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Alonso-Meijide, J.M., Costa, J., García-Jurado, I. *et al.* On egalitarian values for cooperative games with a priori unions. *TOP* **28**, 672–688 (2020). <https://doi.org/10.1007/s11750-020-00553-2>.

Link to published version: <https://doi.org/10.1007/s11750-020-00553-2>

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# On egalitarian values for cooperative games with a priori unions\*

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## Abstract

In this paper we extend the equal division and the equal surplus division values for transferable utility cooperative games to the more general setup of transferable utility cooperative games with a priori unions. In the case of the equal surplus division value we propose three possible extensions. We provide axiomatic characterizations of the new values. Furthermore, we apply the proposed modifications to a particular motivating example and compare the numerical results with those obtained with the original values.

**Keywords:** cooperative games, a priori unions, equal division value, equal surplus division value.

## 1 Introduction

Many economic problems deal with situations in which several agents cooperate to generate benefits or to reduce costs. Cooperative game theory

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\*This work has been supported by the ERDF, the MINECO/AEI grants MTM2017-87197-C3-1-P, MTM2017-87197-C3-3-P, and by the Xunta de Galicia (Grupos de Referencia Competitiva ED431C-2016-015 and ED431C-2017/38 and Centro Singular de Investigación de Galicia ED431G/01). The authors would like to thank two anonymous referees for their helpful suggestions to improve this article.

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studies procedures to allocate the resulting benefits (or costs) among the cooperating agents in those situations.

One of the most commonly used allocating procedures is the Shapley value, introduced in Shapley (1953) and analyzed more recently in Moretti and Patrone (2008) or in Algaba et al. (2019). Very often, however, agents cooperate on the basis of a kind of egalitarian principle according to which the benefits will be shared equitably. For instance, Selten (1972) indicates that egalitarian considerations explain in a successful way observed outcomes in experimental cooperative games.

In recent years, the game theoretical literature has dealt with several egalitarian solutions in cooperative games. For instance, van den Brink (2007) provides a comparison of the equal division value and the Shapley value, and Casajus and Hüttner (2014) compare those two solutions with the equal surplus division value (studied first in Driessen and Funaki, 1991). In van den Brink and Funaki (2009), Chun and Park (2012), van den Brink et al. (2016), Ferrières (2017) and Béal et al. (2019) several axiomatic characterizations of the equal division and equal surplus division values are provided. Ju et al. (2007) introduce and characterize the consensus value, a new solution that somewhat combines the Shapley value and the equal division rule. Dutta and Ray (1989) introduce the egalitarian solution for cooperative games, closely related to Lorenz dominance, that considers cooperating agents who believe in equality as a desirable social goal and negotiate accordingly; this solution was later characterized by Dutta (1990), Klijn et al. (2000) and Arín et al. (2003), and modified by Dietzenbacher et al. (2017).

Another stream of literature in cooperative game theory started in Owen (1977), where a variant of the Shapley value for games with a priori unions is introduced and characterized. In a game with a priori unions there exists a partition of the set of players, whose classes are called unions, that is interpreted as an a priori coalition structure that conditions the negotiation among the players and, consequently, modifies the fair outcome of the negotiation. There is a large literature concerning the Owen value and its applications; just to cite some recent papers, Lorenzo-Freire (2016) provides new axiomatic characterizations of the Owen value, Costa (2016) deals with an application in a cost allocation problem, and Saavedra-Nieves et al. (2018) propose a sampling procedure to approximate it. Not only the Shapley value but also other values have been modified for the case with a priori unions. For instance, Alonso-Meijide and Fiestras-Janeiro (2002) deal with the Banzhaf value for games with a priori unions, Casas-Méndez et al. (2003) introduce the  $\tau$ -value for games with a priori unions, Alonso-Meijide

et al. (2011) study the Deegan-Packel index for simple games with a priori unions, and Hu et al. (2019) introduce an egalitarian efficient extension of the Aumann-Drèze value (Aumann and Drèze, 1974). Finally, the literature of games with a priori unions has developed in many other directions. For instance, Alonso-Meijide et al. (2014) analyze an extension of the Shapley value for games with a priori unions alternative to the Owen value, Vázquez-Brage et al. (1996) and van den Brink et al. (2015) introduce and study values for games with graph-restricted communication and a priori unions, and Hu (2019) deals with a weighted value for games with a priori unions.

In this paper we modify the equal division value and the equal surplus division value for games with a priori unions. In Section 2 we illustrate the interest of our study describing a cost allocation problem that arises in the installation of an elevator in an apartment building. In Section 3 we define and characterize the equal division rule for games with a priori unions. In Section 4 we introduce and characterize three alternative extensions of the equal surplus division rule for games with a priori unions. In Section 5 we include some final remarks.

## 2 An example

In this section we consider an example where the owners of apartments in a building have agreed to install an elevator and share the corresponding costs. This example is inspired by a problem analyzed in Crettez and Deloche (2019) from the point of view of French legislation. The French Law on Apartment Ownership of Buildings does not provide a precise method for sharing the cost of an improvement but indicates that the co-owners must pay “in proportion to the advantages” they will receive. In the case of elevators in France, Crettez and Deloche (2019) indicate that there is a de facto sharing method that they call the *elevator rule*. The elevator rule associates a parameter  $\lambda_i = 1 + (i - 1)/2$  to each floor  $i$  and distributes the total cost among the floors proportionally to those parameters; then the cost allocated to each floor is divided equally among its apartments. In their paper Crettez and Deloche study the elevator rule and other proposals in the spirit of the French legislation.

However, Crettez and Deloche (2019) explain that in other European countries the legislation is based on principles of egalitarian character. For example, in the Netherlands each of the owners of the apartments must “participate for an equal part in the debts and costs which are for account of all apartments owned pursuant to law or the internal arrangements, unless

the internal arrangements provide for another proportion of participation.”

The Spanish Horizontal Property Law 49/1960 (modified by the Act 8/2013) indicates that “to each apartment or local will be attributed a quota of participation in relation to the total of the value of the building (. . .). This quota will serve as a module to determine the participation in the burdens and benefits due to the community.” These quotas generally depend on the surface area of each apartment but can take into account other aspects.

In a particular example, let us see how the Dutch and Spanish rules would share the costs of installing an elevator. Consider the following three-storey building with no apartments or offices on the ground floor: on the first floor there is a single apartment of 180 square meters, on the second floor there are two apartments, one of 100 and other one of 90 square meters, and on the third floor there are three apartments of 60 square meters each. The second floor has a slightly larger area because one of the two apartments on the floor has an additional gallery. Suppose now that the cost of installing the elevator is 120 (in thousands of euros), 50 of which correspond to the machine, 40 to the works to make the hollow of the elevator, and 30 to the works to be done on each floor to allow access to the elevator (10 in each of them). Table 1 below shows the distribution of costs for each of the apartments according to the Dutch and Spanish rules (the latter with quotas for each apartment given by its surface). Notice that both rules are based on egalitarian principles and can be interpreted as the equal division rule; the difference is that in the case of the Dutch rule the subjects that receive the equitable distribution are the apartments, whereas in the case of the Spanish rule the equitable distribution subjects are the quota units.<sup>1</sup> Notice that the same egalitarian spirit of these rules can be maintained despite changing the equitable distribution subjects. For instance, it would be natural to consider a kind of two-step equitable distribution subjects, where the subjects in the first step are the floors and the subjects in the second step are the apartments (in the case of the Dutch rule) or the quota units (in the case of the Spanish rule). This would result in the distribution of costs shown in Table 2 below. Observe that this variation arises from considering that the floors of the building naturally give rise to a structure of a priori unions in the sense of Owen (1977) and, thus, the convenience of extending the equal division value for games with a priori unions emerges spontaneously in this example. We do it formally in Section 3. Table 2 also displays the distribution proposed by the elevator rule that is by definition

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<sup>1</sup>In this example the quota units are the square meters of the apartments. For the approach we adopt to be meaningful, the quota unit numbers must be integers.

a two-step rule.

There are other possible variations of these Dutch and Spanish rules with and without the structure of a priori unions when using the equal surplus division value instead of the equal division value. Thus, the convenience of extending the equal surplus division value for games with a priori unions can also be motivated on the basis of this example. We do it in Section 4, where we also analyse in more depth how the equal surplus division value for games with a priori unions can be applied in the example we have discussed in this section.

	Dutch rule	Spanish rule
3rd floor	20 20 20	13.0909 13.0909 13.0909
2nd floor	20 20	21.8182 19.6364
1st floor	20	39.2727

Table 1: Distribution according to the Dutch and Spanish rules

	Dutch rule	Spanish rule	elevator rule
3rd	13.3333 13.3333 13.3333	13.3333 13.3333 13.3333	17.7777 17.7777 17.7777
2nd	20 20	21.0526 18.9474	20 20
1st	40	40	26.6666

Table 2: Distribution according to the two-step Dutch and Spanish rules and to the elevator rule

### 3 The equal division value for TU-games with a priori unions

In this section we extend the equal division value for TU-games to the more general setup of TU-games with a priori unions. To start with, we introduce the basic concepts and notations we use in this paper.

A transferable utility cooperative game (from now on a TU-game) is a pair  $(N, v)$  where  $N$  is a finite set of  $n$  players, and  $v$  is a map from  $2^N$  to  $\mathbb{R}$  with  $v(\emptyset) = 0$ , that is called the characteristic function of the game. In the sequel,  $\mathcal{G}_N$  will denote the family of all TU-games with player set  $N$  and  $\mathcal{G}$  the family of all TU-games. A value for TU-games is a map  $f$  that assigns

to every TU-game  $(N, v) \in \mathcal{G}$  a vector  $f(N, v) = (f_i(N, v))_{i \in N} \in \mathbb{R}^N$  with  $\sum_{i \in N} f_i(N, v) = v(N)$ .<sup>2</sup>

As it was remarked in the introduction, sometimes agents cooperate on the basis of a kind of egalitarian principle according to which the benefits will be shared equitably. This gives rise to the equal division value  $ED$  that distributes  $v(N)$  equally among the players in  $N$ . Formally, the equal division value  $ED$  is defined for every  $(N, v) \in \mathcal{G}$  and for all  $i \in N$  by

$$ED_i(N, v) = \frac{v(N)}{n}.$$

Now denote by  $P(N)$  the set of all partitions of  $N$ . A TU-game with a priori unions is a triplet  $(N, v, P)$  where  $(N, v) \in \mathcal{G}$  and  $P = \{P_1, \dots, P_m\} \in P(N)$ . The set of TU-games with a priori unions and with player set  $N$  will be denoted by  $\mathcal{G}_N^U$ , and the set of all TU-games with a priori unions by  $\mathcal{G}^U$ . A value for TU-games with a priori unions is a map  $g$  that assigns to every  $(N, v, P) \in \mathcal{G}^U$  a vector  $g(N, v, P) = (g_i(N, v, P))_{i \in N} \in \mathbb{R}^N$  with  $\sum_{i \in N} g_i(N, v, P) = v(N)$ . The next definition provides the natural extension of the equal division value to TU-games with a priori unions.

**Definition 3.1** The equal division value for TU-games with a priori unions  $ED^U$  is defined by

$$ED_i^U(N, v, P) = \frac{v(N)}{mp_k}$$

for all  $i \in N$  and all  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and  $i \in P_k$ ;  $p_k$  denotes the cardinal of  $P_k$ .

Notice that the equal division value for TU-games with a priori unions has been used in the motivating example in Section 2 (see Table 2 and the corresponding comments). Next we provide an axiomatic characterization of this value. We start giving some properties of a value  $g$  for TU-games with a priori unions.

**Additivity (ADD).** A value  $g$  for TU-games with a priori unions satisfies additivity if, for all  $(N, v, P), (N, w, P) \in \mathcal{G}^U$ , it holds that

$$g(N, v + w, P) = g(N, v, P) + g(N, w, P).$$

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<sup>2</sup>Notice that we have included the efficiency in the definition of value. We could have considered it as one more property and then it would have appeared explicitly in the characterizations; nothing relevant would have changed in that case.

Take a TU-game  $(N, v) \in \mathcal{G}^N$  and  $i, j \in N$ . We say that  $i, j$  are indistinguishable in  $v$  if  $v(S \cup i) = v(S \cup j)$  for all  $S \subseteq N \setminus \{i, j\}$ .

**Symmetry within unions (SWU).** A value  $g$  for TU-games with a priori unions satisfies symmetry within unions if, for all  $(N, v, P) \in \mathcal{G}^U$ , all  $P_k \in P$ , and all  $i, j \in P_k$  indistinguishable in  $v$ , it holds that  $g_i(N, v, P) = g_j(N, v, P)$ .

Take  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and denote  $M = \{1, \dots, m\}$ . The quotient game of  $(N, v, P)$  is the TU-game  $(M, v/P)$  where

$$(v/P)(R) = v(\cup_{r \in R} P_r) \quad \text{for all } R \subseteq M.$$

**Symmetry among unions (SAU).** A value  $g$  for TU-games with a priori unions satisfies symmetry among unions if, for all  $(N, v, P) \in \mathcal{G}^U$  and all  $k, l \in M$  indistinguishable in  $v/P$ , it holds that  $\sum_{i \in P_k} g_i(N, v, P) = \sum_{i \in P_l} g_i(N, v, P)$ .

Take a TU-game  $(N, v) \in \mathcal{G}^N$  and  $i \in N$ . We say that  $i$  is a nullifying player in  $v$  if  $v(S \cup i) = 0$  for all  $S \subseteq N$ . In words, a player is nullifying when every coalition containing it receives zero according to the characteristic function.

**Nullifying player property (NPP).** A value  $g$  for TU-games with a priori unions satisfies the nullifying player property if, for all  $(N, v, P) \in \mathcal{G}^U$  and all  $i \in N$  nullifying player in  $v$ , it holds that  $g_i(N, v, P) = 0$ .

An analogous to NPP above is used in van den Brink (2007) to characterize the equal division value for TU-games. In the next theorem, we extend van den Brink's result to TU-games with a priori unions.

**Theorem 3.2**  *$ED^U$  is the unique value for TU-games with a priori unions that satisfies ADD, SWU, SAU and NPP.*

*Proof.* It is immediate to check that  $ED^U$  satisfies ADD, SWU, SAU and NPP. To prove the unicity, consider a value  $g$  for TU-games with a priori unions that satisfies ADD, SWU, SAU and NPP. Fix  $N$  and define for all  $\alpha \in \mathbb{R}$  and all non-empty  $T \subseteq N$  the TU-game  $(N, e_T^\alpha)$  given by  $e_T^\alpha(S) = \alpha$  if  $S = T$  and  $e_T^\alpha(S) = 0$  if  $S \neq T$ . If  $T = N$ , since  $g$  satisfies SWU and SAU, it is clear that  $g_i(N, e_N^\alpha, P) = \frac{\alpha}{mp_k}$  for any  $P = \{P_1, \dots, P_m\}$  and all  $i \in P_k \subseteq N$ , because all players in  $N$  are indistinguishable in  $e_N^\alpha$  and all



players in  $M$  are indistinguishable in  $e_N^\alpha/P$ . If  $T \subset N$  notice that all players in  $N \setminus T$  are nullifying players in  $e_T^\alpha$  and then, since  $g$  satisfies NPP,

$$\sum_{i \in T} g_i(N, e_T^\alpha, P) = \sum_{i \in N} g_i(N, e_T^\alpha, P) = e_T^\alpha(N) = 0$$

for any  $P$ . Then, since  $g$  satisfies SWU and SAU it is not difficult to check that  $g(N, e_T^\alpha, P) = 0$ . Finally, the additivity of  $g$  and the fact that  $v = \sum_{T \subseteq N} e_T^{v(T)}$  imply that

$$g_i(N, v, P) = \sum_{T \subseteq N} g_i(N, e_T^{v(T)}, P) = g_i(N, e_N^{v(N)}, P) = \frac{v(N)}{mp_k} = ED_i^U(N, v, P)$$

for any  $P$  and all  $i \in P_k \subseteq N$ .  $\square$

## 4 The equal surplus division value for TU-games with a priori unions

In this section we extend the equal surplus division value for TU-games to the more general setup of TU-games with a priori unions. To start with, remember that the equal surplus division value  $ESD$  is defined for every  $(N, v) \in \mathcal{G}$  and for all  $i \in N$  by

$$ESD_i(N, v) = v(i) + \frac{v^0(N)}{n},$$

where  $v^0(S) = v(S) - \sum_{i \in S} v(i)$  for all  $S \subseteq N$ . Notice that  $ESD$  is a variant of  $ED$  in which we first allocate  $v(i)$  to each player  $i \in N$ , and then distribute  $v^0(N)$  among the players using  $ED$ .  $ESD$  is a reasonable alternative to  $ED$  for situations where individual benefits and joint benefits are neatly separable. Let us illustrate this with the example of Section 2 (notice that it deals with costs instead of with benefits).

Consider again the three-storey building of Section 2. Assume now that the cost of the machine is 55 and that the owner of the third apartment of the third floor can get a discount of 5. Clearly, the cost of the machine is a joint cost, whereas the cost due to the works to be done on each floor should be paid by the owners of each floor. With respect to the costs of the hollow, assume that there is a fixed cost of 10 and an individual cost of 10 for the owners of the first floor that is incremented by 10 for the owners of the second floor and by an additional 10 for the owners of the third floor. According to this, the cost  $c(i)$  in which each player is involved is:

- $55$  (machine) +  $10$  (floor) +  $40$  (hollow) =  $105$ , for the first and second players of the third floor,
- $50$  (machine) +  $10$  (floor) +  $40$  (hollow) =  $100$ , for the third player of the third floor,
- $55$  (machine) +  $10$  (floor) +  $30$  (hollow) =  $95$ , for the players of the second floor,
- $55$  (machine) +  $10$  (floor) +  $20$  (hollow) =  $85$ , for the players of the first floor.

Now we can compute the equal surplus division value for the game in which the players are the apartments and  $c(N) = 120$  (this is what we call the ES-Dutch rule) and the equal surplus division value for the game in which the players are the quota units and  $c(N) = 120$  (this is what we call the ES-Spanish rule). In the latter case, the discount achieved by the owner of the third apartment of the third floor is divided equally among its quota units, i.e.,  $c(i) = 105 - 5/60$  for each square meter  $i$  in the third apartment of the third floor. Table 3 below displays the distributions of the cost among the apartments using both rules. Notice that these distributions are not satisfactory because they seem to penalize too much the apartments on the third floor, specially the ES-Spanish rule that even proposes that the apartment on the first floor is recompensed if the elevator is installed. The reason for this seems to be that the individual costs in this example actually belong to the floors instead of to the players; consequently it would be more reasonable to use a kind of two-step rule for the equal surplus division value analogous to the two-step rule for the equal division value introduced in Section 2. In other words, this example suggests that we should consider the structure of a priori unions given by the floors and distribute the costs using an extension of the equal surplus division value to TU-games with a priori unions.

	Dutch rule	Spanish rule
3rd floor	27.5 27.5 22.5	613.6364 613.6364 608.6364
2nd floor	17.5 17.5	22.7273 20.4545
1st floor	7.5	-1759.0910

Table 3: Distribution according to the ES-Dutch and ES-Spanish rules

Next we propose three alternative ways for extending the equal surplus division value to TU-games with a priori unions. The first one divides the value of the grand coalition in the quotient game using the equal surplus division value and then divides the amount assigned to each union equally among its members. It maintains the spirit of *ESD* but applies it to the quotient game in order to take into account the unions; then it divides equally within the unions.

**Definition 4.1** The equal surplus division value (one) for TU-games with a priori unions  $ESD1^U$  is defined by

$$ESD1_i^U(N, v, P) = \frac{(v/P)(k)}{p_k} + \frac{(v/P)^0(M)}{mp_k} = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$$

for all  $i \in N$  and all  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and with  $i \in P_k$ .

The second extension divides again the value of the grand coalition in the quotient game using the equal surplus division value; then it distributes the amount  $\frac{v(N) - \sum_{l \in M} v(P_l)}{m}$  equally among the players in each union, and the amount  $v(P_k)$  giving  $v(i)$  to each player  $i \in P_k$  and dividing  $v(P_k) - \sum_{j \in P_k} v(j)$  equally among the players in  $P_k$ . It maintains the spirit of *ESD* and, in some sense, applies it twice: first to the quotient game and second to divide each  $v(P_k)$  among its members.

**Definition 4.2** The equal surplus division value (two) for TU-games with a priori unions  $ESD2^U$  is defined by

$$ESD2_i^U(N, v, P) = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$$

for all  $i \in N$  and all  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and with  $i \in P_k$ .

Finally, the third extension assigns  $v(i)$  to each player  $i$  and then divides  $v^0(N)$  among the players using  $ED^U$ . It maintains the spirit of *ESD* in the sense that it allocates  $v(i)$  to each  $i$ ; then it divides  $v(N) - \sum_{j \in N} v(j)$  equally, first among the unions and then within the unions.

**Definition 4.3** The equal surplus division value (three) for TU-games with a priori unions  $ESD3^U$  is defined by

$$ESD3_i^U(N, v, P) = v(i) + ED^U(N, v^0, P) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{mp_k}$$

for all  $i \in N$  and all  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and with  $i \in P_k$ .

Now we can compute the equal surplus division values one, two and three for the game with a priori unions in which the players are the apartments, the unions are the floors and  $c(N) = 120$  (they are what we call the  $ESD1^U$ ,  $ESD2^U$  and  $ESD3^U$ -Dutch rules) and the equal surplus division values one, two and three for the game with a priori unions in which the players are the quota units, the unions are the floors and  $c(N) = 120$  (they are what we call the  $ESD1^U$ ,  $ESD2^U$  and  $ESD3^U$ -Spanish rules). Tables 4, 5 and 6 below display the distributions of the cost among the apartments using these rules. The results in Tables 4 and 5 seem to be more reasonable than those in Table 3; notice that they slightly penalize the higher floors in comparison with the results in Table 2 (except for the elevator rule). The results in Table 6 are not satisfactory since they penalize too much the apartments on the third floor and allocate a negative cost to the apartment on the first floor; thus, Table 6 shows that  $ESD3^U$  is not appropriate for this example. The reason for this can be that  $ESD3^U$  does not satisfy the quotient game property (QGP).<sup>3</sup> We believe that QGP is especially relevant in this example because, as we have already mentioned, the individual costs here belong to the floors (the unions) more than to the players.

	Dutch rule			Spanish rule		
3rd floor	15.5556	15.5556	15.5556	15.5556	15.5556	15.5556
2nd floor	20.8333	20.8333		21.9298	19.7368	
1st floor	31.6667			31.6667		

Table 4: Distribution according to  $ESD1^U$

	Dutch rule			Spanish rule		
3rd floor	17.2222	17.2222	12.2222	17.2222	17.2222	12.2222
2nd floor	20.8333	20.8333		21.9298	19.7368	
1st floor	31.6667			31.6667		

Table 5: Distribution according to  $ESD2^U$

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<sup>3</sup>A value  $g$  for TU-games with a priori unions satisfies the quotient game property if, for all  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and for its quotient game  $(M, v/P)$ , it holds that  $\sum_{i \in P_k} g_i(N, v, P) = g_k(M, v/P, P^m)$  for all  $P_k \in P$ , where  $P^m = \{\{1\}, \{2\}, \dots, \{m\}\}$ .

	Dutch rule	Spanish rule
3rd floor	53.3333 53.3333 48.3333	508.3333 508.3333 503.3333
2nd floor	17.5 17.5	355.2632 319.7368
1st floor	-70	-2075

Table 6: Distribution according to  $ESD3^U$

In the remainder of this section we study  $ESD1^U$ ,  $ESD2^U$  and  $ESD3^U$  from the point of view of their properties; in particular, we provide axiomatic characterizations of these values. We start by introducing new properties of a value  $g$  for TU-games with a priori unions. Take  $(N, v) \in \mathcal{G}$  and  $i \in N$ . We say that  $i$  is a dummifying player in  $v$  if  $v(S \cup i) = \sum_{j \in S \cup i} v(j)$  for all  $S \subseteq N$ . In words, a player is dummifying when every coalition containing it is inessential according to the characteristic function, in the sense that its value equals the sum of the individual values of its members. Take now a TU-game with a priori unions  $(N, v, P) \in \mathcal{G}^U$  where  $P = \{P_1, \dots, P_m\}$ . We say that  $P_k$  is a dummifying union in  $(v, P)$  if  $k$  is a dummifying player in  $v/P$ . Dummifying players and dummifying unions should play a relevant role in the characterizations of  $ESD1^U$ ,  $ESD2^U$  and  $ESD3^U$  since a property on dummifying players is used in Casajus and Hüttner (2014) for characterizing  $ESD$ . In fact they use the following property (for  $\mathcal{G}$  instead of  $\mathcal{G}^U$ ).

**Dummifying player property (DPP).** A value  $g$  for TU-games with a priori unions satisfies the dummifying player property if, for all  $(N, v, P) \in \mathcal{G}^U$  and all  $i \in N$  dummifying player in  $v$ , it holds that  $g_i(N, v, P) = v(i)$ .

Notice that  $ESD3^U$  satisfies DPP, but neither  $ESD1^U$  nor  $ESD2^U$  satisfy it. In the search of properties that  $ESD1^U$  or  $ESD2^U$  might satisfy, we propose the following variations of DPP and NPP.

**Dummifying union/player property (DUPP).** A value  $g$  for TU-games with a priori unions satisfies the dummifying union/player property if, for all  $(N, v, P) \in \mathcal{G}^U$  and all  $P_k \in P$  dummifying union in  $(v, P)$  with  $i \in P_k$  being a dummifying player in  $v_{P_k}$ ,<sup>4</sup> it holds that  $g_i(N, v, P) = v(i)$ .

**Dummifying union/nullifying player property (DUNPP).** A value  $g$  for TU-games with a priori unions satisfies the dummifying union/nullifying

<sup>4</sup> $v_{P_k}$  denotes the characteristic function of the TU-game  $(P_k, v_{P_k})$ , where  $v_{P_k}(S) = v(S)$  for all  $S \subseteq P_k$ .

player property if, for all  $(N, v, P) \in \mathcal{G}^U$  and all  $P_k \in P$  dummifying union in  $(v, P)$  with  $i \in P_k$  being a nullifying player in  $v_{P_k}$ , it holds that  $g_i(N, v, P) = 0$ .

Now we give parallel characterizations of the three extensions of *ESD* using the properties we have introduced above.

**Theorem 4.4** *ESD1<sup>U</sup> is the unique value for TU-games with a priori unions that satisfies ADD, SWU, SAU and DUNPP.*

*Proof.* It is immediate to check that *ESD1<sup>U</sup>* satisfies ADD, SWU, SAU and DUNPP. To prove the unicity, consider a value  $g$  for TU-games with a priori unions that satisfies ADD, SWU, SAU and DUNPP. Take  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and define the TU-game  $(N, v^1)$  given by

$$v^1(S) = \sum_{P_l \subseteq S} v(P_l) = \sum_{l=1}^m v^{P_l}(S)$$

for all  $S \subseteq N$ , where  $v^{P_l}(S) = v(P_l)$  if  $P_l \subseteq S$  and  $v^{P_l}(S) = 0$  otherwise.

Take  $P_k \in P$ . Since  $g$  is a value for TU-games with a priori unions, then

$$\sum_{i \in N} g_i(N, v^{P_k}, P) = v^{P_k}(N) = v(P_k).$$

All unions  $P_l \in P$  are dummifying unions in  $(v^{P_k}, P)$  and all players  $i \in P_l$ , with  $l \neq k$ , are nullifying players in  $(v^{P_k})_{P_l}$ . By DUNPP,  $g_i(N, v^{P_k}, P) = 0$  for all  $i \notin P_k$ . And since all players in  $P_k$  are indistinguishable in  $v^{P_k}$ , then SWU implies that, for all  $i \in P_k$ ,  $g_i(N, v^{P_k}, P) = \frac{v(P_k)}{p_k}$ . Using the additivity of  $g$ , for all  $i \in P_k$ ,

$$g_i(N, v^1, P) = \frac{v(P_k)}{p_k}. \quad (1)$$

Define now  $v^2 = v - v^1$  and, for all  $\alpha \in \mathbb{R}$  and all non-empty  $T \subseteq N$ ,  $e_T^\alpha$  by  $e_T^\alpha(S) = \alpha$  if  $S = T$  and  $e_T^\alpha(S) = 0$  if  $S \neq T$ . It is clear that  $v^2 = \sum_{T \subseteq N} e_T^{v^2(T)}$ . If  $T = N$ , since all players in  $N$  are indistinguishable in  $e_N^{v^2(N)}$  and all players in  $M$  are indistinguishable in  $e_N^{v^2(N)}/P$ , SWU and SAU imply that, for all  $i \in P_k$ ,

$$g_i(N, e_N^{v^2(N)}, P) = \frac{v^2(N)}{mp_k} = \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}.$$

If  $T \subset N$ , consider two cases:

- Take  $T = \cup_{l \in L} P_l$ , with  $\emptyset \subset L \subset M$ . For all  $P_u \in P$ , if  $T \neq P_u$  then  $e_T^{v^2(T)}(P_u) = 0$  and if  $T = P_u$  then  $e_T^{v^2(T)}(P_u) = v^2(P_u) = 0$ . Hence, it is easy to see that all the unions in  $M \setminus L$  are dummifying unions in  $(e_T^{v^2(T)}, P)$ . Also, since all players in  $N \setminus T$  are nullifying players in  $e_T^{v^2(T)}$ , DUNPP implies that  $g_i(N, e_T^{v^2(T)}, P) = 0$  for all  $i \notin T$ . Notice that since all unions in  $L$  are indistinguishable in  $e_T^{v^2(T)}$ , then by SAU  $\sum_{i \in P_k} g_i(N, e_T^{v^2(T)}, P) = \sum_{i \in P_l} g_i(N, e_T^{v^2(T)}, P)$  for all  $k, l \in L$ ; notice also that since

$$\sum_{i \in T} g_i(N, e_T^{v^2(T)}, P) = \sum_{i \in N} g_i(N, e_T^{v^2(T)}, P) = e_T^{v^2(T)}(N) = 0$$

then  $\sum_{i \in P_k} g_i(N, e_T^{v^2(T)}, P) = 0$  for all  $k \in L$ . To conclude, SWU implies that  $g_i(N, e_T^{v^2(T)}, P) = 0$  for all  $i \in P_k$ , with  $k \in L$ , and therefore for all  $i \in N$ .

- For any other  $T \subset N$  that is not in the previous case, the quotient game  $(M, e_T^{v^2(T)}/P)$  satisfies that  $(e_T^{v^2(T)}/P)(R) = 0$  for all  $R \subseteq M$  and, thus, all the unions in  $P$  are indistinguishable and dummifying unions in  $(e_T^{v^2(T)}, P)$ . If  $i \notin T$ , then  $i$  is a nullifying player in  $e_T^{v^2(T)}$  and DUNPP implies that  $g_i(N, e_T^{v^2(T)}, P) = 0$ . Analogously as in the previous case, SAU and SWU imply that  $g_i(N, e_T^{v^2(T)}, P) = 0$  for all  $i \in T$ .

Now ADD implies that, for all  $i \in P_k$  with  $P_k \in P$ ,

$$g_i(N, v^2, P) = \sum_{T \subseteq N} g_i(N, e_T^{v^2(T)}, P) = \frac{v^2(N)}{mp_k}. \quad (2)$$

Finally, from (1), (2), ADD and  $v = v^1 + v^2$  it is clear that

$$g(N, v, P) = ESD1^U(N, v, P).$$

□

**Theorem 4.5**  $ESD2^U$  is the unique value for TU-games with a priori unions that satisfies ADD, SWU, SAU and DUPP.

*Proof.* It is immediate to check that  $ESD2^U$  satisfies ADD, SWU, SAU and DUPP. To prove the unicity, consider a value  $g$  for TU-games with a priori unions that satisfies ADD, SWU, SAU and DUPP. Take  $(N, v, P) \in \mathcal{G}^U$  with  $P = \{P_1, \dots, P_m\}$  and define  $v^a$ ,  $v^{01}$  and  $v^{02}$  by:

- $v^a(S) = \sum_{i \in S} v(i)$ ,
- $v^{01}(S) = \sum_{P_l \subseteq S} v^0(P_l) = \sum_{l=1}^m v^{0P_l}(S)$ ,
- $v^{02}(S) = v^0(S) - \sum_{P_l \subseteq S} v^0(P_l)$ ,

for all  $S \subseteq N$ , where  $v^{0P_l}(S) = v^0(P_l)$  if  $P_l \subseteq S$  and  $v^{0P_l}(S) = 0$  otherwise.

Since all unions are dummifying in  $(v^a, P)$  and all players are dummifying in  $v^a$ , then DUPP implies that, for all  $i \in N$ ,

$$g_i(N, v^a, P) = v^a(i) = v(i). \quad (3)$$

Take  $P_k \in P$ . Since  $g$  is a value for TU-games with a priori unions, then

$$\sum_{i \in N} g_i(N, v^{0P_k}, P) = v^{0P_k}(N) = v^0(P_k).$$

All unions  $P_l \in P$  are dummifying unions in  $(v^{0P_k}, P)$  and all players  $i \in P_l$ , with  $l \neq k$ , are dummifying players in  $(v^{0P_k})_{P_l}$ . By DUPP,  $g_i(N, v^{0P_k}, P) = v^{0P_k}(i) = 0$  for all  $i \notin P_k$ . And since all players in  $P_k$  are indistinguishable in  $v^{0P_k}$ , then SWU implies that, for all  $i \in P_k$ ,  $g_i(N, v^{0P_k}, P) = \frac{v^0(P_k)}{p_k}$ . Using ADD, for all  $i \in P_k$ ,

$$g_i(N, v^{01}, P) = \frac{v^0(P_k)}{p_k}. \quad (4)$$

Take now into account that  $v^{02} = \sum_{T \subseteq N} e_T^{v^{02}(T)}$ . If  $T = N$ , since all players in  $N$  are indistinguishable in  $e_N^{v^{02}(N)}$  and all players in  $M$  are indistinguishable in  $e_N^{v^{02}(N)}/P$ , SWU and SAU imply that, for all  $i \in P_k$ ,

$$g_i(N, e_N^{v^{02}(N)}, P) = \frac{v^{02}(N)}{mp_k}.$$

If  $T \subset N$ , consider two cases:

- Take  $T = \cup_{l \in L} P_l$ , with  $\emptyset \subset L \subset M$ . Since  $e_T^{v^{02}(T)}(P_u) = 0$  for all  $P_u \in P$  and  $(e_T^{v^{02}(T)}/P)(R) = 0$  for all  $R \subseteq M$  with  $R \cap (M \setminus L) \neq \emptyset$ , all the unions in  $M \setminus L$  are dummifying unions in  $(e_T^{v^{02}(T)}, P)$ . Also, since all players in  $N \setminus T$  are dummifying players in  $e_T^{v^{02}(T)}$ , DUPP implies that  $g_i(N, e_T^{v^{02}(T)}, P) = e_T^{v^{02}(T)}(i) = 0$  for all  $i \notin T$ . Notice that since all unions in  $L$  are indistinguishable in  $e_T^{v^{02}(T)}$ , then by SAU



$\sum_{i \in P_k} g_i(N, e_T^{v^{02}(T)}, P) = \sum_{i \in P_l} g_i(N, e_T^{v^{02}(T)}, P)$  for all  $k, l \in L$ , and notice that since

$$\sum_{i \in T} g_i(N, e_T^{v^{02}(T)}, P) = \sum_{i \in N} g_i(N, e_T^{v^{02}(T)}, P) = e_T^{v^{02}(T)}(N) = 0$$

then  $\sum_{i \in P_k} g_i(N, e_T^{v^{02}(T)}, P) = 0$  for all  $k \in L$ . Hence, SWU implies that  $g_i(N, e_T^{v^{02}(T)}, P) = 0$  for all  $i \in T$ .

- For any other  $T \subset N$  that is not in the previous case, the quotient game  $(M, e_T^{v^{02}(T)}/P)$  satisfies that  $(e_T^{v^{02}(T)}/P)(R) = 0$  for all  $R \subseteq M$  and, thus, all the unions in  $P$  are indistinguishable and dummifying unions in  $(e_T^{v^{02}(T)}, P)$ . If  $i \notin T$ , then  $i$  is a dummifying player in  $e_T^{v^{02}(T)}$  and DUPP implies that  $g_i(N, e_T^{v^{02}(T)}, P) = e_T^{v^{02}(T)}(i) = 0$ . Analogously as in the previous case, SAU and SWU imply that  $g_i(N, e_T^{v^{02}(T)}, P) = 0$  for all  $i \in T$ .

Now ADD implies that, for all  $i \in P_k$  with  $P_k \in P$ ,

$$g_i(N, v^{02}, P) = \sum_{T \subseteq N} g_i(N, e_T^{v^{02}(T)}, P) = \frac{v^{02}(N)}{mp_k}. \quad (5)$$

Finally, from (3), (4), (5), ADD and  $v = v^a + v^{01} + v^{02}$  it is clear that

$$g(N, v, P) = ESD2^U(N, v, P).$$

□

Now we provide a characterization of  $ESD3^U$ . In order to do it we introduce a new property that is a weaker version of SAU.

**Weak symmetry among unions (WSAU).** A value  $g$  for TU-games with a priori unions satisfies weak symmetry among unions if, for all  $(N, v, P) \in \mathcal{G}^U$  with  $v(j) = 0$  for all  $j \in N$ , and for all  $k, l \in M$  indistinguishable in  $v/P$ , it holds that  $\sum_{i \in P_k} g_i(N, v, P) = \sum_{i \in P_l} g_i(N, v, P)$ .

**Theorem 4.6**  $ESD3^U$  is the unique value for TU-games with a priori unions that satisfies ADD, SWU, WSAU and DPP.

*Proof.* It is immediate to check that  $ESD3^U$  satisfies ADD, SWU, WSAU and DPP. To prove the unicity, consider a value  $g$  for TU-games with a priori unions that satisfies ADD, SWU, WSAU and DPP. Take now

$(N, v, P) \in \mathcal{G}^U$  and  $i \in P_k$  with  $P_k \in P$ , and define  $v^a = v - v^0$ . ADD implies that

$$g_i(N, v, P) = g_i(N, v^a, P) + g_i(N, v^0, P). \quad (6)$$

Since all players are dummifying in  $v^a$ , then DPP implies that

$$g_i(N, v^a, P) = v^a(i) = v(i). \quad (7)$$

Now, using for  $(N, v^0)$  analogous arguments as those used in the proof of Theorem 3.2, it is clear that ADD, SWU, WSAU and DPP imply that

$$g_i(N, v^0, P) = ED_i(N, v^0, P). \quad (8)$$

Finally, from (6), (7) and (8) it is clear that

$$g(N, v, P) = ESD3^U(N, v, P).$$

□

It is immediate to prove that  $ESD3^U$  does not satisfy SAU. Since WSAU is a weaker version of SAU, and  $ESD3^U$  is characterized with ADD, SWU, WSAU and DPP, we conclude that there does not exist a value for TU-games with a priori unions satisfying ADD, SWU, SAU and DPP.

To conclude, we indicate that the properties in the theorems of this paper are independent. We prove this in a separate appendix.

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## Appendix

a) Independence of the properties of Theorem 3.2:

- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$  satisfies ADD, SWU and SAU, but not NPP.
- $\varphi_i = \frac{v(N)}{n}$  satisfies ADD, SWU and NPP, but not SAU.
- $\varphi_i = \frac{2v(N)}{mp_k}$  if  $i = \min_{j \in P_k} j$  or  $\varphi_i = \frac{(p_k-2)v(N)}{mp_k(p_k-1)}$  if  $i \in P_k$  and  $i \neq \min_{j \in P_k} j$ , satisfies ADD, SAU and NPP, but not SWU.

- $\varphi_i = \frac{2v(N)}{mp_k|Z_k|}$  if  $i \in Z_k = \{j \in P_k / v(j) = \min_{z \in P_k} v(z)\}$ ,  $\varphi_i = \frac{(p_k-2)v(N)}{mp_k(p_k-|Z_k|)}$  if  $i \in P_k \setminus Z_k$ , satisfies SWU, SAU and NPP, but not ADD.

b) Independence of the properties of Theorem 4.4:

- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{n}$  satisfies ADD, SWU, DUNPP, but not SAU.
- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{2(v(N) - \sum_{l \in M} v(P_l))}{mp_k}$  if  $i = \min_{j \in P_k} j$  or  $\varphi_i = \frac{v(P_k)}{p_k} + \frac{(p_k-2)(v(N) - \sum_{l \in M} v(P_l))}{mp_k(p_k-1)}$  if  $i \in P_k$  and  $i \neq \min_{j \in P_k} j$ , satisfies ADD, SAU and DUNPP, but not SWU.
- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{2(v(N) - \sum_{l \in M} v(P_l))}{mp_k|Z_k|}$  if  $i \in Z_k = \{j \in P_k / v(j) = \min_{z \in P_k} v(z)\}$ ,  $\varphi_i = \frac{v(P_k)}{p_k} + \frac{(p_k-2)(v(N) - \sum_{l \in M} v(P_l))}{mp_k(p_k-|Z_k|)}$  if  $i \in P_k \setminus Z_k$ , satisfies SWU, SAU and DUNPP, but not ADD.
- $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$  satisfies ADD, SWU and SAU, but not DUNPP.

c) Independence of the properties of Theorem 4.5:

- $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{n}$  satisfies ADD, SWU and DUPP, but not SAU.
- $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{2(v(N) - \sum_{l \in M} v(P_l))}{mp_k}$  if  $i = \min_{j \in P_k} j$  or  $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{(p_k-2)(v(N) - \sum_{l \in M} v(P_l))}{mp_k(p_k-1)}$  if  $i \in P_k$  and  $i \neq \min_{j \in P_k} j$ , satisfies ADD, SAU and DUPP, but not SWU.
- $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{2(v(N) - \sum_{l \in M} v(P_l))}{mp_k|Z_k|}$  if  $i \in Z_k = \{j \in P_k / v(j) = \min_{z \in P_k} v(z)\}$ ,  $\varphi_i = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{(p_k-2)(v(N) - \sum_{l \in M} v(P_l))}{mp_k(p_k-|Z_k|)}$  if  $i \in P_k \setminus Z_k$ , satisfies SWU, SAU and DUNPP, but not ADD.
- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$  satisfies ADD, SWU and SAU, but not DUPP.

d) Independence of the properties of Theorem 4.6:

- $\varphi_i = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}$  satisfies ADD, SWU and DPP, but not WSAU.

- $\varphi_i = v(i) + \frac{2(v(N) - \sum_{j \in N} v(j))}{mp_k}$  if  $i = \min_{j \in P_k} j$  or  $\varphi_i = v(i) + \frac{(p_k - 2)(v(N) - \sum_{j \in N} v(j))}{mp_k(p_k - 1)}$  if  $i \in P_k$  and  $i \neq \min_{j \in P_k} j$ , satisfies ADD, WSAU and DPP, but not SWU.
- $\varphi_i = v(i) + \frac{2(v(N) - \sum_{j \in N} v(j))}{mp_k |Z_k|}$  if  $i \in Z_k = \{j \in P_k / v(j) = \min_{z \in P_k} v(z)\}$ ,  
 $\varphi_i = v(i) + \frac{(p_k - 2)(v(N) - \sum_{j \in N} v(j))}{mp_k(p_k - |Z_k|)}$  if  $i \in P_k \setminus Z_k$ , satisfies SWU, WSAU and DPP, but not ADD.
- $\varphi_i = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k}$  satisfies ADD, SWU, WSAU but not DPP.