# On local cross sections in topological abelian groups

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**Abstract**

We introduce the notion of local pseudo-homomorphism between two topological abelian groups. We prove that it is closely related with the widely studied notions of local cross sections and splitting extensions in the category of topological abelian groups. In the final section we present an example of a non-splitting extension of $(\mathbb{R}, \tau_\nu)$ by $\mathbb{R}$, where $\tau_\nu$ is a metrizable group topology on $\mathbb{R}$ weaker than the usual one. This extension admits a local cross section.

**Keywords.** Local cross section, extension of topological abelian groups, local pseudo-homomorphism, quasi-homomorphism, weakened Lie group
**MSC2010 codes.** 22A05, 54H11, 22E41

## 1 Introduction

An extension of topological abelian groups is a short exact sequence

$$E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0,$$

where $\imath$ and $\pi$ are continuous and open homomorphisms when considered as mappings onto their images. We say that the extension splits if there exists a continuous homomorphism $\rho : G \to X$ with $\pi \circ \rho = \mathrm{id}_G$. A mapping (not necessarily a homomorphism) $\rho : G \to X$ which is continuous on some neighborhood of zero and satisfies $\pi \circ \rho = \mathrm{id}_G$ is called a local cross section for $E$.

Local cross sections have been been studied since the 1950s. The first classical result in this direction, due to Gleason, can be formulated in this way: any extension of the form $E : 0 \to H \to X \to G \to 0$, where $G$ is a locally compact group and $H$ is a compact Lie group, admits a local cross section [7]. Some significant advances in these problems have also been obtained outside the class of locally compact groups, see for instance [11] or [2]. Not long ago, cross sections have been studied in connection with extensions involving the maximal precompact topology of an abelian group [5].

In this paper we explore the relationship between local cross sections and the splitting problem for extensions of topological abelian groups. The notion of local pseudohomomorphism is an essential tool for that exploration. We explain the background and terminology in Section 2. In Section 3 we present some sufficient conditions for the existence of local cross sections and establish the precise connection between (splitting) extensions and (approximable) local pseudo-homomorphisms. In Section 4 we apply these results to construct an example of a non-splitting extension $E : 0 \to \mathbb{R} \xrightarrow{\imath} X \xrightarrow{\pi} (\mathbb{R}, \nu) \to 0$ admitting a local cross section.

## 2 Background and terminology

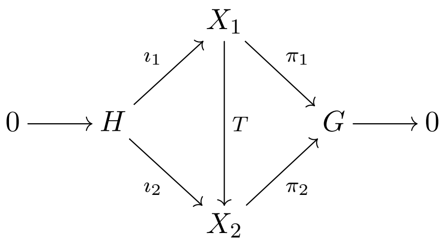
### 2.1 Algebraic extensions of groups

**Definition 2.1.** Let $G$ and $H$ be abelian groups. An *extension* of $G$ by $H$ is a short exact sequence of groups and homomorphisms

$$0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$$

where $X$ is an abelian group and 0 denotes a one-element group.

**Definition 2.2.** Let $G$ and $H$ be abelian groups. Let $E_j : 0 \to H \xrightarrow{\imath_j} X_j \xrightarrow{\pi_j} G \to 0$ $(j = 1, 2)$ be two extensions of $G$ by $H$. We say that $E_1$ and $E_2$ are *equivalent* if there exists a group isomorphism $T : X_1 \to X_2$ for which $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$.




It is an easy consequence of the Five Lemma that any group homomorphism $T : X_1 \to X_2$ making the above diagram commutative is actually an isomorphism.

**Definition 2.3.** We say that the extension of abelian groups $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *splits* if it is equivalent to the canonical extension $0 \to H \xrightarrow{\imath_H} H \times G \xrightarrow{\pi_G} G \to 0$.

The following Proposition characterizes splitting extensions in a convenient way for our purposes. The proof is left to the reader.

**Proposition 2.4.** *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of abelian groups. The following conditions are equivalent:*

*(i)* $E$ *splits.*

*(ii)* *There exists a homomorphism* $P : X \to H$ *with* $P \circ \imath = id_H$.

*(iii)* *There exists a homomorphism* $S : G \to X$ *with* $\pi \circ S = id_G$.

The following known result (see for instance [9, A1.35, A1.14]) characterizes the classes of free abelian groups and divisible abelian groups in terms of their behaviour with respect to splitting extensions:

**Proposition 2.5.** *(a) Let* $H$ *be an abelian group. Then* $H$ *is divisible if and only if every extension of abelian groups of the form* $0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ *splits.*

*(b) Let* $G$ *be an abelian group. Then* $G$ *is free (that is,* $G \cong \mathbb{Z}^{(I)}$ *for some index set* $I$*) if and only if every extension of abelian groups of the form* $0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ *splits.*

## 2.2 Extensions of topological abelian groups

**Definition 2.6.** Let $G$ and $H$ be topological abelian groups. An extension of $G$ by $H$ is a short exact sequence of topological abelian groups and continuous homomorphisms $E : 0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ where $0$ denotes a one-element group, and both $\imath$ and $\pi$ are relatively open.

**Definition 2.7.** Let $G$ and $H$ be topological abelian groups. Let $E_j : 0 \to H \stackrel{\imath_j}{\to} X_j \stackrel{\pi_j}{\to} G \to 0$ $(j = 1, 2)$ be two extensions of $G$ by $H$. We say that $E_1$ and $E_2$ are *equivalent* if there exists a topological isomorphism $T : X_1 \to X_2$ for which $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$.

Actually any continuous group homomorphism $T : X_1 \to X_2$ satisfying $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$ is already a topological isomorphism. This follows from the Five Lemma as above, and Merzon's Lemma [6, Lemma 1] .

**Definition 2.8.** We say that the extension of topological abelian groups $0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ *splits* if it is equivalent to the canonical extension $0 \to H \stackrel{\imath_H}{\to} H \times G \stackrel{\pi_G}{\to} G \to 0$.

**Definition 2.9.** We say that the extension of topological abelian groups $0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ *splits algebraically* if the underlying extension of abelian groups splits in the sense of Definition 2.3.

The following characterization is essential when dealing with extensions of topological abelian groups. It follows from Proposition 2.4 and elementary considerations involving continuity.

**Theorem 2.10.** *Let* $E : 0 \to H \stackrel{\imath}{\to} X \stackrel{\pi}{\to} G \to 0$ *be an extension of topological abelian groups. The following are equivalent:*

*(i)* *E splits.*

*(ii)* *There exists a continuous homomorphism* $P\colon X \to H$ *with* $P \circ \imath = \mathrm{id}_H$.

*(iii)* *There exists a continuous homomorphism* $S\colon G \to X$ *with* $\pi \circ S = \mathrm{id}_G$.

**Definition 2.11.** Let $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ be an extension of topological abelian groups. We say that $E$ admits a *local cross section* if there exists a map $s : G \to X$ such that $\pi \circ s = \mathrm{id}_G$ and $s$ is continuous on a neighborhood of zero in $G$. If this neighborhood of zero can be taken as the whole group $G$, we say that $E$ admits a *global cross section*.

We assume in what follows that any (global or local) cross section $s$ satisfies $s(0) = 0$.

If $X$ and $G$ are topological abelian groups and $\pi : X \to G$ is a quotient homomorphism, we say that $\pi$ admits a local cross section if the canonical extension $0 \to \operatorname{Ker} \pi \to X \to G \to 0$ has a local cross section in the sense of Definition 2.11.

The following simple example shows the difference between local and global cross sections.

**Example 2.12.** The canonical extension $E : 0 \to \mathbb{Z} \overset{\imath}{\to} \mathbb{R} \overset{\pi}{\to} \mathbb{T} \to 0$ admits a local cross section but not a global cross section.

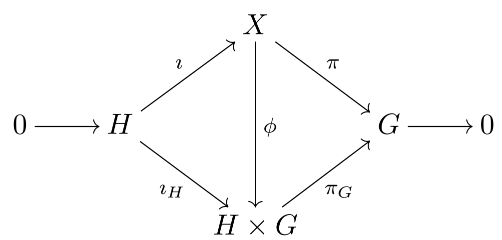
The proof of the following Proposition is straightforward.

**Proposition 2.13.** *Let* $E : 0 \to H \overset{\imath}{\to} X \overset{\pi}{\to} G \to 0$ *be an extension of topological abelian groups. The following are equivalent:*

1. *E has a local cross section.*

2. *There exists a mapping* $r : X \to H$ *continuous on a neighborhood of zero in* $X$ *and such that* $r \circ \imath = \mathrm{id}_H$ *and* $r(x + \imath(h)) = r(x) + h$ *for every* $h \in H$ *and* $x \in X$.

3. *There exists a bijective mapping* $\phi : X \to H \times G$ *such that both* $\phi$ *and* $\phi^{-1}$ *are continuous on a neighborhood of zero, making the following diagram commutative:*



## 3 Local pseudo-homomorphisms and local cross sections

The problem of existence of local cross sections of quotient maps has been thoroughly studied in the locally compact case. It was answered positively for finite dimensional quotient groups by P. S. Mostert in 1953 and for subgroups that are Lie groups by A. M. Gleason in 1950. The following Proposition contains these and some other results about the existence of local cross sections.

**Proposition 3.1.** *An extension* $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *of topological groups admits a local cross section in any of the following cases:*

*(1) [4, Proposition 3.4] $H$ or $G$ discrete*

*(2) ([7, Theorem 4.1]) $X$ locally compact and $H$ a compact Lie group*

*(3) [13, Theorem 8] $X$ locally compact, $G$ finite dimensional*

*(5) [11, Corollary 1.3] $X$ metrizable, $H$ complete and $G$ zero-dimensional.*

Another result involving the existence of local cross sections is the following (see [8] for the definition and properties of locally $k_\omega$ spaces):

**Proposition 3.2.** *Let* $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *be an extension of topological abelian groups where $H$ is compact and $G$ is locally $k_\omega$ and zero-dimensional. Then $E$ has a local cross section.*

*Proof.* It is known that a topological group is locally $k_\omega$ if and only if it has an open subgroup which is a $k_\omega$ space [8, Proposition 5.3]. Let $A$ be an open $k_\omega$ subgroup of $G$. In particular $A$ is zero-dimensional and by [2, Theorem 2.8] there exists a continuous map $r : A \to X$ such that $\pi \circ r = \mathrm{id}_A$. Any mapping $s : G \to X$ extending $r$ in such a way that $\pi \circ s = \mathrm{id}_G$ is a local cross section for $E$. □

Let $G$ and $H$ are topological abelian groups. To every algebraically splitting extension $E : 0 \to H \to X \to G \to 0$ with a local cross section one can associate in a natural way a mapping $\omega : G \to H$ which determines $E$ modulo equivalence of extensions. In particular this mapping carries the information whether or not $E$ splits. We next introduce the relevant definitions.

**Definition 3.3.** Let $G$ and $H$ be topological abelian groups. Let $\omega : G \to H$ be a mapping such that $\omega(0) = 0$. Define $\Delta_\omega : (x, y) \in G \times G \mapsto \omega(x + y) - \omega(x) - \omega(y) \in H$. We say that $\omega$ is

(a) a *quasi-homomorphism* if $\Delta_\omega$ is continuous at $(0, 0)$

(b) a *local pseudo-homomorphism* if $\Delta_\omega$ is continuous on a neighborhood of zero in $G \times G$.

(c) a *pseudo-homomorphism* if $\Delta_\omega$ is continuous on $G \times G$.

The notion of quasi-homomorphism was defined in [3]. Pseudo-homomorphisms were introduced and studied in [1]. The role of pseudo-homomorphisms in relation with cross sections is analogous to the role played by local pseudo-homomorphisms with respect to local cross sections. Approximations to these concepts from other points of view are studied in [4, 10, 12, 11].

**Lemma 3.4.** *Let $G$ and $H$ be topological abelian groups and $\omega : G \to H$ a map with $\omega(0) = 0$. Then $\omega$ is a local pseudo-homomorphism if and only if it satisfies the following properties:*

*(a) The map $\Delta_\omega : (x, y) \in G \times G \mapsto \omega(x + y) - \omega(x) - \omega(y) \in H$ is continuous at $(0, 0)$.*

*(b) There exists $U \in \mathcal{N}_0(G)$ such that if the net $(x_\alpha)$ converges to $x \in U$, then $\omega(x_\alpha) - \omega(x_\alpha - x) \to \omega(x)$.*

*Proof.* If $\omega$ is a local pseudo-homomorphism, (a) is trivially true. Let us prove (b): Let $U$ be a symmetric neighborhood of zero in $G$ such that $\Delta_\omega$ is continuous on $U \times U$. Let $x_\alpha \to x$ in $U$. From the continuity of $\Delta_\omega$ it follows that $\Delta_\omega(x_\alpha, -x) \to \Delta_\omega(x, -x)$. Hence $\omega(x_\alpha - x) - \omega(x_\alpha) - \omega(-x) \to \omega(x - x) - \omega(x) - \omega(-x)$, i. e.

$$\omega(x_\alpha) - \omega(x_\alpha - x) \to \omega(x).$$

Conversely, assume that both (a) and (b) are true. Fix $U \in \mathcal{N}_0(G)$ as in (b). Let us prove that $\Delta_\omega$ is continuous in $V \times V$ where $V$ is a symmetric neighborhood of zero in $G$ such that $V + V \subset U$. Pick two nets $(x_\alpha)_{\alpha \in A} \to x$ in $V$ and $(y_\alpha)_{\alpha \in A} \to y$ in $V$. By condition (b)

$$\begin{aligned} \omega(x_\alpha) - \omega(x_\alpha - x) &\to \omega(x), \\ \omega(y_\alpha) - \omega(y_\alpha - y) &\to \omega(y), \\ \omega(x_\alpha + y_\alpha) - \omega(x_\alpha + y_\alpha - x - y) &\to \omega(x + y) \end{aligned}$$

By condition (a)

$$\omega(x_\alpha - x + y_\alpha - y) - \omega(x_\alpha - x) - \omega(y_\alpha - y) \to 0.$$

Using the structure of a topological group it easily follows that

$$\omega(x_\alpha + y_\alpha) - \omega(x_\alpha) - \omega(y_\alpha) \to \omega(x + y) - \omega(x) - \omega(y).$$

Since $x$ and $y$ are arbitrary elements of $V$, we deduce that $\Delta_\omega$ is continuous on $V \times V$. □

**Definition 3.5.** A local pseudo-homomorphism $\omega$ is said to be approximable if there exists a homomorphism $a : G \to H$ such that $\omega - a$ is continuous on a neighborhood of zero.

In the next Proposition we show that it is enough to require continuity at zero in Definition 3.5.

**Proposition 3.6.** *A local pseudo-homomorphism $\omega : G \to H$ is approximable if there exists a homomorphism $a : G \to H$ such that $\omega - a$ is continuous at* $0$.

*Proof.* Note that $\omega - a$ is a local pseudo-homomorphism and actually $\Delta_{\omega-a} = \Delta_\omega$. Let $U \in \mathcal{N}_0(G)$ be as in Lemma 3.4(b). Let us show that $\omega - a$ is continuous on $U$. Fix a net $(x_\alpha)_{\alpha\in A}$ in $G$ which converges to $x \in U$. By Lemma 3.4(b) we have $(\omega - a)(x_\alpha) - (\omega - a)(x_\alpha - x) \to (\omega - a)(x)$. Since $\omega - a$ is continuous at zero and $x_\alpha \to x$, we deduce $(\omega - a)(x_\alpha) \to (\omega - a)(x)$, as required. □

Local cross sections are closely related with local pseudo-homomorphisms, as we show in the two following propositions:

**Proposition 3.7.** *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. Assume that $E$ splits algebraically and admits a local cross section $\rho : G \to X$. Let $P : X \to H$ be a group homomorphism such that $P \circ \imath = \mathrm{id}_H$.*

*Then $\omega = P \circ \rho$ is a local pseudo-homomorphism. Moreover, the extension $E$ splits if and only if $\omega$ is approximable.*

*Proof.* Assume that $\rho$ is continuous on $W \in \mathcal{N}_0(G)$. Then

$$\begin{aligned}\Delta_\omega(x, y) = \omega(x + y) - \omega(x) - \omega(y) &= P(\rho(x + y) - \rho(x) - \rho(y)) \\ &= \imath^{-1}(\rho(x + y) - \rho(x) - \rho(y))\end{aligned}$$

(because $\rho(x + y) - \rho(x) - \rho(y) \in \operatorname{Ker}\pi = \imath(H)$) and we deduce $\Delta_\omega = \imath^{-1} \circ \Delta_\rho$. It is immediate to show that $\Delta_\rho$ is continuous on $W' \times W'$ for every $W' \in \mathcal{N}_0(G)$ such that $W' + W' \subset W$. Since $\imath^{-1}$ is continuous on $\imath(H)$, we deduce that $\Delta_\omega$ is continuous on $W' \times W'$.

Assume that $E$ splits. Let $S : G \to X$ be a continuous homomorphism such that $\pi \circ S = \mathrm{id}_G$. Note that for every $g \in G$ we have

$$(P \circ \rho - P \circ S)(g) = P(\rho(g) - S(g)) = \imath^{-1}(\rho(g) - S(g))$$

since $\rho(g) - S(g) \in \operatorname{Ker}\pi = \imath(H)$. This clearly implies that $\omega - P \circ S = P \circ \rho - P \circ S$ is continuous on $W$, and in particular $\omega$ is approximable.

Conversely, assume that $\omega = P \circ \rho$ is approximable. Let $a : G \to H$ be a homomorphism such that $P \circ \rho - a = f$ is continuous at zero. Note that every $x \in X$ can be expressed as

$$x = \rho(\pi(x)) + (x - \rho(\pi(x))) = \rho(\pi(x)) + \imath(\imath^{-1}(x - \rho(\pi(x))))$$

since $x - \rho(\pi(x)) \in \operatorname{Ker}\pi = \imath(H)$.

Applying $P$ on both sides we obtain $P(x) = (a + f)(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$. This suggests the definition of $\tilde{P} : X \to H$ as

$$\tilde{P}(x) = P(x) - a(\pi(x)) = f(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$$

for every $x \in X$. From the expression $\tilde{P}(x) = P(x) - a(\pi(x))$ it easily follows that $\tilde{P}$ is a homomorphism and a left inverse for $\imath$. From $\tilde{P}(x) = f(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$ it is clear that $\tilde{P}$ is continuous at zero, hence globally continuous. By Theorem 2.10, $E$ splits. □

Note that by Proposition 3.7, all examples of extensions $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ with local cross sections given in Propositions 3.1 and 3.2 can be presented as examples of local pseudo-homomorphisms $\omega : G \to H$ if we add the hypothesis that the extension splits algebraically. This is the case for instance if the group $H$ is divisible (Proposition 2.4).

The following result is a natural converse to Proposition 3.7.

**Proposition 3.8.** *Let $G$ and $H$ be topological abelian groups and let $\omega : G \to H$ be a local pseudo-homomorphism. There exist an extension of topological abelian groups $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$, a homomorphism $P : X \to H$ and a local cross section $\rho$ for $E$ such that $\omega = P \circ \rho$.*

*Proof.* As mentioned in [3, Lemma 2], it is not difficult to show that the family of sets $\mathcal{W}(V,U) = \{(h,g) \in H \times G \,:\, g \in U,\ h \in \omega(g) + V\}$ where $U \in \mathcal{N}_0(G)$ and $V \in \mathcal{N}_0(H))$ is a basis of neighborhoods of zero for a group topology $\tau_\omega$ on $H \times G$. Let $X$ be the group $(H \times G, \tau_\omega)$. Define $P : X \to H$ as $P(h,g) = h$ and $\rho : G \to X$ as $\rho(g) = (\omega(g), g)$.

Assume that $\Delta_\omega$ is continuous on a neighborhood $S$ of zero in $G \times G$. Let $W \in \mathcal{N}_0(G)$ be such that $W \times (-W) \subset S$. We are going to prove that $\rho$ is continuous on $W$. This means that for every $g \in W$, $V \in \mathcal{N}_0(H)$ and $U \in \mathcal{N}_0(G)$ we need to find $U' \in \mathcal{N}_0(G)$ such that

$$\rho(g + U') \subseteq \rho(g) + \mathcal{W}(V,U).$$

Fix $g \in W$, $V \in \mathcal{N}_0(H)$ and $U \in \mathcal{N}_0(G)$. We may assume that $V$ is symmetric. Consider the pair $(g, -g)$ which, by our choice of $W$, is in $S$. Since $\Delta_\omega$ is continuous at $(g,-g)$ by hypothesis, from $V \in \mathcal{N}_0(H)$ we obtain $\tilde{U} \in \mathcal{N}_0(G)$ with

$$[(g_1, g_2) - (g, -g) \in \tilde{U} \times \tilde{U} \Rightarrow \Delta_\omega(g_1, g_2) - \Delta_\omega(g, -g) \in V].$$

Let $U' \in \mathcal{N}_0(G)$ be such that $U' \subset \tilde{U} \cap U$ and take any $g' \in G$ with $g' - g \in U'$. Let us check that $\rho(g + U') \subseteq \rho(g) + \mathcal{W}(V,U)$. This means that

$$g' - g \in U' \Rightarrow \begin{cases} (1) & g' - g \in U \\ (2) & \omega(g') - \omega(g) \in \omega(g' - g) + V \end{cases}$$

Fix $g' \in G$ with $g' - g \in U'$. Clearly $g'$ satisfies condition (1). To prove (2), consider the element $(g_1, g_2) = (g', -g) \in G \times G$. Note that $(g', -g) - (g, -g) = (g' - g, 0) \in \tilde{U} \times \tilde{U}$ since $U' \subset \tilde{U}$. By our choosing of $\tilde{U}$ we have $\Delta_\omega(g', -g) - \Delta_\omega(g, -g) \in V$. It is easy to check that

$$\Delta_\omega(g', -g) - \Delta_\omega(g, -g) = \omega(g' - g) - \omega(g') + \omega(g) \in V = -V,$$

from which we obtain condition (2).

□

# 4 Weakened topologies on $\mathbb{R}$ providing non-splitting extensions

We devote the remainder of the paper to showing that there exists a non-splitting extension of $(\mathbb{R}, \tau_\nu)$ by $(\mathbb{R}, \tau)$ which admits a local cross section, where $\tau_\nu$ is a metrizable group topology on $\mathbb{R}$, weaker than the usual one. This will also provide an example of a local pseudo-homomorphism that is not approximable. To this end, we will use the technique for weakening Euclidean topologies introduced in [15] by the fourth-named author.

In what follows the notation $\|\cdot\|$ always represents Euclidean norms, and $\tau^n$ stands for the usual topology on $\mathbb{R}^n$. We abbreviate $\tau^1$ to $\tau$. The notation $\mathbb{Z}^{(\mathbb{N})}$ represents the subgroup of the product $\mathbb{Z}^{\mathbb{N}}$ formed by those sequences in $\mathbb{Z}$ with only finitely many nonzero elements. The integer part of the nonnegative real number $x$ is denoted by $\lfloor x \rfloor$.

**Definition 4.1.** A groupnorm $\nu$ on an abelian group $G$ is a mapping $\nu : G \to [0, \infty)$ satisfying the following conditions:

(a) $\nu(g) = 0$ if and only if $g = 0$

(b) $\nu(-g) = \nu(g)$ for every $g \in G$

(c) $\nu(g + h) \leq \nu(g) + \nu(h)$ for every $g, h \in G$.

If $\nu$ is a groupnorm on $G$, the family of sets $\{B_\nu(\varepsilon)\}_{\varepsilon > 0}$, where $B_\nu(\varepsilon) = \{g \in G : \nu(g) < \varepsilon\}$ is a basis of neighborhoods of zero for a metrizable group topology $\tau_\nu$ on $G$.

In order to proceed with our example we need to introduce the following definitions:

**Definition 4.2.** A sequential norming pair (SNP) on $\mathbb{R}^n$ is a pair $(\{v_j\}, \{p_j\})$ where

- $\{v_j\}$ is a sequence of elements of $\mathbb{R}^n$ such that $0 < \|v_j\| \leq \|v_{j+1}\|$ for every $j$,

- $\{p_j\}$ is a sequence of real numbers which converges to zero in the usual topology and such that $0 < p_{j+1} \leq p_j$ for every $j$,

- there exists a positive lower bound for the sequence $\{p_{j+1}\|v_{j+1}\|/\|v_j\|\}$.

For instance, $\{(j!), (1/j)\}$ is a SNP on $\mathbb{R}$.

**Definition 4.3.** Let $(\{v_j\}, \{p_j\})$ be a SNP on $\mathbb{R}^n$ and let $\{y_j\}$ be any sequence in $\mathbb{R}^m$. Consider the sequence $\{(v_j, y_j)\}$ in $\mathbb{R}^{n+m}$. The pair $(\{(v_j, y_j)\}, \{p_j\})$ is said to be an *extended norming pair* (ENP) on $\mathbb{R}^{n+m}$ associated to the SNP $(\{v_j\}, \{p_j\})$ on $\mathbb{R}^n$.

**Facts 4.4.** (a) [15, Proposition 4.1] Let $(\{v_j\}, \{p_j\})$ be a SNP on $\mathbb{R}^n$. The function

$$\nu : x \in \mathbb{R}^n \to \nu(x) = \inf\{\sum |c_j|p_j + \|x - \sum c_j v_j\| : \{c_j\} \in \mathbb{Z}^{(\mathbb{N})}\}$$

is a groupnorm on $\mathbb{R}^n$ which satisfies $\nu(x) \leq \|x\|$ for every $x \in \mathbb{R}^n$ and $\nu(v_j) \leq p_j$ for every $j \in \mathbb{N}$. We will call $\nu$ the groupnorm associated to the SNP $(\{v_j\}, \{p_j\})$.

(b) [14, Proposition 5] Let $(\{(v_j, y_j)\}, \{p_j\})$ be a ENP on $\mathbb{R}^{n+m}$ The function

$$\mu : x \in \mathbb{R}^{n+m} \to \mu(x) = \inf\{\sum |c_j|p_j + \|x - \sum c_j(v_j, y_j)\| : \{c_j\} \in \mathbb{Z}^{(\mathbb{N})}\}$$

is a groupnorm on $\mathbb{R}^{n+m}$ which satisfies $\mu(x) \leq \|x\|$ for every $x \in \mathbb{R}^{n+m}$ and $\mu(v_j, y_j) \leq p_j$ for every $j \in \mathbb{N}$. We will call $\mu$ the groupnorm associated to the ENP $(\{(v_j, y_j)\}, \{p_j\})$.

Let $(\{v_j\}, \{p_j\})$ be a SNP in $\mathbb{R}^n$ and let $\nu$ be its associated groupnorm. For any $x \in \mathbb{R}^n$, there are infinitely many ways to write $x$ in the form $x = \sum c_j v_j + z$, where $\{c_j\} \in \mathbb{Z}^{(\mathbb{N})}$ and $z \in \mathbb{R}^n$, and for each of them we have $\nu(x) \leq \sum |c_j|p_j + \|z\|$. For the SNP $(\{j!\}, \{1/j\})$, for example, we can write 20 as $3(3!) + 1(2!)$ and also as $1(4!) - 2(2!)$. We are interested in the case when there is a unique "best" representation for $x$, in the sense that $\nu(x) = \sum |c_j|p_j + \|z\|$. This happens for those $x$ such that $\nu(x)$ is sufficiently small, as we establish in the following Proposition. Its proof is contained in the proofs of Theorem 8 and Lemma 13 in [14] (especially pp. 57-58).

**Proposition 4.5.** *Let $(\{v_j\}, \{p_j\})$ be a SNP on $\mathbb{R}^n$ such that*

$$p_j \lfloor \frac{\|v_{j+1}\|}{\|v_j\|} - 1 \rfloor \geq 1 \quad \textit{for every } j$$

*and let $\nu$ be its associated groupnorm. There exists $r > 0$ such that any $x \in \mathbb{R}^n$ with $\nu(x) < r$ can be uniquely expressed as $x = \sum c_j v_j + z$ where $\{c_j\} \in \mathbb{Z}^{(\mathbb{N})}$ and $z \in \mathbb{R}^n$ are such that $\sum |c_j|p_j + \|z\| < r$. Moreover, we have $\nu(x) = \sum |c_j|p_j + \|z\|$.*

After these preliminaries, we can now prove the theorem that contains our example.

**Theorem 4.6.** *Let $(\{(v_j, y_j)\}, \{p_j\})$ be an ENP on $\mathbb{R}^{n+m}$ associated to the SNP $(\{v_j\}, \{p_j\})$ on $\mathbb{R}^n$. Assume that*

$$p_j \lfloor \frac{\|v_{j+1}\|}{\|v_j\|} - 1 \rfloor \geq 1 \quad \textit{for every } j.$$

*Let $\nu$ and $\mu$ be the groupnorms associated to the above SNP and ENP, respectively.*

*(a) If $\tau_\nu$ and $\tau_\mu$ are the group topologies induced by the groupnorms $\nu$ and $\mu$, respectively, then the sequence $E : 0 \to (\mathbb{R}^m, \tau^m) \xrightarrow{\imath} (\mathbb{R}^{n+m}, \tau_\mu) \xrightarrow{\pi} (\mathbb{R}^n, \tau_\nu) \to 0$ is an extension of topological abelian groups which admits a local cross section, where*

$$\begin{array}{rcccl} \imath : & (\mathbb{R}^m, \tau^m) & \longrightarrow & (\mathbb{R}^{n+m}, \tau_\mu) \\ & y & \longmapsto & (0, y) \end{array} \qquad \begin{array}{rcccl} \pi : & (\mathbb{R}^{n+m}, \tau_\mu) & \longrightarrow & (\mathbb{R}^n, \tau_\nu) \\ & (x, y) & \longmapsto & x \end{array}$$

*(b) If $n = m = 1$, $\{y_j\}$ does not converge to zero in $\tau$ and $\{y_j/v_j\}$ converges to zero in $\tau$, the extension $E : 0 \to (\mathbb{R}, \tau) \xrightarrow{\imath} (\mathbb{R}^2, \tau_\mu) \xrightarrow{\pi} (\mathbb{R}, \tau_\nu) \to 0$ does not split.*

*Proof.* (a) It is clear that $E$ is an exact sequence. The fact that $\imath$ is an embedding is proved as Proposition 7 in [14]. It follows from the definitions of $\nu$ and $\mu$ that $\nu(\pi(x, y)) = \nu(x) \leq \mu(x, y)$, and thus $\pi$ is continuous. The remaining assertions are a consequence of the following claim:

Claim: *If $r > 0$ is as in Proposition 4.5, the mapping $\rho : (\mathbb{R}^n, \tau_\nu) \to (\mathbb{R}^{n+m}, \tau_\mu)$ defined by*

$$\rho(x) = \begin{cases} \sum c_j(v_j, y_j) + (z, 0) & \textit{if } \nu(x) < r \\ (x, 0) & \textit{otherwise} \end{cases}$$

*is a local cross section for $E$ such that $\mu(\rho(x)) = \nu(x)$ for every $x \in \mathbb{R}^n$ with $\nu(x) < r$. Here $\{c_j\} \in \mathbb{Z}^{(\mathbb{N})}$ and $z \in \mathbb{R}^n$ are uniquely defined by the decomposition $x = \sum c_j v_j + z$ with $\nu(x) = \sum |c_j| p_j + \|z\|$ .*

Proof of the Claim: It is clear that $\pi \circ \rho = \mathrm{id}_{\mathbb{R}^n}$. Fix $x \in \mathbb{R}^n$ with $\nu(x) < r$; let us prove that $\mu(\rho(x)) = \nu(x)$. As noted above, for every $(x, y) \in \mathbb{R}^{n+m}$ we have $\nu(x) \leq \mu(x, y)$. Since $\pi(\rho(x)) = x$, it follows that $\nu(x) \leq \mu(\rho(x))$. On the other hand, the triangle inequality implies that $\mu(\rho(x)) = \mu(\sum c_j(v_j, y_j) + (z, 0)) \leq \mu(\sum c_j(v_j, y_j)) + \mu(z, 0) \leq \sum |c_j| p_j + \|(z, 0)\| = \nu(x)$.

In particular $\rho$ is continuous at zero, which clearly implies that $\pi$ is open. This completes the proof of (a).

(b) To prove (b), it suffices to show that the only continuous homomorphism from $(\mathbb{R}^2, \tau_\mu)$ to $(\mathbb{R}, \tau)$ is the trivial one. Let $f : (\mathbb{R}^2, \tau_\mu) \to (\mathbb{R}, \tau)$ be a continuous homomorphism. Since $\tau_\mu$ is weaker than $\tau^2$, $f$ is continuous with respect to the usual topologies. Hence $f(x, y) = ax + by$ for some $a, b \in \mathbb{R}$. Since $(v_j, y_j) \to 0$ in $\tau_\mu$ and $f$ is continuous, we deduce $av_j + by_j \to 0$ in $(\mathbb{R}, \tau)$. Since $\{|v_j|\}$ is increasing and nonzero, the sequence $(av_j + by_j)/v_j$ converges to zero, too. From this fact and our hypothesis $y_j/v_j \to 0$ it follows that $a = 0$ and hence $by_j \to 0$ which (since $y_j$ does not converge to zero) implies that $b = 0$. □

**Remark 4.7.** An instance where the requirements in Theorem 4.6(b) are fulfilled is $v_j = j!$, $y_j = j$, $p_j = 1/j$.

**Corollary 4.8.** *With the notations and hypotheses of Theorem 4.6(b), the mapping*

$$\omega : (\mathbb{R}, \tau_\nu) \to (\mathbb{R}, \tau), \quad \omega(x) = \begin{cases} \sum c_j y_j & \text{if } \nu(x) < r \\ 0 & \text{otherwise} \end{cases}$$

*is a not approximable, local pseudo-homomorphism.*

*Proof.* This follows at once from Theorem 4.6 and Proposition 3.7. □

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