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A FINITE ELEMENT FORMULATION FOR THE RESOLUTION OF THE UNSTEADY INCOMPRESSIBLE VISCOUS FLOW FOR LOW REYNOLDS NUMBERS.

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ABSTRACT

The following paper shows a Finite Element formulation for the resolution of the –local and convective acceleration terms including- Navier-Stokes equations, which gives analytical response to the problem of viscous, incompressible, unsteady flows. The integration of the resulting non-linear system of first order ordinary differential equations, is made upon a successive approximation algorithm together with an implicit backward time integrating scheme. The interpolation of the spatial domain is made in terms of a Q1/P0 pair (bilinear velocity-constant pressure). The usage of a Bubnov Galerkin formulation in the process of obtaining a weak form implies that flows of a certain velocity need the employment of a very refined spatial mesh so as to avoid numerical instability. For high Reynolds numbers the convection term becomes predominant compared to the diffussion term and a different algorithm (SPGU, GLS), should be introduced. Finally the developed program is checked over some of the most commonly used flow tests and its results on velocity and pressure are shown.

GOVERNING EQUATIONS

The unsteady incompressible Navier-Stokes equations are given by:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} && \text{in } \Omega \\ &\nabla\cdot\mathbf{u} = \mathbf{0} && \text{in } \Omega \end{aligned} \qquad (1.1)$$

for $0 \le t \le T$, (where T is a specified time), together with the initial and boundary conditions:

$$\begin{aligned} &\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \\ &\mathbf{u} = \mathbf{g} && \text{on } \partial\Omega \times (0,T) \end{aligned} \qquad (1.2)$$

where $\mathbf{u}$ is the velocity, p is the pressure, t is the time, ν is the cinematic viscosity, $\mathbf{f}$ is the body force per unit mass and $\nabla, \Delta,$ are the gradient and laplacian tensor operators. The problem consists in finding $\mathbf{u} \in \mathbf{H}_0^{\,1}$ and $p \in S_0^1$ in a time-space domain Ω x(0,*T*).

FINITE ELEMENT FORMULATION

Let V be the subspace of $D(\Omega)$ satisfying the incompressibility constraint: $V = \{\mathbf{u} \in D(\Omega) : \nabla \cdot \mathbf{u} = 0\}$, let H be the closure of V *in* $H_0^1(\Omega)$. If Ω is sufficiently well behaved then $H = \left\{\mathbf{u} \in \left(H^1(\Omega)\right)^n : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\right\}$.
If we apply the weighted-residual argument by taking the scalar product of the momentum equation times an arbitrary test function v satisfying $\nabla \cdot \mathbf{v} = \mathbf{0}$, that is for $\mathbf{v} \in V$ **,**

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\, d\Omega + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\Omega - n \int_\Omega \Delta \mathbf{u} \cdot \mathbf{v}\, d\Omega + \int_\Omega \nabla p \cdot \mathbf{v}\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega \tag{2.1}$$

Applying the divergence theorem and taking into account that $\nabla \cdot \mathbf{v} = \mathbf{0}$ in Ω, we arrive to:

$$\int_\Omega \frac{\P \mathbf{u}}{\P t} \cdot \mathbf{v}\, \mathrm{d}\Omega + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\Omega + n \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, d\Omega - \int_\Omega p \nabla \cdot \mathbf{v}\, \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega \tag{2.2}$$

for all admissible functions $\mathbf{v} \in V$ **.**
Even with $\nabla \cdot \mathbf{v} = 0$ in Ω, we will maintain the pressure term so as to be able to use a mixed formulation rather than the penalized one. This allows us to keep as variables the pressure unknowns.Although it produces a certain increase in computational cost, the penalty parameter formulation is known to be the cause of loss of accuracy for small values and for holding up the convergence of the solution for too large ones.
In the same way, making the scalar product of the continuity equation with an arbitrary test function $q \in L^2(\Omega)/R = P$ **,** the weak expression turns into:

$$\int_\Omega (\nabla \cdot \mathbf{u}) q\, \mathrm{d}\Omega \tag{2.3}$$

The velocity and pressure fields may be approximated now as piecewise polynomials on the discretization by (Q1/P0) Lagrange-Type elements, so they are expressed in terms of the trial functions, $f_n(\mathbf{x})$, $c_m(\mathbf{x})$.

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}_h(\mathbf{x}) = \sum_{n=1}^{N} \mathbf{u}_n f_n(\mathbf{x}) \qquad\qquad p(\mathbf{x}) \approx p_h(\mathbf{x}) = \sum_{m=1}^{M} p_m c_m(\mathbf{x}) \tag{2.4}$$

Replacing (2.4) into (2.2) and (2.3):

$$\int_\Omega \frac{\P \mathbf{u}_h}{\P t} \cdot \mathbf{v}_h\, \mathrm{d}\Omega + \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h d\Omega + n \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h\, d\Omega - \int_\Omega p_h \nabla \cdot \mathbf{v}_h\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\, d\Omega \tag{2.5}$$

$$\int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\, \mathrm{d}\Omega \tag{2.6}$$

For all admissible test functions $\mathbf{v}_h \in V^h$ and $q_h \in P^h$ **.**
Introducing the expansions for $\mathbf{u}_h, p_h$, and integrating, the following non-linear ordinary differential equations system is obtained.

$$\mathbf{M}\,\dot{\mathbf{u}} + \mathbf{C}(\mathbf{u}) + n\mathbf{A}\mathbf{u} - \mathbf{B}\mathbf{p} = \mathbf{F}$$

$$\mathbf{B}^T\mathbf{u} = \mathbf{0} \tag{2.7}$$

In expanded matrix form:

$$\begin{bmatrix} \hat{\mathbf{M}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{M}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_1 \\ \dot{p} \end{bmatrix} + \mathbf{C}(\mathbf{u}) + \begin{bmatrix} n\hat{\mathbf{A}} & \mathbf{0} & \mathbf{B}^x \\ \mathbf{0} & n\hat{\mathbf{A}} & \mathbf{B}^y \\ (\mathbf{B}^x)^T & (\mathbf{B}^y)^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{0} \end{bmatrix} \tag{2.8}$$

where:

$$\hat{\mathbf{A}} = (\hat{A}_{rs}) = \int_{\Omega_h} \frac{\P f_r}{\P x}\frac{\P f_s}{\P x} + \frac{\P f_r}{\P y}\frac{\P f_s}{\P y}\,dx \qquad \hat{\mathbf{M}} = (\hat{M}_{rs}) = \int_{\Omega_h} f_r f_s \,\mathrm{d}\Omega$$

$$\mathbf{B}^x = (B^x_{rt}) = \int_{\Omega_h} \frac{\P f_r}{\P x} c_t\,dx \qquad \mathbf{B}^y = (B^y_{rt}) = \int_{\Omega_h} \frac{\P f_r}{\P y} c_t\,dx$$

$$F^x_r = \int_{\Omega_h} f_1 f_r\,dx \qquad F^y_r = \int_{\Omega_h} f_2 f_r\,dx \tag{2.9}$$

In order to turn equation (2.8) into a linear system of equations we are going to approximate the non-linear term $\mathbf{C}(\mathbf{u})$, by an iterative scheme known as successive-approximation:

$$\mathbf{c}(\mathbf{u}_k) \approx \mathbf{c}(\mathbf{u}_{k-1})\mathbf{u}_k = \int_{\Omega} (\mathbf{u}_h^{k-1}\cdot\nabla)\mathbf{u}_h^k \cdot \mathbf{v}_h\, d\Omega \tag{2.10}$$

Thus (2.7) is written:

$$\mathbf{M}\,\dot{\mathbf{u}}_k + \mathbf{C}\mathbf{u}_k + n\mathbf{A}\mathbf{u}_k - \mathbf{B}\mathbf{p}_k = \mathbf{F}_k$$

$$\mathbf{B}^T\mathbf{u}_k^* = \mathbf{0} \tag{2.11}$$

Where $\mathbf{C}$ is:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \mathbf{C}_{ij} = \int_{\Omega_h} \left[(u_k f_k)\frac{\partial f_j}{\partial x} + (v_k f_k)\frac{\partial f_j}{\partial y} \right] f_i\, d\mathbf{x} \tag{2.12}$$

Finally, the time integration based on an implicit backward scheme, results in the following non-differential linear system:

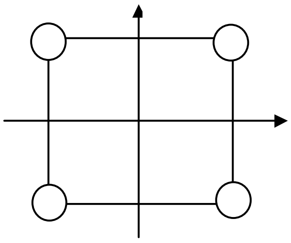
$$\mathbf{M}\left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}\right) + \mathbf{C}\mathbf{u}_{n+1} + n\mathbf{A}\mathbf{u}_{n+1} - \mathbf{B}\mathbf{p}_{n+1} = \mathbf{F}_{n+1} \tag{2.13}$$

$$\mathbf{B}^T\mathbf{u}_{n+1} = \mathbf{0}$$

Which characterises the unsteady viscous incompressible flow.

DISCRETIZATION

The Q1/P0 pair means a quadratic first order approximation for the velocity field and a constant pressure approximation for each basic element. The shape functions for the velocity field N_i are expressed in terms of the local coordinates η, ξ .



$$N_1 = \frac{1}{4}(\xi+1)(\eta+1), \quad N_2 = \frac{-1}{4}(\xi+1)(\eta-1)$$
$$N_3 = \frac{1}{4}(\xi-1)(\eta-1), \quad N_4 = \frac{-1}{4}(\xi-1)(\eta+1) \tag{3.1}$$

The derivatives of the form functions with respect to the global axis coordinates must be calculated so as to be able to constitute the basic matrices, the change of coordinates leads to:

$$\frac{\partial N_i}{\partial x} = \frac{1}{|J(\xi,\eta)|}\left(\frac{\partial N_i}{\partial \xi}\sum_{K=1}^{4} y_k \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta}\sum_{K=1}^{4} y_k \frac{\partial N_k}{\partial \xi}\right) = \frac{\partial N_i^e}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{1}{|J(\xi,\eta)|}\left(-\frac{\partial N_i}{\partial \xi}\sum_{K=1}^{4} x_k \frac{\partial N_k}{\partial \eta} + \frac{\partial N_i}{\partial \eta}\sum_{K=1}^{4} x_k \frac{\partial N_k}{\partial \xi}\right) = \frac{\partial N_i^e}{\partial \eta} \tag{3.2}$$

Combining the former expressions, the matrices M, A, B^x, B^y, C may be now expressed in terms of the derivatives with respect to the local axis coordinates. So, the matrix B^x (for instance) being:

$$\mathbf{B_x} = \iint \frac{1}{|J(\xi,\eta)|}\left(\frac{\partial N_i(\xi,\eta)}{\partial \xi}\sum_{K=1}^{4} y_k \frac{\partial N_k(\xi,\eta)}{\partial \eta} - \frac{\partial N_i(\xi,\eta)}{\partial \eta}\sum_{K=1}^{4} y_k \frac{\partial N_k(\xi,\eta)}{\partial \xi}\right)\left(-\frac{\partial N_i}{\partial \xi}\sum_{K=1}^{4} x_k \frac{\partial N_k}{\partial \eta} + \frac{\partial N_i}{\partial \eta}\sum_{K=1}^{4} x_k \frac{\partial N_k}{\partial \xi}\right) d\xi\, d\eta \tag{3.3}$$

The integration of expression (3.3) and that of the other matrices will be carried out, following a numerical 4-point two-dimensional Gauss integration, where the Gauss points are in $(\pm 0.5773, \pm 0.5773)$, and the weighting function H=1.

$$I = \iint F(\xi,\eta)\, d\xi d\eta = \sum_{i=1}^{2}\sum_{j=1}^{2} H_i H_j F(\xi_i, \eta_j) \tag{3.4}$$

NUMERICAL EXAMPLES

A FORTRAN code has been developed making use of the previously explained formulation. The program has been checked with the well known backward step and driven cavity flow tests and the results for velocity and pressure unknowns are shown bellow

The following figure shows the evolution in time of the behaviour of the streamlines for the backward step problem in which the eddy takes form for flow decreasing boundary conditions.

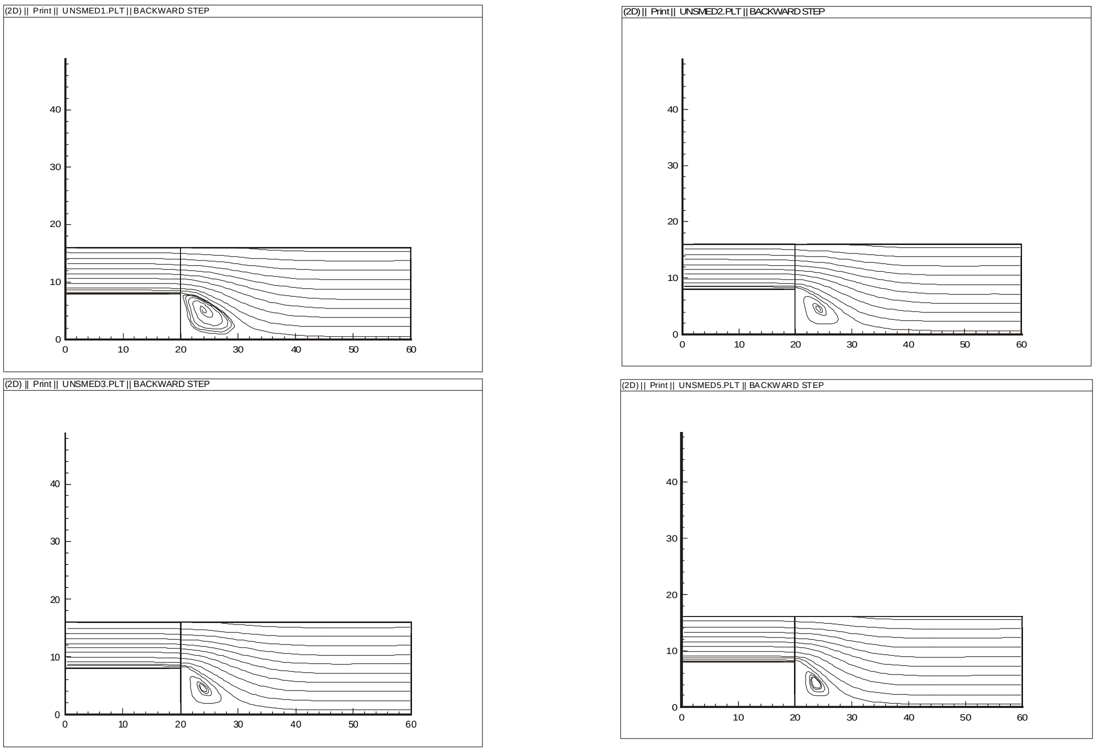


Fig 1. Backward step streamlines varying in time.

The results of the velocity field, both for the backward step and cavity flow are plotted bellow

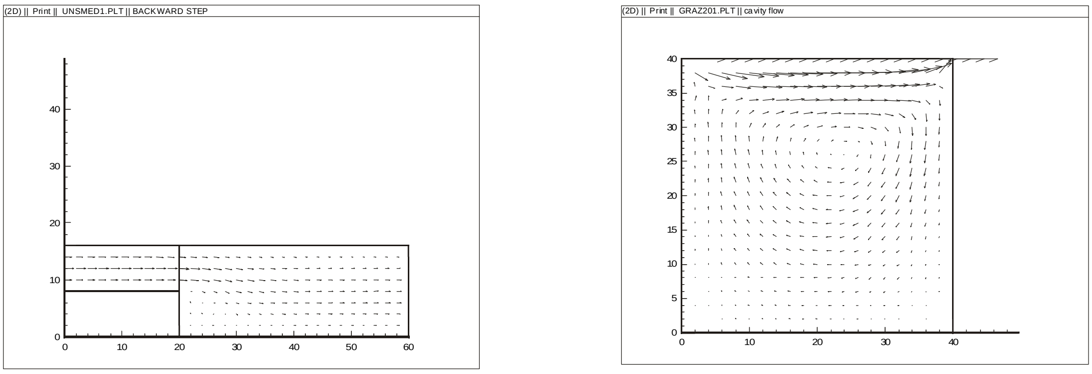


Fig 2. Velocity fields.

Pressure field numerical results in iso-lines and 3D graph

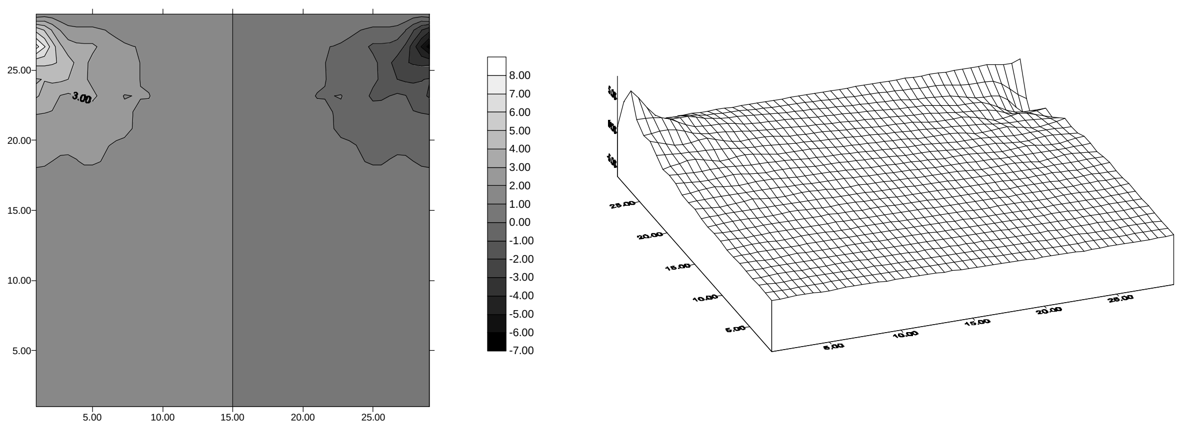


Fig 3 Pressure fields for the cavity flow problem.

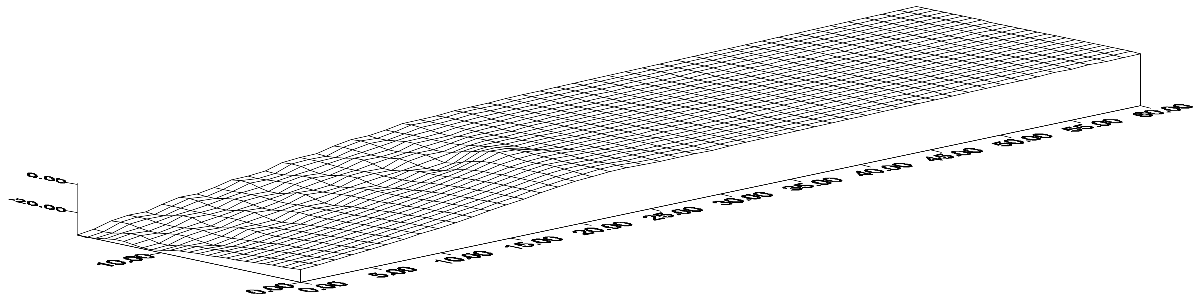


Fig 4 Pressure field for the backward (16x60) step flow

CONCLUSIONS AND FURTHER DEVELOPMENTS.

The results may be considered accurate enough for small Reynolds numbers, with its magnitude being a function of the mesh grade of refinement**.** Once the Reynolds number becomes of considerable magnitude (and so the convective term gets bigger compared with the diffusive term), the upwinding Petrov Galerking formulation together with the inclusion of an artificial streamline diffusion term (SUPG), should provide a computationaly speaking inexpensive and convenient approach. Next, a turbulence-considering scheme and a pattern being able to model transitions in flow, would be implemented.

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