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Pricing pension plans based on average salary without early retirement: partial differential equation modeling and numerical solution

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In this paper, a partial differential equation model for the pricing of pension plans based on average salary is posed by using the dynamic hedging methodology. The existence and uniqueness of solutions for the resulting initial-value problem associated with a Kolmogorov equation is obtained. Moreover, a numerical method based on a Crank–Nicolson characteristics time discretization combined with finite elements to approximate the solution is proposed. Finally, some test examples illustrate the performance of the numerical methods as a tool for pricing these pension plans.

1 INTRODUCTION

Pension funds are an important aspect of finance in countries with advanced economies, and they affect the retirement decisions taken by their populations. Pension plans are generally classified as either defined contribution plans or defined benefit plans (Bodie (1990)). In the first case, at retirement the employee receives an annuity whose value depends on the investment earnings and the total contributions of both employer and employee to the pension plan account of the employee. Moreover, sometimes the employee can decide among the possible investments of the account, so that he/she bears the risk. In defined benefit plans, the pension at retirement depends on several labor-related factors, such as the number of years of service, the salary or the average salary. A traditional defined benefit pension plan

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pays the salary at retirement multiplied by a number of years and by an accrued rate, with the resulting amount being paid as a monthly pension or a lump sum. Final average pay (FAP) schemes, for which the average salary over the last years before retirement determines the benefit, are most common in the United States. In the case of defined benefit pension plans, the employer bears the liability, which is the amount of money that the employer has to set aside to fund the employee's retirement pension in the future. The value of this liability is roughly what we refer to here as the value of the pension plan. Sometimes, however, pension plans also include integration with social security (integrated plans). In typical integrated plans, the pension benefit at retirement is a fraction of the average salary minus the social security benefit. In Perun (2002) it is proved that the proportion of integrated plans is greater in defined benefit schemes (one-third) than in defined contribution schemes (one-quarter).

In this paper we use a mathematical modeling approach to obtain the value of a defined benefit pension plan understood as the value of the liabilities of the plan with an active member. More precisely, we assume that the amount received by the employee depends on the average salary corresponding to a certain number of years before retirement. The departure point in our modeling approach is the consideration that the salary is stochastic, so that the pension plan can be handled as an option on the average salary. The dynamic hedging methodology in option pricing can then be adapted to state a partial differential equation (PDE) model. Indeed, some features appear that are also found in Asian options and bond pricing PDE models (Wilmott *et al* (1993)). In Sherris and Shen (1999), the same kind of models are stated for pension plans depending on the salary at retirement or on the average salary by using a risk-neutral probability approach. Previous works in insurance can be found, for example, in Norberg (1996), Shimko (1989) and Wolthius (1994).

Additionally, we state the existence and uniqueness of solutions by using the methodology developed by Barucci *et al* (2001) for these kinds of Kolmogorov equation (Oleinik and Radkevic (1973)). This methodology is mainly based on Barucci *et al* (2001) for the existence of sub- and supersolutions and Aronson and Besala (1967) for the uniqueness.

Moreover, we provide a numerical method for solving the PDE model by using the techniques developed in Bermúdez *et al* (2006c) for Asian option pricing models. The numerical analysis of the proposed characteristics Crank–Nicolson time discretization was addressed in Bermúdez *et al* (2006a). Moreover, Bermúdez *et al* (2006b) studied its combination with Lagrange finite elements and considered appropriate different quadrature formulas required in the practical implementation of the methods for the fully discretized problem. Both of these works were applied to the general (possibly degenerated) convection–diffusion–reaction equation under certain assumptions. We note that these assumptions are not fully verified by the PDE that we are dealing with. This explains why the predicted orders of convergence are not achieved in the academic example presented here, for which the exact solution can be analytically obtained.

The paper is organized as follows. In Section 2 the mathematical model is posed in terms of a final value problem associated with a Kolmogorov equation. In Section 3 the mathematical analysis of the model is developed to obtain the existence and uniqueness of solutions. Section 4 contains the description of the numerical methods, which are applied after a localization procedure and a variational formulation. In Section 5 some examples are presented to illustrate the performance of the proposed numerical method. In Section 6 some conclusions and directions for future work are given. Finally, Appendix A contains detailed proofs of the theoretical results.

2 MATHEMATICAL MODELING

Let us denote the age of entry of a member in the pension plan by x and the time since the entry by t, so that $t = 0$ corresponds to the recruitment date of a person aged x. As the pension plan is indexed to the salary of the member, let us denote by $S(t; x)$ the wage at time t of a member entering the pension plan with an age x. Following Sherris and Shen (1999), we assume that $S(t; x)$ is governed by the stochastic differential equation (SDE):

$$\mathrm{d}S = \alpha(t, S; x)\,\mathrm{d}t + \sigma(t, S)\,\mathrm{d}Z \tag{2.1}$$

together with the initial condition $S(0; x) = S_0(x)$, and the salary growth rate α depends on the time of entry into the plan, the current salary and the age at entry, where $\sigma(t, S)$ denotes the volatility of the salary and where Z represents a Wiener process.

The model that we are using assumes that uncertainty about the salary only depends on the volatility, and that it follows a diffusion model. This kind of evolution could correspond to an employee having a variable part of his/her salary (perhaps related to his/her bonus or the firm benefits). We also point out that, in real situations, some sudden events could produce abrupt changes in the salary. In that case, a jump-diffusion model turns out to be more appropriate, so the SDE (2.1) could be replaced by the following, for example:

$$\mathrm{d}S = \alpha(t, S; x)\,\mathrm{d}t + \sigma(t, S)\,\mathrm{d}Z + \mathrm{d}\left(\sum_{i=1}^{N} U_i\right) \tag{2.2}$$

where N denotes a Poisson process with parameter λ and U_i is a sequence of square integrable, identically distributed random variables, so that Z, N and U_i are independent. In all processes, we omit the subscript t for simplicity of notation. For the case $\sigma = 0$, the model would include the pure jump case. Although, in this paper, we will restrict ourselves to the SDE model (2.1), we point out that model (2.2) leads to a partial integro-differential equation, so that appropriate numerical techniques, such as those used in D'Halluin *et al* (2005) to cope with jumps in the case of Asian options, could be applied. The analysis and numerical solution of the case (2.2) is being considered by the authors as an extension of the present work.

Let us denote by $V(t, S; x)$ the value at time t of the benefits payable to the member of the plan when he/she is aged $x + t$ and the salary is S. In this section we pose the mathematical model in terms of a PDE to obtain V, when the retirement benefits depend on the continuous arithmetic average of the salary during the last n_y years, and early retirement is not allowed.

Moreover, the payment from the fund is assumed to occur in case of the death of a member, or cancelation of the plan (withdrawal) to move to another plan. Therefore, we assume three possible states of a member of the plan: active (a), dead (d) and withdrawal (w). It is considered that retirement occurs at the final age of service in the pension fund. The transition intensities from active membership to death or cancelation are denoted by $\mu_{\mathrm{d}}(t; x)$ and $\mu_{\mathrm{w}}(t; x)$, respectively. From an actuarial standpoint, these intensities are understood as forces of decrement acting at time t on a member aged $x + t$ and can be expressed in terms of the corresponding transition probabilities from one state to another.

In classic actuarial mathematics, the value of the benefits of the pension plan, V, is understood as the value of the provision that the plan promoter should reserve in order to guarantee the contractual obligations to the active member. By using actuarial arguments, when assuming deterministic benefits paid to the fund and ignoring any contributions, the variation of the value of the retirement benefits from time t to time $t + \mathrm{d}t$ is given by Thiele's differential equation (see, for example, Bowers *et al* (1997)):

$$\mathrm{d}V = r(t)V\,\mathrm{d}t - \sum_{i=\mathrm{d,w}} \mu_i(t; x)[A_i(t, S; x) - V]\,\mathrm{d}t \tag{2.3}$$

where $r(t)$ is the deterministic risk-free interest rate and $A_i(t, S; x)$ denotes the deterministic benefit paid by the fund in case of death ($i = \mathrm{d}$) or withdrawal ($i = \mathrm{w}$). Note that the difference $A_i(t, S; x) - V$ represents the sum-at-risk associated with decrement i, so that:

$$\sum_{i=\mathrm{d,w}} \mu_i(t; x)(A_i(t, S; x) - V)\,\mathrm{d}t \tag{2.4}$$

denotes the expected value of the payments from the fund in the interval $[t, t + \mathrm{d}t]$.

Following Sherris and Shen (1999, Section 5), we assume that:

$$A_i(t, S; x) = \alpha_i S, \quad \alpha_i \geqslant 0, \ i = \mathrm{d, w}$$

so that the death and withdrawal benefits are a constant multiple of the salary, and the transition intensities $\mu_i(t;x) = \mu_i$ are nonnegative constants.

As we are assuming that retirement benefits depend on the continuous arithmetic average of the salary, in an analogous way to the case of Asian options (Wilmott *et al* (1993)), we introduce the following variable representing the cumulative function of the salary of the last n_y years before T_r:

$$I(t;x) = \int_0^t g(\tau, S(\tau;x))\,\mathrm{d}\tau \tag{2.5}$$

with:

$$g(t,S) = \begin{cases} 0 & \text{if } 0 \leqslant t < T_r - n_y \\ h(S) & \text{if } T_r - n_y \leqslant t \leqslant T_r \end{cases} \tag{2.6}$$

where $T_r > n_y$ denotes the retirement date and h is appropriately chosen. In this paper we consider the particular case $h(S) = k_1 S$, with a given accrual constant $k_1 > 0$. As in the case of Asian options, the variation of I in the interval $[t, t + \mathrm{d}t]$ is given by:

$$\begin{aligned} \mathrm{d}I &= I(t + \mathrm{d}t) - I(t) \\ &= \int_t^{t+\mathrm{d}t} g(\tau, S(\tau;x))\,\mathrm{d}\tau \\ &= g(t,S)\,\mathrm{d}t \end{aligned} \tag{2.7}$$

Next, by considering that $V = V(t, S, I; x)$, we can apply Ito's lemma jointly with the Thiele differential equation (2.3) to obtain the variation of V from t to $t + \mathrm{d}t$:

$$\begin{aligned} \mathrm{d}V = {}& \left(\frac{\partial V}{\partial t} + \alpha(t,S;x)\frac{\partial V}{\partial S} + g(t,S)\frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma(t,S)^2\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t \\ & + \sigma(t,S)\frac{\partial V}{\partial S}\,\mathrm{d}Z + \left(\sum_{i=\mathrm{d,w}} \mu_i(t;x)(A_i(t,S;x) - V)\right)\mathrm{d}t \end{aligned} \tag{2.8}$$

where the first two terms on the right-hand side are associated with the stochastic variation of the salary, while the third term is related to the expected payments in $(t, t + \mathrm{d}t)$ due to death or withdrawal.

In Sherris and Shen (1999) the PDE model is obtained by arguing that the risk-adjusted expected change in the liabilities value, after taking into account the benefit cashflows from the fund related to possible death or withdrawal, should be equal to the risk-free interest rate. We apply the dynamic hedging methodology to deduce the PDE model. For this purpose, we consider two pension plans with values V_j, $j = 1, 2$, that pay the quantities $A_{\mathrm{d}}^j(t,S;x)$ and $A_{\mathrm{w}}^j(t,S;x)$ in the cases of death and withdrawal, respectively. Moreover, we assume that the intensities μ_i are independent of j. Therefore, the variation of each plan V_j from t to $t+\mathrm{d}t$ verifies the corresponding equation (2.8). At this point, we proceed in a similar manner to the dynamic hedging methodology used in the case of bonds (Wilmott *et al* (1993)). Thus, by buying one unit of plan V_1 and selling Δ units of plan V_2, the value of the resulting portfolio is:

$$\Pi = V_1 - \Delta V_2$$

Note that the variation of the portfolio value between t and $t + \mathrm{d}t$ is given by:

$$\mathrm{d}\Pi = \mathrm{d}V_1 - \Delta\,\mathrm{d}V_2 = (\cdots)\,\mathrm{d}t + \sigma\left(\frac{\partial V_1}{\partial S} - \Delta\frac{\partial V_2}{\partial S}\right)\mathrm{d}Z \tag{2.9}$$

where $(\cdots)$ contains the drift term. Therefore, Π turns out to be risk-free for the following choice:

$$\Delta = \left(\frac{\partial V_1/\partial S}{\partial V_2/\partial S}\right) \tag{2.10}$$

Moreover, for this choice of Δ, the variation of the risk-free portfolio is given by:

$$\begin{aligned}\mathrm{d}\Pi = \Bigg[&\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2\frac{\partial^2 V_1}{\partial S^2} + g\frac{\partial V_1}{\partial I} + \sum_{i=\mathrm{d,w}}\mu_i(A_i^1 - V_1)\\ &- \Delta\left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2}\sigma^2\frac{\partial^2 V_2}{\partial S^2} + g\frac{\partial V_2}{\partial I} + \sum_{i=\mathrm{d,w}}\mu_i(A_i^2 - V_2)\right)\Bigg]\mathrm{d}t\end{aligned} \tag{2.11}$$

By using the arbitrage-free assumption, this variation is also given by $\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$. So, we obtain the identity:

$$\begin{aligned}&\left(\frac{\partial V_1}{\partial S}\right)^{-1}\left(rV_1 - \frac{\partial V_1}{\partial t} - g\frac{\partial V_1}{\partial I} - \tfrac{1}{2}\sigma^2\frac{\partial^2 V_1}{\partial S^2} - \sum_{i=\mathrm{d,w}}\mu_i(A_i^1 - V_1)\right)\\ &\quad = \left(\frac{\partial V_2}{\partial S}\right)^{-1}\left(rV_2 - \frac{\partial V_2}{\partial t} - g\frac{\partial V_2}{\partial I} - \tfrac{1}{2}\sigma^2\frac{\partial^2 V_2}{\partial S^2} - \sum_{i=\mathrm{d,w}}\mu_i(A_i^2 - V_2)\right)\end{aligned} \tag{2.12}$$

Note that (2.12) holds for any considered pair of pension plans. Then we can introduce the quantity:

$$\beta(t,S,I;x) = \left(\frac{\partial V}{\partial S}\right)^{-1}\left(rV - \frac{\partial V}{\partial t} - g\frac{\partial V}{\partial I} - \tfrac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - \sum_{i=\mathrm{d,w}}\mu_i(A_i - V)\right) \tag{2.13}$$

So, by reordering the terms in (2.13) we obtain the following PDE that governs the value of the benefits of the pension plan:

$$\frac{\partial V}{\partial t} + \beta\frac{\partial V}{\partial S} + g\frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - (\mu_{\mathrm{d}} + \mu_{\mathrm{w}} + r)V = -\mu_{\mathrm{d}}A_{\mathrm{d}} - \mu_{\mathrm{w}}A_{\mathrm{w}} \tag{2.14}$$

which is initially posed on the unbounded domain $(0, T_r) \times \Omega$, with $\Omega = (0, +\infty) \times (0, +\infty)$.

Assuming that, at retirement date T_r, the owner of the pension plan receives a fraction of the average salary during the last n_y years, Equation (2.14) is completed with the final condition:

$$V(T_r, S, I; x) = \frac{a}{n_y} I \tag{2.15}$$

where $a \in (0, 1)$ is a given constant. For simplicity, in the following sections we drop the dependence on the entry age x in all functions (in particular, $V = V(t, S, I)$).

REMARK 2.1 Note that function β can be related to the market price of the risk associated with uncertainty about the salary. More precisely, if we introduce:

$$\lambda = \frac{\alpha - \beta}{\sigma} \tag{2.16}$$

then Equation (2.8) can be equivalently written as:

$$\mathrm{d}V = \left(rV + \lambda\sigma \frac{\partial V}{\partial S} \right) \mathrm{d}t + \sigma \frac{\partial V}{\partial S} \,\mathrm{d}Z \tag{2.17}$$

so that:

$$\mathrm{d}V - rV \,\mathrm{d}t = \sigma \frac{\partial V}{\partial S} (\lambda \,\mathrm{d}t + \mathrm{d}Z) \tag{2.18}$$

Therefore, the extra reward of the pension plan is $\lambda \,\mathrm{d}t$ per assumed risk unit. Then λ can be understood as the market price of the risk associated with uncertainty about the salary.

3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

So far, the mathematical model for the value of the pension plan has been posed as a Cauchy problem associated with the backward-in-time equation (2.14) jointly with the final condition (2.15).

In this section we study the existence and uniqueness of solutions. In what follows we assume that σ and β are proportional to the salary, so that $\sigma(t, S) = \sigma S$ and $\beta(t, S, I) = \theta S$, where $\theta > 0$ and $\sigma > 0$ are given constants. The assumption on σ implies that salary volatility increases with salary, which is a reasonable argument, mainly in the private sector. Additionally, the joint assumption on the expressions of both σ and β implies that the market price of risk is a constant parameter when a lognormal evolution for salaries is considered (ie, $\alpha(t, S; x) = \alpha S$). Moreover, these assumptions allow for the use of the techniques developed in this section to obtain the existence and uniqueness of solutions by appropriately transforming the PDE. Nevertheless, the numerical methods described in the next section could be applied without the need for these simplifying assumptions.

Thus, let us consider the following Kolmogorov operator:

$$\mathcal{L}[V] = \frac{\partial V}{\partial t} + \theta S \frac{\partial V}{\partial S} + g(t,S)\frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r + \mu_{\mathrm{d}} + \mu_{\mathrm{w}})V \tag{3.1}$$

where V denotes a function defined on the domain $(0, T_r) \times \Omega$. Thus, let us consider the Cauchy problem:

$$\left.\begin{aligned} \mathcal{L}[V] &= f && \text{for } (t,S,I) \in (0,T_r) \times \Omega \\ V(T_r, S, I) &= \frac{a}{n_y} I && \text{for } (S,I) \in \Omega \end{aligned}\right\} \tag{3.2}$$

where $f = -(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})S$.

In order to study the existence of solutions we introduce the following change of variables and unknown:

$$y_1 = S, \qquad y_2 = \tfrac{1}{2}\sigma^2 I, \qquad \tau = \tfrac{1}{2}\sigma^2 (T_r - t)$$
$$\phi(\tau, y_1, y_2) = y_1^m \exp(q\tau) V\left(T_r - \frac{2\tau}{\sigma^2}, y_1, \frac{2y_2}{\sigma^2}\right)$$

and the following parameters:

$$m = \frac{\theta}{\sigma^2}, \qquad q = m^2 - m + \frac{2(r + \mu_{\mathrm{d}} + \mu_{\mathrm{w}})}{\sigma^2}, \qquad T = \tfrac{1}{2}\sigma^2 (T_r - 0)$$

After the previous changes, the Cauchy problem (3.2) can be written in terms of the new unknown as:

$$\left.\begin{aligned} \mathcal{L}_1[\phi] &= F && \text{for } (\tau, y_1, y_2) \in (0,T) \times \Omega \\ \phi(0,\cdot) &= \Lambda && \text{for } (y_1, y_2) \in \Omega \end{aligned}\right\} \tag{3.3}$$

where operator $\mathcal{L}_1$ is defined as:

$$\mathcal{L}_1[\phi] = \frac{\partial \phi}{\partial \tau} - y_1^2 \frac{\partial^2 \phi}{\partial y_1^2} - \bar{g}(\tau, y_1)\frac{\partial \phi}{\partial y_2} \tag{3.4}$$

for any function ϕ defined in $(0, T) \times \Omega$, with:

$$\bar{g}(\tau, y_1) = \begin{cases} k_1 y_1 & \text{if } \tau \leqslant \tfrac{1}{2}\sigma^2 n_y \\ 0 & \text{if } \tau > \tfrac{1}{2}\sigma^2 n_y \end{cases} \tag{3.5}$$

The second member function in (3.3) is given by:

$$F(\tau, y_1, y_2) = \frac{2}{\sigma^2}(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}}) \exp(q\tau) y_1^{m+1} \tag{3.6}$$

and the initial condition is:

$$\Lambda(y_1, y_2) = \frac{2a}{\sigma^2 n_y} y_1^m y_2 \tag{3.7}$$

Next, we analyze the existence of solutions for (3.3). As indicated in Barucci *et al* (2001), the presence of the coefficient y_1^2 in the second-order term leads to a kind of degeneracy in the equation that cannot be easily avoided by the usual logarithmic transformation in classical linear Kolmogorov equations. If the change $y_1 = \log(S)$ is applied instead of $y_1 = S$, then the exponential term $k_1 \exp(y_1)$ will appear in the expression (3.5) of $\bar{g}$ for $\tau \leqslant \frac{1}{2}\sigma^2 n_y$ and the degeneracy problem is shifted to the first-order term coefficient. Nevertheless, in Marcozzi (2003), for the case of Asian options, the logarithmic change is applied and the consequent existence results are obtained from the results appearing in Gencev (1963). We follow the ideas of Barucci *et al* (2001), thereby avoiding the use of the logarithmic change in variable S.

In order to state the existence of solutions, for $p \geqslant 1$ we introduce the following functional space related to the solution of problem (3.3):

$$\mathcal{S}^p_{\text{loc}}(\Omega) = \{u \in L^p_{\text{loc}}(\Omega) / \partial_\tau - y_1 \partial^2_{y_1^2} - \bar{g}\partial_{y_2} \in L^p_{\text{loc}}(\Omega)\} \tag{3.8}$$

and, for $\alpha \in (0, 1)$, we consider the Hölder spaces $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$, $\mathcal{C}^{1,\alpha}_{\mathcal{L}_1}(\Omega)$ and $\mathcal{C}^{2,\alpha}_{\mathcal{L}_1}(\Omega)$ defined by the norms (see, for example, Di Francesco *et al* (2008)):

$$\|u\|_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} = \sup_\Omega |u| + \sup_{z,y \in \Omega, z \neq y} \frac{|u(z) - u(y)|}{\|z^{-1} \circ y\|^\alpha} \tag{3.9}$$

$$\|u\|_{\mathcal{C}^{1,\alpha}_{\mathcal{L}_1}(\Omega)} = \|\partial_{y_1} u\|_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} \tag{3.10}$$

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\mathcal{L}_1}(\Omega)} = \|\partial^2_{y_1^2} u\|_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} + \|\partial_{y_2} u - \partial_\tau u\|_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} \tag{3.11}$$

Note that the notation $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$ associates the definition to the particular form of the operator $\mathcal{L}_1$ and that $u \in \mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$ implies that u is Hölder continuous in the usual sense. Moreover, some embedding theorems recalled in Di Francesco *et al* (2008) state the relationships between spaces $\mathcal{S}^p_{\text{loc}}(\Omega)$ and $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$.

In order to state the existence of solutions, we first present a result concerning the existence of solutions for boundary-value problems associated with second-order operators with nonnegative characteristic form (Oleinik and Radkevic (1973)). More precisely, in Manfredini (1997), the following result is stated for the problem:

$$\left.\begin{aligned} \mathcal{L}_1[u] &= G \quad \text{in } Q \\ u &= H \quad \text{on } \partial Q \end{aligned}\right\} \tag{3.12}$$

where Q is an open bounded set in R^3.

THEOREM 3.1 *Let Q be an open bounded set of R^3 such that $\bar{Q} \subset \{y_1 \neq 0\}$, and let $G \in \mathcal{C}(Q)$, $H \in \mathcal{C}(\partial Q)$. Then the problem (3.12) has a classical solution $u \in \mathcal{C}^{2,\alpha}(Q) \cap \mathcal{C}(\bar{Q})$.*

Note that, although the hypothesis $\bar{Q} \subset \{y_1 \neq 0\}$ is not introduced in Manfredini (1997), it is assumed anyway as we cannot guarantee that the coefficient associated with the second-order term is bounded from below by some positive constant (see Barucci *et al* (2001)). Moreover, although only the homogeneous case $G = 0$ is addressed in Manfredini (1997), the extension to $G \in \mathcal{C}(\partial Q)$ is straightforward. In fact, a nonhomogeneous case is stated in Polidoro and Di Francesco (2006, Theorem 4.2). Next, as in Barucci *et al* (2001), we recall a sufficient condition for the regularity of the points at the boundary (see Manfredini (1997, Theorems 6.1 and 6.3)) to be used in problem (3.12).

PROPOSITION 3.2 *Let Ω be an open set of R^3 such that $\bar{\Omega} \subset \{y_1 \neq 0\}$, and let $(t^0, y_1^0, y_2^0) \in \partial\Omega$. Let us consider the problem (3.3). If there exists an outer normal vector $\nu = (\nu_t, \nu_{y_1}, \nu_{y_2})$ such that one of*

(1) $\nu_{y_1} \neq 0$ *or*

(2) $\nu_{y_1} = 0$, *but* $y_1^0 \nu_{y_2} - \nu_t > 0$ *and there exists a positive constant* δ *such that* $(y_1^0)^2 \delta^2 \leqslant y_1^0 \nu_{y_2} - \nu_t$ *and that:*

$$\{(t, y_1, y_2) \in R^3 / (t - t_0 - \delta^2 \nu_t)^2 + \delta^2 (y_1 - y_1^0)^2 \\ + (y_2 - y_2^0 - \delta^2 \nu_{y_2})^2 \leqslant \delta^4\} \subset R^3 - \Omega$$

is satisfied, then (t^0, y_1^0, y_2^0) is a regular point.

Next, we introduce the concepts of supersolutions and subsolutions associated with problem (3.3), which is posed on the unbounded domain $(0, T) \times \Omega$.

DEFINITION 3.3 A supersolution:

$$\bar{\phi} \in \mathcal{S}^p_{\mathrm{loc}}((0, T) \times \Omega) \cap \mathcal{C}((0, T) \times R^2)$$

of problem (3.3) is a function satisfying:

$$\left.\begin{aligned} \mathcal{L}_1[\bar{\phi}] \geqslant F \quad & \text{for } (\tau, y_1, y_2) \in (0, T) \times \Omega \\ \bar{\phi}(0, \cdot) \geqslant \Lambda \quad & \text{for } (y_1, y_2) \in \Omega \end{aligned}\right\} \tag{3.13}$$

Moreover, a subsolution $\underline{\phi}$ to problem (3.3) is defined simply by considering the reverse inequalities in (3.13).

In the following proposition, we shall obtain a supersolution and a subsolution to problem (3.3).

PROPOSITION 3.4 *For $\alpha_1 \geqslant 3$ and $\alpha_2 \geqslant 1$, let:*

$$\bar{\phi}(\tau, y_1, y_2) = \gamma y_1^m y_2 \exp(\tilde{q}\tau) + \frac{2}{\sigma^2}(\mu_\mathrm{d}\alpha_\mathrm{d} + \mu_\mathrm{w}\alpha_\mathrm{w}) y_1^{m+1} \exp((q + \tilde{q})\tau) + \gamma k_1 y_1^{m+1} \exp(\tilde{q}\tau) \tag{3.14}$$

where:

$$q = m^2 - m + \frac{2(r + \mu_\mathrm{d} + \mu_\mathrm{w})}{\sigma^2}, \qquad \tilde{q} = m^2 + (\alpha_1 - 1)m + \alpha_2, \qquad \gamma = \frac{2a}{\sigma^2 n_y}$$

Then, the function $\bar{\phi}$ is a supersolution to problem (3.3). Moreover, the function $\underline{\phi} = 0$ is a subsolution to problem (3.3).

The proof of Proposition 3.4 is included in Appendix A. Next, we adapt the theorem appearing in Barucci *et al* (2001) to our case in order to state the following theorem.

THEOREM 3.5 *Let $\Lambda \in \mathcal{C}(\Omega)$ and $F \in \mathcal{C}((0, T) \times \Omega)$. Let $\bar{\phi}$ and $\underline{\phi}$ be a supersolution and a subsolution of problem (3.3), respectively, such that $\underline{\phi} \leqslant \bar{\phi}$ in $(0, T) \times \Omega$. Then there exists a classical solution ϕ to problem (3.3), such that:*

$$\underline{\phi} \leqslant \phi \leqslant \bar{\phi} \quad \text{in } (0, T) \times \Omega \tag{3.15}$$

The proof of Theorem 3.5 is detailed in Appendix A.

THEOREM 3.6 *There exists a classical solution ϕ to problem (3.3). Moreover, the solution verifies:*

$$0 \leqslant \phi \leqslant \bar{\phi} = \gamma y_1^m y_2 \exp(\tilde{q}\tau) + \frac{2}{\sigma^2}(\mu_\mathrm{d}\alpha_\mathrm{d} + \mu_\mathrm{w}\alpha_\mathrm{w}) y_1^{m+1} \exp((q + \tilde{q})\tau) + \gamma k_1 y_1^{m+1} \exp(\tilde{q}\tau)$$

with:

$$q = m^2 - m + \frac{2(r + \mu_\mathrm{d} + \mu_\mathrm{w})}{\sigma^2}, \qquad \tilde{q} = m^2 + 2m + 1, \qquad \gamma = \frac{2a}{\sigma^2 n_y}$$

The proof of Theorem 3.6 follows directly from Theorem 3.5 and Proposition 3.4 by taking $\alpha_1 = 3$ and $\alpha_2 = 1$.

REMARK 3.7 Note that the choice of supersolutions and subsolutions implies that any solution satisfies $\phi(\tau, 0, y_2) = 0$.

In order to study the uniqueness of solutions, we directly apply the following result taken from Aronson and Besala (1967) to the particular case of problem (3.3). This result has already been used in Gleit (1978) for Black–Scholes equations associated with one-factor models.

THEOREM 3.8 *Let $\Omega \subset R^n$ be an arbitrary unbounded open domain and let L be the differential operator:*

$$L = \sum_{i,j=1}^{n} a_{ij}(\tau, y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n} b_i(t, y)\partial_{y_i} + c - \partial_\tau$$

defined in $(0, T) \times R^n$, such that the coefficients of L satisfy:

$$\sum_{i,j=1}^{n} a_{ij}(\tau, y)\xi_i\xi_j \leqslant 0 \quad \textit{for all } (\tau, y) \in [0, T] \times \Omega,\ \xi \in R^n$$

and:

$$|a_{ij}| \leqslant A(|y|^2 + 1), \qquad |b_i| \leqslant B(|y|^2 + 1)^{1/2}, \qquad |c| \leqslant C$$

in $[0, T] \times \Omega$ for some positive constants A, B and C. If u is a classical solution of $Lu \leqslant 0$ in $[0, T] \times \Omega$, such that:

$$u \geqslant 0 \quad \textit{for } (\tau, y) \in \{[0, T] \times \partial\Omega\} \cup \{\{0\} \times \partial\Omega\}$$

and:

$$u(\tau, y) \leqslant -M \exp\{k \log(|y|^2 + 1) + 1\}^2 \tag{3.16}$$

in $[0, T] \times \Omega$, for some positive constant M and k, then $u \geqslant 0$ in $[0, T] \times \Omega$.

The following result follows directly from the previous theorems and existence results.

COROLLARY 3.9 *There exists a unique classical solution of problem (3.3) such that (3.16) is satisfied.*

4 NUMERICAL METHODS

In this section we introduce the numerical method for solving problem (3.2). First, we point out some difficulties in the numerical solution. On the one hand, the spatial domain Ω is unbounded. Due to this fact, as in the localization technique used in the previous section, domain truncation and boundary conditions are proposed. Note that the particular localization procedure used in the previous section is not practical for numerical purposes. On the other hand, the diffusion matrix is strongly degenerated. So, we propose a combination of the Crank–Nicolson characteristics method for the time discretization and piecewise quadratic finite element method for the spatial discretization on the bounded domain. In the literature, we can find different applications of the classical first-order method of characteristics for the solution of financial problems (see, for example, D'Halluin *et al* (2005) and Vázquez (1998)). Higher-order Crank–Nicolson characteristics methods for general convection–diffusion–reaction equations (eventually degenerated) were recently proposed and analyzed numerically in Bermúdez *et al* (2006a,b). Furthermore, they have been successfully applied to Asian options pricing in Bermúdez *et al* (2006c). In this section we apply the method to the pension plan pricing problem. Although this problem does not satisfy all of the hypotheses required in Bermúdez *et al* (2006c) to obtain a second-order Lagrange–Galerkin scheme, in practice, good numerical results are obtained. It is important to note that, due to the specific expression of the PDE, in the Lagrangian step the characteristic curves associated with the convection term in the equation can be computed exactly, thereby avoiding the use of appropriate ordinary differential equation solvers to approximate the position of the basis point of the characteristics.

4.1 Localization procedure and formulation in a bounded domain

First, we consider a problem posed in a sufficiently large spatial bounded domain, so that the solution in the region of financial interest is not affected by the truncation of the unbounded domain and the required boundary conditions (localization procedure). This procedure was analyzed in Kangro and Nicolaides (2000) for vanilla options and Dirichlet boundary conditions. For this purpose, let us introduce the notation:

$$x_0 = t, \qquad x_1 = S, \qquad x_2 = I \tag{4.1}$$

let us consider both x_1^∞ and x_2^∞ as sufficiently large real numbers suitably chosen, and let:

$$\Omega^* = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$$

with $x_0^\infty = T_r$. Then, let us denote the Lipschitz boundary by $\Gamma^* = \partial\Omega^*$ such that $\Gamma^* = \bigcup_{i=0}^{2}(\Gamma_i^{*,-} \cup \Gamma_i^{*,+})$, where we use the notation:

$$\Gamma_i^{*,-} = \{(x_0, x_1, x_2) \in \Gamma^* \mid x_i = 0\}$$
$$\Gamma_i^{*,+} = \{(x_0, x_1, x_2) \in \Gamma^* \mid x_i = x_i^\infty\}$$

for $i = 0, 1, 2$.

Then, the PDE in problem (3.2) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 V}{\partial x_i x_j} + \sum_{j=0}^{2} b_j \frac{\partial V}{\partial x_j} + b_0 V = f_0 \tag{4.2}$$

where the data involved is defined as follows:

$$B(x_0, x_1, x_2) = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{b}(x_0, x_1, x_2) = (b_j) = \begin{pmatrix} 1 \\ \theta x_1 \\ g(t, x_1) \end{pmatrix} \tag{4.3}$$

$$b_0(x_0, x_1, x_2) = -(r + \mu_{\mathrm{d}} + \mu_{\mathrm{w}}), \qquad f_0(x_0, x_1, x_2) = -(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})x_1 \tag{4.4}$$

Thus, following Oleinik and Radkevic (1973), in terms of the normal vector to the boundary pointing inward Ω^*, $\boldsymbol{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ^*:

$$\Sigma^0 = \left\{ x \in \Gamma^* \Big/ \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\} \tag{4.5}$$

$$\Sigma^1 = \Gamma^* - \Sigma^0 \tag{4.6}$$

$$\Sigma^2 = \left\{ x \in \Sigma_0 \Big/ \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \partial b_{ij} / \partial x_j \right) m_i < 0 \right\} \tag{4.7}$$

As indicated in Oleinik and Radkevic (1973), the boundary conditions at $\Sigma^1 \cup \Sigma^2$ for the so-called first boundary-valued problem associated with (4.2) are required. Note that $\Sigma^1 = \Gamma_1^{*,+}$ and $\Sigma^2 = \Gamma_0^{*,+} \cup \Gamma_2^{*,+}$. Therefore, in addition to the final condition in (3.2) on $\Gamma_0^{*,+}$, we propose the following conditions:

$$\frac{\partial V}{\partial x_1} = 0 \qquad \text{on } \Gamma_1^{*,+} \tag{4.8}$$

$$\frac{\partial V}{\partial x_2} = \frac{a}{n_y} \quad \text{on } \Gamma_2^{*,+} \tag{4.9}$$

At this point, the question of the existence of the solution for the previous problem in the localized domain and the convergence to the solution in the unbounded domain arises. In general, existence is gained for Dirichlet boundary conditions obtained from the initial condition (value at the time horizon in the original financial variables formulation). In Kangro and Nicolaides (2000), a particular study with Dirichlet conditions for the European multiasset vanilla option was developed. In Jaillet *et al* (1990), the existence, uniqueness and convergence for multiasset American options was addressed. Moreover, in Barles *et al* (1995), the use of Dirichlet and Neumann boundary conditions deduced from the payoff function in the framework of viscosity solutions was analyzed. Taking into account the analogies with European-style Asian options, we follow the ideas of Kemma and Vorst (1990) and Bermúdez *et al* (2006c), so we impose Neumann boundary conditions obtained from the payoff. In this way, we avoid imposing that the solution matches the initial condition at the new boundaries of the bounded domain. We note that the convergence analysis to the solution in the unbounded domain is an open problem. In view of the numerical results appearing in a forthcoming section of this paper, numerical evidence of convergence is obtained.

Next, in order to state the problem (3.2) as an equivalent initial-boundary-value problem in divergence form, we distinguish the time-and-space-bounded domains and introduce the following change-of-time variable and the notation for spatial-like variables:

$$\tau = T_r - t, \qquad x_1 = S, \qquad x_2 = I \tag{4.10}$$

in addition to the notation related to the spatial domain:

$$\Omega = (0, x_1^{\infty}) \times (0, x_2^{\infty}), \qquad \Gamma = \bigcup_{i=1}^{2} (\Gamma_{i,-} \cup \Gamma_{i,+})$$

with:

$$\Gamma_{i,-} = \{(x_1, x_2) \in \Gamma / x_i = 0\}, \qquad \Gamma_{i,+} = \{(x_1, x_2) \in \Gamma / x_i = x_i^{\infty}\}$$

for $i = 1, 2$.

Problem (3.2) is then replaced by the following.

Find $V\colon [0, T_r] \times \Omega \to \mathbb{R}$, such that:

$$\frac{\partial V}{\partial \tau} + \boldsymbol{v}\nabla V - \operatorname{div}(A\nabla V) + lV = f \quad \text{in } (0, T_r) \times \Omega \tag{4.11}$$

$$V(0, \cdot) = \varphi \quad \text{in } \Omega \tag{4.12}$$

$$\frac{\partial V}{\partial x_1} = g_1 \quad \text{on } (0, T_r) \times \Gamma_{1,+} \tag{4.13}$$

$$\frac{\partial V}{\partial x_2} = g_2 \quad \text{on } (0, T_r) \times \Gamma_{2,+} \tag{4.14}$$

where the data involved is defined as follows:

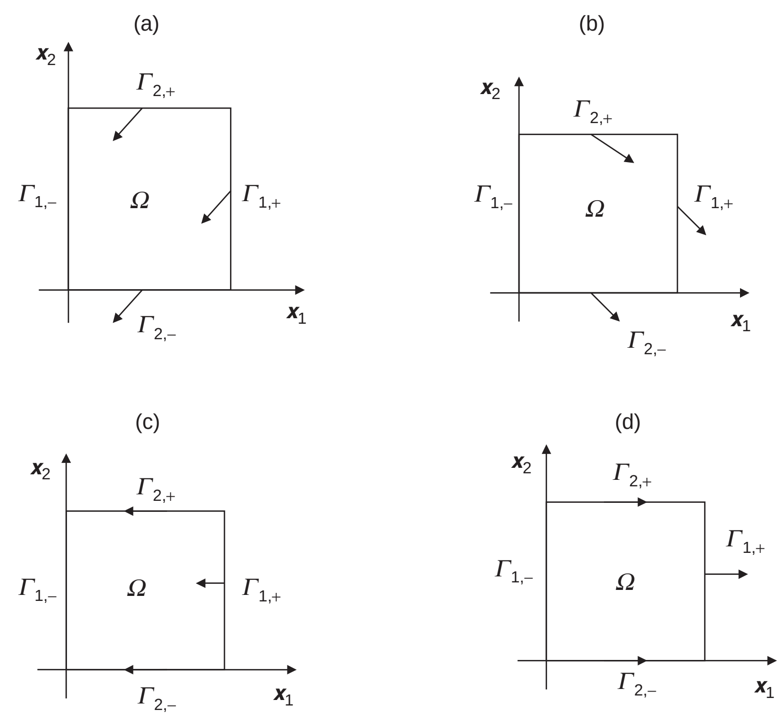
$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{v}(x_1, x_2) = \begin{pmatrix} (\sigma^2 - \theta)x_1 \\ -g(T_r - \tau, x_1) \end{pmatrix} \tag{4.15}$$

$$l(\tau, x_1, x_2) = r + \mu_{\mathrm{d}} + \mu_{\mathrm{w}}, \qquad f(\tau, x_1, x_2) = (\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})x_1 \tag{4.16}$$

$$\varphi(x_1, x_2) = \frac{a}{n_y} x_2, \qquad g_1(\tau, x_1, x_2) = 0, \qquad g_2(\tau, x_1, x_2) = \frac{a}{n_y} \tag{4.17}$$

Figure 1 on the next page shows the qualitative behavior of the velocity field, $\boldsymbol{v}$, at the boundaries for different parameter cases and times. Note that the velocity field

FIGURE 1 Bounded domain and velocity field, $\boldsymbol{v}$, at the boundaries.



(a) $\theta > \sigma^2$ for $\tau \leqslant n_y$. (b) $\theta \leqslant \sigma^2$ for $\tau \leqslant n_y$. (c) $\theta > \sigma^2$ for $\tau > n_y$. (d) $\theta \leqslant \sigma^2$ for $\tau > n_y$.

either enters the domain or vanishes at $\Gamma_{2,+}$, while it is either tangential or it points outward the domain at $\Gamma_{2,-}$. Also, both the diffusion matrix and the velocity field vanish at $\Gamma_{1,-}$. The previously discussed requirements of boundary conditions are closely related to inflow boundaries. Also note that the velocity field is not continuous with respect to the time variable.

4.2 Time discretization

The method of characteristics is used for the time discretization and it is included in the more general setting of upwinding methods, which take into account the local direction of the flux. More precisely, it is based on a finite difference scheme for the discretization of the material derivative, ie, the time derivative along the characteristic lines of the convective part of the equation. In this section we will also introduce the variational formulation for the time-discretized problem.

First, we define the characteristics curve through $\boldsymbol{x} = (x_1, x_2)$ at time $\bar{\tau}$, $X_e(\boldsymbol{x}, \bar{\tau}; s)$, which verifies:

$$\frac{\partial}{\partial s} X_e(\boldsymbol{x}, \bar{\tau}; s) = \boldsymbol{v}(X_e(\boldsymbol{x}, \bar{\tau}; s)), \quad X_e(\boldsymbol{x}, \bar{\tau}; \bar{\tau}) = \boldsymbol{x} \tag{4.18}$$

For $N > 1$, let us consider the time step $\Delta\tau = T_r/N$ and the time mesh points $\tau_n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, N$. The material derivative approximation by the characteristics method is given by:

$$\frac{DV}{D\tau} = \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau}$$

where $X_e^n(\boldsymbol{x}) := X_e(\boldsymbol{x}, \tau^{n+1}; \tau^n)$. In view of the expression of the velocity field and the continuous function g given by expressions (4.15) and (2.6), respectively, the components of $X_e^n(\boldsymbol{x})$ can be analytically computed. More precisely, we distinguish the following two main cases.

- If $\theta \neq \sigma^2$, then $[X_e^n]_1(\boldsymbol{x}) = x_1 \exp((\theta - \sigma^2)\Delta\tau)$ and:

$$[X_e^n]_2(\boldsymbol{x}) = \begin{cases} x_2 & \text{if } n\Delta\tau > n_y \\ \dfrac{k_1 x_1}{\sigma^2 - \theta}(1 - \exp((\theta - \sigma^2)\Delta\tau)) + x_2 & \text{if } n\Delta\tau \leqslant n_y \end{cases}$$

- If $\theta = \sigma^2$, then $[X_e^n]_1(\boldsymbol{x}) = x_1$, and

$$[X_e^n]_2(\boldsymbol{x}) = \begin{cases} x_2 & \text{if } n\Delta\tau > n_y \\ k_1 x_1 \Delta\tau + x_2 & \text{if } n\Delta\tau \leqslant n_y \end{cases}$$

Next, we consider a Crank–Nicolson scheme around $(X_e(\boldsymbol{x}, \tau_{n+1}; \tau), \tau)$ for $\tau = \tau_{n+1/2}$. So, for $n = 0, \dots, N-1$, the time-discretized equation can be written as follows. Find V^{n+1} such that:

$$\begin{aligned} \frac{V^{n+1}(\boldsymbol{x}) - V^n(X_e^n(\boldsymbol{x}))}{\Delta\tau} &- \tfrac{1}{2}\operatorname{div}(A\nabla V^{n+1})(\boldsymbol{x}) \\ - \tfrac{1}{2}\operatorname{div}(A\nabla V^n)(X_e^n(\boldsymbol{x})) &+ \tfrac{1}{2}(lV^{n+1})(\boldsymbol{x}) + \tfrac{1}{2}(lV^n)(X_e^n(\boldsymbol{x})) \\ &= \tfrac{1}{2} f^{n+1}(\boldsymbol{x}) + \tfrac{1}{2} f^n(X_e^n(\boldsymbol{x})) \end{aligned} \tag{4.19}$$

In order to obtain the variational formulation of the semidiscretized problem, we multiply Equation (4.19) by a suitable test function, integrate in Ω, use the classical

Green formula and the following (see Bermúdez *et al* (2006c, Lemma 3.4)):

$$\begin{aligned}\int_{\Omega} \operatorname{div}(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\Gamma} (\boldsymbol{F}_e^n)^{-T}(\boldsymbol{x})\boldsymbol{n}(x)\cdot(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_{\Omega} (\boldsymbol{F}_e^n)^{-1}(\boldsymbol{x})(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\cdot\nabla\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_{\Omega} \operatorname{div}((\boldsymbol{F}_e^n)^{-t}(\boldsymbol{x}))(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{4.20}$$

Note that, in the present case, we have:

$$\int_{\Omega} \operatorname{div}((\boldsymbol{F}_e^n)^{-t}(\boldsymbol{x}))(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

After these steps, we can write a variational formulation for the semidiscretized problem as follows.

Find $V^{n+1} \in H^1(\Omega)$ such that, for all $\Psi \in H^1(\Omega)$:

$$\begin{aligned}&\int_{\Omega} V^{n+1}(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \tfrac{1}{2}\Delta\tau \int_{\Omega} (\boldsymbol{A}\nabla V^{n+1})(\boldsymbol{x})\nabla\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad + \tfrac{1}{2}\Delta\tau \int_{\Omega} l V^{n+1}(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad = \int_{\Omega} V^n(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \tfrac{1}{2}\Delta\tau \int_{\Omega} (\boldsymbol{F}_e^n)^{-1}(\boldsymbol{x})(\boldsymbol{A}\nabla V^n)(X_e^n(\boldsymbol{x}))\nabla\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad - \tfrac{1}{2}\Delta\tau \int_{\Omega} l V^n(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \tfrac{1}{2}\Delta\tau \int_{\Gamma} \tilde{g}^n(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}A_{\boldsymbol{x}} \\ &\qquad + \tfrac{1}{2}\Delta\tau \int_{\Gamma_{1,+}} \bar{g}_1{}^{n+1}(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}A_{\boldsymbol{x}} + \tfrac{1}{2}\Delta\tau \int_{\Omega} f^{n+1}(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \tfrac{1}{2}\Delta\tau \int_{\Omega} f^n(X_e^n(\boldsymbol{x}))\Psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{4.21}$$

where $\boldsymbol{F}_e^n = \nabla X_e^n$ can be analytically computed, $\bar{g}_1^{n+1}(\boldsymbol{x}) = g_1^{n+1}(\boldsymbol{x})a_{11}(\boldsymbol{x}) = 0$ and:

$$\tilde{g}^n(\boldsymbol{x}) := \begin{cases} 0 & \text{on } \Gamma_{1,-} \\ -[(\boldsymbol{F}_e^n)^{-T}]_{12}(\boldsymbol{x})a_{11}(X_e^n(\boldsymbol{x}))\dfrac{\partial V}{\partial x_1}(X_e^n(\boldsymbol{x})) & \text{on } \Gamma_{2,-} \\ [(\boldsymbol{F}_e^n)^{-T}]_{12}(\boldsymbol{x})a_{11}(X_e^n(\boldsymbol{x}))\dfrac{\partial V}{\partial x_1}(X_e^n(\boldsymbol{x})) & \text{on } \Gamma_{2,+} \\ [(\boldsymbol{F}_e^n)^{-T}]_{11}(\boldsymbol{x})a_{11}(X_e^n(\boldsymbol{x}))g_1^n(X_e^n(\boldsymbol{x})) & \text{on } \Gamma_{1,+} \end{cases} \tag{4.22}$$

REMARK 4.1 Note that once the characteristics method for time discretization has been applied, only boundary condition (4.13), which corresponds to the boundary with a nonvanishing diffusive term, is used to obtain the variational formulation. Note that condition (4.14) is mainly motivated by the velocity field entering the domain. Nevertheless, at each time step, the upwinded total derivative term is part of the second member of the discretized equation.

REMARK 4.2 As a result of the application of the characteristics method, we need to evaluate functions at the points $X_e^n(\boldsymbol{x})$, which are obtained by upwinding in the trajectories of the velocity field. Some technical interpolation skills are required when these points are placed outside the domain, the idea being to use the information at the boundaries. For the points that enter the domain through $\Gamma_{1,+}$, we use boundary condition (4.13), while, for those entering through $\Gamma_{2,+}$, we use (4.14).

4.3 Finite element discretization

As mentioned at the beginning of the section, we use the Crank–Nicolson characteristics method for the time discretization jointly with finite elements for spatial discretization. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less than or equal to 2 in each spatial variable and where Σ_T is the subset of nodes of the element T. More precisely, let us define the finite element space V_h as:

$$V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) \colon \phi_{h_T} \in \mathcal{Q}_2,\ \forall T \in \tau_h\} \tag{4.23}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

For a more general equation, under suitable assumptions on the data, the method is analyzed in Bermúdez *et al* (2006b) and proved to be unconditionally stable in the case of exact integration of the integral terms. Also, for academic cases of constant coefficients convection–diffusion and pure convection equations, a study of the different quadrature formulas used to compute the integral terms is carried out. Thus, stability can be proved for the case of trapezoidal or Simpson formulas for the pure convection equation in one spatial dimension. In the presence of an additional diffusive term, the stability region is smaller for lower Peclet numbers, so these formulas turn out to be convenient for convection-dominated problems. These results can be extended to higher spatial dimensions when products of one-dimensional finite element spaces and quadrature formulas are considered. Note that the piecewise quadratic finite elements over quadrangular meshes are a particular case of product finite element spaces. Noting that they do not correspond to the analyzed academic cases, in all presented examples in this paper we use a Simpson quadrature formula to approximate all the integral terms appearing in the fully discretized problem.

TABLE 1 Errors for different meshes and numbers of time steps (NT) at time $\tau = 40$, when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.25$, $\mu_{\mathrm{d}} = 0.025$ and $\mu_{\mathrm{w}} = 0$ are considered.

NT	Mesh 12	Mesh 24	Mesh 48	Mesh 96
100	4.695867×10^{-2}	4.675261×10^{-2}	4.671738×10^{-2}	4.676544×10^{-2}
1 000	6.088001×10^{-3}	5.517705×10^{-3}	5.423675×10^{-3}	5.420425×10^{-3}
10 000	2.592423×10^{-3}	6.473885×10^{-4}	5.812531×10^{-4}	5.483266×10^{-4}
100 000	1.575346×10^{-3}	3.958598×10^{-4}	1.112673×10^{-4}	5.526974×10^{-5}

TABLE 2 Errors for different meshes and numbers of time steps (NT) at time $\tau = 40$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_{\mathrm{d}} = 0.025$ and $\mu_{\mathrm{w}} = 0.2$ are considered.

NT	Mesh 12	Mesh 24	Mesh 48	Mesh 96
100	7.172828×10^{-2}	7.159433×10^{-2}	7.153603×10^{-2}	7.138368×10^{-2}
1 000	1.116242×10^{-2}	1.068611×10^{-2}	1.065671×10^{-2}	1.065913×10^{-2}
10 000	2.230513×10^{-3}	1.191936×10^{-3}	1.115101×10^{-3}	1.080565×10^{-3}
100 000	1.547804×10^{-3}	4.176562×10^{-4}	1.117329×10^{-4}	1.020686×10^{-4}

5 NUMERICAL EXAMPLES

First, we consider an academic test with known analytical solution. More precisely, the appropriate data is imposed, so that the solution is given by:

$$V^e(\tau, \boldsymbol{x}) = \exp(\tau x_1 x_2 \times 10^{-4}), \quad (\tau, \boldsymbol{x}) \in (0, 40) \times \Omega$$

with $\Omega = (0, 40) \times (0, 40)$, for the choice:

$$f(\tau, \boldsymbol{x}) = \begin{cases} \exp(\tau x_1 x_2 \times 10^{-4})(p(\tau, \boldsymbol{x}) - k_1 x_1^2 \tau \times 10^{-4}) & \text{if } \tau \leqslant n_y \\ \exp(\tau x_1 x_2 \times 10^{-4}) p(\tau, \boldsymbol{x}) & \text{if } \tau > n_y \end{cases} \tag{5.1}$$

with:

$$\begin{aligned} p(\tau, \boldsymbol{x}) = x_1 x_2 \times 10^{-4} - \sigma^2 \tau x_1 x_2 \times 10^{-4} \\ - 0.5 \times 10^{-8} \sigma^2 \tau^2 x_1^2 x_2^2 + (\sigma^2 - \theta) \tau x_1 x_2 \times 10^{-4} + l \end{aligned}$$

Initial and boundary condition data in (4.12)–(4.14) are provided by the exact solution.

The computed $l^\infty((0, 40); l^2(\Omega))$ errors for different meshes and numbers of time steps for two different sets of parameters are shown in Table 1 and Table 2. The only

TABLE 3 Finite element method mesh data.

	Number of elements	Number of nodes
Mesh 12	144	625
Mesh 24	576	2 401
Mesh 48	2 304	9 409
Mesh 96	9 216	37 249

TABLE 4 Retirement benefits at mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered.

NT	Mesh 12	Mesh 24	Mesh 48	Mesh 96
100	2.808332	2.808633	2.808697	2.808645
1 000	2.804451	2.804589	2.804606	2.804609
10 000	2.804144	2.804174	2.804182	2.804184
100 000	2.804103	2.804141	2.804141	2.804141

differences in the data are the value of parameters μ_w ($\mu_w = 0$ in Table 1 on the facing page and $\mu_w = 0.2$ in Table 2 on the facing page) and k_1 ($k_1 = 0.25$ in Table 1 on the facing page and $k_1 = 0.5$ in Table 2 on the facing page). The number of nodes and elements of the referred quadratic finite element meshes are shown in Table 3. The qualitative and quantitative results for both sets of parameters are very close. For a sufficiently fine fixed mesh in space, a first-order convergence in time is clearly observed. If the mesh in space is not sufficiently fine, the first-order convergence appears until the spatial error dominates the total error. For a fixed, sufficiently fine mesh in time, a second-order convergence in space is illustrated. Also, if the time mesh is not sufficiently fine, then the second-order convergence appears until the time error dominates the total error. For the academic cases analyzed in Bermúdez *et al* (2006b), a second-order convergence is obtained in both space and time. We note that the academic cases in Bermúdez *et al* (2006b) correspond to constant coefficient equations and certain assumptions on the velocity field. In fact, all theoretical results stated in Bermúdez *et al* (2006a,b) assume that the velocity field is continuous with respect to the time variable, which is not the case in the present example.

After the previous academic test, the results for the first real data set are shown in Table 4. More precisely, at the point $(S, I) = (25, 20)$, the values for different meshes to illustrate the convergence are indicated.

TABLE 5 Retirement benefits value for different domains when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_{\rm d} = 0.025$, $\mu_{\rm w} = 0.2$, $\alpha_{\rm d} = 1$ and $\alpha_{\rm w} = 0$ are considered.

	$(S, I) = (1.2, 15)$	$(S, I) = (2.4, 30)$	$(S, I) = (4.8, 30)$
$\Omega = (0, 40) \times (0, 40)$	0.133451	0.270426	0.546684
$\Omega = (0, 80) \times (0, 80)$	0.133371	0.266911	0.535248
$\Omega = (0, 160) \times (0, 160)$	0.133376	0.266756	0.533692
$\Omega = (0, 320) \times (0, 320)$	0.133375	0.266751	0.533469

It is also important to illustrate the effect of truncation and introduction of boundary conditions in the obtained values in the financially interesting region. For this purpose, we consider that S represents salaries in thousands of unit currencies (for example, $S = 1$ corresponds to €1,000 or US$1,000). Moreover, we note that average salary is actually given by:

$$\bar{S} = \frac{1}{n_y} \int_{T_r - n_y}^{T_r} S(\tau)\,\mathrm{d}\tau = \frac{I}{k_1 n_y} \tag{5.2}$$

With this in mind, in Table 5 we represent the obtained pension plan values for the salaries $S = 1.2, 2.4, 4.8$ (for example, $S = 1.2$ corresponds to 1200 currency units) and the average salaries $\bar{S} = 1, 2$ (corresponding to the values $I = 15$ and 30, respectively). Note the small influence of the location of the boundaries of the truncated domain in the obtained value.

Table 6 on the facing page shows the behavior of the pension plan value at time $t = 0$ in terms of the different involved parameters in the model and for different (S, I) coordinates. First note that, taking expression (5.2) into account, the number of years affects the actual average salary. This explains the choice of coordinate I in the case of $n_y = 15$, in order to maintain the same value of $\bar{S}$ as in the case of $n_y = 30$. First, note that increasing volatility leads to a small increase in benefit of pension plan value. The same occurs with increasing value of a. As expected, an increase in risk-free interest rates leads to lower benefit values. The number of years has almost no influence on the obtained values, as we are maintaining the value of the resulting average salary. Table 7 on page 134 shows the behavior of the pension plan value at time $t = 38$. In order to compare the behavior at time $t = 0$ and $t = 38$, we present Figure 2 on page 135 and Figure 3 on page 135. Note the influence of two factors (salary and average salary) in the second case, while, in the first case, the value of the average salary has a very small influence on the pension plan value. This is because the time $t = 0$ is before the initial date ($T_r - n_y = 10$) which is used to compute the average salary that enters in the payoff function.

TABLE 6 Retirement benefits value at time $t = 0$ for different (S, I) points and parameter values.

(a) $n_y = 30$

r	a	$S = 1.2$, $I = 15$	$S = 1.2$, $I = 22.5$	$S = 2.4$, $I = 30$
$\sigma = 0.1$				
0.025	0.75	0.133451	0.133598	0.270426
0.025	0.95	0.133488	0.133665	0.271463
0.075	0.75	0.109117	0.109176	0.220036
0.075	0.95	0.109124	0.109185	0.220478
$\sigma = 0.2$				
0.025	0.75	0.133636	0.133838	0.270412
0.025	0.95	0.133736	0.134004	0.271634
0.075	0.75	0.109187	0.109265	0.219801
0.075	0.95	0.109217	0.109319	0.220308

(b) $n_y = 15$

r	a	$S = 1.2$, $I = 7.5$	$S = 1.2$, $I = 11.25$	$S = 2.4$, $I = 15$
$\sigma = 0.1$				
0.025	0.75	0.133384	0.133394	0.266849
0.025	0.95	0.133402	0.133415	0.266907
0.075	0.75	0.109098	0.109114	0.218547
0.075	0.95	0.109102	0.109143	0.218623
$\sigma = 0.2$				
0.025	0.75	0.133404	0.133419	0.266844
0.025	0.95	0.133431	0.133448	0.266924
0.075	0.75	0.109103	0.109105	0.218208
0.075	0.95	0.109107	0.109111	0.218223

6 CONCLUSIONS

In this paper the use of a dynamic hedging methodology provides a PDE model associated with a Kolmogorov equation that governs the value of liabilities of a defined benefit pension plan depending on the average salary without early retirement capability. This methodology allows the price of this liability to be interpreted using the framework of option pricing theory. Once the PDE model has been posed, the existence of a solution can be obtained by using analogous tools to those previously used

TABLE 7 Retirement benefits value at time $t = 38$ for different (S, I) points and parameter values.

(a) $n_y = 30$

r	a	$S = 1.2$, $I = 15$	$S = 1.2$, $I = 22.5$	$S = 2.4$, $I = 30$
$\sigma = 0.1$				
0.025	0.75	0.29442368	0.40814817	0.58884736
0.025	0.95	0.36005233	0.50410336	0.72010465
0.075	0.75	0.26883814	0.37174032	0.53767628
0.075	0.95	0.32822141	0.45856416	0.65644281
$\sigma = 0.2$				
0.025	0.75	0.29442379	0.40814829	0.58884759
0.025	0.95	0.36005247	0.50410351	0.72010495
0.075	0.75	0.26883824	0.37174042	0.53767649
0.075	0.95	0.32822153	0.45856429	0.65644307

(b) $n_y = 15$

r	a	$S = 1.2$, $I = 7.5$	$S = 1.2$, $I = 11.25$	$S = 2.4$, $I = 15$
$\sigma = 0.1$				
0.025	0.75	0.31308212	0.42680661	0.62616424
0.025	0.95	0.38368635	0.52773738	0.76737271
0.075	0.75	0.28572099	0.38862318	0.57144199
0.075	0.95	0.34960635	0.47994911	0.69921269
$\sigma = 0.2$				
0.025	0.75	0.31308234	0.42680684	0.62616465
0.025	0.95	0.38368664	0.52773766	0.76737322
0.075	0.75	0.28572121	0.38862338	0.57144237
0.075	0.95	0.34960661	0.47994937	0.69921317

in the literature for arithmetic Asian options pricing models. Also, the uniqueness of solutions can be obtained. Moreover, an appropriate numerical method for solving the model is proposed so that the numerical results can be discussed in terms of the different model parameters.

The theory and numerical methods can be applied to more general conditions. In particular, a more general expression for the benefits at retirement can be easily

FIGURE 2 Retirement benefits at time $t = 0$ when the parameters $T_r = 40$, $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered.

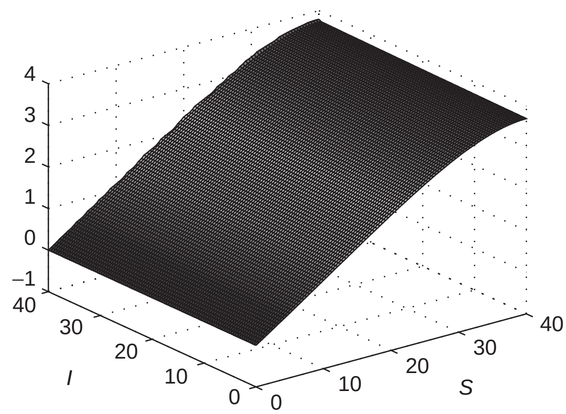
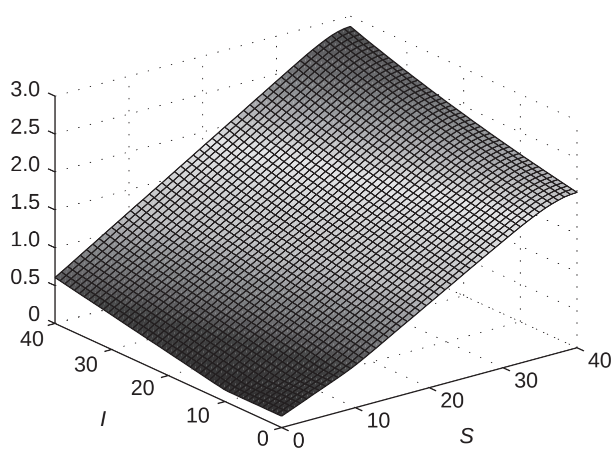


FIGURE 3 Retirement benefits at time $t = 38$ when the parameters $T_r = 40$, $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered.



addressed. For example, a pension plan that guarantees a minimum amount of money or that pays at retirement the maximum among a fixed quantity, a proportion of the final salary and another proportion of the final average salary.

Future work concerning the possibility of early retirement is being considered by the authors. In this case, linear complementarity formulations of the resulting free boundary problem can be analyzed to obtain the existence of solutions, and suitable numerical methods are required to obtain not only the pension plan value but also the regions in which it is optimal to retire before the retirement date, as well as at the optimal retirement boundary. On the other hand, the possibility of including jumps in the salary could also be considered, leading to partial integro-differential equations that require an appropriate treatment of the additional nonlocal integral terms.

APPENDIX A

In this appendix we mainly include the proofs of Proposition 3.4 and Theorem 3.5.

PROOF OF PROPOSITION 3.4 Clearly, $\underline{\phi} = 0$ is a subsolution to problem (3.3).

In order to state the supersolution properties, first we note that, for $\tau = 0$, we obtain:

$$\begin{aligned}\bar{\phi}(0, y_1, y_2) &= \gamma y_1^m y_2 + (\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})\frac{2}{\sigma^2} y_1^{m+1} + \gamma k_1 y_1^{m+1} \\ &\geqslant \gamma y_1^m y_2 \end{aligned} \tag{A.1}$$

So, as:

$$(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})\frac{2}{\sigma^2} y_1^{m+1} \geqslant 0, \qquad \gamma k_1 y_1^{m+1} \geqslant 0$$

the second inequality in (3.13) is satisfied.

Next, in order to verify the first inequality, we calculate:

$$\begin{aligned}\mathcal{L}_1[\bar{\phi}] &= \tilde{q}\gamma y_1^m y_2 \exp(\tilde{q}\tau) + \frac{2}{\sigma^2}(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}}) y_1^{m+1}(q + \tilde{q}) \exp((q + \tilde{q})\tau) \\ &\quad + \tilde{q} k_1 y_1^{m+1} \gamma \exp(\tilde{q}\tau) - \gamma m(m-1) y_1^m y_2 \exp(\tilde{q}\tau) \\ &\quad - \frac{2}{\sigma^2}(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}}) m(m+1) y_1^{m+1} \exp((q + \tilde{q})\tau) \\ &\quad - k_1 m(m+1)\gamma y_1^{m+1} \exp(\tilde{q}\tau) - \bar{g}(\tau, y_1)\gamma y_1^m \exp(\tilde{q}\tau) \end{aligned} \tag{A.2}$$

Then, after some easy computations, we obtain:

$$\begin{aligned}\mathcal{L}_1[\bar{\phi}] &= (\tilde{q} - m(m-1))\gamma y_1^m y_2 \exp(\tilde{q}\tau) \\ &\quad + \frac{2}{\sigma^2}(\mu_{\mathrm{d}}\alpha_{\mathrm{d}} + \mu_{\mathrm{w}}\alpha_{\mathrm{w}})(q + \tilde{q} - m(m+1)) y_1^{m+1} \exp((q + \tilde{q})\tau) \\ &\quad + [\tilde{q} y_1 k_1 - k_1 y_1 m(m+1) - \bar{g}(\tau, y_1)]\gamma y_1^m \exp(\tilde{q}\tau) \end{aligned} \tag{A.3}$$

Now, note first that $\alpha_1 \geqslant 3$ and $\alpha_2 \geqslant 1$ implies that:

$$\tilde{q} - m(m-1) = m^2 + (\alpha_1 - 1)m + \alpha_2 - m^2 + m \geqslant 0$$

Secondly, the conditions on α_1 and α_2 jointly with inequalities $m > 0$ and $2(r + \mu_\mathrm{d} + \mu_\mathrm{w})/\sigma^2 > 0$ imply that:

$$m^2 + (\alpha_1 - 1)m + \alpha_2 \geqslant 0$$
$$m^2 + (\alpha_1 - 3)m + \frac{2(r + \mu_\mathrm{d} + \mu_\mathrm{w})}{\sigma^2} + \alpha_2 \geqslant 1$$

Thus, we have $q + \tilde{q} - m(m+1) \geqslant 1$ and $\tilde{q} \geqslant 0$. On the other hand, as $(\alpha_1 - 2)m + \alpha_2 \geqslant 1$, then $\tilde{q} - m(m+1) - 1 \geqslant 0$, and therefore:

$$\tilde{q} y_1 k_1 - k_1 y_1 m(m+1) - k_1 y_1 \geqslant 0$$

Next, by using that the product of k_1 and y_1 is an upper bound for $\bar{g}(\tau, y_1)$, we have:

$$\tilde{q} y_1 k_1 - k_1 y_1 m(m+1) - \bar{g}(\tau, y_1) \geqslant \tilde{q} y_1 k_1 - k_1 y_1 m(m+1) - k_1 y_1 \geqslant 0$$

Therefore, the inequalities:

$$\tilde{q} - m(m-1) \geqslant 0, \quad q + \tilde{q} - m(m+1) \geqslant 1, \quad \tilde{q} y_1 k_1 - k_1 y_1 m(m+1) - \bar{g}(\tau, y_1) \geqslant 0$$

hold, so the first inequality in (3.13) is satisfied and the proof is concluded. □

Proof of Theorem 3.5 First, we define a sequence of initial-boundary-value problems posed on bounded open subsets of $(0, T) \times \Omega$, Ω_k, such that $\Omega_k \subset \Omega_{k+1}$ and $\cup \Omega_k = (0, T) \times \Omega$. More precisely, we consider the sequence of bounded sets:

$$\Omega_k = (0, T) \times \left(\frac{1}{k+1}, k+1\right) \times \left(\frac{1}{k+1}, k+1\right) \quad \text{with } k \in N$$

and we consider the sequence of cutoff functions $\chi_k \colon (0, +\infty) \times (0, +\infty) \to R$, $\chi_k \in \mathcal{C}(\Omega)$, such that:

$$\chi_k(y_1, y_2) = \begin{cases} 0 & \text{if } (y_1, y_2) \notin \left(\dfrac{1}{k+1}, k+1\right) \times \left(\dfrac{1}{k+1}, k+1\right) \\ 1 & \text{if } (y_1, y_2) \in \left(\dfrac{1}{k}, k\right) \times \left(\dfrac{1}{k}, k\right) \end{cases} \tag{A.4}$$

and $0 \leqslant \chi_k(y_1, y_2) \leqslant 1$ otherwise. In terms of these functions, we define:

$$\underline{\Lambda}_k(\tau, y_1, y_2) = \chi_k(y_1, y_2)\Lambda(y_1, y_2) + (1 - \chi_k(y_1, y_2))\underline{\phi}(\tau, y_1, y_2) \tag{A.5}$$

By Theorem 3.1, there exists a classical solution $u_k \in \mathcal{C}^{2,\alpha}(\Omega_k) \cap \mathcal{C}(\bar{\Omega}_k)$ to problem:

$$\left.\begin{aligned} \mathcal{L}_1[u_k] = F & \quad \text{in } \Omega_k \\ u_k = \underline{\Lambda}_k & \quad \text{on } \partial\Omega_k \end{aligned}\right\} \tag{A.6}$$

Moreover, from the maximum principle, it follows that:

$$\underline{\phi} \leqslant u_k \leqslant \bar{\phi} \quad \text{in } \Omega_k \tag{A.7}$$

Note that functions u_k are not defined outside Ω_k. Next, we use the same arguments as Barucci *et al* (2001) to build a solution for problem (3.3). For this purpose, let us introduce the sets:

$$D_k = \left\{ (t, y_1, y_2) \in \Omega_k \;\middle|\; \frac{T}{3k} < t < \frac{T}{1 - 1/(3k)} \right\} \tag{A.8}$$

so that:

$$(0, T) \times \Omega = \bigcup_{k \in N} D_k$$

Note that, from (A.7), it follows that the sequence $\{u_k\}$ is bounded in $\bar{D}_1$ and, from Barucci *et al* (2001, Proposition 4.1), it is also equicontinuous. Thus, following the Ascoli–Arzela theorem, there exists a subsequence that converges uniformly to some function $v_1 \in \mathcal{C}(\bar{D}_1)$. Moreover, v_1 is a classical solution of (3.3) in D_1 and $\underline{\phi} \leqslant v_1 \leqslant \bar{\phi}$ in D_1. Next, we can apply the same argument to the previous subsequence on the set $\bar{D}_2$, thereby obtaining a new subsequence that converges to $v_2 \in \mathcal{C}(\bar{D}_2)$, so that v_2 is the solution of (3.3) in D_2, verifies that $\underline{\phi} \leqslant v_2 \leqslant \bar{\phi}$ in D_2 and v_2 coincides with v_1 in D_1. The argument can continue by induction and we can define a limit function u as follows. For $(t, y_1, y_2) \in (0, T) \times \Omega$, we choose a natural number n such that $(t, y_1, y_2) \in D_n$ and define $u(t, y_1, y_2) = v_n(t, y_1, y_2)$. In this way, u is well-defined and verifies Equation (3.3) and $\underline{\phi} \leqslant u \leqslant \bar{\phi}$ in $(0, T) \times \Omega$.

It now remains to prove that the function u verifies the boundary condition at $\tau = 0$. For this purpose, we verify that, for any $(y_1^0, y_2^0) \in R^+ \times R^+$, we have:

$$\lim_{(t, y_1, y_2) \to (0, y_1^0, y_2^0)} u(\tau, y_1, y_2) = \Lambda(y_1^0, y_2^0) \tag{A.9}$$

Note that the fact that $u_k(0, y_1^0, y_2^0) = \Lambda(y_1^0, y_2^0)$ for $(0, y_1^0, y_2^0) \in \partial\Omega_k$ does not guarantee the same result for function u at the boundary. Nevertheless, the use of a standard argument of barrier functions in the proof of Proposition 3.2 provides an estimate of the rate of convergence as (t, y_1, y_2) tends to $(0, y_1^0, y_2^0)$, which is uniform with respect to k, allowing us to obtain (A.9). □

Remark A.1 Note that, by construction, $u_{k+1} \geqslant \underline{\Lambda}_k$ so that $\{u_k\}$ is an increasing sequence. On the other hand, if we define:

$$\bar{\Lambda}_k(\tau, y_1, y_2) = \chi_k(y_1, y_2)\Lambda(y_1, y_2) + (1 - \chi_k(y_1, y_2))\bar{\phi}(\tau, y_1, y_2) \tag{A.10}$$

and we consider it as the boundary condition for problem (A.6). We then obtain a decreasing sequence of solutions $\{v_k\}$ converging, uniformly on compact sets, to a solution v of the problem (3.3) that verifies $\underline{\phi} \leqslant v \leqslant \bar{\phi}$. In the case where uniqueness can be proved, both solutions u and v coincide.

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