ON STRONGLY INFLEXIBLE MANIFOLDS

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Abstract. An oriented closed connected N-manifold M is inflexible if it does not admit self-maps of unbounded degree. In addition, if all the maps from any other oriented closed connected N-manifold have bounded degree, then M is said to be strongly inflexible. The existence of simply-connected inflexible manifolds was established by Arkowitz and Lupton. However, the existence of simply-connected strongly inflexible manifolds is still an open question. We provide an algorithm relying on Sullivan models that allows us to prove that all, but one, of the known examples of simply-connected inflexible manifolds are not strongly inflexible.

1. Introduction

Let Mfd_N be the class of oriented closed connected manifolds of dimension N. Given $M \in \mathrm{Mfd}_N$ we are interested in the set

$$\deg(M) \overset{\text{def}}{:=} \{\deg(f) \mid f\colon M \to M \text{ continuous}\} \subset \mathbb{Z}.$$

Constant maps and the identity map show that $0, 1 \in \deg(M)$, and $-1 \in \deg(M)$ unless M is not strongly chiral [16]. Moreover, $\deg(M)$ is a multiplicative semi-group, hence if there exists any $\ell \in \deg(M)$ such that $|\ell| > 1$, then $\deg(M)$ is unbounded. We say that M is *inflexible* if $\deg(M)$ is bounded.

The existence of inflexible manifolds is intimately related to the existence of nontrivial functorial seminorms (see [9] for more details). An important example is the functorial l^1-seminorm, studied by Gromov, which is nontrivial on the fundamental class of oriented closed connected hyperbolic manifolds. Hence, hyperbolic manifolds are inflexible. However, there exist inflexible manifolds where either the l^1-seminorm is trivial or not known [18, 19]. In particular, the l^1-seminorm is trivial on the nonzero homology classes of simply-connected spaces, thus raising Gromov's question [12, 5.35 Remarks.(b)] on whether finite nontrivial functorial seminorms on singular homology exist for simply-connected spaces. A step towards understanding this problem is to ask whether simply-connected inflexible manifolds do exist. The first example in literature of a simply-connected inflexible manifold is established by Arkowitz and Lupton [2, Example 5.1, 5.3 Remarks] using Rational homotopy theory techniques. Every example of a simply-connected inflexible manifold constructed at the time of writing has been built using Sullivan models [1, 6, 7, 8, 9].

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Suppose now that M and M' are in Mfd_N and let

$$\deg(M', M) \stackrel{\text{def}}{:=} \{\deg(f) \,|\, f\colon M' \to M \text{ continuous}\} \subset \mathbb{Z}.$$

Map composition results in a multiplicative action of the semigroup $\deg(M)$ on the set $\deg(M', M)$. Therefore $0 \in \deg(M', M)$, and if there exists a nonzero $d \in \deg(M', M)$ and M is not inflexible, then $\deg(M', M)$ is unbounded. We say that M is *strongly inflexible* if for every N-manifold M' the set $\deg(M', M)$ is bounded. The bound might depend on M', for instance, if we take the connected sum of n copies of M, $M' = \#_n M$, there exists a degree n map $f : M' \to M$. Observe that if M is strongly inflexible, then M is automatically inflexible.

The interesting aspect of a strongly inflexible manifold M, is that a domination Mfd_N-seminorm associated with M can be defined [9, Definition 7.1]

$$v_M(M') \stackrel{\text{def}}{:=} \sup \big\{ |d| \,|\, d \in \deg(M', M) \big\},$$

which can be extended to a nontrivial finite functorial seminorm in singular homology of simply-connected spaces. Hence, if strongly inflexible manifolds were to exist, Gromov's open question [12, 5.35 Remarks.(b)] would be settled in the positive, and mapping degree theorems for simply-connected manifolds could be deduced. However, no example of simply-connected strongly inflexible manifolds are known.

The condition of being inflexible or strongly inflexible can be both translated to the language of Sullivan algebras for simply-connected (or even nilpotent) spaces. Let $(\Lambda V, d)$ be a Sullivan algebra such that its cohomology $H(\Lambda V, d)$ is a Poincaré duality algebra of formal dimension N ([10, Section 38]). Suppose that $V^1 = 0$, and let $\nu \in (\Lambda V)^N$ be a representative of the fundamental class. Then $(\Lambda V, d)$ is inflexible if for every dga morphism

$$\varphi\colon (\Lambda V, d) \to (\Lambda V, d)$$

we have $\deg(\varphi) = 0, \pm 1$, where $H^N([\nu]) = \deg(\varphi)[\nu]$. Moreover, the Sullivan algebra $(\Lambda V, d)$ is strongly inflexible if for every Sullivan algebra $(\Lambda W, d)$ with Poincaré duality cohomology of formal dimension N, the set of mapping degrees

$$(\Lambda V, d) \to (\Lambda W, d)$$

is bounded. In this paper, using the notion of (strongly) inflexibility in dgas and its relation with the (non) existence of positive weights (see Definition 2.1), a criteria for a manifold $M \in \mathrm{Mfd}_N$ to be not strongly inflexible is given:

Theorem A (Theorem 3.6)**.** *Let $M \in \mathrm{Mfd}_N$, $N \geq 4$, with minimal model $(\Lambda V, d)$ and let η be its cohomological fundamental class. Write the rational cohomological fundamental class as $\eta \otimes_{\mathbb{Q}} 1 = [\nu]$. Assume there exists a dga morphism $\psi\colon (\Lambda V, d) \to (\mathcal{A}, d)$ where $(\mathcal{A}, d)$ is a simply-connected finite type dga with positive weight and $H^N(\psi)([\nu]) \neq 0$. Then M is not strongly inflexible.*

Applying Theorem A we prove that *all but one* of the known examples of simply-connected inflexible dgas and manifolds, as given in [1, 2, 6, 7, 8, 9], are not strongly inflexible (see Remark 5.5 for the example we have not yet been able to prove that is not strongly inflexible):

Theorem B (Theorems 5.4 and 6.4, Remark 6.5)**.** *Let $\mathcal{M}$ be one of the inflexible dgas from* [1, Example 3.8]*,* [2, Examples 5.1 and 5.2]*,* [6, Definition 1.1]*,* [7, Definition 4.1]*,* [8, Definition 2.1]*,* [9, Examples I.1–I.4]*. Then, $\mathcal{M}$ is not strongly inflexible. Furthermore, any manifold for which $\mathcal{M}$ is a Sullivan model is not strongly inflexible either.*

These results lead us to raise the following conjecture, that would imply that all finite functorial seminorms vanish on simply-connected manifolds:

Conjecture 1.1. *Simply-connected strongly inflexible manifolds do not exist.*

In [17, Section 5.3] it is asked whether simply-connected closed inflexible N-manifolds can be dominated by a product $X_1 \times X_2$ where one of the factors is not an inflexible manifold. If so, then Conjecture 1.1 would be true.

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2. Weights

We work with differential graded algebras (dga) with coefficients over $\mathbb{Q}$. In this section we discuss the key ingredient in this work, namely, the relationship between weights in a dga and the strong inflexibility property. Our definition of weight is taken from [3, 4].

Definition 2.1. *Let $(\mathcal{A}, d)$ be a connected dga. We say that $(\mathcal{A}, d)$ has a weight ω if each degree component $\mathcal{A}^k$ of $\mathcal{A}$ decomposes as a direct sum*

$$\mathcal{A}^k = \bigoplus_{n\in\mathbb{Z}} {}_n\mathcal{A}^k, \qquad k \geq 0,$$

such that

(i) $d({}_n\mathcal{A}^k) \subset {}_n\mathcal{A}^{k+1}$*, and*
(ii) ${}_n\mathcal{A}^k \wedge {}_m\mathcal{A}^l \subset {}_{n+m}\mathcal{A}^{k+l}$.

Every nonzero $x \in {}_n\mathcal{A} = \bigoplus_{k\geq 0} {}_n\mathcal{A}^k$ is said to have weight n, and it is denoted by $\omega(x) = n$. An element $x \in \mathcal{A}$ is said ω-homogeneous if there exists $n \in \mathbb{Z}$ such that $x \in {}_n\mathcal{A}$.

We now give some notions of weights.

Definition 2.2. *Let $(\mathcal{A}, d)$ be a connected dga with a weight ω.*

(i) *Given $a \in \mathcal{A}$ we say that ω detects a if a is ω-homogeneous and $\omega(a) \neq 0$.*
(ii) *We say that ω is nontrivial if it detects some nonzero element $a \in \mathcal{A}$.*
(iii) *We say that ω is positive if ${}_n\mathcal{A} = 0$ for every $n < 0$ and ${}_0\mathcal{A} = \mathcal{A}^0$.*

Lemma 2.3. *Let $(\mathcal{A}, d)$ be a connected dga with a weight ω. Then, there is an induced weight in $H(\mathcal{A})$, also denoted by ω, given by $w([a]) = w(a)$ for every w-homogeneous element $a \in \mathcal{A}$. Hence, given $x \in H(\mathcal{A}, d)$, there exists a decomposition $x = \sum_{i=0}^{r} [a_i]$, where the elements $a_i \in \mathcal{A}$ are ω-homogeneous cocyles, and $\omega(a_i) = \omega(a_j)$ if and only if $i = j$.*

Proof. Notice that we can decompose $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} {}_n\mathcal{A}$ where each $({}_n\mathcal{A}, d)$ is a differential graded module. Therefore $H(\mathcal{A}, d) = \bigoplus_{n\in\mathbb{Z}} H({}_n\mathcal{A}, d)$ which induces a weight ω in $H(\mathcal{A}, d)$ by defining ${}_nH(\mathcal{A}, d) := H({}_n\mathcal{A}, d)$. □

Corollary 2.4. *Let $(\mathcal{A}, d)$ be a connected dga with Poincaré duality cohomology of formal dimension N. If $(\mathcal{A}, d)$ has a weight ω, then there exists a w-homogeneous cocycle ν such that $[\nu]$ is the fundamental class of $H^N(\mathcal{A})$.*

Proof. According to Lemma 2.3, we have $[\nu] = \sum_{i=0}^{r}[\nu_i]$, where the elements $\nu_i \in \mathcal{A}$ are ω-homogeneous cocycles, and $\omega(\nu_i) = \omega(\nu_j)$ if and only if $i = j$. Since $H^N(\mathcal{A}, d) \cong \mathbb{Q}$, and $H^N(\mathcal{A}, d)$ has an induced weight, it must be that only one i_0 satisfies $[\nu_{i_0}] \neq 0$. Hence $[\nu] = [\nu_{i_0}]$ with ν_{i_0} anf ω-homogeneous cocycle. □

Definition 2.5. *Let $(\mathcal{A}, d)$ be a connected dga. A class $x \in H(\mathcal{A}, d)$ is said flexible if for every $n \in \mathbb{Z}$ there exists a dga morphism $f_n \colon \mathcal{A} \to \mathcal{A}$ such that $H(f_n)(x) = \lambda\, x$ with $\lambda \geq n$.*

The existence of nontrivial weights implies the existence of flexible classes.

Lemma 2.6. *Let $(\mathcal{A}, d)$ be a connected dga, and $[a] \in H(\mathcal{A}, d)$. If $(\mathcal{A}, d)$ has a weight ω that detects the element $a \in \mathcal{A}$, then $[a]$ is a flexible class.*

Proof. For $q \in \mathbb{Q}$, let $\varphi_q \colon (\mathcal{A}, d) \to (\mathcal{A}, d)$ be the following dga morphism: for y an ω-homogeneous element, $\varphi_q(y) = q^{\omega(y)}y$. Otherwise, we decompose $y = \sum_{i=0}^{r} y_i$ into ω-homogeneous elements and $\varphi_q(y) = \sum_{i=0}^{r} q^{\omega(y_i)}y_i$.

Now, for every $n \in \mathbb{Z}$, since ω detects a, $\omega(a) \neq 0$ and therefore there exists $q_n \in \mathbb{Q}$ such that $q_n^{\omega(a)} \geq n$. By defining $f_n = \varphi_{q_n} : \mathcal{A} \to \mathcal{A}$ we get that $H(f_n)([a]) = q_n^{\omega(a)}\,[a]$ with $q_n^{\omega(a)} \geq n$. Hence $[a]$ is a flexible class. □

Corollary 2.7. *Let $(\Lambda V, d)$ be the Sullivan model of a simply-connected dga $(\mathcal{A}, d)$ with Poincaré duality cohomology of formal dimension N. If $(\mathcal{A}, d)$ has a weight ω that detects a representative of the fundamental class, then $(\Lambda V, d)$ is not inflexible.*

Proof. Since ω detects ν, then by Lemma 2.6 the fundamental class $[\nu]$ is a flexible class. Then, for every $n \in \mathbb{Z}$ there exists a dga morphism $f_n \colon \mathcal{A} \to \mathcal{A}$ such that $\deg(f_n) = \lambda \geq n$. Finally, according to the Lifting Lemma [10, Lemma 12.4], f_n lifts to $\widetilde{f_n}$, a self-morphism of $(\Lambda V, d)$ whose degree is $\deg(\widetilde{f_n}) = \deg(f_n) = \lambda \geq n$, and therefore $(\Lambda V, d)$ is not inflexible. □

Proposition 2.8. *Let $(\mathcal{A}, d)$ be a simply-connected, finite type dga. For any nonzero class $[\nu] \in H^N(\mathcal{A}, d)$, $N \geq 4$, there is a dga $(\bar{\mathcal{A}}, d)$ whose cohomology is Poincaré duality of formal dimension N, and a dga morphism*

$$q \colon (\mathcal{A}, d) \to (\bar{\mathcal{A}}, d),$$

such that $H^N(q)([\nu]) \neq 0$ is a fundamental class for $H^*(\bar{\mathcal{A}}, d)$.

Proof. We take a simply-connected finite type CW-complex X whose Sullivan model is isomorphic to $(\mathcal{A}, d)$, [22, Theorem 10.2 (ii)]. Let X' be the $(N+1)$-skeleton of X, which is a finite complex. We have that $H^{\leq N}(X';\mathbb{Q}) \cong H^{\leq N}(X;\mathbb{Q}) \cong H^{\leq N}(\mathcal{A}, d)$. By [21, Théorèmes III.4], the cohomology class $[\nu] \in H^N(X', \mathbb{Q})$ is detected by an oriented closed N-manifold M, or in other words, there exists a map $f\colon M \to X'$ such that $H^N(f,\mathbb{Q})([\nu]) = a[M]_{\mathbb{Q}}$, $a \in \mathbb{Q}$ nontrivial, where $[M]_{\mathbb{Q}}$ denotes the rational cohomological fundamental class of M. Moreover, we can assume that M is simply-connected (see [9, Corollary 3.2]).

Let $(\bar{\mathcal{A}}, d)$ be a model of M. The cohomology of $(\bar{\mathcal{A}}, d)$ is a Poincaré duality algebra of formal dimension N. Then $f'\colon M \to X' \subset X$ is represented by a dga morphism $q\colon (\mathcal{A}, d) \to (\bar{\mathcal{A}}, d)$ such that $H^N(q)([\nu]) \neq 0$. □

Remark 2.9. The assumption of simply-connectedness in Proposition 2.8 is not necessary. The dga $(\mathcal{A}, d)$ can be considered nilpotent. Hence, the space X is nilpotent too and by [9, Theorem 3.1(2)], we can assume that there exists a continuous map $f : M \to X'$ such that $\pi_1(M) \to \pi_1(X')$ is an isomorphism. Therefore M is also nilpotent and it admits a model.

Corollary 2.10. *Let $(\mathcal{A}, d)$ be a connected dga with Poincaré duality cohomology and fundamental class $[\nu] \in H^N(\mathcal{A}, d)$, $N \geq 4$. Suppose that there exists a dga $(\mathcal{A}', d)$ and a morphism $\varphi\colon (\mathcal{A}, d) \to (\mathcal{A}', d)$ such that $H^N(\varphi)([\nu]) \neq 0$. If $(\mathcal{A}', d)$ has a positive weight ω, then $(\mathcal{A}, d)$ is not strongly inflexible.*

Proof. By Lemma 2.3, we decompose $H^N(\varphi)([\nu]) = \sum_{i=0}^{r} [a_i']$ where a_i' are ω-homogeneous cocycles and $\omega(a_i') \neq \omega(a_j')$ if $i \neq j$. Assume that $\omega(a_0') = \max\{\omega(a_i') | i = 0, \dots, r\}$. Notice that since ω is a positive weight, it detects all the ω-homogeneous elements in positive degree.

In the proof of Lemma 2.6 we have shown how to construct, for every $n \in \mathbb{N}$, a morphism $f_n\colon (\mathcal{A}', d) \to (\mathcal{A}', d)$ verifying that $H^N(f_n)([a_i']) = q_n^{\omega(a_i')}[a_i']$ for every i, where q_n is a natural number satisfying that $q_n^{\omega(a_0')} \geq n$.

Now, associated to $(\mathcal{A}', d)$, by Proposition 2.8 there is a dga $(\bar{\mathcal{A}}, d)$ whose cohomology is a Poincaré duality algebra of formal dimension N, and a dga morphism $q\colon (\mathcal{A}', d) \to (\bar{\mathcal{A}}, d)$ such that $H^N(q)([a_0']) \neq 0$ is a fundamental class, let us call it $[\mu]$, for $H^*(\bar{\mathcal{A}}, d)$.

Finally, take $G_n = q \circ f_n \circ \varphi\colon (\mathcal{A}, d) \to (\bar{\mathcal{A}}, d)$, which in cohomology satisfies that:

$$H^N(G_n)([\nu]) = (q_n^{\omega(a_0')} + \sum_{1}^{r} q_n^{\omega(a_i')} \alpha_i)[\mu], \ \alpha_i \in \mathbb{Q}.$$

Now, observe $P(x) = x^{\omega(a_0')} + \sum_{1}^{r} x^{\omega(a_i')}\alpha_i$ is a rational monic polinomial of degree $\omega(a_0')$. As $q_n^{\omega(a_0')} \geq n$, the set $\{\deg(G_n) | \, n \in \mathbb{N}\}$ (which coincides with $\{P(q_n) | \, q_n^{\omega(a_0')} \geq n \in \mathbb{N}\}$) is unbounded. □

Proposition 2.11. *Every 2-step Sullivan algebra admits a positive weight.*

Proof. A 2-step Sullivan algebra is of the form $(\Lambda(V_1 \oplus V_2), d)$ such that $d(V_1) = 0$, and $d(V_2) \subset \Lambda V_1$. Let $\{x_i \mid i \in I\}$ be a degree homogeneous basis of V_1 and $\{y_j \mid j \in J\}$

a degree homogeneous basis of V_2. Then a positive weight ω is defined by declaring the elements of the basis ω-homogeneous of weight

$$\omega(x_i) = |x_i|,\ \omega(y_j) = |y_j| + 1$$

and extending this weight to monomials in $(\Lambda(V_1 \oplus V_2), d)$ according to the rule in Definition 2.1(ii). □

3. Universal spaces

In this section we develop the necessary tools to extend our previous results on non-strong inflexibility from dgas to manifolds. Recall that given p, a prime or zero, a map $f\colon X \to Y$ is said to be a p-equivalence if f induces an isomorphism on $H^*(X;\mathbb{Z}/p) \cong H^*(Y;\mathbb{Z}/p)$. Here $\mathbb{Z}/0 \overset{\text{def}}{:=} \mathbb{Q}$.

Combining [5, 14, 15], a finite CW-complex X is said to be universal if for every p, for any given p-equivalence $k\colon Y \to Z$, and for every map $g\colon X \to Z$, there is a map $h\colon X \to Y$ and there is a p-equivalence $f\colon X \to X$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \overset{k}{\longrightarrow} & Z. \end{array}$$

Universal spaces are characterized by their minimal models ([5, Theorem A]):

Theorem 3.1. *Let X be a simply-connected finite CW-complex, and $(\Lambda V, d)$ be its minimal model. Then X is universal if and only if $(\Lambda V, d)$ admits a positive weight.*

The cohomology of universal spaces is generated by flexible classes.

Theorem 3.2. *Let M be a manifold in Mfd_N with cohomological fundamental class $\eta \in H^N(M;\mathbb{Z})$. Assume that there exists a map $g\colon X \to M$ such that X is a simply-connected finite universal CW-complex, $H^N(X;\mathbb{Q}) \cong \mathbb{Q}$, and $H^N(g;\mathbb{Q})(\eta_{\mathbb{Q}}) \neq 0$, where $\eta_{\mathbb{Q}} = \eta \otimes_{\mathbb{Q}} 1$. Then M is not strongly inflexible.*

Proof. Let $(\Lambda V, d)$ be the Sullivan minimal model of X, which by Theorem 3.1 admits a positive weight ω. Then, by Lemma 2.3, there exists a decomposition of $H^N(g;\mathbb{Q})(\eta_{\mathbb{Q}}) \neq 0$ into ω-homogeneous elements:

$$H^N(g;\mathbb{Q})(\eta_{\mathbb{Q}}) = [v] \in H^N(X;\mathbb{Q}) \cong H^N(\Lambda V, d) \cong \mathbb{Q}.$$

As we have shown in the proof of Lemma 2.6, for any $n \in \mathbb{N}$, we can choose an integer q_n such that $q_n^{\omega(v)} \geq n$. Then there exists a dga morphism:

$$\begin{array}{rccc} f_n\colon & (\Lambda V, d) & \to & (\Lambda V, d) \\ & x & \mapsto & q_n^{\omega(x)} x, \end{array}$$

for every ω-homogeneous element x of $(\Lambda V, d)$.

Let $\zeta_X\colon X \to X_{\mathbb{Q}}$ be the rationalization of X, that is, the localization at $\mathbb{Q}$, and let $\psi_n\colon X_{\mathbb{Q}} \to X_{\mathbb{Q}}$ be the realization of f_n. Using that X is 0-universal, there exists a 0-equivalence $\widetilde{\psi}_n \colon X \to X$ and a map $h_n \colon X \to X$ such that the following diagram homotopy commutes:

$$\begin{array}{ccccc} X & & \overset{\widetilde{\psi}_n}{\dashrightarrow} & & X \\ {\scriptstyle h_n}\downarrow & & & & \downarrow{\scriptstyle \zeta_X} \\ X & \xrightarrow{\zeta_X} & X_{\mathbb{Q}} & \xrightarrow{\psi_n} & X_{\mathbb{Q}}. \end{array} \tag{1}$$

Observe that $\widetilde{\psi}_n$, ψ_n and ζ_X are 0-equivalences, thus h_n is also a 0-equivalence.

Using that the diagram (1) is homotopy commutative, we get that

$$H^N(\widetilde{\psi}_n;\mathbb{Q})([v]) = H^N(\psi_n \circ \zeta_X \circ h_n;\mathbb{Q})([v]) = H^N(h_n;\mathbb{Q})(l_n^{\omega(v)}[v]) = c_n l_n^{w(v)}[v], \tag{2}$$

where c_n is a nonzero integer since h_n is a 0-equivalence.

Let $\nu = H^N(g;\mathbb{Z})(\eta) \in H^N(X;\mathbb{Z})$. By [21, Théorèmes III.4], there exists an N-manifold M' with cohomological fundamental class μ, and a map $\theta \colon M' \to X$ such that $H^N(\theta;\mathbb{Z})(\nu) = k\mu$ for some integer $k \neq 0$. We are going to show that $\deg(M', M)$ is unbounded, thus M is not strongly inflexible.

Indeed, for every $n \in \mathbb{N}$, we can define the composition:

$$G_n \colon M' \xrightarrow{\theta} X \xrightarrow{\widetilde{\psi}_n} X \xrightarrow{g} M.$$

Then,

$$\begin{aligned} H^N(G_n;\mathbb{Z})(\eta) &= H^N(\widetilde{\psi}_n \circ \theta;\mathbb{Z})(H^N(g;\mathbb{Z})(\eta)) \\ &= H^N(\theta;\mathbb{Z})(H^N(\widetilde{\psi}_n;\mathbb{Z})(\nu)) && \text{(we now use equation (2))} \\ &= H^N(\theta,\mathbb{Z})(c_n l_n^{\omega(v)}\nu + t) && \text{(where } t \text{ is some torsion element)} \\ &= c_n l_n^{\omega(v)} H^N(\theta;\mathbb{Z})(\nu) && (H^N(\theta;\mathbb{Z})(t) = 0 \text{ since } H^N(M';\mathbb{Z}) \cong \mathbb{Z}) \\ &= k c_n l_n^{\omega(v)}\mu. \end{aligned}$$

Therefore $|\deg(G_n)| = |k c_n l_n^{\omega(v)}| \geq n$, as k and c_n are nonzero integers. □

Remark 3.3. Observe that from Theorem 3.2 we deduce that a universal manifold $M \in \mathrm{Mfd}_N$ is not inflexible (see also [13, Corollary 4.1]).

Proposition 3.4. *Let $(\mathcal{A}, d)$ be a simply-connected dga with a positive weight ω. Then there exists a minimal model $\rho \colon (\Lambda V, d) \xrightarrow{\sim} (\mathcal{A}, d)$ such that $(\Lambda V, d)$ has a positive weight $\widetilde{\omega}$ with $\omega(\rho(v)) = \widetilde{\omega}(v)$, for every $\widetilde{\omega}$-homogeneous element $v \in \Lambda V$.*

Proof. We construct ρ inductively. Suppose that $\rho : \Lambda V^{<n} \to \mathcal{A}$ is constructed in such a way that, on the one hand, $\Lambda V^{<n}$ admits a positive weight $\widetilde{\omega}$ with $\omega(\rho(v)) = \widetilde{\omega}(v)$, for every $\widetilde{\omega}$-homogeneous element $v \in \Lambda V^{<n}$. And, on the other hand, $H^{<n}(\rho)$ is an isomorphism and $H^n(\rho)$ is a monomorphism.

By construction, the morphism $H(\rho) : H(\Lambda V^{<n}) \to H(\mathcal{A})$ preserves the induced weights in cohomology given by Lemma 2.3. Now, we consider the cokernel

$$Z^n \overset{\text{def}}{:=} \operatorname{coker}\big(H^n(\rho) \colon H^n(\Lambda V^{<n}) \to H^n(\mathcal{A})\big).$$

By the above, it admits a basis $\{z_i\}$ formed by ω-homogeneous elements. This implies that there are cocycles $a_i \in \mathcal{A}^n$ which are ω-homogeneous and $z_i = [a_i]$. Consider also the kernel

$$B^n \overset{\text{def}}{:=} \ker\big(H^{n+1}(\rho)\colon H^{n+1}(\Lambda V^{<n}) \to H^{n+1}(\mathcal{A})\big),$$

which has a basis $\{b_j\}$ formed by $\widetilde{\omega}$-homogeneous elements. Let η_j be $\widetilde{\omega}$-homogeneous elements of $\Lambda^{n+1}V^{<n}$ with $b_j = [\eta_j]$. As $H^{n+1}(\rho)(b_j) = 0 = [\rho(\eta_j)]$, we have that $\rho(\eta_j) = d\psi_j$ for some $\psi_j \in \mathcal{A}$. We can assume that ψ_j is ω-homogeneous and $\omega(\psi_j) = \widetilde{\omega}(\eta_j)$.

Now we define

$$V^n \overset{\text{def}}{:=} Z^n \oplus B^n, \qquad dz_i = 0,\ db_j = \eta_j,$$

declare $\widetilde{\omega}(z_i) = \omega(a_i)$, $\widetilde{\omega}(b_j) = \omega(\psi_j)$ and construct the morphism

$$\rho\colon V^n \to \mathcal{A}$$

given by $\rho(z_i) = a_i$ and $\rho(b_j) = \psi_j$.

By the inductive hypothesis $H^{<n}(\rho)$ is an isomorphism and since $H^n(\Lambda V^{<n+1}) = H^n(\Lambda V^{<n}) \oplus Z^n$ by construction, then $H^n(\rho)$ is an isomorphism too. Moreover, as we are considering simply-connected dgas, $H^{n+1}(\Lambda V^{<n+1}) = H^{n+1}(\Lambda V^{<n})/B^n$ again by construction, hence $H^{n+1}(\rho)$ is injective, which finishes the inductive step. □

Lemma 3.5. *Let $(\mathcal{A}, d)$ be a connected dga with a positive weight ω, and $I \subset \mathcal{A}$ be a differential closed ideal (that is, $dI \subset I$) generated (as a vector space) by ω-homogeneous elements. Then $\widetilde{\mathcal{A}} = \mathcal{A}/I$ is a connected dga with positive weight $\widetilde{\omega}$ defined by $\widetilde{\omega}(\bar{a}) = \omega(a)$, for every ω-homogeneous element $a \in \mathcal{A}$ such that $\bar{a} \neq 0$.*

Proof. Since $I \subset \mathcal{A}$ is a differential closed ideal, then $\widetilde{\mathcal{A}} = \mathcal{A}/I$ is a connected dga, and it only remains to prove that the weight $\widetilde{\omega}$ is well defined.

Assume $\widetilde{\omega}$ is not well defined. Therefore there exist ω-homogeneous elements $a_l \in \mathcal{A}$, $l = 1, 2$, such that $\omega(a_1) \neq \omega(a_2)$ and $\bar{a}_1 = \bar{a}_2 \neq 0$. Then $a_1 - a_2 \in I$, and $a_1 - a_2 = \sum_{i=0}^{r} x_i$ where every $x_i \in I$ is ω-homogeneous, and $\omega(x_i) = \omega(x_j)$ if and only if $i = j$. Moreover, a_l is ω-homogeneous, $l = 1, 2$, and $\omega(a_1) \neq \omega(a_2)$. Hence $a_l = x_{i(l)}$, $l = 1, 2$, and $\bar{a}_1 = \bar{a}_2 = 0$. This is a contradiction as we assumed $\bar{a}_1 = \bar{a}_2 \neq 0$. □

Finally, we obtain an integral version of Corollary 2.10.

Theorem 3.6. *Let $M \in \mathrm{Mfd}_N$, $N \geq 4$, with minimal model $(\Lambda V, d)$ and let η be its cohomological fundamental class. Write the rational cohomological fundamental class as $\eta \otimes_{\mathbb{Q}} 1 = [\nu]$. Assume there exists a dga morphism $\psi\colon (\Lambda V, d) \to (\mathcal{A}, d)$ where $(\mathcal{A}, d)$ is a simply-connected finite type dga with positive weight and $H^N(\psi)([\nu]) \neq 0$. Then M is not strongly inflexible.*

Proof. By Lemma 2.3, $H^N(\psi)([\nu]) = \sum_{i=0}^{r} [a_i]$ where every $a_i \in \mathcal{A}$ is an ω-homogeneous cocycle, and $\omega(a_i) = \omega(a_j)$ if and only if $i = j$. Fix $\widetilde{a} \in \mathcal{A}$, a nontrivial a_i in the decomposition above, take a complement $\mathcal{A}^N = \langle \widetilde{a} \rangle \oplus W$, where W is spanned by ω-homogeneous elements, and define

$$I \overset{\text{def}}{:=} \mathcal{A}^{\geq N+1} \oplus W.$$

Then $I \subset \mathcal{A}$ is a closed differential ideal generated (as a vector space) by ω-homogeneous elements. By Lemma 3.5, $\widetilde{\mathcal{A}} = \mathcal{A}/I$ is a finite type connected dga with positive weight and formal dimension N. Therefore, by Proposition 3.4, $\widetilde{\mathcal{A}}$ admits a minimal model $(\Lambda W, d)$ with positive weight, which is the rational homotopy type of a simply-connected finite CW-complex X, with $H^N(X;\mathbb{Q}) \cong \mathbb{Q}$. Moreover X is universal by Theorem 3.1.

Let us consider the composition $\widetilde{\psi}\colon \Lambda V \xrightarrow{\psi} \mathcal{A} \twoheadrightarrow \widetilde{\mathcal{A}}$, and let $\widetilde{\Psi}\colon X_{\mathbb{Q}} \to M_{\mathbb{Q}}$ be its geometrical realization. Observe that $H^N(\widetilde{\psi})([\nu]) \neq 0$ and so $H^N(\widetilde{\Psi})([\nu]) \neq 0$.

As X is universal, for $\zeta_M\colon M \to M_{\mathbb{Q}}$ the rationalization of M and $\zeta_X\colon X \to X_{\mathbb{Q}}$ the rationalization of X, there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ {\scriptstyle g}\,\Big\downarrow & & \Big\downarrow\,{\scriptstyle \widetilde{\Psi}\circ\zeta_X} \\ M & \xrightarrow{\zeta_M} & M_{\mathbb{Q}} \end{array}$$

where f is a 0-equivalence. We are going to show that $H^N(g;\mathbb{Q})([\nu]) \neq 0$ and therefore, by Theorem 3.2, M is not strongly inflexible:

$$H^N(g\,;\mathbb{Q})([\nu]) = H^N(\widetilde{\Psi} \circ \zeta_X \circ f\,;\mathbb{Q})([\nu]) = H^N(f;\mathbb{Q})(H^N(\widetilde{\Psi})([\nu])) \neq 0.$$

□

4. 3-step Sullivan algebras

Positive weights in Sullivan algebras relate to non inflexibility and non-strong inflexibility properties. We are going to exploit this aspect of positive weights to give a systematic way to check that certain 3-step Sullivan algebras are not strongly inflexible.

Definition 4.1. *We say that a Sullivan algebra*

$$\mathcal{M} = (\Lambda V_1 \otimes \Lambda V_2 \otimes \Lambda V_3, d),$$

is a 3-step algebra if $d(V_2) \subset \Lambda V_1$ and $d(V_3) \subset \Lambda(V_1 \oplus V_2)$.

We are only interested in 3-step Sullivan algebras of the form

$$\mathcal{M} = (\Lambda V_1 \otimes \Lambda V_2 \otimes \Lambda(z), d).$$

For such Sullivan algebras, we are going to describe how to construct a universal dga $\mathcal{B}_{\mathcal{M}}$ admitting positive weights, and a morphism of dgas $\psi\colon \mathcal{M} \to \mathcal{B}_{\mathcal{M}}$. Therefore, we have a sufficient condition, Corollary 2.10, to check that $\mathcal{M}$ is not strongly inflexible.

We first introduce some notions.

Definition 4.2. *A derivation differential graded algebra, ddga for short, is a triple $(\mathcal{A}, d, \theta)$ such that $(\mathcal{A}, d)$ is a dga, and θ is a derivation of degree 0 on $\mathcal{A}$ such that $d\theta = \theta d$.*

Let $(\mathcal{A}, d, \theta)$ and $(\mathcal{A}', d', \theta')$ be ddga. A ddga morphism $f\colon \mathcal{A} \to \mathcal{A}'$ is a dga morphism such that $f \circ \theta = \theta' \circ f$.

Definition 4.3. *Let $(\mathcal{A}, d)$ be a dga. We call the free derivation differential graded algebra generated by A to a ddga $(P(\mathcal{A}), d, \theta)$ equipped with a dga morphism $i\colon \mathcal{A} \to P(\mathcal{A})$ which is* *initial for ddgas morphisms, that is, for any ddga $(\mathcal{A}', d', \theta')$ and any dga map $f\colon \mathcal{A} \to \mathcal{A}'$, there is a unique ddga morphism $\bar{f}\colon P(\mathcal{A}) \to \mathcal{A}'$ such that $f = \bar{f} \circ i$.*

Clearly, as it is defined by a universal property, if it exists, $(P(\mathcal{A}), d, \theta)$ has to be unique up to isomorphism. For its existence, it can be constructed as follows. Let

$$\widehat{P}(\mathcal{A}) = T(\mathcal{A} \oplus \theta\mathcal{A} \oplus \theta^2\mathcal{A} \oplus \ldots)$$

be the tensor algebra on elements of $\mathcal{A}$ and "formal" derivatives $\theta^k\mathcal{A} \cong \theta(\theta^{k-1})\mathcal{A}$, for $k \geq 1$, and $\theta^0\mathcal{A} = \mathcal{A}$. Consider $I \subset \widehat{P}(\mathcal{A})$ the ideal generated by elements $a \otimes b - ab$, and $\theta(ab) - \theta a \otimes b - a \otimes \theta b$, for any pair of elements $a, b \in \mathcal{A}$. This is a differential ideal, that is $\theta(I) \subset I$. Finally, set

$$P(\mathcal{A}) = \widehat{P}(\mathcal{A})/I,$$

and $d(\theta a) = \theta(da)$, for all $a \in \mathcal{A}$.

Unfortunately, the algebra $P(\mathcal{A})$ is not of finite type. To solve this, we define the dot algebra.

Definition 4.4. *Let $(\mathcal{A}, d, \theta)$ be a ddga. We say $\mathcal{A}$ is* dot algebra *if $\theta^2 = 0$.*

There is a notion of free dot algebra generated by a dga:

Definition 4.5. *Let $(\mathcal{A}, d)$ be a dga. We call the* free dot algebra generated by $\mathcal{A}$ *or simply dot algebra of $\mathcal{A}$, to a dot algebra $(\dot{\mathcal{A}}, d, \theta)$ equipped with a dga morphism $i\colon \mathcal{A} \to \dot{\mathcal{A}}$ which is universal for dot algebras, that is for any dot algebra $(\mathcal{A}', d', \theta')$ and any dga map $f\colon \mathcal{A} \to \mathcal{A}'$, there is a unique ddga morphism $\bar{f}\colon \dot{\mathcal{A}} \to \mathcal{A}'$ such that $f = \bar{f} \circ i$.*

The concrete construction of a model is

$$\dot{\mathcal{A}} = P(\mathcal{A})/P^{\geq 2}(\mathcal{A}),$$

which is the quotient of $P(\mathcal{A})$ by the ideal generated by $\theta^j(\mathcal{A})$, $j \geq 2$, and $\theta(\mathcal{A}) \cdot \theta(\mathcal{A})$. We denote

$$\dot{a} = \theta(a).$$

Now let us deal with the situation of 3-step Sullivan algebras

$$(\mathcal{M}, d) = (\Lambda V_1 \otimes \Lambda V_2 \otimes \Lambda(z), d) \tag{3}$$

Let $\mathcal{M}_{[2]} \overset{\text{def}}{:=} \Lambda V_1 \otimes \Lambda V_2 = \Lambda(x_i, y_j)$ be the 2-step dga associated to (3), where the generators of V_1 are denoted by x_i, and the generators of V_2 by y_j. We endow $(\mathcal{M}_{[2]}, d)$ with the positive weight ω given by Proposition 2.11. Hence, we can decompose $dz \in \mathcal{M}_{[2]}$ into ω-homogeneous elements:

$$dz = P_0 + P_1 + \ldots + P_m$$

Note that elements $P_0, P_1, \ldots, P_m$ are all cocycles. If $m = 0$, then dz is ω-homogeneous and we can define $\omega(z) = \omega(P_0)$. Then $\mathcal{M}$ has a positive weight and therefore applying Corollary 2.10 to $\varphi = id_{\mathcal{M}}$ we obtain that $\mathcal{M}$ is not strongly inflexible.

We focus on the case $m = 1$, which is all that we need for our applications. So, we shall assume that $dz = P_0 + P_1$. Then, the dot algebra of $(\mathcal{M}_{[2]}, d) = (\Lambda(V_1 \oplus V_2), d)$ is actually

$$(\dot{\mathcal{M}}_{[2]}, d) = (\mathcal{M}_{[2]} \otimes (\mathbb{Q} \oplus \dot{V}_1 \oplus \dot{V}_2), d) \tag{4}$$

where the new elements $\dot{x}_i, \dot{y}_j$ have differentials given by:

$$d\dot{x}_i = 0,$$
$$d\dot{y}_j = \dot{dy_j}$$

We introduce the following dga:

$$(\mathcal{B}_{\mathcal{M}}, d) \stackrel{\text{def}}{:=} (\dot{\mathcal{M}}_{[2]} \otimes \Lambda(u_1, u_2, u_3), d) \tag{5}$$

where the new elements u_1, u_2, u_3 have differentials given by:

$$du_1 = P_0 + \dot{P}_1,$$
$$du_2 = P_1,$$
$$du_3 = \dot{P}_0.$$

We assign a weight ω to $\dot{\mathcal{M}}_{[2]}$ given by

$$\omega(\dot{v}) = \omega(v) + (\omega(P_0) - \omega(P_1)),$$

for $v \in V_1 \oplus V_2$. That way, $P_0 + \dot{P}_1$ is ω-homogeneous and we can extend ω to a weight in $(\mathcal{B}_{\mathcal{M}}, d)$ as follows:

$$\omega(u_1) = \omega(P_0 + \dot{P}_1),\ \omega(u_2) = \omega(P_1),\ \omega(u_3) = \omega(\dot{P}_0).$$

Remark 4.6. Observe that if $(\mathcal{M}, d)$ in (3) is $(\omega(P_1) - \omega(P_0))$-connected, then ω is a positive weight in $(\mathcal{B}_{\mathcal{M}}, d)$.

Finally consider the dga morphism

$$\begin{array}{rcl} \psi\colon \mathcal{M} = \mathcal{M}_{[2]} \otimes \Lambda(z) & \longrightarrow & \mathcal{B}_{\mathcal{M}} = \dot{\mathcal{M}}_{[2]} \otimes \Lambda(u_1, u_2, u_3), \\ m \in \mathcal{M}_{[2]} & \mapsto & m + \dot{m}, \\ z & \mapsto & u_1 + u_2 + u_3. \end{array} \tag{6}$$

The following result gives a criteria to decide when the morphism ψ above detects the fundamental class of the 3-step Sullivan algebra $\mathcal{M}$ with Poincaré duality cohomology.

Theorem 4.7. *Let $(\mathcal{M}, d) = (\mathcal{M}_{[2]} \otimes \Lambda(z), d)$ be a 3-step Sullivan algebra with Poincaré duality cohomology, where $\mathcal{M}_{[2]}$ is 2-step and endowed with the positive weight ω given by Proposition 2.11, and $d(z) = P_0 + P_1 \in \mathcal{M}_{[2]}$ where P_0 and P_1 are ω-homogeneous. Let $[\nu] \in H^N(\mathcal{M})$, $N \geq 4$, be the fundamental class of $\mathcal{M}$ and assume that the representative $\nu \in \mathcal{M}_{[2]}$. If the following holds*

(1) $\mathcal{M}$ is $(\omega(P_1) - \omega(P_0))$-connected,
(2) There are not any nontrivial $\xi \in H^{N+1-2|dz|}(\mathcal{M}_{[2]})$, such that $\xi \cdot [P_0] = 0 = \xi \cdot [P_1]$, and
(3) For $A, B \in \mathcal{M}_{[2]}^{N-|dz|}$ cocyles such that $[\nu] = [AP_0 + BP_1]$, then $(A - B)\dot{P}_1$ (up to a coboundary) is not in the ideal of $\dot{\mathcal{M}}_{[2]}$ generated by $P_0, \dot{P}_0, P_1$,

then $H^N(\psi)([\nu]) \neq 0$, where ψ is the morphism (6). In particular $\mathcal{M}$ is not strongly inflexible. Moreover, if M is a simply-connected N-manifold for which $\mathcal{M}$ is a model, then M is not strongly inflexible either.

Proof. Suppose that $H^N(\psi)([\nu]) = 0$. Then $\nu + \dot{\nu} = d\chi$ in $\mathcal{B}_{\mathcal{M}}$ and we can write

$$\chi = \Big(\sum_{1\le i<j\le 3} (A_i + \dot{B}_i)u_i + (C_{ij} + \dot{D}_{ij})u_iu_j \Big) + (E + \dot{F})u_1u_2u_3 + (G + \dot{H}),$$

for some elements $A_i, \dot{B}_i, C_{ij}, \dot{D}_{ij}, E, \dot{F}, G, \dot{H}$ in $\dot{\mathcal{M}}_{[2]}$, where the dot means that the element lies in the dot part, and the indexes $i < j$.

Let $n = N - |dz|$. Then we have the following set of equations

$$\begin{aligned} \nu &= (-1)^n(A_1P_0 + A_2P_1) + dG, \\ \dot{\nu} &= (-1)^n(\dot{B}_1P_0 + A_1\dot{P}_1 + \dot{B}_2P_1 + A_3\dot{P}_0) + d\dot{H}, \\ 0 &= dA_1 + (-1)^nC_{12}P_1, \\ 0 &= dA_2 + (-1)^{2n-N+1}C_{12}P_0, \\ 0 &= dC_{12}, \end{aligned} \tag{7}$$

the last three obtained by taking the dot–free coefficients of u_1, u_2, and u_1u_2, respectively, in the equality $\nu + \dot{\nu} = d\chi$.

From the above, $\xi = [C_{12}]$ is a cohomology class with $\xi\cdot[P_0] = 0 = \xi\cdot[P_1]$. By hypothesis (2), $C_{12} = d\eta$ for some η. Hence

$$\begin{aligned} A &= A_1 - (-1)^n\eta P_1, \\ B &= A_2 - (-1)^{2n-N+1}\eta P_0, \end{aligned}$$

are cocycles and $\nu = (-1)^n(AP_0 + BP_1) + dG$.

Now, in (7), apply the dot operator to the first equation to get

$$\dot{\nu} = (-1)^n(\dot{A}_1P_0 + \dot{A}_2P_1 + A_1\dot{P}_0 + A_2\dot{P}_1) + d\dot{G},$$

and compare it with the second equation. Then, we obtain that $(A_1 - A_2)\dot{P}_1$ is up to a coboundary, in the ideal of $\dot{\mathcal{M}}_{[2]}$ generated by P_0, $\dot{P}_0$ and P_1. Clearly $(A - B)\dot{P}_1$ also lies in the same ideal, which is a contradiction with hypothesis (3). Hence, $H(\psi)([\nu]) \neq 0$.

Finally, Remark 4.6 and hypothesis (1) guarantee that the weight ω in $\mathcal{B}_{\mathcal{M}}$ is a positive weight, which implies that ω detects $H(\psi)([\nu])$. Applying Corollary 2.10 we conclude that $\mathcal{M}$ is not strongly inflexible. The last statement follows from Theorem 3.6. □

5. The Arkowitz-Lupton Example

The existence of inflexible manifolds was first established by Arkowitz and Lupton in [2, Example 5.1, Example 5.2]. They gave examples of simply-connected Sullivan algebras with Poincaré duality cohomology that have finitely many homotopy classes of dga endomorphisms. Then, using Barge and Sullivan obstruction theory, they showed that those examples are minimal models of simply-connected manifolds. In particular, these manifolds are inflexible.

The succeeding examples in literature have been built upon [2, Example 5.1] that we now introduce:

Definition 5.1. *(The Arkowitz-Lupton example) Let* $(\mathcal{M}, d) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$ *be the Sullivan algebra with*

$$\begin{aligned} |x_1| &= 8 & dx_1 &= 0 \\ |x_2| &= 10 & dx_2 &= 0 \\ |y_1| &= 33 & dy_1 &= x_1^3 x_2 \\ |y_2| &= 35 & dy_2 &= x_1^2 x_2^2 \\ |y_3| &= 37 & dy_3 &= x_1 x_2^3 \\ |z| &= 119 & dz &= x_1^4 \gamma + x_1^{15} + x_2^{12} \end{aligned}$$

where $\gamma = \alpha\beta$, $\alpha = x_1 y_2 - x_2 y_1$, $\beta = x_1 y_3 - x_2 y_2$. *So* $\gamma = x_1^2 y_2 y_3 - x_1 x_2 y_1 y_3 + x_2^2 y_1 y_2$.

We are going to show that the dga from Definition 5.1 is not strongly inflexible. Observe that it is a 3-step algebra as we introduced in the previous section. With the same notation as in Section 4, the 2-step algebra

$$(\mathcal{M}_{[2]}, d) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d),$$

has a positive weight ω given by Proposition 2.11. The cohomology is

$$\begin{aligned} H(\mathcal{M}_{[2]}) =& \mathbb{Q}\{x_1^n, x_2^m, x_1^n\alpha, x_2^m\alpha, x_1^n\beta, x_2^m\beta, x_1^n\gamma, x_2^m\gamma \mid n, m \geq 0\} \\ &\oplus \mathbb{Q}\{x_1x_2, x_1^2x_2, x_1x_2^2, x_1x_2\alpha, x_1x_2\beta, x_1x_2^2\alpha\}. \end{aligned}$$

Notice that there is a "free" part (first summand) and some extra low degree generators. There are the following coboundaries:

$$d(y_1y_2) = x_1^2x_2\alpha,\ d(y_2y_3) = x_1x_2^2\beta,\ d(y_1y_3) = x_1^2x_2\beta + x_1x_2^2\alpha. \tag{8}$$

We also introduce the dot algebra associated to $\mathcal{M}_{[2]}$, see (4),

$$(\dot{\mathcal{M}}_{[2]}, d) \ = (\mathcal{M}_{[2]} \otimes (\mathbb{Q} \oplus \dot{V}_1 \oplus \dot{V}_2), d) \text{ where}$$

$$\begin{aligned} |\dot{x}_1| &= 8 & d\dot{x}_1 &= 0 \\ |\dot{x}_2| &= 10 & d\dot{x}_2 &= 0 \\ |\dot{y}_1| &= 33 & d\dot{y}_1 &= 3x_1^2\dot{x}_1x_2 + x_1^3\dot{x}_2 \\ |\dot{y}_2| &= 35 & d\dot{y}_2 &= 2x_1\dot{x}_1x_2^2 + 2x_1^2x_2\dot{x}_2 \\ |\dot{y}_3| &= 37 & d\dot{y}_3 &= \dot{x}_1x_2^3 + 3x_1x_2^2\dot{x}_2 \end{aligned}$$

and $\dot{\gamma} = \dot{\alpha}\beta + \alpha\dot{\beta}$, with $\dot{\alpha} = \dot{x}_1y_2 + x_1\dot{y}_2 - \dot{x}_2y_1 - x_2\dot{y}_1$, $\dot{\beta} = \dot{x}_1y_3 + x_1\dot{y}_3 - \dot{x}_2y_2 - x_2\dot{y}_2$. Therefore

$$\begin{aligned} \dot{\gamma} =& 2\dot{x}_1x_1y_2y_3 - \dot{x}_1x_2y_1y_3 - x_1\dot{x}_2y_1y_3 + 2x_2\dot{x}_2y_1y_2 \\ &+ x_1^2\dot{y}_2y_3 - x_1x_2\dot{y}_1y_3 + x_2^2\dot{y}_1y_2 + x_1^2y_2\dot{y}_3 - x_1x_2y_1\dot{y}_3 + x_2^2y_1\dot{y}_2 \end{aligned} \tag{9}$$

We have gathered the necessary elements to apply Theorem 4.7 and prove our main result in this section. For the proof to be more readable, we have moved to the end of this section some technical lemmas.

Theorem 5.2. *The dga* $\mathcal{M}$ *from Definition 5.1 is not strongly inflexible. Furthermore, a manifold* M *for which* $\mathcal{M}$ *is a Sullivan model is not strongly inflexible either.*

Proof. The cohomology $H(\mathcal{M})$ is a Poincaré duality algebra ([2, 5.3 Remarks]) where $\nu = x_1^{26} \in \mathcal{M}_{[2]}$ is the representative of the fundamental class $[\nu] \in H^{208}(\mathcal{M})$, and $dz = P_0 + P_1$, where

$$P_0 = x_1^{15} + x_2^{12},$$
$$P_1 = x_1^4\gamma.$$

We now check that hypothesis (1) – (3) in Theorem 4.7 hold.

It is clear that hypothesis (1) holds since $(\omega(P_1) - \omega(P_0)) = 122 - 120 = 2$, and $\mathcal{M}$ is 7-connected. Moreover, $N + 1 - 2|dz| = 208 + 1 - 240 = -31 < 0$ hence hypothesis (2) holds too.

We are now going to check hypothesis (3). For $N - |dz| = 208 - 120 = 88$, we know that $H^{88}(\mathcal{M}_{[2]}) = \mathbb{Q}\{[x_1^{11}], [\gamma]\}$. Let us write $[\nu] = [AP_0 + BP_1]$, where $A, B \in \mathcal{M}_{[2]}^{88}$ are cocycles. The possible cases are

$$A = x_1^{11} + d\eta_A,$$
$$B = a\gamma + d\eta_B,$$

for $a \in \mathbb{Q}$. Then, we deduce that

$$(A - B)\dot{P}_1 = \left(4x_1^{14}\dot{x}_1\gamma + x_1^{15}\dot{\gamma}\right) + d((\eta_A - \eta_B)\dot{P}_1)$$

using that $\dot{P}_1 = 4x_1^3\dot{x}_1\gamma + x_1^4\dot{\gamma}$, $\gamma^2 = 0$ and $\gamma\dot{\gamma} = 0$. In order to apply Theorem 4.7, we need to check that the element $(A - B)\dot{P}_1$ is not in the ideal

$$I = \langle x_1^{15} + x_2^{12}, 15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2, x_1^4\gamma\rangle + \operatorname{im} d \subset \dot{\mathcal{M}}_{[2]}.$$

Since $4x_1^{14}\dot{x}_1\gamma \in I$, it is enough to check that $X \overset{\text{def}}{:=} x_1^{15}\dot{\gamma} \notin I$.

First, notice that there is a y-gradation in $\dot{\mathcal{M}}_{[2]}$ according to the number of $y_j, \dot{y}_j$, and the differential decreases the degree by one. The y-degree of X is two, so let us assume that $X = Y + d\eta$, for some cocycle Y in I of y-degree equals two. Namely

$$Y = \dot{F}(x_1^{15} + x_2^{12}) + G(15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2) + \dot{H}x_1^4\gamma, \tag{10}$$

where $\dot{F}$, $G, \dot{H}$ are of the form

$$\dot{F} = \dot{F}' + \dot{F}'',\ \dot{F}' = \sum_{1\le i,j\le 3} F'_{ij}y_i\dot{y}_j,\ \dot{F}'' = \sum_{1\le i<j\le 3}\Big(\sum_{1\le k\le 2} F''_{ijk}\dot{x}_k y_i y_j\Big),$$
$$G = \sum_{1\le i<j\le 3} G_{ij}y_iy_j,$$
$$\dot{H} = \sum_{1\le k\le 2} H_k\dot{x}_k,$$

with F'_{ij}, F''_{ijk}, G_{ij}, $H_k \in \mathbb{Q}[x_1, x_2]$. As Y is a cocycle, we have

$$0 = dY = d(\dot{F}' + \dot{F}'')(x_1^{15} + x_2^{12}) + dG(15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2).$$

Look at the $\dot{y}_j$-term in this expression, $j = 1, 2$, to conclude that $\sum_{1\le i\le 3} F'_{ij}y_i$ is a cocycle. Hence, according to degrees, we deduce that

$$\dot{F}' \in W = \mathbb{Q}\{x_2\beta\dot{y}_1, x_2\alpha\dot{y}_2, x_1\beta\dot{y}_2, x_1\alpha\dot{y}_3\}.$$

Recalling that $X = Y + d\eta$, and (8), we see that the $\dot{y}$-part of the exact term $d\eta$ is of the form $x_1x_2 \sum_{1\le j\le 3} (R_j\alpha + S_j\beta)\dot{y}_j$, with R_j, S_j elements in $\mathbb{Q}[x_1, x_2]$.

Looking at the components $x_1\alpha, x_2\alpha, x_1\beta, x_2\beta$, and comparing the $\dot{y}_j$-parts, $j = 1, 2, 3$ of X, Y and $d\eta$, we obtain that:

$$\dot{F}' = x_1\beta\dot{y}_2 + x_1\alpha\dot{y}_3\,.$$

Therefore using (9) and (10), the equality $X = Y + d\eta$ becomes

$$\begin{aligned} x_1^{15}\dot{\gamma} &= x_1^{15}\big(-\beta x_2\dot{y}_1 + (\beta x_1 - \alpha x_2)\dot{y}_2 + \alpha x_1\dot{y}_3\big) + x_1^{15}\big((\beta y_2 + \alpha y_3)\dot{x}_1 - (\beta y_1 + \alpha y_2)\dot{x}_2\big) = \\ &= (x_1\beta\dot{y}_2 + x_1\alpha\dot{y}_3)(x_1^{15} + x_2^{12}) + \dot{F}''(x_1^{15} + x_2^{12}) + G(15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2) + \dot{H}x_1^4\gamma + d\eta, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\dot{F}''(x_1^{15} + x_2^{12}) + G(15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2) + \dot{H}x_1^4\gamma - x_1^{15}\big((\beta y_2 + \alpha y_3)\dot{x}_1 - (\beta y_1 + \alpha y_2)\dot{x}_2\big) = \\ &\quad = -x_1^{15}x_2\beta\dot{y}_1 - x_1^{15}x_2\alpha\dot{y}_2 - x_1x_2^{12}\beta\dot{y}_2 - x_1x_2^{12}\alpha\dot{y}_3 - d\eta\,. \end{aligned}$$

Now use the formulas (8) to get rid of the terms $\dot{y}_j$ at the expense of exact terms, for example $x_1^{15}x_2\beta\dot{y}_1 = d(x_1^{12}y_1\beta\dot{y}_1) - x_1^{12}y_1\beta d\dot{y}_1$. Hence

$$\begin{aligned} &\dot{F}''(x_1^{15} + x_2^{12}) + G(15x_1^{14}\dot{x}_1 + 12x_2^{11}\dot{x}_2) + \dot{H}x_1^4\gamma - x_1^{15}\big((\beta y_2 + \alpha y_3)\dot{x}_1 - (\beta y_1 + \alpha y_2)\dot{x}_2\big) = \\ &\quad = x_1^{12}y_1\beta d\dot{y}_1 + x_1^{12}y_1\alpha d\dot{y}_2 + x_2^9y_3\beta d\dot{y}_2 + x_2^9y_3\alpha d\dot{y}_3 + d\tilde{\eta}\,, \end{aligned} \tag{11}$$

which is an equation in $\mathcal{M}_{[2]} \otimes \mathbb{Q}\{\dot{x}_1, \dot{x}_2\}$. Using the formulas for $d\dot{y}_1, d\dot{y}_2, d\dot{y}_3$ and separating the components in $\dot{x}_1, \dot{x}_2$ independently, we get two equations in $\mathcal{M}_{[2]}$:

$$\begin{aligned} &F_1''(x_1^{15} + x_2^{12}) + 15Gx_1^{14} + H_1x_1^4\gamma - x_1^{15}(\beta y_2 + \alpha y_3) + d\tilde{\eta}_1 = \\ &\quad = 3x_1^{14}x_2y_1\beta + 2x_1^{13}x_2^2y_1\alpha + 2x_1x_2^{11}y_3\beta + x_2^{12}y_3\alpha\,, \\ &F_2''(x_1^{15} + x_2^{12}) + 12Gx_2^{11} + H_2x_1^4\gamma + x_1^{15}(\beta y_1 + \alpha y_2) + d\tilde{\eta}_2 = \\ &\quad = x_1^{15}y_1\beta + 2x_1^{14}x_2y_1\alpha + 2x_1^2x_2^{10}y_3\beta + 3x_1x_2^{11}y_3\alpha. \end{aligned}$$

We look at the first equation modulo γ, x_2^{12}. By Lemma 5.8, $G = e\gamma$ for some $e \in \mathbb{Q}$, and by Lemma 5.6 and Remark 5.7, $d\tilde{\eta}_1$ is in the ideal generated by γ. Hence, the first equation reduces to

$$F_1''(x_1^{15}) - x_1^{15}(\beta y_2 + \alpha y_3) = 3x_1^{14}x_2y_1\beta + 2x_1^{13}x_2^2y_1\alpha + 2x_1x_2^{11}y_3\beta. \tag{12}$$

From Lemma 5.8 we know that the $\dot{x}_1$-part of $\dot{F}''$ is $F_1'' = x_2y_1y_3 + x_1y_2y_3$. Then, the left hand side of (12) is of the form

$$2x_1^{15}x_2y_1y_3 + x_1^{16}y_2y_3\,.$$

The right hand side of (12) modulo γ, x_2^{12} reduces to

$$-3x_1^{15}x_2y_1y_3 + 5x_1^{14}x_2^2y_1y_2\,.$$

Hence, we obtain that

$$x_1^{16}y_2y_3 + 5x_1^{15}x_2y_1y_3 - 5x_1^{14}x_2^2y_1y_2 = 0 \mod (\gamma, x_2^{12}),$$

which by Lemma 5.9 leads to a contradiction. Therefore, the element X is not in the ideal I and hypothesis (3) holds.

Since we have proved that hypothesis (1) – (3) hold, we conclude that neither $\mathcal{M}$ from Definition 5.1 nor any manifold M for which $\mathcal{M}$ is a Sullivan model are strongly inflexible. ☐

Example 5.3. *(Inflexible dgas in literature) We list the collection of inflexible dgas in literature that follow the same pattern as Arkowitz-Lupton's example. In all cases, they are inflexible* 3*-step dgas* $\mathcal{M} = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$*. The sequence of degrees* $\{|x_1|, |x_2|, |y_1|, |y_2|, |y_3|, |z|\}$ *and nontrivial differentials are given in the tables below:*

	[2, Ex. 5.1] (*cf.* [9, Ex. I.3])	[2, Ex. 5.2] (*cf.* [9, Ex. I.4])	[9, Ex. I.1]
Degrees	$\{8, 10, 33, 35, 37, 119\}$	$\{10, 12, 41, 43, 45, 119\}$	$\{2, 4, 9, 11, 13, 35\}$
dy_1	$x_1^3 x_2$	$x_1^3 x_2$	$x_1^3 x_2$
dy_2	$x_1^2 x_2^2$	$x_1^2 x_2^2$	$x_1^2 x_2^2$
dy_3	$x_1 x_2^3$	$x_1 x_2^3$	$x_1 x_2^3$
dz	$x_1^4 \alpha\beta + x_1^{15} + x_2^{12}$	$x_2 \alpha\beta + x_1^{12} + x_2^{10}$	$x_2^2 \alpha\beta + x_1^{18} + x_2^9$
ν	x_1^{26}	x_2^{19}	x_2^{16}

	[9, Ex. I.2]	[6, Def. 1.1] $k \geq 1$	[1, Ex. 3.8]
Degrees	$\{4, 6, 17, 19, 21, 59\}$	$\{10k+8,\ 12k+10,\ 42k+33,\ 44k+35,\ 46k+37,\ 60k^2+38k+39\}$	$\{2, 2, 9, 11, 13, 35\}$
dy_1	$x_1^3 x_2$	$x_1^3 x_2$	$x_1^3 \bar{x}_2$
dy_2	$x_1^2 x_2^2$	$x_1^2 x_2^2$	$x_1^2 \bar{x}_2^2$
dy_3	$x_1 x_2^3$	$x_1 x_2^3$	$x_1 \bar{x}_2^3$
dz	$x_2^2 \alpha\beta + x_1^{15} + x_2^{10}$	$x_1^{6k-6} \alpha\beta + x_1^{6k+5} + x_2^{5k+4}$	$\bar{x}_2^2 \alpha\beta + x_1^{18} + \bar{x}_2^9$
ν	x_2^{18}	x_1^{6k+16}	x_2^{33} *where* $\bar{x}_2 = x_2^2$

The dgas in these tables are all minimal models of simply-connected inflexible manifolds. In Theorem 5.2 we have carried out in detail the proof that the dga from Definition 5.1 (first example of the table) is not strongly inflexible. The other examples can be analogously treated, so following the lines of Theorem 5.2 we obtain:

Theorem 5.4. *Let* $\mathcal{M}$ *be one of the inflexible dgas from Example 5.3. Then,* $\mathcal{M}$ *is not strongly inflexible. Furthermore, any manifold for which* $\mathcal{M}$ *is a Sullivan model is not strongly inflexible either.*

Remark 5.5. The dga constructed in [1, Section 3] is slightly different, since the cohomological fundamental class ν contains a Massey product like element. The dga is $(\Lambda(x_1, x_2, y_1, y_2, y_3, y_4, z), d)$ with

$$
\begin{aligned}
|x_1| &= 4 & dx_1 &= 0 \\
|x_2| &= 6 & dx_2 &= 0 \\
|y_1| &= 27 & dy_1 &= x_1^4 x_2^2 \\
|y_2| &= 29 & dy_2 &= x_1^3 x_2^3 \\
|y_3| &= 31 & dy_3 &= x_1^2 x_2^4 \\
|y_4| &= 75 + 4k & dy_4 &= x_1^{19+k} \\
|z| &= 77 & dz &= x_1 x_2 \alpha\beta + x_2 x_1^{18} + x_2^{13}
\end{aligned}
$$

where $k \geq 0$, and $\alpha = x_1y_2 - x_2y_1$, $\beta = x_1y_3 - x_2y_2$, $\delta = x_2^2y_4 - x_1^{15+k}y_1$ are nontrivial Massey products. A representative of the fundamental class is given by $\nu = x_2^{26}y_4 - x_1^{15+k}x_2^{24}y_1 = x_2^{24}\delta$.

We expect that the methods used in this section prove that this dga is not strongly inflexible. However, showing that the corresponding map (6) satisfies $H(\psi)([\nu])) \neq 0$, needs significantly more calculations and our attempts have not proven fruitful yet.

The end of this section is devoted to prove some technical lemmas that are needed in the proof of Theorem 5.2.

Lemma 5.6. *Let $d\tilde{\eta}$ be the element in* (11). *Then, $d\tilde{\eta}$ is in the ideal generated by the element γ introduced in Definition 5.1.*

Proof. From (11), we know that $d\tilde{\eta} \in \mathcal{M}_{[2]} \otimes (\mathbb{Q} \oplus \dot{V}_1)$ and has y-length two. Observe that since $\dot{x}_i\dot{y}_j = 0$, we can express

$$\tilde{\eta} = \sum_{1\leq i\leq 2} \tilde{\eta}_i\dot{x}_i + \sum_{1\leq j\leq 3} \hat{\eta}_j\dot{y}_j\,,$$

where $\tilde{\eta}_i, \hat{\eta}_j \in \mathcal{M}_{[2]}$. If we apply the differential

$$d\tilde{\eta} = \sum_{1\leq i\leq 2} d\tilde{\eta}_i\dot{x}_i + \sum_{1\leq j\leq 3} d\hat{\eta}_j\dot{y}_j \pm \sum_{1\leq j\leq 3} \hat{\eta}_j d\dot{y}_j\,,$$

we conclude that the elements $d\hat{\eta}_j = 0$ (since $d\tilde{\eta}$ does not contain terms with $\dot{y}_j$) and they are of y-length one. Hence, $\hat{\eta}_j$ are closed elements of y-length two. Reasoning with the degrees of the elements, we obtain that the only possibilities are:

$$\begin{aligned}
\hat{\eta}_1 &= (a_1x_1^2x_2^7 + b_1x_1^7x_2^3)\gamma,\\
\hat{\eta}_2 &= (a_2x_1^3x_2^6 + b_2x_1^8x_2^2)\gamma,\\
\hat{\eta}_3 &= (a_3x_1^4x_2^5 + b_3x_1^9x_2)\gamma,
\end{aligned}$$

for $a_i, b_i \in \mathbb{Q}$.

On the other hand, since the differential decreases by one the y-length, $\tilde{\eta}$ is of y-length three and so $\tilde{\eta}_i$ is also of y-length three, which implies that $\tilde{\eta}_i = R_i(x_1, x_2)y_1y_2y_3$. Hence

$$d\tilde{\eta}_i = R_i(x_1, x_2)d(y_1y_2y_3) = R_i(x_1, x_2)x_1x_2\gamma.$$

Gathering all this information, we deduce that $d\tilde{\eta}$ is in the ideal generated by γ. □

Remark 5.7. Observe that the $\dot{x}_i$-part of $d\tilde{\eta}$ is also in the ideal generated by γ, $i = 1, 2$.

Lemma 5.8. *Let $\dot{F}''$ and G be the terms in* (10). *Then $G = e\gamma$, for some $e \in \mathbb{Q}$, and*

$$\dot{F}'' = x_2\dot{x}_2y_1y_2 + (x_2\dot{x}_1 + 2x_1\dot{x}_2)y_1y_3 + x_1\dot{x}_1y_2y_3.$$

Proof. Recall that $\dot{F}'' = \sum_{1\leq i<j\leq 3}\left(\sum_{1\leq k\leq 2} F''_{ijk}\dot{x}_ky_iy_j\right)$ and that $|\dot{F}''| = 88$. By reasoning with degrees, we get: $F''_{121} = 0$, $F''_{122} = ax_2$, $F''_{131} = bx_2$, $F''_{132} = cx_1$, $F''_{231} = px_1$, $F''_{232} = 0$, $a, b, c, p \in \mathbb{Q}$, so

$$\dot{F}'' = ax_2\dot{x}_2y_1y_2 + (bx_2\dot{x}_1 + cx_1\dot{x}_2)y_1y_3 + px_1\dot{x}_1y_2y_3\,.$$

Applying the differential and using Equation (8), we obtain that

$$\begin{aligned} d\dot{F}'' = &\dot{x}_1\big(- bx_1x_2^4y_1 - px_1^2x_2^3y_2 + (bx_1^3x_2^7 + px_1^3x_2^2)y_3\big) \\ &+ \dot{x}_2\big((-ax_1^2x_2^3 - cx_1^2x_2^3)y_1 + ax_1^3x_2^2y_2 + cx_1^4x_2y_3\big). \end{aligned}$$

We proceed in the same way for $G = \sum_{1\leq i<j\leq 3} G_{ij}y_iy_j$ and $|G| = 88$. By reasoning with degrees, we get $G_{12} = ex_2^2$, $G_{13} = fx_1x_2$ and $G_{23} = gx_1^2$, $e, f, g \in \mathbb{Q}$, so

$$G = ex_1^2y_1y_2 + fx_1x_2y_1y_3 + gx_1^2y_2y_3.$$

Applying the differential, we get that

$$dG = y_1(-e - f)x_1^2x_2^4 + y_2(e - g)x_1^3x_2^3 + y_3(f + g)x_1^4x_2^2\,.$$

We now apply the differential to Equation (11) and compare the $\dot{x}_i$ components of the new equation for $i = 1, 2$. Denote $d\dot{F}''_{|\dot{x}_i}$ the component of $d\dot{F}''$ in $\dot{x}_i$, for $i = 1, 2$. We start by comparing the $\dot{x}_1$-components:

$$\begin{aligned} d\dot{F}''_{|\dot{x}_1}(x_1^{15} + x_2^{12}) + 15dGx_1^{14} = &y_1(-x_1^{16}x_2^4 - x_1x_2^{16}) + y_2(-x_1^{17}x_2^3 - x_1^2x_2^{15}) \\ &+ y_3(2x_1^{18}x_2^2 + 2x_1^3x_2^{14}), \end{aligned}$$

which is an equation in $\mathcal{M}_{[2]}$. Then, using our computation above of $d\dot{F}''$ and dG, we are going to compare the y_i-components of this last equation, for $i = 1, 2, 3$.

Comparing the y_1-terms, we get that:

$$-bx_1^{16}x_2^4 - bx_1x_2^{16} - 15(f + e)x_1^{16}x_2^4 = -x_1^{16}x_2^4 - x_1x_2^{16}\,,$$

so we deduce that $b = 1$ and that $e = -f$. Comparing the y_2-terms, we get that $p = 1$ and $e = g$.

We now compare the $\dot{x}_2$-components of the equation above:

$$\begin{aligned} d\dot{F}''_{|\dot{x}_2}(x_1^{15} + x_2^{12}) + 12dGx_2^{11} = &y_1(-3x_1^{17}x_2^3 - 3x_1^2x_2^{15}) + y_2(x_1^{18}x_2^2 - x_1^3x_2^{14}) \\ &+ y_3(2x_1^{19}x_2 + 2x_1^4x_2^{13}). \end{aligned}$$

Again using our computations above of $d\dot{F}''$ and dG and comparing the y_i-terms of this last equation, for $i = 1, 2, 3$, we obtain that $a = 1$, and $c = 2$, which concludes our proof. □

Lemma 5.9. *The element $Z = x_1^{16}y_2y_3 + 5x_1^{15}x_2y_1y_3 - 5x_1^{14}x_2^2y_1y_2$ is not in the ideal generated by γ and x_2^{12}.*

Proof. Let us suppose that $Z \in \langle \gamma, x_2^{12}\rangle$. Since $|Z| = 200$, by reasoning with degrees, Z can only be expressed as follows:

$$\begin{aligned} Z = &(a_1x_1^{10} + a_2x_2^8 + a_3x_1^5x_2^4 + a_4x_2y_1y_3 + a_5x_1y_2y_3)x_2^{12} \\ &+ (b_1x_1^{14} + b_2x_1^4x_2^8 + b_3x_1^9x_2^4)\gamma \end{aligned}$$

By comparing both sides of the equation, it is clear that $a_1 = a_2 = a_3 = 0$. Therefore

$$\begin{aligned} x_1^{16}y_2y_3 + 5x_1^{15}x_2y_1y_3 - 5x_1^{14}x_2^2y_1y_2 = \\ b_1x_1^{16}y_2y_3 - b_1x_1^{15}x_2y_1y_3 + b_1x_1^{14}x_2^{10}y_1y_2 \\ + b_2x_1^6x_2^8y_2y_3 - b_2x_1^5x_2^9y_1y_3 + b_2x_1^4x_2^{10}y_1y_2 \\ + b_3x_1^{11}x_2^4y_2y_3 - b_3x_1^{10}x_2^5y_1y_3 + b_3x_1^9x_2^6y_1y_2. \end{aligned}$$

From this equation, we get that on the one hand $b_1 = 1$ and, on the other hand $b_1 = -5$, which is a contradiction. □

6. The Costoya-Viruel Example

In [8], the authors construct, for any given finite group Γ, a simply-connected elliptic inflexible manifold whose group of self homotopy equivalences is Γ. To that end, they start with the inflexible dga from Definition 5.1 and add generators corresponding to a certain graph $G = (V, E)$, with set of vertices V and set of edges E. These new generators interrelate in such a way to ensure that the self homotopy equivalences of the dga are the automorphisms of the graph, which happens to be Γ.

To analyse the dga of [8], we need the following result on fibrations.

Lemma 6.1. *Given a commutative diagram of simply-connected dgas*

$$\begin{array}{ccccc} (\Lambda V_1, d_1) & \hookrightarrow & (\Lambda V_2, d_2) & \twoheadrightarrow & (\Lambda V_3, d_3) \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \|{\scriptstyle \psi_3} \\ (\Lambda \widetilde{V}_1, \tilde{d}_1) & \hookrightarrow & (\Lambda \widetilde{V}_2, \tilde{d}_2) & \twoheadrightarrow & (\Lambda V_3, d_3) \end{array} \tag{13}$$

where each $(\Lambda V_i, d_i)$, $i = 1, 2, 3$, has cohomology with Poincaré duality of formal dimension N_i and fundamental class η_i. If $V_2 = V_1 \oplus V_3$, and $\widetilde{V}_2 = \widetilde{V}_1 \oplus V_3$, then $H(\psi_1)(\eta_1) \neq 0$ if and only if $H(\psi_2)(\eta_2) \neq 0$.

Proof. By [10, Section 15(a)], since $V_2 = V_1 \oplus V_3$, and $\widetilde{V}_2 = \widetilde{V}_1 \oplus V_3$, the diagram in (13) is a Sullivan model for a commutative diagram of simply-connected rational spaces

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 \\ {\scriptstyle \Psi_1}\uparrow & & {\scriptstyle \Psi_2}\uparrow & & \|{\scriptstyle \Psi_3} \\ \widetilde{X}_1 & \longleftarrow & \widetilde{X}_2 & \longleftarrow & X_3 \end{array} \tag{14}$$

where the rows are rational fibrations. We proceed by analysing the rational cohomology Serre spectral sequence (Sss) associated to each fibration in the diagram (14), and compare them via the induced maps connecting both rows.

As our spaces are simply-connected, the Sss associated to the top row is

$$E_2^{p,q} = H^p\big(X_1; H^q(X_3;\mathbb{Q})\big) = H^p(X_1;\mathbb{Q}) \otimes H^q(X_3;\mathbb{Q}) \Longrightarrow H^{p+q}(X_2;\mathbb{Q}).$$

Since the rational cohomology of X_i is concentrated in degrees at most N_i, $i = 1, 3$, then the group of highest total degree in the E_2-term is $E_2^{N_1,N_3} = \mathbb{Q}\{\eta_1 \otimes \eta_3\}$, and the class

$\eta_1 \otimes \eta_3$ survives in the E_∞-term. Therefore, $N_2 = N_1 + N_3$ and $\eta_1 \otimes \eta_3$ represents a nontrivial multiple of η_2 in the associated graded vector space given by the Sss.

We now consider $\widetilde{E}_2^{p,q}$, the Sss associated to the bottom row in diagram (14). By means of the edge morphisms, we also know $H(\psi_1)(\eta_1) \otimes H(\psi_3)(\eta_3) = H(\psi_1)(\eta_1) \otimes \eta_3$ represents a nontrivial multiple of $H(\psi_2)(\eta_2)$ in the associated graded vector space given by the $\widetilde{E}_\infty$-term. Thus if $H(\psi_2)(\eta_2) \neq 0$, then $H(\psi_1)(\eta_1) \otimes \eta_3 \neq 0$ and $H(\psi_1)(\eta_1) \neq 0$.

Assume now $H(\psi_1)(\eta_1) \neq 0$, but $H(\psi_2)(\eta_2) = 0$. Then $H(\psi_1)(\eta_1) \otimes \eta_3$ represents the zero class in the $\widetilde{E}_\infty$-term. Since $\widetilde{E}_n^{p,q} = 0$ for $q > N_3$, $H(\psi_1)(\eta_1) \otimes \eta_3$ can only be trivial in the $\widetilde{E}_\infty$-term if there exists $n \in \mathbb{N}$ such that $\widetilde{d}_n\big(H(\psi_1)(\eta_1) \otimes \eta_3\big) \neq 0$ in $\widetilde{E}_n^{N_1-n-1, N_3-n}$. However,

$$\begin{aligned}
\widetilde{d}_n\big(H(\psi_1)(\eta_1) \otimes \eta_3\big) &= (H(\psi_1) \otimes H(\psi_3))\big(d_n(\eta_1 \otimes \eta_3)\big) && \text{(by naturality of edge morphisms)}\\
&= (H(\psi_1) \otimes H(\psi_3))(0) && \text{(since } 0 \neq \eta_1 \otimes \eta_3 \in E_\infty^{N_1,N_3}\text{)}\\
&= 0,
\end{aligned}$$

which is a contradiction. Therefore if $H(\psi_1)(\eta_1) \neq 0$, then $H(\psi_2)(\eta_2) \neq 0$. □

The dga of [8] is defined as follows.

Definition 6.2. *For a finite and simple connected graph $G = (V, E)$ with more than one vertex, we define the minimal Sullivan algebra associated to G as*

$$\mathcal{M}_G = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v \,|\, v \in V), d\Big),$$

where degrees and differential are described by

$$\begin{aligned}
|x_1| &= 8, & dx_1 &= 0,\\
|x_2| &= 10, & dx_2 &= 0,\\
|y_1| &= 33, & dy_1 &= x_1^3 x_2,\\
|y_2| &= 35, & dy_2 &= x_1^2 x_2^2,\\
|y_3| &= 37, & dy_3 &= x_1 x_2^3,\\
|x_v| &= 40, & dx_v &= 0,\\
|z| &= 119, & dz &= y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12},\\
|z_v| &= 119, & dz_v &= x_v^3 + \sum_{(v,w)\in E} x_v x_w x_2^4.
\end{aligned}$$

Theorem 6.3. *Let $G = (V, E)$ be a finite and simply-connected graph with more than one vertex. Then the minimal model $\mathcal{M}_G$ from Definition 6.2 is an elliptic inflexible dga of formal dimension $N = 208 + 80|V|$ and fundamental class $\nu = x_1^{26} \prod_{v \in V} x_v^2$.*

Proof. This follows from in [8] and [9, Proposition I.6]. □

Theorem 6.4. *Let $G = (V, E)$ be a finite and simply-connected graph with more than one vertex, and let $\mathcal{M}_G$ be the minimal model from Definition 6.2. Then there exists a dga morphism $\psi\colon \mathcal{M}_G \to \bar{\mathcal{B}}$ with $\bar{\mathcal{B}}$ a finite type dga with positive weight, and moreover $H(\psi)([\nu]) \neq 0$ where ν is the fundamental class given in Theorem 6.3. Therefore $\mathcal{M}_G$ is* *not strongly inflexible. Furthermore, any manifold for which $\mathcal{M}_G$ is a Sullivan model is not strongly inflexible either.*

Proof. Let $\mathcal{M}_{[2]}$ be the 2-step algebra

$$\mathcal{M}_{[2]} = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3) \otimes \Lambda(x_v, z_v \,|\, v \in V), d\Big) \subset \mathcal{M}_G,$$

and define $P_0 = x_1^{15} + x_2^{12}$, $P_1 = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6$. The algebra $\mathcal{M}_{[2]}$ has a positive weight ω given by Proposition 2.11, and following the arguments of Section 4, there exists a dga $\mathcal{B}_{\mathcal{M}_G} \overset{\text{def}}{:=} (\dot{\mathcal{M}}_{[2]} \otimes \Lambda(u_1, u_2, u_3), d)$, where

$$du_1 = \omega(P_0 + \dot{P}_1),\; du_2 = \omega(P_1),\; du_3 = \omega(\dot{P}_0).$$

such that ω extends to a positive weight on $\mathcal{B}_{\mathcal{M}_G}$ given by

$$\omega(u_1) = \omega(P_0 + \dot{P}_1),\; \omega(u_2) = \omega(P_1),\; \omega(u_3) = \omega(\dot{P}_0).$$

Let $\mathcal{I} \subset \mathcal{B}_{\mathcal{M}_G}$ be the differential ideal generated by the ω-homogeneous elements $\dot{x}_v, \dot{z}_v$. Then, according to Lemma 3.5, the dga $\bar{\mathcal{B}} = \mathcal{B}_{\mathcal{M}_G}/\mathcal{I}$ admits a positive weight $\widetilde{\omega}$. Define the dga morphism

$$\begin{array}{rcl} \psi\colon \mathcal{M}_G & \longrightarrow & \bar{\mathcal{B}}, \\ m \in \mathcal{M}_{[2]} & \mapsto & \bar{m} + \bar{\dot{m}}, \\ z & \mapsto & \bar{u}_1 + \bar{u}_2 + \bar{u}_3, \end{array} \tag{15}$$

and consider the commutative diagram

$$\begin{array}{ccccc} (\mathcal{M}, d) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d) & \hookrightarrow & \mathcal{M}_G & \twoheadrightarrow & \big(\Lambda(x_v, z_v \,|\, v \in V), d_3\big) \\ \downarrow \psi_1 & & \downarrow \psi & & \| \psi_3 \\ (\mathcal{B}_{\mathcal{M}}, d) & \hookrightarrow & \bar{\mathcal{B}} & \twoheadrightarrow & \big(\Lambda(x_v, z_v \,|\, v \in V), d_3\big), \end{array} \tag{16}$$

where $(\mathcal{M}, d)$ is the dga from Definition 5.1, and $\psi_1\colon (\mathcal{M}, d) \to (\mathcal{B}_{\mathcal{M}}, d)$ is the morphism constructed in (6). As we have shown in the proof of Theorem 5.2, the dga $(\mathcal{M}, d)$ satisfies the hypotheses of Theorem 4.7, hence $H(\psi_1)$ maps nontrivially the fundamental class of $H(\mathcal{M})$. Notice that $d_3(x_v) = 0$, and $d_3(z_v) = x_v^3$, thus $\big(\Lambda(x_v, z_v \,|\, v \in V), d_3\big)$ is an elliptic dga, hence with a Poincaré duality cohomology. Then $H(\psi)(\nu) \neq 0$ by Lemma 6.1, and $\mathcal{M}_G$ is not strongly inflexible by Corollary 2.10. □

Remark 6.5. The same arguments and calculations used to prove that the dgas from Definition 6.2, [8, Def. 2.1], are not strongly inflexible, work to show that the dgas from [6, Definition 2.1] and [7, Definition 4.1] are also not strongly inflexible. Hence, as a corollary of Theorem 3.6, we get that any simply-connected manifold admitting one of those dgas as Sullivan minimal model is not a strongly inflexible manifold.

7. Connected sums and strong inflexibility

In this section we deal with more examples of inflexible manifolds that are produced from the manifolds of Sections 5 and 6. Using connected sums and products, infinitely many oriented closed simply-connected inflexible manifolds are constructed in [1] and [9]. We develop here all the tools to prove that building upon not strongly inflexible manifolds produces not strongly inflexible manifolds.

Proposition 7.1. *Suppose that M_1 is an oriented closed manifold which is not strongly inflexible, and let M_2 be any other oriented closed manifold. Then $M_1 \times M_2$ is not strongly inflexible.*

Proof. As M_1 is not strongly inflexible, there exists an oriented closed manifold M' such that $\deg(M', M_1)$ is unbounded. Lef $\lambda > 0$ and $f\colon M' \to M_1$ such that $|\deg(f)| \geq \lambda$. Then $f \times \mathrm{Id}\colon M' \times M_2 \to M_1 \times M_2$ has $\deg(f \times \mathrm{Id}) = \deg(f)$, hence $|\deg(f \times \mathrm{Id})| \geq \lambda$. So $\deg(M' \times M_2, M_1 \times M_2)$ is unbounded, and hence $M_1 \times M_2$ is not strongly inflexible. □

Corollary 7.2. *The manifolds of* [9, Theorem II.5] *are inflexible but not strongly inflexible.*

Proof. They are of the form $\prod_{1 \leq l \leq k} M_l$, where M_l are taken from the examples given in Section 5, which are not strongly inflexible by our previous results. Using Proposition 7.1, we conclude. □

Proposition 7.3. *If M is a simply-connected oriented closed manifold which is not strongly inflexible, then $\#_k M$ is not strongly inflexible for all $k \geq 2$.*

Proof. Let M' be a simply-connected oriented closed manifold such that $\deg(M', M)$ is unbounded. We apply [20, Lemma 3.8.(2)] inductively to M, M' and $\#_k M, \#_k M'$ so that

$$\deg(M', M) \cap \deg(\#_k M', \#_k M) \subseteq \deg(\#_{k+1} M', \#_{k+1} M).$$

Hence $\deg(M', M) \subseteq \deg(\#_k M', \#_k M)$ for all $k \geq 2$. This proves the result. □

Corollary 7.4. *The manifolds of* [9, Example II.4] *are inflexible but not strongly inflexible.*

Finally, in [9, Proposition II.13], the authors construct nonspinable simply-connected inflexible manifolds. To deal with them, we need the following:

Proposition 7.5. *Let M_1, M_2 be oriented closed simply-connected and not strongly inflexible N-manifolds. Then, $M_1 \# M_2$ is not strongly inflexible if there exist:*

(1) A closed oriented simply-connected N-manifold M_2', with $\deg(M_2', M_2) = \mathbb{Z}$; or
(2) Oriented closed simply-connected N-manifolds M_1', M_2' such that for some $s > 0$, $s\mathbb{Z} \subset \deg(M_2', M_2)$ and $\deg(M_1', M_1) \cap s\mathbb{Z}$ is infinite.

Proof. Let us prove the first assertion. Let M_1' be an oriented closed N-manifold such that $\deg(M_1', M_1)$ is unbounded. The result follows from [20, Lemma 3.8.(2)] as $\deg(M_2', M_2) = \mathbb{Z}$, implies $\deg(M_1', M_1) \subset \deg(M_1' \# M_2, M_1 \# M_2)$. Thus $\deg(M_1' \# M_2, M_1 \# M_2)$ is unbounded and $M_1 \# M_2$ is not strongly inflexible.

The second assertion is proved in a similar way. □

Let $s > 0$ be an integer. We have the following improvement of Corollary 2.10 and Theorem 3.6

Proposition 7.6. *Let $(\mathcal{A}, d)$ be a dga with Poincaré duality cohomology algebra and fundamental form $[\nu] \in H^N(\mathcal{A}, d)$, $N \geq 4$. Suppose that there exists a dga $(\mathcal{A}', d')$ with a positive weight, and a morphism $\varphi\colon (\mathcal{A}, d) \to (\mathcal{A}', d')$ such that $H(\varphi)([\nu]) \neq 0$. Then there exists a dga $(\bar{\mathcal{A}}, d)$ whose cohomology is Poincaré duality of formal dimension N, and $\deg(\mathcal{A}, \bar{\mathcal{A}}) \cap s\mathbb{Z}$ is unbounded.*

Moreover, if (M, η) *is a simply-connected* N*-manifold with model* $(\mathcal{A}, d)$ *as above, such that* $\eta \otimes_{\mathbb{Q}} 1 = [\nu]$*, and* $(\mathcal{A}', d')$ *is simply-connected and of finite type, then there exists a manifold* M' *such that* $\deg(M', M) \cap s\mathbb{Z}$ *is unbounded.*

Proof. Following the arguments and notation in proof of Corollary 2.10, there exist a dga $(\bar{\mathcal{A}}, d)$ whose cohomology is Poincaré duality of formal dimension N, fixed numbers $0 \neq \omega(a'_i) \in \mathbb{N}$, and $\alpha_i \in \mathbb{Q}$, for $i = 0, \ldots, r$, such that given any natural number q_n there is a dga morphism $G_n \colon (\mathcal{A}, d) \to (\bar{\mathcal{A}}, d)$ with

$$\deg(G_n) = q_n^{\omega(a'_0)} + \sum_{i=1}^{r} q_n^{\omega(a'_i)} \alpha_i \,.$$

If $t \in \mathbb{Z}$ verifies $t\alpha_i \in \mathbb{Z}$, for every $i = 1, \ldots, r$, then $q_n = nst$, $n \in \mathbb{N}$ makes $\deg(G_n) \in s\mathbb{Z}$. Therefore $\deg(\mathcal{A}, \bar{\mathcal{A}}) \cap s\mathbb{Z}$ is unbounded.

The last assertion follows from Theorem 3.6 that hinges in Theorem 3.2. There, it is shown that there exist a manifold M', and integers $k \neq 0$ and $\omega(v) > 0$, such that for every integer l_n there exists a map $G_n \colon M' \to M$ such that $|\deg(G_n)| = |kc_n l_n^{\omega(v)}|$ for some non zero integer c_n. Therefore

$$\{\deg(G_n) \,|\, l_n \in s\mathbb{Z}\} \subset \deg(M', M) \cap s\mathbb{Z}$$

is unbounded. □

All manifolds in Sections 5 and 6 satisfy the conditions of Proposition 7.6, hence the degrees of maps $\deg(M', M) \cap s\mathbb{Z}$ are unbounded for any $s > 1$.

Corollary 7.7. *The manifolds of* [9, Prop. II.13] *are inflexible but not strongly inflexible.*

Proof. The manifolds of [9, Prop. II.13] are of the form $M^{\times k} \times W$, where M is one of the manifolds in [9, Examples I.2–I.4] (see Example 5.3), and $W = S^{N-2} \tilde{\times} S^2$ is the total space of the sphere bundle of the nontrivial rank $(n-1)$–vector bundle over S^2.

With $s = 2$, there is some M' such that $\deg(M', M^{\times k}) \cap 2\mathbb{Z}$ is unbounded. Now taking the pull-back under the degree 2 map $g : S^2 \to S^2$ of the nontrivial bundle $S^{N-2} \to W \to S^2$ we get a trivial bundle $\tilde{W} = g^*W \cong S^{N-2} \times S^2$. As this has self-maps of any degree, we have that $\deg(\tilde{W}, W) \supset 2\mathbb{Z}$. Applying now Proposition 7.5, we get that $M^{\times k} \times W$ is not strongly inflexible. □

For completeness, we will include some results on degrees of maps for connected sums, using dgas. First, given subsets $A, B \subset \mathbb{Z}$, define

$$A + B \stackrel{\text{def}}{:=} \{a + b \,|\, a \in A, b \in B\} \subset \mathbb{Z}.$$

Lemma 7.8. *Let* M_i*,* $i = 1, 2, 3$*, be (not necessarily simply-connected) closed oriented* N*-manifolds. Then*

$$\deg(M_1, M_3) + \deg(M_2, M_3) \subset \deg(M_1 \# M_2, M_3).$$

Proof. Let $q \colon M_1 \# M_2 \to M_1 \vee M_2$ denote the pinching map. Then for any given maps $f_i \colon M_i \to M_3$, the map $f \colon M_1 \# M_2 \to M_3$ given by the composition $f = (f_1 \vee f_2) \circ q$ verifies $\deg(f) = \deg(f_1) + \deg(f_2)$, and the result follows. □

Note that the inclusion in Lemma 7.8 can be strict. Consider, for example, $M_1 = M_2 = T_2$, the 2-torus, and $M_3 = T_2 \# T_2$. Then $\deg(M_i, M_3) = 0$ for $i = 1, 2$, whereas $1 \in \deg(M_3, M_3)$.

Corollary 7.9. *Let M be a closed oriented and not strongly inflexible N-manifold. Then there exist infinitely many closed oriented N-manifolds M' such that $\deg(M', M)$ is unbounded.*

Proof. Since M is not strongly inflexible, there exists a closed oriented N-manifold M' such that $\deg(M', M)$ is unbounded. Then, according to Lemma 7.8 for any N-manifolds W, $\deg(M', M) \subset \deg(W \# M', M)$, and therefore $\deg(W \# M', M)$ is unbounded too. □

The following is a straightforward consequence of the previous corollary.

Corollary 7.10. *Let M be a closed oriented N-manifold, and let v_M be the domination Mfd_N-seminorm associated with M (see [9, Definition 7.1]). If v_M is not finite, then there exist infinitely many closed oriented N-manifolds M' such that $v_M(M') = \infty$.*

Proof. Recall that given an oriented closed N-manifold M',

$$v_M(M') = \sup\{|d| \mid d \in \deg(M', M)\},$$

thus the result follows from Corollary 7.9. □

Definition 7.11. *Let $\mathcal{A}_i$, $i = 1, 2$, be connected dgas, and let $a_i \in \mathcal{A}_i$ be elements such that $|a_1| = |a_2|$. The connected sum of the pairs $(\mathcal{A}_i, a_i)$, $i = 1, 2$, is the dga*

$$(\mathcal{A}_1, a_1) \# (\mathcal{A}_2, a_2) \stackrel{def}{:=} (\mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2)/I\,,$$

where $\mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2 \stackrel{def}{:=} (\mathcal{A}_1 \oplus \mathcal{A}_2)/\mathbb{Q}\{(1, -1)\}$, and $I \subset \mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2$ is the differential ideal generated by $a_1 - a_2$.

The connected sum of dgas provides a model for the connected sum of manifolds.

Theorem 7.12. *Let M_i, $i = 1, 2$, be simply-connected closed oriented N-manifolds. Let $(\mathcal{M}_i, m_i)$ be the associated rational model with fundamental class. Then $(\mathcal{M}_1, m_1) \# (\mathcal{M}_2, m_2)$ is a rational model of $M_1 \# M_2$.*

Proof. If $(\mathcal{M}_i, m_i)$ is a rational model of M_i, then $\mathcal{M}_1 \oplus_{\mathbb{Q}} \mathcal{M}_2$ is a rational model of $M_1 \vee M_2$ [11, Example 2.47], and therefore dividing by the ideal generated by $m_1 - m_2$ is a rational model of $M_1 \# M_2$ [11, Example 3.6]. □

The connected sum of dgas behaves nicely with positive weights.

Proposition 7.13. *Let $\mathcal{A}_i$, $i = 1, 2$, be connected dgas, and let $a_i \in \mathcal{A}_i$ be elements such that $|a_1| = |a_2|$. If both $\mathcal{A}_1$ and $\mathcal{A}_2$ have positive weights, namely ω_1 and ω_2, such that a_i is ω_i-homogeneous, $i = 1, 2$, then $(\mathcal{A}_1, a_1) \# (\mathcal{A}_2, a_2)$ admits a positive weight.*

Proof. Let ω be the positive weight in $\mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2$ given by

$$\omega(x) = \begin{cases} \omega_2(a_2)\omega_1(x), & \text{if } x \in \mathcal{A}_1 \text{ is } \omega_1\text{-homogeneous}, \\ \omega_1(a_1)\omega_2(x), & \text{if } x \in \mathcal{A}_2 \text{ is } \omega_2\text{-homogeneous}. \end{cases}$$

Then $a_1 - a_2 \in \mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2$ is ω-homogeneous, hence $d(a_1 - a_2)$ is so. Therefore since

$$I = (\mathcal{A}_1 \oplus_{\mathbb{Q}} \mathcal{A}_2) \cdot (a_1 - a_2, d(a_1 - a_2)),$$

the differential closed ideal I is generated (as vector space) by ω-homogeneous elements, and according to Lemma 3.5, ω gives rise to a positive weight on $(\mathcal{A}_1, a_1)\#(\mathcal{A}_2, a_2)$. □

Finally,

Theorem 7.14. *Let (M_i, η_i) be a simply-connected N-manifold with minimal model $\mathcal{M}_i = (\Lambda V_i, d)$, $i = 1, 2$. Write the cohomological fundamental class as $\eta_i = [\nu_i]$. Assume there exist dga morphisms $\psi_i \colon \mathcal{M}_i \to \mathcal{A}_i$, $i = 1, 2$, such that $\mathcal{A}_i$ is a finite type dga that has a positive weight and $H(\psi_i)(\eta_i) \neq 0$. Then $M_1\#M_2$ is not strongly inflexible.*

Proof. Let ω_i be a positive weight on $\mathcal{A}_i$, $i = 1, 2$. We claim $\psi_i(\nu_i)$ may be assumed ω_i-homogeneous. As in the proof of Theorem 3.2, decompose $\psi_i(\nu_i) = \sum_{j=0}^{r} \alpha_j$ into ω_i-homogeneous elements, fix $\widetilde{a}_i \in \mathcal{A}_i$ with nontrivial α_j, ω_i-homogeneous complements $\mathcal{A}_i = \langle \widetilde{a}_i \rangle \oplus W_i$, and define

$$I_i = \mathcal{A}_i^{\geq N+1} \oplus W_i \,.$$

Then by Lemma 3.5, $\widetilde{\mathcal{A}_i} = \mathcal{A}_i / I_i$ is a finite type connected dga with positive weight, and the induced morphism $\widetilde{\psi}_i \colon \mathcal{M}_i \to \widetilde{\mathcal{A}}_i$ maps ν_i to a $\widetilde{\omega}_i$-homogeneous element.

The algebra $\big(\widetilde{\mathcal{A}}_1, \widetilde{\psi}_1(\nu_1)\big)\#\big(\widetilde{\mathcal{A}}_2, \widetilde{\psi}_2(\nu_2)\big)$ inherits a positive weight by Proposition 7.13. Therefore the obvious morphism

$$\widetilde{\psi}_1\#\widetilde{\psi}_2 \colon (\mathcal{M}_1, \nu_1)\#(\mathcal{M}_2, \nu_2) \to \big(\widetilde{\mathcal{A}}_1, \widetilde{\psi}_1(\nu_1)\big)\#\big(\widetilde{\mathcal{A}}_2, \widetilde{\psi}_2(\nu_2)\big)$$

is a morphism from the rational model of $(M_1\#M_2, \eta)$ to a finite type dga that has a positive weight, such that $[(\widetilde{\psi}_1\#\widetilde{\psi}_2)(\nu)] \neq 0$, where $\eta = [\nu]$ is the fundamental class. Finally, according to Theorem 3.6, $(M_1\#M_2, \eta)$ is not strongly inflexible. □

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