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# A MESHLESS NUMERICAL FORMULATION FOR THE SOLUTION OF POTENTIAL PROBLEMS IN ENGINEERING APPLICATIONS

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## Summary

It is widely acknowledged that mesh generation process is one of the big challenges in finite element, finite differences and boundary element computations. Thus, when applying these techniques to some computational mechanics problems involving very complicated or changing geometries, the main shortcoming is the need to discretize the domain. To overcome these troubles, a new family of numerical methods, the “meshless methods”, has been recently introduced to eliminate the grid used in the conventional methods by constructing the approximation entirely in terms of points.

A gridless technique based on the Moving Least Square method combined with a point collocation approach is proposed in this paper for the solution of potential problems in the electrical engineering field.

## Introduction

Advances in numerical methods, together with the development of computer sciences, have represented a significant improvement in the treatment of some engineering problems. However, in some practical applications with moving boundaries and discontinuities or in cases of domains with very complicated geometry, difficulties arise when applying conventional numerical techniques which require the discretization of the whole domain. In fact, since an efficient grid is needed, the mesh generation process frequently becomes the bottle neck.

Hereby, a family of numerical methods where explicit element meshes are unnecessary (meshless methods) has been proposed [1] in the last years. The solution domain is formed by a set of nodal points. Every point has an associated subdomain including its closest points and so, a local approximation is achieved in each node. Thus, for every central node or “star node” this approximation is built with the information provided by its subdomain points.

Although the first meshless methods were developed years ago, the more important theoretical research works about them and their application in computational mechanics have been recently performed [1]. Other authors [2,3] have proposed a method which combines a moving least square approximation with a Point Collocation approach to compute the integral terms and then no auxiliary grid is required. Furthermore, different interpolants can be derived depending on the weighting function. And so, it can be fixed within each subdomain as in the Diffuse Least Square Method (DLS) or it can depend on the point where the approximated value is to be computed (Moving Least Square Method, MLS). A modified Moving Least Square method combined with a Point Collocation scheme is implemented in this paper for solving a problem in the electrical engineering field.

## Modified Moving Least Square Approximations

Moving Least Square approximations consist of a local weighted least square fitting, valid on a small neighbourhood of a point and based on the information provided by its  $n$  closest points (subdomain  $\Omega_k$ ). The local character of the approximation comes from a moving weighting function which takes its maximum value at this point and vanishes outside a surrounding region [2].

The proper definition of the approximation at any point implies that all subdomains  $\Omega_k$  cover all the interpolation domain. Hereby, these subdomains must overlap and the common areas have to include enough nodal points in order to ensure the convergence of the method. Thus, for a given nodal point the selection of the points belonging to its subdomain has been performed according to an effective technique based on the ‘‘four quadrants’’ criterium [2].

Let  $\Omega_k$  be the interpolation domain of a function  $u(\mathbf{x})$ . A local approximation to  $u(\mathbf{x})$  can be derived in the form,

$$u(\mathbf{x}) \cong \hat{u}(\mathbf{x}) = \sum_{i=1}^m p_i(\mathbf{x})\alpha_i = \mathbf{p}^t(\mathbf{x})\boldsymbol{\alpha} \quad (1)$$

where  $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_m]^t$  is a set of unknown coefficients and  $\mathbf{p}(\mathbf{x})$  contains a base of interpolating functions (monomial terms, generally) which order is  $m$  [3].

In the present paper, the base interpolating functions are normalized within each subdomain  $\Omega_k$ . Thus, we can define the normalized coordinates [4] within a subdomain dividing by the maximum distance  $d$  between the star point  $i$  of the subdomain and the surrounding points as,

$$\boldsymbol{\xi}(\mathbf{x}) = \left[ \frac{x - x_i}{d}, \frac{y - y_i}{d}, \frac{z - z_i}{d} \right] \quad (2)$$

On the other hand, function  $u(\mathbf{x})$  can be sampled in the  $n$  points belonging to  $\Omega_k$  in the form,

$$\mathbf{u}^h = \begin{pmatrix} u_1^h \\ \vdots \\ u_n^h \end{pmatrix} \cong \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{p}_1^t \\ \vdots \\ \mathbf{p}_n^t \end{pmatrix} \boldsymbol{\alpha} = \mathbf{S}\boldsymbol{\alpha} \quad (3)$$

being  $u_j^h$  the values of unknown function evaluated in nodal points of subdomain  $\Omega_k$  ( $u_j^h = u(\mathbf{x}_j)$ ,  $j = 1, \dots, n$ ),  $\hat{u}_j = \hat{u}(\mathbf{x}_j)$  their approximated values, and  $\mathbf{p}_j$  contains the normalized base interpolating functions evaluated in  $\boldsymbol{\xi}_j$  (where  $\boldsymbol{\xi}_j = \boldsymbol{\xi}(\mathbf{x}_j)$ ).

In general, the number of subdomain points  $n$  is greater than the order  $m$  of the polynomials base, so  $\mathbf{S}$  is a rectangular matrix and the approximation cannot fit all the  $u_j^h$  values. However, approximated values  $\hat{u}(\mathbf{x})$  can be determined by minimizing the weighted sum of the square differences between the exact value  $u_j^h$  and the approximation  $\hat{u}(\mathbf{x}_j)$  at each nodal point  $\mathbf{x}_j$  belonging to the domain of node  $\mathbf{x}_k$  [3]. The weighting function is usually built in such a way that it equals unity in point  $\mathbf{x}_k$  and vanishes outside domain  $\Omega_k$ , ensuring that the number of  $n$  points within  $\Omega_k$  is equal or greater than order  $m$ .

Thus, the functional to be minimized with respect to  $\boldsymbol{\alpha}$  results in

$$J(\mathbf{x}_k) = \sum_{j=1}^n \omega_k(\mathbf{x}_j, \mathbf{x}_k) (u_j^h - \hat{u}(\mathbf{x}_j))^2 \quad (4)$$

where  $\omega_k(\mathbf{x}_j, \mathbf{x}_k)$  is the weighting function computed in  $\mathbf{x}_j$ , which shape and span depend on the arbitrary position  $\mathbf{x}_k$ .

The performance of this process yields to [5],

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}^h \quad \mathbf{A}(\mathbf{x}) = \mathbf{P}\mathbf{W}(\mathbf{x})\mathbf{P}^t \quad \mathbf{B}(\mathbf{x}) = \mathbf{P}\mathbf{W}(\mathbf{x}) \quad (5)$$

being auxiliary matrices  $\mathbf{P}$  and  $\mathbf{W}(\mathbf{x})$ :

$$\mathbf{P} = [\mathbf{p}(\boldsymbol{\xi}_1) \quad \dots \quad \mathbf{p}(\boldsymbol{\xi}_n)] \quad \mathbf{W}(\mathbf{x}) = \text{diag} [\omega_k(\mathbf{x}_j, \mathbf{x})], \quad j = 1, \dots, n \quad (6)$$

Substitution of (5) in (1) allows to obtain an approximation to function  $u(\mathbf{x})$  in  $\Omega_k$  in the form,

$$u(\mathbf{x}) \cong \hat{u}(\mathbf{x}) = \mathbf{p}^t(\boldsymbol{\xi})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}^h \quad (7)$$

and therefore, one can define the shape functions in  $\mathbf{x}$  as [4],

$$\mathbf{N}^t(\mathbf{x}) = \mathbf{p}^t(\boldsymbol{\xi})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \quad (8)$$

It must be taken into account that the local values of the approximating function do not fit the nodal unknown values,  $\hat{u}(\mathbf{x}_j) \neq u_j^h$ , due to the least square character of the approximation. It must also be pointed out that if  $n = m$ , the FEM type approximation is recovered and no effect of weighting is presented [3]. Besides, if the weighting function is constant and equals the unity, the standard least square method is reproduced.

An important ingredient in the performance of this kind of meshless methods is the weighting function. As mentioned, the local character of the approximation comes from a moving weighting function which takes its maximum value at the point  $\mathbf{x}_k$  and vanishes outside its subdomain ( $\omega_k(\mathbf{x}_i, \mathbf{x}_k) = 1$ ;  $\omega_k(\mathbf{x}_i, \mathbf{x}) \neq 0$ ,  $\mathbf{x} \in \Omega_k$ ;  $\omega_k(\mathbf{x}_i, \mathbf{x}) = 0$ ,  $\mathbf{x} \notin \Omega_k$ ). So, the weight function has compact support. Furthermore, function  $\omega_k(\mathbf{x}_i, \mathbf{x}_k)$  must be smooth, continuous and differentiable in  $\Omega_k$  [1]. However, the definition of a different weighting function for every interpolating point  $\mathbf{x}_k$  is very difficult, since it presents an infinite number of possibilities. To overcome this problem, the weighting functions are defined for every nodal point  $\mathbf{x}_i$  and evaluated in the arbitrary point  $\mathbf{x}_k$ . Therefore,  $\omega_k(\mathbf{x}_i, \mathbf{x}_k)$  may be substituted by  $\omega_i(\mathbf{x}_i, \mathbf{x}_k)$  [2]. In the examples presented in this paper, the truncated gaussian distribution has been used [5].

On the other hand, application of a weighted residual method allows to obtain a variational form of the problem to be solved. As regards the statement of a discretized set of equations, different formulations can be derived according to the test functions. Thus, in this paper a point collocation method (Dirac deltas as test functions) has been implemented in order to take advantage of the meshless character of the approximation. Nevertheless, other approaches based on integral methods have been proposed although they require some kind of auxiliary grid to evaluate the resulting integrals [1].

The validation of this moving least square method with a point collocation approach has been performed for different numerical tests [5]. Furthermore, Oñate *et al.* [2,3] have applied this method with promising results to solve problems in fluid mechanics. As we present in the following section, the meshless character of this formulation may represent an improvement in the treatment of some problems related to the grounding analysis in electrical engineering applications.

## Application to grounding analysis

In order to guarantee the integrity of equipments and the continuity of the service under fault conditions —providing means to carry and dissipate electrical currents into the ground—, and to safeguard that persons working or walking in the surroundings of the grounded installation are not exposed to dangerous electrical shocks, a safe grounding system is required. Therefore, the equivalent electrical resistance of the system must be low enough to assure that fault currents dissipate mainly through the grounding into the earth, meanwhile maximum potential differences between close points on the earth surface must be kept under certain tolerances [6].

Maxwell's Electromagnetic Theory [7] allows to model physical phenomena underlying fault currents dissipation into the earth. Thus, constraining the analysis to the obtention of the electrokinetic steady-state response, and neglecting the inner resistivity of the earthing electrode, the 3D problem associated to an electrical current derivation can be written as

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma} &= 0, & \boldsymbol{\sigma} &= -\underline{\underline{\boldsymbol{\gamma}}}\operatorname{grad}V \text{ in } E; \\ \boldsymbol{\sigma}^t\mathbf{n}_E &= 0 \text{ in } \Gamma_E; & V &= V_\Gamma \text{ in } \bar{\Gamma}; & V &\longrightarrow 0, \text{ if } |\mathbf{x}| \rightarrow \infty; \end{aligned} \quad (9)$$

where  $E$  is the earth,  $\underline{\underline{\boldsymbol{\gamma}}}$  its conductivity tensor,  $\Gamma_E$  the earth surface,  $\mathbf{n}_E$  its normal exterior unit field and  $\Gamma$  the electrode surface [8]. Thus, when the electrode attains a voltage  $V_\Gamma$  (Ground Potential Rise or GPR) relative to a distant grounding point, the solution of this problem gives the potential  $V$  and the current density  $\boldsymbol{\sigma}$  at an arbitrary point  $\mathbf{x}$ . Since  $V$  and  $\boldsymbol{\sigma}$  are proportional to  $V_\Gamma$ , assumption  $V_\Gamma = 1$  is not restrictive at all and it will be used from now on.

It must be pointed out that models describing all variations of soil conductivity in the surroundings of a substation are unaffordable from the technical and economical point of view. A more practical approach to situations where conductivity is not markedly uniform with depth consists of considering the earth stratified in a number of horizontal layers. Therefore, it is widely accepted that two layer earth models should be sufficient to obtain good designs of earthing systems in most practical cases. Thus, conductivity tensor  $\underline{\underline{\boldsymbol{\gamma}}}$  is substituted in the upper layer  $E_1$  by the measured apparent scalar conductivity  $\gamma_1$ . In the same way, the scalar conductivity associated with the lower layer  $E_2$  is  $\gamma_2$ .

In the last three decades, most of the methods proposed for grounding design and computation were founded on semiempirical works or on the basis of intuitive ideas, such as superposition of punctual current sources and error averaging. Although these techniques represented a significant improvement in the area of earthing analysis, some difficulties were reported. For this reason, the authors have recently developed numerical formulations based on the Boundary Element Method for the analysis of grounding grids embedded in uniform soils and in stratified soils. This approach was implemented in a CAD system for grounding analysis of electrical substations and several real problems have been successfully solved in the cases of uniform and two layer soil models [8,9].

However, the application of the Boundary Element Method in the cases of heterogeneous or multi-layer soils implies a considerable computational effort. On the other hand, the specific geometry of earthing systems in practice (a grid of interconnected buried conductors) precludes the use of standard numerical techniques [8], since the obtention of sufficiently accurate results would imply unacceptable computing efforts because the discretization of the domain (the earth) is required. For these reasons, we have turned our attention to investigate the applicability of numerical formulations based on meshless methods [4] for the solution of potential problems in the electrical engineering field.

The example presented in this paper consists of a toroidal electrode horizontally buried to a depth of 7 m. The interior diameter of the ring is 20 m and the electrode diameter is 3 m. The upper layer depth is 14 m. The scalar conductivity associated with the lower layer  $\gamma_2$ , is four times the one

corresponding to the upper layer  $\gamma_1$  ( $\gamma_2 = 4 \gamma_1$ ). Due to the axial symmetry of the problem, solution can be obtained by using a 2D model. This case has been solved with 3019 points obtained by means of the program GEN4U [10]. The base interpolating functions used are linear and all subdomains contain at least five points. Figure 1 shows the nodal point distribution, the contour lines and the potential distribution around the electrode in two cases: assuming the hypothesis of homogeneous and isotropic soil or considering a two-layer model. These numerical results agree significantly with those obtained by using a very dense point distribution and with results of a boundary element program.

## Conclusions

In this paper a modified moving least square interpolation method, combined with a point collocation scheme, is presented. This formulation is applied for the solution of a problem in grounding analysis, consisting of toroidal electrodes. First results obtained for different point distributions, even with a two-layer soil model, are very promising and require a reasonable computational cost.

The proposed method does not require any kind of mesh and therefore, it is possible to derive the solution to boundary value problems from a finite set of points of the domain. Thus, the meshless character of these methods may represent an improvement in the computational analysis of grounding grids in the electrical engineering field in cases in which discretization is unaffordable in practice. Nevertheless, convergence studies and further analysis of mathematical and numerical aspects must be done in order to assess the practical feasibility of this approach.

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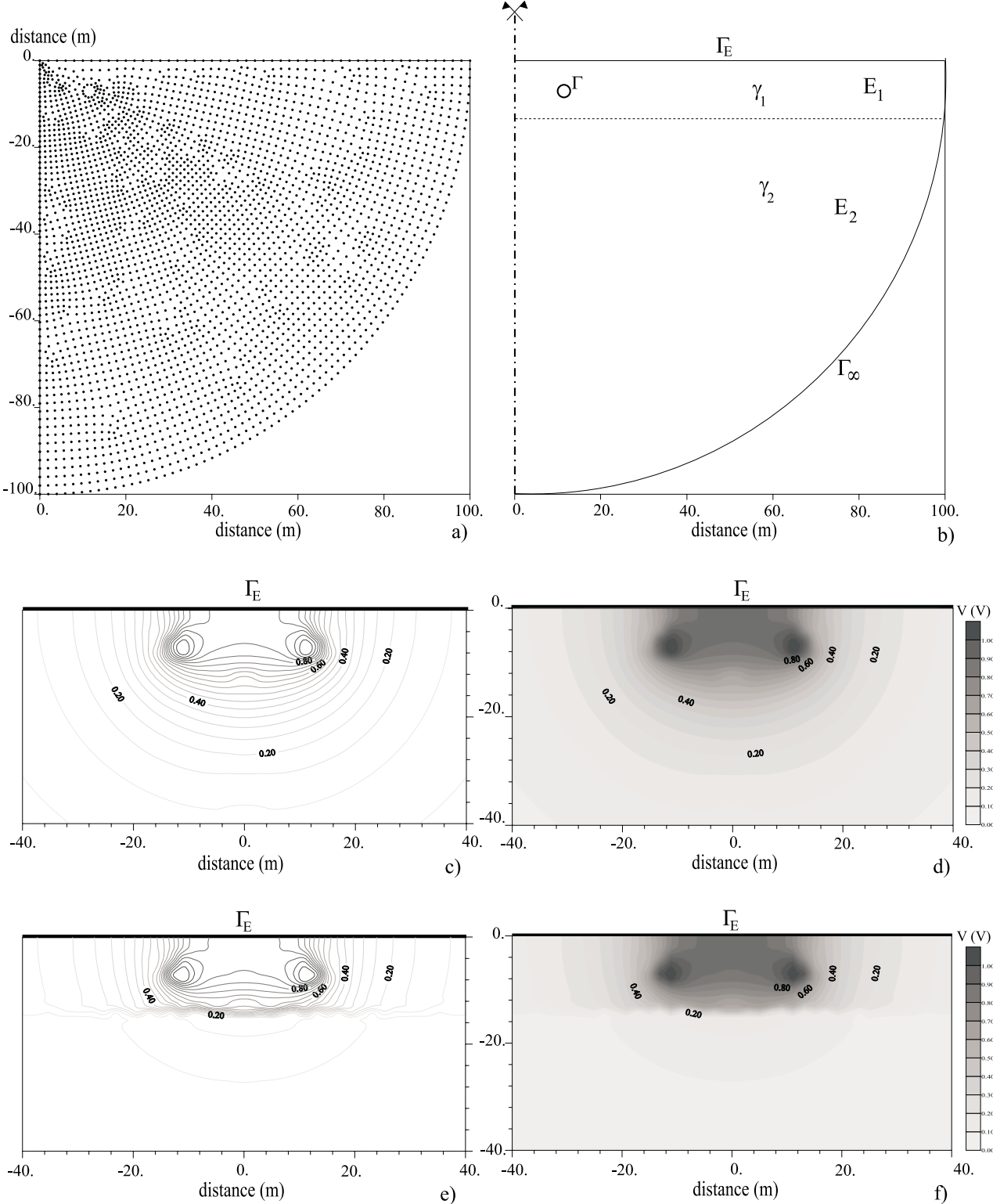


Figure 1.- Toroidal electrode buried to a depth of 7 m: a) Distribution of nodal points (npoin=3019), b) Domain solution scheme, c) Contour lines around the electrode, considering the soil homogeneous and isotropic, d) Potential distribution around the electrode, considering the soil as homogeneous and isotropic, e) Contour lines around the electrode assuming a two-layer model ( $\gamma_2 / \gamma_1 = 4$ ), f) Potential distribution around the electrode assuming a two-layer model ( $\gamma_2 / \gamma_1 = 4$ ).