Mathematical analysis of obstacle problems for pricing fixed-rate mortgages with prepayment and default options

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Abstract

In this paper, we address the mathematical analysis of a partial differential equation model for pricing fixed-rate mortgages with prepayment and default options, where the underlying stochastic factors are the house price and the interest rate. The mathematical model is posed in terms of a sequence of linked complementarity problems, one for each month of the loan life, associated with a uniformly parabolic operator. We study the existence of a strong solution to each one of the obstacle problems.

Keywords: Fixed-rate mortgages, obstacle problem, uniformly parabolic operator, strong solution

1. Introduction

A mortgage is a financial contract between two parts, a borrower and a lender, in which the borrower obtains funds from the lender (a bank or a financial institution, for example) by using a risky asset as a guarantee (collateral), usually a house. In this work, we focus on the mathematical model for fixed rate mortgages with monthly payments. The loan is reimbursed through monthly payments until the cancellation of the debt at maturity date. Thus, the mortgage value is understood as the discounted value of the future monthly payments (without including a possible insurance on the loan by the lender) and the underlying stochastic factors are the interest rate and the house price. The mathematical model is posed in [4, 13] so that prepayment is allowed at any time during the life of the loan and default only can occur at any monthly payment date. Thus, the mathematical model is posed in terms of a sequence of linked complementarity problems associated to a parabolic partial differential equation, one for each month of the loan life. The link between obstacle problems comes from the condition at the end of the month defined as the mortgage value at this date obtained from the obstacle problem for next month. The existence of solution for each obstacle problem in these sequence is an open problem treated in the present paper. In [4, 13] a log-normal process is assumed for house price evolution, so that this value evolves continuously whereas the interest rate dynamics is described by means of the CIR model. More recently, in [3] a jump-diffusion model is proposed to describe the house price dynamics to account for bubble or crisis phenomena in real state markets. The numerical resolution of these problems has been addressed by using different techniques (see [1, 8, 13, 4], for example).

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The main objective of this paper is the mathematical analysis of the obstacle problems involved in the valuation of fixed rate mortgages with prepayment and default options. We propose an approach based on the concept of strong solution. The existence of a strong solution is studied in the framework of uniformly parabolic PDEs with variable coefficients, mainly adapting the results in [11]. The mathematical analysis of obstacle problems related to finance has been addressed in the literature. For example, in [6] the authors prove the existence of a strong solution for an obstacle problem linked to a non-uniformly parabolic operator of Kolmogorov type. This kind of Kolmogorov operators also appear in the complementarity problems that arise in the pricing of other financial products, such as American Asian options [10], pension plans with early retirement [2] or stock loans [12].

This paper is organized as follows. In Section 2 we pose the pricing model in terms of a sequence of complementarity problems. In Section 3 we first prove the existence of a unique strong solution to each obstacle problem in a bounded domain by adapting a penalization technique. Next, we prove the existence of a strong solution to the obstacle problem in the unbounded domain by solving a sequence of complementarity problems in regular bounded domains.

2. Mathematical modelling of the pricing problem

A mortgage can be treated as a derivative financial product, for which the underlying state variables are the house price and the term structure of interest rates. So, we have to model the dynamics of the underlying factors. Under risk neutral probability, the value of the house at time t, H_t, is assumed to follow the stochastic differential equation (see [9]):

$$dH_t = (r-\delta)H_t dt + \sigma_H H_t dX_t^H, \tag{1}$$

where r is the interest rate, δ is the 'dividend-type' per unit service flow provided by the house, σ_H is the house-price volatility and X_t^H is the standard Wiener process associated to the house price. The other source of uncertainty, the stochastic interest rate r_t at time t, is assumed to be a classical Cox-Ingerrsoll-Ross (CIR) process [5], satisfying

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dX_t^r, \tag{2}$$

where κ is the speed of adjustment in the mean reverting process, θ is the long term mean of the short-term interest rate (steady state spot rate), σ_r is the interest-rate volatility parameter and X_t^r is the standard Wiener process associated to the interest rate. Wiener processes, X_t^H and X_t^r can be assumed to be correlated according to $dX_t^H dX_t^r = \rho dt$, where ρ is the instantaneous correlation coefficient.

Following the same notation as in [4], we assume that the mortgage is repaid by a sequence of monthly payments at dates T_m, $m = 1, ..., M$, where M is the number of months of loan life. Assuming that $T_0 = 0$, let $\Delta T_m = T_m - T_{m-1}$ the duration of month m. Moreover, $\tau_m = T_m - t$ denotes the time until the payment date in month m, c is the fixed contract rate and $P(0)$ is the initial loan (i.e. the principal at $T_0 = 0$), the fixed mortgage payment (MP) is given by:

$$MP = \frac{(c/12)(1+c/12)^M P(0)}{(1+c/12)^M - 1}. \tag{3}$$

For $m = 1, ..., M$, the unpaid loan just after the $(m-1)$th payment date is

$$P(m-1) = \frac{((1+c/12)^M - (1+c/12)^{m-1})P(0)}{(1+c/12)^M - 1}. \tag{4}$$

Next, we take into account the prepayment and default options. The option to default only happens at payment dates if the borrower does not pay the monthly amount MP. The option to prepay can be exercised at any time during the life of the loan, if the borrower fully amortizes the mortgage at time τ_m by paying the amount (which includes the total remaining debt plus an early termination penalty):

$$TD(\tau_m) = (1+\Psi)(1+c(\Delta T_m - \tau_m))P(m-1), \tag{5}$$

where $\Delta T_m - \tau_m$ represents the time that has elapsed since the beginning of month m and $\Psi \geq 0$ denotes the prepayment penalty factor.

By using Ito lemma combined dynamic hedging methodology, in [4] the formulation in terms of linked obstacle problems proposed in [13] is justified. Thus, the mortgage valued process, V_t, can be obtained in the form $\bar{V}_t = \bar{V}(t, H_t, r_t)$ where the function $\bar{V}$ satisfies a sequence of PDE problems.

More precisely, for month m, the function $\bar{V}$ is the solution of the following obstacle problem:

$$\begin{cases} \min\{\bar{\mathcal{L}}\bar{V} - \bar{a}\bar{V}, TD - \bar{V}\} = 0, & \text{in } \hat{S}_T = (0, \Delta T_m) \times \mathbb{R}_+^2, \\ \bar{V}(\tau_m = 0, H, r) = \bar{g}, & (H, r) \in \mathbb{R}_+^2, \end{cases} \tag{6}$$

where

$$\bar{\mathcal{L}}\bar{V} = -\frac{\partial \bar{V}}{\partial_{\tau_m}} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 \bar{V}}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial \bar{V}}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 \bar{V}}{\partial r^2} + (r-\delta)H\frac{\partial \bar{V}}{\partial H} + \kappa(\theta - r)\frac{\partial \bar{V}}{\partial r}, \tag{7}$$

$\bar{a} = r$ and $\bar{g}$ denotes the initial condition with respect to τ_m for each month, defined as follows:

- at month M, just before the last payment:

$$\bar{g}(\tau_M = 0, H, r) = \min(MP, H) \tag{8}$$

- at the other payment dates:

$$\bar{g}(\tau_m = 0, H, r) = \min(\bar{V}(\tau_{m+1} = \Delta T_{m+1}, H, r) + MP, H), \quad 1 \leq m \leq M-1. \tag{9}$$

3. Mathematical analysis

3.1. Equivalent problem and properties

In order to prove the existence of solution for the obstacle problem (6), we adapt some results proved in [11] for parabolic PDEs with variable coefficients. First, we make the change of variables to $\mathbf{x} = (x_1, x_2)$, with $x_1 = \ln(H)$ and $x_2 = \sqrt{r}$. So, we can write problem (6) in the equivalent form :

$$\begin{cases} \min\{\mathcal{L}V - aV, TD - V\} = 0, & \text{in } S_T = (0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+, \\ V(\tau_m = 0, \mathbf{x}) = g, & \mathbf{x} \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{10}$$

where $a = a(\tau_m, \mathbf{x}) = x_2^2$, $g(\tau_m, \mathbf{x}) = \bar{g}(\tau_m, \hat{\mathbf{x}}(H, r))$ and

$$\mathcal{L}V \;=\; \frac{1}{2}\sum_{i,j=1}^{2} c_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial_{\tau_m}} \tag{11}$$

with

$$c_{ij}(\tau_m, \mathbf{x}) = \begin{pmatrix} \sigma_H^2 & \frac{1}{2}\rho\sigma_H\sigma_r \\ \frac{1}{2}\rho\sigma_H\sigma_r & \frac{1}{4}\sigma_r^2 \end{pmatrix} \text{ and } b_i(\tau_m, \mathbf{x}) = \begin{pmatrix} (x_2^2 - \delta) - \dfrac{\sigma_H^2}{2} \\ \left(\dfrac{\kappa}{2}(\theta - x_2^2) - \dfrac{\sigma_r^2}{8}\right)/x_2 \end{pmatrix}. \tag{12}$$

Next, we introduce the natural definition of parabolic Hölder spaces for these equations.

Definition 3.1. *For the exponent α $(0 < \alpha < 1)$ and the domain $S_T = (0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+$, we denote by $C_P^\alpha(S_T)$ the space of bounded functions u such that there exists a constant C satisfying*

$$|u(t, x) - u(s, y)| \le C(|t - s|^{\alpha/2} + |x - y|^\alpha), \quad t, s \in (0, \Delta T_m), x, y \in \mathbb{R} \times \mathbb{R}_+.$$

Therefore, the space $C_P^\alpha(S_T)$ can be equipped with the norm

$$\|u\|_{C_P^\alpha(S_T)} = \sup_{(t,x)\in S_T} |u(t, x)| + \sup_{(t,x)\in(S_T),(t,x)\neq(s,y)} \frac{|u(t, x) - u(s, y)|}{|t - s|^{\alpha/2} + |x - y|^\alpha}.$$

Next, we introduce two lemmas. The first one states that operator $\mathcal{L}$ is uniformly parabolic and the second one is about the regularity of the operator coefficients.

Lemma 3.2. *As $|\rho| < 1$, the matrix c_{ij} is symmetric and positive definite. Thus, $\mathcal{L}$ is uniformly parabolic, i.e., there exists a positive constant Λ such that*

$$\Lambda^{-2}|\psi|^2 \le \sum_{i,j=1}^{2} c_{ij}(\tau_m, \mathbf{x})\psi_i\psi_j \le \Lambda^2|\psi|^2, \quad (\tau_m, \mathbf{x}) \in (0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+, \quad \psi \in \mathbb{R}^2$$

Lemma 3.3. *The coefficients c_{ij} are bounded and parabolic Hölder continuous functions with exponent $\alpha = \frac{1}{2}$, that is: $c_{ij} \in \mathcal{C}_P^\alpha((0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+)$, $1 \le i, j \le 2$. Moreover, the coefficients b_i and a are locally bounded and locally Hölder continuous on compact subsets of $(0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+$.*

Next, we introduce the Sobolev spaces and the concept of strong solution.

Definition 3.4. *Let O a domain in $(0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+$ and $1 \le p \le \infty$. We denote by $S^p(O)$ the space of functions $V \in L^p(O)$ such that the weak derivatives*

$$\frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial x_i \partial x_j}, \frac{\partial V}{\partial \tau_m} \in L^p(O) \quad \text{for} \quad i, j = 1, 2.$$

Moreover, $V \in S_{loc}^p(O)$ if $V \in S^p(O_1)$ for each domain O_1 verifying that $\bar{O}_1 \subseteq O$.

Definition 3.5. *A strong solution for the obstacle problem (10) is a function $V \in S_{loc}^1(S_T) \cap C(\bar{S}_T)$ satisfying $\min\{\mathcal{L}V - aV, TD - V\} = 0$ a.e in S_T and the initial condition (8) or (9).*

We define the concept of sub-solution to be used for the existence of a strong solution.

Definition 3.6. *A sub-solution $\underline{V}$ of the obstacle problem (10) is a function $\underline{V} \in S_{loc}^1(S_T) \cap C(\bar{S}_T)$ which satisfies:*

$$\begin{cases} \min\{\mathcal{L}\underline{V} - a\underline{V}, TD - \underline{V}\} \ge 0, & \text{in } S_T \\ \underline{V}(\tau_m = 0, \mathbf{x}) \le g, & \mathbf{x} \in \mathbb{R} \times \mathbb{R}_+. \end{cases} \tag{13}$$

Lemma 3.7. *The function* $\underline{V}(\tau_m, \mathbf{x}) = 0$ *is a sub-solution for the obstacle problem (10).*

Next, we introduce a lemma on the growth and regularity of the obstacle function TD. This condition provides an upper bound for the second order distributional derivatives of TD. Note that any C^2 function, such as for example (5), satisfies the following hypothesis.

Lemma 3.8. *The obstacle TD is a Lipschitz continuous function in $\bar{S}_T$, such that for each compact subset O, $\bar{O} \subseteq S_T$, there exists a constant $\tilde{c} \in \mathbb{R}$ satisfying*

$$\sum_{i,j=1}^{2} \xi_i \xi_j \frac{\partial^2 TD}{\partial x_i \partial x_j} \leq \tilde{c}|\xi|^2 \quad in \quad O, \quad \xi \in \mathbb{R} \times \mathbb{R}_+,$$

in the sense of distributions, i.e.

$$\sum_{i,j=1}^{2} \xi_i \xi_j \int_O TD \frac{\partial^2 \phi}{\partial x_i \partial x_j} \leq \tilde{c}|\xi|^2 \int_O \phi, \qquad for\ all\ \xi \in \mathbb{R}^2\ and\ \phi \in C_0^\infty(O), \ \ \phi \geq 0.$$

Proof of Lemma 3.8. Although we state the result in a general form, note that TD does not depend on x_1 and x_2. Thus, the proof is trivial. □

Next, we need to prove a inequality condition between the functions defining the obstacle and the initial conditions for the obstacle problems.

3.2. Existence and uniqueness of solution in the bounded domain

We study the existence and uniqueness of a strong solution in a bounded domain for the problem:

$$\begin{cases} \min\{\mathcal{L}V - aV, TD - V\} = 0 & \text{in } Q(T_m), \\ V|_{\partial_P Q(T_m)} = \tilde{g}. \end{cases} \tag{14}$$

where $Q(T_m) = (0, \Delta T_m) \times Q$, Q being a bounded domain in $\mathbb{R} \times \mathbb{R}_+$ and $\partial_P Q(T_m) = \partial Q(T_m) \setminus (Q \times \{\Delta T_m\})$ being the parabolic boundary of $Q(T_m)$. As in [6], we use the theory of barrier functions in [7] to prove that the solution of (14) is continuous up to the boundary.

Hypothesis 3.9. *For any point $\zeta \in \partial_P Q(T_m)$, there exists a barrier $w \in C^2(U \cap \overline{Q(T_m)}; \mathbb{R})$, that is*

- $\mathcal{L}w \geq 1 \in U \cap Q(T_m)$,
- $w > 0$ *in* $U \cap \overline{Q(T_m)} \setminus \{\zeta\}$ *and* $w(\zeta) = 0$

where U is a neighbourhood of ζ.

First, we prove the following auxiliary result.

Theorem 3.10. *If Hypothesis 3.9 is satisfied, $\tilde{g} \in C(\partial_P Q(T_m))$, $\tilde{g} \le TD$ and $a \in C \cap L^\infty(Q(T_m))$, then there exists a classical solution V of the problem:*

$$\begin{cases} \mathcal{L}V = h(.,V) & \text{in } Q(T_m), \\ V|_{\partial_P Q(T_m)} = \tilde{g}. \end{cases} \tag{15}$$

where $h = h(.,V)$ is a continuous Lipschitz function in $\overline{Q(T_m)} \times \mathbb{R}$. Moreover, $\inf_{Q(T_m)} |V| \ge 0$.

Proof of Theorem 3.10. Following [11], we use an iterative method starting with $V_0 = 0$ and take into account that $h(.,V) \le c_1(1+|V|)$ for a positive constant c_1 which depends on the L^∞ norm and the Lipschitz constant of h. Let $\tilde{\lambda}$ is the Lipschitz constant of h and define recursively the sequence $(V_j)_{j\in\mathbb{N}}$ as the solution of the problem

$$\begin{cases} \mathcal{L}V_j - \tilde{\lambda}V_j = h(.,V_{j-1}) - \tilde{\lambda}V_{j-1} & \text{in } Q(T_m), \\ V_j|_{\partial_P Q(T_m)} = \tilde{g}. \end{cases} \tag{16}$$

By induction, we prove that V_j is an increasing sequence. From maximum principle we get $V_0 \le V_1$, in fact we have $\mathcal{L}(V_0 - V_1) - \tilde{\lambda}(V_0 - V_1) = \mathcal{L}V_0 - h(.,V_0) = c_1(1+V_0) - h(.,V_0) \ge 0$ and $V_0 \le V_1$ in $\partial_P Q(T_m)$. For a fixed $j \in \mathbb{N}$, using the induction hypothesis $V_{j-1} \le V_j$, we have

$$\mathcal{L}(V_j - V_{j+1}) - \tilde{\lambda}(V_j - V_{j+1}) = h(.,V_{j-1}) - h(.,V_j) - \tilde{\lambda}(V_j - V_{j-1}) \ge 0.$$

By the maximum principle, $V_j = V_{j+1}$ in $\partial_P Q(T_m)$ implies $V_j \le V_{j+1}$ in $Q(T_m)$. Then, we obtain the increasing sequence V_j, with $V_0 \le V_j \le V_{j+1} \le \|TD\|_{L^\infty}$, and passing to the limit when $j \to \infty$ we get

$$\begin{cases} \mathcal{L}V = h(.,V) \text{ in } Q(T_m), \\ V|_{\partial_P Q(T_m)} = \tilde{g}. \end{cases} \tag{17}$$

Next, to prove that $V \in C(\overline{Q(T_m)})$, we use a barrier function technique as follows. First, we fix $\varsigma \in \partial_P Q(T_m)$ and $\varepsilon > 0$. Let U be a neighbourhood of ς, such that $|\tilde{g}(z) - \tilde{g}(\varsigma)| \le \varepsilon$, $\forall z \in U \cap \partial_P Q(T_m)$, and let us consider a barrier function w defined as in the Hypothesis 3.9. As in [6], we set $v^\pm(z) = \tilde{g}(\varsigma) \pm (\varepsilon + k_\varepsilon w(z))$ for a suitably large positive constant k_ε independent of j such that

$$\mathcal{L}(V_j - v^+) \le h(.,V_{j-1}) - \tilde{\lambda}(V_{j-1} - V_j) + k_\varepsilon \ge 0,$$

and $V_j \le v^+$ in $\partial(U \cap Q(T_m))$. By the maximum principle $V_j \le v^+$ in $U \cap Q(T_m)$. Analogously, we have that $V_j \ge v^-$ in $U \cap Q(T_m)$ and when $j \to \infty$ we obtain

$$\tilde{g}(\varsigma) - \varepsilon - k_\varepsilon w(z) \le V(z) \le \tilde{g}(\varsigma) + \varepsilon + k_\varepsilon w(z), \quad z \in U \cap Q(T_m).$$

Then, we have $\tilde{g}(\varsigma) - \varepsilon \le \liminf_{z\to\varsigma} V(z) \le \limsup_{z\to\varsigma} V(z) \le \tilde{g}(\varsigma) + \varepsilon$, $\quad z \in U \cap Q(T_m)$. As ε is arbitrary, we obtain that $V \in C(\overline{Q(T)})$. □

Next, we prove the existence of a strong solution to (14).

Theorem 3.11. *If Hypothesis 3.9 is satisfied, $\tilde{g} \in C(\partial_P Q(T_m))$, $\tilde{g} \le TD$ and $a \in C \cap L^\infty(Q(T_m))$, then there exists a strong solution V to the obstacle problem (14). In addition, there exists a positive constant C, depending on $\mathcal{L}$, O, $Q(T_m)$, p and the L^∞ norms of $\tilde{g}$ and TD, such that $\|V\|_{\mathcal{S}^p(O)} \le C$, where $p \ge 1$ and O is a compact subset of $Q(T_m)$.*

Proof of Theorem 3.11. We adapt the penalization technique used in [11]. Thus, we consider a family of functions $(\beta_\varepsilon)_{\varepsilon\in(0,1)} \in C^\infty(\mathbb{R})$, such that β_ε is a bounded, increasing function with bounded first order derivative and satisfying

$$\beta_\varepsilon(0) = 0, \quad \beta_\varepsilon(s) \geq -\varepsilon \text{ for } s < 0, \quad \textit{and} \quad \lim_{\varepsilon\longrightarrow 0} \beta_\varepsilon(s) = \infty \text{ for } s > 0.$$

Next, for $\gamma \in (0,1)$ we denote by TD^γ and a^γ the regularizations of TD and a, respectively. Since $\tilde{g} \leq TD$ in $\partial_P Q(T_m)$ we have $\tilde{g}^\gamma = \tilde{g} - \tilde{\lambda}\gamma \leq TD^\gamma$ in $\partial_P Q(T_m)$, where $\tilde{\lambda}$ is the TD Lipschitz constant. Then, we pose the following penalized problem:

$$\begin{cases} \mathcal{L}^\gamma V - a^\gamma V = \beta_\varepsilon(V - TD^\gamma) & \text{in } Q(T_m), \\ V|_{\partial_P Q(T_m)} = \tilde{g}^\gamma. \end{cases} \tag{18}$$

Next, we prove that problem (18) has a classical solution $V_{\varepsilon,\gamma}$ by using Theorem 3.10 applied to function $h(.,V) = \beta_\varepsilon(V - TD^\gamma) + a^\gamma V$. Finally, we need to check that

$$|\beta_\varepsilon(V_{\varepsilon,\gamma} - TD^\gamma)| \leq \tilde{c}$$

for a constant $\tilde{c}$ depending on ε and γ. Taking into consideration that $\beta_\varepsilon \geq -\varepsilon$, we only need to check that the penalization function β_ε is bounded from above. For this purpose, let us denote by ς the maximum point of the function $\beta_\varepsilon(V_{\varepsilon,\gamma} - TD^\gamma) \in C(Q(T_m))$ and we assume that $\beta_\varepsilon(V_{\varepsilon,\gamma}(\varsigma) - TD^\gamma(\varsigma)) \geq 0$. If $\varsigma \in \partial_P Q(T_m)$ then $-\varepsilon \leq \beta_\varepsilon(\tilde{g}^\gamma(\varsigma) - TD^\gamma(\varsigma)) \leq 0$. Nevertheless, if $\varsigma \in Q(T_m)$ then $V_{\varepsilon,\gamma} - TD^\gamma$ also assumes the maximum in ς because β_ϵ is an increasing function. Then,

$$\mathcal{L}^\gamma V_{\varepsilon,\gamma}(\varsigma) - \mathcal{L}^\gamma TD^\gamma(\varsigma) \leq 0 \leq a^\gamma(\varsigma)(V_{\varepsilon,\gamma}(\varsigma) - TD^\gamma(\varsigma))$$

By Lemma 3.8, $\mathcal{L}^\gamma TD^\gamma(\varsigma)$ is bounded from above by a constant independent of γ. Moreover, we have that

$$\beta_\varepsilon(V_{\varepsilon,\gamma}(\varsigma) - TD^\gamma(\varsigma)) = \mathcal{L}^\gamma V_{\varepsilon,\gamma}(\varsigma) - a^\gamma(\varsigma)(V_{\varepsilon,\gamma}(\varsigma) \leq \mathcal{L}^\gamma TD^\gamma(\varsigma) - a^\gamma(\varsigma) TD^\gamma(\varsigma) \leq \tilde{c},$$

where $\tilde{c}$ is independent of ε and γ. Finally, the sequence $V_{\epsilon,\gamma}$ converges to the solution V. □

Next, we prove a comparison principle and the uniqueness of solution for the obstacle problem (14). For this purpose, we rewrite Proposition 8.31 in [11] in case we have a sub-solution instead of a super-solution.

Proposition 3.12. *Let V be a strong solution of the obstacle problem (14) and $\underline{V}$ a sub-solution, i.e. $\underline{V} \in S^1_{loc}(Q_n \cup C(\bar{Q}_n))$. If*

$$\begin{cases} \min\{\mathcal{L}\underline{V} - a\underline{V}, TD - \underline{V}\} \geq 0 & \textit{in } Q_n, \\ \underline{V}|_{\partial_P Q_n} \leq \tilde{g}_n \end{cases}$$

then $V \geq \underline{V}$ in Q_n. Therefore, the solution is unique.

Proof of Proposition 3.12. We use a contradiction argument and the maximum principle. If we assume that the open set $F = \{z \in Q_n | V(z) < \underline{V}(z)\}$ is not empty, then $V \leq \underline{V} \leq TD$ in F and

$$\mathcal{L}V - aV = 0, \quad \mathcal{L}\underline{V} - a\underline{V} \geq 0 \quad \text{in} \quad F.$$

The maximum principle applied to $(\underline{V} - V)$ implies that $V \geq \underline{V}$ in F and we get a contradiction. From the maximum principle, uniqueness of solution directly follows. □

3.3. Existence of strong solution in the unbounded domain

Next, we prove the main theorem in the framework of parabolic PDEs with variable coefficients [11].

Theorem 3.13. *If $g \leq TD$ and there exists a sub-solution $\underline{V}$ for the problem (10) then we have a strong solution $V \geq \underline{V}$ in S_T. Moreover, $V \in S^p_{loc}(S_T)$ for every $p \geq 1$.*

Proof of Theorem 3.13. Let $D_{\tilde{\rho}}(x_1, x_2)$ denote the Euclidean ball centered at $(x_1, x_2) \in \mathbb{R}^2$, with radius $\tilde{\rho}$. We consider the sequence of domains $O_n = D_n\left(n + \frac{1}{n}, 0\right) \cap D_n\left(0, n + \frac{1}{n}\right)$ covering $\mathbb{R}\times\mathbb{R}_+$. For any $n \in \mathbb{N}$, the cylinder $Q_n = (0, \Delta T_m) \times O_n$ is a $\mathcal{L}$*-regular domain* in the sense that there exists a barrier function at any point of the parabolic boundary $\partial_P Q_n := \partial Q_n \setminus (\{0\} \times O_n)$. By applying Theorem 3.11 with $\tilde{g} = \tilde{g}_n$, for any $n \in \mathbb{N}$, there exists a strong solution $V_n \in \mathcal{S}^p_{\text{loc}}(Q_n) \cap C(Q_n \cup \partial_P Q_n)$ to problem

$$\begin{cases} \min\{\mathcal{L}V - aV, TD - V\} = 0 & \text{in } Q_n, \\ V|_{\partial_P H_n} = \tilde{g}_n. \end{cases} \tag{19}$$

Moreover, the following estimate holds: for every $p \geq 1$ and $Q \subset\subset Q_n$ there exists a positive constant C, only depending on $Q, Q_n, p, \|TD\|_{L^\infty(Q_n)}$ such that $\|V_n\|_{\mathcal{S}^p(Q)} \leq C$. Next, we consider a sequence of cut-off functions $\chi_n \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, such that $\chi_n = 1$ on O_{n-1}, $\chi_n = 0$ on $\mathbb{R}\times\mathbb{R}_+ \setminus O_n$ and $0 \leq \chi_n \leq 1$. We set

$$\tilde{g}_n(\tau_m, x_1, x_2) = \chi_n(x_1, x_2) g + (1 - \chi_n(x_1, x_2)) \underline{V}(\tau_m, x_1, x_2),$$

where $\underline{V}$ is the sub-solution. By the comparison principle (Proposition 3.12) we have the increasing sequence $\underline{V} \leq V_n \leq V_{n+1} \leq \tilde{g} \leq g \leq TD$.

Then, we can pass to the limit as $n \to \infty$, on compact subsets of $(0, \Delta T_m) \times \mathbb{R} \times \mathbb{R}_+$, to get a strong solution of $\min\{\mathcal{L}V - aV, TD - V\} = 0$ in the space S^p_{loc}. A barrier argument similar to the one used in the proof of Theorem 3.10 shows that V attains the initial condition. □

Remark 3.14. *Although in Theorem 3.13 we assume that $g \leq TD$, for $m = M$ we can proof that this condition holds. More precisely, at the end of the last month, M, which corresponds to $\tau_M = 0$, we have that $g(\tau_M = 0, x_1, x_2) = \min(MP, \exp(x_1)) \leq MP$ and*

$$TD(\tau_M = 0) = (1 + \Psi)(1 + c\Delta T_M)P(M - 1) = (1 + \Psi)MP,$$

where $\Psi \geq 0$. Next, since $MP \leq (1 + \Psi)MP$ we have that $g(\tau_M = 0, x_1, x_2) \leq TD(\tau_M = 0)$.

However, for any month m, where $1 \leq m \leq M - 1$, we only could check the condition $g \leq TD$ numerically. More precisely, we have that

$$\begin{aligned} g(\tau_m = 0, x_1, x_2) &= \min(V(\tau_{m+1} = \Delta T_{m+1}, \exp(x_1), x_2^2) + MP, \exp(x_1)) \\ &\leq V(\tau_{m+1} = \Delta T_{m+1}, \exp(x_1), x_2^2) + MP. \end{aligned}$$

Moreover, we have that

$$V(\tau_{m+1} = \Delta T_{m+1}, \exp(x_1), x_2^2) \leq TD(\tau_{m+1} = \Delta T_{m+1}).$$

So, it only would remain to check that $TD(\tau_{m+1} = \Delta T_{m+1}) + MP \leq TD(\tau_m = 0)$, or equivalently to prove that $(1+\Psi)P(m) + MP \leq (1+\Psi)(1+c\Delta T_m)P(m-1)$. By using the numerical methods proposed in [4] we have numerically checked that these two amounts coincide when $\Psi = 0$ or when $(1 + \Psi)P(m) + MP < (1 + \Psi)(1 + c\Delta T_m)P(m - 1)$ for $\Psi \neq 0$. Thus, numerically we have checked that $g \leq TD(\tau_m = 0)$, for $m = 1, \ldots, M - 1$. However, we could not prove it theoretically, so that we assume this condition in the theorem statement.

Acknowledgements

This work has been partially funded by Spanish MINECO (Project MTM2013-47800-C2-1-P) and Xunta de Galicia grant GRC2014/044, including FEDER funds.The authors also would like to thank one reviewer for the suitable remarks which helped to improve the final version of the article.

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