Journal of Computational Finance **20**(3), 1–27
DOI: 10.21314/JCF.2016.317

Research Paper

Pricing swing options in electricity markets with two stochastic factors using a partial differential equation approach

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(Received October 29, 2015; revised March 29, 2016; accepted March 30, 2016)

ABSTRACT

In this paper, we consider the numerical valuation of swing options in electricity markets based on a two-factor model. These kinds of contracts are modeled as path-dependent options with multiple exercise rights. From a mathematical point of view, the valuation of these products is posed as a sequence of free boundary problems, where two exercise rights are separated by a time period. In order to solve the pricing problem, we propose appropriate numerical methods based on a Crank–Nicolson semi-Lagrangian method combined with biquadratic Lagrange finite elements for the discretization of the partial differential equation. In addition, we use an augmented Lagrangian active set method to cope with the early exercise feature when it appears. Moreover, we derive appropriate artificial boundary conditions to treat the unbounded domain numerically. Finally, we present some numerical results to illustrate the proper behavior of the numerical schemes.

Keywords: swing options; electricity price; augmented Lagrangian active set (ALAS) formulation; semi-Lagrangian method; biquadratic Lagrange finite elements; artificial boundary conditions.

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Print ISSN 1460-1559 | Online ISSN 1755-2850

1 INTRODUCTION

Nowadays, due to the liberalization of electricity markets, electricity prices are determined by the principle of supply and demand. This absence of regularization affects prices by increasing their volatility and introducing uncertainty. Due to this fact, (mainly) companies that buy electricity directly from an exchange are demanding the existence of contracts that would protect them against high prices but also give them the possibility of benefiting from low prices. For an introduction to electricity markets, we refer the reader to, for example, Chapter 2 of the recent book by Aïd (2015).

In this paper, we focus on one standard type of these contracts: swing options. Swing contracts are a kind of path-dependent option that allows the holder to exercise a right multiple times over a period, with the constraint that the two consecutive exercise dates must be separated by a refracting period. That is, not all the rights can be exercised at one time. One example of this right could be to receive the payoff of a call option. Nevertheless, there are other possibilities, such as the consideration of different payoff functions depending on the spot price, such as calls and puts, or calls with different strikes. Swing options are widely offered in the market and used by major energy companies, especially in the electricity and fossil fuel markets. Sometimes the volume of the physical underlying commodity is also a state variable. A very interesting summary of the different features of swing options in practice and their valuation can be found in Eydeland and Wolyniec (2003, Chapter 8), which is devoted to structured products based on fuels and commodities.

As indicated in Carmona and Ludkovski (2009), the first discussions about swing options appeared in energy magazines (Barbieri and Garman 1996), while the first rigorous treatment of this topic was in Jaillet *et al* (2004). The formulation of swing option prices in terms of multiple optimal stopping times represents a relevant step developed in Carmona and Touzi (2008), who related swing and American options. From this relation, the partial differential equation (PDE) approach to price swing options arises (for one exercise, both options are equivalent). Along these lines, we direct the reader to Dahlgren (2005) and Wilhelm and Winter (2008). In Dahlgren (2005), a one-factor Ornstein–Uhlenbeck process for the logarithm of the commodity price is considered, and a system of variational inequalities is posed and numerically solved. In addition, a one-factor model is considered in Wilhelm and Winter (2008). Other alternative models to describe the evolution of electricity prices are presented in Lucia and Schwartz (2002) and Barlow (2002), where the author introduces a nonlinear Ornstein–Uhlenbeck process to model the spot prices; this model is without jumps, but it can incorporate spikes. More recently, in Hambly *et al* (2009), a model with two stochastic factors is considered, although the pricing problem is solved by means of binomial trees. One of the innovative points of the present work is the consideration of numerical methods to solve the PDE formulation associated with a two-factor model for electricity price. Hereby, we consider the stochastic two-factor model proposed in Hambly *et al* (2009).

Financially, swing options can be equivalently handled as a portfolio of American-type options with a waiting period (the so-called refracting period) between the two exercises. From a mathematical point of view, the swing option valuation problem can be posed as a sequence of free boundary problems, one for each right. Since in the obstacle function the value of the contract with one fewer exercise right is involved, an initial boundary value problem (IBVP), restricted to a time interval of length equal to the refracting period, also needs to be solved.

In the literature, different numerical techniques have been employed to obtain the value of swing contracts. Binomial trees are considered in Jaillet *et al* (2004), but only when the underlying is a one-factor, seasonal, mean-reverting process. Also, in Hambly *et al* (2009), a binomial tree method is used when the spot price is modeled as the sum of a deterministic function in order to incorporate seasonality and two stochastic factors, with the possibility of incorporating spikes. In other works, such as Meinshausen and Hambly (2004) and Thanawalla (2005), the valuation of multiple stopping time problems is tackled using Monte Carlo simulation techniques. One of the first numerical solutions of the PDE approach is provided in Dahlgren (2005), in which the domain is truncated to a bounded domain, and homogeneous Neumann boundary conditions are imposed. Additionally, finite elements or finite differences are combined with a projected successive over relaxation (PSOR) algorithm to cope with the early exercise feature. Finite elements are also applied in Wilhelm and Winter (2008) to solve the PDE problem when the spot price only depends on one stochastic factor, whereas in Wegner (2002) the solution of the PDE is discretized using finite difference schemes. Also, in the case of electricity prices with one stochastic factor, swing options have been treated with Fourier-based methods in Zhang and Oosterlee (2013). Jump diffusion processes to describe the evolution of the underlying asset can also be taken into account, thus leading to a partial integro-differential equation. In this setting, a finite difference scheme combined with a dynamic programming technique has been used in Kjaer (2007), and an implicit–explicit finite difference scheme was proposed in Nguyen and Ehrhardt (2012). As indicated in Carmona and Ludkovski (2009), practitioners usually value swing options by simulation techniques; however, the rigorous error analysis associated with many simulation schemes is difficult.

In the present paper, we propose the numerical solution of the two-factor model by means of a Crank–Nicolson characteristics scheme for the time discretization, combined with finite elements for the space discretization. The classical characteristics scheme was first introduced in Pironneau (1982), and first applied in finance to price European and American options in Vázquez (1998). The method was then applied to American–Asian options with jumps in D'Halluin *et al* (2005) and natural gas storage valuation problems in Cheng and Forsyth (2007). The Crank–Nicolson characteristics scheme we propose was first used in Bermúdez *et al* (2006c) to price American–Asian options without jumps, and in other finance-related problems in Calvo-Garrido and Vázquez (2012), Calvo-Garrido *et al* (2013) and Calvo-Garrido and Vázquez (2015). To the best of our knowledge, the numerical solution of PDE models for swing options when two stochastic factors are considered in electricity prices has not yet been addressed in the literature. The mathematical analysis of these discretization schemes has already been treated in Bermúdez *et al* (2006a,b). Further, in order to deal with the inequalities associated with the early exercise feature of swing options, we use the augmented Lagrangian active set (ALAS) algorithm (Kärkkäinen *et al* 2003), which is more efficient than the classical PSOR, alternative duality or penalization methods. More precisely, in the PSOR method, the convergence depends on the relaxation parameter, and it deteriorates as soon as meshes are refined. Penalization methods also rely on the convergence of the penalized problem when the penalization parameter tends to zero. Active set strategies mainly consist of two steps: one to select the set of mesh nodes with active constraints, and another to solve the reduced system associated with the mesh nodes located in the inactive set (Tarvainen 1997). The presence of a reduced system only involving the nodes in the inactive set represents a very competitive advantage with respect to alternative methods, especially when the structure of the discrete problem and the appropriate mesh nodes numbering allows us to efficiently identify the evolution of this inactive set of nodes. Although in the here-proposed ALAS method a parameter is also involved, it only affects the first iteration. In Tarvainen (1997), the PSOR method is compared with different active set methods, while in Bermúdez *et al* (2006d) the authors illustrate the advantages of the ALAS method when compared with an alternative duality method in the pricing of Asian options with arithmetic average and early exercise opportunity. Moreover, in order to obtain a numerical solution to the problem, we need to replace the unbounded domain with a bounded one; hence, appropriate boundary conditions are required. For this purpose, instead of using homogeneous Neumann boundary conditions (empirically motivated by the expression of the payoff function), as proposed in Dahlgren (2005) for the one-factor case, we derive more appropriate artificial boundary conditions (ABCs) based on the work of Halpern (1986).

This paper is organized as follows. In Section 2, we describe the stochastic model for the spot electricity price under consideration, and we state the mathematical problem that governs the valuation of swing contracts on this underlying. In Section 3, we formulate the swing option pricing problem in a bounded domain after a localization procedure. Since we have to supply boundary conditions, we construct appropriate ABCs. Then, we introduce the discretization in time of the problem, using a Crank–Nicolson characteristic scheme, and we state the variational formulation of the time discretized problem in order to apply finite elements. At the end of this section, we describe the ALAS algorithm. Finally, in Section 4, we present some numerical results to illustrate our findings.

2 THE MATHEMATICAL MODEL

2.1 The electricity spot price model

In this section, we introduce the model of Hambly, Howison and Kluge (Hambly *et al* 2009) to describe the stochastic evolution of the electricity spot price. More precisely, under the risk neutral probability measure, the spot electricity price, S_t, is assumed to be the continuous time process

$$S_t = \exp(f(t) + \bar{X}_t + Y_t), \tag{2.1}$$

where f is a deterministic periodic function that represents the seasonality and accounts for regular changes in the prices evolution; $\bar{X}_t$ denotes the Ornstein–Uhlenbeck (OU) process with zero mean-reversion level and mean-reversion speed, $\alpha > 0$. Thus, the following stochastic differential equation (SDE) is satisfied:

$$\mathrm{d}\bar{X}_t = -\alpha \bar{X}_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t, \tag{2.2}$$

where σ denotes the volatility of the process and W_t represents a standard Brownian motion. For the third component Y_t in Hambly *et al* (2009), the following SDE is posed:

$$\mathrm{d}Y_t = -\beta Y_t \,\mathrm{d}t + J_t \,\mathrm{d}Z_t, \tag{2.3}$$

where β is a mean-reversion speed, J_t denotes the jumps size distribution and Z_t is a Poisson process of intensity λ. As in some cases in Hambly *et al* (2009), in the present paper we consider the case without jumps in Y_t dynamics, so that, by choosing $\lambda = 0$, the model can be written as

$$\mathrm{d}Y_t = -\beta Y_t \,\mathrm{d}t. \tag{2.4}$$

The general case with jumps (2.3) leads to a partial integro-differential equation formulation, the numerical solution of which will be the subject of a forthcoming paper by the authors. Keeping this in mind, although this last model for Y_t is deterministic, we prefer to maintain the numerical methodology associated with a two stochastic factor formulation, which can potentially be extended to other two-factor diffusive mean-reverting processes, such as those in Lucia and Schwartz (2002) and Pilipovic (1997).

For convenience, we write the seasonal function f as a time-dependent mean-reversion level of the process $\bar{X}_t$, and we introduce $X_t = \bar{X}_t + f(t)$. Next, we consider the following processes:

$$M_t = \exp(X_t), \qquad N_t = \exp(Y_t),$$

so that $S_t = M_t N_t$, and

$$\begin{aligned} \mathrm{d}M_t &= \alpha(\mu(t) - \ln(M_t))M_t\,\mathrm{d}t + \sigma M_t\,\mathrm{d}W_t, \\ \mathrm{d}N_t &= -\beta\ln(N_t)N_t\,\mathrm{d}t, \end{aligned} \tag{2.5}$$

with

$$\mu(t) = f(t) + \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + f'(t)\right).$$

As indicated in Lucia and Schwartz (2002), the process M_t tends to the mean value of f in the long term. For a given value of M_0, this convergence is faster for larger values of α. There exist other, alternative models, either with one or two stochastic factors, to describe the spot electricity price evolution, such as those presented in Lucia and Schwartz (2002).

2.2 The PDE formulation

The price of any asset whose value is a function of time t and the stochastic factors M_t and N_t (the dynamics of which are described by (2.5)) is given by a stochastic process, $V_t = V(t, M_t, N_t)$, where V denotes a sufficiently smooth function. Then, by using a dynamic hedging methodology similar to the case of pension plans depending on salary (see, for example, Calvo-Garrido and Vázquez 2012), the function V is the solution of a certain PDE problem. Thus, we can apply Itô's lemma (Itô 1951) to obtain the variation of V_t, $\mathrm{d}V_t$, from time t to $t + \mathrm{d}t$ for small $\mathrm{d}t$. Hereafter, we suppress the dependence on t in order to simplify notation. More precisely, we have

$$\begin{aligned} \mathrm{d}V = {} & \left(\frac{\partial V}{\partial t} + \alpha(\mu(t) - \ln(M))M\frac{\partial V}{\partial M} - \beta\ln(N)N\frac{\partial V}{\partial N} + \tfrac{1}{2}\sigma^2 M^2\frac{\partial^2 V}{\partial M^2}\right)\mathrm{d}t \\ & + \sigma M\frac{\partial V}{\partial M}\,\mathrm{d}W. \end{aligned} \tag{2.6}$$

Next, we build a risk-free portfolio Π by buying one unit of the asset V_1 with maturity T_1, and selling Δ units of asset V_2 with maturity T_2. Thus, the resulting portfolio Π reads

$$\Pi = V_1 - \Delta V_2.$$

Note that the variation of the portfolio value between t and $t + \mathrm{d}t$ is given by

$$\mathrm{d}\Pi = \mathrm{d}V_1 - \Delta\,\mathrm{d}V_2 = (\cdots)\,\mathrm{d}t + \sigma M\left(\frac{\partial V_1}{\partial M} - \Delta\frac{\partial V_2}{\partial M}\right)\mathrm{d}W, \tag{2.7}$$

where $(\cdots)$ contains the drift term. Therefore, Π turns out to be risk free for the following choice of Δ:

$$\Delta = \frac{\partial V_1/\partial M}{\partial V_2/\partial M}. \tag{2.8}$$

Moreover, for this choice of Δ, the variation of the risk-free portfolio is given by

$$\mathrm{d}\Pi = \bigg[\frac{\partial V_1}{\partial t} - \beta \ln(N) N \frac{\partial V_1}{\partial N} + \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V_1}{\partial M^2} - \Delta\bigg(\frac{\partial V_2}{\partial t} - \beta \ln(N) N \frac{\partial V_2}{\partial N} + \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V_2}{\partial M^2}\bigg)\bigg]\mathrm{d}t. \qquad (2.9)$$

Next, by using the arbitrage-free assumption, this variation is also given by $\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$, where r is the deterministic risk-free interest rate. Hence, we obtain the identity

$$\bigg(\frac{\partial V_1}{\partial M}\bigg)^{-1}\bigg(rV_1 - \frac{\partial V_1}{\partial t} + \beta \ln(N) N \frac{\partial V_1}{\partial N} - \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V_1}{\partial M^2}\bigg) = \bigg(\frac{\partial V_2}{\partial M}\bigg)^{-1}\bigg(rV_2 - \frac{\partial V_2}{\partial t} + \beta \ln(N) N \frac{\partial V_2}{\partial N} - \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V_2}{\partial M^2}\bigg). \qquad (2.10)$$

Note that (2.10) holds for any pair of assets. Then, we can introduce the quantity

$$\bar{a}(t, M, N) = \bigg(\frac{\partial V}{\partial M}\bigg)^{-1}\bigg(rV - \frac{\partial V}{\partial t} + \beta \ln(N) N \frac{\partial V}{\partial N} - \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V}{\partial M^2}\bigg), \qquad (2.11)$$

where it is convenient to write $\bar{a}(t, M, N) = \alpha(\mu(t) - \ln(M))M$.

By reordering the terms in (2.11), we obtain the following PDE in two spatial dimensions that governs the value of any asset, depending on the two underlying stochastic factors M and N:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 V}{\partial M^2} + \alpha(\mu(t) - \ln(M))M \frac{\partial V}{\partial M} - \beta \ln(N) N \frac{\partial V}{\partial N} - rV = 0. \qquad (2.12)$$

For the particular case of electricity markets in which the payoff $\phi(T, S)$ is a function depending on the electricity price, S, at maturity T, (2.12) is supplied with the final condition

$$V(T, M, N) = \phi(T, MN). \qquad (2.13)$$

2.3 The swing option pricing problem

From a mathematical point of view, swing options in electricity markets can be modeled as financial products with multiple exercises of American type. Moreover, two exercise dates are separated by a constant refracting period $\delta > 0$. As mentioned in Carmona and Touzi (2008), the consideration of this refracting period avoids the exercise of all the rights at once, which would be optimal in the absence of this separation time. That is, without the refracting period δ, the swing option pricing problem could be reduced to the valuation of multiple American options.

Let us consider $p \in \mathbb{N}$ exercise rights. If we denote by $\mathcal{T}_{t,T}$ the set of all stopping times with values in $[0, T]$, and by $\mathcal{T}_{t,\infty}$ the set of all stopping times with values greater than or equal to t, we can define the set of admissible stopping time vectors in the following way (see, for example, Carmona and Touzi 2008; Wilhelm and Winter 2008):

$$\mathcal{T}_t^{(p)} = \{\tau^{(p)} = (\tau_1, \dots, \tau_p) \mid \tau_i \in \mathcal{T}_{t,\infty}, \\ \text{with} \tau_1 \leqslant T \text{ almost surely and } \tau_{i+1} - \tau_i \geqslant \delta \text{ for } i = 1, \dots, p-1\}. \tag{2.14}$$

Note that at least one exercise right of the swing option with maturity T is exercised, but it is not necessary to exercise all the rights. The investor could let an exercise right expire to benefit from better future prices. Thus, not all stopping times of a vector have their values in the interval $[0, T]$.

In Wilhelm and Winter (2008), the risk-free price of a swing option depending on one underlying factor is written as a multiple stopping time problem; it is proven in Carmona and Touzi (2008) that it can be translated to a sequence of single stopping time problems. Analogously, when the electricity price depends on two stochastic factors under a risk-neutral probability measure Q, the price of a swing option with $p \in \mathbb{N}$ exercise rights is given by

$$V^{(p)}(t, M_t, N_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q[\mathrm{e}^{-r(\tau - t)} \Phi^{(p)}(\tau, S_\tau)], \quad p \geqslant 1, \tag{2.15}$$

with $S_t = M_t N_t$ and

$$\Phi^{(p)}(t, S_t) = \begin{cases} \phi(t, S_t) + \mathbb{E}^Q[\mathrm{e}^{-r\delta} V^{(p-1)}(t + \delta, M_{t+\delta}, N_{t+\delta})] & \text{if } t \leqslant T - \delta, \\ \phi(t, S_t) & \text{if } t > T - \delta. \end{cases}$$

Moreover, we start from

$$V^{(0)}(t, M_t, N_t) = 0. \tag{2.16}$$

2.3.1 The free boundary problem

After making the time reversal change of variable $\tau = T - t$, we introduce the function $u^{(p)}(\tau, M, N) = V^{(p)}(T - \tau, M, N)$ so that it solves the following complementarity problem:

$$\begin{aligned} \mathcal{L}[u^{(p)}] \leqslant 0 \quad & \text{in } (0, T) \times \mathbb{R}_+^2, \\ u^{(p)} \geqslant \Psi^{(p)} \quad & \text{in } (0, T) \times \mathbb{R}_+^2, \\ (\mathcal{L}[u^{(p)}])(u^{(p)} - \Psi^{(p)}) = 0 \quad & \text{in } (0, T) \times \mathbb{R}_+^2, \\ u^{(p)}(0, \cdot) = \Psi^{(p)}(0, \cdot) \quad & \text{in } \mathbb{R}_+^2, \end{aligned} \tag{2.17}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}[F] = -\frac{\partial F}{\partial \tau} + \tfrac{1}{2}\sigma^2 M^2 \frac{\partial^2 F}{\partial M^2} + \alpha(\mu(T-\tau) - \ln(M))M\frac{\partial F}{\partial M} - \beta\ln(N)N\frac{\partial F}{\partial N} - rF, \tag{2.18}$$

and the pth reward obstacle function $\Psi^{(p)}$ has the following form:

$$\Psi^{(p)}(\tau, S) = \begin{cases} \phi(T-\tau, S) + w^{\tau,(p-1)}(\delta, M, N) & \text{for } \tau \in [\delta, T], \\ \phi(T-\tau, S) & \text{for } \tau \in [0, \delta). \end{cases} \tag{2.19}$$

In (2.19), $w^{\tau,(p-1)}(\delta, M, N)$ denotes the value of the swing option with one fewer exercise right.

In order to obtain the value of $w^{\tau,(p-1)}(\delta, M, N)$ for $\tau \in [\delta, T]$, when $p = 1$, we note that

$$w^{\tau,(0)}(t, M, N) = 0 \quad \text{for } (t, M, N) \in [0, \delta] \times \mathbb{R}_+^2.$$

When $p > 1$, however, we need to solve the following PDE problem:

$$\left.\begin{aligned} \mathcal{L}[w^{\tau,(p-1)}] = 0 \quad & \text{in } (0, \delta) \times \mathbb{R}_+^2, \\ w^{\tau,(p-1)}(0, \cdot) = u^{(p-1)}(\tau - \delta, \cdot) \quad & \text{in } \mathbb{R}_+^2, \end{aligned}\right\}$$

where $\mathcal{L}$ is given by (2.18).

Note that, due to the constant refracting period, the reward function (2.19) can be equivalently written as

$$\Psi^{(p)}(\tau, S) = \begin{cases} \phi(T-\tau, S) + w^{\tau,(p-1)}(\delta, M, N) & \text{for } \tau \geqslant (p-1)\delta, \\ \Psi^{(p-1)}(\tau, S) & \text{for } \tau < (p-1)\delta. \end{cases} \tag{2.21}$$

That is, in a period of length $(p-1)\delta$, we can only exercise $(p-1)$ rights, due to the refracting period. That is why the value of the reward function with p exercise rights is equal to the value with $(p-1)$ rights at any time $\tau < (p-1)\delta$.

3 THE NUMERICAL METHODS

In order to obtain a numerical approach to the value of a swing option with $p \in \mathbb{N}$ exercise rights, we need to solve a free boundary problem for each value of p. Additionally, for $p > 1$, in order to obtain the value of the reward function $\Psi^{(p)}(\tau, S)$ associated with each complementarity problem (2.17), the solution for certain times of an initial value problem is required. For the numerical solution of the PDEs (2.17) and (2.20), we propose a Crank–Nicolson characteristics time discretization scheme combined with a piecewise biquadratic Lagrange finite element method. Thus, first a localization technique is used to cope with the initial formulation in an unbounded domain. For the additional inequality constraints associated with the complementarity problem (2.17), we propose a mixed formulation and an ALAS technique.

3.1 Localization procedure and formulation in a bounded domain

In this section, we replace the unbounded domain with a bounded one. In order to determine the required boundary conditions for the associated PDE problems, we follow Oleinik and Radkevic (1973), which is based on the theory proposed in Fichera (1960). More recently, this theory was also applied to degenerated parabolic PDEs, which appear in finance in Bucková *et al* (2015). Let us introduce the notation

$$x_0 = \tau, \quad x_1 = M \quad \text{and} \quad x_2 = N, \tag{3.1}$$

and let us consider both x_1^{∞} and x_2^{∞} to be large enough, suitably chosen real numbers. Let

$$\Omega = (0, x_0^{\infty}) \times (0, x_1^{\infty}) \times (0, x_2^{\infty}),$$

with $x_0^{\infty} = T$. Then, let us denote the Lipschitz boundary by $\Gamma = \partial\Omega$, such that $\Gamma = \bigcup_{i=0}^{2}(\Gamma_i^- \cup \Gamma_i^+)$, where

$$\begin{aligned} \Gamma_i^- &= \{(x_0, x_1, x_2) \in \Gamma \mid x_i = 0\}, \\ \Gamma_i^+ &= \{(x_0, x_1, x_2) \in \Gamma \mid x_i = x_i^{\infty}\}, \quad i = 0, 1, 2. \end{aligned}$$

Then, the operator defined in (2.18) can be written in the form

$$\mathcal{L}[F] = \sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 F}{\partial x_i x_j} + \sum_{j=0}^{2} b_j \frac{\partial F}{\partial x_j} + b_0 F, \tag{3.2}$$

where the involved data is given by

$$B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

$$\vec{b} = (b_j) = \begin{pmatrix} -1 \\ g(x_0, x_1) \\ h(x_2) \end{pmatrix}, \quad b_0 = -r, \tag{3.4}$$

where

$$g(x_0, x_1) = \begin{cases} 0 & \text{if } x_1 = 0, \\ \alpha(\mu(T - x_0) - \ln(x_1))x_1 & \text{if } x_1 \neq 0, \end{cases}$$

$$h(x_2) = \begin{cases} 0 & \text{if } x_2 = 0, \\ -\beta \ln(x_2) x_2 & \text{if } x_2 \neq 0. \end{cases}$$

Thus, following Oleinik and Radkevic (1973), in terms of the normal vector to the boundary pointing inward Ω, $\vec{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma^0 = \left\{ x \in \Gamma \;\middle|\; \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \quad \Sigma^1 = \Gamma - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 \;\middle|\; \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

As indicated in Oleinik and Radkevic (1973), the boundary conditions at $\Sigma^1 \cup \Sigma^2$ for the so-called first boundary value problem associated with (3.2) are required. Note that $\Sigma^1 = \Gamma_1^+$ and $\Sigma^2 = \Gamma_0^-$. Therefore, in addition to an initial condition (see Section 2.3.1), we need to impose a boundary condition on Γ_1^+. For this purpose, in order to construct an ABC on this boundary, we replace the operator (2.18) in the right exterior domain (ie, for $x_1 > x_1^\infty$) with the following:

$$\bar{\mathcal{L}}[F] = -\frac{\partial F}{\partial \tau} + \tfrac{1}{2}\sigma^2 (x_1^\infty)^2 \frac{\partial^2 F}{\partial x_1^2} + \alpha(\mu - \ln(x_1^\infty)) x_1^\infty \frac{\partial F}{\partial x_1} - rF, \tag{3.5}$$

where we assume that the coefficients are constant, and there is no dependency on the variable x_2. Next, by applying the Laplace method, we can write the Laplace-transformed right ABC as

$$\hat{F}_{x_1}(x_1^\infty, s) = \left(\frac{b}{2a} - \frac{1}{2a} \sqrt[+]{b^2 + 4(c+s)a} \right) \hat{F}(x_1^\infty, s), \tag{3.6}$$

where

$$a = \tfrac{1}{2}\sigma^2 (x_1^\infty)^2, \qquad b(T-\tau) = \alpha(\ln(x_1^\infty) - \mu(T-\tau)) x_1^\infty, \qquad c = r,$$

and s is the dual variable of the Laplace transform. Here, $\sqrt[+]{\cdots}$ denotes the branch of the square root with a positive real part. In what follows, for simplicity we drop the time dependence in the notation for b. Also note that if we neglect the seasonality ($f = 0$), as in the forthcoming Example 1, neither μ nor b depends on time.

Taking into account the approach of Halpern (1986), we use a first-order Taylor approximation for small values of a of the square root term in (3.6). This leads to the following transformed boundary condition:

$$\hat{F}_{x_1}(x_1^\infty, s) \approx \left(\frac{b - |b|}{2a} - \frac{c+s}{|b|} \right) \hat{F}(x_1^\infty, s). \tag{3.7}$$

Finally, using an inverse Laplace transformation, for $b > 0$ we obtain the following first-order ABC:

$$\frac{\partial F}{\partial \tau} + b \frac{\partial F}{\partial x_1} + cF = 0 \quad \text{on } \Gamma_1^+. \tag{3.8}$$

Taking into account the previous change of spatial variables, we write (2.20) in divergence form in the bounded spatial domain $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$. Thus, the IBVP takes the following form.

Find $w^{\tau,(p-1)}\colon [0,\delta] \times \Omega \to \mathbb{R}$, such that

$$\frac{\partial w^{\tau,(p-1)}}{\partial t} + \vec{v} \cdot \nabla w^{\tau,(p-1)} - \mathrm{Div}(A \nabla w^{\tau,(p-1)}) + l w^{\tau,(p-1)} = \tilde{f} \quad \text{in } (0,\delta) \times \Omega, \tag{3.9}$$

$$w^{\tau,(p-1)}(0,\cdot) = u^{(p-1)}(\tau - \delta, \cdot) \quad \text{in } \Omega, \tag{3.10}$$

$$\frac{\partial w^{\tau,(p-1)}}{\partial t} + \alpha(\ln(x_1^\infty) - \mu) x_1^\infty \frac{\partial w^{\tau,(p-1)}}{\partial x_1} + l w^{\tau,(p-1)} = 0 \quad \text{on } (0,\delta) \times \Gamma_1^+. \tag{3.11}$$

Further, for the complementarity problem associated with the swing option value, denoting by P the Lagrange multiplier, we can pose the following mixed formulation.

Find $u^{(p)}\colon [0,T] \times \Omega \to \mathbb{R}$, such that

$$\frac{\partial u^{(p)}}{\partial \tau} + \vec{v} \cdot \nabla u^{(p)} - \mathrm{Div}(A \nabla u^{(p)}) + l u^{(p)} + P = \tilde{f} \quad \text{in } (0,T) \times \Omega, \tag{3.12}$$

with the complementarity conditions

$$u^{(p)} \geqslant \Psi^{(p)}, \quad P \leqslant 0, \quad (u^{(p)} - \Psi^{(p)}) P = 0 \quad \text{in } (0,T) \times \Omega \tag{3.13}$$

and the initial and boundary conditions

$$u^{(p)}(0,\cdot) = \Psi^{(p)}(0,\cdot) \quad \text{in } \Omega, \tag{3.14}$$

$$\frac{\partial u^{(p)}}{\partial \tau} + \alpha(\ln(x_1^\infty) - \mu) x_1^\infty \frac{\partial u^{(p)}}{\partial x_1} + l u^{(p)} = 0 \quad \text{on } (0,T) \times \Gamma_1^+. \tag{3.15}$$

For both problems, the involved data is defined as follows:

$$A = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} \tilde{g}(\tau, x_1) \\ \tilde{h}(x_2) \end{pmatrix}, \qquad l = r, \qquad \tilde{f} = 0,$$

$$\tilde{g}(\tau, x_1) = \begin{cases} 0 & \text{if } x_1 = 0, \\ (\sigma^2 - \alpha(\mu(T-\tau) - \ln(x_1))) x_1 & \text{if } x_1 \neq 0, \end{cases}$$

$$\tilde{h}(x_2) = \begin{cases} 0 & \text{if } x_2 = 0, \\ \beta \ln(x_2) x_2 & \text{if } x_2 \neq 0. \end{cases}$$

3.2 Time discretization

First, we define the characteristics curve through $\boldsymbol{x} = (x_1, x_2)$ at time $\bar{\tau}$, $X(\boldsymbol{x}, \bar{\tau}; s)$, which satisfies

$$\frac{\partial}{\partial s} X(\boldsymbol{x}, \bar{\tau}; s) = \vec{v}(X(\boldsymbol{x}, \bar{\tau}; s)), \quad X(\boldsymbol{x}, \bar{\tau}; \bar{\tau}) = \boldsymbol{x}. \tag{3.16}$$

In order to discretize in time the material derivative in the complementarity problem (3.12), let us consider a number of time steps $\bar{N}$, the time step $\Delta\tau = T/\bar{N}$ and the time mesh points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \bar{N}$. In order to obtain the initial condition for solving the problem (3.9), the time discretization has to be chosen such that $\delta/\Delta\tau \in \mathbb{N}$. So, we should choose $\bar{N}$ as a multiple of T/δ. In the discretization of the material derivative in the initial value problem (3.9), we consider a number of time steps equal to $\delta/\Delta\tau$.

The material derivative approximation by the characteristics method for both problems is given by

$$\frac{DF}{D\tau} = \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau},$$

where $F = u^{(p)}, w^{\tau,(p-1)}$ and $X^n(\boldsymbol{x}) := X(\boldsymbol{x}, \tau^{n+1}; \tau^n)$. For the case of $f = 0$, the components of $X^n(\boldsymbol{x})$ can be computed analytically:

$$X_1^n(\boldsymbol{x}) = \begin{cases} x_1 & \text{if } x_1 = 0, \\ \exp[(\exp(-\alpha\Delta\tau)(\sigma^2 + \alpha\ln(x_1)) - \sigma^2)/\alpha] & \text{if } x_1 \neq 0, \end{cases}$$

$$X_2^n(\boldsymbol{x}) = \begin{cases} x_2 & \text{if } x_2 = 0, \\ \exp(\ln(x_2)\exp(-\beta\Delta\tau)) & \text{if } x_2 \neq 0. \end{cases}$$

However, for the general case in which it is not possible to compute the characteristics curves analytically, some numerical ordinary differential equation (ODE) solvers can be used (see, for example, Bermúdez *et al* 2006a).

Next, we consider a Crank–Nicolson scheme around $(X(\mathbf{x}, \tau^{n+1}; \tau), \tau)$ for $\tau = \tau^{n+(1/2)}$. So, the time discretized equation for $F = u^{(p)}, w^{\tau,(p-1)}$ and $P = 0$ can be written as follows.

Find F^{n+1} such that

$$\begin{aligned} \frac{F^{n+1}(\boldsymbol{x}) - F^n(X^n(\boldsymbol{x}))}{\Delta\tau} &- \tfrac{1}{2}\operatorname{Div}(A\nabla F^{n+1})(\boldsymbol{x}) \\ &- \tfrac{1}{2}\operatorname{Div}(A\nabla F^n)(X^n(\boldsymbol{x})) + \tfrac{1}{2}(lF^{n+1})(\boldsymbol{x}) + \tfrac{1}{2}(lF^n)(X^n(\boldsymbol{x})) = 0. \end{aligned} \tag{3.17}$$

Moreover, we also discretize the artificial boundary condition on Γ_1^+:

$$\frac{F^{n+1}(\boldsymbol{x}) - F^n(\hat{X}^n(\boldsymbol{x}))}{\Delta\tau} + \tfrac{1}{2}(cF^{n+1})(\boldsymbol{x}) + \tfrac{1}{2}(cF^n)(\hat{X}^n(\boldsymbol{x})) = 0, \tag{3.18}$$

where $\hat{X}^n(\boldsymbol{x}) = (-b\Delta\tau + x_1, x_2)^T$ in the case of $f = 0$.

Thus,

$$F^{n+1}(\boldsymbol{x}) = \frac{1 - c\Delta\tau/2}{1 + c\Delta\tau/2} F^n(\hat{X}^n(\boldsymbol{x})) \quad \text{on } \Gamma_1^+. \tag{3.19}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply (3.17) by a suitable test function, integrate in Ω and use the classical Green formula as well as the following (Nogueiras 2005):

$$\begin{aligned}\int_\Omega \operatorname{Div}(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_\Gamma (\nabla X^n)^{-T}(\boldsymbol{x})\boldsymbol{n}(x)(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega (\nabla X^n)^{-1}(\boldsymbol{x})(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\nabla\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega \operatorname{Div}((\nabla X^n)^{-T}(\boldsymbol{x}))(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \qquad (3.20)\end{aligned}$$

Note that, when $f = 0$, we have

$$\operatorname{Div}((\nabla X^n)^{-T}(\boldsymbol{x})) = \begin{pmatrix} \dfrac{1}{e_1}(\exp(\alpha\Delta\tau) - 1) \\ \dfrac{1}{e_2}(\exp(\beta\Delta\tau) - 1)\end{pmatrix}, \qquad (3.21)$$

where

$$e_1 = \exp\left[\frac{\exp(-\alpha\Delta\tau)(\sigma^2 + \alpha\ln(x_1)) - \sigma^2}{\alpha}\right]$$

and

$$e_2 = \exp(\ln(x_2)\exp(-\beta\Delta\tau)).$$

In the general case, $\operatorname{Div}((\nabla X^n)^{-T}(\boldsymbol{x}))$ needs to be approximated. After the previous steps, we can write a variational formulation for the time discretized problem as follows.

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$ such that $\psi = 0$ on Γ_1^+,

$$\begin{aligned}&\int_\Omega F^{n+1}(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{\Delta\tau}{2}\int_\Omega (\boldsymbol{A}\nabla F^{n+1})(\boldsymbol{x})\nabla\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \frac{\Delta\tau}{2}\int_\Omega lF^{n+1}(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega F^n(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{\Delta\tau}{2}\int_\Omega (\nabla X^n)^{-1}(\boldsymbol{x})(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\nabla\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad - \frac{\Delta\tau}{2}\int_\Omega lF^n(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad - \frac{\Delta\tau}{2}\int_\Omega \operatorname{Div}((\nabla X^n)^{-T}(\boldsymbol{x}))(\boldsymbol{A}\nabla F^n)(X^n(\boldsymbol{x}))\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad (3.22)\end{aligned}$$

where ∇X^n can be computed analytically in some cases. At other times, it needs to be approximated (see, for example, Bermúdez *et al* 2006a).

3.3 Finite elements discretization

For the spatial discretization, we consider $\{\tau_h\}$, a quadrangular mesh of the domain Ω. Let $(T_1, \mathcal{Q}_2, \Sigma_{T_1})$ be a family of piecewise biquadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T_1 \in \tau_h$, with degree less than or equal to two in each spatial variable; Σ_{T_1} is the subset of nodes of the element T_1. More precisely, let us define the finite elements space F_h as

$$F_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) \colon \phi_{h_{T_1}} \in \mathcal{Q}_2, \text{ for all } T_1 \in \tau_h\}, \tag{3.23}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of piecewise continuous functions on $\bar{\Omega}$.

3.4 ALAS algorithm

Here, the ALAS algorithm proposed in Kärkkäinen *et al* (2003) is applied to the fully discretized in time and space mixed formulations (3.12) and (3.13). More precisely, after this full discretization procedure, the discrete problem can be written in the following form:

$$M_h u_h^{(p),n} + P_h^n = b_h^{n-1}, \tag{3.24}$$

with the discrete complementarity conditions

$$u_h^{(p),n} \geqslant \Psi_h^{(p),n}, \qquad P_h^n \leqslant 0, \qquad (u_h^{(p),n} - \Psi_h^{(p),n}) P_h^n = 0, \tag{3.25}$$

where P_h^n denotes the vector of the multiplier values, and $\Psi_h^{(p),n}$ denotes the vector of the nodal values defined by the function $\Psi^{(p)}$.

The basic iteration of the ALAS algorithm consists of two steps. In the first step, the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not). In the second step, a reduced linear system associated with the inactive part is solved. Thus, we use the algorithm for unilateral problems, which are based on the augmented Lagrangian formulation.

First, for any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} := \{1, 2, \dots, N_{\text{dof}}\}$, let us denote by $[\mathcal{M}_h]_{\mathcal{I}\mathcal{I}}$ the principal minor of the matrix $\mathcal{M}_h$, and by $[\mathcal{M}_h]_{\mathcal{I}\mathcal{J}}$ the co-diagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, for each time τ_n, the ALAS algorithm computes not only $u_h^{(p),n}$ and P_h^n but also a decomposition $\mathcal{N} = \mathcal{J}^n \cup \mathcal{I}^n$, such that

$$\begin{aligned} \mathcal{M}_h u_h^{(p),n} + P_h^n &= b_h^{n-1}, \\ [P_h^n]_j + \gamma [u_h^{(p),n} - \Psi^{(p)}]_j &\leqslant 0, \quad \text{for all } j \in \mathcal{J}^n, \\ [P_h^n]_i &= 0, \quad \text{for all } i \in \mathcal{I}^n, \end{aligned} \tag{3.26}$$

for a given positive parameter γ. In the above equations, $\mathcal{I}^n$ and $\mathcal{J}^n$ are the inactive and active sets at time t_n, respectively. More precisely, the iterative algorithm builds sequences $\{u_{h,m}^{(p),n}\}_m$, $\{P_{h,m}^n\}_m$, $\{\mathcal{I}_m^n\}_m$ and $\{\mathcal{J}_m^n\}_m$, converging to $u_h^{(p),n}$, P_h^n, $\mathcal{I}^n$ and $\mathcal{J}^n$ by means of the following procedure.

(1) Initialize $u_{h,0}^{(p),n} = \Psi_h^{(p),n}$ and $P_{h,0}^n = \min\{b_h^n - \mathcal{M}_h u_{h,0}^{(p),n}, 0\} \leqslant 0$. Choose $\gamma > 0$. Set $m = 0$.

(2) Compute

$$\begin{aligned} Q_{h,m}^n &= \min\{0, P_{h,m}^n + \gamma(u_{h,m}^{(p),n} - \Psi_{h,m}^{(p),n})\}, \\ \mathcal{J}_m^n &= \{j \in \mathcal{N},\ [Q_{h,m}^n]_j < 0\}, \\ \mathcal{I}_m^n &= \{i \in \mathcal{N},\ [Q_{h,m}^n]_i = 0\}. \end{aligned}$$

(3) If $m \geqslant 1$ and $J_m^n = J_{m-1}^n$, then convergence is achieved. Stop.

(4) Let $u^{(p)}$ and P be the solution of the linear system

$$\begin{gathered} \mathcal{M}_h u^{(p)} + P = b_h^{n-1}, \\ P = 0 \quad \text{on } \mathcal{I}_m^n \quad \text{and} \quad u^{(p)} = \Psi_{h,m}^{(p),n} \quad \text{on } \mathcal{J}_m^n. \end{gathered} \tag{3.27}$$

Set $u_{h,m+1}^{(p),n} = u^{(p)}$, $P_{h,m+1}^n = \min\{0, P\}$, $m = m + 1$ and go to (2).

It is important to note that, instead of solving the full linear system in (4), for $\mathcal{I} = \mathcal{I}_m^n$ and $\mathcal{J} = \mathcal{J}_m^n$, the following reduced system on the inactive set is solved:

$$\begin{aligned} [\mathcal{M}_h]_{\mathcal{I}\mathcal{I}}[u^{(p)}]_{\mathcal{I}} &= [b^{n-1}]_{\mathcal{I}} - [\mathcal{M}_h]_{\mathcal{I}\mathcal{J}}[\Psi^{(p)}]_{\mathcal{J}}, \\ [u^{(p)}]_{\mathcal{J}} &= [\Psi^{(p)}]_{\mathcal{J}}, \\ P &= b^{n-1} - \mathcal{M}_h V. \end{aligned} \tag{3.28}$$

In Kärkkäinen *et al* (2003), the authors proved the convergence of the algorithm in a finite number of steps for a Stieltjes matrix (ie, a real symmetric positive definite matrix with negative off-diagonal entries (see Varga 1962)) and a suitable initialization (the same we consider in this paper). They also proved that $\mathcal{I}_m \subset \mathcal{I}_{m+1}$. Nevertheless, a Stieltjes matrix can only be obtained for linear elements, and never for the here-used biquadratic elements, because we have some positive off-diagonal entries arising from the stiffness matrix (actually, we use a lumped mass matrix). However, we have obtained good results using the ALAS algorithm with biquadratic finite elements.

Concerning the efficient solution of the different reduced systems that appear at each iteration of the ALAS algortithm, as in Nogueiras (2005), we order the mesh nodes from right to left, and from bottom to top, so that we obtain a matrix $\mathcal{M}_h$ of the complete system with N_{x_2} blocks of dimension N_{x_1}. Indeed, each set of nodes with the same x_2 coordinate gives rise to a block in the matrix. Thus, for each block (x_2 coordinate) we have either all the nodes in the inactive set or only the first $n(x_2) < N_{x_1}$ nodes in the inactive set. The full matrix $\mathcal{M}_h$ is factorized only once outside the ALAS loop (we use a Cholesky factorization); at each ALAS iteration, we solve the N_{x_2} linear systems with variable dimension.

4 NUMERICAL RESULTS

In this section, we show some numerical results to illustrate the performance of the numerical methods by comparing them with some examples in the literature. Note that this paper is the first to consider the numerical solution of the PDE associated with a two-factor model for electricity prices. Thus, we mainly compare our results with the example in Hambly *et al* (2009) that considers two factors and a binomial method, as well as with the extension to two factors of the one-factor stochastic model solved in Wilhelm and Winter (2008) with finite elements.

Concerning the numerical convergence of the Crank–Nicolson characteristics discretization method, first note that the numerical analysis for the initial boundary value problem under rather general conditions on a PDE operator has been developed in Bermúdez *et al* (2006a,b), where second-order convergence in space and time is theoretically proved. However, in the present nonlinear problem, we need to combine the method with the ALAS algorithm, so that the second-order convergence cannot be obtained, as also happens, for example, in Calvo-Garrido and Vázquez (2015). However, we prefer to maintain Crank–Nicolson characteristics, which results in a slightly better accuracy than a possible alternative fully implicit method.

4.1 Example 1

First, as in Hambly *et al* (2009), we consider the valuation of a swing option with up to $p = 20$ exercise rights, in which the rights correspond to the payoff of a call option. For this purpose, we need to specify a set of parameters related to the market values of the data involved in the underlying factors, the initial conditions of the stochastic processes and the parameters of the payoff function. All of these are taken from Hambly *et al* (2009) and shown in Table 1. We have chosen these parameters in order to compare the results we obtain with those in Hambly *et al* (2009), in which a binomial method has been used. Moreover, concerning the numerical methods, we select the parameters collected in Table 2. The finite element mesh corresponds to a constant mesh step in each direction and 24×24 finite elements (ie, $\Delta x_i = x_i^{\infty}/24$, $i = 1, 2$). Note that, as we consider $f = 0$, thus neglecting seasonality, b does not depend on time in Example 1.

In Figure 1, we show the value per exercise right of the swing option when the maturity of the contract is one year, and a right can be exercised at most once per day (ie, the refracting period δ is one day). Moreover, we consider that the time step $\Delta\tau$ is also one day. For this example, in Figure 2, we represent the approximate location of the free boundary at origination (ie, $t = 0$) when $p = 2$. In the white region, it is optimal to exercise the option, whereas the black region corresponds to the non-early exercise region. In Figure 3, we show the exercise value (or obstacle) at origination for the same swing contract and data set. In Figure 4, the swing option value at origination

TABLE 1 Fixed parameters of the model for Example 1 (cf. Hambly *et al* 2009).

Market parameters of the underlying factors	
Speed of mean-reversion process M, α	7
Volatility, σ	1.4
Speed of mean-reversion process N, β	200
Interest rate, r	0
Seasonality, f	0
Initial conditions	
Initial value of M, M_0	1
Initial value of N, N_0	1
Payoff function parameters	
Payoff, $\phi(T, S)$	$(S - K)_+$
Strike, K	1

TABLE 2 Parameters of the numerical methods in Example 1.

Computational domain	
x_1^∞	3K
x_2^∞	3K
ABC	
Coefficient b	20.13
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
ALAS algorithm	
Parameter γ	10000

is shown. Note that in the white region of Figure 2, the value of the swing option in Figure 4 coincides with the exercise value represented in Figure 3. In addition, we present some results of just changing $p = 2$ to $p = 6$ in the previous data. More precisely, in Figures 5 and 6, we observe that the exercise region is not too greatly affected by this change, while the solution changes mainly due to the change in the new exercise value function.

FIGURE 1 Value per right of a swing option with one year to delivery in Example 1.

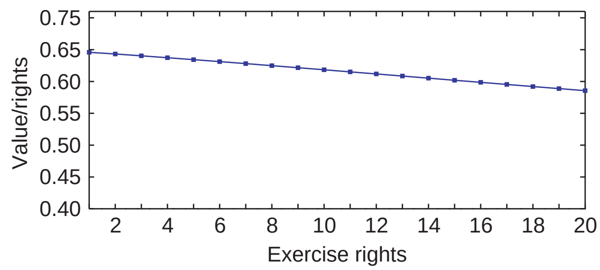
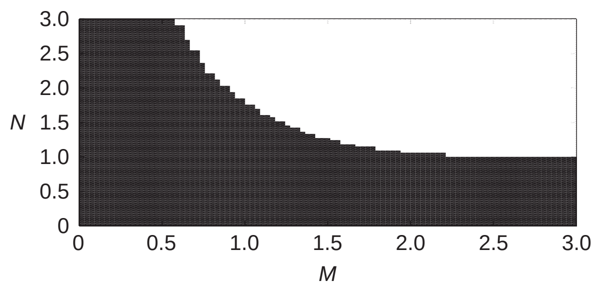


FIGURE 2 Approximated free boundary in the grid at origination of a swing option with $p = 2$ rights and one year to delivery in Example 1.



Exercise region in white and non-exercise region in black.

Next, in Figure 7, we present the value per exercise right of the swing option when the maturity of the contract is two months, the refracting period δ is one day and the time step $\Delta\tau$ coincides with the refracting period. Finally, in Figure 8, we consider that the option has ten exercise opportunities per day (ie, the refracting period is 0.1 days) and the delivery period is six days. The time step $\Delta\tau$ is equal to the refracting period.

Figures 1, 7 and 8 are in full agreement with the analogous ones appearing in Hambly *et al* (2009) and Kluge (2006), which are obtained using binomial methods. More precisely, the results in Figure 1 agree with those for the case without jumps in Hambly *et al* (2009, Figure 10) (see also Kluge 2006, Figure 4.8); the results in Figure 7 are the same as those without jumps in the bottom-right graph of Hambly *et al* (2009, Figure 11) (equivalently in Kluge 2006, Figure 4.8); and our results in Figure 8 agree with those in Kluge (2006, Figure 4.11). Further, we observe that the

FIGURE 3 Obstacle at origination of a swing option with $p = 2$ rights and one year to delivery in Example 1.

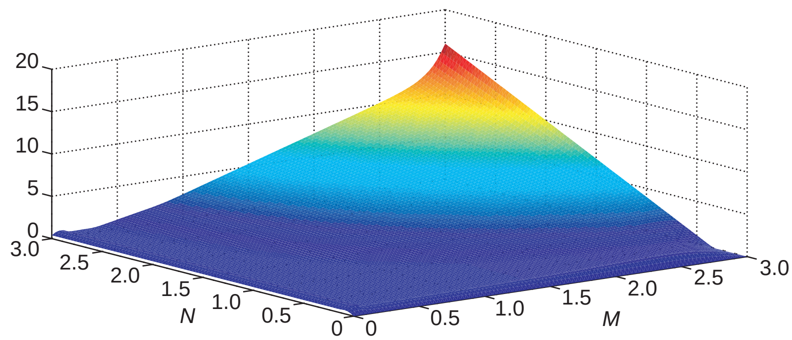
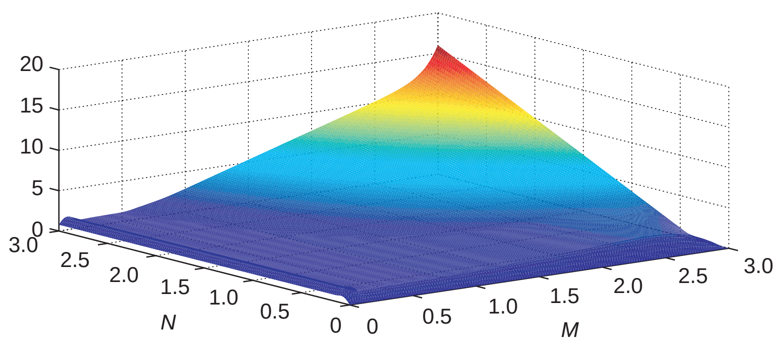


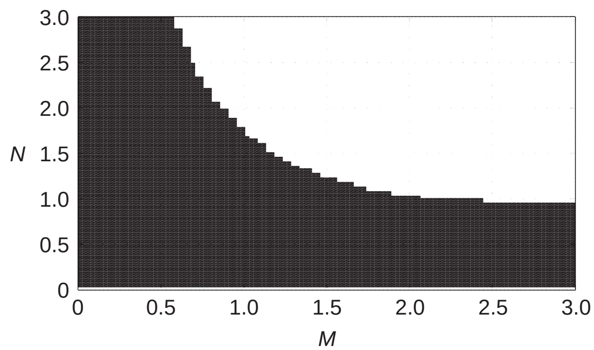
FIGURE 4 Swing option value at origination with $p = 2$ rights and one year to delivery in Example 1.



price per exercise right decreases with the number of exercise rights. This is what we expected, because p swing options with one exercise right (which are equivalent to p American options) give more flexibility, as you can exercise all the rights at once; consequently, its price must be higher than the price of one swing option with p exercise rights. In Figure 8, the difference between two values per exercise right is smaller due to the value of the refracting period. As expected, when the value of the refracting period decreases, the value of a swing option with p exercise rights tends to the value of p American options with one exercise right.

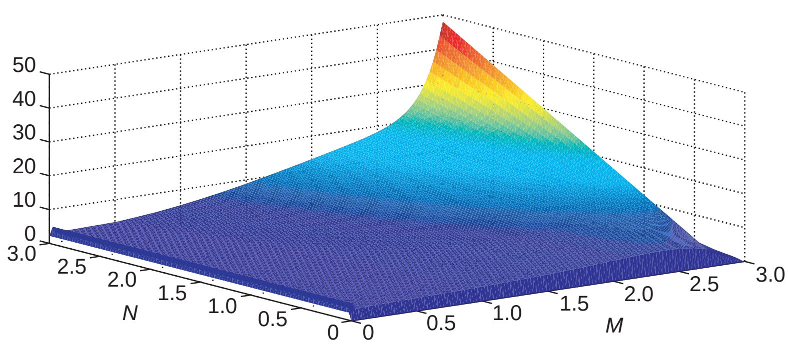
Concerning the computational cost of the numerical experiments related to Exam-

FIGURE 5 Approximated free boundary in the grid at origination of a swing option with $p = 6$ rights and one year to delivery in Example 1.



Exercise region in white and non-exercise region in black.

FIGURE 6 Swing option value at origination with $p = 6$ rights and one year to delivery in Example 1.



ple 1, for the results presented in Figure 8, the computing time ranges from 23 seconds (for $p = 1$) up to 86 seconds (for $p = 4$) in a computer with Intel Core I5-2400 CPU @ 3.10 GHz with 4GB of RAM. The same order of computational times has been analogously observed Example 2.

4.2 Example 2

In this section, unlike in Example 1, we show some cases in which the seasonality function and the interest rate are different from zero. For this purpose, we consider a

FIGURE 7 Value per right of a swing option with sixty days to delivery in Example 1.

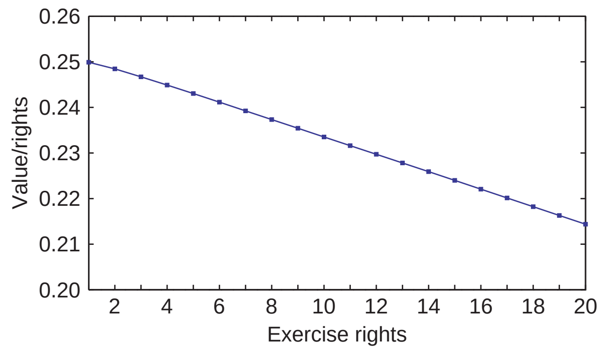
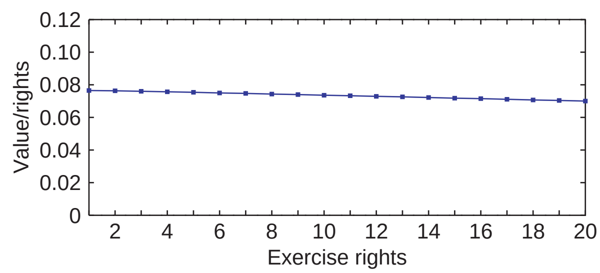


FIGURE 8 Value per right of a swing option with six days to delivery in Example 1.

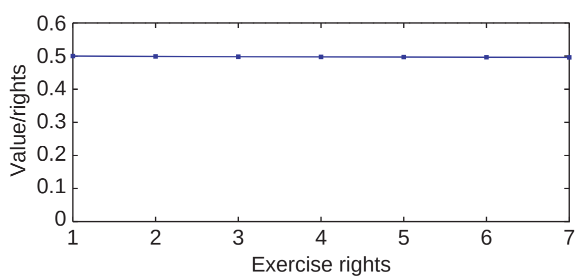


swing option with up to $p = 7$ rights, a maturity of one year and a refracting period of 0.1 years. Moreover, we consider the values for the parameters involved in the underlying factors, which appear in Table 3. Most of these are taken from Wilhelm and Winter (2008) for a one-factor stochastic model for electricity prices; these, in turn, are taken from Lucia and Schwartz (2002) and are obtained from daily electricity spot and future price experimental observations. In order to pose a two-factor model, we consider different nonzero values for the parameter β. For the numerical solution, we again consider the parameters in Table 2, except the coefficient b of the ABC, which in this case depends on time and is always greater than zero. In this example, the time step is $\Delta\tau = 0.01$. In Figures 9 and 10, we show the value of this option per exercise right when $\beta = 0.2$ and $\beta = 2$, respectively. In Figure 11, we represent its value for $\beta = 20$. As illustrated by these three figures, we can observe that the value of the swing option decreases when we increase the value of the mean-reversion

TABLE 3 Fixed parameters of the model with seasonality in Example 2.

Market parameters of the underlying factors	
Speed of mean-reversion process M, α	0.016
Volatility, σ	0.086
Speed of mean-reversion process N, β	0.2, 2, 20
Interest rate, r	0.05
Seasonality, f	$4.867 + 0.306 \cos((t + 0.836)(2\pi/365))$
Initial conditions	
Initial value of M, M_0	1
Initial value of N, N_0	1.5
Payoff function parameters	
Payoff, $\phi(T, S)$	$(S - K)_+$
Strike, K	1

FIGURE 9 Value per right of a swing option with one year to delivery when $\beta = 0.2$ in Example 2.



parameter β. Note that an increase in β implies a decrease in the asset value and, therefore, in the value of the call swing option. This property has also been observed in Hambly *et al* (2009), where jumps are additionally included in the dynamics of Y_t.

5 CONCLUSIONS

In this paper, we have considered the valuation of swing options in electricity markets by numerically solving a PDE-based formulation. While the case of electricity prices driven by one stochastic factor has been considered in the literature with PDE

FIGURE 10 Value per right of a swing option with one year to delivery when $\beta = 2$ in Example 2.

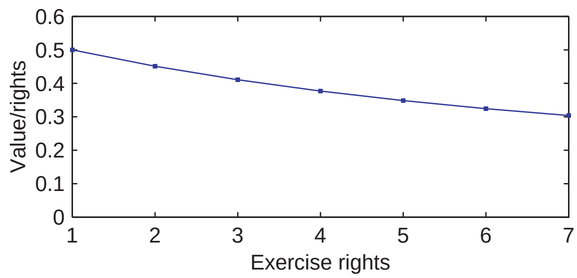
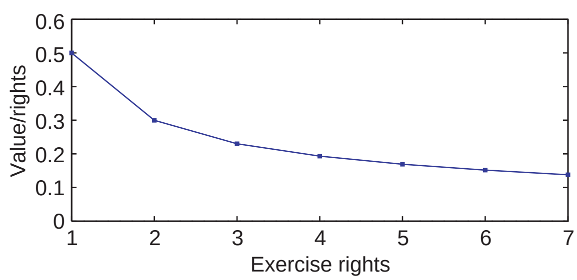


FIGURE 11 Value per right of a swing option with one year to delivery when $\beta = 20$ in Example 2.



methods, we have successfully addressed the case with two stochastic factors. Indeed, another novelty relies on the consideration of ABCs instead of the not-so-well-justified homogeneous Neumann boundary conditions already used in the one-factor case.

The swing option mainly consists of a path-dependent option with multiple exercise rights. The right consists of receiving the payoff of a call option. The valuation problem has been posed as a sequence of free boundary problems, one for each right. In addition, an initial value problem has to be solved due to the fact that the value of a swing option with one fewer exercise is involved in the definition of the obstacle function.

In order to obtain a numerical solution to the problem, we have proposed appropriate numerical methods based on Lagrange–Galerkin formulations combined with the ALAS algorithm to deal with the early exercise feature. As we have to confine the unbounded domain, appropriate artificial boundary conditions are constructed.

Finally, we show some numerical results in order to illustrate the behavior of the proposed methods.

It is important to note some advantages of numerical methods for PDE valuation of swing options with respect to alternative Monte Carlo or lattice methods. The numerical methods provide the surface of swing prices at origination for the whole set of electricity spot prices, while the alternative approaches obtain one swing option price for each spot price. Also, the methods we propose exhibit a clear advantage in the computation of the exercise boundary and exercise region simultaneously with the computation of swing option prices for a set of electricity spot prices. The use of Monte Carlo or lattice methods for this purpose would require a lot of additional computation. In future work, we plan to incorporate possible spikes in the electricity prices. For this purpose, jump diffusion processes are required to describe the evolution of the underlying factors, thus leading to partial integro-differential equation problems instead of PDE ones.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper. This work has been partly funded by MINECO of Spain (Project MTM2013-47800-C2-1-P), by MICINN of Spain (FPI grant BES-2011-044746) and by the German Federal Ministry of Education and Research (Project 57049700 2014 DAAD).

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