Jump-diffusion models with two stochastic factors for pricing swing options in electricity markets with partial-integro differential equations [1]

M. C. Calvo-Garrido[a], M. Ehrhardt[b], C. Vázquez[a,2]

[a] Department of Mathematics and CITIC, University of A Coruña, Campus Elviña s/n, 15071 A Coruña, Spain
[b] Lehrstuhl für Angewandte Mathematik und Numerische Analysis, Fakultät 4-Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gauss-Strasse 20, 42119 Wuppertal, Germany

Abstract

In this paper we consider the valuation of swing options with the possibility of incorporating spikes in the underlying electricity price. This kind of contracts are modelled as path dependent options with multiple exercise rights. From the mathematical point of view the valuation of these products is posed as a sequence of free boundary problems where two consecutive exercise rights are separated by a time period. Due to the presence of jumps, the complementarity problems are associated with a partial-integro differential operator. In order to solve the pricing problem, we propose appropriate numerical methods based on a Crank-Nicolson semi-Lagrangian method for the time discretization of the differential part of the operator, jointly with the explicit treatment of the integral term by using the Adams-Bashforth scheme and combined with biquadratic Lagrange finite elements for space discretization. In addition, we use an augmented Lagrangian active set method to cope with the early exercise feature. Moreover, we employ appropriate artificial boundary conditions to treat the unbounded domain numerically. Finally, we present some numerical results in order to illustrate the proper behaviour of the numerical schemes.

Keywords: Swing options; electricity price; jump-diffusion models; Augmented Lagrangian Active Set(ALAS) formulation; semi-Lagrangian method; biquadratic Lagrange finite elements; artificial boundary conditions.

1 Introduction

Swing contracts are a kind of path-dependent option that allows the holder to exercise a right multiple times over a period, with the constraint that the two consecutive exercise dates must

[1] This work has been partially funded by MINECO of Spain (Project MTM2016-76497-R), Xunta de Galicia grants GRC2014/044 and ED431C 2018/33, including FEDER funds, and Bilateral German-Spanish Project HiPeCa: High Performance Calibration and Computation in Finance, Programme Acciones Conjuntas Hispano-Alemanas funded by German DAAD and the Fundación Universidad

[2] Corresponding author: carlosv@udc.es

Email addresses: mcalvog@udc.es (M.C. Calvo-Garrido), ehrhardt@math.uni-wuppertal.de (M. Ehrhardt), carlosv@udc.es (C. Vázquez)

be separated by a refracting period. That is, two rights can not be exercised at the same time. One example of this right could be to receive the payoff of a call option. Nevertheless, there are other possibilities, such as the consideration of different payoff functions depending on the spot price, such as calls and puts, or calls with different strikes. In this work we focus on the valuation of swing options where the underlying is the electricity price. In order to model some properties of the electricity prices observed in the markets, such as spikes, it is necessary to adopt jump-diffusion models. According to [17], these sudden changes in the prices appear when the maximum supply is approached by the current demand. Concerning risk management and pricing, it is very important to treat properly these features of the electricity prices. The consideration of jumps in the electricity prices is the main innovative point of the present paper. In the literature, there are several examples of the valuation of financial derivatives when the underlying assets follow a jump-diffusion process (see [30], for example). Among them, in this work we assume that the spike sizes follow an exponential distribution [17], or the spikes are governed by Merton [26] and Kou [23] jump-diffusion models. The consideration of jump-diffusion models leads to a partial-integro differential equation (PIDE) problem.

In the absence of jumps, different numerical techniques have been employed to deal with the one-factor swing option valuation problem. Binomial trees are considered in [19], while Monte Carlo simulation techniques are employed in [25] and [33]. As indicated in [7], the numerical resolution of the PDE approach with finite elements or finite differences has been carried out in [11], [34] and [35]. In the literature, also Fourier-based methods have been employed to treat this valuation problem in [36].

Taking into account the possibility of jumps in the electricity price, the two-factor swing option valuation problem is solved by using binomial trees in [17]. When jump-diffusion processes are considered, a partial-integro differential equation arises. Thus, in [21] the one-factor PIDE is solved by using a finite-difference scheme combined with a dynamic programming technique, while in [27] an implicit-explicit finite difference scheme is proposed. The finite difference method jointly with the approximation of the integral term by means of a recursion formula is also implemented in [5] for the one factor problem under Kou jump-diffusion model.

In the present paper, we propose the numerical solution of the PIDE model that governs the valuation of swing contracts depending on two stochastic factor with the possibility of incorporating spikes. Our previous article [7] was based on a model without jumps. Now, as we take jumps into account, we have to switch from A PDE to a PIDE approach, which is more realistic. Concerning the numerical methods for solving PIDE problems arising in finance, in [13] the authors propose a semi-Lagrangian method for pricing American Asian options assuming jump-diffusion models for the underlying asset while in [9] an implicit finite difference method to obtain the value of options on two assets under jump-diffusion processes is considered. Moreover, in order to avoid the solution of a linear system with a dense matrix, they combine a fixed point iteration with a FFT technique. In [1], the authors solve the PIDE to obtain the value of European vanilla options under Merton and Kou jump-diffusion models for the underlying. Recently, in [2] a model with one stochastic factor without jumps is considered in a probabilistic approach to the problem (so that only one spatial dimension would arise in the corresponding PDE approach, without integral term). Also, in [2] the holder has to exercise all the rights (unlike in our approach). In this different setting, the authors obtain the involved free boundaries (two in this case) as solution of integral equations, that they solve numerically and they can obtain qualitative properties for the free boundaries. We understand that this analysis is much more difficult in our case with two stochastic factors and remains an open problem.

In the presence of jumps, when the integral term is treated implicitly, the numerical schemes employed to solve the PIDE lead to a dense matrix, being necessary to employ appropriate methods to solve the system as the one proposed in [31]. In contrast when treating explicitly the integral term either by using a time discretization scheme which involves the solution of the previous time step [10] or of the two previous time steps [32] the same matrix as in the absence of jumps is maintained. In our previous work [6] we have chosen the first approach, whereas in this work we implement the second one, treating explicitly the integral term by taking into account the solution in the two previous time steps.

In order to solve the complementarity problems that arise in the valuation of products of American type, there exists different methods proposed in the literature. For example, in [12] a penalty method is proposed and in [18] an operator splitting method is presented.

In this paper we propose a Lagrange-Galerkin method for time and space discretization [3, 4], combined with an Augmented Lagrangian Active Set (ALAS) algorithm [20], which is more efficient than the classical PSOR, or alternative duality or penalization methods (see [7], for more detail). Moreover, we implement the explicit treatment of the integral term proposed in [32] (i.e. the Adams-Bashforth scheme).

This paper is organized as follows. In Section 2, we describe the stochastic model for the evolution of the electricity price under jump diffusion-processes we consider, and we state the PIDE problem that governs the valuation of swing options on this underlying. In Section 3, we formulate the swing option pricing problem with jumps in a bounded domain after a localization procedure. Next, we introduce the discretization in time of the problem, using a Crank-Nicolson characteristic scheme for the differential part of the operator jointly with the explicit treatment of the integral term, and we state the variational formulation of the discretized problem in order to apply finite elements. Finally, in Section 4, we present some numerical results and in Section 5 we finish with some conclusions.

2 The mathematical model under jump-diffusion processes

2.1 The electricity spot price model incorporating spikes

In our previous work [7], we considered a model without jumps to describe the dynamics of the electricity price. Nevertheless, in this paper we take into account the possibility of spikes in the prices, which is more realistic to represent the behaviour of the electricity prices in the markets. Thus, in this section we introduce the model to describe the stochastic evolution of the electricity spot price in the presence of jumps. More precisely, under the risk neutral probability measure, the spot electricity price, S_t, is assumed to be the continuous time process

$$S_t = \exp(f(t) + \bar{X}_t + Y_t), \tag{1}$$

where f is a deterministic periodic function that represents the seasonality and accounts for regular changes in the prices evolution, $\bar{X}_t$ denotes the Ornstein-Uhlenbeck (OU) process with zero mean-reversion level and mean-reversion speed, $\alpha > 0$. Thus, the following stochastic differential equation (SDE) is satisfied by the process $\bar{X}_t$:

$$d\bar{X}_t \;=\; -\alpha\bar{X}_t dt \;+\; \sigma dW_t\,, \tag{2}$$

where σ denotes the volatility of the process and W_t represents a standard Brownian motion. In order to incorporate spikes, for the third component, Y_t, we consider the following SDE:

$$dY_t \;=\; -(\beta Y_t + \lambda\kappa)dt + J_t dZ_t\,, \tag{3}$$

where β is the mean-reversion speed, J_t denotes the jump size distribution, Z_t is a compensated Poisson process of intensity λ and $\kappa = \mathbb{E}[\exp(J_t) - 1]$. Note that the mean-reversion rate of the jump process, Y_t, can be different to the one of the process $\bar{X}_t$ and even higher, thus allowing us to model spikes mainly in markets with low mean-reversion speed. For convenience, we write the seasonal function f as a time-dependent mean reversion level of the process $\bar{X}_t$ and we introduce $X_t = \bar{X}_t + f(t)$. For convenience, we consider the following processes

$$M_t = \exp(X_t), \quad N_t = \exp(Y_t),$$

so that $S_t = M_t N_t$ and the following SDEs are satisfied

$$\begin{aligned} dM_t &= \alpha\left(\mu(t) - \ln(M_t)\right) M_t dt \;+\; \sigma M_t dW_t\,, \\ dN_t &= -(\beta\ln(N_t) + \lambda\kappa)N_t dt + \left(\exp(J_t) - 1\right) N_t dZ_t\,, \end{aligned} \tag{4}$$

with

$$\mu(t) = f(t) + \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + f'(t)\right).$$

In the absence of jumps, there exist other alternative models, either with one or two stochastic factors, to describe the spot electricity price evolution as the ones presented in [24].

2.2 The PIDE formulation

In the case without jumps, by using a dynamic hedging technique in [7], a partial differential equation (PDE) model for pricing any asset depending on the stochastic factors M_t and N_t (the dynamics of which are described without jumps in [7]) is posed. In the here considered jump-diffusion models, if we denote by $V_t = V(t, M_t, N_t)$ the price of any asset whose value is a function of time t and the stochastic factors M_t and N_t (the dynamics of which are described by equations (4)) then standard techniques based on Ito formulas for jump-diffusion process prove that the function V satisfies the following PIDE (see [10], for example):

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 V}{\partial M^2} + \alpha\left(\mu(t) - \ln(M)\right) M \frac{\partial V}{\partial M} - \beta\ln(N) N \frac{\partial V}{\partial N} - rV \\ &+ \int_{-\infty}^{\infty} \lambda\left[V\left(t, M, N\exp(z)\right) - V(t, M, N) - N\left(\exp(z) - 1\right)\frac{\partial V(t, M, N)}{\partial N}\right]\nu(z)dz = 0. \end{aligned} \tag{5}$$

Moreover, in order to complete the model, we must also specify the distribution of the jump sizes. For this purpose, we will consider either the Merton model [26] or the Kou model [23]. More precisely, under Merton model J follows the normal distribution ($N(\mu_j, \gamma_j^2)$), with the density function

$$\nu(z) = \nu_m(z) = \frac{1}{\gamma_j\sqrt{2\pi}} \exp\left(-\frac{(z-\mu_j)^2}{2\gamma_j^2}\right), \tag{6}$$

where μ_j is the mean jump size and γ_j is the standard deviation of the jump size, whereas under Kou model J corresponds to a distribution with double-exponential density function

$$\nu(z) = \nu_k(z) = \begin{cases} q_k\alpha_2 \exp(\alpha_2 z), & z < 0 \\ p_k\alpha_1 \exp(-\alpha_1 z), & z \geq 0, \end{cases} \tag{7}$$

where p_k, q_k, α_1 and α_2 are positive constants such that $p_k + q_k = 1$ and $\alpha_1 > 1$. Note that, p_k and q_k represent the probabilities of upward and downward jumps, respectively.

Since $\nu(z)$ is the probability density function of the jump amplitude J, then

$$\int_{-\infty}^{\infty} \nu(z)dz = 1.$$

Moreover, we can compute the expectations for the Merton and Kou models

$$\mathbb{E}_m[\exp(J)] = \int_{-\infty}^{\infty} \exp(z)\nu_m(z)dz = e^{\mu_j+\gamma_j^2/2},$$

$$\mathbb{E}_k[\exp(J)] = \int_{-\infty}^{\infty} \exp(z)\nu_k(z)dz = \frac{p_k\alpha_1}{\alpha_1 - 1} + \frac{q_k\alpha_2}{\alpha_2 + 1}.$$

Therefore, the PIDE (5) can be written in the equivalent form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 V}{\partial M^2} + \alpha\left(\mu(t) - \ln(M)\right) M \frac{\partial V}{\partial M} - \left(\beta \ln(N) + \lambda\kappa\right) N \frac{\partial V}{\partial N} - (r+\lambda)V + \lambda \int_{-\infty}^{\infty} V\left(t, M, N\exp(z)\right)\nu(z)dz = 0. \tag{8}$$

where $\kappa = e^{\mu_j+\gamma_j^2/2} - 1$ or $\kappa = \frac{p_k\alpha_1}{\alpha_1-1} + \frac{q_k\alpha_2}{\alpha_2+1} - 1$ for the Merton or Kou models, respectively.

Note that with respect to the PDE model in [7], there is an additional integral term in Equation (8) due to the presence of jumps. This term makes the PIDE more difficult to solve than the corresponding PDE. In Section 3.4 we show how to discretize this integral term in order to find a numerical solution of the PIDE problem.

For the particular case of electricity markets in which the payoff $\phi(T, S)$ is a function depending on the electricity price, S, at maturity T, Equation (8) is supplied with the final condition

$$V(T, M, N) = \phi(T, MN), \tag{9}$$

where the function ϕ denotes the payoff of the contract.

2.3 The swing option pricing problem under jump diffusion models

From a mathematical point of view, swing options in electricity markets can be modelled as financial products with multiple exercises of American type. Moreover, two consecutive exercise dates are separated by a constant refracting period $\delta > 0$. As it is mentioned in [8], the consideration of this refracting period avoids the exercise of all the rights at once, which would be optimal in the absence of this separation time. That is, without the refracting period δ, the swing option pricing problem could be reduced to the valuation of multiple American options.

Let us consider $p \in \mathbb{N}$ exercise rights. If we denote by $\mathcal{T}_{t,T}$ the set of all stopping times with values in $[0, T]$ and by $\mathcal{T}_{t,\infty}$ the set of all stopping times with values greater or equal than t, then we can define the set of admissible stopping time vectors in the following way (see [8] or [35], for example)

$$\mathcal{T}_t^{(p)} = \{\tau^{(p)} = (\tau_1, ..., \tau_p) | \tau_i \in \mathcal{T}_{t,\infty} \text{ with } \tau_1 \leq T \text{ a.s. and } \tau_{i+1} - \tau_i \geq \delta \text{ for } i = 1, ..., p-1\}. \quad (10)$$

Note that at least one exercise right of the swing option with maturity T is exercised, although it is not necessary to exercise all the rights. The investor could let an exercise right expire to get a benefit from better future prices. Thus, not all stopping times of a stopping time vector have their values in the interval $[0, T]$.

In summary, at any time $t \in [0, T]$, the owner of the option has two possibilities: either he/she decides to exercise one right receiving the exercise value, so that after waiting the refracting time, since $t + \delta$ he/she will have a swing option with one exercise right less, or to maintain the option with the current number of exercise rights.

In [35] the risk free price of a swing option depending on one underlying factor without jumps is written as a multiple stopping time problem and it is proved in [8] that it can be translated into a sequence of single stopping time problems. Analogously, in the presence of jumps and when the electricity price depends on two stochastic factors under a risk neutral probability measure Q, the price of a swing option with $p \in \mathbb{N}$ exercise rights at time t is denoted by $V^{(p)}(t, M_t, N_t)$ and is given by

$$V^{(p)}(t, M_t, N_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q \left[e^{-r(\tau - t)} \Phi^{(p)}(\tau, S_\tau) \right], \quad p \geq 1, \quad (11)$$

with $S_t = M_t N_t$ and

$$\Phi^{(p)}(t, S_t) = \begin{cases} \phi(t, S_t) + \mathbb{E}^Q \left[e^{-r\delta} V^{(p-1)}(t + \delta, M_{t+\delta}, N_{t+\delta}) \right] & \text{if } t \leq T - \delta \\ \phi(t, S_t) & \text{if } t > T - \delta. \end{cases} \quad (12)$$

Note that as indicated by expression (12), the exercise value at time t of the swing option with p rights $\Phi^{(p)}(t, S_t)$ depends on the value of the swing option with $p - 1$ exercise rights $V^{(p-1)}$ along the time interval $[t + \delta, T]$, when the remaining $p - 1$ rights could be exercised. Moreover, if $t + \delta > T$ then no additional rights can be exercised due to the refracting period required after the last exercised right.

Therefore, by starting from

$$V^{(0)}(t, M_t, N_t) = 0, \tag{13}$$

we recursively solve the problem for $m = 1, \dots, p$ exercise rights.

Note that for $p = 1$ we recover the classical expression for the price of an American option, which only involves one exercise right.

2.3.1 The free boundary problem under jump-diffusion models

In the presence of spikes in the electricity price, after making the time reversal change of variable, $\tau = T - t$, we introduce the function $u^{(p)}(\tau, M, N) = V^{(p)}(T - \tau, M, N)$ so that it solves the following complementarity problem:

$$\begin{aligned} \mathcal{L}[u^{(p)}] \leq 0 \quad & \text{in } (0,T) \times \mathbb{R}_+^2, \\ u^{(p)} \geq \Psi^{(p)} \quad & \text{in } (0,T) \times \mathbb{R}_+^2, \\ (\mathcal{L}[u^{(p)}])(u^{(p)} - \Psi^{(p)}) = 0 \quad & \text{in } (0,T) \times \mathbb{R}_+^2, \\ u^{(p)}(0,.) = \Psi^{(p)}(0,.) \quad & \quad\quad \text{in } \mathbb{R}_+^2, \end{aligned} \tag{14}$$

where the operator $\mathcal{L}$ is defined by

$$\begin{aligned} \mathcal{L}[F] = -\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 F}{\partial M^2} + \alpha\left(\mu(T-\tau) - \ln(M)\right) M \frac{\partial F}{\partial M} - (\beta \ln(N) + \lambda\kappa) N \frac{\partial F}{\partial N} \\ -(r+\lambda)F + \lambda \int_{-\infty}^{\infty} F(\tau, M, N\exp(z))\nu(z)dz \end{aligned} \tag{15}$$

and the p^{th} reward obstacle function $\Psi^{(p)}$ in (12) is transformed into:

$$\Psi^{(p)}(\tau, S) = \begin{cases} \phi(T-\tau, S) + \mathbb{E}^Q\left[e^{-r\delta} u^{(p-1)}(T-\tau+\delta, M_{T-\tau+\delta}, N_{T-\tau+\delta})\right] & \text{for } \tau \in [\delta, T] \\ \phi(T-\tau, S) & \text{for } \tau \in [0, \delta). \end{cases}$$

From the Feynman-Kac theorem we can write the expectation appearing in (16) in terms of the solution of an appropriate PDE problem. More precisely, we denote

$$w^{\tau,(p-1)}(\delta, M, N) = \mathbb{E}^Q\left[e^{-r\delta} u^{(p-1)}(T-\tau+\delta, M_{T-\tau+\delta}, N_{T-\tau+\delta})\right], \tag{16}$$

so that the dependence on τ is reflected in the superindex of $w^{\tau,(p-1)}$. From Feynman-Kac theorem, the function $w^{\tau,(p-1)}$ satisfies the PDE problem

$$\begin{aligned} \mathcal{L}[w^{\tau,(p-1)}] = 0 \quad & \text{in } (0,\delta) \times \mathbb{R}_+^2, \\ w^{\tau,(p-1)}(0,\cdot) = u^{(p-1)}(\tau-\delta,\cdot) \quad & \quad\quad \text{in } \mathbb{R}_+^2, \end{aligned} \tag{17}$$

where $\mathcal{L}$ is given by (15).

In terms of $w^{\tau,(p-1)}$, the reward function for the swing option with p exercise rights can be written as

$$\Psi^{(p)}(\tau, S) = \begin{cases} \phi(T-\tau, S) + w^{\tau,(p-1)}(\delta, M, N) & \text{for } \tau \in [\delta, T] \\ \phi(T-\tau, S) & \text{for } \tau \in [0, \delta). \end{cases} \tag{18}$$

In order to obtain the value of $w^{\tau,(p-1)}(\delta, M, N)$ for $\tau \in [\delta, T]$ when $p = 1$, we note that

$$w^{\tau,(0)}(t, M, N) = 0 \quad \text{for } (t, M, N) \in [0, \delta] \times \mathbb{R}_+^2,$$

whereas when $p > 1$ we need to solve the corresponding PDE problem (17).

Note that due to the constant refracting period, the reward function (18) can be equivalently written as

$$\Psi^{(p)}(\tau, S) = \begin{cases} \phi(T-\tau, S) + w^{\tau,(p-1)}(\delta, M, N) & \text{for } \tau \geq (p-1)\delta \\ \Psi^{(p-1)}(\tau, S) & \text{for } \tau < (p-1)\delta. \end{cases} \tag{19}$$

That is, in a period of length $(p-1)\delta$ we only can exercise $(p-1)$ rights due to the refracting period. That is why the value of the reward function with p exercise rights is equal to the value with $(p-1)$ rights at any time $\tau < (p-1)\delta$.

As in the simpler and special case of American options ($p = 1$), once the complementarity problem has been solved we obtain not only the price of the swing option for each time but also the region where it is optimal to exercise

$$\Omega^0 = \{(\tau, M, N)/u^{(p)}(\tau, M, N) = \Psi^{(p)}(\tau, MN)\}$$

and the one where it is optimal to maintain the option

$$\Omega^+ = \{(\tau, M, N)/u^{(p)}(\tau, M, N) > \Psi^{(p)}(\tau, MN)\}.$$

Both regions are separated by the so called optimal exercise boundary Σ, which can be defined as

$$\Sigma = \partial\Omega^+ \cap \partial\Omega^0,$$

where ∂C denotes the boundary of the set C. Note that the free boundary Σ depends on τ, M and N, so that it can be represented by a curve in the MN-plane for each value of τ. For simplicity, we use $\Sigma = \Sigma(\tau, M, N)$.

Note that when the regularity of the solution and the free boundary can be proved, as in the case of American options ($p = 1$), then the following *smooth pasting* conditions hold at this optimal exercise boundary:

$$u^{(p)} = \Psi^{(p)} \quad \text{and} \quad \frac{\partial u^{(p)}}{\partial M} = \frac{\partial \Psi^{(p)}}{\partial M} \quad \text{on} \quad \Sigma\,. \tag{20}$$

However, the regularity remains an open problem in the case of swing options for $p > 1$. In fact, the regularity of the solution and the free boundary for the swing option with $p > 1$ exercise rights depends on the regularity of the reward function, which in turn depends on the regularity of the swing option price with $p-1$ exercise rights, thus leading to a recursive dependence in the number of rights.

3 Numerical methods

As in the case without jumps, in order to obtain a numerical approximation of the value of a swing option with $p \in \mathbb{N}$ exercise rights, we need to solve a free boundary problem for each value of p. Additionally, for $p > 1$, in order to obtain the value of the reward function, $\Psi^{(p)}(\tau, S)$, associated with each complementarity problem (14), the solution for certain times of an initial value problem is required. The difference is that the problems are linked to an integro-differential operator. For the numerical solution of the PIDEs (14) and (17), we propose a Crank-Nicolson characteristics time discretization scheme combined with a piecewise quadratic Lagrange finite element method and we treat explicitly in time the integral term, which is discretized with a suitable quadrature formula. For the additional inequality constraints associated with the complementarity problem (14), we propose a mixed formulation and an Augmented Lagrangian Active Set (ALAS) technique.

3.1 Localization procedure and formulation in a bounded domain

In a similar way to the case without spikes, first, we need to approximate the unbounded domain in which the PIDE is posed, by a bounded one and we impose boundary conditions on the boundaries where they are required. Moreover, we localize the domain of integration in the integral term that appears with the presence of jumps in the electricity price. Let us introduce the notation:

$$x_1 = M, \quad x_2 = N \quad \text{and} \quad \bar{x}_2 = \ln(x_2), \tag{21}$$

and let us consider both x_1^∞ and x_2^∞ to be large enough suitably chosen real numbers. Let

$$\Omega = (0, x_1^\infty) \times (0, x_2^\infty).$$

Then, let us denote the Lipschitz boundary by $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{i=1}^2 (\Gamma_i^- \cup \Gamma_i^+)$, where

$$\Gamma_i^- = \{(x_1, x_2) \in \Gamma \mid x_i = 0\}, \quad \Gamma_i^+ = \{(x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\}, \quad i = 1, 2.$$

Following [29] which is based on the theory proposed by Fichera in [15] and as it is said in [7], we need to impose a boundary condition on Γ_1^+. On this boundary we impose the artificial boundary condition (ABC) derived in [7], taking into account the approach of [16]. After the previous change of spatial variables we write the equation (17) in divergence form in the bounded spatial domain $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$. Thus, the IBVP takes the following form:

Find $w^{\tau,(p-1)} : [0, \delta] \times \Omega \to \mathbb{R}$ such that

$$\frac{\partial w^{\tau,(p-1)}}{\partial t} + \vec{v} \cdot \nabla w^{\tau,(p-1)} - Div(A \nabla w^{\tau,(p-1)}) + l w^{\tau,(p-1)}$$

$$-\lambda \int_{z_{min}}^{z_{max}} \bar{w}^{\tau,(p-1)}(\tau, x_1, \bar{x}_2 + z)\nu(z)dz = \tilde{f} \qquad \text{in } (0, \delta) \times \Omega\,, \tag{22}$$

$$w^{\tau,(p-1)}(0, .) = u^{(p-1)}(\tau - \delta, .) \qquad \text{in } \Omega, \tag{23}$$

$$\frac{\partial w^{\tau,(p-1)}}{\partial t} + b\frac{\partial w^{\tau,(p-1)}}{\partial x_1} + c w^{\tau,(p-1)} = 0 \qquad \text{on } (0, \delta) \times \Gamma_1^+\,, \tag{24}$$

where $\bar{w}^{\tau,(p-1)}(\tau, x_1, \bar{x}_2 + z) = w^{\tau,(p-1)}(\tau, x_1, \exp(\bar{x}_2 + z))$, $b = \alpha\,(\ln(x_1^\infty) - \mu)\,x_1^\infty$ and $c = l$. Furthermore, for the complementarity problem associated with the swing option value, denoting by P the Lagrange multiplier, we can pose the mixed formulation:

Find $u^{(p)} : [0,T] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &\frac{\partial u^{(p)}}{\partial \tau} + \vec{v} \cdot \nabla u^{(p)} - Div(A \nabla u^{(p)}) + l u^{(p)} \\ &-\lambda \int_{z_{min}}^{z_{max}} \bar{u}^{(p)}(\tau, x_1, \bar{x}_2 + z)\nu(z)dz + P = \tilde{f} \qquad \text{in } (0,T) \times \Omega\,, \end{aligned} \tag{25}$$

with the complementarity conditions

$$u^{(p)} \geq \Psi^{(p)}, \quad P \leq 0, \quad (u^{(p)} - \Psi^{(p)})P = 0 \quad \text{in } (0,T) \times \Omega \tag{26}$$

where $\bar{u}^{(p)}(\tau, x_1, \bar{x}_2 + z) = u^{(p)}(\tau, x_1, \exp(\bar{x}_2 + z))$ and we consider the initial and boundary conditions

$$u^{(p)}(0,.) = \Psi^{(p)}(0,.) \qquad \text{in } \Omega, \tag{27}$$

$$\frac{\partial u^{(p)}}{\partial \tau} + b\frac{\partial u^{(p)}}{\partial x_1} + c u^{(p)} = 0 \qquad \text{on } (0,T) \times \Gamma_1^+\,. \tag{28}$$

with $b = \alpha\,(\ln(x_1^\infty) - \mu)\,x_1^\infty$ and $c = l$. For both problems, the involved data is defined as follows:

$$A = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \tilde{g}(\tau, x_1) \\ \tilde{h}(x_2) \end{pmatrix}, \quad l = r, \quad \tilde{f} = 0,$$

$$\tilde{g}(\tau, x_1) = \begin{cases} 0 & \text{if } x_1 = 0 \\ (\sigma^2 - \alpha\,(\mu(T-\tau) - \ln(x_1)))\,x_1 & \text{if } x_1 \neq 0, \end{cases}$$

$$\tilde{h}(x_2) = \begin{cases} 0 & \text{if } x_2 = 0 \\ (\beta \ln(x_2) + \lambda\kappa)x_2 & \text{if } x_2 \neq 0. \end{cases}$$

Remark 3.1 *Note that the differential part of the PIDE is defined in the domain* $[0, x_1^\infty] \times [0, x_2^\infty]$, *using the discrete grid* $0 = x_{2_0}, x_{2_1}, ..., x_{2_q} = x_2^\infty$. *Since* $\ln(x_{2_0}) = -\infty$, *we choose* $z_{min} = \ln(x_{2_1})$ *and* $z_{max} = \ln(x_{2_q})$ *as it is proposed in [14].*

3.2 Time discretization

The method of characteristics is based on a finite differences scheme for the discretization of the material derivative. The material derivative is given by:

$$\frac{DF}{D\tau} = \frac{\partial F}{\partial \tau} + \vec{v} \cdot \nabla F. \tag{29}$$

First, we define the characteristics curve through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, $X(\mathbf{x}, \bar{\tau}; s)$, which satisfies:

$$\frac{\partial}{\partial s}X(\mathbf{x},\bar{\tau};s)=\vec{v}(X(\mathbf{x},\bar{\tau};s)),\quad X(\mathbf{x},\bar{\tau};\bar{\tau})=\mathbf{x}. \tag{30}$$

In order to discretize in time the material derivative in the complementarity problem (25), let us consider a number of time steps $\bar{N}$, the time step $\Delta\tau=T/\bar{N}$ and the time mesh points $\tau^n=n\Delta\tau$, $n=0,\frac{1}{2},1,\frac{3}{2},\ldots,\bar{N}$. In order to obtain the initial condition for solving the problem (22) the time discretization has to be chosen such that $\delta/\Delta\tau\in\mathbb{N}$. So, we should choose $\bar{N}$ as a multiple of T/δ. In the discretization of the material derivative in the initial value problem (22) we consider a number of time steps equal to $\delta/\Delta\tau$.

The material derivative approximation by the characteristics method for both problems is given by:

$$\frac{DF}{D\tau}=\frac{F^{n+1}-F^n\circ X^n}{\Delta\tau},$$

where $F=u^{(p)},w^{\tau,(p-1)}$ and $X^n(\mathbf{x}):=X(\mathbf{x},\tau^{n+1};\tau^n)$. For the case of $f=0$ (i.e., without seasonality effect), the components of $X^n(\mathbf{x})$ can be computed analytically:

$$\begin{aligned}
X_1^n(\mathbf{x}) &= \begin{cases} x_1 & \text{if } x_1=0\\ \exp\left[(\exp(-\alpha\Delta\tau)(\sigma^2+\alpha\ln(x_1))-\sigma^2)/\alpha\right] & \text{if } x_1\neq 0,\end{cases}\\
X_2^n(\mathbf{x}) &= \begin{cases} x_2 & \text{if } x_2=0\\ \exp[((\beta\ln(x_2)+\lambda\kappa)\exp(-\beta\Delta\tau)-\lambda\kappa)/\beta] & \text{if } x_2\neq 0.\end{cases}
\end{aligned}$$

However, for the general case where it is not possible to compute the characteristics curves analytically, some numerical ODE solvers can be used (see [3], for example).

Next, we consider a Crank-Nicolson method around $\big(X(\mathbf{x},\tau^{n+1};\tau),\tau\big)$ for $\tau=\tau^{n+\frac{1}{2}}$ for the discretization in time of the differential part of the operator and we treat explicitly the integral term by using the Adams-Bashforth (AB) scheme proposed in [32]. Note that for the first time step, the AB scheme is reduced to the explicit scheme proposed in [10]. So, the time discretized equation for $F=u^{(p)},w^{\tau,(p-1)}$ and $P=0$ can be written as follows:

Find F^{n+1} such that:

$$\begin{aligned}
&\frac{F^{n+1}(\mathbf{x})-F^n(X^n(\mathbf{x}))}{\Delta\tau}-\frac{1}{2}Div(A\nabla F^{n+1})(\mathbf{x})-\frac{1}{2}Div(A\nabla F^n)(X^n(\mathbf{x}))\\
&+\frac{1}{2}(l\,F^{n+1})(\mathbf{x})+\frac{1}{2}(l\,F^n)(X^n(\mathbf{x}))-\lambda\int_{z_{min}}^{z_{max}}\frac{3\bar{F}^n(x_1,\bar{x}_2+z)-\bar{F}^{n-1}(x_1,\bar{x}_2+z)}{2}\nu(z)dz = 0.
\end{aligned} \tag{31}$$

where $\bar{F}^n(x_1,\bar{x}_2+z)=F^n(x_1,\exp\bar{x}_2+z)$ and $\bar{F}^{n-1}(x_1,\bar{x}_2+z)=F^{n-1}(x_1,\exp\bar{x}_2+z)$. Note that the integral term is evaluated at the previous time steps, so that the presence of a full matrix in the linear systems associated with the fully discretized problems is avoided.

Moreover, we also discretize the artificial boundary condition (28) on Γ_1^+ as follows:

$$\frac{F^{n+1}(\mathbf{x})-F^n(\hat{X}^n(\mathbf{x}))}{\Delta\tau}+\frac{1}{2}(c\,F^{n+1})(\mathbf{x})+\frac{1}{2}(c\,F^n)(\hat{X}^n(\mathbf{x})) = 0 \tag{32}$$

where $\hat{X}^n(\mathbf{x}) = (-b\Delta\tau + x_1, x_2)^T$ in the case of $f = 0$.

Thus,

$$F^{n+1}(\mathbf{x}) = \frac{1 - c\Delta\tau/2}{1 + c\Delta\tau/2} F^n(\hat{X}^n(\mathbf{x})) \quad on \quad \Gamma_1^+. \tag{33}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply (31) by a suitable test function, integrate in Ω, use the classical Green formula and the following one [28]:

$$\begin{aligned}\int_\Omega Div\left(\mathbf{A}\nabla F^n\right)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \;&=\; \int_\Gamma (\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x) \cdot (\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \\ &- \int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x})) \cdot \nabla\psi(\mathbf{x})d\mathbf{x} \\ &- \int_\Omega Div\left((\nabla X^n)^{-T}(\mathbf{x})\right)(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}\end{aligned} \tag{34}$$

Note that, when $f = 0$, we have:

$$Div\left((\nabla X^n)^{-T}(\mathbf{x})\right) = \begin{pmatrix} \dfrac{1}{e_1}(\exp(\alpha\Delta\tau) - 1) \\ \\ \dfrac{1}{e_2}(\exp(\beta\Delta\tau) - 1) \end{pmatrix}, \tag{35}$$

where $e_1 = \exp\left[(\exp(-\alpha\Delta\tau)(\sigma^2 + \alpha\ln(x_1)) - \sigma^2)/\alpha\right]$ and $e_2 = \exp[((\beta\ln(x_2) + \lambda\kappa)\exp(-\beta\Delta\tau) - \lambda\kappa)/\beta]$. In the general case $Div((\nabla X^n)^{-T}(\mathbf{x}))$ needs to be approximated. After the previous steps, we can write a variational formulation for the time discretized problem as follows:

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$ such that $\psi = 0$ on Γ_1^+:

$$\begin{aligned}&\int_\Omega F^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega (\mathbf{A}\nabla F^{n+1})(\mathbf{x})\nabla\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega lF^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \\ &= \int_\Omega F^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\nabla\psi(\mathbf{x})d\mathbf{x} \\ &-\frac{\Delta\tau}{2}\int_\Omega lF^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} + \lambda\Delta\tau\int_\Omega\left[\int_{z_{min}}^{z_{max}} \frac{3\bar{F}^n(x_1, \bar{x}_2 + z) - \bar{F}^{n-1}(x_1, \bar{x}_2 + z)}{2}\nu(z)dz\right]\psi(\mathbf{x})d\mathbf{x} \\ &-\frac{\Delta\tau}{2}\int_\Omega Div\left((\nabla X^n)^{-T}(\mathbf{x})\right)(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x},\end{aligned} \tag{36}$$

where ∇X^n can be computed analytically in some cases. Otherwise it needs to be numerically approximated (see [3], for example).

3.3 Finite elements discretization and nonlinear terms

For the spatial discretization we consider piecewise biquadratic Lagrange finite elements. For the numerical integration of the terms appearing in this finite elements discretization, we use a 9-points quadrature formula which implies a lumped mass matrix computation when dealing with this term.

In order to deal with the nonlinearities in the free boundary problem that governs the valuation of the swing options, we apply to the mixed formulation (25)-(26) the ALAS algorithm proposed in [20] and explained in detail for the case of swing options without spikes in the electricity price in [7]. This algorithm is applied to the fully discretized in time and space problem and computes sequences that converge to the swing option value, to the Lagrange multiplier associated with the inequality constraint and to the early exercise and non-early exercise regions, thus allowing also to identify an approximation to the optimal exercise boundary.

3.4 Approximation of the integral term

In order to approximate the integral term that appears in the PIDE due to the presence of jumps in the electricity price, we use a suitable numerical integration procedure. More precisely, we use the classical composite trapezoidal rule with $m+1$ points in the following way:

$$\int_{z_{min}}^{z_{max}} G(x_1,\bar{x}_2+z)\nu(z)dz \approx \frac{h}{2}\Big[G(x_1,\bar{x}_2+z_{min})\nu(z_{min})+G(x_1,\bar{x}_2+z_{max})\nu(z_{max}) \\ +2\sum_{j=1}^{m-1} G(x_1,\bar{x}_2+k_j)\nu(k_j)\Big]$$

where $G(x_1,\bar{x}_2+z) = \frac{3\bar{F}^n(x_1,\bar{x}_2+z)-\bar{F}^{n-1}(x_1,\bar{x}_2+z)}{2}$, $h = \frac{z_{max}-z_{min}}{m}$, $k_j = z_{min}+jh$ for $j = 1,...,m-1$.

4 Numerical results

In this section we show some numerical results to illustrate the performance of the numerical methods, by comparing them with some examples in the literature with and without the presence of jumps in the electricity price. Note that this paper is the first one to consider the numerical solution of the PIDE associated with a two factor model for electricity prices. Thus, we mainly compare our results with the example in [17] considering two factors and a binomial method.

4.1 Example 1

In this example we consider as in [17] that the dynamics of the stochastic process N_t under a risk neutral probability measure is given in terms of the Poisson process, $\tilde{Z}$, instead of the compensated one, in the following way:

$$dN_t = -\beta\ln(N_t)N_t dt + (\exp(J_t)-1)N_t d\tilde{Z}_t\,. \quad (37)$$

Moreover, we assume that $J \sim \exp(1/\mu_j)$ with density function

$$\nu(z) = \frac{1}{\mu_j} \exp\left(-\frac{1}{\mu_j} z\right) \mathbb{1}_{z \geq 0}.$$

Thus, the integro-differential operator in (15) is written as

$$\begin{aligned} \mathcal{L}[F] = -\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 F}{\partial M^2} + \alpha\left(\mu(T-\tau) - \ln(M)\right) M \frac{\partial F}{\partial M} - \beta \ln(N) N \frac{\partial F}{\partial N} \\ -(r+\lambda)F + \lambda \int_0^\infty F\left(\tau, M, N\exp(z)\right)\nu(z)dz. \end{aligned} \tag{38}$$

In the present example, we consider as in [17] the valuation of a swing option with up to $p = 20$ exercise rights where the rights correspond to the payoff of a call option. Moreover, we take into account the cases with spikes ($\lambda = 4$) or without spikes ($\lambda = 0$) in the electricity price. For this purpose, we need to specify a set of parameters, related to the market values of the data involved in the underlying factors, the initial conditions of the stochastic processes and the parameters of the payoff function. All of them are taken from [17] and are shown in Table 1. We have chosen these parameters in order to compare the results we obtain with the ones in [17] where an alternative binomial method has been used. Moreover, concerning the numerical methods we select the parameters collected in Table 2. Note that, as we consider $f = 0$, thus neglecting seasonality, b does not depend on time in this particular example.

Market parameters of the underlying factors	
Speed of mean reversion process M, α	7
Volatility, σ	1.4
Speed of mean reversion process N, β	200
Interest rate, r	0
Seasonality, f	0
Parameter of Poisson process, λ	0, 4
Parameter of jump size distribution, μ_j	0.4
Initial conditions	
Initial value of M, M_0	1
Initial value of N, N_0	1
Payoff function parameters	
Payoff, $\phi(T,S)$	$(S-K)_+$
Strike, K	1

Table 1: Fixed parameters of the model for Example 1, cf. [17].

At the top of Figure 1 we show the value per exercise right with and without spikes of the swing option when the maturity of the contract is one year and a right can be exercised at most once per day (i.e. the refracting period δ is one day). Moreover, we consider that the time step $\Delta\tau$ is also one day.

Computational domain	
x_1^∞	4K
x_2^∞	4K
ABC	
Coefficient b	20.13
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
ALAS algorithm	
Parameter γ	10000

Table 2: Parameters of the numerical methods in Example 1.

Next, in the middle of Figure 1 we present the value per exercise right (with and without spikes) of the swing option when the maturity of the contract is two months, the refracting period δ is one day and the time step $\Delta\tau$ coincides with the refracting period. Finally, at the bottom of Figure 1 we consider that the option has ten exercise opportunities per day (i.e. the refracting period is 0.1 days) and that the delivery period is six days. The time step $\Delta\tau$ is equal to the refracting period.

All results in Figure 1 are in full agreement with the analogous ones appearing in [17] and [22] obtained with binomial methods. More precisely, the results at the top of Figure 1 agree with those in [17], Figure 10 (see also [22], Figure 4.8); the results in the middle of Figure 1 are the same as those in the bottom-right graph of [17], Figure 11 (equivalently in [22], Figure 4.9); and our results in at the bottom of Figure 1 agree with those in [22], Figure 4.11. Furthermore, we can observe how the price per exercise right (with and without spikes) decreases with the number of exercise rights. It is what it is expected because p swing options with one exercise right (that would be equivalent to p American options) give more flexibility because you can exercise all the rights at once and consequently its price must be higher than the price of one swing option with p exercise rights. At the bottom of Figure 1, the difference between two values per exercise right (with and without spikes) is smaller due to the value of the refracting period. As expected, when the value of the refracting period decreases the value of a swing option with p exercise rights tends to the value of p American options with 1 exercise right. In the presence of jumps the value of the swing option is higher than without spikes because the risk increases and the difference between both values is more significant for small numbers of exercise rights. According to [17], this is due to the option with more exercise rights will be used to protect from high prices due to the diffusive part and less due to the jump part.

Concerning the computational cost and its comparison with the costs associated to the binomial tree technique proposed in [17], we first note that the binomial tree approach provides the swing option price for only one initial spot price while the here proposed numerical methods compute at the same time the swing option prices for all spot prices associated to the nodes of the finite elements mesh. As the whole swing option price curve is available in this latter

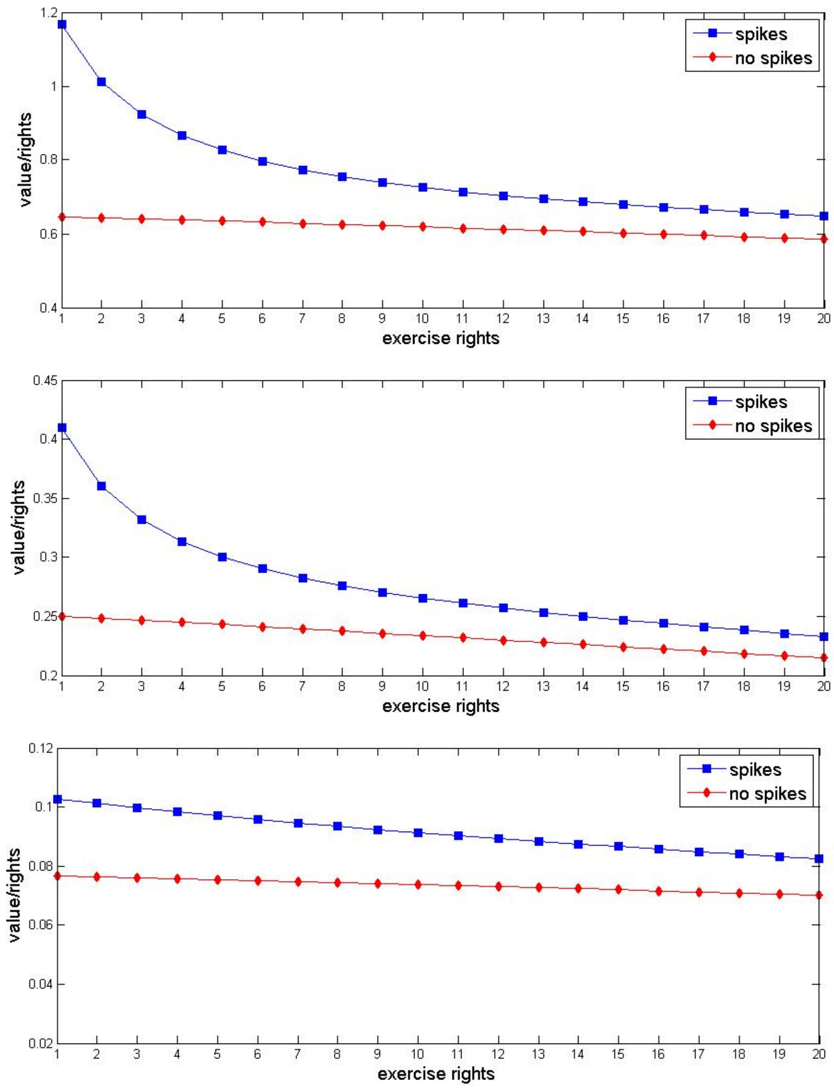

Figure 1: Value per right of a swing option with 1 year (top), 60 days (middle) and 6 days (bottom) to delivery in Example 1.

approach, we can easily analyze the dependence of the swing price on the initial spot prices. Also some Greeks could be easily obtained from the computed swing prices on the mesh. Another important advantage of the PIDE approach followed in this article is that the optimal exercise boundary can be obtained without any additional computational cost, while the binomial trees approach would require the use of several initial spot prices to get a reasonable approximation of the exercise boundary, thus multiplying the computational cost for one price by the required number of spot prices to be used. Note that the knowledge of the exercise boundary and the regions where it is optimal to exercise or to maintain the option is specially relevant in the case of options with multiple exercise rights.

After the previous relevant consideration to analyze the comparison between both methods, we report that to obtain the results in Figure 1 for a mesh of 2401 nodes (i.e. 2401 spot price values) the complete calculus requires around 45 minutes using an Intel Core I5-2400 CPU 3.10 GHz with 4GB of RAM. In [17] the authors report that for the same calculus in Figure 1 with a binomial tree technique (from a couple of values X_0 and Y_0, so that $S_0 = 1$) their implementation requires 10 minutes in a Intel P4, 3.4 GHz. In view of this example, it seems that both techniques have advantages and disadvantages. Thus, when just one price is required it seems that the binomial tree approach is more competitive. However, if we need a wider information, such as the dependence on the swing option price on the spot electricity price, swing price variation with the number of rights, some swing price sensitivities or the optimal exercise boundary, the use of PIDE models and the here proposed numerical methods seems more convenient. Finally, note that in both methods the computational time highly depends on some numerical parameters (time steps, number of mesh nodes, etc) that can be usually chosen to get a balance between accuracy and real time response requirements.

4.2 Example 2

In this section, unlike Example 1, we show some cases in which the seasonality function and the interest rate are different from zero and we consider the possibility of jumps in the electricity price following Merton and Kou models. For this purpose, we consider a swing option with up to $p = 7$ rights, maturity 1 year and refracting period 0.1 years. Moreover, we consider the values for the parameters involved in the underlying factors which appear in Table 3. Most of them are taken from [35] for a one factor stochastic model for electricity prices, which in turn are taken from [24] and are obtained from experimental observations of daily electricity spot and future prices experimental observations. In order to pose a two factor model we consider different nonzero values for the parameter β. For the numerical solution we consider again the parameters in Table 2, except the coefficient b of the ABC, that in this case depends on time and it is always greater than zero. In this example, the time step is $\Delta\tau = 0.01$. Moreover, in order to compare the results obtained with Merton and Kou models we need a certain matching between the density functions of the normal distribution (Merton) and of the double-exponential distribution (Kou). For this purpose, we consider the parameters involved in the jump-diffusion models which are proposed in [14].

In Figure 2 we show the value of this option per exercise right with and without spikes for the values $\beta = 0.2$, $\beta = 2$ and $\beta = 20$, from top to bottom. Taking into account the results in Figure 2, we can observe that the value of the swing option, with and without spikes, decreases when we increase the value of the mean reversion parameter β. As it is indicated in [7] and according to [17], an increase in β implies a decrease in the asset value and, therefore, in the value of the call swing option.

For the case of $\beta = 0.2$, in Figure 3 we represent the approximate location of the free boundary at origination (i.e. $t = 0$) when $p = 2$ and under Merton and Kou jump diffusion models, as well as for the case without spikes. As τ is fixed, the free boundary is a curve in the MN-plane. Note that the free boundaries for the different models exhibit a vertical straight line graph for a particular small value of M. In this sense, it seems that the slope of the free boundary curve blows up and the curve turns into a straight line. On the other hand, the consideration of spikes and its modelling does not affect too much the optimal exercise boundary and the results for both jump diffusion models are close each other. Next, for Merton model in Figure 4 we show the evolution of the free boundary near expirity. For $\tau = 0$ the exact free boundary is shown, while for the rest of the values the numerical approximation of the respective free boundaries are exhibited. Again, for fixed times near maturity we observe that each corresponding free boundary curve in the MN-plane turns into a vertical straight line for a particular small value of M.

Next, in Figure 5, for $M = 1$ we show the monotonicity of the free boundary with respect to τ by plotting the points $(\tau, S_\tau) = (\tau, 1, N_\tau)$ of the free boundary for the cases $p = 2$ and $p = 4$. As in the model treated in [35], a strictly increasing function in the interval $[(p-1)\delta, T]$ is obtained for a swing call option.

Finally, in Figures 6 and 7 the swing option value at origination ($t = 0$) is shown for the cases with spikes under Merton or Kou models, respectively.

Market parameters of the underlying factors	
Speed of mean reversion process M, α	0.016
Volatility, σ	0.086
Speed of mean reversion process N, β	0.2, 2, 20
Interest rate, r	0.05
Seasonality, f	$4.867 + 0.306\cos\left((t + 0.836)\frac{2\pi}{365}\right)$
Parameter of Poisson process, λ	0.8
Mean of jump size (Merton), μ_j	-0.1
Standard deviation of jump size (Merton), γ_j	0.45
Probability of upward jump (Kou), p_k	0.3445
Parameter (Kou), α_1	3.0465
Parameter (Kou), α_2	3.0775
Initial conditions	
Initial value of M, M_0	1
Initial value of N, N_0	1.5
Payoff function parameters	
Payoff, $\phi(T, S)$	$(S - K)_+$
Strike, K	1

Table 3: Fixed parameters of the model with seasonality in Example 2.

5 Conclusions

In this paper we have considered the valuation of swing options in electricity markets where the electricity price dynamics is described by a jump-diffusion model. The assumption of jump-

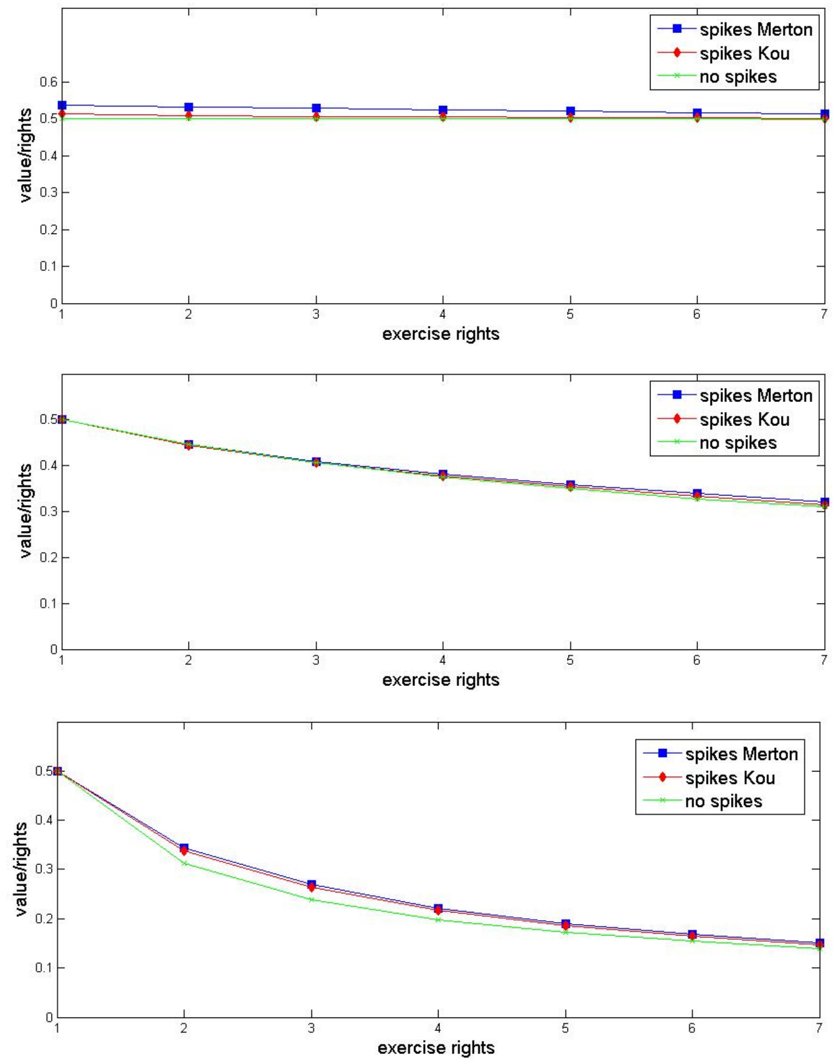

Figure 2: Value per right of a swing option with 1 year to delivery when $\beta = 0.2$ (top), $\beta = 2$ (middle) and $\beta = 20$ (bottom) in Example 2.

diffusion models seems reasonable to capture some features of the electricity prices observed in the markets. The consideration of spikes in the electricity prices leads to a PIDE problem, which has to be solved with specially designed numerical methods.

The swing option mainly consists of a path dependent option with multiple exercise rights. The right consists of receiving the payoff of a call option. The valuation problem has been posed as a sequence of free boundary problems associated with an integro-differential operator, one for each right. Additionally, an initial value problem linked to the same operator has to be solved due to the fact that the value of a swing option with one exercise less is involved in the definition of the obstacle function.

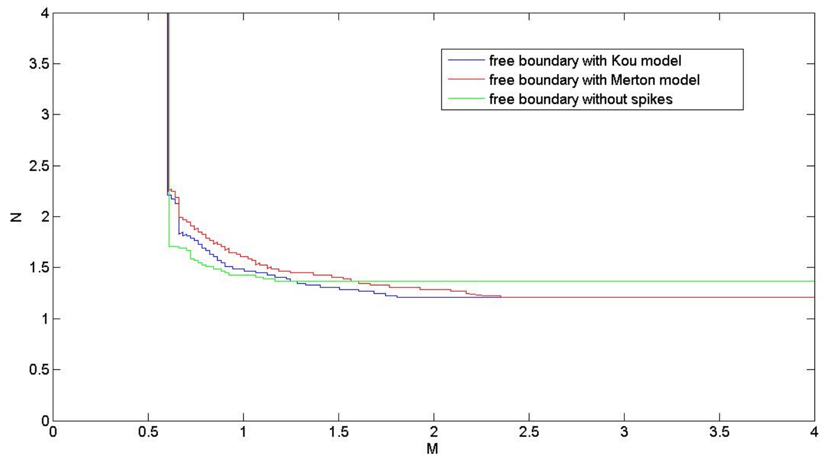

Figure 3: Approximated free boundaries of a swing option at $t = 0$ in Example 2 with Merton model, Kou model and without spikes (for $p = 2$ and $\beta = 0.2$).

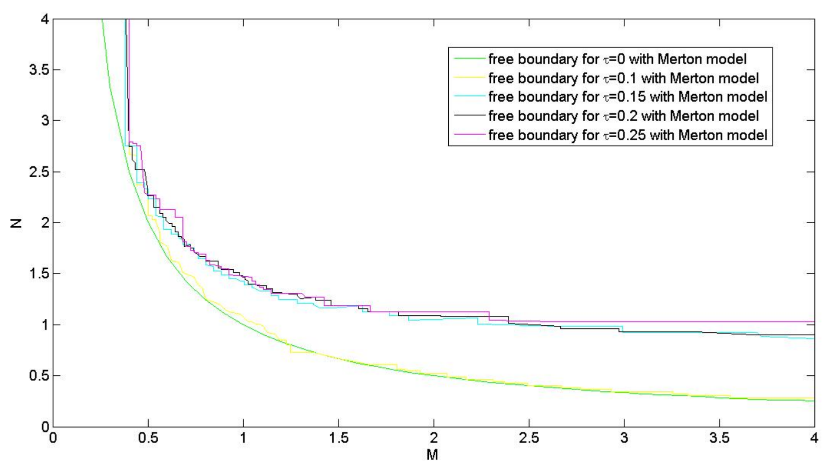

Figure 4: Approximated free boundaries of a swing option near maturity in Example 2 with Merton model (for $p = 2$ and $\beta = 0.2$).

In order to obtain a numerical solution of the problem, we have proposed appropriate numerical methods based on Lagrange-Galerkin formulations combined with the ALAS algorithm to deal with the early exercise feature. Moreover, the integral term that arises due to the presence of jumps is explicitly treated. Finally, we show some numerical results in order to illustrate the behaviour of the proposed methods and the quantitative and qualitative properties of the solutions, as well as the difference in the swing value with and without spikes in the electricity prices. It is observed that the presence of spikes increases the value of the swing option although it does not affect too much the optimal exercise boundary.

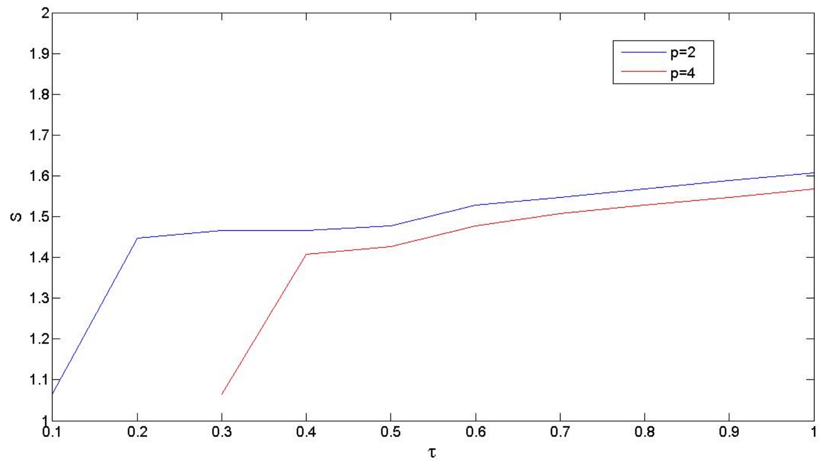

Figure 5: Evolution of the free boundary with respect to τ of the swing option in Example 2 with Merton model (for $p = 2$, 4 and $\beta = 0.2$).

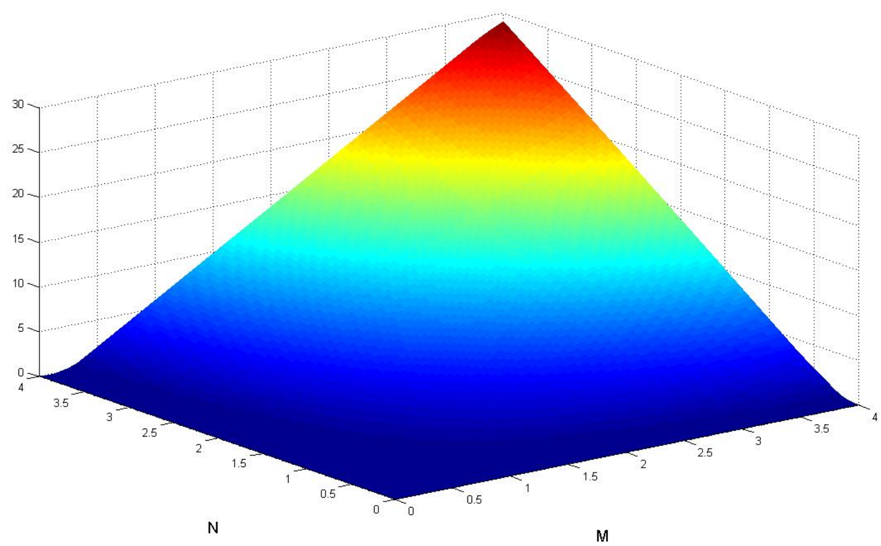

Figure 6: Swing option value at $t = 0$ in Example 2 with $p = 2$ and $\beta = 0.2$ for Merton model

Acknowledgements

The authors thank one of the referees, whose remarks improved substantially the final version of this article.

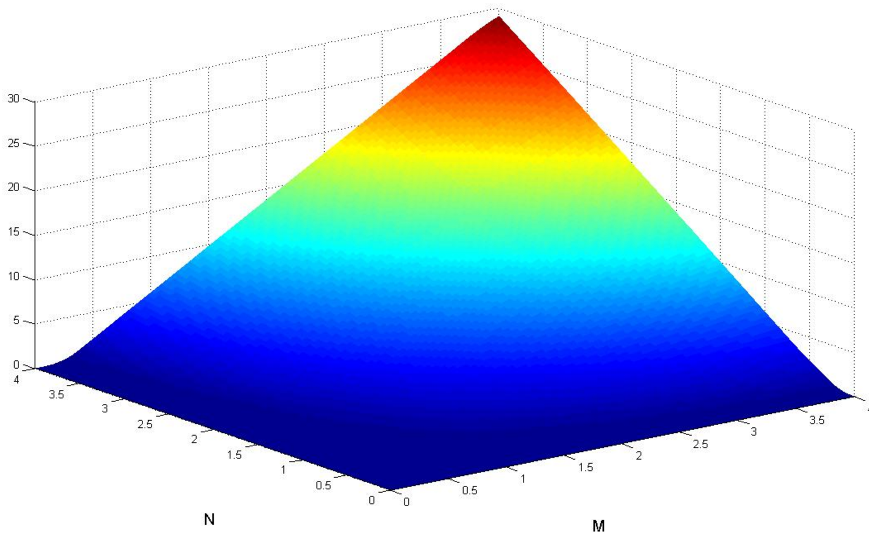

Figure 7: Swing option value at $t = 0$ in Example 2 with $p = 2$ and $\beta = 0.2$ for Kou model

References

[1] A. Almendral and C. W. Oosterlee, Numerical valuation of options with jumps in the underlying, Applied Numerical Mathematics, 53 (2005), 1-18.

[2] T. de Angelis and Y. Kitapbayev, On the optimal exercise boundaries of swing put options, Mathematics of Operations Research, 43 (2018), 252-274.

[3] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part I: Time discretization, SIAM Journal on Numerical Analysis, 44 (2006), 1829–1853.

[4] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part II: Fully discretized scheme and quadrature formulas, SIAM Journal on Numerical Analysis, 44 (2006), 1854–1876.

[5] A. Bodea, Valuation of swing options in electricity commodity markets, Phd Thesis, University of Heidelberg, (2012).

[6] M. C. Calvo-Garrido and C. Vázquez, Effects of jump-diffusion models for the house price dynamics in the pricing of fixed-rate mortgages, insurance and coinsurance, Applied Mathematics and Computation, 271 (2015), 730-742.

[7] M. C. Calvo-Garrido, M. Ehrhardt and C. Vázquez, Pricing swing options in electricity markets with two stochastic factors using a partial differential equation approach, Journal of Computational Finance, 20 (2017), 81-107.

[8] R. Carmona and N. Touzi, Optimal multiple stopping and valuation of swing options, Mathematical Finance, 18 (2008), 239–268.

[9] S. S. Clift and P. A. Forsyth, Numerical solution of two asset jump diffusion models for option valuation, Applied Numerical Mathematics, 58 (2008), 743-782.

[10] R. Cont and P. Tankov, Financial Modelling With Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, CRC Press, (2004).

[11] M. Dahlgren, A continuous time model to price commodity-based swing options, Review of Derivatives Research, 8 (2005), 27–47.

[12] Y. D'Halluin, P.A. Forsyth and G. Labahn, A penalty method for American options with jump diffusion processes, Numerische Mathematik, 97 (2004), 321-352.

[13] Y. D'Halluin, P.A. Forsyth and G. Labahn, A semi-Lagrangian approach for American Asian options under jump diffusion, SIAM Journal on Scientific Computing, 27 (2005), 315-345.

[14] Y. D'Halluin, P. A. Forsyth and K. R. Vetzal, Robust numerical methods for contingent claims under jump diffusion processes, IMA Journal of Numerical Analysis, 25 (2005), 87-112.

[15] G. Fichera, On a Unified theory of boundary value problems for elliptic-parabolic equations of second order in boundary value problems, ed. R.E. Langer, University of Wisconsin Press, (1960).

[16] L. Halpern, Artificial Boundary Conditions for the Linear Advection Diffusion Equation, Mathematics of Computation, 46 (1986), 425–438.

[17] B. M. Hambly, S. Howison and T. Kluge, Modelling spikes and pricing swing options in electricity markets, Quantitative Finance, 9 (2009), 937–949.

[18] S. Ikonen and J. Toivanen, Operator splitting methods for American option pricing, Applied Mathematics Letters, 17 (2004), 809-814.

[19] P. Jaillet, E. I. Ronn and S. Tompaidis, Valuation of commodity-based swing options, Management Science, 50 (2004), 909–921.

[20] T. Kärkkäinen, K. Kunisch and P. Tarvainen, Augmented Lagrangian Active Set methods for obstacle problems, Journal of Optimization Theory and Applications, 119 (2003), 499–533.

[21] M. Kjaer, Pricing swing options in a mean reverting model with jumps, Applied Mathematical Finance, 15 (2007), 479–502.

[22] T. Kluge, Pricing swing options and other electricity derivatives, PhD Thesis, University of Oxford, (2006).

[23] S. G. Kou, A jump-diffusion model for option pricing, Managerial Science, 48 (2002), 1086-1101.

[24] J. Lucia and E. Schwartz, Electricity prices and power derivatives: Evidence from the nordic power exchange, Review of Derivatives Research, 5(1) (2002), 5–50.

[25] N. Meinshausen and B. M. Hambly, Monte Carlo methods for the valuation of multiple-exercise options, Mathematical Finance, 14 (2004), 557–583.

[26] R. C. Merton, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 3 (1976), 125-144.

[27] M.H. Nguyen and M. Ehrhardt, Modelling and Numerical Valuation of Power Derivatives in Energy Markets, Advances in Applied Mathematics and Mechanics, 4(3) (2012), 259–293.

[28] M. R. Nogueiras, Numerical analysis of second order Lagrange-Galerkin schemes. Application to option pricing problems, PhD. Thesis, University of Santiago de Compostela (2005).

[29] O. A. Oleinik and E. V. Radkevic, Second order equations with nonnegative characteristic form, American Mathematical Society, Providence, Rhode Island, Plenum Press, New York-London, (1973).

[30] A. Pascucci, PDE and martingale methods in option pricing, Bocconi & Springer Series, Springer-Verlag, Milan, 2011.

[31] S. Salmi and J. Toivanen, An iterative method for pricing American options under jump-diffusion models, Applied Numerical Mathematics, 61 (2011), 821-831.

[32] S. Salmi and J. Toivanen, IMEX schemes for pricing options under jump-diffusion models, Applied Numerical Mathematics, 84 (2014), 33-45.

[33] R. T. Thanawalla, Valuation of swing options using an extended least squares Monte Carlo algorithm, PhD. Thesis, Heriot-Watt University, England, (2005).

[34] T. Wegner, Swing options and seasonality of power prices, Master Thesis, University of Oxford, (2002).

[35] M. Wilhelm and C. Winter, Finite element valuation of swing options, Journal of Computational Finance, 11 (2008), 107-132.

[36] B. Zhang and C. W. Oosterlee, An efficient pricing algorithm for swing options based on Fourier cosine expansions, Journal of Computational Finance, 16(4) (2013), 1-32.