

The second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor variation are involved in the above weak form. Since the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are conjugate magnitudes (table 7.1), it seems reasonable to adopt the equation that defines the relation between both magnitudes as the constitutive equation.

Moreover, the equation that defines the Green-Lagrange strain tensor by means of the displacement gradient tensor was already defined in section 4.6.2. Therefore, this equation is the compatibility equation adopted in this analysis.

The constitutive and the compatibility equations are exposed in detail in the following subsections.

Compatibility equation

On the one hand, the compatibility equation is the one that defines the strain field by means of the displacement field.

The definition of the Green-Lagrange strain tensor in terms of the displacement gradient tensor was stated in (4.73) as:

$$\mathbf{E}_{G,\mathcal{L}} = \frac{1}{2} \left[\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T + \left(\frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right)^T \frac{\partial \mathbf{u}_\mathcal{L}}{\partial \mathbf{r}_0} \right] \quad (10.61)$$

According to the previous definition, the strain tensor remains constant if a translation is applied to the solid. Moreover, as demonstrated in section 4.6.4, it also remains invariant when a rotation occurs. Therefore, it can be concluded that the Green-Lagrange strain tensor remains constant if a rigid motion (rotation and/or translation) is applied.

This is an important property to deal with a large displacement analysis, since it can be guaranteed that no additional strains will appear when a rigid motion is applied. Hence, the Green-Lagrange strain tensor turns out to be a suitable strain tensor if a nonlinear analysis is carried out.

Constitutive equation

On the other hand, the equation that defines the relation between the strain field and its corresponding stress one is the constitutive equation.

As stated in section 7.6, the second Piola-Kirchhoff stress tensor remains invariant if a rigid motion is applied to the solid. Moreover, it turns out to be work conjugate with the Green-Lagrange strain tensor (table 7.1). That is, the double product between both tensors gives the work per unit volume developed by the internal forces during the deformation process. Therefore, this tensor seems to be an appropriate choice to represent the stress field in nonlinear theory.

In order to set the constitutive equation, the equation that states the mathematical relation between the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor has to be defined.

$$\mathbf{S}_\mathcal{L} = \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}) \quad (10.62)$$

10.10.4. Unknowns

The main unknown to solve is the displacement field that the solid experiments. Once this field is known, the compatibility equation (10.61) allows to define the strain field by means of the displacement field. And finally, the constitutive equation (10.62) defines the stress field corresponding to the previous strain field.

$$\mathbf{u}_\mathcal{L} \implies \mathbf{E}_{G,\mathcal{L}}(\mathbf{u}_\mathcal{L}) \implies \mathbf{S}_\mathcal{L}(\mathbf{E}_{G,\mathcal{L}}(\mathbf{u}_\mathcal{L})) \quad (10.63)$$

Once the above fields are known, the computation of the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{g}_{R,0,\mathcal{L}} = \mathbf{P}_\mathcal{L} \mathbf{n}_0 \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.64)$$

To obtain this reaction, the first Piola-Kirchhoff stress tensor has to be calculated. Equation (3.50) defines the second Piola-Kirchhoff stress tensor by means of the first one. Therefore, the first Piola-Kirchhoff stress tensor can be obtained as:

$$\mathbf{S}_\mathcal{L} = \mathbf{F}_\mathcal{L}^{-1} \mathbf{P}_\mathcal{L} \iff \mathbf{P}_\mathcal{L} = \mathbf{F}_\mathcal{L} \mathbf{S}_\mathcal{L} \quad (10.65)$$

10.10.5. Switch to vector notation

The Voigt notation (section A.14) allows to reach an equivalent Lagrangian left-hand side expressed in vector notation, instead of the tensor one obtained in equation (10.60).

The vectorial form of the second Piola-Kirchhoff stress tensor is:

$$\mathbf{S}_\mathcal{L} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \implies \bar{\mathbf{S}}_\mathcal{L} = \begin{pmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{13} \\ S_{23} \end{pmatrix} \quad (10.66)$$

And the vectorial form of Green-Lagrange strain tensor variation turns out to be:

$$\delta \mathbf{E}_{G,\mathcal{L}} = \begin{bmatrix} \delta E_{11} & \delta E_{12} & \delta E_{13} \\ \delta E_{12} & \delta E_{22} & \delta E_{23} \\ \delta E_{13} & \delta E_{23} & \delta E_{33} \end{bmatrix} \implies \delta \bar{\mathbf{E}}_{G,\mathcal{L}} = \begin{pmatrix} \delta E_{11} \\ \delta E_{22} \\ \delta E_{33} \\ 2\delta E_{12} \\ 2\delta E_{13} \\ 2\delta E_{23} \end{pmatrix} \quad (10.67)$$

According to this notation, the double dot product between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor variation is equal to the scalar product between their equivalent vectorial forms.

$$\mathbf{S}_{\mathcal{L}} : \delta \mathbf{E}_{G,\mathcal{L}} = \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} \quad (10.68)$$

Therefore, the left-hand side of the Lagrangian weak form (10.60) can be equivalently written as:

$$\iiint_{\Omega_0} \mathbf{S}_{\mathcal{L}} : \delta \mathbf{E}_{G,\mathcal{L}} d\Omega_0 = \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \quad (10.69)$$

10.11. Equivalent Lagrangian weak form statement

If the equivalent vectorial expression obtained in (10.69) is adopted, the Lagrangian weak form exposed in section 10.10.3 can be equivalently written as exposed below.

The main objective is to obtain the displacement field that verifies the essential boundary condition, and the reaction that appears on the surface where the essential boundary condition is applied (if needed).

$$\mathbf{u}_{\mathcal{L}}(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_{\mathcal{L}}(\mathbf{r}_0) = \mathbf{u}_{\mathcal{L}}^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (10.70)$$

$$\mathbf{g}_{R,0,\mathcal{L}}(\mathbf{r}_0) \quad \mathbf{r}_0 \in \Gamma_0^u \quad (10.71)$$

The above unknowns verify that:

$$\begin{aligned} & \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 = \\ & = \iiint_{\Omega_0} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 \quad \forall \delta \mathbf{u}_{\mathcal{L}} \in H_{\delta u} \end{aligned} \quad (10.72)$$

where the compatibility equation (10.61) and the constitutive one (10.62), are now defined according to their vector notation. The vectorial form of the Green-Lagrange strain tensor defined in (4.109) is recalled.

$$\begin{aligned} \bar{\mathbf{S}}_{\mathcal{L}} &= \bar{\mathbf{S}}_{\mathcal{L}}(\bar{\mathbf{E}}_{G,\mathcal{L}}) & \mathbf{r}_0 \in \Omega_0 \\ \bar{\mathbf{E}}_{G,\mathcal{L}} &= \left(\mathbf{A}_C + \frac{1}{2} \mathbf{A} \right) \bar{\mathbf{J}}_{\mathcal{L}} & \mathbf{r}_0 \in \Omega_0 \end{aligned} \quad (10.73)$$

10.11.1. Alternative left-hand side

The vectorial form of the Green-Lagrange strain tensor variation was defined in (4.112) as:

$$\delta \bar{\mathbf{E}}_{G,\mathcal{L}} = (\mathbf{A}_C + \mathbf{A}) \delta \bar{\mathbf{J}}_{\mathcal{L}} \quad (10.74)$$

Where the vectorial form of the displacement gradient tensor variation can be expressed, in terms of a differential operator, as:

$$\delta \bar{\mathbf{J}}_{\mathcal{L}} = \begin{pmatrix} \frac{\partial \delta u_1}{\partial r_{0,1}} \\ \frac{\partial \delta u_1}{\partial r_{0,2}} \\ \frac{\partial \delta u_1}{\partial r_{0,3}} \\ \frac{\partial \delta u_2}{\partial r_{0,1}} \\ \frac{\partial \delta u_2}{\partial r_{0,2}} \\ \frac{\partial \delta u_2}{\partial r_{0,3}} \\ \frac{\partial \delta u_3}{\partial r_{0,1}} \\ \frac{\partial \delta u_3}{\partial r_{0,2}} \\ \frac{\partial \delta u_3}{\partial r_{0,3}} \end{pmatrix} = \begin{bmatrix} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \end{bmatrix} \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{Bmatrix} = \boldsymbol{\partial}_0 \delta \mathbf{u}_{\mathcal{L}} \quad (10.75)$$

On the other hand, an approach to the test functions is defined by linearly combining the functions that compose the test functions basis.

$$\begin{aligned} \delta \mathbf{u}_{\mathcal{L}} &\approx \delta \mathbf{u}_{\mathcal{L}}^h = \sum_{j=1}^{\eta} \underbrace{[\delta u_{j,\mathcal{L}} \mathbf{I}]}_{\boldsymbol{\Omega}_{j,\mathcal{L}}} \boldsymbol{\beta}_j \\ &= \sum_{j=1}^{\eta} \boldsymbol{\Omega}_{j,\mathcal{L}} \boldsymbol{\beta}_j \\ &= [\boldsymbol{\Omega}_{1,\mathcal{L}} \quad \cdots \quad \boldsymbol{\Omega}_{\eta,\mathcal{L}}] \begin{Bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_{\eta} \end{Bmatrix} \\ &= \boldsymbol{\Omega}_{\mathcal{L}} \boldsymbol{\beta} \quad \delta \mathbf{u}_{\mathcal{L}}^h \in H_{\delta \mathbf{u}}^h \subset H_{\delta \mathbf{u}} \end{aligned} \quad (10.76)$$

The initial displacement field approximation (9.67) defined in linear theory is considered to be the null function. Consequently, the displacement field approximation is

defined by means of the trial functions, as:

$$\begin{aligned}
 \mathbf{u}_\mathcal{L} &\approx \mathbf{u}_\mathcal{L}^h = \sum_{i=1}^{\eta} \underbrace{[\phi_{i,\mathcal{L}} \mathbf{I}]}_{\phi_{i,\mathcal{L}}} \boldsymbol{\alpha}_i \\
 &= \sum_{i=1}^{\eta} \phi_{i,\mathcal{L}} \boldsymbol{\alpha}_i \\
 &= [\phi_{1,\mathcal{L}} \quad \cdots \quad \phi_{\eta,\mathcal{L}}] \begin{Bmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_\eta \end{Bmatrix} \\
 &= \phi_\mathcal{L} \boldsymbol{\alpha} \quad \mathbf{u}_\mathcal{L}^h \in H_u^h \subset H_u
 \end{aligned} \tag{10.77}$$

The Bubnov-Galerkin approach is adopted, so the test and trial functions are equivalent.

$$\boldsymbol{\Omega}_\mathcal{L} = \phi_\mathcal{L} \iff H_{\delta u}^h = H_u^h \tag{10.78}$$

Hence, the test functions are finally defined as:

$$\delta \mathbf{u}_\mathcal{L} \approx \boldsymbol{\Omega}_\mathcal{L} \boldsymbol{\beta} = \phi_\mathcal{L} \boldsymbol{\beta} \tag{10.79}$$

The above approximation is now substituted into the vectorial form of the displacement gradient tensor variation (10.75).

$$\left. \begin{array}{l} \delta \bar{\mathbf{J}}_\mathcal{L} = \boldsymbol{\partial}_0 \delta \mathbf{u}_\mathcal{L} \\ \delta \mathbf{u}_\mathcal{L} \approx \phi_\mathcal{L} \boldsymbol{\beta} \end{array} \right\} \implies \begin{cases} \delta \bar{\mathbf{J}}_\mathcal{L} \approx \boldsymbol{\partial}_0 (\phi_\mathcal{L} \boldsymbol{\beta}) \\ = (\boldsymbol{\partial}_0 \phi_\mathcal{L}) \boldsymbol{\beta} \\ = \mathbf{G}_0 \boldsymbol{\beta} \end{cases} \tag{10.80}$$

The tensor \mathbf{G}_0 is defined by means of the tensor $\phi_\mathcal{L}$ (10.77), and its components turn out to be:

$$\begin{aligned}
 \mathbf{G}_0 &= \boldsymbol{\partial}_0 \phi_\mathcal{L} \\
 &= \boldsymbol{\partial}_0 [\phi_{1,\mathcal{L}} \quad \cdots \quad \phi_{\eta,\mathcal{L}}] \\
 &= [\boldsymbol{\partial}_0 \phi_{1,\mathcal{L}} \quad \cdots \quad \boldsymbol{\partial}_0 \phi_{\eta,\mathcal{L}}] \\
 &= [\mathbf{G}_0^{(1)} \quad \cdots \quad \mathbf{G}_0^{(\eta)}]
 \end{aligned} \tag{10.81}$$

Where:

$$\begin{aligned}
 \mathbf{G}_0^{(k)} &= \partial_0 \phi_{k,\mathcal{L}} \\
 &= \begin{bmatrix} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \end{bmatrix} \begin{bmatrix} \phi_{k,\mathcal{L}} & 0 & 0 \\ 0 & \phi_{k,\mathcal{L}} & 0 \\ 0 & 0 & \phi_{k,\mathcal{L}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \end{bmatrix}
 \end{aligned} \tag{10.82}$$

The substitution of the vectorial form of the displacement gradient tensor variation (10.80) into the vectorial form of the Green-Lagrange strain tensor variation (10.74) leads to:

$$\left. \begin{aligned} \delta \bar{\mathbf{E}}_{G,\mathcal{L}} &= (\mathbf{A}_C + \mathbf{A}) \delta \bar{\mathbf{J}}_{\mathcal{L}} \\ \delta \bar{\mathbf{J}}_{\mathcal{L}} &\approx \mathbf{G}_0 \beta \end{aligned} \right\} \implies \begin{cases} \delta \bar{\mathbf{E}}_{G,\mathcal{L}} \approx (\mathbf{A}_C + \mathbf{A}) (\mathbf{G}_0 \beta) \\ = [(\mathbf{A}_C + \mathbf{A}) \mathbf{G}_0] \beta \\ = \mathbf{B} \beta \end{cases} \tag{10.83}$$

Where the tensor that defines the above relation is:

$$\begin{aligned}
 \mathbf{B} &= (\mathbf{A}_C + \mathbf{A}) \mathbf{G}_0 \\
 &= \mathbf{A}_C \mathbf{G}_0 + \mathbf{A} \mathbf{G}_0 \\
 &= \mathbf{B}_{L,0} + \mathbf{B}_{N,0}
 \end{aligned} \tag{10.84}$$

Two different terms compose this tensor. The first one is $\mathbf{B}_{L,0}$, which is the linear term, since it does not depend on the displacement field.

$$\begin{aligned}
 \mathbf{B}_{L,0} &= \mathbf{A}_C \mathbf{G}_0 \\
 &= \mathbf{A}_C [\mathbf{G}_0^{(1)} \quad \dots \quad \mathbf{G}_0^{(n)}] \\
 &= [\mathbf{A}_C \mathbf{G}_0^{(1)} \quad \dots \quad \mathbf{A}_C \mathbf{G}_0^{(n)}] \\
 &= [\mathbf{B}_{L,0}^{(1)} \quad \dots \quad \mathbf{B}_{L,0}^{(n)}]
 \end{aligned} \tag{10.85}$$

Its components can be defined if the tensor \mathbf{A}_C (4.107), and the definition of the tensor \mathbf{G}_0 stated in (10.82) and (10.81), are recalled.

$$\begin{aligned}
 \mathbf{B}_{L,0}^{(k)} = \mathbf{A}_C \mathbf{G}_0^{(k)} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \end{bmatrix} \tag{10.86}
 \end{aligned}$$

The second term $\mathbf{B}_{N,0}$ is the nonlinear one. This component does depend on the displacement field, since \mathbf{A} depends on the displacement gradient tensor.

$$\begin{aligned}
 \mathbf{B}_{N,0} &= \mathbf{A} \mathbf{G}_0 \\
 &= \mathbf{A} [\mathbf{G}_0^{(1)} \quad \cdots \quad \mathbf{G}_0^{(n)}] \\
 &= [\mathbf{A} \mathbf{G}_0^{(1)} \quad \cdots \quad \mathbf{A} \mathbf{G}_0^{(n)}] \\
 &= [\mathbf{B}_{N,0}^{(1)} \quad \cdots \quad \mathbf{B}_{N,0}^{(n)}] \tag{10.87}
 \end{aligned}$$

In order to simplify the notation, the components of the displacement gradient

tensor are defined as:

$$\mathbf{J}_{\mathcal{L}} = [J_{ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \quad J_{ij} = \frac{\partial u_i}{\partial r_{0,j}} \quad (10.88)$$

And the definitions of \mathbf{A} and \mathbf{G}_0 , exposed in (4.108) and (10.81), are recalled. If these definitions are taken into account, the components of the nonlinear term become:

$$\mathbf{B}_{N,0}^{(k)} = \mathbf{A} \mathbf{G}_0^{(k)}$$

$$= \begin{bmatrix} J_{11} & 0 & 0 & J_{21} & 0 & 0 & J_{31} & 0 & 0 \\ 0 & J_{12} & 0 & 0 & J_{22} & 0 & 0 & J_{32} & 0 \\ 0 & 0 & J_{13} & 0 & 0 & J_{23} & 0 & 0 & J_{33} \\ J_{12} & J_{11} & 0 & J_{22} & J_{21} & 0 & J_{32} & J_{31} & 0 \\ J_{13} & 0 & J_{11} & J_{23} & 0 & J_{21} & J_{33} & 0 & J_{31} \\ 0 & J_{13} & J_{12} & 0 & J_{23} & J_{22} & 0 & J_{33} & J_{32} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \end{bmatrix}$$

$$= \begin{bmatrix} J_{11} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & J_{21} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} & J_{31} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} \\ J_{12} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & J_{22} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & J_{32} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \\ J_{13} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{23} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{33} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \\ J_{12} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{11} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & J_{22} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{21} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} & J_{32} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{31} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} \\ J_{13} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{11} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{23} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{21} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{33} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,1}} + J_{31} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \\ J_{13} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} + J_{12} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{23} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} + J_{22} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} & J_{33} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,2}} + J_{32} \frac{\partial \phi_{k,\mathcal{L}}}{\partial r_{0,3}} \end{bmatrix} \quad (10.89)$$

At this point, the substitution of the vectorial form of the Green-Lagrange strain tensor variation obtained in (10.83) and (10.84), into the left-hand side of the La-

grangian weak form exposed in (10.72), leads to the following alternative expression.

$$\begin{aligned}
 \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 &= \iiint_{\Omega_0} (\mathbf{B} \boldsymbol{\beta})^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \\
 &= \boldsymbol{\beta}^T \iiint_{\Omega_0} \mathbf{B}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \\
 &= \boldsymbol{\beta}^T \iiint_{\Omega_0} \left[(\mathbf{A}_C + \mathbf{A}) \mathbf{G}_0 \right]^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \\
 &= \boldsymbol{\beta}^T \iiint_{\Omega_0} \mathbf{G}_0^T (\mathbf{A}_C + \mathbf{A})^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0
 \end{aligned} \tag{10.90}$$

10.11.2. Right-hand side simplification

On the one hand, the test functions were defined according to the Bubnov-Galerkin method in (10.79) as:

$$\delta \mathbf{u}_{\mathcal{L}} = \boldsymbol{\phi}_{\mathcal{L}} \boldsymbol{\beta} \tag{10.91}$$

If the above definition is substituted, the right-hand side of the Lagrangian weak form (10.72) becomes:

$$\begin{aligned}
 &\iiint_{\Omega_0} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \iint_{\Gamma_0^u} \delta \mathbf{u}_{\mathcal{L}}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 = \\
 &= \boldsymbol{\beta}^T \iiint_{\Omega_0} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \boldsymbol{\beta}^T \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 + \boldsymbol{\beta}^T \iint_{\Gamma_0^u} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0
 \end{aligned} \tag{10.92}$$

On the other hand, the trial functions are defined null over the surface where the essential boundary condition is applied. This condition allows to get rid of the reaction, since it does not appear in the weak form. If it is required, this unknown can be computed once the problem is solved, as exposed in section 10.10.4.

$$\boldsymbol{\phi}_{\mathcal{L}}(\mathbf{r}_0) = \mathbf{0} \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad \implies \quad \boldsymbol{\beta}^T \iint_{\Gamma_0^u} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{g}_{R,0,\mathcal{L}} d\Gamma_0 = 0 \tag{10.93}$$

Therefore, the Lagrangian right-hand side of the weak form can be finally reduced to:

$$\boldsymbol{\beta}^T \iiint_{\Omega_0} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \boldsymbol{\beta}^T \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 \tag{10.94}$$

10.11.3. Final Lagrangian weak form

The equivalent Lagrangian left-hand side (10.90) is now introduced into the Lagrangian weak form (10.72), and the trial functions are defined according to the condition (10.93), so the reaction is not involved in the equation. The Lagrangian weak form is then reduced to:

$$\boldsymbol{\beta}^T \iiint_{\Omega_0} \mathbf{B}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 = \boldsymbol{\beta}^T \iiint_{\Omega_0} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \boldsymbol{\beta}^T \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}_{\mathcal{L}}^T \mathbf{g}_{0,\mathcal{L}} d\Gamma_0 \tag{10.95}$$

The left-hand side depends on the tensor \mathbf{B} , which was defined in (10.84), and on the second Piola-Kirchhoff stress tensor. Both tensors depend on the displacement field. Hence, this side turns out to depend on the displacement field.

Nevertheless, the right-hand side does not depend on it. It depends on the trial functions and the external forces. Both magnitudes depend on the reference configuration geometry, and not on the displacement field.

$$\underbrace{\beta^x \iiint_{\Omega_0} \mathbf{B}^x \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0}_{f(\mathbf{u}_{\mathcal{L}})} = \underbrace{\beta^x \iiint_{\Omega_0} \phi_{\mathcal{L}}^x \mathbf{b}_{\mathcal{L}} \rho_{\mathcal{L}}^0 d\Omega_0 + \beta^x \iint_{\Gamma_0^{\sigma}} \phi_{\mathcal{L}}^x \mathbf{g}_{0,\mathcal{L}} d\Gamma_0}_P \quad (10.96)$$

Therefore, the above equation turns out to be a nonlinear equation whose unknown is the displacement field. This nonlinear equation has to be solved in order to obtain the displacement field that the solid undergoes.

$$f(\mathbf{u}_{\mathcal{L}}) = P \quad (10.97)$$

As stated in section 4.7, an incremental loading procedure has to be applied in order to properly carry out a nonlinear analysis. Hence, each load step has its corresponding Lagrangian weak form (10.96). In the following section, the incremental loading process and its implications are extensively analysed.

10.12. Incremental loading process

The external forces are applied according to the incremental loading process proposed in section 4.7, and the total approach developed in section 4.7.1 is adopted. That is, the initial material domain becomes the reference configuration, and all the variables are defined with respect to this initial domain.

10.12.1. Previous known deformed material domain

Let's consider the load step $t + \Delta t$ of the incremental loading procedure. At this point, the previous one has been already solved, so the material domain corresponding to the load step t is completely defined. The deformation vector that defines the position of a material particle that belongs to this domain turns out to be:

$$\underbrace{\mathbf{r}_{\mathcal{L}}^t(\mathbf{r}_0)}_{\text{known}} = \mathbf{r}_0 + \underbrace{\mathbf{u}_{\mathcal{L}}^t(\mathbf{r}_0)}_{\text{known}} \quad (10.98)$$

And its corresponding deformation gradient tensor, defined by means of the displacement gradient tensor, is:

$$\mathbf{F}_{\mathcal{L}}^t = \frac{d\mathbf{r}_{\mathcal{L}}^t}{d\mathbf{r}_0} = \mathbf{I} + \mathbf{J}_{\mathcal{L}}^t \quad \mathbf{J}_{\mathcal{L}}^t = \frac{d\mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \quad (10.99)$$

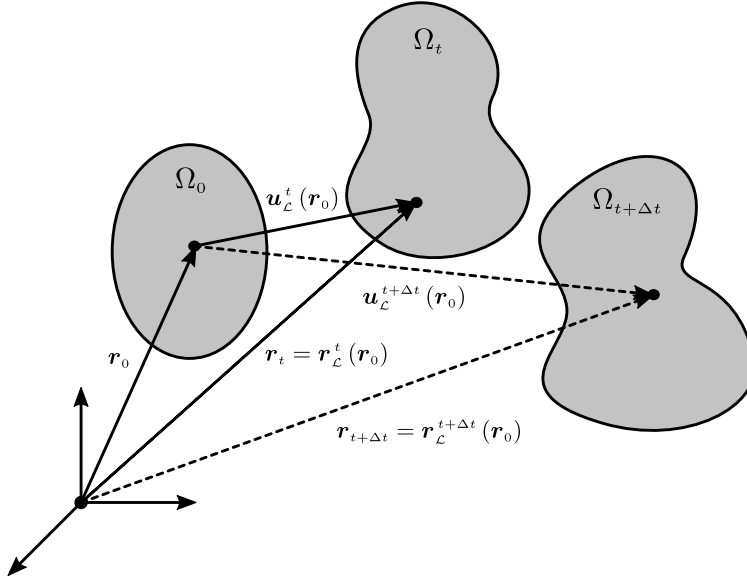


Figure 10.3. Total Lagrangian approach: reference configuration and material domains corresponding to two consecutive load steps.

10.12.2. Current unknown deformed material domain

Once the load step t is solved, an increment is added to the external forces, and the main aim is now to obtain the displacement field corresponding to this new load level. The displacement field caused by this new loading state completely defines its corresponding material domain.

The displacement field corresponding to the current load step $t + \Delta t$ is defined as the previous displacement field plus an increment, which is unknown.

$$\underbrace{\mathbf{u}_L^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{u}_L^t(\mathbf{r}_0) + \underbrace{\Delta \mathbf{u}_L^t(\mathbf{r}_0)}_{\text{unknown}} \quad (10.100)$$

And the deformation vector that defines the position of a material particle that belongs to this material domain is:

$$\underbrace{\mathbf{r}_L^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{r}_0 + \underbrace{\mathbf{u}_L^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} \quad (10.101)$$

If the decomposition of the displacement field (10.100) is taken into account, and the deformation vector of the previous load step (10.98) is recalled, the deformation

vector of the current one becomes:

$$\begin{aligned}
 \mathbf{r}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0) &= \mathbf{r}_0 + \mathbf{u}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0) \\
 &= \mathbf{r}_0 + \left[\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) + \Delta \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) \right] \\
 &= \left[\mathbf{r}_0 + \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) \right] + \Delta \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) \\
 &= \mathbf{r}_\mathcal{L}^t(\mathbf{r}_0) + \Delta \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)
 \end{aligned} \tag{10.102}$$

Moreover, the displacement field decomposition (10.100) allows to define the displacement gradient tensor as the sum of the previous one plus an increment.

$$\begin{aligned}
 \mathbf{J}_\mathcal{L}^{t+\Delta t} &= \frac{d\mathbf{u}_\mathcal{L}^{t+\Delta t}}{d\mathbf{r}_0} = \frac{d}{d\mathbf{r}_0}(\mathbf{u}_\mathcal{L}^t + \Delta \mathbf{u}_\mathcal{L}^t) \\
 &= \frac{d\mathbf{u}_\mathcal{L}^t}{d\mathbf{r}_0} + \frac{d\Delta \mathbf{u}_\mathcal{L}^t}{d\mathbf{r}_0} \\
 &= \mathbf{J}_\mathcal{L}^t + \Delta \mathbf{J}_\mathcal{L}^t
 \end{aligned} \tag{10.103}$$

And the deformation gradient tensor can also be defined as the one corresponding to the previous load step plus an increment.

$$\begin{aligned}
 \mathbf{F}_\mathcal{L}^{t+\Delta t} &= \frac{d\mathbf{r}_\mathcal{L}^{t+\Delta t}}{d\mathbf{r}_0} = \frac{d}{d\mathbf{r}_0}(\mathbf{r}_\mathcal{L}^t + \Delta \mathbf{u}_\mathcal{L}^t) \\
 &= \frac{d\mathbf{r}_\mathcal{L}^t}{d\mathbf{r}_0} + \frac{d\Delta \mathbf{u}_\mathcal{L}^t}{d\mathbf{r}_0} \\
 &= \mathbf{F}_\mathcal{L}^t + \Delta \mathbf{F}_\mathcal{L}^t
 \end{aligned} \tag{10.104}$$

If equations (10.103) and (10.104) are compared, it can be concluded that the deformation gradient tensor increment is equivalent to the displacement gradient tensor increment.

$$\Delta \mathbf{F}_\mathcal{L}^t = \Delta \mathbf{J}_\mathcal{L}^t = \frac{d\Delta \mathbf{u}_\mathcal{L}^t}{d\mathbf{r}_0} \tag{10.105}$$

In addition, the Lagrangian weak form corresponding to load step $t + \Delta t$ turns out to be:

$$\beta^T \iiint_{\Omega_0} \mathbf{B}^T \bar{\mathbf{S}}_\mathcal{L}^{t+\Delta t} d\Omega_0 = \beta^T \iiint_{\Omega_0} \phi_\mathcal{L}^T \mathbf{b}_\mathcal{L}^{t+\Delta t} \rho_\mathcal{L}^0 d\Omega_0 + \beta^T \iint_{\Gamma_0^\sigma} \phi_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}}^{t+\Delta t} d\Gamma_0 \tag{10.106}$$

The above equation is analogous to one presented in (10.95), but the superscripts $t + \Delta t$ are added to indicate the current load step. This is the nonlinear equation to be solved in order to obtain the displacement field corresponding to the current load step.

$$f(\mathbf{u}_\mathcal{L}^{t+\Delta t}) = P^{t+\Delta t} \tag{10.107}$$

10.13. Iterative solution procedure

As obtained in the previous section, the weak form corresponding to the current load step $t + \Delta t$ is the nonlinear equation to be solved in order to obtain the displacement field that the solid experiments after the application of the external incremental forces.

To simplify the notation throughout this section, the subscript \mathcal{L} that indicates that the magnitudes are described according to their Lagrangian description is omitted. The nonlinear function to be solved is then reduced to:

$$f(\mathbf{u}^{t+\Delta t}) = P^{t+\Delta t} \quad (10.108)$$

Where:

$$P^{t+\Delta t} = \beta^T \iiint_{\Omega_0} \phi^T \mathbf{b}^{t+\Delta t} \rho^0 d\Omega_0 + \beta^T \iint_{\Gamma_\sigma^0} \phi^T \mathbf{g}_0^{t+\Delta t} d\Gamma_0 \quad (10.109)$$

Equation (10.108) can be equivalently rewritten as:

$$g(\mathbf{u}^{t+\Delta t}) = f(\mathbf{u}^{t+\Delta t}) - P^{t+\Delta t} = 0 \quad (10.110)$$

Where the nonlinear function g can be interpreted as a residual, and the displacement field corresponding to the current load step is the value that cancels this residual. Therefore, the displacement field turns out to be the root of the following nonlinear equation:

$$\left. \begin{array}{l} g(\mathbf{u}) = f(\mathbf{u}) - P^{t+\Delta t} \\ \mathbf{u} = \mathbf{u}^{t+\Delta t} \end{array} \right\} \implies g(\mathbf{u}^{t+\Delta t}) = 0 \quad (10.111)$$

This root can be obtained if an iterative method is applied. For instance, the Newton-Raphson method is a suitable procedure to take into consideration. This method guarantees quadratic convergence, if the initial solution approximation is close enough to the root.

The incremental load after each load step is assumed to be so small that the displacement fields corresponding to consecutive load step can be considered to be close enough. If this assumption is fulfilled, the displacement field obtained in the previous load step can be adopted to start the iterative procedure. This allows to start the iterative method from a close approximation to the root, and the quadratic convergence should be guaranteed.

10.13.1. Newton-Raphson method

The Newton-Raphson method can be applied to obtain the root. The iterative procedure is applied as follows.

- To start the iterative method, an initial approximation has to be selected. The displacement field corresponding to the previous load step is adopted as the initial

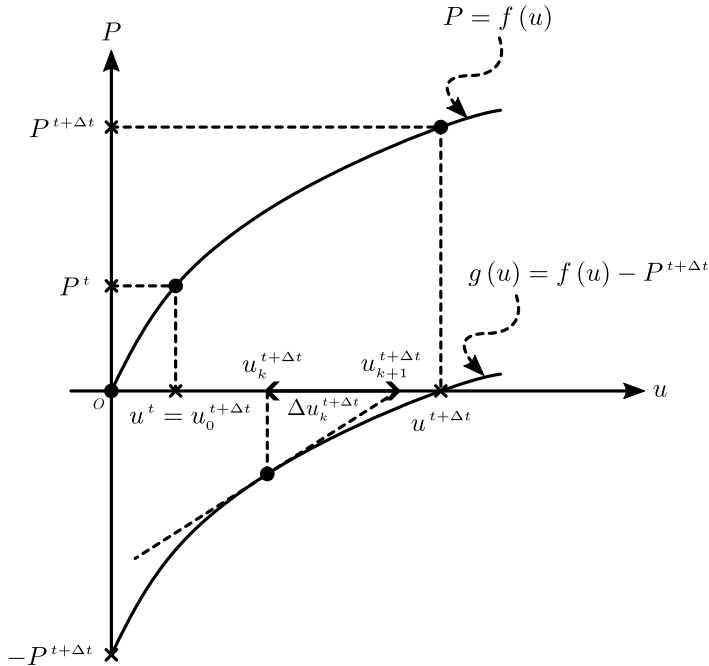


Figure 10.4. Graphical interpretation of an iterative step of the Newton-Raphson method applied to a one-dimensional case.

solution approximation. This choice allows to start the iterative procedure from a close root approximation.

$$\text{Initial approximation: } \mathbf{u}_0^{t+\Delta t} = \mathbf{u}^t \quad (10.112)$$

- It is then considered that the displacement field corresponding to the k -th iteration has already been computed. But its corresponding residual is not zero, so the iterative method has to continue iterating to improve the result.

$$\text{Iteration } k: \mathbf{u}_k^{t+\Delta t} \implies g(\mathbf{u}_k^{t+\Delta t}) \neq 0 \implies \text{? } \mathbf{u}_{k+1}^{t+\Delta t} \text{?} \quad (10.113)$$

- At this point, the next iteration is computed by adding an increment to the displacement field obtained in the previous one.

$$\text{Iteration } k + 1: \mathbf{u}_{k+1}^{t+\Delta t} = \mathbf{u}_k^{t+\Delta t} + \underbrace{\Delta \mathbf{u}_k^{t+\Delta t}}_{\text{?}} \quad (10.114)$$

To calculate the increment of the displacement field between two consecutive iterations, the new residual g is imposed to be zero. The residual is also defined according to its Taylor series expansion about the displacement field obtained from the previous

iteration.

$$g(\mathbf{u}_{k+1}^{t+\Delta t}) = \begin{cases} = 0 \\ = g(\mathbf{u}_k^{t+\Delta t}) + \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} + \mathcal{O}(\|\Delta \mathbf{u}_k^{t+\Delta t}\|^2) \end{cases} \quad (10.115)$$

As the increment of the displacement field between two consecutive iterations is assumed to be small, it can be concluded that:

$$\|\Delta \mathbf{u}_k^{t+\Delta t}\| \ll 1 \quad \implies \quad g(\mathbf{u}_{k+1}^{t+\Delta t}) \approx g(\mathbf{u}_k^{t+\Delta t}) + \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} = 0 \quad (10.116)$$

Or equivalently:

$$\frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} \approx -g(\mathbf{u}_k^{t+\Delta t}) \quad (10.117)$$

Hence, the equation to be solved in order to obtain the increment of the displacement field between two consecutive iterations turns out to be:

$$\left. \begin{aligned} \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} &\approx -g(\mathbf{u}_k^{t+\Delta t}) \\ g(\mathbf{u}) &= f(\mathbf{u}) - P^{t+\Delta t} \\ \frac{dg}{d\mathbf{u}}(\mathbf{u}) &= \frac{df}{d\mathbf{u}}(\mathbf{u}) \end{aligned} \right\} \implies \frac{df}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} \approx P^{t+\Delta t} - f(\mathbf{u}_k^{t+\Delta t}) \quad (10.118)$$

Once the above equation is solved, and the displacement field increment is known, this increment is added to the previous displacement field. And the displacement field corresponding to the current iteration turns out to be:

$$\mathbf{u}_{k+1}^{t+\Delta t} = \mathbf{u}_k^{t+\Delta t} + \Delta \mathbf{u}_k^{t+\Delta t} \quad (10.119)$$

10.13.2. Convergence criterion

A criterion is required to stop the iterative procedure. It can be based on the difference between the displacement field obtained after consecutive iterations. The absolute and relative difference between them are defined as:

$$\begin{aligned} \|\mathbf{u}_{k+1}^{t+\Delta t} - \mathbf{u}_k^{t+\Delta t}\| &= \|\Delta \mathbf{u}_k^{t+\Delta t}\| \\ \frac{\|\mathbf{u}_{k+1}^{t+\Delta t} - \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} &= \frac{\|\Delta \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} \end{aligned} \quad (10.120)$$

And the maximum admitted absolute and relative differences are defined by the parameters E_u and ε_u , respectively. These parameters are user defined and depend on the precision required.

$$\begin{aligned} \|\Delta \mathbf{u}_k^{t+\Delta t}\| &\leq E_u \\ \frac{\|\Delta \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} &\leq \varepsilon_u \iff \|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\| \end{aligned} \quad (10.121)$$

The above inequations can be gathered, and a reasonable convergence criterion applied to the displacement field is shown below.

$$\|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \max \left\{ \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\|, E_u \right\} \quad (10.122)$$

The first term of the above criterion might be too small if the displacement field experimented by the solid is small. To achieve convergence only based on this criterion becomes not possible. Thus, the addition of the second term becomes necessary to define a suitable convergence criterion. In case the displacement field is small, this second value is the one that will rule the criterion.

But the convergence criterion can not be based exclusively on the difference of the displacement field obtained in consecutive iterations. Convergence may not have been reached, even if this difference is small. To solve this inconvenience, the same criterion can be applied to the residual obtained after each iteration. In this case, the maximum residual, and the maximum relative residual calculated with respect to the one corresponding to the initial approximation, are defined by the parameters E_g and ε_g .

$$\frac{|g(\mathbf{u}_{k+1}^{t+\Delta t})|}{|g(\mathbf{u}^t)|} \leq \varepsilon_g \iff |g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq \varepsilon_g |g(\mathbf{u}^t)| \quad (10.123)$$

The above inequations can be gathered, and the residual criterion becomes:

$$|g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq \max \left\{ \varepsilon_g |g(\mathbf{u}^t)|, E_g \right\} \quad (10.124)$$

Therefore, if the displacement convergence criterion (10.122) and the residual one (10.124) are taken into account, the convergence criterion to stop the iterative procedure turns out to be:

$$\text{convergence } \checkmark \iff \begin{cases} \|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \max \left\{ \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\|, E_u \right\} \\ \text{and} \\ |g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq \max \left\{ \varepsilon_g |g(\mathbf{u}^t)|, E_g \right\} \end{cases} \quad (10.125)$$

10.14. Newton-Raphson equation

The equation to be solved at each iteration of the Newton-Raphson method was defined in (10.118) as:

$$\frac{df}{d\mathbf{u}_\mathcal{L}}(\mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}) \Delta \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}) \quad (10.126)$$

Where the scalar function f represents the left-hand side of the Lagrangian weak form, and was defined in (10.72) as:

$$f(\mathbf{u}_\mathcal{L}) = \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_\mathcal{L} d\Omega_0 \quad (10.127)$$

To compute the gradient of the above scalar function, it is taken into account that the reference material domain does not depend on the displacement field. Thus, the derivative can be taken inside the integral.

$$\begin{aligned} \frac{df}{d\mathbf{u}_\varepsilon} &= \frac{d}{d\mathbf{u}_\varepsilon} \left(\iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \bar{\mathbf{S}}_\varepsilon d\Omega_0 \right) \\ &= \iiint_{\Omega_0} \frac{d}{d\mathbf{u}_\varepsilon} \left(\delta \bar{\mathbf{E}}_{G,\varepsilon}^T \bar{\mathbf{S}}_\varepsilon \right) d\Omega_0 \end{aligned} \quad (10.128)$$

And the derivative of the above scalar product can be calculated according to the result obtained in (B.10), where the gradient of a generic scalar product is computed.

$$\begin{aligned} \frac{df}{d\mathbf{u}_\varepsilon} &= \iiint_{\Omega_0} \frac{d}{d\mathbf{u}_\varepsilon} \left(\delta \bar{\mathbf{E}}_{G,\varepsilon}^T \bar{\mathbf{S}}_\varepsilon \right) d\Omega_0 \\ &= \iiint_{\Omega_0} \left(\bar{\mathbf{S}}_\varepsilon^T \frac{d(\delta \bar{\mathbf{E}}_{G,\varepsilon})}{d\mathbf{u}_\varepsilon} + \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \frac{d\bar{\mathbf{S}}_\varepsilon}{d\mathbf{u}_\varepsilon} \right) d\Omega_0 \\ &= \iiint_{\Omega_0} \bar{\mathbf{S}}_\varepsilon^T \frac{d(\delta \bar{\mathbf{E}}_{G,\varepsilon})}{d\mathbf{u}_\varepsilon} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \frac{d\bar{\mathbf{S}}_\varepsilon}{d\mathbf{u}_\varepsilon} d\Omega_0 \end{aligned} \quad (10.129)$$

The scalar product between this gradient and the displacement field increment is now computed.

$$\begin{aligned} \frac{df}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon &= \left(\iiint_{\Omega_0} \bar{\mathbf{S}}_\varepsilon^T \frac{d(\delta \bar{\mathbf{E}}_{G,\varepsilon})}{d\mathbf{u}_\varepsilon} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \frac{d\bar{\mathbf{S}}_\varepsilon}{d\mathbf{u}_\varepsilon} d\Omega_0 \right) \Delta \mathbf{u}_\varepsilon \\ &= \iiint_{\Omega_0} \bar{\mathbf{S}}_\varepsilon^T \frac{d(\delta \bar{\mathbf{E}}_{G,\varepsilon})}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \frac{d\bar{\mathbf{S}}_\varepsilon}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon d\Omega_0 \\ &= \iiint_{\Omega_0} \bar{\mathbf{S}}_\varepsilon^T \underbrace{\left(\frac{d(\delta \bar{\mathbf{E}}_{G,\varepsilon})}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon \right)}_{\Delta(\delta \bar{\mathbf{E}}_{G,\varepsilon})} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \underbrace{\left(\frac{d\bar{\mathbf{S}}_\varepsilon}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon \right)}_{\Delta \bar{\mathbf{S}}_\varepsilon} d\Omega_0 \\ &= \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\varepsilon}^T \Delta \bar{\mathbf{S}}_\varepsilon d\Omega_0 + \iiint_{\Omega_0} \bar{\mathbf{S}}_\varepsilon^T \Delta(\delta \bar{\mathbf{E}}_{G,\varepsilon}) d\Omega_0 \end{aligned} \quad (10.130)$$

Let's focus on the second term. The Voigt notation (A.14) is applied to move from vector to tensor notation, since this addend is easier to manipulate if it is expressed in tensor notation.

The vectorial form of the second Piola-Kirchhoff stress tensor becomes:

$$\bar{\mathbf{S}}_\varepsilon = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{13} \\ S_{23} \end{Bmatrix} \implies \mathbf{S}_\varepsilon = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \quad (10.131)$$

And the vectorial form of the Green-Lagrange strain tensor also recovers its tensor definition.

$$\Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) = \begin{pmatrix} \Delta(\delta E_G)_{11} \\ \Delta(\delta E_G)_{22} \\ \Delta(\delta E_G)_{33} \\ 2\Delta(\delta E_G)_{12} \\ 2\Delta(\delta E_G)_{13} \\ 2\Delta(\delta E_G)_{23} \end{pmatrix} \implies \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) = \begin{bmatrix} \Delta(\delta E_G)_{11} & \Delta(\delta E_G)_{12} & \Delta(\delta E_G)_{13} \\ \Delta(\delta E_G)_{12} & \Delta(\delta E_G)_{22} & \Delta(\delta E_G)_{23} \\ \Delta(\delta E_G)_{13} & \Delta(\delta E_G)_{23} & \Delta(\delta E_G)_{33} \end{bmatrix} \quad (10.132)$$

This tensor notation allows to rewrite the scalar product between both vectors as the double dot product between their equivalent tensor expressions.

$$\bar{\mathbf{S}}_{\mathcal{L}}^T \Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) = \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} \quad (10.133)$$

Therefore, equation (10.130) can be rewritten as:

$$\begin{aligned} \frac{df}{d\mathbf{u}_{\mathcal{L}}} \Delta\mathbf{u}_{\mathcal{L}} &= \iiint_{\Omega_0} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta\bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \bar{\mathbf{S}}_{\mathcal{L}}^T \Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) d\Omega_0 \\ &= \iiint_{\Omega_0} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta\bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \end{aligned} \quad (10.134)$$

The first term of the above scalar product is the so-called material component, and the second one turns out to be the geometric component. Both components are extensively analysed in the following subsections, and arise when applying the Newton-Raphson method to solve the Lagrangian weak form.

10.14.1. Material component

The first term of the previous equation is the so-called material component, since it depends on the constitutive tensor of the material. This dependency will be proved later on in this section.

$$\iiint_{\Omega_0} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta\bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \quad (10.135)$$

This component depends on the vectorial form of the second Piola-Kirchhoff stress tensor increment. The gradients of the displacement field increment are assumed to be small at each iteration of the Newton-Raphson procedure. If this assumption is fulfilled, the increment of the vectorial form of the second Piola-Kirchhoff stress tensor was defined in (7.127) as:

$$\Delta\bar{\mathbf{S}}_{\mathcal{L}} \approx \mathbf{C}_2 \Delta\bar{\mathbf{E}}_{G,\mathcal{L}} \quad \text{with} \quad \mathbf{C}_2 = \frac{d\bar{\mathbf{S}}_{\mathcal{L}}}{d\bar{\mathbf{E}}_{G,\mathcal{L}}} \quad (10.136)$$

Moreover, the vectorial form of the Green-Lagrange strain tensor increment was defined in (4.114) as:

$$\Delta\bar{\mathbf{E}}_{G,\mathcal{L}} = (\mathbf{A}_C + \mathbf{A}) \Delta\bar{\mathbf{J}}_{\mathcal{L}} \quad (10.137)$$

But the vectorial form of the displacement gradient tensor increment can be defined by means of the displacement field increment as:

$$\Delta \bar{\mathbf{J}}_{\mathcal{L}} = \begin{Bmatrix} \frac{\partial \Delta u_1}{\partial r_{0,1}} \\ \frac{\partial \Delta u_1}{\partial r_{0,2}} \\ \frac{\partial \Delta u_1}{\partial r_{0,3}} \\ \frac{\partial \Delta u_2}{\partial r_{0,1}} \\ \frac{\partial \Delta u_2}{\partial r_{0,2}} \\ \frac{\partial \Delta u_2}{\partial r_{0,3}} \\ \frac{\partial \Delta u_3}{\partial r_{0,1}} \\ \frac{\partial \Delta u_3}{\partial r_{0,2}} \\ \frac{\partial \Delta u_3}{\partial r_{0,3}} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r_{0,1}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,2}} & 0 & 0 \\ \frac{\partial}{\partial r_{0,3}} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_{0,1}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,2}} & 0 \\ 0 & \frac{\partial}{\partial r_{0,3}} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_{0,1}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,2}} \\ 0 & 0 & \frac{\partial}{\partial r_{0,3}} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{Bmatrix} = \partial_0 \Delta \mathbf{u}_{\mathcal{L}} \quad (10.138)$$

And the displacement field increment can be approximated by the trial functions as:

$$\begin{aligned} \Delta \mathbf{u}_{\mathcal{L}} &\approx \Delta \mathbf{u}_{\mathcal{L}}^h = \sum_{i=1}^{\eta} \underbrace{[\phi_{i,\mathcal{L}} \mathbf{I}]}_{\phi_{i,\mathcal{L}}} \Delta \boldsymbol{\alpha}_i \\ &= \sum_{i=1}^{\eta} \phi_{i,\mathcal{L}} \Delta \boldsymbol{\alpha}_i \\ &= [\phi_{1,\mathcal{L}} \quad \cdots \quad \phi_{\eta,\mathcal{L}}] \begin{Bmatrix} \Delta \boldsymbol{\alpha}_1 \\ \vdots \\ \Delta \boldsymbol{\alpha}_{\eta} \end{Bmatrix} \\ &= \phi_{\mathcal{L}} \Delta \boldsymbol{\alpha} \end{aligned} \quad (10.139)$$

The substitution of the above approximation into the vectorial form of the displacement gradient tensor increment (10.138) leads to:

$$\left. \begin{array}{l} \Delta \bar{\mathbf{J}}_{\mathcal{L}} = \partial_0 \Delta \mathbf{u}_{\mathcal{L}} \\ \Delta \mathbf{u}_{\mathcal{L}} \approx \phi_{\mathcal{L}} \Delta \boldsymbol{\alpha} \end{array} \right\} \implies \begin{cases} \Delta \bar{\mathbf{J}}_{\mathcal{L}} \approx \partial_0 (\phi_{\mathcal{L}} \Delta \boldsymbol{\alpha}) \\ = (\partial_0 \phi_{\mathcal{L}}) \Delta \boldsymbol{\alpha} \\ = \mathbf{G}_0 \Delta \boldsymbol{\alpha} \end{cases} \quad (10.140)$$

Therefore, the vectorial form of the Green-Lagrange strain tensor increment (10.137)

finally becomes:

$$\left. \begin{aligned} \Delta \bar{\mathbf{E}}_{G,\ell} &= (\mathbf{A}_C + \mathbf{A}) \Delta \bar{\mathbf{J}}_\ell \\ \Delta \bar{\mathbf{J}}_\ell &\approx \mathbf{G}_0 \Delta \boldsymbol{\alpha} \end{aligned} \right\} \implies \begin{cases} \Delta \bar{\mathbf{E}}_{G,\ell} \approx (\mathbf{A}_C + \mathbf{A}) (\mathbf{G}_0 \Delta \boldsymbol{\alpha}) \\ = [(\mathbf{A}_C + \mathbf{A}) \mathbf{G}_0] \Delta \boldsymbol{\alpha} \\ = \mathbf{B} \Delta \boldsymbol{\alpha} \end{cases} \quad (10.141)$$

Where the tensor that defines the above relation is:

$$\mathbf{B} = (\mathbf{A}_C + \mathbf{A}) \mathbf{G}_0 \quad (10.142)$$

The substitution of equations (10.83), (10.136) and (10.141) into the primal definition of the material component (10.135) leads to:

$$\left. \begin{aligned} \delta \bar{\mathbf{E}}_{G,\ell} &\approx \mathbf{B} \boldsymbol{\beta} \\ \Delta \bar{\mathbf{S}}_\ell &\approx \mathbf{C}_2 \Delta \bar{\mathbf{E}}_{G,\ell} \\ \Delta \bar{\mathbf{E}}_{G,\ell} &\approx \mathbf{B} \Delta \boldsymbol{\alpha} \end{aligned} \right\} \implies \begin{cases} \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\ell}^T \Delta \bar{\mathbf{S}}_\ell d\Omega_0 \approx \iiint_{\Omega_0} (\mathbf{B} \boldsymbol{\beta})^T (\mathbf{C}_2 (\mathbf{B} \Delta \boldsymbol{\alpha})) d\Omega_0 \\ = \boldsymbol{\beta}^T \left(\iiint_{\Omega_0} \mathbf{B}^T \mathbf{C}_2 \mathbf{B} d\Omega_0 \right) \Delta \boldsymbol{\alpha} \end{cases} \quad (10.143)$$

As proved in the previous result, this term depends on the constitutive tensor \mathbf{C}_2 . For this reason, this component is usually known as the material component.

10.14.2. Geometric component

The second term of the equation (10.134) is the so-called geometric component, since it depends on the stress state and the geometry of the reference material domain. This dependence will be proved later on in this section.

$$\iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\ell}) : \mathbf{S}_\ell d\Omega_0 \quad (10.144)$$

The increment of the Green-Lagrange strain tensor variation was defined in (4.101) as:

$$\Delta(\delta \mathbf{E}_{G,\ell}) = \frac{1}{2} \left(\delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell + \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \right) \quad (10.145)$$

The above equation holds if the gradients of the displacement field increments are small. As stated in the previous section, the gradients of the displacement field increment are assumed to be small at each step of the iterative procedure. Hence, the application of this equation is justified.

If the increment of the Green-Lagrange strain tensor variation (10.145) is introduced into the definition of the geometric component (10.144), and the double dot product

property exposed in (A.64) is taken into account, the geometric component becomes:

$$\begin{aligned}
 & \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 = \\
 &= \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} + \Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 + \frac{1}{2} \iiint_{\Omega_0} \left(\Delta \mathbf{J}_{\mathcal{L}}^T \delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 + \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right)^T : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 + \frac{1}{2} \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0
 \end{aligned} \tag{10.146}$$

Moreover, the integrand of the above equation can be rewritten in a more convenient form, as:

$$\begin{aligned}
 & \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \iiint_{\Omega_0} \left(\delta \mathbf{J}_{\mathcal{L}}^T \Delta \mathbf{J}_{\mathcal{L}} \right) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\
 &= \iiint_{\Omega_0} \begin{bmatrix} \frac{\partial \delta u_1}{\partial r_{0,1}} & \frac{\partial \delta u_1}{\partial r_{0,2}} & \frac{\partial \delta u_1}{\partial r_{0,3}} \\ \frac{\partial \delta u_2}{\partial r_{0,1}} & \frac{\partial \delta u_2}{\partial r_{0,2}} & \frac{\partial \delta u_2}{\partial r_{0,3}} \\ \frac{\partial \delta u_3}{\partial r_{0,1}} & \frac{\partial \delta u_3}{\partial r_{0,2}} & \frac{\partial \delta u_3}{\partial r_{0,3}} \end{bmatrix}^T \begin{bmatrix} \frac{\partial \Delta u_1}{\partial r_{0,1}} & \frac{\partial \Delta u_1}{\partial r_{0,2}} & \frac{\partial \Delta u_1}{\partial r_{0,3}} \\ \frac{\partial \Delta u_2}{\partial r_{0,1}} & \frac{\partial \Delta u_2}{\partial r_{0,2}} & \frac{\partial \Delta u_2}{\partial r_{0,3}} \\ \frac{\partial \Delta u_3}{\partial r_{0,1}} & \frac{\partial \Delta u_3}{\partial r_{0,2}} & \frac{\partial \Delta u_3}{\partial r_{0,3}} \end{bmatrix} : \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} d\Omega_0 \\
 &= \iiint_{\Omega_0} \left\{ \begin{array}{l} \frac{\partial \delta u_1}{\partial r_{0,1}} \\ \frac{\partial \delta u_1}{\partial r_{0,2}} \\ \frac{\partial \delta u_1}{\partial r_{0,3}} \\ \frac{\partial \delta u_2}{\partial r_{0,1}} \\ \frac{\partial \delta u_2}{\partial r_{0,2}} \\ \frac{\partial \delta u_2}{\partial r_{0,3}} \\ \frac{\partial \delta u_3}{\partial r_{0,1}} \\ \frac{\partial \delta u_3}{\partial r_{0,2}} \\ \frac{\partial \delta u_3}{\partial r_{0,3}} \end{array} \right\} \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{11} & S_{12} & S_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{12} & S_{22} & S_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{13} & S_{23} & S_{33} \end{bmatrix} \left\{ \begin{array}{l} \frac{\partial \Delta u_1}{\partial r_{0,1}} \\ \frac{\partial \Delta u_1}{\partial r_{0,2}} \\ \frac{\partial \Delta u_1}{\partial r_{0,3}} \\ \frac{\partial \Delta u_2}{\partial r_{0,1}} \\ \frac{\partial \Delta u_2}{\partial r_{0,2}} \\ \frac{\partial \Delta u_2}{\partial r_{0,3}} \\ \frac{\partial \Delta u_3}{\partial r_{0,1}} \\ \frac{\partial \Delta u_3}{\partial r_{0,2}} \\ \frac{\partial \Delta u_3}{\partial r_{0,3}} \end{array} \right\} d\Omega_0 \\
 &= \iiint_{\Omega_0} \delta \bar{\mathbf{J}}_{\mathcal{L}}^T \hat{\mathbf{S}}_{\mathcal{L}} \Delta \bar{\mathbf{J}}_{\mathcal{L}} d\Omega_0
 \end{aligned} \tag{10.147}$$

Where $\hat{\mathbf{S}}_{\mathcal{L}}$ is composed by three second Piola-Kirchhoff stress tensors, located in the diagonal of a diagonal tensor.

The substitution of the vectorial forms of both the displacement gradient tensor variation (10.80) and the displacement gradient tensor increment (10.140) into the primal expression of the geometric component (10.144) leads to:

$$\left. \begin{array}{l} \delta \bar{\mathbf{J}}_{\mathcal{L}} \approx \mathbf{G}_0 \beta \\ \Delta \bar{\mathbf{J}}_{\mathcal{L}} \approx \mathbf{G}_0 \Delta \alpha \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 = \iiint_{\Omega_0} \delta \bar{\mathbf{J}}_{\mathcal{L}}^T \hat{\mathbf{S}}_{\mathcal{L}} \Delta \bar{\mathbf{J}}_{\mathcal{L}} d\Omega_0 \\ \approx \iiint_{\Omega_0} (\mathbf{G}_0 \beta)^T \hat{\mathbf{S}}_{\mathcal{L}} (\mathbf{G}_0 \Delta \alpha) d\Omega_0 \\ = \beta^T \left(\iiint_{\Omega_0} \mathbf{G}_0^T \hat{\mathbf{S}}_{\mathcal{L}} \mathbf{G}_0 d\Omega_0 \right) \Delta \alpha \end{array} \right. \quad (10.148)$$

According to this result, this component depends both on the second Piola-Kirchhoff stress tensor and the tensor \mathbf{G}_0 , which contains the derivatives of the trial functions with respect to the reference material domain. Therefore, this term turns out to depend on both the stress field and the geometry of the reference configuration. For this reason, it is usually known as the geometric component.

10.14.3. Tangent stiffness

The sum of the material component (10.143) and the geometric one (10.148) leads to the definition of the tangent stiffness.

$$\begin{aligned} \frac{df}{d\mathbf{u}_{\mathcal{L}}} \Delta \mathbf{u}_{\mathcal{L}} &= \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \\ &= \beta^T \left(\iiint_{\Omega_0} \mathbf{B}^T \mathbf{C}_2 \mathbf{B} d\Omega_0 + \iiint_{\Omega_0} \mathbf{G}_0^T \hat{\mathbf{S}}_{\mathcal{L}} \mathbf{G}_0 d\Omega_0 \right) \Delta \alpha \quad (10.149) \\ &= \beta^T (\mathbf{K}_M + \mathbf{K}_G) \Delta \alpha \\ &= \beta^T \mathbf{K}_T \Delta \alpha \end{aligned}$$

Where \mathbf{K}_T is usually known as the tangent stiffness matrix, which is composed by two terms: the material component \mathbf{K}_M and the geometric stiffness matrix \mathbf{K}_G .

$$\begin{aligned} \mathbf{K}_T &= \mathbf{K}_M + \mathbf{K}_G \\ \mathbf{K}_M &= \iiint_{\Omega_0} \mathbf{B}^T \mathbf{C}_2 \mathbf{B} d\Omega_0 \\ \mathbf{K}_G &= \iiint_{\Omega_0} \mathbf{G}_0^T \hat{\mathbf{S}}_{\mathcal{L}} \mathbf{G}_0 d\Omega_0 \end{aligned} \quad (10.150)$$

Material component

The material component of the tangent stiffness was defined in (10.150) as:

$$\mathbf{K}_M = \iiint_{\Omega_0} \mathbf{B}^T \mathbf{C}_2 \mathbf{B} d\Omega_0 \quad (10.151)$$

This component depends on the constitutive tensor, that is, it depends on the mechanical properties of the material.

If it is compared to the linear stiffness matrix (9.108), it can be concluded that both matrices have similar structure. But the linear one is constant, and does not depend on the displacement field. However, this one does depend on the displacement field, since \mathbf{B} (10.142) depends on it.

The substitution of the tensor \mathbf{B} , according to its decomposition outlined in (10.84), leads to:

$$\begin{aligned} \mathbf{K}_M &= \iiint_{\Omega_0} \mathbf{B}^T \mathbf{C}_2 \mathbf{B} d\Omega_0 \\ &= \iiint_{\Omega_0} (\mathbf{B}_{L,0} + \mathbf{B}_{N,0})^T \mathbf{C}_2 (\mathbf{B}_{L,0} + \mathbf{B}_{N,0}) d\Omega_0 \\ &= \iiint_{\Omega_0} \mathbf{B}_{L,0}^T \mathbf{C}_2 \mathbf{B}_{L,0} d\Omega_0 + \iiint_{\Omega_0} \mathbf{B}_{L,0}^T \mathbf{C}_2 \mathbf{B}_{N,0} d\Omega_0 + \\ &\quad + \left(\iiint_{\Omega_0} \mathbf{B}_{L,0}^T \mathbf{C}_2 \mathbf{B}_{N,0} d\Omega_0 \right)^T + \iiint_{\Omega_0} \mathbf{B}_{N,0}^T \mathbf{C}_2 \mathbf{B}_{N,0} d\Omega_0 \end{aligned} \quad (10.152)$$

Therefore, the material component is stated by three different terms:

$$\begin{aligned} \mathbf{K}_M &= \mathbf{K}'_M + \mathbf{K}''_M + (\mathbf{K}''_M)^T + \mathbf{K}'''_M \\ \mathbf{K}'_M &= \iiint_{\Omega_0} \mathbf{B}_{L,0}^T \mathbf{C}_2 \mathbf{B}_{L,0} d\Omega_0 \\ \mathbf{K}''_M &= \iiint_{\Omega_0} \mathbf{B}_{L,0}^T \mathbf{C}_2 \mathbf{B}_{N,0} d\Omega_0 \\ \mathbf{K}'''_M &= \iiint_{\Omega_0} \mathbf{B}_{N,0}^T \mathbf{C}_2 \mathbf{B}_{N,0} d\Omega_0 \end{aligned} \quad (10.153)$$

Geometric stiffness

The second component of the tangent stiffness is the so-called geometric stiffness, and it was defined in (10.150) as:

$$\mathbf{K}_G = \iiint_{\Omega_0} \mathbf{G}_0^T \hat{\mathbf{S}}_{\mathcal{L}} \mathbf{G}_0 d\Omega_0 \quad (10.154)$$

On the one hand, this component depends on the second Piola-Kirchhoff stress tensor. That is, it depends on the stress state that the solid undergoes. On the other hand, it also depends on \mathbf{G}_0 , which contains the derivatives of the trial functions with respect to the reference material domain. Therefore, this term also depends on the geometry of the reference configuration.

10.14.4. Iterative procedure overview

To sum up, the equation to be solved at each step of the Newton-Raphson method is the one presented in (10.126).

$$\frac{df}{d\mathbf{u}_\mathcal{L}}(\mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}) \Delta \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}) \quad (10.155)$$

As obtained in (10.149), the left-hand side of the above equation can be equivalently written as:

$$\frac{df}{d\mathbf{u}_\mathcal{L}}(\mathbf{u}_\mathcal{L}) \Delta \mathbf{u}_\mathcal{L} = \beta^T \mathbf{K}_T(\mathbf{u}_\mathcal{L}) \Delta \boldsymbol{\alpha} \quad (10.156)$$

And the two addends that compose the right-hand side were defined in (10.109), and (10.96), respectively. The definition of the tensor \mathbf{B} obtained in (10.142) is also recalled.

$$\begin{aligned} P^{t+\Delta t} &= \beta^T \left(\iiint_{\Omega_0} \boldsymbol{\phi}_\mathcal{L}^T \mathbf{b}_\mathcal{L}^{t+\Delta t} \rho_\mathcal{L}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}}^{t+\Delta t} d\Gamma_0 \right) \\ f(\mathbf{u}_\mathcal{L}) &= \beta^T \iiint_{\Omega_0} \mathbf{B}^T \bar{\mathbf{S}}_\mathcal{L} d\Omega_0 = \beta^T \iiint_{\Omega_0} \mathbf{G}_0^T (\mathbf{A}_C + \mathbf{A})^T \bar{\mathbf{S}}_\mathcal{L} d\Omega_0 \end{aligned} \quad (10.157)$$

The substitution of (10.156) and (10.157) into (10.155) leads to the system of linear equations to be solved in order to obtain the increment of the displacement field corresponding to the iteration $k + 1$.

$$\begin{aligned} \mathbf{K}_T(\mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}) \Delta \boldsymbol{\alpha}_k &= \iiint_{\Omega_0} \boldsymbol{\phi}_\mathcal{L}^T \mathbf{b}_\mathcal{L}^{t+\Delta t} \rho_\mathcal{L}^0 d\Omega_0 + \iint_{\Gamma_0^\sigma} \boldsymbol{\phi}_\mathcal{L}^T \mathbf{g}_{0,\mathcal{L}}^{t+\Delta t} d\Gamma_0 \\ &\quad - \left[\iiint_{\Omega_0} \mathbf{G}_0^T (\mathbf{A}_C + \mathbf{A})^T \bar{\mathbf{S}}_\mathcal{L} d\Omega_0 \right] \Big|_{\mathbf{u}_\mathcal{L} = \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t}} \end{aligned} \quad (10.158)$$

Where the components of the tangent stiffness matrix were defined in detail in section 10.14.3. Moreover, the discretization methodology and the criteria taken into account to define the trial functions are the same as those exposed in linear theory. They can be checked in section 9.17 and 9.18, respectively.

Once the above linear system is solved, the displacement field is updated, and the convergence criterion (10.125) is verified. If convergence is achieved, the iterative procedure is stopped, and the last iterative result is adopted as the displacement field corresponding to the current load step. If not, the process outlined in this section is repeated in the next iteration.

$$\begin{aligned} \Delta \boldsymbol{\alpha}_k &\implies \Delta \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t} = \boldsymbol{\phi}_\mathcal{L} \Delta \boldsymbol{\alpha}_k \\ &\implies \mathbf{u}_{k+1,\mathcal{L}}^{t+\Delta t} = \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t} + \Delta \mathbf{u}_{k,\mathcal{L}}^{t+\Delta t} \\ &\implies \text{Convergence?} \quad \begin{cases} \text{Yes} &\implies \mathbf{u}_{k+1,\mathcal{L}}^{t+\Delta t} = \mathbf{u}_\mathcal{L}^{t+\Delta t} \quad (\text{stop}) \\ \text{No} &\implies \mathbf{u}_{k+2,\mathcal{L}}^{t+\Delta t} = \mathbf{u}_{k+1,\mathcal{L}}^{t+\Delta t} + \Delta \mathbf{u}_{k+1,\mathcal{L}}^{t+\Delta t} \quad (\text{continue}) \end{cases} \end{aligned} \quad (10.159)$$

10.14.5. Convergence problems

The Newton-Raphson iterative procedure fails to converge if the tangent stiffness matrix is singular. Figure 10.4 represents a single one degree of freedom equilibrium path. The tangent stiffness becomes singular when the slope of the tangent becomes zero. Hence, the Newton-Raphson technique may fail to converge if passing these limit points is required. Many techniques have been proposed to overcome this convergence problem.

The first authors who worked on this topic were Wempner and Riks. Wempner introduced a generalized arc-length method in order to facilitate the incremental calculations near limit points [Wempner, 1971], while Riks focused on the computation of nonlinear equilibrium paths with continuation through limit points [Riks, 1979].

Later on, Crisfield modified Riks' approach in order to obtain a suitable algorithm for its use in conjunction with the finite element method, and improved the convergence characteristics of the previous methods [Crisfield, 1981]. Ramm reviewed the prior methods, and discussed the necessary modifications for their implementation [Ramm, 1981].

The arc-length method, also known as a path following technique, evolved into a commonly used tool in nonlinear finite element analysis. Many authors proposed variations of this method, such as Forde and Stiemer, who presented a new general formulation, with a simplified procedure and a reduction in computational effort [Forde & Stiemer, 1987]. Fafard and Massicotte worked on another variation called the modified Crisfield-Ramm method, based on the former methods proposed by Crisfield and Ramm [Fafard & Massicotte, 1993]. This improved procedure combines the advantages of its two parent methods.

However, even the arc-length methods can fail to converge if they have to deal with very sharp variations of the tangent stiffness at the contours of limit points. To overcome these sharp variations, a novel algorithm was proposed in Hellweg & Crisfield [1998].

The original algorithm, and its subsequent modifications, are included in Memon & Su [2004], where the developments achieved in the previous two decades are reviewed.

10.15. Strain and stress fields update

Once the convergence has been achieved, the displacement field of the current load step $t + \Delta t$ is known. The next step is to compute its corresponding strain and stress fields.

Before solving this load step, the previous one was completely defined. Hence, the displacement, strain, and stress fields corresponding to the load step t are:

$$\mathbf{u}_L^t(\mathbf{r}_0) \implies \mathbf{E}_{G,\mathcal{L}}^t(\mathbf{u}_L^t(\mathbf{r}_0)) \implies \mathbf{S}_L^t\left(\mathbf{E}_{G,\mathcal{L}}^t(\mathbf{u}_L^t(\mathbf{r}_0))\right) \quad (10.160)$$

Let's move then to the next load step $t + \Delta t$, where the displacement field is obtained by applying the iterative procedure exposed in the previous section. The difference

between the displacement field of consecutive load steps defines the displacement field increment.

$$\Delta \mathbf{u}_{\mathcal{L}}^t = \mathbf{u}_{\mathcal{L}}^{t+\Delta t} - \mathbf{u}_{\mathcal{L}}^t \quad (10.161)$$

If the gradients of the displacement field increment are small, the Green-Lagrange strain tensor increment, and the second Piola-Kirchhoff stress tensor increment can be computed according to equations (4.114) and (7.127).

$$\left\| \frac{d\Delta \mathbf{u}_{\mathcal{L}}^t}{d\mathbf{r}_0} \right\| \ll \|\mathbf{I}\| \quad \Rightarrow \quad \begin{cases} \Delta \bar{\mathbf{E}}_{G,\mathcal{L}}^t = (\mathbf{A}_C + \mathbf{A}) \Delta \bar{\mathbf{J}}_{\mathcal{L}}^t \\ \Delta \bar{\mathbf{S}}_{\mathcal{L}}^t = \mathbf{C}_2(\bar{\mathbf{E}}_{G,\mathcal{L}}^t) \Delta \bar{\mathbf{E}}_{G,\mathcal{L}}^t \end{cases} \quad (10.162)$$

Therefore, the updated strain and stress fields are obtained by adding the above increments to the previous results.

$$\begin{aligned} \bar{\mathbf{E}}_{G,\mathcal{L}}^{t+\Delta t} &= \bar{\mathbf{E}}_{G,\mathcal{L}}^t + \Delta \bar{\mathbf{E}}_{G,\mathcal{L}}^t \\ \bar{\mathbf{S}}_{\mathcal{L}}^{t+\Delta t} &= \bar{\mathbf{S}}_{\mathcal{L}}^t + \Delta \bar{\mathbf{S}}_{\mathcal{L}}^t \end{aligned} \quad (10.163)$$

10.16. Overview and conclusions

10.16.1. Nonlinear analysis strategy

In nonlinear analysis, the solid is supposed to experiment large displacements and large displacement gradients.

- On the one hand, the assumption of large displacements implies that the reference configuration and the deformed domain can not be considered coincident. Thus, the equilibrium equations have to be imposed over the deformed material domain, which is the unknown to be solved. As the initial and deformed configurations are not considered equivalent, the determinant of the deformation gradient tensor is not equal to one, and the Lagrangian and Eulerian descriptions of a given magnitude are not coincident.
- On the other hand, the displacement gradients are assumed to be large, so the approximate polar decomposition of the deformation gradient tensor is not acceptable. To properly decompose the deformation gradient tensor, an eigenvalue problem has to be solved. Once this problem is solved, the rotation tensor and the strain tensor can be defined, and the deformation gradient tensor can be decomposed as the product between both tensors. Furthermore, the Cauchy stress tensor can be expressed by means of both tensors.

This work is focused on static analysis. That is, the solid adopts a deformed configuration in static equilibrium after the application of the external loads. Therefore, both the equilibrium of forces and moments have to be fulfilled. It is recalled that the Cauchy stress tensor is symmetric if the angular momentum conservation is verified.

If the angular momentum is conserved, it can be stated that the sum of all torques is zero, and the equilibrium of moments is fulfilled.

The external applied forces are supposed to not depend on the displacement field. This assumption can be made when dealing with usual load cases, such as usage structural overload or self-weight loads.

In addition, the mass is also considered to not vary. Therefore, the mass conservation leads to the equation that defines the value of the density field over time. The density turns out to be defined by means of both the initial density field and the determinant of the deformation gradient tensor.

Besides the equilibrium equations, the boundary conditions have to be properly defined to obtain the mathematical model. Two types of boundary conditions are differentiated: the essential boundary conditions and the natural ones. The essential boundary conditions are based on the definition of the displacement field on a specific solid surface, whereas the natural boundary conditions define the stress vector on the surface where the external surface forces are applied.

Finally, a compatibility equation and a constitutive one have to be stated in order to complete the mathematical model. The compatibility equation defines the strain field by means of the displacement field, whereas the constitutive equation states the stress field in terms of the strain field.

10.16.2. Total Lagrangian nonlinear finite element analysis

The original form of the problem is composed by the equations that compose the mathematical model, written in terms of the residuals of both the equilibrium equation and the natural boundary condition. This original form is usually known as the strong form, and the weighted residual method is then applied to obtain its corresponding weak form.

Under the assumptions made in nonlinear analysis, the weak form turns out to be an Eulerian one. That is, it is composed by integrals defined over the unknown deformed domain, and the magnitudes involved in it are described according to their Eulerian description. This is an important issue, since a computation over an unknown material domain can not be performed. To overcome this inconvenience, the Eulerian weak form can be manipulated in order to obtain its equivalent Lagrangian form.

According to the result obtained in this chapter, the equivalent Lagrangian weak form is composed by integrals defined over the well-known initial configuration, and the magnitudes are described according to their Lagrangian description. The second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor variation are involved in the resulting Lagrangian weak form.

Since the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are conjugate magnitudes, it seems reasonable to adopt the equation that defines the relation between both magnitudes as the constitutive equation. Both magnitudes remain constant if a rigid motion (rotation and/or translation) is applied to the solid. This is an important property to deal with a large displacement analysis, since it can

be guaranteed that no additional strains nor stresses will appear when a rigid motion is applied.

Therefore, the Green-Lagrange strain tensor is adopted to describe the strain field, and the second Piola-Kirchhoff stress tensor represents its corresponding stress field. And the equation that defines the Green-Lagrange strain tensor in terms of the displacement gradient tensor becomes then the compatibility equation, whereas the equation that states the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor is the constitutive equation adopted in this nonlinear analysis.

The Lagrangian weak form turns out to be a nonlinear equation whose unknown is the displacement field. This nonlinear equation has to be solved in order to obtain the displacement field that the solid undergoes. An incremental loading procedure is applied in order to properly carry out the nonlinear analysis. The external forces are applied according to the incremental loading process, and the initial material domain becomes the reference configuration with respect to which all variables are defined.

The main unknown to obtain is the displacement field that the solid experiments. Once the displacement field is known, the compatibility equation allows to define its corresponding strain field, and the constitutive equation leads to the stress field corresponding to the previous strain field. Then, the reaction that appears on the surface where the essential boundary condition is applied can be calculated, if needed.

The root of the nonlinear equation can be obtained by means of an iterative method. The Newton-Raphson method is a suitable option that guarantees quadratic convergence if the initial solution approximation is close enough to the root. The incremental load after each load step is assumed to be so small that the displacement field corresponding to consecutive load steps can be considered to be close enough. If this assumption is fulfilled, the displacement field obtained in the previous load step can be adopted to start the iterative procedure at each load step. This allows to start the iterative method from a close approximation to the root, and the quadratic convergence should be guaranteed.

The iterative procedure is stopped according to a given criterion, which can be based on both the difference between the displacement field obtained after consecutive iterations and the residual obtained after each iteration.

The Newton-Raphson method leads to a system of linear equations which has to be solved to obtain the displacement field corresponding to each step of the iterative process. The matrix of this system of equations is the so-called tangent stiffness, which is composed by the sum of two terms: the material and the geometric components. The material component turns out to depend on the mechanical properties of the material, whereas the geometric component depends on both the stress state and the geometry of the reference material domain.

Once the system of linear equations is solved, the displacement field can be updated, and the convergence criterion has to be verified. If convergence is achieved, the iterative procedure is stopped, and the last iterative result is adopted as the displacement field

corresponding to the current load step. If not, the process is repeated until reaching convergence.

It should be noted that the Newton-Raphson method may fail to converge if the tangent stiffness matrix becomes singular. To deal with these limit points, many techniques have been proposed to successfully pass them. A review of these techniques has also been outlined in this chapter.

Updated Lagrangian finite element analysis

11.1. Introduction

The finite element formulation derived in the previous chapter is usually known as the Total Lagrangian one, since the initial material domain is adopted as the reference configuration along the entire incremental loading process.

This consideration implies that the Lagrangian weak form that leads to the finite element formulation turns out to be composed by integrals defined over the undeformed configuration, which is well-known. Moreover, the magnitudes involved in it are described with respect to the initial configuration. That is, they are defined according to their Lagrangian description.

In this chapter, the updated approach is adopted instead of the total one. According to this updated point of view, the last computed material domain becomes the new reference configuration. Once each load step is solved, its corresponding material domain becomes known, and this configuration is taken as the reference one for the next load step.

11.2. Conceptual problem and analysis hypotheses

In this chapter, the same problem as the one stated in the previous chapters is faced. The conceptual problem was described in section 9.2.

On the one hand, the definition of the initial configuration and the deformed material domain were presented in sections 9.3 and 10.2, respectively. Furthermore, two types of boundary conditions are taken into consideration: the essential boundary conditions (section 10.6.1), and the natural ones (section 10.6.2).

On the other hand, the structural analysis is carried out under the assumption that the solid experiments large displacements and large displacement gradients after the

application of the external forces. The implications of these hypotheses are detailed in sections 10.3.1 and 10.3.2, respectively.

11.3. Eulerian weak form

For this study, the Eulerian weak form obtained in section 10.9.2 of the previous chapter holds.

According to this weak form, the main aim is to obtain the displacement field that verifies the essential boundary condition. Moreover, the computation of the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}_\varepsilon(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_\varepsilon(\mathbf{r}_0) = \mathbf{u}_\varepsilon^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_0^u \quad (11.1)$$

$$\mathbf{g}_{R,\varepsilon}(\mathbf{r}) \quad \mathbf{r} \in \Gamma^u \quad (11.2)$$

The above unknowns verify the following equation:

$$\begin{aligned} & \iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) d\Omega = \\ & = \iiint_{\Omega} \boldsymbol{\omega}_\varepsilon^T \mathbf{b}_\varepsilon \rho_\varepsilon d\Omega + \iint_{\Gamma^\sigma} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_\varepsilon d\Gamma + \iint_{\Gamma^u} \boldsymbol{\omega}_\varepsilon^T \mathbf{g}_{R,\varepsilon} d\Gamma \quad \forall \boldsymbol{\omega}_\varepsilon \in H_\omega \end{aligned} \quad (11.3)$$

This equation is composed by integrals defined over the deformed material domain, which is unknown, and the magnitudes are described according to their Eulerian description.

11.3.1. Equivalent left-hand side

The left-hand side of the Eulerian weak form obtained in the previous section is:

$$\iiint_{\Omega} \text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) d\Omega \quad (11.4)$$

An equivalent expression of the above integrand can be obtained, if the property of the trace operator stated in (A.81) is applied:

$$\text{Tr} \left(\frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} \boldsymbol{\sigma}_\varepsilon \right) = \frac{d\boldsymbol{\omega}_\varepsilon}{d\mathbf{r}} : \boldsymbol{\sigma}_\varepsilon^T \quad (11.5)$$

Let's consider the test functions as a displacement field variation, compatible with the essential boundary conditions. The modified displacement field and its corresponding displacement gradient tensor are:

$$\begin{aligned} \boldsymbol{\omega}_\varepsilon = \delta \mathbf{u}_\varepsilon & \implies \mathbf{u}'_\varepsilon = \mathbf{u}_\varepsilon + \delta \mathbf{u}_\varepsilon \\ & \implies \mathbf{J}'_\varepsilon = \frac{d\mathbf{u}'_\varepsilon}{d\mathbf{r}_0} = \frac{d\mathbf{u}_\varepsilon}{d\mathbf{r}_0} + \frac{d\delta \mathbf{u}_\varepsilon}{d\mathbf{r}_0} = \mathbf{J}_\varepsilon + \delta \mathbf{J}_\varepsilon \end{aligned} \quad (11.6)$$

Let's also consider that the gradients of the displacement field variation are small. Hence, the gradient tensor of the displacement field variation fulfils the following condition.

$$\|\delta \mathbf{J}_\mathcal{L}\| \ll \|\mathbf{I}\| \quad (11.7)$$

The equivalence between the Lagrangian and the Eulerian descriptions of the modified displacement field is:

$$\mathbf{u}'_\mathcal{L} = \mathbf{u}_\mathcal{L} + \delta \mathbf{u}_\mathcal{L} \quad \Longleftrightarrow \quad \mathbf{u}'_\mathcal{E} \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0)} = (\mathbf{u}_\mathcal{E} + \delta \mathbf{u}_\mathcal{E}) \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0)} \quad (11.8)$$

Since the gradients of the displacement field variation fulfil condition (11.7), it can be stated that the gradients of the Eulerian displacement field variation are small, too. To prove it, the equation that defines the derivative of a given Lagrangian magnitude with respect to the reference position vector (1.20) and the definition of the deformation gradient tensor by means of the displacement gradient tensor (1.12) are recalled, as well as the assumption that the solid behaves with large displacement gradients.

$$\left. \begin{array}{l} \delta \mathbf{J}_\mathcal{L} = \frac{d\delta \mathbf{u}_\mathcal{L}}{d\mathbf{r}_0} = \frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_\mathcal{L}(\mathbf{r}_0)} \mathbf{F}_\mathcal{L} \\ \|\delta \mathbf{J}_\mathcal{L}\| \ll \|\mathbf{I}\| \\ \mathbf{F}_\mathcal{L} = \mathbf{I} + \mathbf{J}_\mathcal{L} \\ \|\mathbf{J}_\mathcal{L}\| \ll \|\mathbf{I}\| \end{array} \right\} \Rightarrow \|\mathbf{F}_\mathcal{L}\| \ll \|\mathbf{I}\| \quad \Rightarrow \quad \left\| \frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} \right\| \ll \|\mathbf{I}\| \quad (11.9)$$

On the other hand, the gradient tensor of the Eulerian displacement field variation can be decomposed as the sum of a symmetric tensor and a skew-symmetric one. If the Eulerian displacement field variation is interpreted as a displacement variation applied on the deformed geometry, and taking into account that its gradients are small, the symmetric term can be interpreted as an infinitesimal strain tensor variation defined with respect to the deformed geometry.

$$\frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} = \delta \mathcal{E}_\mathcal{E} + \delta \mathcal{W}_\mathcal{E} \quad \Rightarrow \quad \left\{ \begin{array}{l} \delta \mathcal{E}_\mathcal{E} = \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} + \left(\frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} \right)^T \right] = [\delta \mathcal{E}_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \delta \mathcal{W}_\mathcal{E} = \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} - \left(\frac{d\delta \mathbf{u}_\mathcal{E}}{d\mathbf{r}} \right)^T \right] = [\delta \mathcal{W}_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right. \quad (11.10)$$

Where the components of the symmetric component and the skew-symmetric one verify that:

$$\begin{aligned} \delta \mathcal{E}_\mathcal{E}^T = \delta \mathcal{E}_\mathcal{E} &\quad \Longleftrightarrow \quad \delta \mathcal{E}_{ji} = \delta \mathcal{E}_{ij} \\ \delta \mathcal{W}_\mathcal{E}^T = -\delta \mathcal{W}_\mathcal{E} &\quad \Longleftrightarrow \quad \delta \mathcal{W}_{ji} = \begin{cases} -\delta \mathcal{W}_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \end{aligned} \quad (11.11)$$

If the previous decomposition and the symmetry of the Cauchy stress tensor are taken into account, as well as the nullity of the double dot product between a skew-

symmetric tensor and a symmetric one, the integrand (11.5) can be reduced to:

$$\text{Tr} \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \boldsymbol{\sigma}_\varepsilon \right) = \frac{d\delta \mathbf{u}_\varepsilon}{dr} : \boldsymbol{\sigma}_\varepsilon^T = (\delta \boldsymbol{\mathcal{E}}_\varepsilon + \delta \boldsymbol{\mathcal{W}}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon = \delta \boldsymbol{\mathcal{E}}_\varepsilon : \boldsymbol{\sigma}_\varepsilon + \underbrace{\delta \boldsymbol{\mathcal{W}}_\varepsilon : \boldsymbol{\sigma}_\varepsilon}_{=0} = \delta \boldsymbol{\mathcal{E}}_\varepsilon : \boldsymbol{\sigma}_\varepsilon \quad (11.12)$$

The integration of the above equation over the deformed material domain leads to an equivalent expression of the left-hand side (11.4).

$$\iiint_\Omega \text{Tr} \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \boldsymbol{\sigma}_\varepsilon \right) d\Omega = \iiint_\Omega \delta \boldsymbol{\mathcal{E}}_\varepsilon : \boldsymbol{\sigma}_\varepsilon d\Omega \quad (11.13)$$

11.3.2. Switch to vector notation

The magnitudes involved in the previous equivalent left-hand side are tensor ones. However, it is convenient to switch to vector notation since this one is easier to deal with.

The test function gradient tensor was decomposed in (11.10) into the sum of a symmetric and a skew-symmetric tensor. Its symmetric component was defined as:

$$\delta \boldsymbol{\mathcal{E}}_\varepsilon = \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\varepsilon}{dr} + \left(\frac{d\delta \mathbf{u}_\varepsilon}{dr} \right)^T \right] \quad (11.14)$$

Where the gradient tensor of the Eulerian displacement field variation is:

$$\frac{d\delta \mathbf{u}_\varepsilon}{dr} = \begin{bmatrix} \frac{\partial \delta u_1}{\partial r_1} & \frac{\partial \delta u_1}{\partial r_2} & \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} & \frac{\partial \delta u_2}{\partial r_2} & \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} & \frac{\partial \delta u_3}{\partial r_2} & \frac{\partial \delta u_3}{\partial r_3} \end{bmatrix} \quad (11.15)$$

Hence, the components of the symmetric term turn out to be:

$$\delta \boldsymbol{\mathcal{E}}_\varepsilon = \begin{bmatrix} \frac{\partial \delta u_1}{\partial r_1} & \frac{1}{2} \left(\frac{\partial \delta u_1}{\partial r_2} + \frac{\partial \delta u_2}{\partial r_1} \right) & \frac{1}{2} \left(\frac{\partial \delta u_1}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_1} \right) \\ \frac{1}{2} \left(\frac{\partial \delta u_1}{\partial r_2} + \frac{\partial \delta u_2}{\partial r_1} \right) & \frac{\partial \delta u_2}{\partial r_2} & \frac{1}{2} \left(\frac{\partial \delta u_2}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_2} \right) \\ \frac{1}{2} \left(\frac{\partial \delta u_1}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_1} \right) & \frac{1}{2} \left(\frac{\partial \delta u_2}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_2} \right) & \frac{\partial \delta u_3}{\partial r_3} \end{bmatrix} \quad (11.16)$$

To move from tensor to vector notation, the Voigt notation (section A.14) is applied.

Thus, the vectorial form of the above symmetric tensor is:

$$\delta \mathcal{E}_\varepsilon = \begin{bmatrix} \delta \mathcal{E}_{11} & \delta \mathcal{E}_{12} & \delta \mathcal{E}_{13} \\ \delta \mathcal{E}_{12} & \delta \mathcal{E}_{22} & \delta \mathcal{E}_{23} \\ \delta \mathcal{E}_{13} & \delta \mathcal{E}_{23} & \delta \mathcal{E}_{33} \end{bmatrix} \implies \delta \bar{\mathcal{E}}_\varepsilon = \begin{Bmatrix} \delta \mathcal{E}_{11} \\ \delta \mathcal{E}_{22} \\ \delta \mathcal{E}_{33} \\ 2 \delta \mathcal{E}_{12} \\ 2 \delta \mathcal{E}_{13} \\ 2 \delta \mathcal{E}_{23} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \\ \frac{\partial \delta u_1}{\partial r_2} + \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_2} \end{Bmatrix} \quad (11.17)$$

And the vectorial expression of the Cauchy stress tensor is:

$$\boldsymbol{\sigma}_\varepsilon = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \implies \bar{\boldsymbol{\sigma}}_\varepsilon = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \quad (11.18)$$

This equivalent vector notation allows to replace the double dot product between both tensors by the scalar product between their vectorial forms.

$$\delta \mathcal{E}_\varepsilon : \boldsymbol{\sigma}_\varepsilon = \delta \bar{\mathcal{E}}_\varepsilon^T \bar{\boldsymbol{\sigma}}_\varepsilon \quad (11.19)$$

Consequently, the left-hand side obtained in (11.13) can be equivalently written as:

$$\iiint_\Omega \delta \mathcal{E}_\varepsilon : \boldsymbol{\sigma}_\varepsilon d\Omega = \iiint_\Omega \delta \bar{\mathcal{E}}_\varepsilon^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \quad (11.20)$$

11.3.3. Equivalent Eulerian weak form

The interpretation of the test functions as a compatible displacement field variation leads to the following equivalent Eulerian weak form.

The main aim is to obtain the displacement field that verifies the essential boundary condition. In addition, the computation of the reaction that appears on the surface where the essential boundary condition is applied may also be required.

$$\mathbf{u}_\varepsilon(\mathbf{r}_0) \in H_u \quad | \quad \mathbf{u}_\varepsilon(\mathbf{r}_0) = \mathbf{u}_\varepsilon^0(\mathbf{r}_0) \quad \forall \mathbf{r}_0 \in \Gamma_u^0 \quad (11.21)$$

$$\mathbf{g}_{R,\varepsilon}(\mathbf{r}) \quad \mathbf{r} \in \Gamma_u \quad (11.22)$$

Where the above unknowns verify that:

$$\begin{aligned}
 & \iiint_{\Omega} \delta \bar{\boldsymbol{\mathcal{E}}}_{\varepsilon}^T \bar{\boldsymbol{\sigma}}_{\varepsilon} d\Omega = \\
 & = \iiint_{\Omega} \delta \mathbf{u}_{\varepsilon}^T \mathbf{b}_{\varepsilon} \rho_{\varepsilon} d\Omega + \iint_{\Gamma^{\sigma}} \delta \mathbf{u}_{\varepsilon}^T \mathbf{g}_{\varepsilon} d\Gamma + \iint_{\Gamma^u} \delta \mathbf{u}_{\varepsilon}^T \mathbf{g}_{R,\varepsilon} d\Gamma \quad \forall \delta \mathbf{u}_{\varepsilon} \in H_{\delta u}
 \end{aligned} \tag{11.23}$$

The above equation is the result of substituting the equivalent left-hand side obtained in (11.20) into the Eulerian weak form presented in (11.3).

11.3.4. Equivalent left-hand side in vectorial form

In the previous sections, the change from tensor to vector notation was outlined. The next objective is to obtain an equivalent expression of the Eulerian left-hand side presented in the previous subsection.

On the one hand, the vector $\delta \bar{\boldsymbol{\mathcal{E}}}_{\varepsilon}$ can be expressed by means of the tensor \mathbf{A}_C , which was introduced while defining the vectorial form of the infinitesimal strain tensor (5.70).

$$\delta \bar{\boldsymbol{\mathcal{E}}}_{\varepsilon} = \left\{ \begin{array}{c} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \\ \frac{\partial \delta u_1}{\partial r_2} + \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_3} + \frac{\partial \delta u_3}{\partial r_2} \end{array} \right\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_2} \\ \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_3}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \end{array} \right\} = \mathbf{A}_C \delta \bar{\mathbf{J}}_t \tag{11.24}$$

Where $\delta \bar{\mathbf{J}}_t$ is the vectorial form of the Eulerian test function gradient tensor, which is composed by the rows of the gradient tensor organized in a single column. The subscript t indicates that the derivatives are taken with respect to the deformed geometry corresponding to a given load step t . That is, the current deformed geometry is

assumed to correspond to the load level t .

$$\delta \mathbf{J}_t = \frac{d\delta \mathbf{u}_\varepsilon}{d\mathbf{r}} = \begin{bmatrix} \frac{\partial \delta u_1}{\partial r_1} & \frac{\partial \delta u_1}{\partial r_2} & \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} & \frac{\partial \delta u_2}{\partial r_2} & \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} & \frac{\partial \delta u_3}{\partial r_2} & \frac{\partial \delta u_3}{\partial r_3} \end{bmatrix} \implies \delta \bar{\mathbf{J}}_t = \begin{pmatrix} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_2} \\ \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_3}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \end{pmatrix} \quad (11.25)$$

The above vector can be defined by means of a differential operator which contains derivatives with respect to the deformed geometry. The subscript t is added again to clarify that the derivatives contained in the operator are taken with respect to the deformed geometry corresponding to a given load step t .

$$\delta \bar{\mathbf{J}}_t = \begin{pmatrix} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_2} \\ \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_3}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \end{pmatrix} = \begin{bmatrix} \frac{\partial}{\partial r_1} & 0 & 0 \\ \frac{\partial}{\partial r_2} & 0 & 0 \\ \frac{\partial}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_1} & 0 \\ 0 & \frac{\partial}{\partial r_2} & 0 \\ 0 & \frac{\partial}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_1} \\ 0 & 0 & \frac{\partial}{\partial r_2} \\ 0 & 0 & \frac{\partial}{\partial r_3} \end{bmatrix} \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} = \boldsymbol{\partial}_t \delta \mathbf{u}_\varepsilon \quad (11.26)$$

On the other hand, the Eulerian test functions can be approximated by a linear

combination of the functions that compose the test functions basis.

$$\begin{aligned}
 \delta \mathbf{u}_\varepsilon &\approx \delta \mathbf{u}_\varepsilon^h = \sum_{j=1}^{\eta} \underbrace{[\delta u_{j,\varepsilon} \mathbf{I}]}_{\boldsymbol{\Omega}_{j,\varepsilon}} \boldsymbol{\beta}_j \\
 &= \sum_{j=1}^{\eta} \boldsymbol{\Omega}_{j,\varepsilon} \boldsymbol{\beta}_j \\
 &= [\boldsymbol{\Omega}_{1,\varepsilon} \quad \cdots \quad \boldsymbol{\Omega}_{\eta,\varepsilon}] \begin{Bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_\eta \end{Bmatrix} \\
 &= \boldsymbol{\Omega}_\varepsilon \boldsymbol{\beta} \quad \delta \mathbf{u}_\varepsilon^h \in H_{\delta u}^h \subset H_{\delta u}
 \end{aligned} \tag{11.27}$$

And the Eulerian description of the displacement field can be approximated by means of the trial functions defined according to their Eulerian description.

$$\begin{aligned}
 \mathbf{u}_\varepsilon &\approx \mathbf{u}_\varepsilon^h = \sum_{i=1}^{\eta} \underbrace{[\phi_{i,\varepsilon} \mathbf{I}]}_{\boldsymbol{\phi}_{i,\varepsilon}} \boldsymbol{\alpha}_i \\
 &= \sum_{i=1}^{\eta} \phi_{i,\varepsilon} \boldsymbol{\alpha}_i \\
 &= [\phi_{1,\varepsilon} \quad \cdots \quad \phi_{\eta,\varepsilon}] \begin{Bmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_\eta \end{Bmatrix} \\
 &= \boldsymbol{\phi}_\varepsilon \boldsymbol{\alpha} \quad \mathbf{u}_\varepsilon^h \in H_u^h \subset H_u
 \end{aligned} \tag{11.28}$$

If the Bubnov-Galerkin approach is applied, the functions adopted as the trial functions are also chosen to define the test ones.

$$\boldsymbol{\Omega}_\varepsilon = \boldsymbol{\phi}_\varepsilon \iff H_{\delta u}^h = H_u^h \tag{11.29}$$

According to this approach, the test functions (11.27) become:

$$\delta \mathbf{u}_\varepsilon \approx \boldsymbol{\Omega}_\varepsilon \boldsymbol{\beta} = \boldsymbol{\phi}_\varepsilon \boldsymbol{\beta} \tag{11.30}$$

Therefore, the vector $\delta \bar{\mathbf{J}}_t$ (11.26) can be approximated as:

$$\left. \begin{aligned} \delta \bar{\mathbf{J}}_t &= \boldsymbol{\partial}_t \delta \mathbf{u}_\varepsilon \\ \delta \mathbf{u}_\varepsilon &\approx \boldsymbol{\phi}_\varepsilon \boldsymbol{\beta} \end{aligned} \right\} \implies \begin{cases} \delta \bar{\mathbf{J}}_t \approx \boldsymbol{\partial}_t (\boldsymbol{\phi}_\varepsilon \boldsymbol{\beta}) \\ = (\boldsymbol{\partial}_t \boldsymbol{\phi}_\varepsilon) \boldsymbol{\beta} \\ = \mathbf{G}_t \boldsymbol{\beta} \end{cases} \tag{11.31}$$

The tensor \mathbf{G}_t that establishes the above relation is defined by means of ϕ_ε , which was introduced in (11.28). This tensor can be equivalently defined as:

$$\begin{aligned}
 \mathbf{G}_t &= \partial_t \phi_\varepsilon \\
 &= \partial_t [\phi_{1,\varepsilon} \quad \cdots \quad \phi_{n,\varepsilon}] \\
 &= [\partial_t \phi_{1,\varepsilon} \quad \cdots \quad \partial_t \phi_{n,\varepsilon}] \\
 &= [\mathbf{G}_t^{(1)} \quad \cdots \quad \mathbf{G}_t^{(n)}]
 \end{aligned} \tag{11.32}$$

Where:

$$\begin{aligned}
 \mathbf{G}_t^{(k)} &= \partial_t \phi_{k,\varepsilon} \\
 &= \begin{bmatrix} \frac{\partial}{\partial r_1} & 0 & 0 \\ \frac{\partial}{\partial r_2} & 0 & 0 \\ \frac{\partial}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_1} & 0 \\ 0 & \frac{\partial}{\partial r_2} & 0 \\ 0 & \frac{\partial}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_1} \\ 0 & 0 & \frac{\partial}{\partial r_2} \\ 0 & 0 & \frac{\partial}{\partial r_3} \end{bmatrix} \begin{bmatrix} \phi_{k,\varepsilon} & 0 & 0 \\ 0 & \phi_{k,\varepsilon} & 0 \\ 0 & 0 & \phi_{k,\varepsilon} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 & 0 \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & 0 & 0 \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} \end{bmatrix}
 \end{aligned} \tag{11.33}$$

The substitution of the vectorial form of the test function gradient tensor (11.31) into the vector $\delta \bar{\mathcal{E}}_\varepsilon$ (11.24) leads to:

$$\left. \begin{aligned} \delta \bar{\mathcal{E}}_\varepsilon &= \mathbf{A}_C \delta \bar{\mathbf{J}}_t \\ \delta \bar{\mathbf{J}}_t &\approx \mathbf{G}_t \beta \end{aligned} \right\} \implies \begin{cases} \delta \bar{\mathcal{E}}_\varepsilon \approx \mathbf{A}_C (\mathbf{G}_t \beta) \\ = (\mathbf{A}_C \mathbf{G}_t) \beta \\ = \mathbf{B}_t \beta \end{cases} \tag{11.34}$$

The tensor \mathbf{B}_t that defines the above relation depends on \mathbf{G}_t , which was introduced in (11.32) and (11.33). This tensor can be equivalently written as:

$$\begin{aligned}
 \mathbf{B}_t &= \mathbf{A}_C \mathbf{G}_t \\
 &= \mathbf{A}_C [\mathbf{G}_t^{(1)} \quad \cdots \quad \mathbf{G}_t^{(n)}] \\
 &= [\mathbf{A}_C \mathbf{G}_t^{(1)} \quad \cdots \quad \mathbf{A}_C \mathbf{G}_t^{(n)}] \\
 &= [\mathbf{B}_t^{(1)} \quad \cdots \quad \mathbf{B}_t^{(n)}]
 \end{aligned} \tag{11.35}$$

Where:

$$\begin{aligned}
 \mathbf{B}_t^{(k)} = \mathbf{A}_C \mathbf{G}_t^{(k)} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 & 0 \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & 0 & 0 \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 & 0 \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & 0 \\ 0 & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} & 0 \\ \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_1} \\ 0 & \frac{\partial \phi_{k,\varepsilon}}{\partial r_3} & \frac{\partial \phi_{k,\varepsilon}}{\partial r_2} \end{bmatrix} \quad (11.36)
 \end{aligned}$$

Finally, the substitution of the vector $\delta \bar{\mathcal{E}}_\varepsilon$ defined in (11.34) and (11.35) into the left-hand side of the Eulerian weak form (11.23) leads to the following equivalent left-hand side.

$$\left. \begin{array}{l} \delta \bar{\mathcal{E}}_\varepsilon \approx \mathbf{B}_t \boldsymbol{\beta} \\ \mathbf{B}_t = \mathbf{A}_C \mathbf{G}_t \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \iiint_\Omega \delta \bar{\mathcal{E}}_\varepsilon^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \approx \iiint_\Omega (\mathbf{B}_t \boldsymbol{\beta})^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \\ = \boldsymbol{\beta}^T \iiint_\Omega \mathbf{B}_t^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \\ = \boldsymbol{\beta}^T \iiint_\Omega (\mathbf{A}_C \mathbf{G}_t)^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \\ = \boldsymbol{\beta}^T \iiint_\Omega \mathbf{G}_t^T \mathbf{A}_C^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \end{array} \right. \quad (11.37)$$

11.4. Incremental loading process

The external forces are applied according to the incremental loading process proposed in section 4.7, and the updated approach developed in section 4.7.3 is adopted. That is, the material domain computed after the last load step becomes the reference configuration, and the magnitudes have to be defined with respect to this material domain.

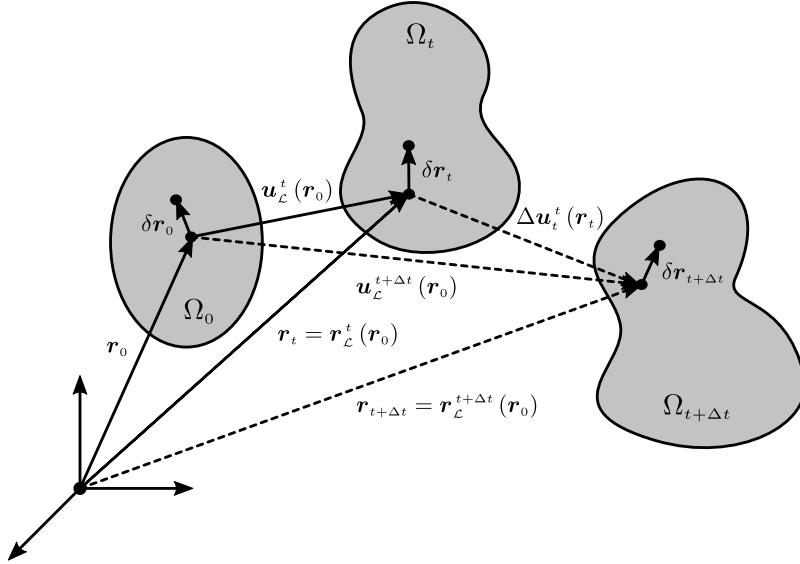


Figure 11.1. Incremental loading process: initial configuration and material domains corresponding to two consecutive load steps.

11.4.1. Reference configuration

If the updated approach is adopted, the material domain defined after the last load step becomes the new reference configuration. Hence, at the load level $t + \Delta t$, the material domain defined at the load step t becomes the reference domain. The vector that defines the position of a material particle that belongs to this domain is:

$$\mathbf{r}_t \in \Omega_t \quad (11.38)$$

And the weak form corresponding to this load step turns out to be:

$$\iiint_{\Omega_t} (\delta \bar{\boldsymbol{\varepsilon}}_t)^T \bar{\boldsymbol{\sigma}}_t d\Omega_t = \iiint_{\Omega_t} \delta \mathbf{u}_t^T \mathbf{b}_t \rho_t d\Omega_t + \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T \mathbf{g}_t d\Gamma_t + \iint_{\Gamma_t^u} \delta \mathbf{u}_t^T \mathbf{g}_{R,\varepsilon}^t d\Gamma_t \quad (11.39)$$

Where $\delta \mathbf{u}_t$ is a compatible displacement variation applied to the displacement field that corresponds to the load step t (\mathbf{u}_t^t). The superscript t is added to the above magnitudes to indicate the load level they correspond to.

The vector $\delta\bar{\mathcal{E}}_\varepsilon^t$ is the vectorial form of the following tensor:

$$\delta\mathcal{E}_\varepsilon^t = \frac{1}{2} \left[\frac{d\delta\mathbf{u}_\varepsilon}{dr_t} + \left(\frac{d\delta\mathbf{u}_\varepsilon}{dr_t} \right)^T \right] \quad (11.40)$$

And the vectorial form of the above tensor was obtained in (11.34) and (11.35) as:

$$\delta\bar{\mathcal{E}}_\varepsilon^t = \mathbf{B}_t \boldsymbol{\beta} \quad \text{with} \quad \mathbf{B}_t = \mathbf{A}_C \mathbf{G}_t \quad (11.41)$$

11.4.2. Unknown deformed material domain

Once the load step t is solved, the incremental external forces are applied and the load step $t + \Delta t$ becomes the current one. The deformation vector that defines the position of a given material particle that belongs to the unknown material domain is:

$$\mathbf{r}_{t+\Delta t} = \underbrace{\mathbf{r}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{r}_0 + \underbrace{\mathbf{u}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} \quad (11.42)$$

The above unknown displacement vector can be decomposed as the sum of the displacement field corresponding to the previous load step plus an increment (figure 11.1).

$$\underbrace{\mathbf{u}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) + \underbrace{\Delta\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)}_{\text{unknown}} \quad (11.43)$$

Consequently, the deformation vector (11.42) becomes:

$$\mathbf{r}_{t+\Delta t} = \underbrace{\mathbf{r}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{r}_0 + \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) + \underbrace{\Delta\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)}_{\text{unknown}} \quad (11.44)$$

But the geometry of the previous material domain is completely defined by the deformation vector corresponding to the previous load step.

$$\mathbf{r}_t = \mathbf{r}_\mathcal{L}^t(\mathbf{r}_0) = \mathbf{r}_0 + \mathbf{u}_\mathcal{L}^t(\mathbf{r}_0) \quad (11.45)$$

Hence, the deformation vector of the current load step (11.44) becomes:

$$\mathbf{r}_{t+\Delta t} = \underbrace{\mathbf{r}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)}_{\text{unknown}} = \mathbf{r}_\mathcal{L}^t(\mathbf{r}_0) + \underbrace{\Delta\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)}_{\text{unknown}} \quad (11.46)$$

The above magnitudes are defined according to its Lagrangian description. That is, they are defined with respect to the initial configuration. However, if the updated approach is adopted, the material domain Ω_t becomes the new reference one. The introduction of equation (11.45) into the above equation, as a change of variable, allows its definition with respect to the new reference material domain.

$$\mathbf{r}_{t+\Delta t} = \underbrace{\mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)}_{\mathbf{r}_\mathcal{L}^{t+\Delta t}(\mathbf{r}_0)} \Big|_{\mathbf{r}_t = \mathbf{r}_\mathcal{L}^t(\mathbf{r}_0)} = \left[\mathbf{r}_t + \Delta\mathbf{u}_t^t(\mathbf{r}_t) \right] \Big|_{\mathbf{r}_t = \mathbf{r}_\mathcal{L}^t(\mathbf{r}_0)} = \underbrace{\mathbf{r}_\mathcal{L}^t(\mathbf{r}_0) + \Delta\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)}_{\mathbf{r}_\mathcal{L}^t(\mathbf{r}_0) + \Delta\mathbf{u}_\mathcal{L}^t(\mathbf{r}_0)} \quad (11.47)$$

Therefore, the deformation vector can be finally defined with respect to the previous material domain as:

$$\mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) = \mathbf{r}_t + \Delta \mathbf{u}_t^t(\mathbf{r}_t) \quad (11.48)$$

The subscript t is added to the above magnitudes to clarify that they are defined with respect to the material domain corresponding to the load step t .

Material vector transformation between consecutive load steps

The tensor that rules the geometric transformation of a material vector between the current unknown material domain and the previous computed one, can be obtained as shown below.

$$\begin{aligned} \delta \mathbf{r}_{t+\Delta t} &= \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t + \delta \mathbf{r}_t) - \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) \\ &= \left[\mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) + \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t}(\mathbf{r}_t) \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \right] - \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) \\ &= \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t}(\mathbf{r}_t) \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \end{aligned} \quad (11.49)$$

Therefore, the tensor that rules this geometric transformation turns out to be:

$$\delta \mathbf{r}_{t+\Delta t} = \mathbf{F}_t^{t+\Delta t} \delta \mathbf{r}_t + \mathcal{O}(\|\delta \mathbf{r}_t\|^2) \quad \text{with} \quad \begin{cases} \mathbf{F}_t^{t+\Delta t} = \frac{d\mathbf{r}_t^{t+\Delta t}}{d\mathbf{r}_t} \\ = \frac{d}{d\mathbf{r}_t}(\mathbf{r}_t + \Delta \mathbf{u}_t^t) \\ = \mathbf{I} + \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \end{cases} \quad (11.50)$$

The main hypothesis is to consider that the gradients of the displacement field increment are small.

$$\left\| \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right\| \ll \|\mathbf{I}\| \quad (11.51)$$

This assumption implies that the displacement increments between consecutive load steps may be large, while the gradients of these increments are necessarily small. Since the gradients are small, the approximate polar decomposition (section 5.3) can be applied to decompose the tensor that rules the incremental geometric transformation presented in (11.50). This decomposition implies that its corresponding strain field is infinitesimal. The strain effect is neglected, so only the effect of the rotation between consecutive load steps is taken into account.

$$\mathbf{F}_t^{t+\Delta t} \approx \mathcal{R}_t^{t+\Delta t} [\mathbf{I} + \boldsymbol{\varepsilon}_t^{t+\Delta t}] \approx \mathcal{R}_t^{t+\Delta t} \quad (11.52)$$

As proved in (5.24) and (5.36), the above infinitesimal rotation tensor has to be orthogonal and represent a proper rotation. Thus, it fulfils the following conditions:

$$(\mathcal{R}_t^{t+\Delta t})^{-1} \approx (\mathcal{R}_t^{t+\Delta t})^T \quad \text{with} \quad \det(\mathcal{R}_t^{t+\Delta t}) \approx 1 \quad (11.53)$$

As demonstrated in section 1.12, the determinant of the tensor that rules the geometric transformation can be physically interpreted as the proportion which represents the differential volume corresponding to the current unknown material domain with respect to the previous computed one. This definition, as well as the property exposed in (11.53), lead to the conclusion that the volume variation between consecutive load steps can be neglected.

$$F_t^{t+\Delta t} = \det \left(\underbrace{\mathbf{F}_t^{t+\Delta t}}_{\mathcal{R}_t^{t+\Delta t}} \right) = \frac{d\Omega_{t+\Delta t}}{d\Omega_t} \approx 1 \quad \iff \quad d\Omega_{t+\Delta t} \approx d\Omega_t \quad (11.54)$$

On the other hand, tensor (11.52) and its determinant state the relation between the differential areas corresponding to consecutive material domains, as proved in section 1.13.

$$\begin{aligned} d\Gamma_{t+\Delta t} &= F_t^{t+\Delta t} (\mathbf{F}_t^{t+\Delta t})^{-T} d\Gamma_t \\ &\approx \mathcal{R}_t^{t+\Delta t} d\Gamma_t \end{aligned} \quad (11.55)$$

Both magnitudes also define the first Piola-Kirchhoff stress tensor, which was extensively defined in section 3.4.

$$\begin{aligned} \mathbf{P}_t^{t+\Delta t} &= F_t^{t+\Delta t} \boldsymbol{\sigma}_t^{t+\Delta t} (\mathbf{F}_t^{t+\Delta t})^{-T} \\ &\approx \boldsymbol{\sigma}_t^{t+\Delta t} \mathcal{R}_t^{t+\Delta t} \end{aligned} \quad (11.56)$$

11.5. Weak form of the current load step

The weak form corresponding to the current load step $t + \Delta t$ turns out to be:

$$\begin{aligned} &\iiint_{\Omega_{t+\Delta t}} (\delta \bar{\boldsymbol{\mathcal{E}}}_\varepsilon^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_\varepsilon^{t+\Delta t} d\Omega_{t+\Delta t} = \\ &= \iiint_{\Omega_{t+\Delta t}} \delta \mathbf{u}_\varepsilon^T \mathbf{b}_\varepsilon^{t+\Delta t} \rho_\varepsilon^{t+\Delta t} d\Omega_{t+\Delta t} + \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_\varepsilon^{t+\Delta t} d\Gamma_{t+\Delta t} + \iint_{\Gamma_{t+\Delta t}^u} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_{R,\varepsilon}^{t+\Delta t} d\Gamma_{t+\Delta t} \end{aligned} \quad (11.57)$$

Where $\delta \mathbf{u}_\varepsilon$ is a compatible displacement field variation applied to the displacement field that corresponds to the load step $t + \Delta t$ ($\mathbf{u}_\varepsilon^{t+\Delta t}$). This equation is equivalent to the one presented in (11.23), adding the superscripts $t + \Delta t$ to clarify the load level.

The vector $\delta \bar{\boldsymbol{\mathcal{E}}}_\varepsilon^{t+\Delta t}$ is the vectorial form of the following tensor:

$$\delta \boldsymbol{\mathcal{E}}_\varepsilon^{t+\Delta t} = \frac{1}{2} \left[\frac{d\delta \mathbf{u}_\varepsilon}{d\mathbf{r}_{t+\Delta t}} + \left(\frac{d\delta \mathbf{u}_\varepsilon}{d\mathbf{r}_{t+\Delta t}} \right)^T \right] \quad (11.58)$$

And its vectorial form was defined in (11.34) and (11.35) as:

$$\delta \bar{\boldsymbol{\mathcal{E}}}_\varepsilon^{t+\Delta t} = \mathbf{B}_{t+\Delta t} \boldsymbol{\beta} \quad \text{with} \quad \mathbf{B}_{t+\Delta t} = \mathbf{A}_C \mathbf{G}_{t+\Delta t} \quad (11.59)$$

Where $\mathbf{B}_{t+\Delta t}$ contains the derivatives of the Eulerian trial functions with respect to the deformed geometry that corresponds to the material domain $\Omega_{t+\Delta t}$.

11.5.1. Right-hand side

The right-hand side of the previous weak form is composed by integrals defined over an unknown deformed material domain.

$$\iiint_{\Omega_{t+\Delta t}} \delta \mathbf{u}_\varepsilon^T \mathbf{b}_\varepsilon^{t+\Delta t} \rho_\varepsilon^{t+\Delta t} d\Omega_{t+\Delta t} + \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_\varepsilon^{t+\Delta t} d\Gamma_{t+\Delta t} + \iint_{\Gamma_{t+\Delta t}^u} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_{R,\varepsilon}^{t+\Delta t} d\Gamma_{t+\Delta t} \quad (11.60)$$

In the following subsections, mathematical manipulations are applied in order to obtain integrals defined over the material domain corresponding to the previous load step, which is completely defined.

The external applied forces are involved in this side of the weak form equation. Let's consider that they do not depend on the displacement field. If the external forces do depend on the displacement field, their treatment has to be faced differently.

Change of the integration domain

The right-hand side is composed by integrals defined over the unknown material domain. In this section, these integrals are manipulated to change the integration domain and obtain integral expressions defined over the configuration computed in the previous load step. As the updated approach is adopted, the material domain corresponding to the previous load step becomes the new reference configuration.

According to the result obtained in (11.54), the volume variation between consecutive load steps can be neglected. Thus, consecutive differential volumes can be considered equivalent. Furthermore, as the mass is assumed to remain constant, the density corresponding to consecutive load steps can be considered equivalent too.

$$d\Omega_{t+\Delta t} \approx d\Omega_t \iff \frac{dm}{\rho_t^{t+\Delta t}} \approx \frac{dm}{\rho_t^t} \iff \rho_t^t \approx \rho_t^{t+\Delta t} \quad (11.61)$$

To change the integration domain of the first addend of the right-hand side (11.60), the methodology proposed in section 1.14 is applied. In addition, the assumptions that there is neither density change nor volume variation between consecutive load steps are taken into account.

$$\begin{aligned} \iiint_{\Omega_{t+\Delta t}} \delta \mathbf{u}_\varepsilon^T \mathbf{b}_\varepsilon^{t+\Delta t} \rho_\varepsilon^{t+\Delta t} d\Omega_{t+\Delta t} &= \iiint_{\Omega_t} \left(\delta \mathbf{u}_\varepsilon^T \mathbf{b}_\varepsilon^{t+\Delta t} \rho_\varepsilon^{t+\Delta t} \right) \Big|_{\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)} F_t^{t+\Delta t} d\Omega_t \\ &\approx \iiint_{\Omega_t} \delta \mathbf{u}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^{t+\Delta t} d\Omega_t \\ &\approx \iiint_{\Omega_t} \delta \mathbf{u}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t \end{aligned} \quad (11.62)$$

The same strategy is applied to the second addend, hence, the methodology presented in section 1.14 is applied. Moreover, the equation that defines the area variation (11.55) and the stress vector definition (3.35) are taken into account, as well

as the definition of the first Piola-Kirchhoff stress tensor with respect to the previous computed material domain (11.56).

$$\begin{aligned}
 \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_\varepsilon^{t+\Delta t} d\Gamma_{t+\Delta t} &= \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T (\boldsymbol{\sigma}_\varepsilon^{t+\Delta t} \mathbf{n}_{t+\Delta t}) d\Gamma_{t+\Delta t} \\
 &= \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T \boldsymbol{\sigma}_\varepsilon^{t+\Delta t} (d\Gamma_{t+\Delta t} \mathbf{n}_{t+\Delta t}) \\
 &= \iint_{\Gamma_{t+\Delta t}^\sigma} \delta \mathbf{u}_\varepsilon^T \boldsymbol{\sigma}_\varepsilon^{t+\Delta t} d\Gamma_{t+\Delta t} \\
 &= \iint_{\Gamma_t^\sigma} \left(\delta \mathbf{u}_\varepsilon^T \boldsymbol{\sigma}_\varepsilon^{t+\Delta t} \right) \Big|_{\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)} \left(\mathbf{F}_t^{t+\Delta t} (\mathbf{F}_t^{t+\Delta t})^{-T} d\Gamma_t \right) \\
 &\approx \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T \left(\boldsymbol{\sigma}_t^{t+\Delta t} \mathbf{R}_t^{t+\Delta t} \right) (d\Gamma_t \mathbf{n}_t) \\
 &= \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T \mathbf{P}_t^{t+\Delta t} (d\Gamma_t \mathbf{n}_t) \\
 &= \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T (\mathbf{P}_t^{t+\Delta t} \mathbf{n}_t) d\Gamma_t \\
 &= \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t
 \end{aligned} \tag{11.63}$$

The vector $\mathbf{g}_{t,t}^{t+\Delta t}$ was introduced in the above result. It represents the differential force acting on $\Gamma_{t+\Delta t}^\sigma$ per unit area of the previous computed solid surface Γ_t^σ . Equation (3.44), where the definition of the first Piola-Kirchhoff stress vector arises, is recalled to clarify its physical interpretation. This equation is rewritten indicating the current load level and clarifying that the reference configuration is the one corresponding to the previous load step t .

$$\left. \begin{aligned}
 d\mathbf{f}_t^{t+\Delta t} &= \mathbf{P}_t^{t+\Delta t} d\Gamma_t \\
 &= \mathbf{P}_t^{t+\Delta t} (d\Gamma_t \mathbf{n}_t) \\
 &= (\mathbf{P}_t^{t+\Delta t} \mathbf{n}_t) d\Gamma_t \\
 &= \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t
 \end{aligned} \right\} \implies \mathbf{g}_{t,t}^{t+\Delta t} = \frac{d\mathbf{f}_t^{t+\Delta t}}{d\Gamma_t} \quad \forall \mathbf{r}_t \in \Gamma_t^\sigma \tag{11.64}$$

The manipulation of the third addend is analogous to the one performed in (11.63). Then, this term becomes:

$$\iint_{\Gamma_{t+\Delta t}^u} \delta \mathbf{u}_\varepsilon^T \mathbf{g}_{R,\varepsilon}^{t+\Delta t} d\Gamma_{t+\Delta t} = \iint_{\Gamma_t^u} \delta \mathbf{u}_t^T \mathbf{g}_{R,t,t}^{t+\Delta t} d\Gamma_t \tag{11.65}$$

Where $\mathbf{g}_{R,t,t}^{t+\Delta t}$ represents the differential force acting on the deformed solid surface where the essential boundary condition is applied, per unit area of the previous solid surface Γ_t^u .

$$\mathbf{g}_{R,t,t}^{t+\Delta t} = \frac{d\mathbf{f}_t^{t+\Delta t}}{d\Gamma_t} \quad \forall \mathbf{r}_t \in \Gamma_t^u \tag{11.66}$$

The addition of equations (11.62), (11.63), and (11.65) defines an equivalent right-hand side of the weak form, where the integrals are defined over the reference material domain previously defined.

$$\iiint_{\Omega_t} \delta \mathbf{u}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \iint_{\Gamma_t^\sigma} \delta \mathbf{u}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t + \iint_{\Gamma_t^u} \delta \mathbf{u}_t^T \mathbf{g}_{R,t,t}^{t+\Delta t} d\Gamma_t \quad (11.67)$$

Equivalent right-hand side

According to the Bubnov-Galerkin method, the test functions were defined in (11.30) as:

$$\delta \mathbf{u}_\varepsilon = \phi_\varepsilon \beta \quad (11.68)$$

The application of the change of variable (11.48) to the above test functions leads to their equivalent definition with respect to the reference material domain.

$$\delta \mathbf{u}_\varepsilon \Big|_{\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)} = (\phi_\varepsilon \beta) \Big|_{\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)} \iff \delta \mathbf{u}_t = \phi_t \beta \quad (11.69)$$

Therefore, the right-hand side (11.67) becomes:

$$\beta^T \iiint_{\Omega_t} \phi_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \beta^T \iint_{\Gamma_t^\sigma} \phi_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t + \beta^T \iint_{\Gamma_t^u} \phi_t^T \mathbf{g}_{R,t,t}^{t+\Delta t} d\Gamma_t \quad (11.70)$$

Furthermore, the imposition of the trial function nullity over the surface where the essential boundary condition is applied allows to get rid of the term that contains the reaction. Then,

$$\phi_t(\mathbf{r}_t) = \mathbf{0} \quad \forall \mathbf{r}_t \in \Gamma_t^u \implies \beta^T \iint_{\Gamma_t^u} \phi_t^T \mathbf{g}_{R,t,t}^{t+\Delta t} d\Gamma_t = 0 \quad (11.71)$$

Therefore, the right-hand side of the weak form is finally reduced to:

$$\beta^T \iiint_{\Omega_t} \phi_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \beta^T \iint_{\Gamma_t^\sigma} \phi_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t \quad (11.72)$$

11.5.2. Left-hand side

In this section, the focus is on the left-hand side of the Eulerian weak form (11.57), which is an integral defined over the unknown material domain.

The integration domain is again changed by applying the methodology presented in section 1.14. The change of variable (11.48) is applied to the integrand, and the equivalence between consecutive differential volumes (11.54) is recalled. This set of operations allows to obtain an integral defined over the previous computed material domain, as well as magnitudes defined with respect to this known configuration.

$$\begin{aligned} \iiint_{\Omega_{t+\Delta t}} (\delta \bar{\boldsymbol{\varepsilon}}_\varepsilon^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_\varepsilon^{t+\Delta t} d\Omega_{t+\Delta t} &= \iiint_{\Omega_t} [(\delta \bar{\boldsymbol{\varepsilon}}_\varepsilon^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_\varepsilon^{t+\Delta t}] \Big|_{\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t)} F_t^{t+\Delta t} d\Omega_t \\ &\approx \iiint_{\Omega_t} (\delta \bar{\boldsymbol{\varepsilon}}_t^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_t^{t+\Delta t} d\Omega_t \end{aligned} \quad (11.73)$$

According to the change of variable (11.48) applied in the above equation, the position of a material particle that belongs to the unknown material domain depends on the increment of the displacement field.

$$\mathbf{r}_{t+\Delta t} = \mathbf{r}_t^{t+\Delta t}(\mathbf{r}_t) = \mathbf{r}_t + \underbrace{\Delta \mathbf{u}_t^t(\mathbf{r}_t)}_{\text{unknown}} \quad (11.74)$$

The above increment defines the displacement field corresponding to the current load step, with respect to the updated reference material domain.

$$\mathbf{u}_t^{t+\Delta t}(\mathbf{r}_t) = \mathbf{u}_t^t(\mathbf{r}_t) + \Delta \mathbf{u}_t^t(\mathbf{r}_t) \quad (11.75)$$

Therefore, the displacement field increment can be equivalently written as the difference between consecutive displacement fields. The displacement field corresponding to the previous load step is assumed to be known, since it was computed previously, whereas the one that corresponds to the current load step is an unknown.

$$\Delta \mathbf{u}_t^t(\mathbf{r}_t) = \underbrace{\mathbf{u}_t^{t+\Delta t}(\mathbf{r}_t)}_{\text{unknown}} - \mathbf{u}_t^t(\mathbf{r}_t) \quad (11.76)$$

Consequently, it can be concluded that the equivalent left-hand side (11.73) depends on the displacement field corresponding to the current load step as:

$$f(\mathbf{u}_t^{t+\Delta t}) = \iiint_{\Omega_t} (\delta \bar{\boldsymbol{\epsilon}}_t^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_t^{t+\Delta t} d\Omega_t \quad (11.77)$$

11.5.3. Equivalent weak form

The left and right-hand sides obtained in (11.73) and (11.72), respectively, are now gathered. This equivalent weak form is finally defined with respect to the previous material domain, which is completely known.

$$\underbrace{\iiint_{\Omega_t} (\delta \bar{\boldsymbol{\epsilon}}_t^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}_t^{t+\Delta t} d\Omega_t}_{f(\mathbf{u}_t^{t+\Delta t})} = \underbrace{\beta^T \iiint_{\Omega_t} \boldsymbol{\phi}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \beta^T \iint_{\Gamma_t^\sigma} \boldsymbol{\phi}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t}_{P^{t+\Delta t}} \quad (11.78)$$

The left-hand side turns out to depend on the displacement field that the solid experiments after the application of the current load step, whereas the right-hand side is a constant that depends on the value of the external forces. Hence, the above equation is the nonlinear equation to be solved in order to obtain the displacement field that the solid undergoes.

This nonlinear equation needs to be solved by iterative methods, and can be stated as:

$$f(\mathbf{u}_t^{t+\Delta t}) = P^{t+\Delta t} \quad (11.79)$$

11.6. Iterative solution procedure

As exposed in the previous section, the weak form corresponding to the current load step $t + \Delta t$ is a nonlinear equation whose solution leads to the displacement field that the solid experiments after the application of the external incremental forces.

To simplify the notation throughout this section, the subscript t that indicates that the magnitudes are described with respect to the material domain corresponding to the previous load step is omitted. Therefore, the nonlinear equation (11.79) is reduced to:

$$f(\mathbf{u}^{t+\Delta t}) = P^{t+\Delta t} \quad (11.80)$$

Where:

$$\begin{aligned} f(\mathbf{u}^{t+\Delta t}) &= \iiint_{\Omega_t} (\delta \bar{\boldsymbol{\varepsilon}}^{t+\Delta t})^T \bar{\boldsymbol{\sigma}}^{t+\Delta t} d\Omega_t \\ P^{t+\Delta t} &= \beta^T \iiint_{\Omega_t} \boldsymbol{\phi}^T \mathbf{b}^{t+\Delta t} \rho^t d\Omega_t + \beta^T \iint_{\Gamma_t^\sigma} \boldsymbol{\phi}^T \mathbf{g}_t^{t+\Delta t} d\Gamma_t \end{aligned} \quad (11.81)$$

Equation (11.80) can be equivalently rewritten as:

$$g(\mathbf{u}^{t+\Delta t}) = f(\mathbf{u}^{t+\Delta t}) - P^{t+\Delta t} = 0 \quad (11.82)$$

Where the nonlinear function g can be interpreted as a residual, and the displacement field corresponding to the current load step turns out to be the value that cancels this residual. That is, this displacement field is the root of the following nonlinear equation.

$$\left. \begin{aligned} g(\mathbf{u}) &= f(\mathbf{u}) - P^{t+\Delta t} \\ f(\mathbf{u}) &= \iiint_{\Omega_t} \delta \bar{\boldsymbol{\varepsilon}}^T \bar{\boldsymbol{\sigma}} d\Omega_t \\ \mathbf{u} &= \mathbf{u}^{t+\Delta t} \end{aligned} \right\} \implies g(\mathbf{u}^{t+\Delta t}) = 0 \quad (11.83)$$

This root can be obtained through an iterative method. For instance, the Newton-Raphson method is a suitable procedure to take into consideration. This method guarantees quadratic convergence, if the initial solution approximation is close enough to the root.

Let's consider that the incremental load after each load step is so small, that the displacement fields corresponding to consecutive load step can be considered to be close enough. If this assumption is fulfilled, the displacement field obtained in the previous load step can be adopted to start the iterative procedure. This choice allows to start the iterative method from a close approximation to the root, and the quadratic convergence should be guaranteed.

11.6.1. Newton-Raphson method

The iterative procedure of the Newton-Raphson method is applied as follows.

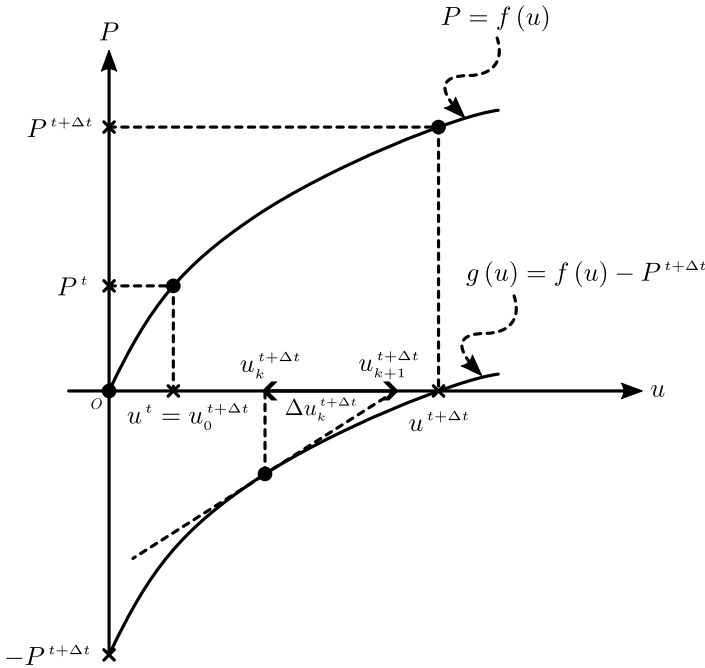


Figure 11.2. Graphical interpretation of an iterative step of the Newton-Raphson method applied to a one-dimensional case.

- An initial approximation is chosen to start the iterative method. The displacement field corresponding to the previous load step is adopted as the initial solution approximation. This choice allows to start the iterative procedure from a close root approximation.

$$\text{Initial approximation: } \mathbf{u}_0^{t+\Delta t} = \mathbf{u}^t \quad (11.84)$$

- Let's consider that the displacement field corresponding to the k -th iteration is already computed, but the residual is not zero. Hence, the iterative method has to continue iterating to improve the result.

$$\text{Iteration } k: \mathbf{u}_k^{t+\Delta t} \implies g(\mathbf{u}_k^{t+\Delta t}) \neq 0 \implies i \mathbf{u}_{k+1}^{t+\Delta t} ? \quad (11.85)$$

- At this point, the next iteration is computed by adding an increment to the displacement field obtained in the previous one.

$$\text{Iteration } k + 1: \mathbf{u}_{k+1}^{t+\Delta t} = \mathbf{u}_k^{t+\Delta t} + \underbrace{\Delta \mathbf{u}_k^{t+\Delta t}}_{i?} \quad (11.86)$$

To calculate the displacement field increment between two consecutive iterations, the nullity of the new residual g is imposed. Moreover, the residual is defined according

to its Taylor series expansion about the displacement field obtained in the previous iteration.

$$g(\mathbf{u}_{k+1}^{t+\Delta t}) = \begin{cases} = 0 \\ = g(\mathbf{u}_k^{t+\Delta t}) + \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} + \mathcal{O}(\|\Delta \mathbf{u}_k^{t+\Delta t}\|^2) \end{cases} \quad (11.87)$$

As the displacement field increment between two consecutive iterations is supposed to be small, it can be stated that:

$$\|\Delta \mathbf{u}_k^{t+\Delta t}\| \ll 1 \quad \implies \quad g(\mathbf{u}_{k+1}^{t+\Delta t}) \approx g(\mathbf{u}_k^{t+\Delta t}) + \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} = 0 \quad (11.88)$$

Or equivalently:

$$\frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} \approx -g(\mathbf{u}_k^{t+\Delta t}) \quad (11.89)$$

Therefore, the equation whose resolution leads to the displacement field increment between consecutive iterations turns out to be:

$$\left. \begin{aligned} \frac{dg}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} &\approx -g(\mathbf{u}_k^{t+\Delta t}) \\ g(\mathbf{u}) &= f(\mathbf{u}) - P^{t+\Delta t} \\ \frac{dg}{d\mathbf{u}}(\mathbf{u}) &= \frac{df}{d\mathbf{u}}(\mathbf{u}) \end{aligned} \right\} \implies \frac{df}{d\mathbf{u}}(\mathbf{u}_k^{t+\Delta t}) \Delta \mathbf{u}_k^{t+\Delta t} \approx P^{t+\Delta t} - f(\mathbf{u}_k^{t+\Delta t}) \quad (11.90)$$

Once the above equation is solved and the displacement field increment is known, this increment is added to the displacement field corresponding to the previous iteration. This calculation defines the displacement field corresponding to the current iteration.

$$\mathbf{u}_{k+1}^{t+\Delta t} = \mathbf{u}_k^{t+\Delta t} + \Delta \mathbf{u}_k^{t+\Delta t} \quad (11.91)$$

11.6.2. Convergence criterion

A criterion is required to stop the iterative procedure. On the one hand, it can be based on the difference between the displacement field obtained in consecutive iterations. The absolute difference between consecutive displacement fields, and its relative difference with respect to the last computed displacement field, are:

$$\begin{aligned} \|\mathbf{u}_{k+1}^{t+\Delta t} - \mathbf{u}_k^{t+\Delta t}\| &= \|\Delta \mathbf{u}_k^{t+\Delta t}\| \\ \frac{\|\mathbf{u}_{k+1}^{t+\Delta t} - \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} &= \frac{\|\Delta \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} \end{aligned} \quad (11.92)$$

The maximum admitted absolute and relative differences are defined by means of the parameters E_u and ε_u , respectively. These parameters are user defined and depend on the precision required.

$$\begin{aligned} \|\Delta \mathbf{u}_k^{t+\Delta t}\| &\leq E_u \\ \frac{\|\Delta \mathbf{u}_k^{t+\Delta t}\|}{\|\mathbf{u}_{k+1}^{t+\Delta t}\|} &\leq \varepsilon_u \iff \|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\| \end{aligned} \quad (11.93)$$

The above inequations can be gathered, and a reasonable convergence criterion applied to the displacement field is shown below.

$$\|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \max \left\{ \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\|, E_u \right\} \quad (11.94)$$

The first term of the above criterion might be too small if the displacement field experimented by the solid is small. To achieve convergence only based on this criterion becomes not possible. Thus, the addition of the second term becomes necessary to define a suitable convergence criterion. In case the displacement field is small, this second value is the one that will rule the criterion.

On the other hand, the convergence criterion can not be based exclusively on the difference between the displacement field obtained in consecutive iterations. Convergence may not have been reached, even if this difference is small. To circumvent this inconvenience, the same criterion can be applied to the residual obtained after each iteration. In this case, the maximum residual, and the maximum relative residual calculated with respect to the one corresponding to the initial approximation, are defined by the parameters E_g and ε_g , respectively.

$$\frac{|g(\mathbf{u}_{k+1}^{t+\Delta t})|}{|g(\mathbf{u}_0^{t+\Delta t})|} = \frac{|g(\mathbf{u}_{k+1}^{t+\Delta t})|}{|g(\mathbf{u}^t)|} \leq \varepsilon_g \iff |g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq E_g |g(\mathbf{u}^t)| \quad (11.95)$$

If the above inequations are gathered, the residual criterion becomes:

$$|g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq \max \left\{ \varepsilon_g |g(\mathbf{u}^t)|, E_g \right\} \quad (11.96)$$

Therefore, if the displacement convergence criterion (11.94) and the residual one (11.96) are taken into account, the convergence criterion to stop the iterative procedure is finally defined as:

$$\text{convergence } \checkmark \iff \begin{cases} \|\Delta \mathbf{u}_k^{t+\Delta t}\| \leq \max \left\{ \varepsilon_u \|\mathbf{u}_{k+1}^{t+\Delta t}\|, E_u \right\} \\ \text{and} \\ |g(\mathbf{u}_{k+1}^{t+\Delta t})| \leq \max \left\{ \varepsilon_g |g(\mathbf{u}^t)|, E_g \right\} \end{cases} \quad (11.97)$$

11.7. Newton-Raphson first iteration

The equation that defines the iterative procedure was presented in (11.90) as:

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_{k,t}^{t+\Delta t}) \Delta \mathbf{u}_{k,t}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{k,t}^{t+\Delta t}) \quad k = 0, 1, 2, \dots \quad (11.98)$$

In the above equation, the subscripts t are added again. They indicate the reference configuration with respect to which the magnitudes are defined. In this case, the reference material domain is the one corresponding to the load step t .

The value $k = 0$ leads to the equation that corresponds to the first iteration.

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_{0,t}^{t+\Delta t}) \Delta \mathbf{u}_{0,t}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{0,t}^{t+\Delta t}) \quad (11.99)$$

An initial root approximation is needed to start the iterative procedure. As stated before, the displacement field corresponding to the previous load step is the one adopted to start iterating.

$$\mathbf{u}_{0,t}^{t+\Delta t} = \mathbf{u}_t^t \quad (11.100)$$

Hence, the first step of the iterative method (11.99) can be rewritten as follows:

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_t^t) \Delta \mathbf{u}_{0,t}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_t^t) \quad (11.101)$$

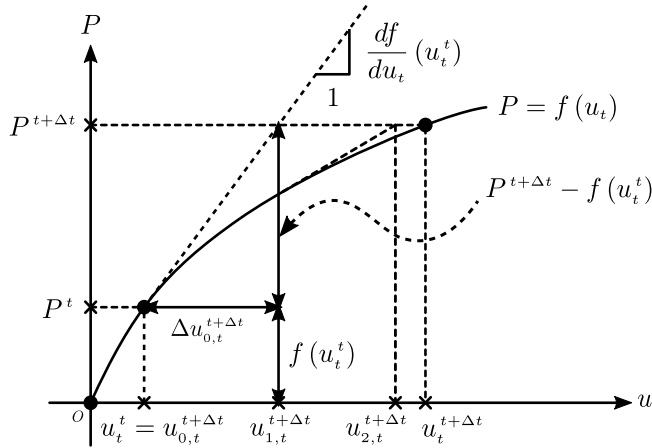


Figure 11.3. Newton-Raphson first iteration applied to a one-dimensional case.

11.7.1. Right-hand side

The equation that defines the first iteration of the iterative procedure was defined in (11.101). Its right-hand side is composed by the following two terms:

$$P^{t+\Delta t} - f(\mathbf{u}_t^t) \quad (11.102)$$

On the one hand, the constant $P^{t+\Delta t}$ was defined in (11.81) as:

$$P^{t+\Delta t} = \beta^T \iiint_{\Omega_t} \phi_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \beta^T \iint_{\Gamma_t^\sigma} \phi_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t \quad (11.103)$$

On the other hand, the nonlinear function f was defined in (11.83) as:

$$f(\mathbf{u}_t) = \iiint_{\Omega_t} \delta \bar{\boldsymbol{\varepsilon}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t \quad (11.104)$$

Where the tensorial form of the vector $\delta\bar{\mathcal{E}}_t$ turns out to be:

$$\delta\mathcal{E}_t = \delta\mathcal{E}_\varepsilon \Big|_{\mathbf{r}=\mathbf{r}(r_t)} = \frac{1}{2} \left[\frac{d\delta\mathbf{u}_\varepsilon}{d\mathbf{r}} + \left(\frac{d\delta\mathbf{u}_\varepsilon}{d\mathbf{r}} \right)^T \right] \Big|_{\mathbf{r}=\mathbf{r}(r_t)} \quad (11.105)$$

And the change of variable applied in the above equation is analogous to the one presented in (11.48), without particularizing the deformation vector for a given load step.

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(\mathbf{r}_t) = \mathbf{r}_t + \Delta\mathbf{u}_t(\mathbf{r}_t) \\ &= \mathbf{r}_t + \left[\mathbf{u}_t(\mathbf{r}_t) - \mathbf{u}_t^t(\mathbf{r}_t) \right] \end{aligned} \quad (11.106)$$

But the scalar function (11.104) has to be evaluated at the displacement field corresponding to the previous load step $\mathbf{u}_t = \mathbf{u}_t^t$. If this displacement field is substituted into the change of variable (11.106), the deformation vector \mathbf{r} becomes the one corresponding to the load step t . Moreover, the tensor $\delta\mathcal{E}_t$ turns out to be analogous to the one presented in (11.40), and its vectorial form was already defined in (11.41).

$$\begin{aligned} \mathbf{u}_t = \mathbf{u}_t^t &\implies \mathbf{r} = \mathbf{r}_t \\ \implies \delta\mathcal{E}_t &= \frac{1}{2} \left[\frac{d\delta\mathbf{u}_\varepsilon}{d\mathbf{r}_t} + \left(\frac{d\delta\mathbf{u}_\varepsilon}{d\mathbf{r}_t} \right)^T \right] = \delta\mathcal{E}_t^t \\ \implies \delta\bar{\mathcal{E}}_t^t &= \mathbf{B}_t \boldsymbol{\beta} \quad \text{with} \quad \mathbf{B}_t = \mathbf{A}_C \mathbf{G}_t \end{aligned} \quad (11.107)$$

Therefore, the evaluation of the scalar function (11.104) at the displacement field corresponding to the previous load step leads to:

$$\begin{aligned} f(\mathbf{u}_t^t) &= \iiint_{\Omega_t} (\delta\bar{\mathcal{E}}_t^t)^T \bar{\boldsymbol{\sigma}}_t^t d\Omega_t \\ &= \beta^T \iiint_{\Omega_t} \mathbf{B}_t^T \bar{\boldsymbol{\sigma}}_t^t d\Omega_t \\ &= \beta^T \iiint_{\Omega_t} \mathbf{G}_t^T \mathbf{A}_C^T \bar{\boldsymbol{\sigma}}_t^t d\Omega_t \end{aligned} \quad (11.108)$$

Equations (11.103) and (11.108) completely define the right-hand side presented in (11.102).

11.7.2. Left-hand side

According to (11.101), the left-hand side of the equation that defines the first step of the iterative procedure turns out to be:

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_t^t) \Delta\mathbf{u}_{0,t}^{t+\Delta t} \quad (11.109)$$

Where the scalar function f was defined in (11.83) as:

$$f(\mathbf{u}_t) = \iiint_{\Omega_t} \delta\bar{\mathcal{E}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t \quad (11.110)$$

Therefore, the gradient of the above scalar function has to be computed to define the left-hand side presented in (11.109).

$$\frac{df}{d\mathbf{u}_t} = \frac{d}{d\mathbf{u}_t} \left(\iiint_{\Omega_t} \delta \bar{\mathbf{E}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t \right) \quad (11.111)$$

Since the deformed material domain Ω_t depends on the displacement field, the derivative can not be taken inside the integral. To overcome this inconvenient, a result obtained in the previous chapter is recalled. As proved in (10.47) and (10.69), the above scalar function can be equivalently expressed with respect to the initial material domain as:

$$\underbrace{\iiint_{\Omega_t} \delta \bar{\mathbf{E}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t}_{f(\mathbf{u}_t)} = \underbrace{\iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0}_{h(\mathbf{u}_{\mathcal{L}})} \quad (11.112)$$

Although both expressions are equivalent, they depend on different displacement field descriptions. The left-hand side depends on the displacement field defined with respect to the material domain Ω_t , whereas the right-hand side depends on the displacement field defined with respect to the initial configuration (Lagrangian description).

Consequently, if a displacement field increment is applied, it can be stated that:

$$\Delta f = \Delta h \quad \iff \quad \frac{df}{d\mathbf{u}_t} \Delta \mathbf{u}_t = \frac{dh}{d\mathbf{u}_{\mathcal{L}}} \Delta \mathbf{u}_{\mathcal{L}} \quad (11.113)$$

The gradient of the scalar function h is easier to calculate than the gradient of the scalar function f since the initial material domain does not depend on the displacement field, and the derivative can now be taken inside the integral. The result obtained in (B.9) is taken into account, where the gradient of a generic scalar product is computed. Thus,

$$\begin{aligned} \frac{dh}{d\mathbf{u}_{\mathcal{L}}} &= \frac{d}{d\mathbf{u}_{\mathcal{L}}} \left(\iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \right) \\ &= \iiint_{\Omega_0} \frac{d}{d\mathbf{u}_{\mathcal{L}}} \left(\delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} \right) d\Omega_0 \\ &= \iiint_{\Omega_0} \left(\bar{\mathbf{S}}_{\mathcal{L}}^T \frac{d(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})}{d\mathbf{u}_{\mathcal{L}}} + \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \frac{d\bar{\mathbf{S}}_{\mathcal{L}}}{d\mathbf{u}_{\mathcal{L}}} \right) d\Omega_0 \\ &= \iiint_{\Omega_0} \bar{\mathbf{S}}_{\mathcal{L}}^T \frac{d(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})}{d\mathbf{u}_{\mathcal{L}}} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \frac{d\bar{\mathbf{S}}_{\mathcal{L}}}{d\mathbf{u}_{\mathcal{L}}} d\Omega_0 \end{aligned} \quad (11.114)$$

And the scalar product between the above gradient and the displacement field

increment leads to the following result.

$$\begin{aligned}
 \frac{dh}{d\mathbf{u}_\mathcal{L}} \Delta \mathbf{u}_\mathcal{L} &= \left(\iiint_{\Omega_0} \bar{\mathbf{S}}_\mathcal{L}^T \frac{d(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})}{d\mathbf{u}_\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \frac{d\bar{\mathbf{S}}_\mathcal{L}}{d\mathbf{u}_\mathcal{L}} d\Omega_0 \right) \Delta \mathbf{u}_\mathcal{L} \\
 &= \iiint_{\Omega_0} \bar{\mathbf{S}}_\mathcal{L}^T \frac{d(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})}{d\mathbf{u}_\mathcal{L}} \Delta \mathbf{u}_\mathcal{L} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \frac{d\bar{\mathbf{S}}_\mathcal{L}}{d\mathbf{u}_\mathcal{L}} \Delta \mathbf{u}_\mathcal{L} d\Omega_0 \\
 &= \iiint_{\Omega_0} \bar{\mathbf{S}}_\mathcal{L}^T \underbrace{\left(\frac{d(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})}{d\mathbf{u}_\mathcal{L}} \Delta \mathbf{u}_\mathcal{L} \right)}_{\Delta(\delta \bar{\mathbf{E}}_{G,\mathcal{L}})} d\Omega_0 + \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \underbrace{\left(\frac{d\bar{\mathbf{S}}_\mathcal{L}}{d\mathbf{u}_\mathcal{L}} \Delta \mathbf{u}_\mathcal{L} \right)}_{\Delta \bar{\mathbf{S}}_\mathcal{L}} d\Omega_0 \\
 &= \iiint_{\Omega_0} \delta \bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta \bar{\mathbf{S}}_\mathcal{L} d\Omega_0 + \iiint_{\Omega_0} \bar{\mathbf{S}}_\mathcal{L}^T \Delta(\delta \bar{\mathbf{E}}_{G,\mathcal{L}}) d\Omega_0
 \end{aligned} \tag{11.115}$$

The above equation is defined in vector notation. In order to manipulate it, it is convenient to switch to its equivalent tensorial form.

On the one hand, the vectorial expression of the Green-Lagrange strain tensor variation is exchanged for its equivalent tensor definition. The same tensor transformation is applied to the vectorial form of the second Piola-Kirchhoff stress tensor increment.

$$\begin{aligned}
 \delta \bar{\mathbf{E}}_{G,\mathcal{L}} = \begin{Bmatrix} \delta E_{11} \\ \delta E_{22} \\ \delta E_{33} \\ 2 \delta E_{12} \\ 2 \delta E_{13} \\ 2 \delta E_{23} \end{Bmatrix} &\implies \delta \mathbf{E}_{G,\mathcal{L}} = \begin{bmatrix} \delta E_{11} & \delta E_{12} & \delta E_{13} \\ \delta E_{12} & \delta E_{22} & \delta E_{23} \\ \delta E_{13} & \delta E_{23} & \delta E_{33} \end{bmatrix} \\
 \Delta \bar{\mathbf{S}}_\mathcal{L} = \begin{Bmatrix} \Delta S_{11} \\ \Delta S_{22} \\ \Delta S_{33} \\ \Delta S_{12} \\ \Delta S_{13} \\ \Delta S_{23} \end{Bmatrix} &\implies \Delta \mathbf{S}_\mathcal{L} = \begin{bmatrix} \Delta S_{11} & \Delta S_{12} & \Delta S_{13} \\ \Delta S_{12} & \Delta S_{22} & \Delta S_{23} \\ \Delta S_{13} & \Delta S_{23} & \Delta S_{33} \end{bmatrix}
 \end{aligned} \tag{11.116}$$

On the other hand, the vectorial form of the second Piola-Kirchhoff stress tensor is exchanged for its equivalent tensor definition, and the same tensor transformation is applied to the increment of the Green-Lagrange strain tensor variation defined in

vectorial form.

$$\begin{aligned} \bar{\mathbf{S}}_{\mathcal{L}} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{13} \\ S_{23} \end{Bmatrix} &\implies \mathbf{S}_{\mathcal{L}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \\ \Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) = \begin{Bmatrix} \Delta(\delta E_G)_{11} \\ \Delta(\delta E_G)_{22} \\ \Delta(\delta E_G)_{33} \\ 2\Delta(\delta E_G)_{12} \\ 2\Delta(\delta E_G)_{13} \\ 2\Delta(\delta E_G)_{23} \end{Bmatrix} &\implies \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) = \begin{bmatrix} \Delta(\delta E_G)_{11} & \Delta(\delta E_G)_{12} & \Delta(\delta E_G)_{13} \\ \Delta(\delta E_G)_{12} & \Delta(\delta E_G)_{22} & \Delta(\delta E_G)_{23} \\ \Delta(\delta E_G)_{13} & \Delta(\delta E_G)_{23} & \Delta(\delta E_G)_{33} \end{bmatrix} \end{aligned} \quad (11.117)$$

This notation equivalence allows to replace the scalar product of their vectorial forms by the double dot product of their tensor definitions.

$$\begin{aligned} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta\bar{\mathbf{S}}_{\mathcal{L}} &= \delta\mathbf{E}_{G,\mathcal{L}} : \Delta\mathbf{S}_{\mathcal{L}} \\ \bar{\mathbf{S}}_{\mathcal{L}}^T \Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) &= \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} \end{aligned} \quad (11.118)$$

Consequently, the scalar product computed in (11.115) can be equivalently expressed as:

$$\begin{aligned} \frac{dh}{d\mathbf{u}_{\mathcal{L}}} \Delta\mathbf{u}_{\mathcal{L}} &= \iiint_{\Omega_0} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \Delta\bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \bar{\mathbf{S}}_{\mathcal{L}}^T \Delta(\delta\bar{\mathbf{E}}_{G,\mathcal{L}}) d\Omega_0 \\ &= \iiint_{\Omega_0} \delta\mathbf{E}_{G,\mathcal{L}} : \Delta\mathbf{S}_{\mathcal{L}} d\Omega_0 + \iiint_{\Omega_0} \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \end{aligned} \quad (11.119)$$

As stated in (11.113), the scalar product can be equivalently computed by means of the scalar functions f or h . The computation by means of function h is the one performed in this section, and leads to the following result:

$$h = \iiint_{\Omega_0} \delta\bar{\mathbf{E}}_{G,\mathcal{L}}^T \bar{\mathbf{S}}_{\mathcal{L}} d\Omega_0 \implies \begin{cases} \frac{dh}{d\mathbf{u}_{\mathcal{L}}} \Delta\mathbf{u}_{\mathcal{L}} = \iiint_{\Omega_0} \delta\mathbf{E}_{G,\mathcal{L}} : \Delta\mathbf{S}_{\mathcal{L}} d\Omega_0 \\ \quad + \iiint_{\Omega_0} \Delta(\delta\mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 \end{cases} \quad (11.120)$$

The election of the function f has to drive to a result with analogous structure to the one obtained with the above scalar product. Therefore, it can be stated that:

$$f = \iiint_{\Omega_t} \delta\bar{\mathcal{E}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t \implies \begin{cases} \frac{df}{d\mathbf{u}_t} \Delta\mathbf{u}_t = \iiint_{\Omega_t} \delta\mathcal{E}_t : \Delta\boldsymbol{\sigma}_t d\Omega_t \\ \quad + \iiint_{\Omega_t} \Delta(\delta\mathcal{E}_t) : \boldsymbol{\sigma}_t d\Omega_t \end{cases} \quad (11.121)$$

The definition of the terms that compose the above scalar product is performed in the following subsections.

11.7.3. Material component

As stated in section 10.14.1, the first component of the scalar product (11.120) is the so-called material component, since it turns out to depend on the constitutive tensor of the material. The integral expression that defines this component is:

$$\iiint_{\Omega_0} \delta \mathbf{E}_{G,\mathcal{L}} : \Delta \mathbf{S}_{\mathcal{L}} d\Omega_0 \quad (11.122)$$

This integrand is composed by two tensors: the Green-Lagrange strain tensor variation and the second Piola-Kirchhoff stress tensor increment.

On the one hand, the second Piola-Kirchhoff stress tensor increment was defined in (7.101) as:

$$\left. \begin{aligned} \Delta \mathbf{S}_{\mathcal{L}} = \mathbf{C}_4 : \Delta \mathbf{E}_{G,\mathcal{L}} &= \left[\Delta S_{ij} \right]_{\substack{i=1,2,3 \\ j=1,2,3}} \\ \mathbf{C}_4 &= \left[C_{ijkl} \right]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,2,3 \\ l=1,2,3}} \\ \Delta \mathbf{E}_{G,\mathcal{L}} &= \left[\Delta E_{kl} \right]_{\substack{k=1,2,3 \\ l=1,2,3}} \end{aligned} \right\} \implies \Delta S_{ij} = \sum_{k,l} C_{ijkl} \Delta E_{kl} \quad (11.123)$$

The above equation was derived under the assumption that the gradients of the displacement field increment are small. At each iteration of the iterative procedure, the gradients of the displacement field increment are assumed to be small. Therefore, the above equation can be properly applied in this context. Furthermore, the second Piola-Kirchhoff stress tensor increment was expressed by means of the Green-Lagrange strain tensor increment, which was defined in (4.94) as:

$$\left. \begin{aligned} \Delta \mathbf{E}_{G,\mathcal{L}} &= \mathbf{F}_{\mathcal{L}}^T \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \right] \Bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \quad \mathbf{F}_{\mathcal{L}} = \left[\Delta E_{kl} \right]_{\substack{k=1,2,3 \\ l=1,2,3}} \\ \Delta \mathcal{E}_{\varepsilon} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} &= \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \right] \Bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} = \left[\Delta \mathcal{E}_{rs} \right]_{\substack{r=1,2,3 \\ s=1,2,3}} \\ & \quad \mathbf{F}_{\mathcal{L}} = \left[F_{ij} \right]_{\substack{i=1,2,3 \\ j=1,2,3}} \end{aligned} \right\} \implies$$

$$\implies \Delta E_{kl} = \sum_{r,s} F_{rk} \Delta \mathcal{E}_{rs} F_{sl} \quad (11.124)$$

Thus, the components of the second Piola-Kirchhoff stress tensor increment defined in (11.123) become:

$$\Delta S_{ij} = \sum_{k,l} C_{ijkl} \left(\sum_{r,s} F_{rk} \Delta \mathcal{E}_{rs} F_{sl} \right) \quad (11.125)$$

On the other hand, the Green-Lagrange strain tensor variation can be defined according to the expression obtained in (4.88) as:

$$\left. \begin{aligned} \delta \mathbf{E}_{G,\mathcal{L}} &= \mathbf{F}_{\mathcal{L}}^T \frac{1}{2} \left[\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \right] \Bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} & \mathbf{F}_{\mathcal{L}} &= [\delta E_{ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \\ \delta \mathcal{E}_{\varepsilon} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} &= \frac{1}{2} \left[\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \right] \Bigg|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} & &= [\delta \mathcal{E}_{pq}]_{\substack{p=1,2,3 \\ q=1,2,3}} \\ & & \mathbf{F}_{\mathcal{L}} &= [F_{ij}]_{\substack{i=1,2,3 \\ j=1,2,3}} \end{aligned} \right\} \implies \quad (11.126)$$

$$\implies \delta E_{ij} = \sum_{p,q} F_{pi} \delta \mathcal{E}_{pq} F_{qj}$$

The substitution of the components of the Green-Lagrange strain tensor variation (11.126) and those of the Piola-Kirchhoff stress tensor increment (11.125) into the double dot product between both tensors leads to:

$$\begin{aligned} \delta \mathbf{E}_{G,\mathcal{L}} : \Delta \mathbf{S}_{\mathcal{L}} &= \sum_{i,j} \delta E_{ij} \Delta S_{ij} \\ &= \sum_{i,j} \left(\sum_{p,q} F_{pi} \delta \mathcal{E}_{pq} F_{qj} \right) \left(\sum_{k,l} C_{ijkl} \left(\sum_{r,s} F_{rk} \Delta \mathcal{E}_{rs} F_{sl} \right) \right) \\ &= F_{\mathcal{L}} \sum_{p,q,r,s} \delta \mathcal{E}_{pq} \underbrace{\left(\frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \right)}_{D'_{pqrs}} \Delta \mathcal{E}_{rs} \\ &= F_{\mathcal{L}} \sum_{p,q} \delta \mathcal{E}_{pq} \left(\sum_{r,s} D'_{pqrs} \Delta \mathcal{E}_{rs} \right) \\ &= F_{\mathcal{L}} \delta \mathcal{E}_{\varepsilon} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} : \left(\mathbf{D}'_{4,\mathcal{L}} : \Delta \mathcal{E}_{\varepsilon} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \right) \\ &= F_{\mathcal{L}} \left(\delta \mathcal{E}_{\varepsilon} : \mathbf{D}'_{4,\varepsilon} : \Delta \mathcal{E}_{\varepsilon} \right) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \end{aligned} \quad (11.127)$$

Where $\mathbf{D}'_{4,\mathcal{L}}$ is the so-called fourth order spatial constitutive tensor [Crisfield et al., 1991b; Capaldi, 2012; Kim, 2014; Bonet et al., 2016], whose components turn out to be defined in terms of the components of both the fourth order constitutive tensor \mathbf{C}_4 (section 7.6.2) and the deformation gradient tensor.

$$\mathbf{D}'_{4,\mathcal{L}} = [D'_{pqrs}]_{\substack{p=1,2,3 \\ q=1,2,3 \\ r=1,2,3 \\ s=1,2,3}} \quad D'_{pqrs} = \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \quad (11.128)$$

According to the definition of the above components, it is straightforward to prove that the major symmetry is fulfilled. To prove it, the major symmetry of the tensor \mathbf{C}_4

demonstrated in section 7.6.3 is taken into account.

$$\left. \begin{aligned} D'_{pqrs} &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \\ &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{rk} F_{sl} C_{kl ij} F_{pi} F_{qj} = D'_{rspq} \end{aligned} \right\} \iff D'_{pqrs} = D'_{rspq} \quad (11.129)$$

Furthermore, the minor symmetries of the tensor C_4 (section 7.6.3) allow to prove that the minor symmetries are also verified.

$$\left. \begin{aligned} D'_{pqrs} &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \\ &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijlk} F_{sl} F_{rk} = D'_{pqsr} \\ &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{qj} F_{pi} C_{jikl} F_{rk} F_{sl} = D'_{qprs} \end{aligned} \right\} \iff D'_{pqrs} = D'_{pqsr} = D'_{qprs} \quad (11.130)$$

The integration of equation (11.127) over the initial configuration allows to obtain an equivalent definition of the material component. To change the integration domain, the methodology presented in section 1.14 is taken into account. This methodology allows moving from integrals defined over the deformed domain, to integrals defined over the initial reference configuration. Note that, in this particular case, the inverse transformation is required. That is, the change from an integral defined over the reference material domain to an equivalent one over the deformed configuration is carried out.

$$\begin{aligned} \iiint_{\Omega_0} \delta \mathbf{E}_{G,\mathcal{L}} : \Delta \mathbf{S}_{\mathcal{L}} d\Omega_0 &= \iiint_{\Omega_0} \left(\delta \mathcal{E}_{\varepsilon} : \mathbf{D}'_{4,\varepsilon} : \Delta \mathcal{E}_{\varepsilon} \right) \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} F_{\mathcal{L}} d\Omega_0 \\ &= \iiint_{\Omega} \delta \mathcal{E}_{\varepsilon} : \mathbf{D}'_{4,\varepsilon} : \Delta \mathcal{E}_{\varepsilon} d\Omega \end{aligned} \quad (11.131)$$

And the comparison of the above result with the first term of the scalar product obtained in (11.121) allows to introduce the Eulerian definition of the Cauchy stress tensor increment. Thus, the above material component can be rewritten as:

$$\begin{aligned} \iiint_{\Omega_0} \delta \mathbf{E}_{G,\mathcal{L}} : \Delta \mathbf{S}_{\mathcal{L}} d\Omega_0 &= \iiint_{\Omega} \delta \mathcal{E}_{\varepsilon} : \underbrace{(\mathbf{D}'_{4,\varepsilon} : \Delta \mathcal{E}_{\varepsilon})}_{\Delta \boldsymbol{\sigma}_{\varepsilon}} d\Omega \\ &= \iiint_{\Omega} \delta \mathcal{E}_{\varepsilon} : \Delta \boldsymbol{\sigma}_{\varepsilon} d\Omega \end{aligned} \quad (11.132)$$

The magnitudes involved in the above integral are tensor ones. It is convenient to move to their equivalent vectorial forms by applying the Voigt notation exposed in

section A.14.

$$\delta \mathcal{E}_\varepsilon = \begin{bmatrix} \delta \mathcal{E}_{11} & \delta \mathcal{E}_{12} & \delta \mathcal{E}_{13} \\ \delta \mathcal{E}_{12} & \delta \mathcal{E}_{22} & \delta \mathcal{E}_{23} \\ \delta \mathcal{E}_{13} & \delta \mathcal{E}_{23} & \delta \mathcal{E}_{33} \end{bmatrix} \implies \delta \bar{\mathcal{E}}_\varepsilon = \begin{Bmatrix} \delta \mathcal{E}_{11} \\ \delta \mathcal{E}_{22} \\ \delta \mathcal{E}_{33} \\ 2 \delta \mathcal{E}_{12} \\ 2 \delta \mathcal{E}_{13} \\ 2 \delta \mathcal{E}_{23} \end{Bmatrix} \quad (11.133)$$

$$\Delta \sigma_\varepsilon = \begin{bmatrix} \Delta \sigma_{11} & \Delta \sigma_{12} & \Delta \sigma_{13} \\ \Delta \sigma_{12} & \Delta \sigma_{22} & \Delta \sigma_{23} \\ \Delta \sigma_{13} & \Delta \sigma_{23} & \Delta \sigma_{33} \end{bmatrix} \implies \Delta \bar{\sigma}_\varepsilon = \begin{Bmatrix} \Delta \sigma_{11} \\ \Delta \sigma_{22} \\ \Delta \sigma_{33} \\ \Delta \sigma_{12} \\ \Delta \sigma_{13} \\ \Delta \sigma_{23} \end{Bmatrix}$$

The double dot product between both tensors can be now substituted by the scalar product between their equivalent vectorial forms.

$$\delta \mathcal{E}_\varepsilon : \Delta \sigma_\varepsilon = \delta \bar{\mathcal{E}}_\varepsilon^T \Delta \bar{\sigma}_\varepsilon \quad (11.134)$$

Consequently, the material component (11.132) can be equivalently expressed as:

$$\iiint_\Omega \delta \mathcal{E}_\varepsilon : \Delta \sigma_\varepsilon d\Omega = \iiint_\Omega \delta \bar{\mathcal{E}}_\varepsilon^T \Delta \bar{\sigma}_\varepsilon d\Omega \quad (11.135)$$

The adoption of the vector notation forces the definition of the vectorial form of the Eulerian Cauchy stress tensor increment. Its tensor expression was introduced in (11.132) and turns out to be defined by means of the fourth order spatial constitutive tensor. Hence, its equivalent vectorial form is defined in terms of the second order spatial constitutive tensor.

$$\Delta \sigma_\varepsilon = \mathbf{D}'_{4,\varepsilon} : \Delta \mathcal{E}_\varepsilon \iff \Delta \bar{\sigma}_\varepsilon = \mathbf{D}'_{2,\varepsilon} \Delta \bar{\mathcal{E}}_\varepsilon \quad (11.136)$$

The Voigt notation (section A.14) is again applied to switch from tensor magnitudes to their equivalent vectorial forms.

$$\Delta \mathcal{E}_\varepsilon = \begin{bmatrix} \Delta \mathcal{E}_{11} & \Delta \mathcal{E}_{12} & \Delta \mathcal{E}_{13} \\ \Delta \mathcal{E}_{12} & \Delta \mathcal{E}_{22} & \Delta \mathcal{E}_{23} \\ \Delta \mathcal{E}_{13} & \Delta \mathcal{E}_{23} & \Delta \mathcal{E}_{33} \end{bmatrix} \implies \Delta \bar{\mathcal{E}}_\varepsilon = \begin{Bmatrix} \Delta \mathcal{E}_{11} \\ \Delta \mathcal{E}_{22} \\ \Delta \mathcal{E}_{33} \\ 2 \Delta \mathcal{E}_{12} \\ 2 \Delta \mathcal{E}_{13} \\ 2 \Delta \mathcal{E}_{23} \end{Bmatrix} \quad (11.137)$$

$$\Delta \sigma_\varepsilon = \begin{bmatrix} \Delta \sigma_{11} & \Delta \sigma_{12} & \Delta \sigma_{13} \\ \Delta \sigma_{12} & \Delta \sigma_{22} & \Delta \sigma_{23} \\ \Delta \sigma_{13} & \Delta \sigma_{23} & \Delta \sigma_{33} \end{bmatrix} \implies \Delta \bar{\sigma}_\varepsilon = \begin{Bmatrix} \Delta \sigma_{11} \\ \Delta \sigma_{22} \\ \Delta \sigma_{33} \\ \Delta \sigma_{12} \\ \Delta \sigma_{13} \\ \Delta \sigma_{23} \end{Bmatrix}$$

Thus, the vectorial equation (11.136) becomes:

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{Bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{13} & D'_{14} & D'_{15} & D'_{16} \\ D'_{21} & D'_{22} & D'_{23} & D'_{24} & D'_{25} & D'_{26} \\ D'_{31} & D'_{32} & D'_{33} & D'_{34} & D'_{35} & D'_{36} \\ D'_{41} & D'_{42} & D'_{43} & D'_{44} & D'_{45} & D'_{46} \\ D'_{51} & D'_{52} & D'_{53} & D'_{54} & D'_{55} & D'_{56} \\ D'_{61} & D'_{62} & D'_{63} & D'_{64} & D'_{65} & D'_{66} \end{bmatrix} \begin{Bmatrix} \Delta\mathcal{E}_{11} \\ \Delta\mathcal{E}_{22} \\ \Delta\mathcal{E}_{33} \\ 2\Delta\mathcal{E}_{12} \\ 2\Delta\mathcal{E}_{13} \\ 2\Delta\mathcal{E}_{23} \end{Bmatrix} \quad (11.138)$$

According to its equivalent tensor definition, the components of the above vectorial equation can be equivalently computed by means of the 81 components that compose the fourth order spatial constitutive tensor, as shown below.

$$\Delta\sigma_{\varepsilon} = \mathbf{D}'_{4,\varepsilon} : \Delta\mathcal{E}_{\varepsilon} \iff \Delta\sigma_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 D'_{ijkl} \Delta\mathcal{E}_{kl} \quad (11.139)$$

The tensor $\Delta\mathcal{E}_{\varepsilon}$, which was defined in (11.126), turns out to be symmetric. In addition, the spatial constitutive tensor possesses minor symmetries, as proved in (11.130). If these symmetries are taken into account, the above components become:

$$\begin{aligned} \Delta\sigma_{ij} &= \sum_{k=1}^3 \sum_{l=1}^3 D'_{ijkl} \Delta\mathcal{E}_{kl} \\ &= D'_{ij11} \Delta\mathcal{E}_{11} + D'_{ij12} \Delta\mathcal{E}_{12} + D'_{ij13} \Delta\mathcal{E}_{13} + \\ &\quad + D'_{ij21} \Delta\mathcal{E}_{21} + D'_{ij22} \Delta\mathcal{E}_{22} + D'_{ij23} \Delta\mathcal{E}_{23} + \\ &\quad + D'_{ij31} \Delta\mathcal{E}_{31} + D'_{ij32} \Delta\mathcal{E}_{32} + D'_{ij33} \Delta\mathcal{E}_{33} \\ &= D'_{ij11} \Delta\mathcal{E}_{11} + D'_{ij22} \Delta\mathcal{E}_{22} + D'_{ij33} \Delta\mathcal{E}_{33} + \\ &\quad + D'_{ij12} (2\Delta\mathcal{E}_{12}) + D'_{ij13} (2\Delta\mathcal{E}_{13}) + D'_{ij23} (2\Delta\mathcal{E}_{23}) \end{aligned} \quad (11.140)$$

That is, the 81 tensor components required in (11.139) are reduced to 36.

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{Bmatrix} = \begin{bmatrix} D'_{1111} & D'_{1122} & D'_{1133} & D'_{1112} & D'_{1113} & D'_{1123} \\ D'_{2211} & D'_{2222} & D'_{2233} & D'_{2212} & D'_{2213} & D'_{2223} \\ D'_{3311} & D'_{3322} & D'_{3333} & D'_{3312} & D'_{3313} & D'_{3323} \\ D'_{1211} & D'_{1222} & D'_{1233} & D'_{1212} & D'_{1213} & D'_{1223} \\ D'_{1311} & D'_{1322} & D'_{1333} & D'_{1312} & D'_{1313} & D'_{1323} \\ D'_{2311} & D'_{2322} & D'_{2333} & D'_{2312} & D'_{2313} & D'_{2323} \end{bmatrix} \begin{Bmatrix} \Delta\mathcal{E}_{11} \\ \Delta\mathcal{E}_{22} \\ \Delta\mathcal{E}_{33} \\ 2\Delta\mathcal{E}_{12} \\ 2\Delta\mathcal{E}_{13} \\ 2\Delta\mathcal{E}_{23} \end{Bmatrix} \quad (11.141)$$

If the major symmetry (11.129) of the fourth order constitutive tensor is also taken into account, the number of components is finally reduced to 21.

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{Bmatrix} = \begin{bmatrix} D'_{1111} & D'_{1122} & D'_{1133} & D'_{1112} & D'_{1113} & D'_{1123} \\ & D'_{2222} & D'_{2233} & D'_{2212} & D'_{2213} & D'_{2223} \\ & & D'_{3333} & D'_{3312} & D'_{3313} & D'_{3323} \\ & & & D'_{1212} & D'_{1213} & D'_{1223} \\ & \text{sym} & & & D'_{1313} & D'_{1323} \\ & & & & & D'_{2323} \end{bmatrix} \begin{Bmatrix} \Delta\mathcal{E}_{11} \\ \Delta\mathcal{E}_{22} \\ \Delta\mathcal{E}_{33} \\ 2\Delta\mathcal{E}_{12} \\ 2\Delta\mathcal{E}_{13} \\ 2\Delta\mathcal{E}_{23} \end{Bmatrix} \quad (11.142)$$

Therefore, the second order spatial constitutive tensor turns out to be symmetric.

$$(\mathbf{D}'_{2,\varepsilon})^T = \mathbf{D}'_{2,\varepsilon} \quad \Longleftrightarrow \quad D'_{ji} = D'_{ij} \quad (11.143)$$

And its components, defined by means of the fourth order ones, are:

$$\begin{bmatrix} D'_{11} & D'_{12} & D'_{13} & D'_{14} & D'_{15} & D'_{16} \\ & D'_{22} & D'_{23} & D'_{24} & D'_{25} & D'_{26} \\ & & D'_{33} & D'_{34} & D'_{35} & D'_{36} \\ & & & D'_{44} & D'_{45} & D'_{46} \\ \text{sym} & & & & D'_{55} & D'_{56} \\ & & & & & D'_{66} \end{bmatrix} = \begin{bmatrix} D'_{1111} & D'_{1122} & D'_{1133} & D'_{1112} & D'_{1113} & D'_{1123} \\ & D'_{2222} & D'_{2233} & D'_{2212} & D'_{2213} & D'_{2223} \\ & & D'_{3333} & D'_{3312} & D'_{3313} & D'_{3323} \\ & & & D'_{1212} & D'_{1213} & D'_{1223} \\ \text{sym} & & & & D'_{1313} & D'_{1323} \\ & & & & & D'_{2323} \end{bmatrix} \quad (11.144)$$

Up to this point, the second order spatial constitutive tensor is completely defined. Hence, the definition of the vector $\Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon$ is required to complete the vectorial form of the Eulerian Cauchy stress tensor increment (11.136).

The tensor $\Delta \boldsymbol{\varepsilon}_\varepsilon$ was introduced in (11.124) as:

$$\Delta \boldsymbol{\varepsilon}_\varepsilon = \frac{1}{2} \left[\frac{d\Delta \mathbf{u}_\varepsilon}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_\varepsilon}{d\mathbf{r}} \right)^T \right] \quad \frac{d\Delta \mathbf{u}_\varepsilon}{d\mathbf{r}} = \begin{bmatrix} \frac{\partial \Delta u_1}{\partial r_1} & \frac{\partial \Delta u_1}{\partial r_2} & \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} & \frac{\partial \Delta u_2}{\partial r_2} & \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} & \frac{\partial \Delta u_3}{\partial r_2} & \frac{\partial \Delta u_3}{\partial r_3} \end{bmatrix} \quad (11.145)$$

Where the components of the above tensor are:

$$\Delta \boldsymbol{\varepsilon}_\varepsilon = \begin{bmatrix} \frac{\partial \Delta u_1}{\partial r_1} & \frac{1}{2} \left(\frac{\partial \Delta u_1}{\partial r_2} + \frac{\partial \Delta u_2}{\partial r_1} \right) & \frac{1}{2} \left(\frac{\partial \Delta u_1}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_1} \right) \\ \frac{1}{2} \left(\frac{\partial \Delta u_1}{\partial r_2} + \frac{\partial \Delta u_2}{\partial r_1} \right) & \frac{\partial \Delta u_2}{\partial r_2} & \frac{1}{2} \left(\frac{\partial \Delta u_2}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_2} \right) \\ \frac{1}{2} \left(\frac{\partial \Delta u_1}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_1} \right) & \frac{1}{2} \left(\frac{\partial \Delta u_2}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_2} \right) & \frac{\partial \Delta u_3}{\partial r_3} \end{bmatrix} \quad (11.146)$$

And its vectorial form, obtained by applying the Voigt notation (section A.14),

turns out to be:

$$\Delta \mathcal{E}_\varepsilon = \begin{bmatrix} \Delta \mathcal{E}_{11} & \Delta \mathcal{E}_{12} & \Delta \mathcal{E}_{13} \\ \Delta \mathcal{E}_{12} & \Delta \mathcal{E}_{22} & \Delta \mathcal{E}_{23} \\ \Delta \mathcal{E}_{13} & \Delta \mathcal{E}_{23} & \Delta \mathcal{E}_{33} \end{bmatrix} \Rightarrow \Delta \bar{\mathcal{E}}_\varepsilon = \begin{pmatrix} \Delta \mathcal{E}_{11} \\ \Delta \mathcal{E}_{22} \\ \Delta \mathcal{E}_{33} \\ 2 \Delta \mathcal{E}_{12} \\ 2 \Delta \mathcal{E}_{13} \\ 2 \Delta \mathcal{E}_{23} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \\ \frac{\partial \Delta u_1}{\partial r_2} + \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_2} \end{pmatrix} \quad (11.147)$$

Furthermore, the above vectorial form can be equivalently written by means of the tensor \mathbf{A}_C , which was introduced in (5.70) to define the vectorial form of the infinitesimal strain tensor.

$$\Delta \bar{\mathcal{E}}_\varepsilon = \begin{pmatrix} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \\ \frac{\partial \Delta u_1}{\partial r_2} + \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_3} + \frac{\partial \Delta u_3}{\partial r_2} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \\ \frac{\partial \Delta u_1}{\partial r_2} \\ \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_3}{\partial r_2} \end{pmatrix} = \mathbf{A}_C \Delta \bar{\mathbf{J}}_t \quad (11.148)$$

Where the vector $\Delta \bar{\mathbf{J}}_t$ is made up of the components of the Eulerian displacement

gradient tensor, whose rows are organized in a single column.

$$\Delta \mathbf{J}_t = \frac{d\Delta \mathbf{u}_\varepsilon}{d\mathbf{r}} = \begin{bmatrix} \frac{\partial \Delta u_1}{\partial r_1} & \frac{\partial \Delta u_1}{\partial r_2} & \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} & \frac{\partial \Delta u_2}{\partial r_2} & \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} & \frac{\partial \Delta u_3}{\partial r_2} & \frac{\partial \Delta u_3}{\partial r_3} \end{bmatrix} \implies \Delta \bar{\mathbf{J}}_t = \left\{ \begin{array}{c} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_2} \\ \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_3}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \end{array} \right\} \quad (11.149)$$

The above vector can be rewritten by means of a differential operator. The subscript t is added to the operator in order to clarify that the derivatives are taken with respect to the deformed geometry corresponding to a given load step t .

$$\Delta \bar{\mathbf{J}}_t = \left\{ \begin{array}{c} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_2} \\ \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_3}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \end{array} \right\} = \begin{bmatrix} \frac{\partial}{\partial r_1} & 0 & 0 \\ \frac{\partial}{\partial r_2} & 0 & 0 \\ \frac{\partial}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_1} & 0 \\ 0 & \frac{\partial}{\partial r_2} & 0 \\ 0 & \frac{\partial}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_1} \\ 0 & 0 & \frac{\partial}{\partial r_2} \\ 0 & 0 & \frac{\partial}{\partial r_3} \end{bmatrix} \left\{ \begin{array}{c} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{array} \right\} = \partial_t \Delta \mathbf{u}_\varepsilon \quad (11.150)$$

On the other hand, the Eulerian displacement field increment can be approximated

by means of the Eulerian trial functions.

$$\begin{aligned}
 \Delta \mathbf{u}_\varepsilon &\approx \Delta \mathbf{u}_\varepsilon^h = \sum_{j=1}^{\eta} \underbrace{[\phi_{j,\varepsilon} \mathbf{I}]}_{\phi_{j,\varepsilon}} \Delta \boldsymbol{\alpha}_j \\
 &= \sum_{j=1}^{\eta} \phi_{j,\varepsilon} \Delta \boldsymbol{\alpha}_j \\
 &= [\phi_{1,\varepsilon} \quad \cdots \quad \phi_{\eta,\varepsilon}] \begin{Bmatrix} \Delta \boldsymbol{\alpha}_1 \\ \vdots \\ \Delta \boldsymbol{\alpha}_\eta \end{Bmatrix} \\
 &= \phi_\varepsilon \Delta \boldsymbol{\alpha}
 \end{aligned} \tag{11.151}$$

If the above approximation is introduced into (11.150), the vector $\Delta \bar{\mathbf{J}}_t$ becomes:

$$\left. \begin{aligned} \Delta \bar{\mathbf{J}}_t &= \partial_t \Delta \mathbf{u}_\varepsilon \\ \Delta \mathbf{u}_\varepsilon &\approx \phi_\varepsilon \Delta \boldsymbol{\alpha} \end{aligned} \right\} \implies \begin{cases} \Delta \bar{\mathbf{J}}_t \approx \partial_t (\phi_\varepsilon \Delta \boldsymbol{\alpha}) \\ = (\partial_t \phi_\varepsilon) \Delta \boldsymbol{\alpha} \\ = \mathbf{G}_t \Delta \boldsymbol{\alpha} \end{cases} \tag{11.152}$$

And the vector $\Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon$, which was defined in (11.148) by means of the above vector, can be finally approximated as:

$$\left. \begin{aligned} \Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon &= \mathbf{A}_C \Delta \bar{\mathbf{J}}_t \\ \Delta \bar{\mathbf{J}}_t &\approx \mathbf{G}_t \Delta \boldsymbol{\alpha} \end{aligned} \right\} \implies \begin{cases} \Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon \approx \mathbf{A}_C (\mathbf{G}_t \Delta \boldsymbol{\alpha}) \\ = (\mathbf{A}_C \mathbf{G}_t) \Delta \boldsymbol{\alpha} \\ = \mathbf{B}_t \Delta \boldsymbol{\alpha} \end{cases} \tag{11.153}$$

The above result can be substituted into (11.136), so the vectorial form of the Cauchy stress tensor increment is completely defined. In addition, the vector $\delta \bar{\boldsymbol{\varepsilon}}_\varepsilon$ was defined in (11.34). If these results are taken into account, the material component (11.135) can be finally approximated as:

$$\left. \begin{aligned} \delta \bar{\boldsymbol{\varepsilon}}_\varepsilon &\approx \mathbf{B}_t \boldsymbol{\beta} \\ \Delta \bar{\boldsymbol{\sigma}}_\varepsilon &= \mathbf{D}'_{2,\varepsilon} \Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon \\ \Delta \bar{\boldsymbol{\varepsilon}}_\varepsilon &\approx \mathbf{B}_t \Delta \boldsymbol{\alpha} \end{aligned} \right\} \implies \begin{cases} \iiint \delta \bar{\boldsymbol{\varepsilon}}_\varepsilon^T \Delta \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \approx \iiint (\mathbf{B}_t \boldsymbol{\beta})^T (\mathbf{D}'_{2,\varepsilon} (\mathbf{B}_t \Delta \boldsymbol{\alpha})) d\Omega \\ = \boldsymbol{\beta}^T \left(\iiint \mathbf{B}_t^T \mathbf{D}'_{2,\varepsilon} \mathbf{B}_t d\Omega \right) \Delta \boldsymbol{\alpha} \end{cases} \tag{11.154}$$

According to the previous result, this term turns out to depend on the spatial constitutive tensor $\mathbf{D}'_{2,\varepsilon}$. For this reason, this component is usually known as the material component.

Spatial constitutive tensor

The spatial constitutive tensor was introduced in (11.132) as the tensor that states the linear relation between the Eulerian Cauchy stress tensor increment $\Delta \boldsymbol{\sigma}_\varepsilon$ and the

tensor $\Delta \boldsymbol{\mathcal{E}}_\varepsilon$ defined in (11.124).

$$\Delta \boldsymbol{\sigma}_\varepsilon = \boldsymbol{D}'_{4,\varepsilon} : \Delta \boldsymbol{\mathcal{E}}_\varepsilon \quad \text{with} \quad \Delta \boldsymbol{\mathcal{E}}_\varepsilon = \frac{1}{2} \left[\frac{d\Delta \boldsymbol{u}_\varepsilon}{dr} + \left(\frac{d\Delta \boldsymbol{u}_\varepsilon}{dr} \right)^T \right] \quad (11.155)$$

As the gradients of the displacement field increment are considered to be small at each step of the iterative procedure, the above tensor verifies that:

$$\|\Delta \boldsymbol{\mathcal{E}}_\varepsilon\| \ll \|\boldsymbol{I}\| \quad (11.156)$$

The above condition allows to conclude that the incremental equation (11.155) is the incremental form of the following constitutive equation:

$$\boldsymbol{\sigma}_\varepsilon = \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\mathcal{E}}_\varepsilon) \quad \text{with} \quad \boldsymbol{\mathcal{E}}_\varepsilon = \frac{1}{2} \left[\frac{d\boldsymbol{u}_\varepsilon}{dr} + \left(\frac{d\boldsymbol{u}_\varepsilon}{dr} \right)^T \right] \quad (11.157)$$

The demonstration is analogous to the one performed in section 7.6.1, where the incremental constitutive equation that corresponds to the nonlinear relation between the second Piola-Kirchhoff stress tensor ($\boldsymbol{S}_\mathcal{L}$) and the Green-Lagrange strain tensor ($\boldsymbol{E}_{G,\mathcal{L}}$) is derived.

The equivalent Lagrangian description of the incremental equation (11.155) turns out to be:

$$\Delta \boldsymbol{\sigma}_\mathcal{L} = \boldsymbol{D}'_{4,\mathcal{L}} : \Delta \boldsymbol{\mathcal{E}}_\varepsilon \Big|_{\boldsymbol{r}=\boldsymbol{r}_\mathcal{L}(\boldsymbol{r}_0)} \quad (11.158)$$

Where the Lagrangian description of the fourth order spatial constitutive tensor was defined in (11.128) by means of the components of both the fourth order constitutive tensor \boldsymbol{C}_4 and the deformation gradient tensor.

$$\boldsymbol{D}'_{4,\mathcal{L}} = [D'_{pqrs}]_{\substack{p=1,2,3 \\ q=1,2,3 \\ r=1,2,3 \\ s=1,2,3}} \quad D'_{pqrs} = \frac{1}{F_\mathcal{L}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \quad (11.159)$$

Spatial constitutive tensor in an infinitesimal strain context

When dealing with an infinitesimal strain field, the definition of the spatial constitutive tensor is simplified. If the strain field is infinitesimal, the strain effect can be neglected, so the deformation gradient tensor can be reduced to the rotation tensor.

$$\left. \begin{aligned} \boldsymbol{F}_\mathcal{L} = \boldsymbol{R}_\mathcal{L} [\boldsymbol{I} + \boldsymbol{E}_\mathcal{L}] \\ \|\boldsymbol{E}_\mathcal{L}\| \ll \boldsymbol{I} \end{aligned} \right\} \implies \boldsymbol{F}_\mathcal{L} \approx \boldsymbol{R}_\mathcal{L} = [R_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (11.160)$$

As the rotation tensor has to be orthogonal, the following conditions have to be fulfilled.

$$\boldsymbol{R}_\mathcal{L}^T = \boldsymbol{R}_\mathcal{L}^{-1} \iff \left\{ \begin{aligned} \boldsymbol{R}_\mathcal{L}^T \boldsymbol{R}_\mathcal{L} &= \boldsymbol{I} \\ \boldsymbol{R}_\mathcal{L} \boldsymbol{R}_\mathcal{L}^T &= \boldsymbol{I} \end{aligned} \right\} \iff \left\{ \begin{aligned} \sum_{k=1}^n R_{ki} R_{kj} &= \delta_{ij} \\ \sum_{k=1}^n R_{ik} R_{jk} &= \delta_{ij} \end{aligned} \right\} \quad (11.161)$$

Moreover, the rotation tensor has to represent a proper rotation. This implies that the determinant of the deformation gradient tensor has to be equal to one, as stated in (4.15). Therefore, this deformation process does not produce volume variation.

$$F_{\mathcal{L}} = \det(\mathbf{F}_{\mathcal{L}}) \approx \det(\mathbf{R}_{\mathcal{L}}) = 1 \quad (11.162)$$

On the other hand, if the strain field is infinitesimal, the St Venant-Kirchhoff constitutive model (section 7.7) can be applied. According to this model, the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor can be accurately approximated by the linear relation stated in (7.151). And the tensor that defines this linear relation is the fourth order constitutive tensor \mathbf{C}_4 defined in (7.152). The components of this tensor turn out to be equivalent to the ones that compose the linear constitutive tensor \mathbf{D}_4 (7.82), which states the relation between the Cauchy stress tensor and the infinitesimal strain tensor.

$$\mathbf{C}_4 = [C_{ijkl}]_{\substack{i=1,2,3 \\ j=1,2,3 \\ k=1,3,3 \\ l=1,3,3}} \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (11.163)$$

If the above components are substituted into the components of the fourth order spatial constitutive tensor (11.159), and the conditions (11.161) are taken into account, these components are reduced to:

$$\begin{aligned} D'_{pqrs} &= \frac{1}{F_{\mathcal{L}}} \sum_{i,j,k,l} F_{pi} F_{qj} C_{ijkl} F_{rk} F_{sl} \\ &\approx \sum_{i,j,k,l} \left[\lambda R_{pi} R_{qj} \delta_{ij} \delta_{kl} R_{rk} R_{sl} + \mu (R_{pi} R_{qj} \delta_{ik} \delta_{jl} R_{rk} R_{sl} + R_{pi} R_{qj} \delta_{il} \delta_{jk} R_{rk} R_{sl}) \right] \\ &= \lambda \left(\sum_{i,j} R_{pi} R_{qj} \delta_{ij} \right) \left(\sum_{k,l} R_{rk} R_{sl} \delta_{kl} \right) \\ &\quad + \mu \left[\left(\sum_{i,k} R_{pi} R_{rk} \delta_{ik} \right) \left(\sum_{j,l} R_{qj} R_{sl} \delta_{jl} \right) + \left(\sum_{i,l} R_{pi} R_{sl} \delta_{il} \right) \left(\sum_{j,k} R_{qj} R_{rk} \delta_{jk} \right) \right] \\ &= \lambda \left(\sum_i R_{pi} R_{qi} \right) \left(\sum_k R_{rk} R_{sk} \right) \\ &\quad + \mu \left[\left(\sum_i R_{pi} R_{ri} \right) \left(\sum_j R_{qj} R_{sj} \right) + \left(\sum_i R_{pi} R_{si} \right) \left(\sum_j R_{qj} R_{rj} \right) \right] \\ &= \lambda \delta_{pq} \delta_{rs} + \mu (\delta_{pr} \delta_{qs} + \delta_{ps} \delta_{qr}) \\ &= C_{pqrs} \end{aligned} \quad (11.164)$$

Therefore, if the strain field is infinitesimal, it can be concluded that the spatial constitutive tensor is constant and equivalent to the fourth order constitutive tensor \mathbf{C}_4 . Furthermore, if the strain field is infinitesimal, \mathbf{C}_4 is also equivalent to the linear elastic constitutive tensor \mathbf{D}_4 . Thus,

$$\mathbf{D}'_4 = \mathbf{C}_4 = \mathbf{D}_4 \quad (11.165)$$

11.7.4. Geometric component

The second term of the scalar product (11.120) is the so-called geometric component, since it depends on the stress state and the geometry of the reference material domain. This dependence will be proved later on in this section. The integral that defines this term is:

$$\iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\ell}) : \mathbf{S}_\ell d\Omega_0 \quad (11.166)$$

This term is defined by means of the second Piola-Kirchhoff stress tensor and the increment of the Green-Lagrange strain tensor variation.

The second Piola-Kirchhoff stress tensor was defined in (3.48) as:

$$\mathbf{S}_\ell = F_\ell \left(\mathbf{F}_\ell^{-1} \boldsymbol{\sigma}_\ell \mathbf{F}_\ell^{-T} \right) \quad (11.167)$$

And the increment of the Green-Lagrange strain tensor variation, was obtained in (4.101) as:

$$\Delta(\delta \mathbf{E}_{G,\ell}) = \frac{1}{2} \left(\delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell + \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \right) \quad (11.168)$$

The above equation holds if the gradients of the displacement field increments are small. As stated before, this assumption is made at each step of the iterative procedure. Hence, this equation can be properly applied.

In order to compute the double dot product between both tensors, the properties exposed in (A.79) and (A.81) are applied, and the symmetry of the Cauchy stress tensor is taken into account. Thus, the double dot product between both tensors leads to the following result.

$$\begin{aligned} \Delta(\delta \mathbf{E}_{G,\ell}) : \mathbf{S}_\ell &= \frac{1}{2} \left(\delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell + \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \right) : F_\ell \left(\mathbf{F}_\ell^{-1} \boldsymbol{\sigma}_\ell \mathbf{F}_\ell^{-T} \right) \\ &= \frac{F_\ell}{2} \text{Tr} \left(\left(\delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell + \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \right) \left(\mathbf{F}_\ell^{-1} \boldsymbol{\sigma}_\ell \mathbf{F}_\ell^{-T} \right) \right) \\ &= \frac{F_\ell}{2} \text{Tr} \left(\mathbf{F}_\ell^{-T} \left(\delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell + \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \right) \mathbf{F}_\ell^{-1} \boldsymbol{\sigma}_\ell \right) \\ &= \frac{F_\ell}{2} \text{Tr} \left(\left(\mathbf{F}_\ell^{-T} \delta \mathbf{J}_\ell^T \Delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} + \mathbf{F}_\ell^{-T} \Delta \mathbf{J}_\ell^T \delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} \right) \boldsymbol{\sigma}_\ell \right) \\ &= \frac{F_\ell}{2} \left(\left(\delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} \right)^T \left(\Delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} \right) + \left(\Delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} \right)^T \left(\delta \mathbf{J}_\ell \mathbf{F}_\ell^{-1} \right) \right) : \boldsymbol{\sigma}_\ell \end{aligned} \quad (11.169)$$

Furthermore, the increment and the variation of the displacement gradient tensor can be equivalently expressed by means of the deformation gradient tensor. To obtain these equations, the derivative of a given magnitude with respect to the reference position vector (1.20) is applied. From these equations, the terms which are involved

in the above double dot product can be obtained.

$$\begin{aligned} \delta \mathbf{J}_{\mathcal{L}} &= \frac{d\delta \mathbf{u}_{\mathcal{L}}}{d\mathbf{r}_0} = \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \mathbf{F}_{\mathcal{L}} & \iff & \delta \mathbf{J}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-1} = \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \\ \Delta \mathbf{J}_{\mathcal{L}} &= \frac{d\Delta \mathbf{u}_{\mathcal{L}}}{d\mathbf{r}_0} = \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \mathbf{F}_{\mathcal{L}} & \iff & \Delta \mathbf{J}_{\mathcal{L}} \mathbf{F}_{\mathcal{L}}^{-1} = \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \end{aligned} \quad (11.170)$$

Consequently, these terms can be substituted into (11.169), and the double dot product becomes:

$$\Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} = \mathbf{F}_{\mathcal{L}} \left\{ \frac{1}{2} \left[\left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right] : \boldsymbol{\sigma}_{\varepsilon} \right\} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \quad (11.171)$$

The integration of the above double dot product over the initial configuration leads to an equivalent definition of the geometric component. To change the integration domain and get an integral defined over the deformed domain, the methodology presented in section 1.14 is recalled. This methodology allows moving from integrals defined over the deformed domain, to integrals defined over the initial reference configuration. However, the inverse transformation is required. That is, the change from an integral defined over the reference material domain to an equivalent one over the deformed configuration is carried out.

$$\begin{aligned} & \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 = \\ &= \iiint_{\Omega_0} \left\{ \frac{1}{2} \left[\left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right] : \boldsymbol{\sigma}_{\varepsilon} \right\} \Big|_{\mathbf{r}=\mathbf{r}_{\mathcal{L}}(\mathbf{r}_0)} \mathbf{F}_{\mathcal{L}} d\Omega_0 \quad (11.172) \\ &= \iiint_{\Omega} \frac{1}{2} \left[\left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right] : \boldsymbol{\sigma}_{\varepsilon} d\Omega \end{aligned}$$

And the comparison of the above result with the scalar product (11.121) allows to introduce the definition of the tensor $\Delta(\delta \boldsymbol{\mathcal{E}}_{\varepsilon})$.

$$\begin{aligned} & \left. \begin{aligned} \Delta(\delta \boldsymbol{\mathcal{E}}_{\varepsilon}) &= \frac{1}{2} \left[\left(\frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} + \left(\frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right)^T \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \right] \\ \Delta \mathbf{J}_t &= \frac{d\Delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \\ \delta \mathbf{J}_t &= \frac{d\delta \mathbf{u}_{\varepsilon}}{d\mathbf{r}} \end{aligned} \right\} \implies \quad (11.173) \\ & \implies \Delta(\delta \boldsymbol{\mathcal{E}}_{\varepsilon}) = \frac{1}{2} \left(\Delta \mathbf{J}_t^T \delta \mathbf{J}_t + \delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) \end{aligned}$$

Since the Cauchy stress tensor is symmetric, the double dot product property proved in (A.64) can be applied, which leads to the following equivalent definition of the

geometric component.

$$\begin{aligned}
 \iiint_{\Omega_0} \Delta(\delta \mathbf{E}_{G,\mathcal{L}}) : \mathbf{S}_{\mathcal{L}} d\Omega_0 &= \iiint_{\Omega} \Delta(\delta \mathcal{E}_{\varepsilon}) : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t + \Delta \mathbf{J}_t^T \delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega + \frac{1}{2} \iiint_{\Omega} \left(\Delta \mathbf{J}_t^T \delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega + \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right)^T : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega + \frac{1}{2} \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega
 \end{aligned} \tag{11.174}$$

The substitution of the tensors that compose the above integrand allows to rewrite the geometric component and obtain a more convenient definition.

$$\begin{aligned}
 &\iiint_{\Omega} \Delta(\delta \mathcal{E}_{\varepsilon}) : \boldsymbol{\sigma}_{\varepsilon} d\Omega = \\
 &= \iiint_{\Omega} \left(\delta \mathbf{J}_t^T \Delta \mathbf{J}_t \right) : \boldsymbol{\sigma}_{\varepsilon} d\Omega \\
 &= \iiint_{\Omega} \begin{bmatrix} \frac{\partial \delta u_1}{\partial r_1} & \frac{\partial \delta u_1}{\partial r_2} & \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} & \frac{\partial \delta u_2}{\partial r_2} & \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} & \frac{\partial \delta u_3}{\partial r_2} & \frac{\partial \delta u_3}{\partial r_3} \end{bmatrix}^T \begin{bmatrix} \frac{\partial \Delta u_1}{\partial r_1} & \frac{\partial \Delta u_1}{\partial r_2} & \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} & \frac{\partial \Delta u_2}{\partial r_2} & \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} & \frac{\partial \Delta u_3}{\partial r_2} & \frac{\partial \Delta u_3}{\partial r_3} \end{bmatrix} : \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} d\Omega \\
 &= \iiint_{\Omega} \left\{ \begin{array}{l} \frac{\partial \delta u_1}{\partial r_1} \\ \frac{\partial \delta u_1}{\partial r_2} \\ \frac{\partial \delta u_1}{\partial r_3} \\ \frac{\partial \delta u_2}{\partial r_1} \\ \frac{\partial \delta u_2}{\partial r_2} \\ \frac{\partial \delta u_2}{\partial r_3} \\ \frac{\partial \delta u_3}{\partial r_1} \\ \frac{\partial \delta u_3}{\partial r_2} \\ \frac{\partial \delta u_3}{\partial r_3} \end{array} \right\}^T \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{12} & \sigma_{22} & \sigma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{13} & \sigma_{23} & \sigma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{11} & \sigma_{12} & \sigma_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{12} & \sigma_{22} & \sigma_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \left\{ \begin{array}{l} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_2} \\ \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_3}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \end{array} \right\} d\Omega \\
 &= \iiint_{\Omega} \delta \bar{\mathbf{J}}_t^T \bar{\boldsymbol{\sigma}}_{\varepsilon} \Delta \bar{\mathbf{J}}_t d\Omega
 \end{aligned} \tag{11.175}$$

Where $\hat{\sigma}_\varepsilon$ is a banded tensor composed by three Eulerian Cauchy stress tensors located at the diagonal positions.

On the other hand, the vector $\Delta \bar{\mathbf{J}}_t$ can be defined by means of a differential operator, as shown below.

$$\Delta \bar{\mathbf{J}}_t = \begin{Bmatrix} \frac{\partial \Delta u_1}{\partial r_1} \\ \frac{\partial \Delta u_1}{\partial r_2} \\ \frac{\partial \Delta u_1}{\partial r_3} \\ \frac{\partial \Delta u_2}{\partial r_1} \\ \frac{\partial \Delta u_2}{\partial r_2} \\ \frac{\partial \Delta u_2}{\partial r_3} \\ \frac{\partial \Delta u_3}{\partial r_1} \\ \frac{\partial \Delta u_3}{\partial r_2} \\ \frac{\partial \Delta u_3}{\partial r_3} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r_1} & 0 & 0 \\ \frac{\partial}{\partial r_2} & 0 & 0 \\ \frac{\partial}{\partial r_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial r_1} & 0 \\ 0 & \frac{\partial}{\partial r_2} & 0 \\ 0 & \frac{\partial}{\partial r_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial r_1} \\ 0 & 0 & \frac{\partial}{\partial r_2} \\ 0 & 0 & \frac{\partial}{\partial r_3} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{Bmatrix} = \partial_t \Delta \mathbf{u}_\varepsilon \quad (11.176)$$

And the Eulerian displacement field increment was approximated in (11.151) by means of the trial functions as:

$$\Delta \mathbf{u}_\varepsilon \approx \phi_\varepsilon \Delta \boldsymbol{\alpha} \quad (11.177)$$

If the above approximation is introduced into (11.176), the vector $\Delta \bar{\mathbf{J}}_t$ becomes:

$$\left. \begin{array}{l} \Delta \bar{\mathbf{J}}_t = \partial_t \Delta \mathbf{u}_\varepsilon \\ \Delta \mathbf{u}_\varepsilon \approx \phi_\varepsilon \Delta \boldsymbol{\alpha} \end{array} \right\} \implies \begin{cases} \Delta \bar{\mathbf{J}}_t \approx \partial_t (\phi_\varepsilon \Delta \boldsymbol{\alpha}) \\ = (\partial_t \phi_\varepsilon) \Delta \boldsymbol{\alpha} \\ = \mathbf{G}_t \Delta \boldsymbol{\alpha} \end{cases} \quad (11.178)$$

Where the tensor \mathbf{G}_t was previously defined in (11.32).

The substitution of the vectors $\delta \bar{\mathbf{J}}_t$ and $\Delta \bar{\mathbf{J}}_t$ obtained in (11.31) and (11.178), respectively, into the geometric component (11.175) is now possible. By doing so, the geometric component becomes:

$$\left. \begin{array}{l} \delta \bar{\mathbf{J}}_t \approx \mathbf{G}_t \boldsymbol{\beta} \\ \Delta \bar{\mathbf{J}}_t \approx \mathbf{G}_t \Delta \boldsymbol{\alpha} \end{array} \right\} \implies \begin{cases} \iiint \Delta (\delta \mathcal{E}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon d\Omega = \iiint \delta \bar{\mathbf{J}}_t^T \hat{\boldsymbol{\sigma}}_\varepsilon \Delta \bar{\mathbf{J}}_t d\Omega \\ \approx \iiint (\mathbf{G}_t \boldsymbol{\beta})^T \hat{\boldsymbol{\sigma}}_\varepsilon (\mathbf{G}_t \Delta \boldsymbol{\alpha}) d\Omega \\ = \boldsymbol{\beta}^T \left(\iiint \mathbf{G}_t^T \hat{\boldsymbol{\sigma}}_\varepsilon \mathbf{G}_t d\Omega \right) \Delta \boldsymbol{\alpha} \end{cases} \quad (11.179)$$

According to the above definition, this component depends on the Cauchy stress tensor and the tensor \mathbf{G}_t , which contains the derivatives of the trial functions with respect to the material domain that corresponds to load step t . Therefore, this term turns out to depend on the stress field and the geometry of the reference configuration. For this reason, it is usually known as the geometric component.

11.7.5. Tangent stiffness

The addition of the material component (11.154) and the geometric one (11.179) leads to the definition of the tangent stiffness.

$$\begin{aligned}
 \frac{df}{d\mathbf{u}_\varepsilon} \Delta \mathbf{u}_\varepsilon &= \iiint_\Omega \delta \bar{\boldsymbol{\varepsilon}}_\varepsilon^T \Delta \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega + \iiint_\Omega \Delta(\delta \boldsymbol{\varepsilon}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon d\Omega \\
 &= \boldsymbol{\beta}^T \left(\iiint_\Omega \mathbf{B}_t^T \mathbf{D}'_{2,\varepsilon} \mathbf{B}_t d\Omega + \iiint_\Omega \mathbf{G}_t^T \hat{\boldsymbol{\sigma}}_\varepsilon \mathbf{G}_t d\Omega \right) \Delta \boldsymbol{\alpha} \quad (11.180) \\
 &= \boldsymbol{\beta}^T (\mathbf{K}_M + \mathbf{K}_G) \Delta \boldsymbol{\alpha} \\
 &= \boldsymbol{\beta}^T \mathbf{K}_T \Delta \boldsymbol{\alpha}
 \end{aligned}$$

Where \mathbf{K}_T is usually known as the tangent stiffness matrix, which is composed by two terms: the material component \mathbf{K}_M and the geometric stiffness matrix \mathbf{K}_G .

$$\begin{aligned}
 \mathbf{K}_T &= \mathbf{K}_M + \mathbf{K}_G \\
 \mathbf{K}_M &= \iiint_\Omega \mathbf{B}_t^T \mathbf{D}'_{2,\varepsilon} \mathbf{B}_t d\Omega \\
 \mathbf{K}_G &= \iiint_\Omega \mathbf{G}_t^T \hat{\boldsymbol{\sigma}}_\varepsilon \mathbf{G}_t d\Omega
 \end{aligned} \quad (11.181)$$

Material component

The material component of the tangent stiffness was defined as:

$$\mathbf{K}_M = \iiint_\Omega \mathbf{B}_t^T \mathbf{D}'_{2,\varepsilon} \mathbf{B}_t d\Omega \quad (11.182)$$

Its structure is similar to the linear stiffness matrix (9.108), with some major differences.

- On the one hand, the tensor \mathbf{B}_t includes the derivatives of the trial functions with respect to the geometry corresponding to load step t , while in linear analysis, these derivatives are taken with respect to the initial configuration.
- On the other hand, the constitutive tensor involved in this component is the spatial constitutive tensor $\mathbf{D}'_{2,\varepsilon}$, instead of the linear constitutive one \mathbf{D}_2 defined in (7.94).

Geometric stiffness

The second component of the tangent stiffness is the so-called geometric stiffness, which was defined in (11.181) as:

$$\mathbf{K}_G = \iiint_{\Omega} \mathbf{G}_t^T \hat{\boldsymbol{\sigma}}_\varepsilon \mathbf{G}_t d\Omega \quad (11.183)$$

Its structure is quite similar to the one obtained with the total Lagrangian approach in (10.154). However, their comparison leads to the following major differences.

- The Cauchy stresses are involved instead of the second Piola-Kirchhoff ones.
- And the derivatives of the trial functions are taken with respect to the geometry corresponding to load step t , which is the new reference configuration, instead of being taken with respect to the initial one.

11.7.6. Iterative procedure overview

To sum up, the equation to be solved at each step of the Newton-Raphson method is the one presented in (11.98).

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_{k,t}^{t+\Delta t}) \Delta \mathbf{u}_{k,t}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{k,t}^{t+\Delta t}) \quad k = 0, 1, 2, \dots \quad (11.184)$$

As obtained in (11.180), the left-hand side of the above equation can be equivalently written as:

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_t) \Delta \mathbf{u}_t = \boldsymbol{\beta}^T \mathbf{K}_T(\mathbf{u}_t) \Delta \boldsymbol{\alpha} \quad (11.185)$$

And the two addends that compose the right-hand side of equation (11.184) were defined in (11.103) and (11.104), respectively.

$$\begin{aligned} P^{t+\Delta t} &= \boldsymbol{\beta}^T \left(\iiint_{\Omega_t} \boldsymbol{\phi}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \iint_{\Gamma_t^\sigma} \boldsymbol{\phi}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t \right) \\ f(\mathbf{u}_t) &= \iiint_{\Omega_t} \delta \bar{\boldsymbol{\varepsilon}}_t^T \bar{\boldsymbol{\sigma}}_t d\Omega_t \end{aligned} \quad (11.186)$$

Modified Newton-Raphson method

The value $k = 0$ applied to (11.184) leads to the equation that defines the first iteration.

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_{0,t}^{t+\Delta t}) \Delta \mathbf{u}_{0,t}^{t+\Delta t} = P^{t+\Delta t} - f(\mathbf{u}_{0,t}^{t+\Delta t}) \quad (11.187)$$

The initial approximation needs to be defined to start the iterative procedure. As stated in previous sections, the displacement field corresponding to the previous load step is adopted as the initial root approximation.

$$\mathbf{u}_{0,t}^{t+\Delta t} = \mathbf{u}_t^t \quad (11.188)$$

And the terms that depend on the above displacement field are defined below. The evaluation of the function f at the above displacement field was defined in (11.108) as:

$$f(\mathbf{u}_{0,t}^{t+\Delta t}) = f(\mathbf{u}_t^t) = \beta^T \iiint_{\Omega_t} \mathbf{G}_t^T \mathbf{A}_C^T \bar{\boldsymbol{\sigma}}_t^t d\Omega_t \quad (11.189)$$

And the left-hand side of the equation (11.187) becomes:

$$\frac{df}{d\mathbf{u}_t}(\mathbf{u}_{0,t}^{t+\Delta t}) \Delta \mathbf{u}_{0,t}^{t+\Delta t} = \frac{df}{d\mathbf{u}_t}(\mathbf{u}_t^t) \Delta \mathbf{u}_{0,t}^{t+\Delta t} = \beta^T \mathbf{K}_T(\mathbf{u}_t^t) \Delta \boldsymbol{\alpha}_0 \quad (11.190)$$

Where the components of the above tangent stiffness, according to their definition presented in (11.181), turn out to be:

$$\begin{aligned} \mathbf{K}_T(\mathbf{u}_t^t) &= \mathbf{K}_M(\mathbf{u}_t^t) + \mathbf{K}_G(\mathbf{u}_t^t) \\ \mathbf{K}_M(\mathbf{u}_t^t) &= \iiint_{\Omega_t} \mathbf{B}_t^T \mathbf{D}'_{2,t} \mathbf{B}_t d\Omega_t \\ \mathbf{K}_G(\mathbf{u}_t^t) &= \iiint_{\Omega_t} \mathbf{G}_t^T \hat{\boldsymbol{\sigma}}_t^t \mathbf{G}_t d\Omega_t \end{aligned} \quad (11.191)$$

The above results allow to obtain an equivalent equation that defines the first iteration (11.187), as shown below.

$$\begin{aligned} \mathbf{K}_T(\mathbf{u}_t^t) \Delta \boldsymbol{\alpha}_0 &= \iiint_{\Omega_t} \boldsymbol{\phi}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \iint_{\Gamma_t^\sigma} \boldsymbol{\phi}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t \\ &\quad - \iiint_{\Omega_t} \mathbf{G}_t^T \mathbf{A}_C^T \bar{\boldsymbol{\sigma}}_t^t d\Omega_t \end{aligned} \quad (11.192)$$

If the modified Newton-Raphson method is applied, the tangent stiffness is only computed once, and the same stiffness is maintained over the entire iterative process. This method avoids to update the tangent stiffness at each iterative step, so a big amount of computational effort is saved. Nevertheless, more iterations might be needed to reach convergence.

The decision of applying this simplified method or the general one has to be made. As a general rule, it is worth trying the modified Newton-Raphson method. If convergence issues arise, the general Newton-Raphson method, in which the computation of the tangent stiffness has to be made at each iterative step, has to be taken into account.

The equation that defines the remaining iterations of the modified Newton-Raphson method is:

$$\begin{aligned} \mathbf{K}_T(\mathbf{u}_t^t) \Delta \boldsymbol{\alpha}_k &= P^{t+\Delta t} - f(\mathbf{u}_{k,t}^{t+\Delta t}) \\ &= \iiint_{\Omega_t} \boldsymbol{\phi}_t^T \mathbf{b}_t^{t+\Delta t} \rho_t^t d\Omega_t + \iint_{\Gamma_t^\sigma} \boldsymbol{\phi}_t^T \mathbf{g}_{t,t}^{t+\Delta t} d\Gamma_t \\ &\quad - \left[\iiint_{\Omega} \mathbf{G}^T \mathbf{A}_C^T \bar{\boldsymbol{\sigma}}_\varepsilon d\Omega \right] \Big|_{\mathbf{u}_t = \mathbf{u}_{k,t}^{t+\Delta t}} \quad k = 1, 2, 3, \dots \end{aligned} \quad (11.193)$$

Therefore, the only term that has to be updated in order to continue iterating is the scalar function f , since the integral is defined over the last computed material domain. This material domain is defined according to the following equation:

$$\begin{aligned} \mathbf{r}_{k,t}^{t+\Delta t}(\mathbf{r}_t) &= \mathbf{r}_t + \Delta \mathbf{u}_{k,t}^{t+\Delta t}(\mathbf{r}_t) \\ &= \mathbf{r}_t + \left[\mathbf{u}_{k,t}^{t+\Delta t}(\mathbf{r}_t) - \mathbf{u}_t^t(\mathbf{r}_t) \right] \end{aligned} \quad (11.194)$$

Once the system of linear equations (11.193) is solved, the displacement field is updated and the convergence criterion (10.125) is checked. If convergence is achieved, the iterative procedure is stopped, and the last iterative result is adopted as the displacement field corresponding to the current load step. If not, the next iteration is carried out by repeating the procedure exposed in this section.

$$\begin{aligned} \Delta \alpha_k &\implies \Delta \mathbf{u}_{k,t}^{t+\Delta t} = \phi_t \Delta \alpha_k \\ &\implies \mathbf{u}_{k+1,t}^{t+\Delta t} = \mathbf{u}_{k,t}^{t+\Delta t} + \Delta \mathbf{u}_{k,t}^{t+\Delta t} \\ &\implies \text{Convergence?} \quad \begin{cases} \text{Yes} &\implies \mathbf{u}_{k+1,t}^{t+\Delta t} = \mathbf{u}_t^{t+\Delta t} \quad (\text{stop}) \\ \text{No} &\implies \mathbf{u}_{k+2,t}^{t+\Delta t} = \mathbf{u}_{k+1,t}^{t+\Delta t} + \Delta \mathbf{u}_{k+1,t}^{t+\Delta t} \quad (\text{continue}) \end{cases} \end{aligned} \quad (11.195)$$

Note that the discretization methodology, as well as the criteria to define the trial functions, are the same as those exposed in linear theory. They can be checked in sections 9.17 and 9.18, respectively.

11.8. Strain and stress fields update

Once the convergence has been achieved, the displacement field of the current load step $t + \Delta t$ is known. The next step is to compute its corresponding strain and stress fields.

Before solving this load step, the previous one was completely defined. Hence, the displacement, strain, and stress fields corresponding to the load step t are:

$$\mathbf{u}_t^t(\mathbf{r}_t) \implies \mathcal{E}_t^t(\mathbf{u}_t^t(\mathbf{r}_t)) \implies \boldsymbol{\sigma}_t^t(\mathcal{E}_t^t(\mathbf{u}_t^t(\mathbf{r}_t))) \quad (11.196)$$

Let's move then to the next load step $t + \Delta t$, where the displacement field is obtained by applying the iterative procedure exposed in the previous section. The difference between the displacement field of consecutive load steps defines the displacement field increment.

$$\Delta \mathbf{u}_t^t = \mathbf{u}_t^{t+\Delta t} - \mathbf{u}_t^t \quad (11.197)$$

As the gradients of the displacement field increment are small, the strain tensor increment and the Cauchy stress tensor increment can be computed according to equations (11.148) and (11.136), respectively.

$$\left\| \frac{d\Delta \mathbf{u}_t^t}{d\mathbf{r}_t} \right\| \ll \|\mathbf{I}\| \implies \begin{cases} \Delta \bar{\boldsymbol{\epsilon}}_t^t = \mathbf{A}_C \Delta \bar{\mathbf{J}}_t^t \\ \Delta \bar{\boldsymbol{\sigma}}_t^t = \mathbf{D}'_{2,t}(\bar{\boldsymbol{\epsilon}}_t^t) \Delta \bar{\boldsymbol{\epsilon}}_t^t \end{cases} \quad (11.198)$$

Therefore, the updated strain and stress fields are obtained by adding the above increments to the previous results.

$$\begin{aligned}\bar{\boldsymbol{\epsilon}}_t^{t+\Delta t} &= \bar{\boldsymbol{\epsilon}}_t^t + \Delta \bar{\boldsymbol{\epsilon}}_t^t \\ \bar{\boldsymbol{\sigma}}_t^{t+\Delta t} &= \bar{\boldsymbol{\sigma}}_t^t + \Delta \bar{\boldsymbol{\sigma}}_t^t\end{aligned}\tag{11.199}$$

11.9. Overview and conclusions

11.9.1. Nonlinear analysis strategy

This chapter is again focused on the static analysis of solids subjected to external loads. The external forces are supposed to be common loads that do not depend on the displacement field. The solid adopts a deformed configuration in static equilibrium after the application of the external loads.

A nonlinear analysis is again carried out in order to obtain the structural behaviour. That is, the solid is supposed to behave with large displacements and large displacement gradients.

- As the displacements that the solid undergoes are large, the reference configuration and the deformed domain can not be considered coincident. Therefore, the equilibrium equations have to be imposed over the unknown deformed material domain. Moreover, the volume variation has to be taken into account, and the Lagrangian and Eulerian descriptions of a given magnitude are not coincident.
- The displacement gradients are also assumed to be large, so the approximate polar decomposition of the deformation gradient tensor can not be applied in this context. Consequently, an eigenvalue problem has to be solved to properly decompose the deformation gradient tensor. Once this problem is solved, both the rotation and the strain tensors can be defined, and their product states the polar decomposition of the deformation gradient tensor. Furthermore, the stress field experimented by the solid can also be defined by means of both tensors.

As the structural response is nonlinear, a given load state could have multiple solutions. Thus, the total load can not be applied in only one step. If the order in which the external loads are applied is not taken into account, a structural behaviour that does not correspond to the real one may be obtained. And the superposition principle which is commonly applied in linear analysis does no longer hold.

To overcome the issues of dealing with a nonlinear structural response, an incremental loading process can be carried out. Moreover, an incremental loading procedure is a suitable method, since the weak form that leads to the finite element formulation turns out to be a nonlinear equation that depends on the displacement field. This equation has to be solved by an iterative method, which needs to start iterating from a close approximation to the solution. The application of a small incremental load at each step allows to start the iterative procedure from a close approximation to the displacement field of the current load step.

11.9.2. Updated Lagrangian finite element analysis

The finite element formulation derived in the previous chapter is usually known as the total Lagrangian one. This approach is based on the adoption of the initial material domain as the reference configuration along the entire incremental loading process. The weak form that leads to the finite element formulation turns out to be a Lagrangian one. That is, it turns out to be composed by integrals defined over the well-known undeformed configuration, and the magnitudes involved in it are described according to their Lagrangian descriptions.

However, the updated approach is now adopted instead of the total one. According to this updated point of view, the last computed material domain becomes the new reference configuration. Once each load step is solved, the last computed material domain becomes the reference one for the next load step.

The derivation of the finite element formulation corresponding to this updated approach is analogous to the one presented in the previous chapter. The weak form of the current load step is composed by integrals defined over the unknown material domain, and the magnitudes are described according to their Eulerian description. The real challenge is to manipulate this weak form to obtain integrals defined over the material domain corresponding to the previous load step, which has become the new reference configuration, as well as magnitudes described with respect to this domain. The resulting weak form turns out to be a nonlinear equation which needs to be solved iteratively. The application of an iterative method in conjunction with the finite element method leads to the obtention of a system of linear equations, which has to be solved in order to obtain the displacement field at each iteration.

The matrix that defines this system of linear equations is the so-called tangent stiffness, since it can be geometrically interpreted as the tangent of the nonlinear structural response. This stiffness turns out to be composed by the sum of two components: the material component and the geometric stiffness. The material component of the tangent stiffness turns out to depend on the mechanical properties of the material, whereas the geometric component depends on both the stress state and the geometry of the reference material domain.

The tangent stiffness can be calculated only once, if the same tangent stiffness is maintained over the entire iterative process. This method avoids to update the tangent stiffness at each iterative step, so a big amount of computational effort is saved. Nevertheless, more iterations might be needed to reach convergence. The decision of applying this simplified method or the general one has to be made. As a general rule, it is worth trying the modified method. If convergence issues arise, the general one has to be taken into account, in which the computation of the tangent stiffness has to be made at each iteration.

Both the total and the updated approaches lead to analogous system of linear equations. Nevertheless, the following major differences between the components of the tangent stiffness can be outlined.

- On the one hand, the material component of the tangent stiffness is similar to the linear stiffness matrix. However, the derivatives of the trial functions are taken with respect to the geometry of the previous load step, instead of being taken with respect to the initial configuration. Moreover, this term is defined by means of the spatial constitutive tensor, instead of the linear constitutive one.
- On the other hand, the geometric stiffness is quite similar to the one obtained in the Total Lagrangian approach. Nevertheless, the Cauchy stresses are involved instead of the second Piola-Kirchhoff ones. And the derivatives of the trial functions are taken with respect to the geometry of the previous load step, instead of being taken with respect to the initial one.

Once the system of linear equations is solved, the displacement field can be updated, and the convergence criterion has to be verified. If convergence is achieved, the iterative procedure is stopped, and the last iterative result is adopted as the displacement field corresponding to the current load step. If not, the process is repeated until reaching convergence.

Note that the iterative method may fail to converge if the tangent stiffness matrix becomes singular. To deal with these limit points, many techniques have been proposed to successfully pass them. A review of these techniques can be checked in the previous chapter.

Conclusions and future research

12.1. Conclusions

In this section, the most relevant conclusions obtained along this work are presented. The first subsection contains the main general conclusions. The second one is dedicated to the principal conclusions obtained after proposing a general formulation for the nonlinear solid mechanics, whereas the third one is focused on the conclusions that can be drawn from the derivation of the nonlinear finite element formulations.

12.1.1. General conclusions

One of the main contributions of this work is the proposition of a novel unifying formulation for nonlinear solid mechanics.

- The assumptions made about the magnitude of both the displacements and the displacement gradients are very important, since they define the theoretical framework of the structural analysis. Most references in the existing literature do not clearly identify the implications of these assumptions.
 - This formulation allows to clearly identify the implications of each assumption in order to properly define the mathematical models that governs the structural behaviour both in linear and nonlinear analysis.
- A novel, simple, and clear nomenclature is proposed.
- Only the necessary magnitudes and equations that allow to describe the presented deformation process are stated in order to achieve a detailed formulation for nonlinear solid mechanics.
 - This formulation is based on only two main equations that completely characterize the solid behaviour.

- The equation that defines the motion of a given particle that belongs to the initial configuration. This one defines the solid motion.
- And the equation that rules the finite or infinitesimal geometric transformation experimented by a given material vector. This one governs the change of volume, orientation, and shape experimented by the solid.
- The incremental approach that defines the incremental geometric transformation experimented by a given material vector between two consecutive time steps is also derived.
 - This formulation can be extended to describe the incremental loading process usually applied in nonlinear analysis, if the incremental load at each step is small enough.
 - This approach is also suitable for describing viscous behaviour.

The proposed nonlinear solid mechanics formulation establishes the basic principles that allow to derive a novel subsequent finite element formulation. Another main contribution of this thesis is to present the complete derivation of two nonlinear finite element formulations.

- Many references address the derivation of the nonlinear finite element formulations. Nevertheless, there is no consensus about a common nomenclature and notation. Moreover, the hypotheses made along these derivations are not clearly specified or are not even stated.
 - A novel derivation which clarifies these formulations is proposed, in order to deeply understand the underlying physics and the essence of the algorithms. The intermediate hypotheses are clearly identified, and the origin and definition of the matrices that compose the algorithms is extensively analysed.
 - A detailed guideline that facilitates the deep comprehension of this powerful technology is proposed.
- Some classical nonlinear finite element formulations are written entirely in index notation, which is convenient for coding and implementing the algorithms into a computer software. In this work, the use of index notation is avoided whenever possible for facilitating the understanding of the concepts. If required, it is applied as an intermediate step to reach a final tensor notation.
- The derivation of static finite element formulations is presented. However, the nonlinear solid mechanics principles proposed in this thesis also allow the derivation of the corresponding dynamic formulation.
- The nonlinear analysis is carried out according to an incremental loading process. At each loading step, the finite element method in conjunction with an iterative

method lead to the resolution of a linear system of equations at each iteration. The matrix of this system is the tangent stiffness, which varies at each iteration.

- For saving computing effort, the tangent stiffness can be calculated only once at the first iteration, so the same stiffness is maintained over the entire iterative process. This modified method allows to save a big amount of computation effort. Nevertheless, more iterations might be needed to reach convergence.
 - The decision of applying this simplified method or the general one has to be made. As a general rule, it is worth trying the modified method.
 - The iterative method may fail to converge if the tangent stiffness matrix becomes singular. To deal with these limit points, many techniques have been proposed to successfully pass them. An overview of these techniques has also been presented.
- As a general rule, the Total Lagrangian finite element formulation is preferable to the updated one, since the reference domain does not need to be updated at each step of the incremental loading process. Moreover, to define the magnitudes with respect to the initial configurations along the entire loading process is more convenient than modifying the reference at each load step. In addition, the total approach does not require the use of the spatial constitutive tensor, which leads to less computational effort.

12.1.2. Conclusions derived from the solid mechanics formulation

In this subsection, some of the most relevant conclusions that can be obtained from the derivation of the nonlinear solid mechanics formulation proposed in this thesis are summarized.

The deformation vector and its corresponding gradient tensor are the most important magnitudes to describe the deformation of a solid media subjected to external forces.

- The solid motion is described by the deformation vector, which is defined by means of the displacement vector. Therefore, if the displacement field is known, the deformed material vector can be defined over time, and the solid motion is completely defined.
- The deformation gradient tensor turns out to rule the geometric transformation of a given material vector over time. Thus, this tensor is the one that governs the change of volume, orientation, and shape. It contains the required information to properly define the displacement field, as well as its corresponding strain and stress fields.

- The determinant of the deformation gradient tensor governs the volume variation of a given differential volume defined on the initial configuration.
 - A novel derivation to obtain the determinant of the deformation gradient tensor is proposed. The classical proof is based on the vector product, which is a tensor operation only defined in a three-dimensional space. The demonstration proposed in this thesis is based on the metric tensor, and the use of the vector product is not required.

The balance equations of mass, linear momentum, and angular momentum rule the solid structural behaviour. These equations govern the value of the density field over time, and state the dynamic equilibrium of forces and torques.

- In Solid Mechanics, it is not common to deal with mass sources. So, it is usually considered that the mass remains constant. Under the assumption of mass conservation, the density of the media turns out to be defined by means of the initial density field and the value of determinant of the deformation gradient tensor.
- The balance equations also allow to obtain the equations that rule the static equilibrium. A solid at static equilibrium fulfils the conservation of both the linear momentum and the angular momentum.
 - The imposition of the linear momentum conservation leads to the equation that rules the static equilibrium of forces.
 - The material is assumed to not been capable of absorbing angular momentum per unit volume, so the angular momentum is conserved. And the angular momentum conservation applied to its corresponding balance equation leads to the equation that governs the equilibrium of torques. This equation turns out to be reduced to the symmetry of the Cauchy stress tensor.

The external loads cause a displacement field, which produces internal stresses. The stress vector at a given material particle is defined with respect to a specific plane. Cauchy's definition leads to state the stress vector as the product between the Cauchy stress tensor and the normal unit vector that defines the plane with respect to which the stress is defined. And the Cauchy stress tensor turns out to be composed by the components of the stress vectors which are defined with respect to the Cartesian planes.

- If the initial configuration is adopted as the reference domain, instead of the deformed one as in the Cauchy's proposal, alternative stress vectors arise which fulfil interesting properties to deal with a nonlinear analysis.

- An alternative stress vector can be defined as the product between the first Piola-Kirchhoff stress tensor and the normal unit vector corresponding to the initial configuration, that identifies the plane with respect to which the stress is calculated. This definition is useful to convert integral weak forms defined over the unknown deformed configuration to integrals defined over a known material domain.
- Another alternative stress vector can be stated as the product between the second Piola-Kirchhoff stress tensor and the normal unit corresponding to the initial configuration, that identifies the plane with respect to which the stress is calculated. Although this vector has no clear physical interpretation, it is defined by means of a symmetric tensor that verifies some interesting properties. The application of a rigid motion (translation and/or rotation) does not alter the value of the second Piola-Kirchhoff stress tensor. Therefore, this tensor becomes a suitable candidate to properly represent the stress field when a nonlinear analysis is carried out.

The deformation gradient tensor is a powerful tool to analyse solids that behave with large displacements and/or large displacement gradients, since it rules the geometric transformation of a given material vector over time.

- Its polar decomposition leads to a clearer physical interpretation of the geometric transformation. The polar decomposition allows to decompose the deformation gradient tensor as the product between the rotation and the strain tensors. In order to define both tensors, an eigenvalue problem has to be solved.
 - Firstly, the strain tensor modifies the modulus and direction of the material vector, by means of a pure stretch transformation.
 - Then, the rotation tensor rotates the previous modified material vector to orientate it according to the deformed geometry.
- This geometric transformation turns out to be infinitesimal if the displacement gradients are small. Note that this consideration does not imply a small displacement behaviour.
 - If this assumption is fulfilled, the approximate polar decomposition can be accurately applied to decompose the deformation gradient tensor. Therefore, the polar decomposition is completely posed only by means of the displacement gradient tensor, and there is no need to solve an eigenvalue problem. This is a major advantage, since less computational effort is required.
 - The deformation gradient tensor is then defined as the product between the infinitesimal rotation tensor and the infinitesimal strain. Both tensors cause an infinitesimal geometric transformation.

- The deformation gradient tensor states the geometric transformation experienced by a given material vector between $t = 0$ and a given instant of time t . This geometric transformation can be finite or infinitesimal, depending on the magnitude of the displacement gradients. However, the incremental approach can also be adopted, which is focused on the incremental geometric transformation experienced by a material vector between two consecutive infinitesimal time steps.
 - This geometric transformation between t and $t + dt$ is ruled by the velocity gradient tensor.
 - As this incremental geometric variation turns out to be infinitesimal, the approximate polar decomposition can be accurately applied to decompose the tensor that governs this transformation. Therefore, this tensor can be defined as the product between an incremental rotation and an incremental strain.
 - This formulation also rules the incremental loading process applied in nonlinear analysis, if the incremental load at each load step is small enough. If an incremental loading process is applied, the time variable vanishes and becomes a parameter that indicates the current load level.

The computation of the difference between the square of the norm of a given material vector and the square of the norm of its initial geometry leads to the definition of the Green-Lagrange strain tensor. This tensor fulfils some interesting properties to deal with nonlinear analyses.

- This tensor does not vary if a rotation is applied to the deformed solid. Its definition allows to get rid of the effect of the rotation tensor that defines the applied rotation. That is, it is able to only take into account the strain effect in order to define the strain field.
- It turns out to be work conjugate with the second Piola-Kirchhoff stress tensor. That is, their double dot product leads to the work per unit volume developed by the internal forces during the deformation process. The second Piola-Kirchhoff stress tensor also remains invariant if a rigid rotation is applied to the solid. Therefore, both tensors are a suitable pair of magnitudes to represent the strain and stress fields. Consequently, a constitutive equation that relate both magnitudes can be defined.

This work is focused on elastic solids. The value of the stress field of an elastic solid only depends on the value of the deformation gradient tensor at a given instant of time.

- If the external loads are removed, the initial configuration is completely recovered.
- If the polar decomposition of the deformation gradient tensor is recalled, it can be stated that the stress tensor depends on both the rotation and the strain tensors.

If the strain field is infinitesimal, the linear elasticity theory can be accurately applied.

- This assumption does not necessarily imply neither small displacements nor small displacement gradients.
- If the strain field is infinitesimal and the displacement gradients are small, the Cauchy stress tensor can be defined according to the Lamé's equation. That is, the Cauchy stress tensor turns out to be defined only by means of the infinitesimal strain tensor, and the relation between both magnitudes turns out to be linear. The linear relation between both magnitudes can also be defined by means of the linear constitutive tensor.

If the strain field is not infinitesimal, the linear elasticity does no longer hold, and a proper definition of the Cauchy stress tensor by means of both the rotation and the strain tensors has to be stated. However, more suitable tensor magnitudes can be adopted to define the strain and stress fields, such as the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor.

- Both tensors remain invariant if a rigid motion (rotation and/or translation) is applied to the solid, and turn out to be work conjugate magnitudes.
- If the solid experiments large displacements, with large or small displacement gradients, these tensors can be adopted to properly describe the strain field and its corresponding stress field. Therefore, a constitutive equation that defines their mathematical relation has to be stated.

As the nonlinear analysis is carried out according to an incremental loading procedure, it is also essential to define the incremental constitutive equations.

- To define the second Piola-Kirchhoff stress tensor increment by means of the Green-Lagrange strain tensor increment, the gradients of the displacement field increments are assumed to be small. If this assumption is fulfilled, the relation between both incremental magnitudes can be stated as a linear relation defined by means of a constitutive tensor.

If the strain field is infinitesimal, even if the displacements and/or the displacement gradients are large, the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor can be accurately approximated as a linear relation defined by means of the linear constitutive tensor. This is the so-called St. Venant-Kirchhoff constitutive equation, which is one of the simplest constitutive models.

12.1.3. Conclusions derived from nonlinear finite element formulations

Here are some of the main conclusions that can be drawn from the derivation of the finite element formulations proposed in this work.

- When the weak forms are stated, the test functions are interpreted as a displacement field variation compatible with the essential boundary conditions. Moreover, the gradients of this displacement field variations are also assumed to be small. Without these considerations, the physical interpretation of some tensors that arise when the weak forms are posed becomes not possible, and to follow the derivation becomes difficult and not intuitive at all.
 - To consider the test functions as a displacement field variation is also the base of the principle of virtual work, which is commonly applied in a big amount of reference textbooks where the derivation of nonlinear finite element formulations is faced.
 - To differently approach these derivations, the calculation of the virtual work of both the external and the internal forces is avoided. Consequently, no energetic terms are defined to state the weak forms.
- To numerically solve a nonlinear analysis, an incremental loading procedure has to be applied, and the displacement field that corresponds to each load step has to be solved iteratively.
- The incremental loading procedure is based on some hypotheses about the magnitude of the displacement field increment that defines consecutive material domains.
 - The displacement field increment between consecutive load steps may be large, but the gradients of the displacement field increment between consecutive load steps are assumed to be small.
- An iterative method is applied at each step of the incremental procedure to obtain the displacement field of each load step. In order to guarantee convergence, the procedure needs to start iterating from a close approximation to the solution.
 - The incremental load applied at each step is assumed to be so small, that the solution corresponding to the previous load step can be considered close enough to the solution of the current load step.
 - The displacement field increments between consecutive iterations are assumed to be small. However, as mentioned before, the displacement field increment between consecutive load steps may be large.

- Two approaches can be adopted, that differ on the material domain they take as a reference configuration along the incremental loading process.
 - The total approach is based on the adoption of the initial material configuration as the reference along the entire incremental process.
 - Whereas, the updated one adopts the last computed material domain as the new reference for the current load step.

- Both approaches in conjunction with the iterative procedure lead to a system of linear equations that has to be solved at each iteration. The matrix of this system is the so-called tangent stiffness, since it can be interpreted as the tangent to the nonlinear structural response. The tangent stiffness turns out to be composed by two terms, regardless the adopted approach.
 - The material component, which turns out to depend on the mechanical properties of the material.
 - And the geometric stiffness, which depends on both the stress state and the geometry of the reference material domain.

12.2. Future research

Up to now, the main conclusions that can be extracted from the novel nonlinear solid mechanics formulation and its subsequent finite element formulations have been summarized. Some guidelines are now presented to further extend this research work in the near future.

- To face the derivation of the dynamic nonlinear finite element formulation.
 - The nonlinear solid mechanics formulation proposed in this thesis also allows to address the derivation of the dynamic finite element formulation.

- The classical derivation of the Cauchy stress vector is based on the application of the linear momentum balance over a given finite tetrahedron that belongs to the deformed configuration, which later becomes infinitesimal.
 - A novel derivation based on the imposition of the linear momentum balance over a domain which depends on a given parameter can be proposed. This parameter is later forced to tend to zero, and the definition of the stress vector should arise. This derivation is more generic and avoids the use of specific geometrical entities, such as the classical tetrahedron.

- Convert an existing linear finite element code into a nonlinear one by applying the minimum number of modifications.

- A guideline to obtain this conversion can be proposed.
 - If the strain field is infinitesimal, this conversion should not be too difficult, as the linear constitutive tensor can be adopted to describe the relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor. Therefore, the tensor that defines the linear constitutive equations turns out to be equivalent to the one that define the nonlinear one, and the definition of the constitutive equation is straightforward to obtain.
 - Many linear analysis applications can be run in its corresponding nonlinear version, and the accuracy of the linear results can be checked.
 - Later on, this nonlinear analysis code can be implemented into a structural optimization algorithm, in order to reduce the amount of material needed under displacement and/or stress constraints
 - A parametric structural analysis under the nonlinear analysis framework could also be addressed. This analysis is based on the Proper Generalized Decomposition (PGD) [Chinesta et al., 2013]. This turns out to be a powerful tool, since it allows to obtain the structural response in terms of certain parameters. For instance, the structural behaviour can be obtained by means of the Young's modulus. This implies that the structural response of different materials can be obtained without the need to repeat the analysis for each one of them.
- The application of artificial intelligence techniques to obtain the nonlinear structural response of solids that behave with large displacement and/or large displacement gradients.
- The equilibrium equations can be written in terms of a residual. For a series of arbitrary displacement fields which are compatible with the essential boundary conditions, their corresponding residuals have to be computed. Then, with the data set composed by the arbitrary displacement fields and their corresponding residuals, a neural network can be trained in order to obtain the equation which defines the residual by means of the displacement field. In general, this equation turns out to be a nonlinear one, and the displacement field that corresponds to the equilibrium configuration cancels the residual. Therefore, the root of the nonlinear equation turns out to define the deformed configuration, and it can be calculated by means of an iterative method.
 - Alternatively, the inverse relation can be obtained from the same data set. That is, the neural network will lead to the equation that defines the displacement field by means of the residual. In this particular case, the displacement field that corresponds to the equilibrium configuration is easier to obtain, since the evaluation of this function at the point of null residual is

only required. With this approach, the application of an iterative procedure is not required.

- Finally, more complex constitutive models have to be proposed, in order to properly analyse solids that do not experiment infinitesimal strains.
 - In this thesis, only the St. Venant-Kirchhoff constitutive model is extensively analysed. This simple model can only be accurately applied in an infinitesimal strain context. In this case, the constitutive tensor can be considered constant and equivalent to the linear constitutive one.
 - If the strain field is not infinitesimal, the St. Venant-Kirchhoff model can no longer be applied and more complex constitutive models have to be considered. In this case, the constitutive tensor is no longer constant and depends on the displacement field undergone by the solid.

Tensor algebra

A.1. First order tensors

A first order tensor turns out to be a vector. A vector can be defined in a n dimensional space as a linear combination of the vectors that compose a particular basis. These vectors are defined with respect to the canonical basis as follows:

$$\vec{e}_k = \mathbf{e}_k = \{e_{k,l}\}_{l=1,\dots,n} \quad k = 1, \dots, n \quad (\text{A.1})$$

In this work, an orthonormal basis is adopted as a reference. Under this assumption, a generic vector can be expressed with respect to the canonical basis as shown below.

$$\vec{\mathbf{a}} = \sum_{k=1}^n \mathbf{e}_k a_k \quad \iff \quad \vec{\mathbf{a}} = \mathbf{E} \mathbf{a} \quad \text{where} \quad \begin{cases} \mathbf{E} = [\mathbf{e}_1 & \cdots & \mathbf{e}_n] \\ \mathbf{a} = \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} \end{cases} \quad (\text{A.2})$$

As the reference basis is an orthonormal one, the matrix \mathbf{E} which is composed by the vectors arranged in columns has to be equal to the identity matrix. Therefore, the definition of a vector with respect to an orthonormal basis can be reduced to:

$$\mathbf{E} = \mathbf{I} \quad \implies \quad \vec{\mathbf{a}} = \mathbf{a} = \{a_k\}_{k=1,\dots,n} \quad (\text{A.3})$$

The previous result implies that the components of the vector are equal to the vector itself. This conclusion only holds when dealing with orthonormal basis.

A.1.1. Transpose

The transpose operator is an operator that can be applied to a first order tensor, and transforms a column vector into a row vector.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \{a_k\}_{k=1,\dots,n} \implies \mathbf{a}^T = [a_1 \ \cdots \ a_n] = [a_k]_{k=1,\dots,n} \quad (\text{A.4})$$

A.1.2. Scalar product

The scalar product is a tensor operation that involves two vectors to produce a scalar. It is also called dot product, since it is represented by a single dot between both vectors. This dot represents the index contraction experimented by the components of both vectors.

$$c = \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \sum_{k=1}^n a_k b_k \quad (\text{A.5})$$

According to the above definition, the scalar product is commutative. That is, the order of the vectors does not affect the result.

$$\mathbf{b} \cdot \mathbf{a} = \sum_{k=1}^n b_k a_k = \sum_{k=1}^n a_k b_k = \mathbf{a} \cdot \mathbf{b} \quad (\text{A.6})$$

A.1.3. Norm

The length of a given vector is adopted to define its norm. According to the Pythagoras' theorem, this length can be computed as:

$$\|\mathbf{a}\| = \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \quad (\text{A.7})$$

If the definition of the scalar product (A.5) is recalled, the norm can also be defined as:

$$\|\mathbf{a}\| = \left(\sum_{k=1}^n a_k^2 \right)^{1/2} = (\mathbf{a} \cdot \mathbf{a})^{1/2} \quad (\text{A.8})$$

A.1.4. Scalar product in a three-dimensional space

Once the norm of a given vector is defined, an alternative equation to compute the scalar product between two vectors embedded in a three-dimensional space can be defined.

Let's consider that the vector \mathbf{b} is decomposed into the sum of the following vectors.

$$\mathbf{b} = \mathbf{b}^p + \mathbf{b}^n \quad \text{where} \quad \begin{cases} \mathbf{b}^p = \lambda \mathbf{a}, & \lambda \in \mathbb{R} \\ \mathbf{b}^n \cdot \mathbf{a} = 0 \end{cases} \quad (\text{A.9})$$

The previous conditions imply that \mathbf{b}^p is parallel to \mathbf{a} . Moreover, if the scalar product of perpendicular vectors is assumed to be null, the vector \mathbf{b}^n turns out to be perpendicular to \mathbf{b}^p and \mathbf{a} (figure A.1).

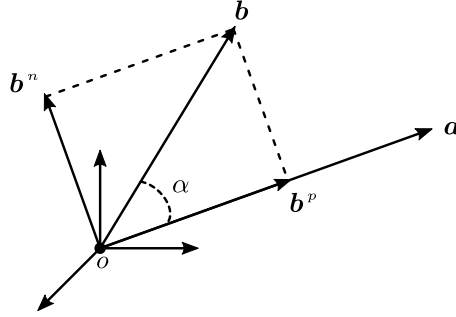


Figure A.1. Decomposition of the vectors involved in a scalar product computation.

The substitution of the above decomposition into the scalar product leads to:

$$c = \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{b}^p + \mathbf{b}^n) = \mathbf{a} \cdot (\lambda \mathbf{a} + \mathbf{b}^n) = \lambda \mathbf{a} \cdot \mathbf{a} + \underbrace{\mathbf{a} \cdot \mathbf{b}^n}_{=0} = \lambda \mathbf{a} \cdot \mathbf{a} \quad (\text{A.10})$$

The value of the constant λ can be obtained from the previous result as:

$$\lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \quad (\text{A.11})$$

Furthermore, the cosine of the angle composed by both vectors turns out to be:

$$\cos(\alpha) = \frac{\|\mathbf{b}^p\|}{\|\mathbf{b}\|} = \frac{(\mathbf{b}^p \cdot \mathbf{b}^p)^{1/2}}{(\mathbf{b} \cdot \mathbf{b})^{1/2}} = \frac{\lambda (\mathbf{a} \cdot \mathbf{a})^{1/2}}{(\mathbf{b} \cdot \mathbf{b})^{1/2}} = \lambda \frac{\|\mathbf{a}\|}{\|\mathbf{b}\|} \quad (\text{A.12})$$

The substitution of the constant λ obtained in (A.11) into the above equation leads to the following result.

$$\cos(\alpha) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (\text{A.13})$$

Therefore, the scalar product can be finally computed as shown below.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\alpha) \quad (\text{A.14})$$

A.1.5. Vector product

The vector product is a tensor operation that involves two vectors to produce another vector as a result. This operation is only defined in a three-dimensional space, and it is also known as the cross product. It is defined as follows:

$$\mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \det \left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \right) \quad (\text{A.15})$$

Where the vectors \vec{i} , \vec{j} , and \vec{k} are the vectors that compose the canonical basis in a three-dimensional space.

It can be proven that the norm of the resulting vector can be computed by means of the norm of both vectors and the angle α between them, as:

$$\|\mathbf{c}\| = \|\mathbf{a} \wedge \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\alpha) \quad (\text{A.16})$$

If the determinant defined in (A.15) is computed, the components of the resulting vector turn out to be:

$$\mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \vec{i}(a_2 b_3 - a_3 b_2) + \vec{j}(a_3 b_1 - a_1 b_3) + \vec{k}(a_1 b_2 - a_2 b_1) \quad (\text{A.17})$$

This operation can also be defined in matrix notation, by defining the following antisymmetric matrix composed by the components of the vector \mathbf{a} .

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \longrightarrow \mathbf{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad \mathbf{A}^T = -\mathbf{A} \quad (\text{A.18})$$

The product between the above matrix and the vector \mathbf{b} leads to the same vector as the one obtained computing the determinant defined in (A.15).

$$\mathbf{c} = \mathbf{A} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{Bmatrix} \quad (\text{A.19})$$

On the other hand, the above components can also be obtained as:

$$c_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} a_j b_k \quad i = 1, 2, 3 \quad (\text{A.20})$$

Where ε_{ijk} are the components of the Levi-Civita symbol, which are defined as:

$$\varepsilon_{ijk} = \begin{cases} = +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), \text{ or } (2, 3, 1), \text{ or } (3, 1, 2) \\ = -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), \text{ or } (2, 1, 3), \text{ or } (1, 3, 2) \\ = 0 & \text{if } i = j, \text{ or } j = k, \text{ or } i = k \end{cases} \quad (\text{A.21})$$

A.1.6. Some vector product properties

Since this tensor operation is defined as a determinant, some properties can be stated if the properties of this operator are taken into account. For instance, the vector product of two parallel vectors is the null vector, as the determinant of a matrix with two identical rows has to be zero.

$$\mathbf{a} \wedge (\lambda \mathbf{a}) = \lambda \det \left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix} \right) = \mathbf{0} \quad (\text{A.22})$$

In addition, if two rows of a determinant are exchanged, the same result with opposite sign is obtained. Thus, if the vector order is exchanged, the opposite vector is obtained. That is, the vector product holds the anticommutative property.

$$\mathbf{b} \wedge \mathbf{a} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} = -\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = -(\mathbf{a} \wedge \mathbf{b}) \quad (\text{A.23})$$

Another interesting particular case occurs when one of the vectors turn out to be the sum of two different vectors.

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ (b_1 + c_1) & (b_2 + c_1) & (b_3 + c_3) \end{pmatrix} \quad (\text{A.24})$$

If the components of a row are composed by the sum of two elements, the determinant can be separated into the sum of two determinants, composed by the first and second term, respectively. The other rows remain identical to the original one.

$$\begin{aligned} & \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ (b_1 + c_1) & (b_2 + c_2) & (b_3 + c_3) \end{pmatrix} = \\ & = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} + \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \end{aligned} \quad (\text{A.25})$$

Hence, the vector product holds the distributive property, and the vector product can be computed as:

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c} \quad (\text{A.26})$$

On the other hand, it is possible to deal with vectors that depend on a given variable. The resulting vector of their vector product will also depend on the same variable.

$$\mathbf{c}(x) = \mathbf{a}(x) \wedge \mathbf{b}(x) \quad (\text{A.27})$$

The derivative of the above vector with respect to the variable x is another vector. Its components are defined as the derivatives of the components obtained in (A.17) with respect to this variable.

$$\frac{d\mathbf{c}}{dx} = \frac{d}{dx}(\mathbf{a} \wedge \mathbf{b}) = \left\{ \begin{array}{l} \frac{dc_1}{dx} \\ \frac{dc_2}{dx} \\ \frac{dc_3}{dx} \end{array} \right\} = \left\{ \begin{array}{l} \frac{d}{dx}(a_2 b_3 - a_3 b_2) \\ \frac{d}{dx}(a_3 b_1 - a_1 b_3) \\ \frac{d}{dx}(a_1 b_2 - a_2 b_1) \end{array} \right\} \quad (\text{A.28})$$

In order to obtain the equation that defines the above vector, the previous derivatives are computed.

$$\frac{d\mathbf{c}}{dx} = \left\{ \begin{array}{l} \frac{da_2}{dx} b_3 - \frac{da_3}{dx} b_2 \\ \frac{db_1}{dx} a_3 - \frac{db_3}{dx} a_1 \\ \frac{da_1}{dx} b_2 - \frac{da_2}{dx} b_1 \end{array} \right\} + \left\{ \begin{array}{l} a_2 \frac{db_3}{dx} - a_3 \frac{db_2}{dx} \\ b_1 \frac{da_3}{dx} - b_3 \frac{da_1}{dx} \\ a_1 \frac{db_2}{dx} - a_2 \frac{db_1}{dx} \end{array} \right\} \quad (\text{A.29})$$

And the above components can be equivalently expressed as the sum of the following determinants.

$$\frac{d\mathbf{c}}{dx} = \det \left(\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{da_1}{dx} & \frac{da_2}{dx} & \frac{da_3}{dx} \\ b_1 & b_2 & b_3 \end{array} \right) + \left(\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ \frac{db_1}{dx} & \frac{db_2}{dx} & \frac{db_3}{dx} \end{array} \right) \quad (\text{A.30})$$

Therefore, the derivative can be computed as follows:

$$\frac{d}{dx}(\mathbf{a} \wedge \mathbf{b}) = \frac{d\mathbf{a}}{dx} \wedge \mathbf{b} + \mathbf{a} \wedge \frac{d\mathbf{b}}{dx} \quad (\text{A.31})$$

A.1.7. Triple product

The triple product is a tensor operation that combines the vector product and the scalar product, to generate a scalar. As the vector product is involved, it can only be defined in a three-dimensional space. It is defined as:

$$d = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} \quad (\text{A.32})$$

According to the result obtained in (A.19), the components of a vector product can be computed as:

$$\mathbf{a} \wedge \mathbf{b} = \left\{ \begin{array}{l} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{array} \right\} \quad (\text{A.33})$$

And the scalar product between the above vector and the third vector involved in the triple product leads to:

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \wedge \mathbf{b})^T \mathbf{c} = c_1 (a_2 b_3 - a_3 b_2) + c_2 (a_3 b_1 - a_1 b_3) + c_3 (a_1 b_2 - a_2 b_1) \quad (\text{A.34})$$

The previous result can also be obtained as the determinant of the following matrix.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \det \left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right) \quad (\text{A.35})$$

Moreover, the property (A.97) of the determinant operator can be applied. According to this property, the determinant of a second order tensor is equivalent to the

determinant of its transpose.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \det \left([\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \right) \quad (\text{A.36})$$

Consequently, the triple product of three specific vectors can be calculated as the determinant of the matrix composed by the vectors organized into columns.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \det \left([\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \right) \quad (\text{A.37})$$

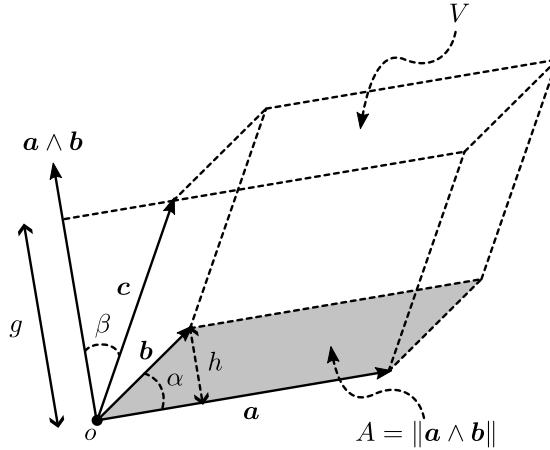


Figure A.2. Geometrical interpretation of the triple product.

To geometrically interpret this operation, the definition of the scalar product stated in (A.14) is applied.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \|\mathbf{a} \wedge \mathbf{b}\| \underbrace{\|\mathbf{c}\| \cos(\beta)}_g = \|\mathbf{a} \wedge \mathbf{b}\| \times g \quad (\text{A.38})$$

Where g turns out to be the height of the parallelepiped composed by the three vectors involved in the triple product. Furthermore, the modulus of the vector $\mathbf{a} \wedge \mathbf{b}$ can be computed as stated in (A.16), as:

$$\|\mathbf{a} \wedge \mathbf{b}\| = \|\mathbf{a}\| \underbrace{\|\mathbf{b}\| \sin(\alpha)}_h = \|\mathbf{a}\| \times h \quad (\text{A.39})$$

Where h is the height of the parallelepiped composed by the vectors \mathbf{a} and \mathbf{b} . Consequently, the modulus of the vector product (A.16) can be interpreted as the area of the parallelepiped composed by the two vectors involved in the vector product.

$$\|\mathbf{a} \wedge \mathbf{b}\| = \|\mathbf{a}\| \times h = A \quad (\text{A.40})$$

And the triple product can be interpreted as the volume of the parallelepiped composed by the three vectors.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \|\mathbf{a} \wedge \mathbf{b}\| \times \|\mathbf{c}\| \cos(\beta) = A \times g = V \quad (\text{A.41})$$

A.1.8. Tensor product

The tensor product is a tensor operation that involves two vectors and produces a second order tensor as a result. This operation is also called dyadic product, and it is defined as:

$$\mathbf{C} = \mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^T = [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.42})$$

Where the components of the resulting second order tensor are defined as:

$$C_{ij} = c_i b_j \quad (\text{A.43})$$

A.2. Second order tensors

The following equation is the one that defines a second order tensor with respect to the basis defined in (A.1).

$$\mathbf{A} \approx \sum_{k=1}^n \sum_{l=1}^n (\mathbf{e}_k \otimes \mathbf{e}_l) A_{kl} \quad (\text{A.44})$$

And the components of the above tensor can be gathered in the following matrix:

$$\mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.45})$$

If the vectors that compose the reference basis are orthonormal, the matrix corresponding to each one of the addends that compose equation (A.44) turns out to be:

$$\mathbf{B} = \mathbf{e}_k \otimes \mathbf{e}_l = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad B_{ij} = \begin{cases} = 1 & \text{if } i = k \text{ and } j = l \\ = 0 & \text{if } i \neq k \text{ or } j \neq l \end{cases} \quad (\text{A.46})$$

That is, the components of the above matrix are null, except for one which is equal to one. Therefore, if the sum of all the addends that compose the second order tensor (A.44) is performed, it can be checked that the tensor and the matrix that gathers its components are equivalent.

$$\mathbf{A} \approx \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.47})$$

The above conclusion only holds when an orthonormal basis is adopted as a reference.

A.2.1. Unit tensor

The second order unit tensor is defined as a matrix whose diagonal components are equal to one, and the non-diagonal components are zero. It can be defined by means of the Kronecker delta, as shown below.

$$\mathbf{I} = [\delta_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \delta_{ij} = \begin{cases} = 1 & ; \quad i = j \\ = 0 & ; \quad i \neq j \end{cases} \quad (\text{A.48})$$

A.2.2. Norm

The result of applying a second order tensor to a vector is another vector, whose norm and direction are different from the original ones. That is, the second order tensor is the one that modifies the norm and direction of the primal vector. Therefore, a suitable measure of its modulus can be defined by means of the unit vector that experiments the biggest norm variation when the second order tensor is applied.

$$\|\mathbf{A}\| = \max_{\mathbf{x} \neq \mathbf{0}} \left\| \mathbf{A} \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A} \mathbf{x}\|}{\|\mathbf{x}\|} \quad (\text{A.49})$$

The norm of the second order unit tensor is straightforward to obtain, since the unit tensor does not modify the norm and direction of a vector.

$$\|\mathbf{I}\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{I} \mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|} = 1 \quad (\text{A.50})$$

A.2.3. Transpose

The transpose of a second order tensor is obtained by permuting the order of the components indices, that is, the rows become columns and vice versa. Thus, the transpose of the tensor defined in (A.47) turns out to be:

$$\mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \Longrightarrow \quad \mathbf{A}^T = [A_{ji}]_{\substack{j=1,\dots,n \\ i=1,\dots,n}} \quad (\text{A.51})$$

A.2.4. Multiplication

The multiplication of two second order tensors leads to another second order tensor as a result.

$$\left. \begin{array}{l} \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right\} \quad \Longrightarrow \quad \mathbf{C} = \mathbf{A} \mathbf{B} = [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.52})$$

Where the components of the resulting tensor are defined as:

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj} \quad (\text{A.53})$$

A.2.5. Transpose of a multiplication

The multiplication of two second order tensors was defined in the previous section. In this section, the transpose of this multiplication is analysed.

$$\left. \begin{array}{l} \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right\} \quad \Longrightarrow \quad \left\{ \begin{array}{l} \mathbf{C} = \mathbf{A} \mathbf{B} = [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad C_{ij} = \sum_{k=1}^n A_{ik} B_{kj} \\ \mathbf{D} = \mathbf{C}^T = (\mathbf{A} \mathbf{B})^T = [D_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right. \quad (\text{A.54})$$

Where the components of the resulting second order tensor turn out to be:

$$D_{ij} = C_{ji} = \sum_{k=1}^n A_{jk} B_{ki} = \sum_{k=1}^n B_{ki} A_{jk} \quad (\text{A.55})$$

In addition, the components of their transposes are defined below.

$$\begin{aligned} \mathbf{E} = \mathbf{A}^T &= [E_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} & E_{ij} &= A_{ji} \\ \mathbf{F} = \mathbf{B}^T &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} & F_{ij} &= B_{ji} \end{aligned} \quad (\text{A.56})$$

Therefore, the components defined in (A.55) become:

$$D_{ij} = \sum_{k=1}^n B_{ki} A_{jk} = \sum_{k=1}^n F_{ik} E_{kj} \quad (\text{A.57})$$

The above components turn out to be the corresponding ones to the result of multiplying both transposes. Consequently, it can be concluded that:

$$\mathbf{D} = \mathbf{F} \mathbf{E} \quad \iff \quad (\mathbf{A} \mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T \quad (\text{A.58})$$

A.2.6. Square

Let's consider the following second order square tensor.

$$\mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \mathbf{A}^T = \mathbf{A} \quad (\text{A.59})$$

The square of the above tensor is defined as:

$$\mathbf{B} = \mathbf{A}^2 = \mathbf{A} \mathbf{A} = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.60})$$

Where the components are defined as shown below.

$$B_{ij} = \sum_{k=1}^n A_{ik} A_{kj} \quad (\text{A.61})$$

A.2.7. Double dot product

The double dot product is a tensor operation that implies the contraction of two indices. The double dot product between two second order tensors leads to a scalar, since the contraction of both indices is performed. This double contraction is symbolized by a double dot, and the operation is defined as shown below.

$$\left. \begin{aligned} \mathbf{A} &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{aligned} \right\} \implies c = \mathbf{A} : \mathbf{B} = \sum_{k=1}^n \sum_{l=1}^n A_{kl} B_{kl} \quad (\text{A.62})$$

The order of the tensors does not modify the result, that is, the double dot product holds the commutative property.

$$\begin{aligned} \mathbf{B} : \mathbf{A} &= \sum_{k=1}^n \sum_{l=1}^n B_{kl} A_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n A_{kl} B_{kl} = \mathbf{A} : \mathbf{B} \end{aligned} \quad (\text{A.63})$$

An interesting property arises when considering that the tensor \mathbf{B} is symmetric.

$$\left. \begin{aligned} \mathbf{A} &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{C} = \mathbf{A}^T &= [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{aligned} \quad \left. \begin{aligned} B_{ji} &= B_{ij} \\ C_{ij} &= A_{ji} \end{aligned} \right\} \implies \left\{ \begin{aligned} \mathbf{C} : \mathbf{B} &= \sum_{k=1}^n \sum_{l=1}^n C_{kl} B_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n A_{lk} B_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n A_{lk} B_{lk} = \mathbf{A} : \mathbf{B} \end{aligned} \right. \quad (\text{A.64})$$

Therefore, if the tensor \mathbf{B} is symmetric, the following property can be taken into account.

$$\mathbf{A}^T : \mathbf{B} = \mathbf{A} : \mathbf{B} \quad (\text{A.65})$$

A.3. Product between a second order tensor and a first order tensor

When the calculation of the product between a second order tensor and a first order one is performed, only one index can be contracted. Consequently, this operation generates a first order tensor as a result.

$$\left. \begin{aligned} \mathbf{A} &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{b} &= \{b_i\}_{i=1,\dots,n} \end{aligned} \right\} \implies \mathbf{c} = \mathbf{A} \mathbf{b} = \{c_i\}_{i=1,\dots,n} \quad (\text{A.66})$$

The components of the resulting vector are defined as follows.

$$c_i = \sum_{k=1}^n A_{ik} b_k \quad (\text{A.67})$$

A.4. Product between the transpose of a first order tensor and a second order tensor

The product of the transpose of a first order tensor and a second order tensor leads to a row vector.

$$\mathbf{c} = \mathbf{b}^T \mathbf{A} = [c_i]_{i=1,\dots,n} \quad (\text{A.68})$$

Where the components of the above row vector are defined as:

$$c_i = \sum_{k=1}^n b_k A_{ki} \quad (\text{A.69})$$

A.5. Transpose of the product between a second order tensor and a first order tensor

As stated in section A.3, the product between a second order tensor and a first order tensor generates a first order tensor. Therefore, the transpose of this result turns out to be a row vector.

$$\left. \begin{aligned} \mathbf{A} &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{b} &= \{b_i\}_{i=1,\dots,n} \\ \mathbf{c} = \mathbf{A}\mathbf{b} &= \{c_i\}_{i=1,\dots,n} \\ c_i &= \sum_{k=1}^n A_{ik} b_k \end{aligned} \right\} \implies \mathbf{c}^T = (\mathbf{A}\mathbf{b})^T = [c_i]_{i=1,\dots,n} \quad (\text{A.70})$$

Where the components of the previous row vector are defined as:

$$c_i = \sum_{k=1}^n A_{ik} b_k = \sum_{k=1}^n b_k A_{ik} \quad (\text{A.71})$$

On the other hand, the transpose of the tensor \mathbf{A} turns out to be:

$$\mathbf{D} = \mathbf{A}^T = [D_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad D_{ij} = A_{ji} \quad (\text{A.72})$$

Thus, the components (A.71) become:

$$c_i = \sum_{k=1}^n b_k A_{ik} = \sum_{k=1}^n b_k D_{ki} \quad (\text{A.73})$$

And according to equation (A.69), it can be concluded that the above components correspond to the following tensor operation.

$$\mathbf{c}^T = \mathbf{b}^T \mathbf{D} \quad \iff \quad (\mathbf{A}\mathbf{b})^T = \mathbf{b}^T \mathbf{A}^T \quad (\text{A.74})$$

A.6. Trace of a second order tensor

The trace is an operator that computes the sum of the components of the tensor that have repeated indices.

$$\mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \implies \quad c = \text{Tr}(\mathbf{A}) = \sum_{k=1}^n A_{kk} \quad (\text{A.75})$$

It can also be interpreted as the double dot product between the tensor itself and the second order unit tensor, which was defined in (A.48).

$$\text{Tr}(\mathbf{A}) = \sum_{k=1}^n A_{kk} = \sum_{k=1}^n \sum_{l=1}^n A_{kl} \delta_{kl} = \mathbf{A} : \mathbf{I} \quad (\text{A.76})$$

A.6.1. Some trace properties

The trace of the transpose of a given second order tensor (section A.2.3) turns to be equal to the trace of the tensor without being transposed.

$$\left. \begin{array}{l} \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} = \mathbf{A}^T = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right\} \begin{array}{l} B_{ij} = A_{ji} \end{array} \implies \left\{ \begin{array}{l} \text{Tr}(\mathbf{A}^T) = \sum_{k=1}^n B_{kk} \\ = \sum_{k=1}^n A_{kk} = \text{Tr}(\mathbf{A}) \end{array} \right. \quad (\text{A.77})$$

Moreover, the application of the trace operator to the sum of two second order tensors is equivalent to the sum of the traces of both tensors.

$$\begin{aligned} \text{Tr}(\mathbf{A} + \mathbf{B}) &= \sum_{k=1}^n (A_{kk} + B_{kk}) \\ &= \sum_{k=1}^n A_{kk} + \sum_{k=1}^n B_{kk} = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B}) \end{aligned} \quad (\text{A.78})$$

The multiplication between two second order tensors was exposed in section A.2.4. If the trace operator is applied to this resulting tensor, the result does not depend on the order in which both tensors are multiplied.

$$\begin{aligned} \text{Tr}(\mathbf{A}\mathbf{B}) &= \sum_{l=1}^n \sum_{k=1}^n (A_{lk} B_{kl}) \\ &= \sum_{k=1}^n \sum_{l=1}^n (B_{kl} A_{lk}) = \text{Tr}(\mathbf{B}\mathbf{A}) \end{aligned} \quad (\text{A.79})$$

On the other hand, the double dot product between two second order tensors (section A.2.7) can also be defined by means of the trace operator, as shown below.

$$\left. \begin{array}{l} \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} = [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{C} = \mathbf{B}^T = [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{array} \right\} \begin{array}{l} C_{ij} = B_{ji} \end{array} \implies \left\{ \begin{array}{l} \mathbf{A} : \mathbf{B} = \sum_{k=1}^n \sum_{l=1}^n A_{kl} B_{kl} \\ = \sum_{k=1}^n \sum_{l=1}^n A_{kl} C_{lk} \\ = \text{Tr}(\mathbf{A}\mathbf{C}) \\ = \text{Tr}(\mathbf{A}\mathbf{B}^T) \end{array} \right. \quad (\text{A.80})$$

If the properties exposed in (A.77) and (A.79) are taken into account, and the transpose of the multiplication between two second order tensors defined in (A.58) is applied, the above property can alternatively be expressed as:

$$\begin{aligned}
 \mathbf{A} : \mathbf{B} &= \text{Tr}(\mathbf{A}\mathbf{B}^T) = \text{Tr}(\mathbf{B}^T\mathbf{A}) \\
 &= \text{Tr}\left((\mathbf{A}\mathbf{B}^T)^T\right) = \text{Tr}\left((\mathbf{B}^T\mathbf{A})^T\right) \\
 &= \text{Tr}(\mathbf{B}\mathbf{A}^T) = \text{Tr}(\mathbf{A}^T\mathbf{B})
 \end{aligned} \tag{A.81}$$

A.7. Double dot product between a third order tensor and a second order tensor

The double dot product between a third order tensor and a second order tensor implies the contraction of two indices. Consequently, the result of this tensor operation is a first order tensor

$$\left. \begin{aligned}
 \mathbf{A} &= [A_{ijk}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n}} \\
 \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\
 \mathbf{c} &= \{c_i\}_{i=1,\dots,n}
 \end{aligned} \right\} \implies \mathbf{c} = \mathbf{A} : \mathbf{B} = \{c_i\}_{i=1,\dots,n} \tag{A.82}$$

Where the components of the resulting second order tensor are defined as:

$$c_i = \sum_{j=1}^n \sum_{k=1}^n A_{ijk} B_{jk} \tag{A.83}$$

A.8. Double dot product between a fourth order tensor and a second order tensor

The double dot product between a fourth order tensor and a second order one implies the contraction of two indices. Therefore, this tensor operation leads to a second order tensor.

$$\left. \begin{aligned}
 \mathbf{A} &= [A_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} \\
 \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\
 \mathbf{C} &= [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}
 \end{aligned} \right\} \implies \mathbf{C} = \mathbf{A} : \mathbf{B} = [C_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \tag{A.84}$$

Where the components of the resulting second order tensor are defined as:

$$C_{ij} = \sum_{k=1}^n \sum_{l=1}^n A_{ijkl} B_{kl} \tag{A.85}$$

A.9. Tensor product between two second order tensors

The tensor product can also be computed between two second order tensors. In this case, this tensor operation leads to a fourth order tensor.

$$\left. \begin{aligned} \mathbf{A} &= [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{B} &= [B_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \end{aligned} \right\} \implies \mathbf{C} = \mathbf{A} \otimes \mathbf{B} = [C_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} \quad (\text{A.86})$$

Where the components of the resulting fourth order tensor are defined as:

$$C_{ijkl} = A_{ij}B_{kl} \quad (\text{A.87})$$

A.10. Fourth order unit tensors

There are two types of fourth order unit tensors, which are defined as:

$$\left. \begin{aligned} \mathbf{I}_4 &= [\alpha_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} & \alpha_{ijkl} &= \delta_{ik} \delta_{jl} \\ \bar{\mathbf{I}}_4 &= [\beta_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} & \beta_{ijkl} &= \delta_{il} \delta_{jk} \end{aligned} \right\} \quad (\text{A.88})$$

Let's consider the following second order tensor, which can be decomposed as the sum of a symmetric and a skew-symmetric component.

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = \mathbf{A}^{\text{sym}} + \mathbf{A}^{\text{skw}} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.89})$$

The double dot product between the first fourth order unit tensor presented in (A.88) and the above second order unit tensor does not alter the value of the second order unit tensor.

$$\left. \begin{aligned} \mathbf{I}_4 : \mathbf{A} &= [\gamma_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \gamma_{ij} &= \sum_{k=1}^n \sum_{l=1}^n \alpha_{ijkl} A_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n \delta_{ik} \delta_{jl} A_{kl} = A_{ij} \end{aligned} \right\} \implies \mathbf{I}_4 : \mathbf{A} = \mathbf{A} \quad (\text{A.90})$$

However, if the second fourth unit tensor is the one involved in the above operation,

the result turns out to be the transpose of the second order tensor.

$$\left. \begin{aligned} \bar{\mathbf{I}}_4 : \mathbf{A} &= [\varphi_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \varphi_{ij} &= \sum_{k=1}^n \sum_{l=1}^n \beta_{ijkl} A_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n \delta_{il} \delta_{jk} A_{kl} = A_{ji} \end{aligned} \right\} \implies \bar{\mathbf{I}}_4 : \mathbf{A} = \mathbf{A}^T \quad (\text{A.91})$$

Another important fourth order tensor is the one defined as the tensor product between two second order unit tensors. The tensor product between two second order tensors was presented in section A.9.

$$\mathbf{I}_2 \otimes \mathbf{I}_2 = [\varepsilon_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} \quad \varepsilon_{ijkl} = \delta_{ij} \delta_{kl} \quad (\text{A.92})$$

The double dot product of the above tensor and a second order tensor is:

$$\left. \begin{aligned} (\mathbf{I}_2 \otimes \mathbf{I}_2) : \mathbf{A} &= [\kappa_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \kappa_{ij} &= \sum_{k=1}^n \sum_{l=1}^n \varepsilon_{ijkl} A_{kl} \\ &= \sum_{k=1}^n \sum_{l=1}^n \delta_{ij} \delta_{kl} A_{kl} \\ &= \delta_{ij} \sum_{k=1}^n A_{kk} = \delta_{ij} \text{Tr}(\mathbf{A}) \end{aligned} \right\} \implies (\mathbf{I}_2 \otimes \mathbf{I}_2) : \mathbf{A} = \text{Tr}(\mathbf{A}) \mathbf{I}_2 \quad (\text{A.93})$$

On the other hand, the first fourth order unit tensor can be decomposed as the sum of a symmetric tensor plus a skew-symmetric one.

$$\mathbf{I}_4 = \frac{1}{2}(\mathbf{I}_4 + \bar{\mathbf{I}}_4) + \mathbf{I}_4^{\text{skw}} = \frac{1}{2}(\mathbf{I}_4 - \bar{\mathbf{I}}_4) = \mathbf{I}_4^{\text{sym}} + \mathbf{I}_4^{\text{skw}} \quad (\text{A.94})$$

Where the components of the above unit tensors turn out to be:

$$\begin{aligned} \mathbf{I}_4^{\text{sym}} &= \frac{1}{2}(\mathbf{I}_4 + \bar{\mathbf{I}}_4) = [\lambda_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} & \lambda_{ijkl} &= \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ \mathbf{I}_4^{\text{skw}} &= \frac{1}{2}(\mathbf{I}_4 - \bar{\mathbf{I}}_4) = [\mu_{ijkl}]_{\substack{i=1,\dots,n \\ j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} & \mu_{ijkl} &= \frac{1}{2}(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \end{aligned} \quad (\text{A.95})$$

The computation of the double dot product between the above symmetric term and the second order tensor defined in (A.89) leads to its corresponding symmetric

component. Nevertheless, if the above skew-symmetric tensor is the one involved in the tensor operation, the result turns out to be the skew-symmetric component of the second order tensor.

$$\begin{aligned} \mathbf{I}_4^{\text{sym}} : \mathbf{A} &= \frac{1}{2}(\mathbf{I}_4 + \bar{\mathbf{I}}_4) : \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) = \mathbf{A}^{\text{sym}} \\ \mathbf{I}_4^{\text{skw}} : \mathbf{A} &= \frac{1}{2}(\bar{\mathbf{I}}_4 - \mathbf{I}_4) : \mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = \mathbf{A}^{\text{skw}} \end{aligned} \quad (\text{A.96})$$

A.11. Some properties of the determinant of a second order tensor

In this section, some properties of the determinant operator are summed up. On the one hand, it can be demonstrated that the determinant of a second order square tensor turns out to be equivalent to the determinant of its transpose.

$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \quad (\text{A.97})$$

It can also be proven that the determinant of the multiplication between two second order tensors is equivalent to the product between the determinant of both tensors.

$$\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B}) \quad (\text{A.98})$$

On the other hand, the determinant of a diagonal second order tensor turns out to be equivalent to the product of its diagonal components.

$$\det(\mathbf{A}) = \det\left(\begin{bmatrix} A_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{nn} \end{bmatrix}\right) = \prod_{k=1}^n A_{kk} \quad (\text{A.99})$$

A.12. Derivative of a determinant with respect to a given parameter

Let's consider the following non-singular square matrix, whose components depend on a given parameter p .

$$\mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad A_{ij} = A_{ij}(p) \quad (\text{A.100})$$

Its derivative, with respect to the parameter p is defined as the following second order tensor.

$$\frac{\partial \mathbf{A}}{\partial p} = \left[\frac{\partial A_{ij}}{\partial p} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{A.101})$$

Since the matrix defined in (A.100) is non-singular, its inverse exists and can be calculated. The mathematical condition that guarantees the existence of this inverse is

the non-nullity of the determinant. The determinant can be calculated along the i -th row, as:

$$A = \det(\mathbf{A}) = \sum_{k=1}^n A_{ik} a_{ik} \neq 0 \quad i \in [1, \dots, n] \quad (\text{A.102})$$

Where a_{ik} is the cofactor corresponding to the component A_{ik} , which is defined by means of the minor M_{ij} . This minor is defined as the determinant of the resulting matrix by removing the i -th row and the j -th column.

$$a_{ik} = \text{cofactor}(A_{ik}) = (-1)^{i+j} M_{ij} \quad (\text{A.103})$$

The derivative of the previous determinant with respect to the parameter p can be calculated by applying the chain rule.

$$\frac{\partial A}{\partial p} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial A}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial p} \quad (\text{A.104})$$

In order to manipulate and simplify the above derivative, the inverse of the second order tensor defined in (A.100) is calculated.

$$\mathbf{B} = \mathbf{A}^{-1} = [B_{ij}]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \quad (\text{A.105})$$

Where its components are defined in terms of the cofactors as:

$$B_{ji} = \frac{1}{A} a_{ij} \quad (\text{A.106})$$

On the other hand, the derivative of the determinant with respect to a given component is involved in equation (A.104). If the computation of the determinant performed in (A.102) is taken into account, this derivative becomes:

$$\frac{\partial A}{\partial A_{ij}} = \frac{\partial}{\partial A_{ij}} \left(\sum_{k=1}^n A_{ik} a_{ik} \right) = a_{ij} \quad (\text{A.107})$$

Moreover, from equation (A.106), the cofactor can be equivalently expressed as:

$$a_{ij} = A B_{ji} \quad (\text{A.108})$$

Therefore, equation (A.107) becomes:

$$\frac{\partial A}{\partial A_{ij}} = A B_{ji} \quad (\text{A.109})$$

And the substitution of the above result into the equation that defines the derivative (A.104), leads to:

$$\frac{\partial A}{\partial p} = A \sum_{j=1}^n \sum_{i=1}^n B_{ji} \frac{\partial A_{ij}}{\partial p} = A \sum_{i=1}^n \sum_{j=1}^n \frac{\partial A_{ij}}{\partial p} B_{ji} \quad (\text{A.110})$$

The above equation is expressed in index notation. If the definition of the trace operator is recalled, it can be equivalently written as:

$$\frac{\partial A}{\partial p} = A \text{Tr} \left(\mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial p} \right) = A \text{Tr} \left(\frac{\partial \mathbf{A}}{\partial p} \mathbf{A}^{-1} \right) \quad (\text{A.111})$$

A.13. Rotation tensor

The equation that defines the scalar product between two vectors was defined in section A.1.4 as:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\alpha) \quad (\text{A.112})$$

According to the above equation, the scalar product turns out to depend on the norm of both vectors and the angle between them.

On the one hand, let's consider that a rotation is applied to both vectors. The components of the rotated vectors are obtained by applying a rotation tensor \mathbf{R} to the original ones. As it is a rotation, the norm of both vectors do not change. The rotation only modifies their direction.

$$\begin{aligned} \mathbf{a}' &= \mathbf{R}\mathbf{a} &\longrightarrow & \|\mathbf{a}'\| = \|\mathbf{a}\| \\ \mathbf{b}' &= \mathbf{R}\mathbf{b} &\longrightarrow & \|\mathbf{b}'\| = \|\mathbf{b}\| \end{aligned} \quad (\text{A.113})$$

Furthermore, the rotation does not modify the angle between them. Hence, the scalar product of the rotated vectors is defined as:

$$\mathbf{a}' \cdot \mathbf{b}' = \|\mathbf{a}'\| \|\mathbf{b}'\| \cos(\alpha) \quad (\text{A.114})$$

But the norm of the rotated vectors is equivalent to the norm of the original one. Thus, the above scalar product becomes:

$$\mathbf{a}' \cdot \mathbf{b}' = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\alpha) = \mathbf{a} \cdot \mathbf{b} \quad (\text{A.115})$$

Therefore, it can be concluded that a rotation does not vary the scalar product. That is, it does not modify neither angles nor distances.

On the other hand, the scalar product between the rotated vectors can be equivalently expressed according to equation (A.5), as shown below.

$$\begin{aligned} \mathbf{a}' \cdot \mathbf{b}' &= (\mathbf{R}\mathbf{a}) \cdot (\mathbf{R}\mathbf{b}) \\ &= (\mathbf{R}\mathbf{a})^T (\mathbf{R}\mathbf{b}) = \mathbf{a}^T \mathbf{R}^T \mathbf{R} \mathbf{b} \end{aligned} \quad (\text{A.116})$$

Consequently, if the scalar product is not modified when a rotation is applied, it can be concluded that the rotation tensor has to be orthogonal.

$$\left. \begin{aligned} \mathbf{a}' \cdot \mathbf{b}' &= \mathbf{a}^T \mathbf{R}^T \mathbf{R} \mathbf{b} \\ \mathbf{a}' \cdot \mathbf{b}' &= \mathbf{a}^T \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \end{aligned} \right\} \implies \mathbf{R}^T \mathbf{R} = \mathbf{I} \quad \iff \mathbf{R}^{-1} = \mathbf{R}^T \quad (\text{A.117})$$

A.14. Voigt notation

A criterion is needed to represent symmetric second order tensors as first order tensors, since it is easier to manipulate and code vector operations than tensor ones. In Solid Mechanics, the Voigt notation is usually applied. According to this notation, a different rule is applied to define the vectorial form of strain and stress tensors.

The vectorial form of a stress tensor is composed by the diagonal components, followed by the non-diagonal ones.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \implies \bar{\boldsymbol{\sigma}} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \quad (\text{A.118})$$

And the vectorial form of a strain tensor has the same structure as the above definition, but the non-diagonal components are multiplied by 2.

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \implies \bar{\mathbf{E}} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{Bmatrix} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2 E_{12} \\ 2 E_{13} \\ 2 E_{23} \end{Bmatrix} \quad (\text{A.119})$$

The main advantage of this notation is the equivalence between the double dot product between both tensors and the scalar product of their vectorial definition.

$$\left. \begin{aligned} \mathbf{E} : \boldsymbol{\sigma} &= \sum_{i=1}^3 \sum_{j=1}^3 E_{ij} \sigma_{ij} \\ \bar{\mathbf{E}}^T \bar{\boldsymbol{\sigma}} &= \sum_{k=1}^3 E_k \sigma_k = \sum_{i=1}^3 \sum_{j=1}^3 E_{ij} \sigma_{ij} \end{aligned} \right\} \implies \mathbf{E} : \boldsymbol{\sigma} = \bar{\mathbf{E}}^T \bar{\boldsymbol{\sigma}} \quad (\text{A.120})$$

Field operators

B.1. Gradient of a scalar field

Let's consider the following scalar field:

$$f = f(\mathbf{r}) \quad \mathbf{r} = \{r_i\}_{i=1,\dots,n} \quad (\text{B.1})$$

Where f is a scalar function that depends on the vector variable \mathbf{r} . Let's also assume that the vector is defined in a n -dimensional space.

The gradient of the previous scalar field is defined as a row vector whose components are the derivatives of the scalar function with respect to the components of the vector \mathbf{r} .

$$\frac{df}{d\mathbf{r}} = \left[\frac{\partial f}{\partial r_1} \quad \dots \quad \frac{\partial f}{\partial r_n} \right] = \left[\frac{\partial f}{\partial r_i} \right]_{i=1,\dots,n} \quad (\text{B.2})$$

B.1.1. Gradient of the scalar product

Let's now consider the following particular scalar field, which turns out to be the scalar product between two different vector fields.

$$\left. \begin{aligned} \mathbf{a}(\mathbf{r}) &= \{a_i(\mathbf{r})\}_{i=1,\dots,m} \\ \mathbf{b}(\mathbf{r}) &= \{b_i(\mathbf{r})\}_{i=1,\dots,m} \\ \mathbf{r} &= \{r_i\}_{i=1,\dots,n} \end{aligned} \right\} \implies \begin{aligned} f(\mathbf{r}) &= \mathbf{a}(\mathbf{r}) \cdot \mathbf{b}(\mathbf{r}) \\ &= \mathbf{a}(\mathbf{r})^T \mathbf{b}(\mathbf{r}) = \sum_{k=1}^m a_k b_k \end{aligned} \quad (\text{B.3})$$

According to the definition presented in (B.2), the gradient of the compound field is:

$$\frac{df}{d\mathbf{r}} = \left[\frac{\partial f}{\partial r_1} \quad \dots \quad \frac{\partial f}{\partial r_n} \right] \quad (\text{B.4})$$

Where the components of the above vector are:

$$\frac{\partial f}{\partial r_i} = \frac{\partial}{\partial r_i} \left(\sum_{k=1}^m a_k b_k \right) = \sum_{k=1}^m \left(\frac{\partial a_k}{\partial r_i} b_k + a_k \frac{\partial b_k}{\partial r_i} \right) \quad (\text{B.5})$$

In addition, the components of the following vectors are involved in the above equation.

$$\begin{aligned}\frac{\partial \mathbf{a}}{\partial r_i} &= \left\{ \frac{\partial a_k}{\partial r_i} \right\}_{k=1, \dots, m} \\ \frac{\partial \mathbf{b}}{\partial r_i} &= \left\{ \frac{\partial b_k}{\partial r_i} \right\}_{k=1, \dots, m}\end{aligned}\tag{B.6}$$

Therefore, the components (B.5) can be equivalently written as:

$$\frac{\partial f}{\partial r_i} = \mathbf{b}^T \frac{\partial \mathbf{a}}{\partial r_i} + \mathbf{a}^T \frac{\partial \mathbf{b}}{\partial r_i}\tag{B.7}$$

And the gradient (B.4) becomes:

$$\frac{df}{d\mathbf{r}} = \begin{pmatrix} \frac{\partial f}{\partial r_1} \\ \vdots \\ \frac{\partial f}{\partial r_n} \end{pmatrix}^T = \begin{pmatrix} \mathbf{b}^T \frac{\partial \mathbf{a}}{\partial r_1} + \mathbf{a}^T \frac{\partial \mathbf{b}}{\partial r_1} \\ \vdots \\ \mathbf{b}^T \frac{\partial \mathbf{a}}{\partial r_n} + \mathbf{a}^T \frac{\partial \mathbf{b}}{\partial r_n} \end{pmatrix}^T = \mathbf{b}^T \left[\frac{\partial \mathbf{a}}{\partial r_1} \quad \dots \quad \frac{\partial \mathbf{a}}{\partial r_n} \right] + \mathbf{a}^T \left[\frac{\partial \mathbf{b}}{\partial r_1} \quad \dots \quad \frac{\partial \mathbf{b}}{\partial r_n} \right]\tag{B.8}$$

The gradient of a given vector function is defined in (B.14). According to this definition, the gradients of the vector functions \mathbf{a} and \mathbf{b} turn out to be:

$$\begin{aligned}\frac{d\mathbf{a}}{d\mathbf{r}} &= \begin{bmatrix} \frac{\partial a_1}{\partial r_1} & \dots & \frac{\partial a_1}{\partial r_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_m}{\partial r_1} & \dots & \frac{\partial a_m}{\partial r_n} \end{bmatrix} = \left[\frac{\partial \mathbf{a}}{\partial r_1} \quad \dots \quad \frac{\partial \mathbf{a}}{\partial r_n} \right] \\ \frac{d\mathbf{b}}{d\mathbf{r}} &= \begin{bmatrix} \frac{\partial b_1}{\partial r_1} & \dots & \frac{\partial b_1}{\partial r_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_m}{\partial r_1} & \dots & \frac{\partial b_m}{\partial r_n} \end{bmatrix} = \left[\frac{\partial \mathbf{b}}{\partial r_1} \quad \dots \quad \frac{\partial \mathbf{b}}{\partial r_n} \right]\end{aligned}\tag{B.9}$$

Thus, the gradient (B.4) can be expressed by means of the above gradients, as:

$$\frac{df}{d\mathbf{r}} = \mathbf{b}^T \frac{d\mathbf{a}}{d\mathbf{r}} + \mathbf{a}^T \frac{d\mathbf{b}}{d\mathbf{r}}\tag{B.10}$$

B.2. Derivative of a scalar field with respect to a second order tensor

Let's consider the following scalar field that depends on a tensor variable:

$$f = f(\mathbf{A}) \quad \mathbf{A} = [A_{ij}]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}\tag{B.11}$$

The derivative of the above scalar field with respect to the tensor variable is defined as the following tensor.

$$\frac{df}{d\mathbf{A}} = \begin{bmatrix} \frac{\partial f}{\partial A_{11}} & \cdots & \frac{\partial f}{\partial A_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial A_{n1}} & \cdots & \frac{\partial f}{\partial A_{nn}} \end{bmatrix} = \left[\frac{\partial f}{\partial A_{ij}} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{B.12})$$

B.3. Gradient of a vector field

Let's consider the following vector field. Both the vector function and the vector variable on which it depends are defined in a n -dimensional space.

$$\mathbf{f} = \mathbf{f}(\mathbf{r}) = \{f_i(\mathbf{r})\}_{i=1,\dots,n} \quad \mathbf{r} = \{r_i\}_{i=1,\dots,n} \quad (\text{B.13})$$

The gradient of the previous vector function is defined as a column vector, whose components turn out to be the gradients of its components. In addition, these components can be rewritten according to the definition of the gradient of a scalar field stated in (B.2).

$$\frac{d\mathbf{f}}{d\mathbf{r}} = \left\{ \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{r}} \\ \vdots \\ \frac{\partial f_n}{\partial \mathbf{r}} \end{bmatrix} \right\} = \begin{bmatrix} \frac{\partial f_1}{\partial r_1} & \cdots & \frac{\partial f_1}{\partial r_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial r_1} & \cdots & \frac{\partial f_n}{\partial r_n} \end{bmatrix} = \left[\frac{\partial \mathbf{f}}{\partial r_1} \quad \cdots \quad \frac{\partial \mathbf{f}}{\partial r_n} \right] = \left[\frac{\partial f_i}{\partial r_j} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{B.14})$$

B.3.1. Gradient of the product between a scalar field and a vector field

The resulting vector field of the product between a scalar field and a vector one is an interesting particular case, since its gradient is computed frequently along this document.

$$\left. \begin{array}{l} \mathbf{h}(\mathbf{r}) = \{h_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{r} = \{r_i\}_{i=1,\dots,n} \end{array} \right\} \implies \begin{array}{l} \mathbf{f}(\mathbf{r}) = g(\mathbf{r}) \mathbf{h}(\mathbf{r}) = \{f_i(\mathbf{r})\}_{i=1,\dots,n} \\ f_i(\mathbf{r}) = g(\mathbf{r}) h_i(\mathbf{r}) \end{array} \quad (\text{B.15})$$

The gradient of the above vector field is defined as:

$$\frac{d\mathbf{f}}{d\mathbf{r}} = \left[\frac{\partial f_i}{\partial r_j} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \frac{\partial f_i}{\partial r_j} = \frac{\partial (g h_i)}{\partial r_j} = \frac{\partial g}{\partial r_j} h_i + g \frac{\partial h_i}{\partial r_j} \quad (\text{B.16})$$

On the other hand, the gradient of the vector field \mathbf{h} is:

$$\frac{d\mathbf{h}}{d\mathbf{r}} = \left[\frac{\partial h_i}{\partial r_j} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{B.17})$$

And the resulting tensor of the tensor product between \mathbf{h} and the gradient of the scalar field g is:

$$\mathbf{h} \otimes \frac{dg}{d\mathbf{r}} = \mathbf{h} \frac{dg}{d\mathbf{r}} = \left[h_i \frac{\partial g}{\partial r_j} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{B.18})$$

Thus, if the tensors defined in (B.17) and (B.18) are taken into account, the gradient of the vector field defined in (B.15) can be finally defined as:

$$\frac{d\mathbf{f}}{d\mathbf{r}} = \mathbf{h} \frac{dg}{d\mathbf{r}} + g \frac{d\mathbf{h}}{d\mathbf{r}} = \left[\frac{\partial f_i}{\partial r_j} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \frac{\partial f_i}{\partial r_j} = h_i \frac{\partial g}{\partial r_j} + g \frac{\partial h_i}{\partial r_j} \quad (\text{B.19})$$

B.4. Divergence of a vector field

The divergence is an operator which is applied to vector fields. The divergence of the vector field presented in (B.13) is defined as:

$$\left. \begin{array}{l} \mathbf{f}(\mathbf{r}) = \{f_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{r} = \{r_i\}_{i=1,\dots,n} \end{array} \right\} \implies \text{div}(\mathbf{f}) = \left[\frac{\partial}{\partial r_1} \quad \dots \quad \frac{\partial}{\partial r_n} \right] \left\{ \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array} \right\} = \sum_{k=1}^n \frac{\partial f_k}{\partial r_k} \quad (\text{B.20})$$

According to the previous definition, it can also be defined as the trace operator applied to the gradient tensor of the vector field. This gradient tensor was previously defined in (B.14).

$$\text{div}(\mathbf{f}) = \text{Tr} \left(\frac{d\mathbf{f}}{d\mathbf{r}} \right) = \sum_{k=1}^n \frac{\partial f_k}{\partial r_k} \quad (\text{B.21})$$

B.4.1. Divergence of the product between a scalar field and a vector field

Let's apply the divergence operator to the following compound field, where a is a scalar field and \mathbf{g} is a vector one.

$$\left. \begin{array}{l} \mathbf{g}(\mathbf{r}) = \{g_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{r} = \{r_i\}_{i=1,\dots,n} \end{array} \right\} \implies \begin{array}{l} \mathbf{f}(\mathbf{r}) = a(\mathbf{r}) \mathbf{g}(\mathbf{r}) = \{f_i(\mathbf{r})\}_{i=1,\dots,n} \\ f_i(\mathbf{r}) = a(\mathbf{r}) g_i(\mathbf{r}) \end{array} \quad (\text{B.22})$$

The divergence of the above vector field turns out to be:

$$\text{div}(\mathbf{f}) = \sum_{k=1}^n \frac{\partial f_k}{\partial r_k} = \sum_{k=1}^n \frac{\partial (a g_k)}{\partial r_k} = \sum_{k=1}^n \frac{\partial a}{\partial r_k} g_k + a \sum_{k=1}^n \frac{\partial g_k}{\partial r_k} \quad (\text{B.23})$$

If the definitions of the gradient of a scalar field and the divergence of a vector field presented in (B.2) and (B.21), respectively, are taken into account, the divergence of the previous compound field can be finally computed as shown below.

$$\text{div}(a \mathbf{g}) = \frac{\partial a}{\partial \mathbf{r}} \mathbf{g} + a \text{div}(\mathbf{g}) \quad (\text{B.24})$$

B.4.2. Divergence of the product between a second order tensor and a first order tensor

Another interesting vector field is the one composed by the product between a second order tensor and a first order one.

$$\left. \begin{aligned} \mathbf{A}(\mathbf{r}) &= [A_{ij}(\mathbf{r})]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \mathbf{b}(\mathbf{r}) &= \{b_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{r} &= \{r_i\}_{i=1,\dots,n} \end{aligned} \right\} \implies \begin{aligned} \mathbf{c}(\mathbf{r}) &= \mathbf{A}(\mathbf{r}) \mathbf{b}(\mathbf{r}) = \{c_j(\mathbf{r})\}_{j=1,\dots,n} \\ c_j(\mathbf{r}) &= \sum_{k=1}^n A_{jk} b_k \end{aligned} \quad (\text{B.25})$$

The application of the divergence operator to the above vector field leads to the following result:

$$\begin{aligned} \operatorname{div}(\mathbf{c}) &= \sum_{j=1}^n \frac{\partial c_j}{\partial r_j} = \sum_{j=1}^n \left[\frac{\partial}{\partial r_j} \left(\sum_{k=1}^n A_{jk} b_k \right) \right] \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial r_j} (A_{jk} b_k) \\ &= \sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial A_{jk}}{\partial r_j} b_k + A_{jk} \frac{\partial b_k}{\partial r_j} \right) \\ &= \sum_{k=1}^n \left[b_k \left(\sum_{j=1}^n \frac{\partial A_{jk}}{\partial r_j} \right) \right] + \sum_{k=1}^n \sum_{j=1}^n A_{jk} \frac{\partial b_k}{\partial r_j} \end{aligned} \quad (\text{B.26})$$

In the above result, the components of the resulting vector of applying the divergence operator to the tensor \mathbf{A}^T are involved in the first term. The divergence of a tensor field is defined in the following section.

$$\operatorname{div}(\mathbf{A}^T) = \left\{ \sum_{j=1}^n \frac{\partial A_{jk}}{\partial r_j} \right\}_{k=1,\dots,n} \quad (\text{B.27})$$

Moreover, the second term turns out to be the double dot product between \mathbf{A}^T and the gradient of the vector field \mathbf{b} .

$$\mathbf{A}^T : \frac{d\mathbf{b}}{d\mathbf{r}} = \sum_{k=1}^n \sum_{j=1}^n A_{jk} \frac{\partial b_k}{\partial r_j} \quad (\text{B.28})$$

Therefore, the divergence of the vector field defined in (B.25) is finally defined as:

$$\operatorname{div}(\mathbf{A} \mathbf{b}) = \mathbf{b}^T \operatorname{div}(\mathbf{A}^T) + \mathbf{A}^T : \frac{d\mathbf{b}}{d\mathbf{r}} \quad (\text{B.29})$$

Furthermore, if the double dot property stated in (A.81) is applied, the above definition can be rewritten as follows.

$$\operatorname{div}(\mathbf{A} \mathbf{b}) = \mathbf{b}^T \operatorname{div}(\mathbf{A}^T) + \operatorname{Tr} \left(\mathbf{A} \frac{d\mathbf{b}}{d\mathbf{r}} \right) \quad (\text{B.30})$$

B.5. Divergence of a tensor field

The divergence operator can also be applied to a tensor field. Let's consider the following tensor field defined in a n -dimensional space. Its components depend on a vector variable composed by n components.

$$\mathbf{S}(\mathbf{r}) = [S_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad S_{ij} = S_{ij}(\mathbf{r}) \quad \mathbf{r} = \{r_i\}_{i=1,\dots,n} \quad (\text{B.31})$$

Its divergence is defined as a vector, whose components are the result of applying the divergence operator to each one of the rows that compose the above tensor. If the rows of the second order tensor are defined as:

$$\mathbf{S} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \quad \Longrightarrow \quad \mathbf{a}_i = \{S_{ij}\}_{j=1,\dots,n} \quad i = 1, \dots, n \quad (\text{B.32})$$

The divergence of the tensor field turns out to be:

$$\text{div}(\mathbf{S}) = \{\text{div}(\mathbf{a}_i)\}_{i=1,\dots,n} = \left\{ \sum_{k=1}^n \frac{\partial S_{ik}}{\partial r_k} \right\}_{i=1,\dots,n} \quad (\text{B.33})$$

B.5.1. Divergence of the tensor product between two first order tensors

The divergence of the tensor field that corresponds to the tensor product between two vector fields is an interesting particular case. The tensor field is defined as:

$$\left. \begin{array}{l} \mathbf{a}(\mathbf{r}) = \{a_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{b}(\mathbf{r}) = \{b_i(\mathbf{r})\}_{i=1,\dots,n} \\ \mathbf{r} = \{r_i\}_{i=1,\dots,n} \end{array} \right\} \Longrightarrow \quad \begin{array}{l} \mathbf{S} = \mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^T = [S_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ S_{ij} = a_i b_j \end{array} \quad (\text{B.34})$$

According to (B.33), the divergence of the above tensor field turns out to be:

$$\begin{aligned} \text{div}(\mathbf{S}) &= \left\{ \sum_{k=1}^n \frac{\partial S_{ik}}{\partial r_k} \right\}_{i=1,\dots,n} \\ &= \left\{ \sum_{k=1}^n \frac{\partial (a_i b_k)}{\partial r_k} \right\}_{i=1,\dots,n} \\ &= \left\{ \sum_{k=1}^n \frac{\partial a_i}{\partial r_k} b_k + a_i \sum_{k=1}^n \frac{\partial b_k}{\partial r_k} \right\}_{i=1,\dots,n} \end{aligned} \quad (\text{B.35})$$

Furthermore, the product between the gradient of the vector field \mathbf{a} and the vector field \mathbf{b} , as well as the divergence of the vector field \mathbf{b} , are:

$$\frac{d\mathbf{a}}{d\mathbf{r}} \mathbf{b} = \left\{ \sum_{k=1}^n \frac{\partial a_i}{\partial r_k} b_k \right\}_{i=1,\dots,n} \quad \text{div}(\mathbf{b}) = \sum_{k=1}^n \frac{\partial b_k}{\partial r_k} \quad (\text{B.36})$$

Therefore, the divergence computed in (B.35) can be finally expressed as:

$$\operatorname{div}(\mathbf{a} \mathbf{b}^T) = \frac{d\mathbf{a}}{d\mathbf{r}} \mathbf{b} + \mathbf{a} \operatorname{div}(\mathbf{b}) \quad (\text{B.37})$$

B.6. Derivative of a tensor field with respect to a second order tensor

Let's consider the following second order tensor field that depends on a second order tensor variable:

$$\mathbf{F}(\mathbf{A}) = [F_{ij}(\mathbf{A})]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \mathbf{A} = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (\text{B.38})$$

The derivative of the above tensor with respect to its corresponding second order tensor variable is defined as a fourth order tensor, as:

$$\left. \begin{aligned} \frac{d\mathbf{F}}{d\mathbf{A}} &= \left[\frac{dF_{ij}}{d\mathbf{A}} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \\ \frac{dF_{ij}}{d\mathbf{A}} &= \left[\frac{\partial F_{ij}}{\partial A_{kl}} \right]_{\substack{k=1,\dots,n \\ l=1,\dots,n}} \end{aligned} \right\} \implies \frac{d\mathbf{F}}{d\mathbf{A}} = \left[\frac{\partial F_{ij}}{\partial A_{kl}} \right]_{\substack{j=1,\dots,n \\ k=1,\dots,n \\ l=1,\dots,n}} \quad (\text{B.39})$$

B.7. Divergence theorem

The divergence operator leads to the divergence theorem, which is frequently applied in Solid Mechanics. In this appendix, the divergence theorem applied to a given vector field, as well as its application to tensor fields, is stated. It is also known as the Gauss-Ostrogradsky theorem or as the Gauss theorem.

B.7.1. Divergence theorem applied to a vector field

Let's consider a given closed domain Ω and its boundary $\partial\Omega$. Let's also assume that a vector field is defined within this domain.

The divergence theorem states that the flux of the vector field through the boundary of the domain is equivalent to the integration of the vector field divergence over the whole domain. Thus,

$$\iint_{\Gamma=\partial\Omega} \mathbf{f}^T \mathbf{n} d\Gamma = \iiint_{\Omega} \operatorname{div}(\mathbf{f}) d\Omega \quad (\text{B.40})$$

B.7.2. Divergence theorem applied to a tensor field

This theorem can be extended to tensor fields. Let's consider the following one:

$$\mathbf{S} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \quad (\text{B.41})$$

In order to obtain its tensor statement, the integration of the divergence definition exposed in (B.33) is performed over the whole domain.

$$\mathbf{div}(\mathbf{S}) = \{\mathbf{div}(\mathbf{a}_i)\}_{i=1,\dots,n} \implies \iiint_{\Omega} \mathbf{div}(\mathbf{S}) \, d\Omega = \left\{ \iiint_{\Omega} \mathbf{div}(\mathbf{a}_i) \, d\Omega \right\}_{i=1,\dots,n} \quad (\text{B.42})$$

The components of the above vector can be substituted by their equivalent expression according to the divergence theorem presented in (B.40)

$$\iiint_{\Omega} \mathbf{div}(\mathbf{S}) \, d\Omega = \left\{ \begin{array}{c} \iint_{\partial\Omega} \mathbf{a}_1^T \mathbf{n} \, d\Gamma \\ \vdots \\ \iint_{\partial\Omega} \mathbf{a}_n^T \mathbf{n} \, d\Gamma \end{array} \right\} = \iint_{\partial\Omega} \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \mathbf{n} \, d\Gamma = \iint_{\partial\Omega} \mathbf{S} \mathbf{n} \, d\Gamma \quad (\text{B.43})$$

Therefore, when dealing with a tensor field, the divergence theorem becomes:

$$\iint_{\Gamma=\partial\Omega} \mathbf{S} \mathbf{n} \, d\Gamma = \iiint_{\Omega} \mathbf{div}(\mathbf{S}) \, d\Omega \quad (\text{B.44})$$

B.8. Taylor series expansion

The Taylor series expansion of a given function is defined as an infinite sum of terms that are expressed in terms of its corresponding derivatives, which are evaluated at a single point. If the point where the derivatives are evaluated is the origin of coordinates, the series expansion is also known as the Maclaurin series expansion.

The terms that compose the infinite sum are defined in terms of the field operators exposed in this appendix, and the tensor operations presented in the previous one are also recalled.

B.8.1. Scalar function of a scalar variable

The Taylor series expansion of a given scalar function that depends on a scalar variable is presented below.

$$f(x) = f(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{(k)}f}{dx^k}(x_0) (x - x_0)^k \quad (\text{B.45})$$

The above series expansion can be rewritten in terms of the linear term. The last addend represents the error made if the nonlinear terms are not taken into account.

$$f(x) = f(x_0) + \frac{df}{dx}(x_0) (x - x_0) + \mathcal{O}(|x - x_0|^2) \quad (\text{B.46})$$

B.8.2. Scalar function of a vector variable

If the scalar function depends on a vector variable, the series expansion becomes:

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{(k)}f}{d\mathbf{x}^k}(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0)^k \\ &= f(\mathbf{x}_0) + \frac{df}{d\mathbf{x}}(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \mathcal{O}(\|\mathbf{x} - \mathbf{x}_0\|^2) \end{aligned} \quad (\text{B.47})$$

B.8.3. Vector function of a scalar variable

Let's now consider vector functions. In case the function depends on a scalar variable, its corresponding Taylor series expansion turns out to be:

$$\begin{aligned} \mathbf{f}(x) &= \mathbf{f}(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{(k)}\mathbf{f}}{dx^k}(x_0) (x - x_0)^k \\ &= \mathbf{f}(x_0) + \frac{d\mathbf{f}}{dx}(x_0) (x - x_0) + \mathcal{O}(|x - x_0|^2) \end{aligned} \quad (\text{B.48})$$

B.8.4. Vector function of a vector variable

If the vector function depends on a vector variable, the series expansion becomes:

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{(k)}\mathbf{f}}{d\mathbf{x}^k}(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0)^k \\ &= \mathbf{f}(\mathbf{x}_0) + \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \mathcal{O}(\|\mathbf{x} - \mathbf{x}_0\|^2) \end{aligned} \quad (\text{B.49})$$

B.8.5. Tensor function of a tensor variable

Finally, let's consider a given tensor function that depends on a tensor variable. Its series expansion is defined as:

$$\begin{aligned} \mathbf{F}(\mathbf{X}) &= \mathbf{F}(\mathbf{X}_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{(k)}\mathbf{F}}{d\mathbf{X}^k}(\mathbf{X}_0) : (\mathbf{X} - \mathbf{X}_0)^k \\ &= \mathbf{F}(\mathbf{X}_0) + \frac{d\mathbf{F}}{d\mathbf{X}}(\mathbf{X}_0) : (\mathbf{X} - \mathbf{X}_0) + \mathcal{O}(\|\mathbf{X} - \mathbf{X}_0\|^2) \end{aligned} \quad (\text{B.50})$$

Calculus of variations

C.1. Relevant space of functions

The Lebesgue space of functions L^2 is the one composed by the square-integrable functions.

$$\mathcal{L}^2(\Omega) = \left\{ f \mid \iiint_{\Omega} f^2 d\Omega < \infty \right\} \quad (\text{C.1})$$

And the following Sobolev's space belongs to the above space of functions. The functions contained in this space, as well as their m first partial derivatives, turn out to be square-integrable functions.

$$\mathcal{W}^{m,2}(\Omega) = \left\{ f \in L^2(\Omega) \mid D^{\alpha} f \in L^2(\Omega), \alpha = 0, \dots, m \right\} \in \mathcal{L}^2(\Omega) \quad (\text{C.2})$$

The above Sobolev's space turns out to be equivalent to the following Hilbert's space of functions.

$$\mathcal{H}^m(\Omega) = \mathcal{W}^{m,2}(\Omega) \quad (\text{C.3})$$

Where the value $m = 1$ leads to the following particular Hilbert's space of functions, that belongs to the \mathcal{C}^1 class of functions. The functions contained in this space, as well as their first derivatives, are continuous.

$$\mathcal{H}^1(\Omega) = \left\{ f \in L^2(\Omega), f' \in L^2(\Omega) \right\} \in \mathcal{C}^1 \quad (\text{C.4})$$

C.2. Localization theorem

The localization theorem is one of the fundamental lemmas of the calculus of variations. It allows dealing with a differential equation instead of dealing with an integral expression.

C.2.1. Localization theorem applied to scalar functions

Let's consider the following continuous function defined over a material domain in a n -dimensional space.

$$\psi(\mathbf{r}) \in \mathcal{C}^0(\Omega) \quad (\text{C.5})$$

The localization theorem states that:

$$\iiint_{\Omega} \omega(\mathbf{r}) \psi(\mathbf{r}) d\Omega = 0 \quad \forall \omega \in H_{\omega} \quad \Longrightarrow \quad \psi(\mathbf{r}) = 0 \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (\text{C.6})$$

Where ω are the so-called test functions. It can be proven that the test functions have to be null over the boundary domain, and they belong to the space of functions \mathcal{H}^1 . That is, the test functions, as well as their first derivatives, have to be square-integrable functions.

$$H_{\omega} = \{\omega(\mathbf{r})\} \quad \text{that verify} \quad \begin{cases} \omega(\mathbf{r}) = 0 & \forall \mathbf{r} \in \partial\Omega \\ \omega(\mathbf{r}) \in \mathcal{H}^1(\Omega) \end{cases} \quad (\text{C.7})$$

Therefore, when dealing with an integral expression as the one defined in (C.6), it can be concluded that the function (C.5) has to be null over the interior of the domain.

Proof by contradiction

Let's consider the following primal assumption:

$$\iiint_{\Omega} \omega(\mathbf{r}) \psi(\mathbf{r}) d\Omega = 0 \quad (\text{C.8})$$

Let's also consider a given point \mathbf{z} that belongs to the interior of the domain. If the scalar function is evaluated at this particular point, the function is positive.

$$\mathbf{z} \in \overset{\circ}{\Omega} \quad | \quad \psi(\mathbf{z}) > 0 \quad (\text{C.9})$$

If the scalar function is continuous, the existence of an open ball where the scalar function is positive can be stated.

$$\psi(\mathbf{r}) \in \mathcal{C}^1(\Omega) \quad \Longrightarrow \quad \exists \delta > 0 \quad | \quad \psi(\mathbf{r}) > 0 \quad \forall \mathbf{r} \in \overset{\circ}{B}(\mathbf{z}, \delta) \quad (\text{C.10})$$

On the other hand, the test function can be defined as:

$$\omega(\mathbf{r}) = \begin{cases} = 0 & \text{if } |\mathbf{x} - \mathbf{z}| \geq \delta \\ = (\delta^2 - |\mathbf{x} - \mathbf{z}|^2)^{2n} & \text{if } |\mathbf{x} - \mathbf{z}| < \delta \end{cases} \quad (\text{C.11})$$

Where the value of n determine the Hilbert's space to which the test function belongs

$$\begin{aligned} n = 1 & \Longrightarrow \omega(\mathbf{r}) \in \mathcal{H}^1(\Omega) \\ n > 1 & \Longrightarrow \omega(\mathbf{r}) \in \mathcal{H}^n(\Omega) \end{aligned} \quad (\text{C.12})$$

The computation of the product between the scalar function and the above test function, and the integration of this result over the whole material domain, leads to:

$$\iiint_{\Omega} \psi(\mathbf{r}) \omega(\mathbf{r}) d\Omega = \iiint_{B(\mathbf{z}, \delta)} \underbrace{\psi(\mathbf{r})}_{>0} \underbrace{(\delta^2 - |\mathbf{x} - \mathbf{z}|^2)^{2n}}_{>0} d\Omega > 0 \quad (\text{C.13})$$

The scalar function and the test function are positive in the interior of the open ball. Moreover, both functions are square integrable, so the integration of their product leads to a finite value. According to the primal assumption (C.8), this result turns out to be a contradiction. Therefore, it can be concluded that the scalar function has to be null when evaluated at the point \mathbf{z} , if the initial assumption (C.8) is fulfilled.

$$\psi(\mathbf{z}) = 0 \quad \forall \mathbf{z} \in \overset{\circ}{\Omega} \quad (\text{C.14})$$

C.2.2. Localization theorem applied to vector functions

This principle can be extended to vector functions. Let's now consider the following vector function, whose components are continuous functions defined over the domain Ω .

$$\boldsymbol{\psi}(\mathbf{r}) = \{\psi_i(\mathbf{r})\}_{i=1, \dots, n} \quad \psi_i \in \mathcal{C}^0(\Omega) \quad (\text{C.15})$$

In this particular case, the localization theorem states that:

$$\iiint_{\Omega} \boldsymbol{\omega}^T(\mathbf{r}) \boldsymbol{\psi}(\mathbf{r}) d\Omega = 0 \quad \forall \boldsymbol{\omega} \in H_{\boldsymbol{\omega}} \quad \implies \quad \boldsymbol{\psi}(\mathbf{r}) = \mathbf{0} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (\text{C.16})$$

And the following subspace is the one that contains the vectorial test functions.

$$H_{\boldsymbol{\omega}} = \{\boldsymbol{\omega}(\mathbf{r})\} \quad \boldsymbol{\omega}(\mathbf{r}) = \{\omega_i(\mathbf{r})\}_{i=1, \dots, n} \quad (\text{C.17})$$

This theorem holds, if the components of the above vectorial test function fulfil two conditions: they have to be null over the boundary and be n times differentiable over the domain.

$$\begin{cases} \omega_i(\mathbf{r}) = 0 & \forall \mathbf{r} \in \partial\Omega \\ \omega_i(\mathbf{r}) \in \mathcal{C}^n(\Omega) \end{cases} \quad (\text{C.18})$$

The fulfilment of the above conditions allows to apply the theorem stated in section C.2.1 to each one of the addends of the scalar product.

$$\begin{aligned} \iiint_{\Omega} \boldsymbol{\omega}^T(\mathbf{r}) \boldsymbol{\psi}(\mathbf{r}) d\Omega &= \sum_{k=1}^n \iiint_{\Omega} \omega_k(\mathbf{r}) \psi_k(\mathbf{r}) d\Omega = 0 \quad \implies \\ \implies \psi_i &= 0 \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \\ \implies \boldsymbol{\psi} &= \{\psi_i\}_{i=1, \dots, n} = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \end{aligned} \quad (\text{C.19})$$

Therefore, when dealing with an integral expression as the one exposed in (C.16), it can be concluded that the vector function (C.15) has to be null over the interior of the domain.

C.3. Weighted residual method

The other fundamental lemma of the calculus of variations is the weighted residual principle. This method allows dealing with an integral expression which is equivalent to a differential one. Once this integral equation is defined, the finite element method can be applied.

C.3.1. Weighted residual method applied to scalar functions

Let's define the following scalar function that depends on a given vector variable, which is null over the interior of the domain.

$$\psi(\mathbf{r}) = 0 \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (\text{C.20})$$

The weighted residual method consists in multiplying the previous function by a scalar test function and integrate the result over the whole domain.

$$\iiint_{\Omega} \omega(\mathbf{r}) \psi(\mathbf{r}) d\Omega = 0 \quad \forall \omega \in H_{\omega} \quad (\text{C.21})$$

Where ω are the so-called test functions that verify:

$$H_{\omega} = \{\omega(\mathbf{r})\}, \quad \begin{cases} \omega(\mathbf{r}) = 0 & \forall \mathbf{r} \in \partial\Omega \\ \omega(\mathbf{r}) \in \mathcal{C}^n(\Omega) \end{cases} \quad (\text{C.22})$$

In conclusion, when dealing with a differential equation as the one defined in (C.20), an equivalent integral one can be obtained by applying the weighted residual method presented in this section.

C.3.2. Weighted residual method applied to vector functions

This method can be extended to vector functions. Let's now consider the following null vector function defined over the interior of the domain.

$$\boldsymbol{\psi}(\mathbf{r}) = \bar{\mathbf{0}} \quad \forall \mathbf{r} \in \overset{\circ}{\Omega} \quad (\text{C.23})$$

In this particular case, the weighted residual method have to be applied as follows. Firstly, the scalar product between the vector test function and the above vector function is computed. Then, the result is integrated over the whole domain. By doing so, an equivalent integral expression is obtained.

$$\iiint_{\Omega} \boldsymbol{\omega}^T(\mathbf{r}) \boldsymbol{\psi}(\mathbf{r}) d\Omega = 0 \quad \forall \boldsymbol{\omega} \in H_{\boldsymbol{\omega}} \quad (\text{C.24})$$

The following subspace is the one that contains the vector test functions.

$$H_{\boldsymbol{\omega}} = \{\boldsymbol{\omega}(\mathbf{r})\} \quad \boldsymbol{\omega}(\mathbf{r}) = \{\omega_i(\mathbf{r})\}_{i=1, \dots, n}, \quad \begin{cases} \omega_i(\mathbf{r}) = 0 & \forall \mathbf{r} \in \partial\Omega \\ \omega_i(\mathbf{r}) \in \mathcal{C}^n(\Omega) \end{cases} \quad (\text{C.25})$$

The fulfilment of the above conditions allows to apply the methodology stated in section C.3.1 to each one of the addends that compose equation (C.24).

$$\iiint_{\Omega} \boldsymbol{\omega}^T(\mathbf{r}) \boldsymbol{\psi}(\mathbf{r}) d\Omega = \sum_{k=1}^n \iiint_{\Omega} \omega_k(\mathbf{r}) \psi_k(\mathbf{r}) d\Omega = 0 \quad (\text{C.26})$$

Therefore, when dealing with a vector function as the one exposed in (C.23), an equivalent integral equation can be defined by applying the methodology presented in this section.

Hyper-volume of a hyper-parallelepiped

D.1. Vector definition with respect to a given basis

Let's consider a n -dimensional space where a basis composed by n linearly independent vectors is defined. The components of these vectors with respect to the canonical basis are:

$$\vec{e}_i = \mathbf{e}_i = \begin{Bmatrix} e^1_i \\ \vdots \\ e^n_i \end{Bmatrix} \iff \mathbf{E} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n] = \begin{bmatrix} e^1_1 & \cdots & e^1_n \\ \vdots & \ddots & \vdots \\ e^n_1 & \cdots & e^n_n \end{bmatrix} \quad (\text{D.1})$$

Therefore, a given vector $\vec{\mathbf{u}}$ can be defined as a linear combination of the vectors that compose the above basis, as:

$$\vec{\mathbf{u}} = \vec{e}_i u^i \iff \vec{\mathbf{u}} = \mathbf{E} \mathbf{u} \quad \text{with} \quad \mathbf{u} = \begin{Bmatrix} u^1 \\ \vdots \\ u^n \end{Bmatrix} \quad (\text{D.2})$$

The above left equation expresses the linear combination in index notation. This notation is ruled by the Einstein Summation Convention [[Einstein, 1916](#)], where lower and upper repeated indexes imply summation. On the right, the same equation is expressed according to the equivalent matrix notation.

D.1.1. Change of basis

Let's assume that the reference basis is modified. The new reference basis is composed by another set of n linearly independent vectors. Each one of these vectors can

be expressed as a linear combination of the ones that compose the original basis as:

$$\begin{aligned} \vec{e}'_\alpha = \vec{e}_i c^i_\alpha &\iff \begin{aligned} \mathbf{E}' &= [\vec{e}'_1 \ \dots \ \vec{e}'_n] \\ &= \begin{bmatrix} e^1_1 & \dots & e^1_n \\ \vdots & \ddots & \vdots \\ e^n_1 & \dots & e^n_n \end{bmatrix} \begin{bmatrix} c^1_1 & \dots & c^1_n \\ \vdots & \ddots & \vdots \\ c^n_1 & \dots & c^n_n \end{bmatrix} = \mathbf{E} \mathbf{C} \end{aligned} \end{aligned} \quad (\text{D.3})$$

Where the coefficients c^i_α can be interpreted as the components of the vectors that compose the new basis, defined with respect to the original one.

D.1.2. Inverse change of basis

The change of basis presented in the previous section can be inverted.

$$\vec{e}_i = \vec{e}'_\alpha \gamma^\alpha_i \iff \mathbf{E} = \mathbf{E}' \mathbf{C}^{-1} \quad (\text{D.4})$$

Where the coefficients γ^i_j are the components of the tensor \mathbf{C}^{-1} .

$$\mathbf{C}^{-1} = [\gamma^i_j]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad \gamma^i_j = \frac{1}{\det(\mathbf{C})} \text{cofactor}(c^j_i) \quad (\text{D.5})$$

And they verify that:

$$\begin{cases} \gamma^\alpha_i c^i_\beta = \delta^\alpha_\beta \\ c^i_\alpha \gamma^\alpha_j = \delta^i_j \end{cases} \iff \begin{cases} \mathbf{C}^{-1} \mathbf{C} = \mathbf{I} \\ \mathbf{C} \mathbf{C}^{-1} = \mathbf{I} \end{cases} \quad (\text{D.6})$$

D.1.3. Change of basis of the contravariant components

If the reference basis is modified, the vector components also vary. In this subsection, this variation is defined.

The vector definition with respect to the original basis, as well as the vectors that compose the modified basis, were previously defined as:

$$\begin{cases} \vec{u} = \vec{e}_i u^i \\ \vec{e}'_\alpha = \vec{e}_i c^i_\alpha \end{cases} \iff \begin{cases} \vec{u} = \mathbf{E} \mathbf{u} \\ \mathbf{E}' = \mathbf{E} \mathbf{C} \end{cases} \quad (\text{D.7})$$

The same vector can be equivalently defined with respect to the new basis, as:

$$\vec{u} = \vec{e}'_\alpha u'^\alpha \iff \vec{u} = \mathbf{E}' \mathbf{u}' \quad (\text{D.8})$$

The substitution of the new basis definition (D.7) into the above equation leads to:

$$\begin{aligned} \vec{u} &= \vec{e}'_\alpha u'^\alpha & \vec{u} &= \mathbf{E}' \mathbf{u}' \\ &= (\vec{e}_i c^i_\alpha) u'^\alpha & &= (\mathbf{E} \mathbf{C}) \mathbf{u}' \\ &= \vec{e}_i \underbrace{(c^i_\alpha u'^\alpha)}_{u^i} & \iff &= \mathbf{E} \underbrace{(\mathbf{C} \mathbf{u}')}_{\mathbf{u}} \\ &= \vec{e}_i u^i & &= \mathbf{E} \mathbf{u} \end{aligned} \quad (\text{D.9})$$

Therefore, the equation that defines the original components by means of the modified ones turns out to be:

$$u^i = c^i_{\alpha} u'^{\alpha} \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{C} \mathbf{u}' \quad (\text{D.10})$$

And the inversion of the above equation allows to define the new components by means of the primal ones.

$$u'^{\alpha} = \gamma^{\alpha}_i u^i \quad \Longleftrightarrow \quad \mathbf{u}' = \mathbf{C}^{-1} \mathbf{u} \quad (\text{D.11})$$

According to the above equation, it can be stated that the expression of a given vector with respect to a different basis implies the variation of the vector components by means of the inverse of the tensor \mathbf{C} . As exposed in (D.3), this tensor is the one involved in the definition of the modified basis. Consequently, it can be concluded that the vector components experiment the opposite variation than the vectors that compose the original basis. This fact justifies denominating these components as the contravariant ones, since their variation is contrary to the variation experimented by the vectors that compose the original basis.

D.2. Scalar product

The scalar product is a tensor operation that involves two vectors and yields a scalar.

Let's consider the two following vectors:

$$\begin{cases} \vec{\mathbf{u}} = \vec{\mathbf{e}}_i u^i \\ \vec{\mathbf{v}} = \vec{\mathbf{e}}_j v^j \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \vec{\mathbf{u}} = \mathbf{E} \mathbf{u} \\ \vec{\mathbf{v}} = \mathbf{E} \mathbf{v} \end{cases} \quad (\text{D.12})$$

The scalar product between the above vectors is defined as:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= (\vec{\mathbf{e}}_i u^i) \cdot (\vec{\mathbf{e}}_j v^j) \\ &= u^i \underbrace{(\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j)}_{g_{ij}} v^j \\ &= u^i g_{ij} v^j \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= (\mathbf{E} \mathbf{u}) \cdot (\mathbf{E} \mathbf{v}) \\ &= (\mathbf{E} \mathbf{u})^T (\mathbf{E} \mathbf{v}) \\ &= \mathbf{u}^T \underbrace{(\mathbf{E}^T \mathbf{E})}_{\mathbf{G}} \mathbf{v} \\ &= \mathbf{u}^T \mathbf{G} \mathbf{v} \end{aligned} \quad (\text{D.13})$$

Where the tensor \mathbf{G} involved in the scalar product computation is the so-called metric tensor.

D.2.1. Metric tensor

The components g_{ij} that compose the above tensor turn out to be the doubly covariant components of the metric tensor.

$$\mathbf{G} = \mathbf{E}^T \mathbf{E} = \left[g_{ij} \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad g_{ij} = \vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \mathbf{e}_i^T \mathbf{e}_j \quad (\text{D.14})$$

According to the above definition, the metric tensor turns out to be symmetric and positive definite.

$$g_{ji} = \vec{e}_j \cdot \vec{e}_i = \vec{e}_i \cdot \vec{e}_j = g_{ij} \quad \Longleftrightarrow \quad \mathbf{G}^T = (\mathbf{E}^T \mathbf{E})^T = \mathbf{E}^T \mathbf{E} = \mathbf{G} \quad (\text{D.15})$$

$$\mathbf{x}^T \mathbf{G} \mathbf{x} = \mathbf{x}^T (\mathbf{E}^T \mathbf{E}) \mathbf{x} = (\mathbf{E} \mathbf{x})^T (\mathbf{E} \mathbf{x}) = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2 > 0 \quad \forall \vec{x} \neq \mathbf{0} \quad (\text{D.16})$$

Note that, if the basis is composed by orthonormal vectors, the metric tensor becomes the second order unit tensor.

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad \Longleftrightarrow \quad \mathbf{G} = \mathbf{I} \quad (\text{D.17})$$

D.2.2. Change of basis of the metric tensor

If the reference basis is modified, the components of the metric tensor vary. In this subsection, this variation is defined.

Let's recall the expression of the metric tensor defined with respect to a given basis (D.14) and the definition of a new basis exposed in (D.3).

$$\begin{cases} g_{ij} = \vec{e}_i \cdot \vec{e}_j \\ \vec{e}'_\alpha = \vec{e}_i c^i_\alpha \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \mathbf{G} = \mathbf{E}^T \mathbf{E} \\ \mathbf{E}' = \mathbf{E} \mathbf{C} \end{cases} \quad (\text{D.18})$$

The metric tensor defined with respect to the modified basis turns out to be:

$$g'_{\alpha\beta} = \vec{e}'_\alpha \cdot \vec{e}'_\beta \quad \Longleftrightarrow \quad \mathbf{G}' = (\mathbf{E}')^T \mathbf{E}' \quad (\text{D.19})$$

The substitution of the new basis definition (D.18) into the above equation leads to:

$$\begin{aligned} g'_{\alpha\beta} &= \vec{e}'_\alpha \cdot \vec{e}'_\beta & \mathbf{G}' &= (\mathbf{E}')^T \mathbf{E}' \\ &= (\vec{e}_i c^i_\alpha) \cdot (\vec{e}_j c^j_\beta) & &= (\mathbf{E} \mathbf{C})^T (\mathbf{E} \mathbf{C}) \\ &= c^i_\alpha \underbrace{(\vec{e}_i \cdot \vec{e}_j)}_{g_{ij}} c^j_\beta & \Longleftrightarrow &= \mathbf{C}^T \underbrace{(\mathbf{E}^T \mathbf{E})}_{\mathbf{G}} \mathbf{C} \end{aligned} \quad (\text{D.20})$$

Therefore, the equation that defines the new components of the metric tensor by means of the initial ones is:

$$g'_{\alpha\beta} = c^i_\alpha g_{ij} c^j_\beta \quad \Longleftrightarrow \quad \mathbf{G}' = \mathbf{C}^T \mathbf{G} \mathbf{C} \quad (\text{D.21})$$

And the inversion of the above equation leads to the equation that defines the original metric tensor by means of the modified one.

$$g_{ij} = \gamma^\alpha_i g'_{\alpha\beta} \gamma^\beta_j \quad \Longleftrightarrow \quad \mathbf{G} = \mathbf{C}^{-T} \mathbf{G}' \mathbf{C}^{-1} \quad (\text{D.22})$$

According to the result obtained in (D.21), it can be stated that the definition of the metric tensor with respect to a modified basis implies the variation of its components by means of the tensor \mathbf{C} . As exposed in (D.3), this tensor is the one involved in the definition of the modified basis. Therefore, the metric tensor suffers the same variation

as the vectors that compose the initial basis. Moreover, since the tensor \mathbf{C} appears twice in the definition of the new metric tensor, it is commonly said that its components suffer a doubly covariant variation.

This fact justifies denominating these components as the doubly covariant ones, since they experiment the same variation as the one experimented by the vectors that compose the original basis.

D.3. Hyper-parallelepiped definition

Let's consider a given basis, composed by the following ν linearly independent vectors defined in a n -dimensional space. These vectors compose a basis of a linear variety whose dimension is $\nu \leq n$.

$$\left\{ \vec{\mathbf{h}}_k \right\}_{k=1, \dots, \nu}, \quad \vec{\mathbf{h}}_k \in \mathbb{R}^n, \quad \nu \leq n \quad (\text{D.23})$$

Let's also define the following k -dimensional hyper-parallelepiped embedded in a n -dimensional space. This geometrical entity is also known as a paralleletope and its corresponding hyper-volume would be represented by V_k .

$$P_k \equiv \left\{ \vec{\mathbf{r}} = \sum_{j=1}^k \vec{\mathbf{h}}_j r^j, \quad 0 \leq r^j \leq 1 \quad \forall j \right\} \quad (\text{D.24})$$

The vector $\vec{\mathbf{r}}$ which defines the above paralleletope is defined with respect to the basis composed by k linearly independent vectors of the previous basis, which, in general, are not orthonormal. Therefore, this basis is assumed to be the primal one, and the components turn out to be the contravariant ones. Lower and upper repeated indices imply summation, according to the Einstein Summation Convention [Einstein, 1916].

D.4. Hyper-parallelepiped embedded in a three-dimensional space

In a three-dimensional space ($n = 3$), the physical interpretation of the above paralleletope is quite simple, specially for the values $k = 1, 2, 3$, which are presented in the following subsections.

D.4.1. Segment

When $k = 1$, the paralleletope P_1 turns out to be a segment.

$$P_1 \equiv \left\{ \vec{\mathbf{r}} = \vec{\mathbf{h}}_1 r^1, \quad 0 \leq r^1 \leq 1 \right\} \quad (\text{D.25})$$

And the hyper-volume is reduced to the norm of the single vector that compose the basis.

$$V_1 = L = \|\vec{\mathbf{h}}_1\| = \sqrt{\vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_1} \quad (\text{D.26})$$

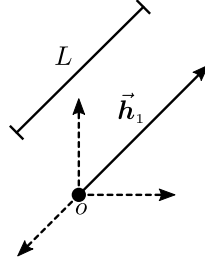


Figure D.1. Segment in a 3-d space (hyper-parallelepiped with $n = 3$ and $k = 1$).

D.4.2. Parallelogram

When $k = 2$, the paralleletope P_2 becomes a parallelogram.

$$P_2 \equiv \left\{ \vec{r} = \sum_{j=1}^2 \vec{h}_j r^j \quad , \quad 0 \leq r^j \leq 1 \quad \forall j \right\} \quad (\text{D.27})$$

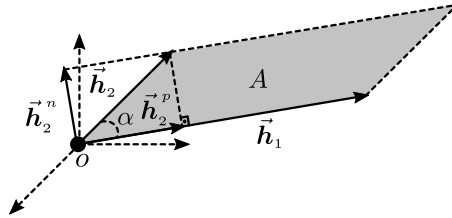


Figure D.2. Parallelogram in a 3-d space (hyper-parallelepiped with $n = 3$ and $k = 2$).

And its corresponding hyper-volume turns out to be the parallelogram area, which can be calculated as the modulus of the vector product between the two vectors that compose the basis. This modulus can be computed according to the equation presented in (A.16).

$$V_2 = A = \|\vec{h}_1 \wedge \vec{h}_2\| = \underbrace{\|\vec{h}_1\|}_{V_1=L} \underbrace{\|\vec{h}_2\| \sin(\alpha)}_g \quad (\text{D.28})$$

Where g is the norm of the component of \vec{h}_2 that is perpendicular to \vec{h}_1 , or equivalently, the norm of the vector \vec{h}_2^n . Moreover, the norm of \vec{h}_1 is the hyper-volume of the paralleletope defined in the previous subsection.

D.4.3. Parallelepiped

Finally, if k adopts the value $k = 3$, the paralleletope P_3 becomes a parallelepiped.

$$P_3 \equiv \left\{ \vec{r} = \sum_{j=1}^3 \vec{h}_j r^j \quad , \quad 0 \leq r^j \leq 1 \quad \forall j \right\} \quad (\text{D.29})$$

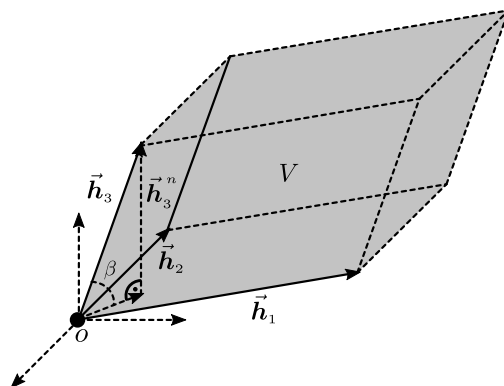


Figure D.3. Parallelepiped in a 3-d space (hyper-parallelepiped with $n = 3$ and $k = 3$).

And the volume of the above parallelepiped can be computed as the triple product between the vectors that compose the basis. Or, equivalently, as the basis area times the parallelepiped height.

$$V_3 = V = (\vec{h}_1 \wedge \vec{h}_2) \cdot \vec{h}_3 = \underbrace{A}_{V_2} \underbrace{\|\vec{h}_3\| |\sin(\beta)|}_h \quad (\text{D.30})$$

Where h represents the parallelepiped height, and can be computed as the norm of the component of \vec{h}_3 that is perpendicular to \vec{h}_1 and \vec{h}_2 . That is, h turns out to be the norm of the vector \vec{h}_3^n . Furthermore, the basis area of the parallelepiped turns out to be equivalent to the area of the parallelepiped defined in the previous subsection.

D.5. Hyper-parallelepiped embedded in n-dimensional space

In this section, the extension of the previous one to a n -dimensional space is presented.

D.5.1. One-dimensional hyper-parallelepiped

With a single vector composing the basis, only a one-dimensional parallelepiped can be defined, whose corresponding hyper-volume turns out to be the length of the segment.

$$V_1 = L = \|\vec{h}_1\| = \sqrt{\vec{h}_1 \cdot \vec{h}_1} \quad (\text{D.31})$$

D.5.2. Two-dimensional hyper-parallelepiped

If the basis is increased and composed by two vectors, the hyper-parallelepiped becomes a two-dimensional one. Its hyper-volume turns out to be its corresponding

area, which can be calculated as follows.

$$V_2 = A = \underbrace{\|\vec{h}_1\|}_{V_1=L} \underbrace{\|\vec{h}_2\| |\sin(\alpha)|}_g = V_1 \times g \quad (\text{D.32})$$

Where g is the norm of the component of \vec{h}_2 that is perpendicular to \vec{h}_1 , and the norm of \vec{h}_1 is the hyper-volume of the previous one-dimensional parallelepiped.

D.5.3. Three-dimensional hyper-parallelepiped

If the basis is finally composed by three vectors, the hyper-parallelepiped becomes a three-dimensional one, whose hyper-volume turns out to be its corresponding volume. This volume can be computed as the basis area times the parallelepiped height, as shown below.

$$V_3 = V = \underbrace{A}_{V_2} \underbrace{\|\vec{h}_3\| |\sin(\beta)|}_h = V_2 \times h \quad (\text{D.33})$$

Where h represents the parallelepiped height, and can be computed as the norm of the component of \vec{h}_3 that is perpendicular to \vec{h}_1 and \vec{h}_2 . Furthermore, the basis area turns out to be equivalent to the area of the parallelepiped defined before.

D.5.4. Higher dimensional hyper-parallelepipeds

Let's consider the following hyper-parallelepiped embedded in a n -dimensional space, defined by means of $k + 1$ linearly independent vectors.

$$\left\{ \vec{h}_j \in \mathbb{R}^n \right\}_{j=1, \dots, k+1} \longrightarrow P_{k+1} \equiv \left\{ \vec{r} = \sum_{j=1}^{k+1} \vec{h}_j r^j \quad , \quad 0 \leq r^j \leq 1 \quad \forall j \right\} \quad (\text{D.34})$$

Let's also define the following k -dimensional linear variety. The vector $\vec{\lambda}$ is defined with respect to the basis composed by k linearly independent vectors of the previous basis.

$$\mathcal{V}_k \equiv \left\{ \vec{\lambda} = \sum_{j=1}^k \vec{h}_j \lambda^j \right\} \quad (\text{D.35})$$

Under the above assumptions, the vector \vec{h}_{k+1} can be expressed as the sum of the following two components.

$$\vec{h}_{k+1} = \vec{h}_{k+1}^p + \vec{h}_{k+1}^n \quad (\text{D.36})$$

- Where \vec{h}_{k+1}^p is defined as the linear combination of the vectors that compose the basis of the linear variety \mathcal{V}_k (D.35).

$$\vec{h}_{k+1}^p = \sum_{j=1}^k \vec{h}_j p^j \quad (\text{D.37})$$

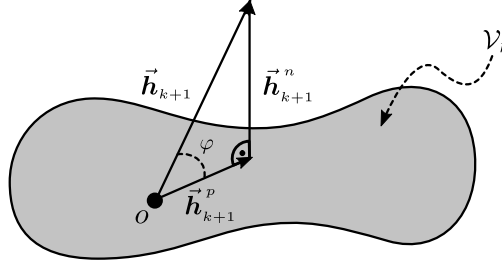


Figure D.4. Hyper-parallelepiped embedded in a n -dimensional space.

- And \vec{h}_{k+1}^n is defined orthogonal with respect to the vectors that compose this basis.

$$\vec{h}_{k+1}^n \cdot \vec{h}_i = 0 \quad i = 1, \dots, k \quad (\text{D.38})$$

According to the results obtained in the previous subsections, it seems reasonable to state that the hyper-volume of the hyper-parallelepiped defined in (D.34) can be computed as:

$$V_{k+1} = V_k \underbrace{\|\vec{h}_{k+1}\|}_{\vec{h}_{k+1}^n} |\sin(\varphi)| = V_k \|\vec{h}_{k+1}^n\| \quad (\text{D.39})$$

Where φ is the angle between \vec{h}_{k+1}^p and \vec{h}_{k+1}^n (figure D.4).

D.6. Hyper-volume of a hyper-parallelepiped

D.6.1. Theorem

Let's consider the following k -dimensional hyper-parallelepiped embedded in a n -dimensional space.

$$\left\{ \vec{h}_j \in \mathbb{R}^n \right\}_{j=1, \dots, k} \longrightarrow P_k \equiv \left\{ \vec{r} = \sum_{j=1}^k \vec{h}_j r^j \quad , \quad 0 \leq r^j \leq 1 \quad \forall j \right\} \quad (\text{D.40})$$

And let's define the following metric tensor:

$$\mathbf{G}_k = [g_{ij}]_{\substack{i=1, \dots, k \\ j=1, \dots, k}} \quad g_{ij} = \vec{h}_i \cdot \vec{h}_j \quad (\text{D.41})$$

In this section, it is proved that the square root of the determinant of the above metric tensor is equivalent to the hyper-volume of the hyper-parallelepiped P_k .

$$V_k = \sqrt{\det(\mathbf{G}_k)} \quad (\text{D.42})$$

Application to a three-dimensional space

If the paralleletores defined in a three-dimensional space are recalled (section D.4), it can be checked that the results obtained according to the previous theorem are consistent with the content presented previously.

For a one-dimensional case, the computation of the determinant of the metric tensor and the calculation of its square root leads to the length of the segment.

$$\left. \begin{array}{l} V_1 = \sqrt{\det(\mathbf{G}_1)} \\ \mathbf{G}_1 = [g_{11}] \end{array} \right\} \implies V_1 = \sqrt{g_{11}} = \sqrt{\vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_1} = \|\vec{\mathbf{h}}_1\| \quad (\text{D.43})$$

If the basis is increased with an additional vector, the square root of the determinant of the metric tensor leads to the area of the parallelogram.

$$\left. \begin{array}{l} V_2 = \sqrt{\det(\mathbf{G}_2)} \\ \mathbf{G}_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \\ \vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_2 = \|\vec{\mathbf{h}}_1\| \|\vec{\mathbf{h}}_2\| \cos(\alpha) \end{array} \right\} \implies \begin{aligned} V_2 &= \sqrt{g_{11} g_{22} - g_{12} g_{21}} \\ &= \sqrt{(\vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_1)(\vec{\mathbf{h}}_2 \cdot \vec{\mathbf{h}}_2) - (\vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_2)^2} \\ &= \sqrt{\|\vec{\mathbf{h}}_1\|^2 \|\vec{\mathbf{h}}_2\|^2 - (\vec{\mathbf{h}}_1 \cdot \vec{\mathbf{h}}_2)^2} \\ &= \sqrt{\|\vec{\mathbf{h}}_1\|^2 \|\vec{\mathbf{h}}_2\|^2 - \|\vec{\mathbf{h}}_1\|^2 \|\vec{\mathbf{h}}_2\|^2 \cos^2(\alpha)} \\ &= \|\vec{\mathbf{h}}_1\| \|\vec{\mathbf{h}}_2\| \sqrt{1 - \cos^2(\alpha)} \\ &= \|\vec{\mathbf{h}}_1\| \|\vec{\mathbf{h}}_2\| \sin(\alpha) \\ &= \|\vec{\mathbf{h}}_1 \wedge \vec{\mathbf{h}}_2\| \end{aligned} \quad (\text{D.44})$$

And finally, if the basis is composed by three vectors, the hyper-parallelepiped becomes a parallelepiped, and the theorem leads to its corresponding volume.

$$\left. \begin{array}{l} V_3 = \sqrt{\det(\mathbf{G}_3)} \\ \mathbf{G}_3 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \\ = \begin{bmatrix} \vec{\mathbf{h}}_1^T \\ \vec{\mathbf{h}}_2^T \\ \vec{\mathbf{h}}_3^T \end{bmatrix} \begin{bmatrix} \vec{\mathbf{h}}_1 & \vec{\mathbf{h}}_2 & \vec{\mathbf{h}}_3 \end{bmatrix} \\ = \mathbf{H}^T \mathbf{H} \end{array} \right\} \implies \begin{aligned} V_3 &= \sqrt{\det(\mathbf{G}_3)} \\ &= \sqrt{\det(\mathbf{H}^T \mathbf{H})} \\ &= \sqrt{\det^2(\mathbf{H})} \\ &= |\det(\mathbf{H})| \\ &= (\vec{\mathbf{h}}_1 \wedge \vec{\mathbf{h}}_2) \cdot \vec{\mathbf{h}}_3 \end{aligned} \quad (\text{D.45})$$

Therefore, in the particular case of dealing with a three-dimensional space, the theorem leads to correct results.

D.6.2. Theorem proof

The following equation, which is defined in terms of the value k , can be proven by induction.

$$V_k = \sqrt{\det(\mathbf{G}_k)} \quad k = 1, 2, \dots \quad (\text{D.46})$$

The above equation is assumed to be true for $k = 1$.

$$V_1 = \sqrt{\det(\mathbf{G}_1)} \quad (\text{D.47})$$

The next step of the mathematical induction is based on demonstrating equation (D.46) for $k + 1$.

$$V_{k+1} = \sqrt{\det(\mathbf{G}_{k+1})} \quad (\text{D.48})$$

Where the above metric tensor can be expressed by means of the previous one, as:

$$\mathbf{G}_{k+1} = \begin{bmatrix} \mathbf{G}_k & \mathbf{g}_{k+1} \\ \mathbf{g}_{k+1}^T & g_{k+1,k+1} \end{bmatrix} \quad \text{with} \quad \mathbf{g}_{k+1} = \begin{Bmatrix} \mathbf{g}_{1,k+1} \\ \mathbf{g}_{2,k+1} \\ \vdots \\ \mathbf{g}_{k,k+1} \end{Bmatrix} \quad (\text{D.49})$$

According to the result obtained in (D.39), the hyper-volume of the parallelepiped P_{k+1} turns out to be:

$$V_{k+1} = V_k \|\vec{\mathbf{h}}_{k+1}^n\| \quad (\text{D.50})$$

If equations (D.36) and (D.37) are taken into account, the above vector $\vec{\mathbf{h}}_{k+1}^n$ can be expressed as:

$$\begin{aligned} \vec{\mathbf{h}}_{k+1}^n &= \vec{\mathbf{h}}_{k+1} - \vec{\mathbf{h}}_{k+1}^p \\ &= \vec{\mathbf{h}}_{k+1} - \sum_{j=1}^k \mathbf{h}_j p^j \end{aligned} \quad (\text{D.51})$$

Moreover, this vector fulfils condition (D.38), so:

$$\vec{\mathbf{h}}_{k+1}^n \cdot \vec{\mathbf{h}}_i = 0 \quad i = 1, \dots, k \quad (\text{D.52})$$

And the substitution of equation (D.51) into the above equation leads to the following result.

$$\begin{aligned} \vec{\mathbf{h}}_{k+1}^n \cdot \vec{\mathbf{h}}_i &= \left(\vec{\mathbf{h}}_{k+1} - \sum_{j=1}^k \mathbf{h}_j p^j \right) \cdot \vec{\mathbf{h}}_i \\ &= \underbrace{\vec{\mathbf{h}}_{k+1} \cdot \vec{\mathbf{h}}_i}_{g_{k+1,i}=g_{i,k+1}} - \sum_{j=1}^k \underbrace{(\vec{\mathbf{h}}_j \cdot \vec{\mathbf{h}}_i)}_{g_{ji}=g_{ij}} p^j \\ &= g_{i,k+1} - \sum_{j=1}^k g_{ij} p^j = 0 \quad i = 1, \dots, k \end{aligned} \quad (\text{D.53})$$

From the above result, it can be stated that:

$$\sum_{j=1}^k g_{ij} p^j = g_{i,k+1} \quad i = 1, \dots, k \quad \iff$$

$$\iff \mathbf{G}_k \mathbf{p} = \mathbf{g}_{k+1} \quad \text{with} \quad \mathbf{p} = \begin{Bmatrix} p^1 \\ \vdots \\ p^k \end{Bmatrix} \quad (\text{D.54})$$

So, the vector \mathbf{p} can be defined as:

$$\mathbf{p} = \mathbf{G}_k^{-1} \mathbf{g}_{k+1} \quad (\text{D.55})$$

On the other hand, the square of the norm of the vector $\vec{\mathbf{h}}_{k+1}^n$ turns out to be:

$$\begin{aligned} \|\vec{\mathbf{h}}_{k+1}^n\|^2 &= \vec{\mathbf{h}}_{k+1}^n \cdot \vec{\mathbf{h}}_{k+1}^n \\ &= \left(\vec{\mathbf{h}}_{k+1} - \sum_{j=1}^k \vec{\mathbf{h}}_j p^j \right) \cdot \left(\vec{\mathbf{h}}_{k+1} - \sum_{l=1}^k \vec{\mathbf{h}}_l p^l \right) \\ &= \underbrace{\vec{\mathbf{h}}_{k+1} \cdot \vec{\mathbf{h}}_{k+1}}_{g_{k+1,k+1}} - \sum_{l=1}^k \underbrace{(\vec{\mathbf{h}}_{k+1} \cdot \vec{\mathbf{h}}_l)}_{g_{k+1,l}} p^l - \sum_{j=1}^k \underbrace{(\vec{\mathbf{h}}_j \cdot \vec{\mathbf{h}}_{k+1})}_{g_{j,k+1}=g_{k+1,j}} p^j + \sum_{j=1}^k \sum_{l=1}^k \underbrace{(\vec{\mathbf{h}}_j \cdot \vec{\mathbf{h}}_l)}_{g_{jl}} p^j p^l \\ &= g_{k+1,k+1} - \sum_{l=1}^k g_{k+1,l} p^l - \sum_{j=1}^k g_{k+1,j} p^j + \sum_{j=1}^k \sum_{l=1}^k p^j p^l g_{jl} \\ &= g_{k+1,k+1} - 2 \mathbf{g}_{k+1}^T \mathbf{p} + \mathbf{p}^T \mathbf{G}_k \mathbf{p} \end{aligned} \quad (\text{D.56})$$

The vector \mathbf{p} obtained in (D.55) can be substituted into the above equation. Thus,

$$\begin{aligned} \|\vec{\mathbf{h}}_{k+1}^n\|^2 &= g_{k+1,k+1} - 2 \mathbf{g}_{k+1}^T \mathbf{p} + \mathbf{p}^T \mathbf{G}_k \mathbf{p} \\ &= g_{k+1,k+1} - 2 \mathbf{g}_{k+1}^T (\mathbf{G}_k^{-1} \mathbf{g}_{k+1}) + (\mathbf{g}_{k+1}^T \mathbf{G}_k^{-T}) \mathbf{G}_k (\mathbf{G}_k^{-1} \mathbf{g}_{k+1}) \\ &= g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{G}_k^{-1} \mathbf{g}_{k+1} \end{aligned} \quad (\text{D.57})$$

And the substitution of the above equation into the square of equation (D.50) leads to:

$$\begin{aligned} V_{k+1}^2 &= V_k^2 \|\vec{\mathbf{h}}_{k+1}^n\|^2 \\ &= \det(\mathbf{G}_k) (g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{G}_k^{-1} \mathbf{g}_{k+1}) \end{aligned} \quad (\text{D.58})$$

As it is demonstrated in the following subsection, the above result turns out to be equivalent to the following determinant.

$$\det(\mathbf{G}_{k+1}) = \det(\mathbf{G}_k) (g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{G}_k^{-1} \mathbf{g}_{k+1}) \quad (\text{D.59})$$

Consequently, equation (D.58) becomes:

$$V_{k+1} = \sqrt{\det(\mathbf{G}_{k+1})} \quad (\text{D.60})$$

And it can be concluded that equation (D.46) is correct and can be applied to obtain the hyper-volume of a given hyper-parallelepiped.

D.6.3. Determinant of the metric tensor

Let's recall the metric tensor exposed in (D.49).

$$\mathbf{G}_{k+1} = \begin{bmatrix} \mathbf{G}_k & \mathbf{g}_{k+1} \\ \mathbf{g}_{k+1}^T & g_{k+1,k+1} \end{bmatrix} \quad (\text{D.61})$$

According to the Cholesky factorization, the tensor \mathbf{G}_k can be decomposed as:

$$\mathbf{G}_k = \mathbf{L}_k \mathbf{L}_k^T \quad (\text{D.62})$$

Where \mathbf{L}_k is a lower triangular matrix.

Analogously, the metric tensor (D.61) can be decomposed as:

$$\mathbf{G}_{k+1} = \mathbf{L}_{k+1} \mathbf{L}_{k+1}^T \quad \text{with} \quad \mathbf{L}_{k+1} = \begin{bmatrix} \mathbf{L}_k & \mathbf{0} \\ \mathbf{l}_{k+1}^T & l_{k+1,k+1} \end{bmatrix} \quad (\text{D.63})$$

The above decomposition can be equivalently written as:

$$\begin{aligned} \mathbf{G}_{k+1} &= \begin{bmatrix} \mathbf{L}_k & \mathbf{0} \\ \mathbf{l}_{k+1}^T & l_{k+1,k+1} \end{bmatrix} \begin{bmatrix} \mathbf{L}_k^T & \mathbf{l}_{k+1} \\ \mathbf{0}^T & l_{k+1,k+1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{L}_k \mathbf{L}_k^T & \mathbf{L}_k \mathbf{l}_{k+1} \\ \mathbf{l}_{k+1}^T \mathbf{L}_k^T & \mathbf{l}_{k+1}^T \mathbf{l}_{k+1} + l_{k+1,k+1}^2 \end{bmatrix} \end{aligned} \quad (\text{D.64})$$

If the above tensor and its initial definition (D.61) are compared, it can be concluded that:

$$\begin{aligned} \mathbf{L}_k \mathbf{l}_{k+1} = \mathbf{g}_{k+1} &\iff \mathbf{l}_{k+1} = \mathbf{L}_k^{-1} \mathbf{g}_{k+1} \\ g_{k+1,k+1} = \mathbf{l}_{k+1}^T \mathbf{l}_{k+1} + l_{k+1,k+1}^2 &\iff l_{k+1,k+1}^2 = g_{k+1,k+1} - \mathbf{l}_{k+1}^T \mathbf{l}_{k+1} \end{aligned} \quad (\text{D.65})$$

From the above equations, it can be also stated that:

$$\begin{aligned} l_{k+1,k+1}^2 &= g_{k+1,k+1} - \mathbf{l}_{k+1}^T \mathbf{l}_{k+1} \\ &= g_{k+1,k+1} - (\mathbf{L}_k^{-1} \mathbf{g}_{k+1})^T (\mathbf{L}_k^{-1} \mathbf{g}_{k+1}) \\ &= g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{L}_k^{-T} \mathbf{L}_k^{-1} \mathbf{g}_{k+1} \\ &= g_{k+1,k+1} - \mathbf{g}_{k+1}^T (\mathbf{L}_k \mathbf{L}_k^T)^{-1} \mathbf{g}_{k+1} \\ &= g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{G}_k^{-1} \mathbf{g}_{k+1} \end{aligned} \quad (\text{D.66})$$

If the Cholesky factorization of the tensors \mathbf{G}_k and \mathbf{G}_{k+1} , as well as the structure of the lower triangular tensor \mathbf{L}_{k+1} shown in (D.63) are taken into account, the computation of the following determinant can be faced as:

$$\begin{aligned} \det(\mathbf{G}_{k+1}) &= \det(\mathbf{L}_{k+1} \mathbf{L}_{k+1}^T) \\ &= \left(\det(\mathbf{L}_{k+1}) \right)^2 \\ &= \left(\det(\mathbf{L}_k) \right)^2 l_{k+1,k+1}^2 \\ &= \det(\mathbf{L}_k \mathbf{L}_k^T) l_{k+1,k+1}^2 \\ &= \det(\mathbf{G}_k) l_{k+1,k+1}^2 \end{aligned} \quad (\text{D.67})$$

And the substitution of the result obtained in (D.66) into the above equation, allows to obtain the following equivalent definition of the determinant.

$$\det(\mathbf{G}_{k+1}) = \det(\mathbf{G}_k)(g_{k+1,k+1} - \mathbf{g}_{k+1}^T \mathbf{G}_k^{-1} \mathbf{g}_{k+1}) \quad (\text{D.68})$$

Therefore, the assumption made in (D.59) is correct, and the hyper-volume of a given hyper-parallelepiped can be properly computed according to equation (D.60).

Resumen extendido en castellano

E.1. Introducción

El método de los elementos finitos es una tecnología bien conocida que permite obtener una aproximación al comportamiento estructural real de un medio sólido continuo sometido a fuerzas externas. Su uso está ampliamente extendido en ingeniería civil y en muchos otros campos, como la ingeniería naval o la aeronáutica.

Esta formulación puede obtenerse bajo el marco de los análisis lineal o no lineal. Si se supone que los desplazamientos y sus correspondientes gradientes son pequeños, el análisis se simplifica considerablemente, y resulta realizarse bajo los supuestos de la teoría lineal. Sin embargo, si los desplazamientos y/o los gradientes de los desplazamientos se consideran grandes, surge el análisis no lineal.

Como ambos análisis se basan en supuestos diferentes, conducen a respuestas estructurales completamente distintas. Y la exactitud de los resultados depende de la precisión de las hipótesis realizadas. Es decir, si la estructura no experimenta pequeños desplazamientos o pequeños gradientes de desplazamiento, el análisis lineal conduce a resultados inaceptables que difieren significativamente del comportamiento real.

Antes de llevar a cabo una simulación estructural, el ingeniero tiene que decidir, basándose en su experiencia e intuición, si los supuestos lineales son correctos. Si la respuesta estructural real no verifica las hipótesis lineales, hay que descartar el análisis lineal y realizar uno no lineal para obtener resultados precisos.

Por lo tanto, las hipótesis adoptadas acerca de la magnitud tanto de los desplazamientos como de los gradientes de los desplazamientos son muy importantes, ya que definen el marco teórico del análisis estructural. Es necesario definir claramente las implicaciones de cada supuesto. En la literatura existente, la mayoría de las referencias no identifica claramente las implicaciones de estos supuestos. Por lo tanto, uno de los principales objetivos de este trabajo es identificarlas claramente y definir adecuadamente los modelos matemáticos lineales y no lineales que rigen el comportamiento estructural asociado a cada análisis.

Para lograr este objetivo, se propone una formulación unificadora de la mecánica de sólidos lineal y no lineal completa y detallada. Esta formulación permite describir y comprender completamente la deformación que experimenta un sólido elástico a lo largo del tiempo. Se propone una nomenclatura novedosa, sencilla y clara para enunciar adecuadamente los principios de la mecánica de sólidos y las ecuaciones estrictamente necesarias que describen este proceso de deformación.

Una vez que los modelos matemáticos están bien planteados, se puede aplicar el método de los elementos finitos. Se presenta una obtención original completa tanto en teoría lineal como no lineal. Se presenta también el desarrollo lineal para compararlo con su versión no lineal.

Una de las principales diferencias entre ambas formulaciones radica en la forma de aplicar las fuerzas externas. En general, la formulación lineal conduce a un comportamiento lineal, mientras que la no lineal conduce a uno no lineal. Mientras la respuesta sea lineal, la carga total puede aplicarse en un solo paso, y el principio de superposición de cargas de la teoría lineal puede aplicarse adecuadamente. Sin embargo, este principio ya no se puede aplicar cuando se trata con un comportamiento no lineal. Si la respuesta es no lineal, un estado de carga dado tiene múltiples soluciones posibles. Por lo tanto, la carga total no puede aplicarse en un solo paso, y hay que tener en cuenta el historial de carga para obtener la solución correcta. Para solventar estos inconvenientes, las cargas externas suelen aplicarse según un proceso de carga incremental.

Esta estrategia incremental es en realidad un procedimiento adecuado, ya que la respuesta estructural correspondiente a cada paso de carga debe resolverse de forma iterativa. Este procedimiento necesita empezar a iterar desde una aproximación cercana a la solución. Si las cargas incrementales son lo suficientemente pequeñas, el resultado del paso de carga anterior puede adoptarse para iniciar el procedimiento iterativo, y la convergencia debería estar garantizada.

Numerosos libros de texto de referencia y trabajos de investigación abordan la obtención de las formulaciones de elementos finitos no lineales. Sin embargo, no existe consenso sobre una nomenclatura y notación comunes. Además, las hipótesis formuladas a lo largo de estos desarrollos no se especifican claramente o ni siquiera se enuncian. Por lo tanto, para comprender completamente la física subyacente y la esencia de los algoritmos propuestos, se hace necesaria una visión más detallada que lo aclare.

En esta tesis se hace un gran esfuerzo por identificar claramente las hipótesis intermedias y analizar ampliamente el origen y la composición de las matrices que surgen en el análisis no lineal. Se elabora una guía detallada que facilita el aprendizaje profundo de esta potente tecnología. Este trabajo plantea una formulación unificadora, clara y completa en el campo del análisis no lineal, para que la extensión de algunas líneas de investigación que hasta ahora se han llevado a cabo en teoría lineal sea posible.

E.2. Deformación de un medio sólido sometido a cargas externas

Para el estudio del movimiento de un medio sólido continuo sometido a fuerzas externas se adopta el enfoque Lagrangiano. Es decir, todas las magnitudes se definen con respecto a la configuración inicial de referencia, que es conocida.

Se obtienen dos ecuaciones principales: la ecuación que define el movimiento de una partícula material y la ecuación que describe la transformación geométrica de un vector material a lo largo del tiempo. Estas ecuaciones son importantes, ya que describen el movimiento del sólido, así como el cambio de volumen, orientación y forma que experimenta debido a las cargas externas.

Por una parte, el vector movimiento (“deformation vector” en inglés) es el que describe el movimiento del sólido. Este vector describe la posición de una partícula material a lo largo del tiempo a través de su posición inicial y del vector desplazamiento.

Por otro lado, la transformación geométrica experimentada por un vector material resulta depender del tensor gradiente de movimientos, el cual, a su vez, puede definirse mediante el tensor gradiente de desplazamientos. De acuerdo con este resultado, se puede concluir que el tensor gradiente de desplazamientos es el que contiene la información necesaria para definir el cambio de volumen, orientación y forma experimentado por el sólido. Esta información es necesaria para definir adecuadamente el campo de desplazamiento que experimenta el sólido, así como sus correspondientes campos de deformación y tensión.

Otro de los objetivos principales es el de definir una nomenclatura novedosa, sencilla y clara, así como manejar las magnitudes y ecuaciones estrictamente necesarias. El objetivo final es alcanzar una visión general de la mecánica de sólidos no lineal completa y detallada, para poder plantear más adelante una novedosa obtención de la formulación de elementos finitos no lineales.

E.3. Ecuaciones de balance

Se han deducido las ecuaciones que establecen el equilibrio de masa, momento lineal y momento angular. Las ecuaciones de balance de estas tres magnitudes rigen el comportamiento estructural de un medio sólido continuo sometido a cargas externas.

Por una parte, la ecuación de balance de masa es la ecuación que gobierna el valor de la densidad a lo largo del tiempo. La variación de la densidad del medio está provocada por las fuentes de masa y la variación de volumen a lo largo del proceso de deformación. Como primera aproximación, se considera la existencia de fuentes de masa. Sin embargo, más adelante en este trabajo se dejarán de considerar y se adoptará la hipótesis de que se cumple la conservación de la masa.

Por otro lado, la ecuación de equilibrio del momento lineal resulta ser la ecuación que gobierna el equilibrio dinámico de fuerzas. La variación temporal del momento lineal está causada por las fuerzas externas aplicadas, así como por el campo de velocidades

de la fuente de masa. La ecuación de equilibrio del momento angular, a su vez, es la que rige el equilibrio dinámico de los momentos. Como el momento angular se define en función del momento lineal, su variación temporal se origina por las mismas causas que producen la variación del momento lineal.

Una vez definida la variación temporal de una magnitud dada (masa, momento lineal o momento angular), es sencillo obtener su correspondiente ecuación de equilibrio escrita en forma integral. Y al operar sobre esta forma integral, se llega a su forma diferencial equivalente. Además, se pueden obtener las versiones Lagrangiana o Euleriana de estas ecuaciones de balance, dependiendo de la descripción de las magnitudes adoptada.

E.4. Campo de tensiones

El vector tensión se define con respecto a un plano determinado, por lo que pueden definirse un número infinito de vectores tensión en un punto que pertenece a la configuración deformada. Se define como el diferencial de fuerza que actúa sobre la configuración deformada por unidad de área diferencial. Obsérvese que esta fuerza diferencial depende del plano considerado, y que el área diferencial pertenece a este plano.

Según esta definición, el vector tensión resulta ser una fuerza por unidad de superficie que depende del punto material y del tiempo. Además, también depende del vector normal que define el plano, ya que la dirección de la fuerza de tracción depende del plano considerado.

El balance de momento lineal aplicado a un tetraedro que pertenece al dominio material deformado conduce a la definición del vector tensión de Cauchy. Esta ecuación define el vector tensión como el producto entre el tensor de tensiones de Cauchy y el vector unitario normal que define el plano con respecto al cual se define el vector tensión.

En un punto dado del material, el tensor de tensiones de Cauchy resulta estar compuesto por las componentes de los vectores tensión definidos con respecto a los planos cartesianos. Este tensor representa el campo de tensiones experimentado por el sólido, definido como fuerzas internas por unidad de superficie. Si se aplica una rotación al dominio deformado, se modifican las componentes del tensor de tensiones de Cauchy.

Pueden definirse tensores de tensiones alternativos, si se modifica la configuración de referencia. Al definir el vector tensión de Cauchy, el dominio deformado es la configuración de referencia. Es decir, las tensiones se definen como fuerzas por unidad de área deformada. Sin embargo, si el dominio inicial pasa a ser el de referencia, surge el primer tensor de tensiones de Piola-Kirchhoff. Este tensor de tensiones permite definir un vector tensión referido al dominio inicial, que es conocido. Este vector tensión alternativo se define como el diferencial de fuerza que actúa sobre la configuración deformada por unidad de área inicial diferencial. La principal desventaja de este tensor

es su no simetría, y su uso para representar el campo de tensiones suele rechazarse, ya que conduce a matrices de rigidez no simétricas.

Si el diferencial de fuerza se multiplica por la inversa del tensor gradiente de movimientos, surge el segundo tensor de tensiones de Piola-Kirchhoff. Aunque este tensor define otro tipo de vector tensión que no tiene una interpretación física clara, cumple algunas propiedades interesantes y útiles cuando nos enfrentamos a un comportamiento estructural de grandes desplazamientos. Define un vector que representa una fuerza de tracción diferencial modificada (el producto entre la inversa del tensor gradiente de movimientos y el diferencial de fuerza) por unidad de área diferencial inicial. Obsérvese que el dominio inicial se adopta de nuevo como referencia. El segundo tensor de tensiones de Piola-Kirchhoff resulta ser simétrico y no varía si se aplica una rotación a un dominio deformado dado.

E.5. Campo de deformaciones finitas

Para comprender la deformación que experimenta un sólido en las proximidades de un determinado punto, se analiza en detalle la transformación geométrica de un vector material. Esta transformación resulta estar gobernada por el tensor gradiente de movimientos, que, a su vez, depende del tensor gradiente de desplazamientos. Por tanto, se puede concluir que el tensor gradiente de desplazamientos contiene la información necesaria para definir el cambio de volumen, orientación y forma que experimenta el sólido. Esta información es necesaria para definir correctamente el campo de desplazamientos, así como el de deformaciones y tensiones.

La descomposición polar del tensor gradiente de movimientos conduce a una interpretación física más clara de la transformación geométrica sufrida por el vector material. Permite descomponer el tensor gradiente de movimientos como el producto entre el tensor de rotación finita y el tensor de deformación finita. En primer lugar, el tensor de deformación finita modifica el módulo y la dirección del vector material, a través de una elongación pura. A continuación, el tensor de rotación finito rota el vector material previamente modificado para orientarlo de acuerdo con la geometría deformada.

También puede considerarse una descomposición polar alternativa. Esta descomposición divide el tensor de deformación finita como el producto entre el tensor de inflación y el tensor de distorsión. El tensor de inflación rige el cambio de volumen en las proximidades de una partícula dada, mientras que el tensor de distorsión es el que gobierna el proceso de distorsión sin variación de volumen.

Otro tensor que cumple propiedades interesantes es el tensor de deformaciones de Green-Lagrange. Este tensor surge cuando se calcula la diferencia entre el cuadrado de la norma de un determinado vector material en un instante de tiempo dado y el cuadrado de la norma de su geometría inicial. El tensor de deformaciones de Green-Lagrange no varía si se aplica una rotación al sólido. Esta es una propiedad importante cuando se trata con un sólido que se comporta con grandes desplazamientos y/o grandes

gradientes de desplazamiento, ya que una rotación rígida no produce una variación del campo de deformaciones. Además, este tensor resulta ser conjugado del segundo tensor de tensiones de Piola-Kirchhoff, que también es invariante cuando se aplican rotaciones. Así, su producto contracto conduce al trabajo por unidad de volumen desarrollado por las fuerzas internas durante el proceso de deformación. Estos tensores constituyen un par de magnitudes adecuadas para representar los campos de deformación y tensión cuando se realiza un análisis no lineal. En consecuencia, es necesario definir una ecuación constitutiva que relacione ambas magnitudes.

En un análisis no lineal se suele adoptar un proceso de carga incremental. Dado que un estado de carga dado tiene múltiples soluciones posibles, la carga total no puede aplicarse en un solo paso, y hay que tener en cuenta el historial de cargas para alcanzar la respuesta estructural correcta. Además, el conjunto de ecuaciones no lineales que rigen el comportamiento estructural suele resolverse mediante un procedimiento iterativo que necesita partir de una aproximación cercana a la solución. El proceso de carga incremental permite partir de una aproximación cercana, así como tener en cuenta el historial de cargas.

Por último, pueden adoptarse dos enfoques de análisis, en función del dominio material que se tome como configuración de referencia. El enfoque total referencia todas las magnitudes a la configuración inicial durante todo el proceso de carga incremental, mientras que el actualizado actualiza el dominio de referencia en cada paso de carga. Es decir, el dominio material calculado en el paso de carga anterior se convierte en el de referencia en cada paso incremental.

E.6. Campo de deformaciones infinitesimales

La transformación geométrica de un vector material resulta ser infinitesimal si los gradientes de los desplazamientos son pequeños. Nótese que una estructura que se comporta con pequeños gradientes de desplazamientos puede sufrir grandes desplazamientos. Por tanto, puede concluirse que una respuesta estructural con pequeños gradientes de desplazamientos no implica un comportamiento con pequeños desplazamientos.

Si los gradientes de los desplazamientos son pequeños, puede aplicarse con precisión la descomposición polar aproximada para descomponer el tensor gradiente de movimientos. Esta descomposición define el tensor gradiente de movimientos como el producto entre el tensor de rotación infinitesimal y la deformación infinitesimal. Ambos tensores resultan estar definidos mediante el tensor gradiente de desplazamientos. Por lo tanto, si los gradientes de los desplazamientos son pequeños, no es necesario resolver un problema de valores propios para obtener la descomposición polar del tensor gradiente de movimientos, como pasa en el caso finito. Esto supone una gran ventaja, ya que implica un menor coste computacional.

La descomposición polar aclara la interpretación física de la transformación geométrica sufrida por el vector material. En primer lugar, la deformación infinitesimal modifica el módulo y la dirección del vector material a través de una elongación infinitesimal

pura. A continuación, el tensor de rotación infinitesimal rota el vector material previamente modificado para orientarlo según la geometría deformada. Ambos efectos resultan ser transformaciones geométricas infinitesimales.

Además, se puede plantear una descomposición polar alternativa. Esta descomposición divide el efecto de la deformación infinitesimal en una inflación infinitesimal, seguida de una distorsión infinitesimal. La inflación se rige por el tensor de inflación infinitesimal, que provoca una variación de volumen en las proximidades de una partícula dada. Mientras que el efecto de distorsión se rige por el tensor de distorsión infinitesimal, que produce una distorsión sin variación de volumen.

La descomposición polar del tensor gradiente de movimientos es una potente herramienta para analizar sólidos que experimentan grandes desplazamientos y/o grandes gradientes de desplazamientos. Si los gradientes de los desplazamientos son pequeños, el análisis estructural se simplifica y la descomposición polar puede obtenerse directamente a través del tensor gradiente de desplazamientos.

E.7. Planteamiento incremental

Hasta ahora se ha analizado ampliamente la transformación geométrica experimentada por un vector material dado entre $t = 0$ y t . Tanto si la variación es finita como infinitesimal, ambos cambios geométricos resultan estar regidos por el tensor gradiente de movimientos. La descomposición polar permite definir el tensor gradiente de movimientos como el producto entre un tensor de rotación y un tensor de deformación.

No obstante, el enfoque incremental se centra en la transformación geométrica incremental entre dos pasos de tiempo infinitesimales consecutivos. La transformación geométrica experimentada por un vector material entre t y $t + dt$ resulta estar regida por el tensor gradiente de velocidad. Como esta variación geométrica es infinitesimal, se puede aplicar con precisión la descomposición polar aproximada para descomponer el tensor que gobierna esta transformación geométrica incremental. Según la descomposición aproximada, este tensor puede definirse como el producto entre una rotación incremental y una deformación incremental.

Además, la deformación incremental puede definirse de forma equivalente como una inflación incremental seguida de una distorsión incremental. La inflación incremental rige la variación incremental de volumen en las proximidades de un punto dado, mientras que la distorsión incremental no implica variación de volumen.

Este enfoque incremental puede ampliarse para comprender el proceso de carga incremental que suele llevarse a cabo en el análisis no lineal. Cuando se trata de un procedimiento de carga incremental, la variable temporal t se convierte en una variable que indica el paso de carga actual. Si la carga incremental es lo suficientemente pequeña en cada paso del proceso de carga, la transformación geométrica incremental de un vector material dado se rige según esta formulación.

E.8. Ecuaciones constitutivas

En este trabajo se analiza el comportamiento estructural de los sólidos elásticos. El valor del campo de tensiones de un sólido elástico solo depende del valor del tensor gradiente de movimientos. Si se tiene en cuenta la descomposición polar del tensor gradiente de movimientos, se puede afirmar de forma equivalente que el tensor de tensiones depende tanto del tensor de rotación finita como del tensor de deformación finita. Además, el tensor de tensiones no depende de los valores anteriores del tensor gradiente de movimientos. Es decir, solo depende del valor del tensor gradiente de movimientos en un instante de tiempo dado. Por tanto, si se eliminan las cargas externas aplicadas, se recupera completamente la configuración inicial.

La definición del tensor de tensiones se simplifica en la teoría de la elasticidad lineal. La hipótesis principal consiste en considerar un campo de deformaciones infinitesimal, lo que no implica necesariamente ni pequeños desplazamientos ni pequeños gradientes de desplazamientos. Esta hipótesis permite simplificar la definición del tensor de tensiones de Cauchy, ya que el término que depende del campo de deformaciones puede linealizarse en términos del tensor de deformaciones infinitesimales. Además, si los gradientes de los desplazamientos son pequeños, el efecto de la rotación infinitesimal puede despreciarse, y el tensor de tensiones de Cauchy puede definirse finalmente sólo mediante el tensor de deformaciones infinitesimales. Por lo tanto, la ecuación que relaciona ambas magnitudes se puede linealizar, y el tensor que define esta relación lineal es el llamado tensor constitutivo lineal.

Si no se cumplen los supuestos de la elasticidad lineal, hay que establecer una definición adecuada del tensor de tensiones de Cauchy mediante el tensor de rotación y el tensor de deformaciones. No obstante, pueden adoptarse magnitudes tensoriales más adecuadas para definir los campos de deformación y tensión.

Tanto el segundo tensor de tensiones de Piola-Kirchhoff como el tensor de deformaciones de Green-Lagrange permanecen invariantes si se aplica al sólido un movimiento rígido (rotación y/o traslación). Esta es una propiedad importante a tener en cuenta cuando se realiza un análisis de grandes desplazamientos. Si el sólido experimenta grandes desplazamientos, con gradientes de desplazamientos grandes o pequeños, estos tensores pueden adoptarse para describir adecuadamente el campo de deformaciones y su correspondiente campo de tensiones. Por lo tanto, es necesario enunciar una ecuación constitutiva que defina su relación matemática.

Para desarrollar un análisis no lineal, es esencial definir la ecuación constitutiva incremental, ya que el análisis se llevará a cabo mediante un procedimiento de carga incremental. Por lo tanto, hay que establecer la ecuación que define el incremento del segundo tensor de tensiones de Piola-Kirchhoff en función del incremento del tensor de deformaciones de Green-Lagrange. Si se supone que los gradientes del incremento del campo de desplazamientos entre pasos de carga consecutivos son pequeños, su correspondiente incremento de deformaciones de Green-Lagrange resulta ser pequeño. Así, la relación entre ambas magnitudes tensoriales puede definirse a través de una

relación lineal definida por el tensor constitutivo.

Si el campo de deformaciones es infinitesimal, incluso si los desplazamientos y/o los gradientes de los desplazamientos son grandes, la relación entre el segundo tensor de tensiones de Piola-Kirchhoff y el tensor de deformaciones de Green-Lagrange puede aproximarse con precisión a través de una relación lineal definida por el tensor constitutivo lineal. Es decir, el mismo tensor constitutivo que establece la relación lineal entre el tensor de tensiones de Cauchy y el tensor de deformaciones infinitesimales define la relación entre el segundo tensor de tensiones de Piola-Kirchhoff y el tensor de deformaciones de Green-Lagrange. Se trata de la denominada ecuación constitutiva de St. Venant-Kirchhoff, que es uno de los modelos constitutivos más sencillos.

E.9. Ecuaciones de equilibrio estático

En Mecánica de Sólidos, no es habitual tratar con fuentes de masa, por lo que se suele considerar que la masa no varía. Si se tiene en cuenta el supuesto de conservación de la masa, la ecuación que define el valor de la densidad a lo largo del tiempo puede obtenerse a partir de la forma diferencial Lagrangiana de la ecuación de balance de masa. La densidad del medio resulta depender de la densidad inicial y del determinante del tensor gradiente de movimientos.

Además, si se analiza el equilibrio estático, hay que verificar el equilibrio de fuerzas y momentos.

Bajo los supuestos de equilibrio estático y conservación de la masa, la forma diferencial Euleriana de la ecuación de balance de momento lineal conduce a la ecuación que rige el equilibrio estático de fuerzas. Esta ecuación puede obtenerse de forma equivalente si se tiene en cuenta la conservación del momento lineal. Es decir, la derivada temporal del momento lineal tiene que ser cero. La sustitución de esta derivada temporal en la forma integral Euleriana de la ecuación de balance de momento lineal permite obtener alternativamente la ecuación que rige el equilibrio estático de fuerzas.

Además, se considera que el material no es capaz de absorber momento angular por unidad de volumen. Por tanto, se verifica la conservación del momento angular, y la variación temporal del momento angular tiene que ser nula. La sustitución de esta derivada temporal en la forma integral Euleriana de la ecuación de balance del momento angular conduce a la simetría del tensor de tensiones de Cauchy. En consecuencia, puede concluirse que el tensor de tensiones de Cauchy tiene que ser simétrico si se cumple la conservación del momento angular.

E.10. Formulación lineal de elementos finitos

E.10.1. Estrategia de análisis lineal

Se consideran dos tipos de condiciones de contorno. Las primeras suelen conocerse como condiciones de contorno esenciales, y se basan en la definición del campo de

desplazamientos en una superficie específica del sólido. El segundo tipo son las denominadas condiciones de contorno naturales, que definen el vector tensión en la superficie sobre la que se aplican las cargas externas.

Si el análisis se realiza en teoría lineal, se adoptan dos hipótesis principales que simplifican considerablemente el análisis.

- Se considera que los desplazamientos que experimenta el sólido son pequeños. Esta hipótesis permite considerar que el dominio inicial y el deformado son coincidentes. Así, las ecuaciones de equilibrio pueden imponerse sobre la configuración de referencia, que es conocida. Esto implica también que no hay variación de volumen entre ambas configuraciones.
- Además, se supone que los gradientes de los desplazamientos son pequeños. Esta suposición permite aplicar la descomposición polar aproximada para descomponer el tensor gradiente de movimientos mediante el tensor gradiente de desplazamientos. Sin esta suposición, hay que resolver un problema de valores propios para obtener su descomposición polar. Esta simplificación resulta ser una gran ventaja, ya que reduce el coste computacional.

No obstante, si el comportamiento estructural real no verifica los supuestos del análisis lineal, los resultados obtenidos con esta teoría no se corresponden con la respuesta estructural real.

Como se supone que la masa y el volumen permanecen constantes, también se considera que la densidad no varía. Por lo tanto, la densidad no es una incógnita a calcular si se adopta el enfoque lineal, ya que la densidad inicial es conocida y permanece constante.

También se han deducido las ecuaciones que rigen el equilibrio estático, tanto la ecuación que rige el equilibrio de fuerzas como la que establece el equilibrio de momentos. El tensor de tensiones de Cauchy resulta ser simétrico si se verifica la conservación del momento angular. La conservación del momento angular implica que la suma de todos los momentos es cero. Por tanto, la ecuación que rige el equilibrio de momentos se reduce a la simetría del tensor de tensiones de Cauchy.

La ecuación de compatibilidad es la que define el campo de tensiones en función del campo de desplazamientos. Según la descomposición polar aproximada, el campo de deformaciones puede aproximarse a través del tensor de deformaciones infinitesimales, que se define completamente mediante el tensor gradiente de desplazamientos. Como los gradientes de los desplazamientos son pequeños y el campo de deformaciones depende del tensor gradiente de desplazamientos, el campo de deformaciones resulta ser infinitesimal en el análisis lineal.

Y la ecuación que define la relación entre los campos de tensiones y deformaciones es la ecuación constitutiva. Como el campo de deformaciones es infinitesimal, y los gradientes de los desplazamientos son pequeños, el efecto de la rotación puede despreciarse para definir el tensor de tensiones de Cauchy. En consecuencia, el tensor de tensiones

puede definirse completamente a través del tensor de deformaciones infinitesimales. Es decir, se puede aplicar la ecuación de Lamé para definir la ecuación constitutiva.

Todas las ecuaciones mencionadas anteriormente componen el modelo matemático que permite obtener la respuesta estructural bajo los supuestos del análisis lineal. La principal incógnita a resolver es el campo de desplazamientos que experimenta el sólido. Una vez conocido, la ecuación de compatibilidad permite calcular el campo de deformaciones a partir del campo de desplazamientos. Y por último, la ecuación constitutiva define el campo de tensiones que corresponde al campo de deformaciones anterior. Además, una vez definido el campo de tensiones, y en caso de que sea necesario, se puede calcular la reacción que aparece en la superficie donde se aplica la condición de contorno esencial.

E.10.2. Análisis lineal por el método de los elementos finitos

El análisis lineal por elementos finitos permite obtener el campo de desplazamientos experimentado por el sólido tras la aplicación de las fuerzas externas, bajo los supuestos de pequeños desplazamientos y pequeños gradientes de desplazamientos.

Se obtienen tanto la forma fuerte como su correspondiente forma débil. La definición de la base de funciones de prueba permite construir una aproximación del campo de desplazamientos, que puede definirse como una aproximación inicial más un término adicional que corrige la anterior. En general, la diferencia entre la solución exacta y la aproximación inicial del campo de desplazamientos no puede ser generada por el subespacio compuesto por la base de funciones de prueba. Por lo tanto, la aproximación del campo de desplazamientos no suele verificar la forma débil. Para obtener una solución que sí verifique la forma débil, se propone una base de funciones de test que permita generar una función de test aproximada que conduzca a una forma débil aproximada.

El problema de equilibrio original se sustituye entonces por un sistema de ecuaciones lineales. Su matriz correspondiente suele denominarse matriz de rigidez, y el vector independiente es el denominado vector de fuerzas. Si la aproximación inicial cumple la condición de contorno esencial, y las funciones de prueba son nulas sobre la superficie donde se aplica esta condición de contorno, la aproximación del campo de desplazamientos verifica automáticamente la condición de contorno esencial. Además, si las funciones de prueba son nulas sobre la superficie donde se aplica la condición esencial de contorno, la reacción no interviene en la formulación.

Se suele aplicar el método Bubnov-Galerkin. Este método se basa en la aplicación de las mismas funciones para definir las funciones de prueba y de test, lo que conduce a matrices de rigidez simétricas.

Una vez definido completamente el sistema de ecuaciones lineales resultante, el dominio material se divide en un conjunto de elementos finitos. Esto permite reescribir las integrales como la suma de las integrales definidas sobre el dominio correspondiente a cada elemento finito. Por lo tanto, las componentes de la matriz de rigidez y el vector de fuerzas se convierten en la suma de la contribución de cada elemento finito.

Además, cada función de prueba puede definirse asociada a un nodo determinado. La función adopta un valor unitario en su nodo correspondiente, mientras que se hace nula en los restantes. Esta condición simplifica la interpretación física del resultado obtenido del sistema de ecuaciones lineales, ya que resulta representar el desplazamiento experimentado por el nodo. Por otra parte, las funciones de prueba y las de test pueden definirse con soporte local. Si se definen nulas sobre una gran cantidad de elementos finitos, se obtienen algunas ventajas computacionales, como una reducción significativa del tiempo de cálculo y de la memoria de almacenamiento requerida.

Por último, las funciones de prueba deben cumplir una condición esencial que suele denominarse partición de la unidad para poder abordar correctamente las traslaciones. Si no se cumple esta condición, una traslación implica la existencia de un campo de tensiones, lo cual es incorrecto y no tiene sentido físico.

E.11. Formulación no lineal de elementos finitos

E.11.1. Estrategia de análisis no-lineal

En análisis no lineal, se asume que el sólido experimenta grandes desplazamientos y grandes gradientes de desplazamientos.

- La hipótesis de grandes desplazamientos implica que la configuración de referencia y el dominio deformado no pueden considerarse coincidentes. Así, las ecuaciones de equilibrio deben imponerse sobre el dominio deformado, que es la incógnita a resolver. Como las configuraciones inicial y deformada no se consideran equivalentes, el determinante del tensor gradiente de movimientos no es igual a uno, y las descripciones Lagrangiana y Euleriana de una magnitud dada no son coincidentes.
- Además, se supone que los gradientes de los desplazamientos son grandes, por lo que la descomposición polar aproximada del tensor gradiente de movimientos no es aceptable. Para descomponer adecuadamente el tensor gradiente de movimientos, hay que resolver un problema de valores propios. Una vez resuelto este problema, pueden definirse el tensor de rotación y el tensor de deformación, y el tensor gradiente de movimientos puede descomponerse como el producto entre ambos tensores. Además, el tensor de tensiones de Cauchy puede expresarse en función de ambos tensores.

Este trabajo se centra en el análisis estático. Es decir, el sólido adopta una configuración deformada en equilibrio estático tras la aplicación de las cargas externas. Por lo tanto, deben cumplirse tanto el equilibrio de fuerzas como el de momentos. Se recuerda que el tensor de tensiones de Cauchy es simétrico si se verifica la conservación del momento angular. Si se conserva el momento angular, puede afirmarse que la suma de todos los momentos es cero, y se cumple el equilibrio de momentos.

Se supone que las fuerzas externas aplicadas no dependen del campo de desplazamiento. Esta suposición puede hacerse cuando se trata de casos de carga habituales, como la sobrecarga estructural de uso o las cargas de peso propio.

Además, también se considera que la masa no varía. Por lo tanto, la conservación de la masa conduce a la ecuación que define el valor de la densidad a lo largo del tiempo. La densidad resulta estar definida tanto por el valor inicial de la densidad como por el determinante del tensor gradiente de movimientos.

Además de las ecuaciones de equilibrio, hay que definir adecuadamente las condiciones de contorno para obtener el modelo matemático. Se distinguen dos tipos de condiciones de contorno: las condiciones de contorno esenciales y las naturales. Las condiciones de contorno esenciales se basan en la definición del campo de desplazamientos en una superficie específica del sólido, mientras que las condiciones de contorno naturales definen el vector tensión en la superficie sobre la que se aplican las fuerzas superficiales externas.

Por último, hay que enunciar una ecuación de compatibilidad y otra constitutiva para completar el modelo matemático. La ecuación de compatibilidad define el campo de deformaciones en función del campo de desplazamientos, mientras que la ecuación constitutiva establece el campo tensiones en función del campo de deformaciones.

Como la respuesta estructural no es lineal, un estado de carga dado podría tener múltiples soluciones. Por lo tanto, la carga total no puede aplicarse en un solo paso de carga. Si no se tiene en cuenta el orden de aplicación de las cargas externas, puede obtenerse un comportamiento estructural que no se corresponda con el real. Además, el principio de superposición que se aplica habitualmente en el análisis lineal deja de ser válido. Para solventar los problemas que plantea el tratamiento de una respuesta estructural no lineal, se plantea un proceso de carga incremental.

Además, un procedimiento de carga incremental es un método adecuado, ya que la forma débil que conduce a la formulación de elementos finitos resulta ser una ecuación no lineal que depende del campo de desplazamientos. Esta ecuación tiene que resolverse a través de un método iterativo, que necesita empezar a iterar desde una aproximación cercana a la solución. La aplicación de una pequeña carga incremental en cada paso permite adoptar la solución del paso de carga anterior como aproximación inicial a la solución del paso de carga de actual.

E.11.2. Formulación Lagrangiana total de elementos finitos

La forma original del problema está compuesta por las ecuaciones que componen el modelo matemático, escritas en términos de los residuos tanto de la ecuación de equilibrio como de la condición de contorno natural. Esta forma original suele denominarse forma fuerte, y a continuación se aplica el método de los residuos ponderados para obtener su correspondiente forma débil.

Bajo los supuestos del análisis no lineal, la forma débil resulta ser Euleriana. Es decir, está compuesta por integrales definidas sobre el dominio deformado desconocido, y las magnitudes que intervienen en ella se describen según su descripción Euleriana.

Este es un problema importante, ya que no se puede realizar un cálculo sobre un dominio material desconocido. Para solventar este inconveniente, la forma débil Euleriana puede reescribirse en una forma Lagrangiana equivalente.

Según el resultado obtenido, la forma débil Lagrangiana equivalente está compuesta por integrales definidas sobre la configuración inicial conocida, y las magnitudes se describen de acuerdo con su descripción Lagrangiana. En la forma débil Lagrangiana resultante intervienen el segundo tensor de tensiones de Piola-Kirchhoff y la variación del tensor de deformaciones de Green-Lagrange.

Dado que el segundo tensor de tensiones de Piola-Kirchhoff y el tensor de deformaciones de Green-Lagrange son magnitudes conjugadas, parece razonable adoptar como ecuación constitutiva la ecuación que define la relación entre ambas magnitudes. Ambas magnitudes permanecen constantes si se aplica un movimiento rígido (rotación y/o traslación) al sólido. Esta es una propiedad importante para abordar un análisis de grandes desplazamientos, ya que se puede garantizar que no aparecerán tensiones ni deformaciones adicionales cuando se aplique un movimiento de sólido rígido.

Por lo tanto, se adopta el tensor de deformaciones de Green-Lagrange para describir el campo de deformaciones, y el segundo tensor de tensiones de Piola-Kirchhoff representa su correspondiente campo de tensiones. Y la ecuación que define el tensor de deformaciones de Green-Lagrange en función del tensor gradiente de desplazamiento se convierte entonces en la ecuación de compatibilidad, mientras que la ecuación que establece la relación entre el segundo tensor de tensiones de Piola-Kirchhoff y el tensor de deformaciones de Green-Lagrange es la ecuación constitutiva adoptada en este análisis no lineal.

La forma débil Lagrangiana resulta ser una ecuación no lineal cuya incógnita es el campo de desplazamiento. Esta ecuación no lineal debe resolverse para obtener el campo de desplazamiento que experimenta el sólido. Para realizar correctamente el análisis no lineal se aplica un procedimiento de carga incremental. Las fuerzas externas se aplican de acuerdo con el proceso de carga incremental, y el dominio material inicial se convierte en la configuración de referencia con respecto a la cual se definen todas las variables.

La principal incógnita a obtener es el campo de desplazamiento que experimenta el sólido. Una vez conocido el campo de desplazamiento, la ecuación de compatibilidad permite definir su correspondiente campo de deformaciones, y la ecuación constitutiva conduce al campo de tensiones correspondiente al campo de deformaciones anterior. A continuación, se puede calcular, si es necesario, la reacción que aparece en la superficie donde se aplica la condición de contorno esencial.

La raíz de la ecuación no lineal puede obtenerse a través de un método iterativo. El método Newton-Raphson es una opción adecuada que garantiza la convergencia cuadrática si la aproximación inicial es cercana a la raíz. Se supone que la carga incremental después de cada paso de carga es tan pequeña que los campos de desplazamientos asociados a pasos de carga consecutivos pueden considerarse lo suficientemente cercanos. Si se cumple esta suposición, el campo de desplazamiento obtenido en el paso

de carga anterior puede adoptarse para iniciar el procedimiento iterativo en cada paso de carga. Esto permite iniciar el método iterativo desde una aproximación cercana a la raíz, y la convergencia cuadrática debería estar garantizada.

El procedimiento iterativo se detiene según un determinado criterio, que puede basarse tanto en la diferencia entre el campo de desplazamiento obtenido tras iteraciones sucesivas como en el residuo obtenido tras cada iteración.

El método Newton-Raphson conduce a un sistema de ecuaciones lineales que hay que resolver para obtener el campo de desplazamiento correspondiente a cada paso del proceso iterativo. La matriz que compone este sistema es la denominada rigidez tangente, que está compuesta por la suma de dos términos: la componente material y la geométrica. La componente material depende de las propiedades mecánicas del material, y la componente geométrica depende tanto del estado tensional como de la geometría del dominio de referencia.

Una vez resuelto el sistema de ecuaciones lineales, puede actualizarse el campo de desplazamiento, y debe verificarse el criterio de convergencia. Si se alcanza la convergencia, se detiene el procedimiento iterativo, y el último resultado se adopta como el campo de desplazamiento correspondiente al paso de carga actual. En caso contrario, se repite el proceso hasta alcanzar la convergencia.

Cabe señalar que el método Newton-Raphson puede no converger si la matriz de rigidez tangente es singular. Para hacer frente a estos puntos límite, se han propuesto técnicas numéricas para superar con éxito estos puntos límite.

E.11.3. Formulación Lagrangiana actualizada de elementos finitos

La formulación Lagrangiana total de elementos finitos se basa en la adopción del dominio inicial como configuración de referencia a lo largo de todo el proceso de carga incremental. La forma débil que conduce a la formulación de elementos finitos resulta ser Lagrangiana. Es decir, está compuesta por integrales definidas sobre la configuración no deformada que es conocida, y las magnitudes que intervienen en ella se describen según sus descripciones Lagrangianas.

Sin embargo, ahora se adopta el enfoque actualizado en lugar del total. Según este punto de vista actualizado, el último dominio calculado se convierte en la nueva configuración de referencia. Una vez resuelto cada paso de carga, el último dominio calculado se convierte en el de referencia para el siguiente paso de carga.

La obtención de la formulación de elementos finitos correspondiente a este enfoque actualizado es análoga a la total. La forma débil del paso de carga actual está compuesta por integrales definidas sobre el dominio material desconocido, y las magnitudes se describen según su descripción Euleriana. El verdadero reto consiste en reescribir esta forma débil para obtener integrales definidas sobre el dominio material correspondiente al paso de carga anterior, que se ha convertido en la nueva configuración de referencia, así como magnitudes descritas con respecto a este dominio. La forma débil resultante resulta ser una ecuación no lineal que debe resolverse de forma iterativa. La aplicación de un método iterativo junto con el método de los elementos finitos conduce a la

obtención de un sistema de ecuaciones lineales, que ha de ser resuelto para obtener el campo de desplazamientos en cada iteración.

La matriz que define este sistema de ecuaciones lineales es la denominada rigidez tangente, ya que geoméricamente puede interpretarse como la tangente de la respuesta estructural no lineal. Esta rigidez resulta estar compuesta por la suma de dos componentes: la componente material y la rigidez geométrica. La componente material de la rigidez tangente depende de las propiedades mecánicas del material, mientras que la componente geométrica depende tanto del estado tensional como de la geometría del dominio de referencia.

La rigidez tangente puede calcularse una sola vez, si se mantiene la misma rigidez tangente durante todo el proceso iterativo. Este método evita actualizar la rigidez tangente en cada paso iterativo, por lo que se ahorra una gran cantidad de coste computacional. No obstante, pueden ser necesarias más iteraciones para alcanzar la convergencia. Hay que decidir si se aplica este método simplificado o el general. Por regla general, merece la pena probar el método modificado. Si surgen problemas de convergencia, hay que tener en cuenta el general, en el que el cálculo de la rigidez tangente debe llevarse a cabo en cada paso del proceso iterativo.

Tanto el enfoque total como el actualizado conducen a sistemas de ecuaciones lineales análogos. No obstante, cabe destacar las siguientes diferencias entre las componentes de la rigidez tangente:

- La componente material de la rigidez tangente es similar a la matriz de rigidez lineal. Sin embargo, las derivadas de las funciones de prueba se toman con respecto a la geometría del paso de carga anterior, en lugar de tomarse con respecto a la configuración inicial. Además, este término se define mediante el tensor constitutivo espacial, en lugar del constitutivo lineal.
- Además, la rigidez geométrica es bastante similar a la obtenida con el planteamiento Lagrangiano Total. Sin embargo, intervienen las tensiones de Cauchy en lugar de las segundas de Piola-Kirchhoff. Y las derivadas de las funciones de prueba se toman con respecto a la geometría del paso de carga anterior, en lugar de tomarse con respecto a la inicial.

Una vez resuelto el sistema de ecuaciones lineales, hay que actualizar el campo de desplazamientos, y debe verificarse el criterio de convergencia. Si se alcanza la convergencia, se detiene el procedimiento iterativo, y el último resultado se adopta como el campo de desplazamiento correspondiente al paso de carga actual. En caso contrario, se repite el proceso hasta alcanzar la convergencia.

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