Method for Pricing Renewable Energy Certificates

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Abstract: In this work we present one valuation method for Renewable Energy Certificates (RECs). Starting from a system of FBSDEs and using Ito lemma, we propose a mathematical model based on a semilinear PDE with two stochastic factors. The main novelty comes from the use of the Bermúdez-Moreno algorithm to deal the non-linear convective term in the PDE. This duality algorithm is based on the Yosida regularization of non-linear maximal monotone operators. The resulting linear problem is discretized by using a characteristics method combined with a second order implicit finite differences scheme. We show illustrative results of the performance of the proposed model and the numerical method.

1 Introduction

In recent years, several governments have developed environmental policies for promoting renewable energy sources. Many countries have already adopted Renewable Portfolio Standards (RPSs) and trading of renewable energy certificates (RECs). Markets for tradable RECs can be used to encourage the growth of a particular type of renewable energy, as in the case of Solar Renewable Energy Certificates (SRECs) (see M. Coulon, J. Khazaei, W. B. Powell (2015) and A. Shrivats, S. Jaimungal (2020)). In the present work, assuming that the price of the certificate depends on two stochastic factors which are the accumulated green certificates and the renewable energy production rate, we present the PDE model that governs the valuation of such financial instruments and we propose an appropriate numerical method for its solution.

For the numerical solution of the nonlinear PDE problem, we first apply the Bermúdez-Moreno algorithm proposed in A. Bermúdez, C. Moreno (1981) to deal with the nonlinear convective term. This duality method is based on the approximation of a nonlinear maximal monotone operator by means of its Yosida regularization. In order to solve the obtained linearized problem, we use numerical methods based on semi-Lagrangian schemes in the direction without diffusion while an implicit second order finite differences scheme is applied in the direction with diffusion term. Finally, several numerical examples are presented to illustrate the good performance of the method and model.

2 Mathematical modelling

In what follows, for a fixed time horizon representing the end of the compliance period $T > 0$, we assume that the source of randomness in the model is given by one-dimensional Wiener process $W = (W_t)_{0 \leqslant t \leqslant T}$. We assume that this Wiener process is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and it is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geqslant 0\}$.

Assuming that the price of a green certificate depends on two stochastic factors, which are the renewable generation rate, G_t, and the number of accumulated green certificates, B_t, we aim to address the model for pricing a REC.

First, let us denote by $\tilde{G}_t = \ln(G_t)$ the Ornstein-Uhlenbeck (OU) process, which satisfies the following stochastic differential equation (SDE):

$$d\tilde{G}_t = \alpha_g \left(f(t) + \frac{\beta_g}{\alpha_g} P_t - \tilde{G}_t \right) dt + \sigma_g dW_t, \tag{32.1}$$

assuming that $G_{t_0} = g_0$, and where α_g is the mean reversion speed of the process, P_t is the certificate price, β_g is the parameter which controls the level of immediate feedback from the price of the certificate, σ_g is the volatility and W_t is the Wiener process governing the stochastic part of the equation. Moreover, taking into account that weather conditions strongly affect the production of energy in the renewable power generation, we introduce a deterministic function $f(t)$ representing the seasonality effect. A common choice is to use a combination of trigonometric functions as it is proposed in M. Coulon, J. Khazaei, W. B. Powell (2015). Furthermore, in the OU process (32.1), the mean reversion level is linear in P_t plus seasonality. Now, the production of renewable energy G_t can be written as

$$G_t = \exp\left(\tilde{G}_t\right).$$

Secondly, we introduce the dynamics of the number of accumulated renewable energy certificates, B_t,

$$dB_t = G_t dt.$$

Note that B_t is non-negative and non-decreasing, and we assume that $B_{t_0} = 0$, where t_0 is the beginning of the compliance period.

3 Statement of the pricing PDE

If we denote by $P = P(t, \tilde{G}, B)$ the price of the renewable energy certificate at time t, by using a dynamic hedging technique and applying Itô's Lemma (see K. Itô (1951)), we can derive the following nonlinear PDE whose solution is the price of the REC:

$$\mathcal{L}[P] = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma_g^2 \frac{\partial^2 P}{\partial \tilde{G}^2} + \alpha_g \left(f(t) - \tilde{G} \right) \frac{\partial P}{\partial \tilde{G}} + \beta_g P \frac{\partial P}{\partial \tilde{G}} + \exp(\tilde{G}) \frac{\partial P}{\partial B} - rP = 0, \tag{32.2}$$

where r is the constant risk free interest rate.

Assuming that the number of life years of the certificate is denoted by γ and the maturity of the certificate is T, the PDE problem associated to (32.2) is initially formulated in the unbounded domain $(T - \gamma, T) \times (-\infty, \infty) \times (0, \infty)$.

For the particular case of one single period (i.e. one year, $\gamma = 1$), the payoff at the expiry date of the certificate, T, is a decreasing function in the number of accumulated green certificates at maturity, B, and depends on the requirement on the percentage of energy obtained from renewables at maturity, R_T. Thus, in order to state the PDE problem, Equation (32.2) is completed with the final condition

$$P(T, \tilde{G}, B) = \pi_T \mathbb{1}_{\{B < R_T\}}, \tag{32.3}$$

where π_T is the penalty amount π at time T.

Moreover, there exists the possibility of extending the problem to multiple periods. In that case, at each compliance date, T^i, for $i = 1, ..., \gamma - 1$, a jump condition must be applied. Thus, when the obligation is set the value of the certificate is given by

$$P(T^i, \tilde{G}, B) = \max\left(\pi_{T^i} \mathbb{1}_{\{B < R_i\}}, P\left(T^i_+, \tilde{G}, \max(0, B - R_i) \right) \right). \tag{32.4}$$

Note that T^i corresponds to the end of the i-th life year of the certificate and R_i is the requirement at that time.

The study of the existence and uniqueness of solution for the nonlinear PDE problems defined by (32.2)-(32.3) or by (32.2)-(32.4) remains as an open problem as it is mentioned in M. A. Baamonde-Seoane, M. C. Calvo-Garrido, M. Coulon, C. Vázquez (2021) and M.A. Baamonde-Seoane, M.C. Calvo-Garrido, C. Vázquez (2023).

4 Numerical techniques

4.1 The duality algorithm

As previously pointed out, the PDE problem (32.2) presents a non-linear convective term. One possibility to deal with this non-linearity is based on the Bermúdez-Moreno algorithm involving the Yosida regularization of non-linear maximal monotone operators (see A. Bermúdez, C. Moreno (1981)).

For this purpose, let us introduce the maximal monotone operator m, defined by

$$m(P) = \begin{cases} 0, & \text{if } P < 0 \\ P^2, & \text{if } P \geqslant 0, \end{cases}$$

so that

$$P\frac{\partial P}{\partial \tilde{G}} \approx \frac{1}{2}\frac{\partial m(P)}{\partial \tilde{G}}.$$

Therefore, the equation (32.2) can be written in the form:

$$\frac{\partial P}{\partial t} + \frac{\sigma_g^2}{2}\frac{\partial^2 P}{\partial \tilde{G}^2} + \alpha_g \left(f(t) - \tilde{G}\right)\frac{\partial P}{\partial \tilde{G}} + \frac{\beta_g}{2}\frac{\partial m(P)}{\partial \tilde{G}} + \exp(\tilde{G})\frac{\partial P}{\partial B} - rP = 0. \tag{32.5}$$

Following the duality technique introduced in A. Bermúdez, C. Moreno (1981), in terms of the constant parameter $\omega > 0$, we introduce the new additional unknown θ, defined by

$$\theta = (m - I\omega)(P)$$

where I denotes the identity operator.

Next, by using the Bermúdez-Moreno lemma, we have the equivalence

$$\theta = m(P) - \omega P \Leftrightarrow \theta = m_\lambda^\omega(P + \lambda\theta), \tag{32.6}$$

where m_λ^ω denotes the Yosida approximation of $m - I\omega$ with parameter λ. For convergence purposes, we impose the relation $2\lambda\omega = 1$ in the choice of the parameters λ and ω. Under this constraint, the Yosida approximation can be analytically computed and is given by

$$m_\lambda^\omega\left(P + \frac{\theta}{2\omega}\right) = \begin{cases} -\theta - 2\omega P, & \text{if } P + \frac{\theta}{2\omega} \leqslant 0, \\ \theta + 2\omega P + \omega^2 - \omega\sqrt{4\theta + 8\omega P + \omega^2}, & \text{if } P + \frac{\theta}{2\omega} \geqslant 0. \end{cases}$$

Next, if we introduce the linear differential operator

$$\mathcal{L}[P] = \frac{\partial P}{\partial t} + \frac{\sigma_g^2}{2}\frac{\partial^2 P}{\partial \tilde{G}^2} + \alpha_g \left(f(t) - \tilde{G}\right)\frac{\partial P}{\partial \tilde{G}} + \frac{\beta_g \omega}{2}\frac{\partial P}{\partial \tilde{G}} + \exp(\tilde{G})\frac{\partial P}{\partial B} - rP, \tag{32.7}$$

the equation (32.5) can be rewritten in the form:

$$\mathcal{L}[P] = -\frac{\beta_g}{2}\frac{\partial\theta}{\partial\tilde{G}}. \tag{32.8}$$

Moreover, from the equivalence stated in (32.6), the equation (32.8) is coupled with the following non-linear equation:

$$\theta = m_\lambda^\omega(P + \lambda\omega).$$

4.2 Formulation of the PDE problem in a bounded domain

In order to apply numerical discretization using finite differences, it is necessary to define the bounded domain of the PDE problem.

Let $\Omega = (T-\gamma, T) \times \mathbb{R} \times (0, +\infty)$ be the initial unbounded domain. Moreover, let $\bar{\Omega} = (T-\gamma, T) \times (0, \hat{b}) \times (-\bar{g}, \bar{g})$ be the truncated bounded domain where $\hat{b}$ and $\bar{g}$ are real numbers, which are influenced by the requirement of the payoff function and the jump conditions at compliance dates. Now, we introduce the changes $\hat{B} = \frac{B}{\hat{b}}$ and $\hat{G} = \frac{\tilde{G}}{\hat{g}}$ with $\hat{g} = 2\bar{g}$, so the bounded spatial domain $\Omega^* = (0,1) \times (0,1)$ in the new variables $(t, \hat{B}, \hat{G})$, whose boundary can be decomposed as $\Gamma = \bigcup_{i=1}^{2}(\Gamma_i^- \cup \Gamma_i^+)$ where

$$\Gamma_i^- = \{(y_1, y_2) \in \Gamma |\, y_i = 0\},\ \ \Gamma_i^+ = \{(y_1, y_2) \in \Gamma |\, y_i = 1\},\ i = 1, 2.$$

Next, as in M. A. Baamonde-Seoane, M. C. Calvo-Garrido, M. Coulon, C. Vázquez (2021), we follow the methodology introduced by O. A. Oleinik (1973) to obtain the boundaries where it is necessary to impose boundary conditions. On those boundaries, we will impose homogeneous Neumann boundary conditions.

4.3 Discretization of the PDE

In order to choose an appropriate time discretization scheme for the PDE (32.8), we note that the linear differential operator (32.7) is degenarate. Thus, we follow the idea first proposed in Y. d'Halluin, P. A. Forsyth, G. Labahn (2005), which consists of choosing a semi-Lagrangian method in the direction without diffusion combined with a Crank-Nicolson finite differences scheme in the direction with diffusion.

For the time discretization, we first consider the change of time variable $\tau = T - t$, where τ represents the time to maturity. Therefore, equation (32.8) can be equivalently written in the domain as follows

$$\frac{\mathrm{D}P}{\mathrm{D}\tau} - \mathcal{A}P = 0, \tag{32.9}$$

where

$$\frac{\mathrm{D}P}{\mathrm{D}\tau} = \frac{\partial P}{\partial \tau} - \hat{b}\exp(\hat{G}\hat{g} - \bar{g})\frac{\partial P}{\partial \hat{B}},$$

$$\mathcal{A}P = \frac{\hat{g}^2\sigma_g^2}{2}\frac{\partial^2 P}{\partial \hat{G}^2} + \hat{g}\alpha_g\left(f(T-\tau) - \left(\hat{G}\hat{g} - \bar{g}\right) + \frac{\beta_g\omega}{2\alpha_g}\right)\frac{\partial P}{\partial \hat{G}} + \frac{\hat{g}\beta}{2}\frac{\partial\theta}{\partial\hat{G}} - rP.$$

Next, we introduce the approximation for the material derivative:

$$\frac{\mathrm{D}P}{\mathrm{D}\tau} \approx \frac{P\left(\tau^{n+1}, \hat{B}, \hat{G}\right) - P\left(\tau^n, \chi^n\left(\hat{B}, \hat{G}\right), \hat{G}\right)}{\Delta\tau},$$

where $\chi^n\left(\hat{B},\hat{G}\right)=\chi(\tau^n)=\hat{B}+\Delta\tau\hat{b}\exp(\hat{G}\hat{g}-\bar{g})$ is the solution for $n=0,1,\ldots,N_T-1$ and represents the position at time τ^n of the point placed at $(\hat{B},\hat{G})$ at time τ^{n+1} and moving according to the velocity field $v=-\hat{b}\exp(\hat{G}\hat{g}-\bar{g})$.

By using a Crank-Nicolson scheme ($\hat{\theta}=0.5$ in the so called $\hat{\theta}$-method) for the second order differential term $\mathcal{A}P$ in equation (32.9), we obtain:

$$\begin{aligned}\frac{P^{n+1}-P^n\circ\chi^n}{\Delta\tau}-\frac{\hat{\theta}\hat{g}^2\sigma_g^2}{2}\frac{\partial^2P^{n+1}}{\partial\hat{G}^2}-\frac{(1-\hat{\theta})\hat{g}^2\sigma_g^2}{2}\frac{\partial^2\left(P^n\circ\chi^n\right)}{\partial\hat{G}^2}\\ -\hat{\theta}\hat{g}\alpha_g\left(f(T-\tau)-\left(\hat{G}\hat{g}-\bar{g}\right)+\frac{\beta_g\omega}{2\alpha_g}\right)\frac{\partial P^{n+1}}{\partial\hat{G}}\\ -(1-\hat{\theta})\hat{g}\alpha_g\left(f(T-\tau)-\left(\hat{G}\hat{g}-\bar{g}\right)+\frac{\beta_g\omega}{2\alpha_g}\right)\frac{\partial\left(P^n\circ\chi^n\right)}{\partial\hat{G}}\\ +r\hat{\theta}P^{n+1}+r(1-\hat{\theta})\left(P^n\circ\chi^n\right)=\frac{\hat{\theta}\hat{g}\beta_g}{2}\frac{\partial\theta^{n+1}}{\partial\hat{G}}+\frac{(1-\hat{\theta})\hat{g}\beta_g}{2}\frac{\partial\theta^n}{\partial\hat{G}}.\end{aligned}\tag{32.10}$$

At each time step, the equation (32.10) is coupled with the following non-linear relation between P^{n+1} and θ^{n+1}:

$$\theta^{n+1}=m_\lambda^\omega(P^{n+1}+\lambda\omega^{n+1}).\tag{32.11}$$

Next, we propose a fixed point algorithm to approximate the solution of the non-linear problem (32.10)-(32.11).

At each fixed point iteration, the full discretization of problem can be written as follows:

$$\begin{aligned}\frac{P_{i,j}^{n+1,k+1}-P_{i,j}^n\circ\chi^n}{\Delta\tau}&-\frac{\hat{\theta}\hat{g}^2\sigma_g^2}{2}\left(\frac{P_{i,j+1}^{n+1,k+1}-2P_{i,j}^{n+1,k+1}+P_{i,j-1}^{n+1,k+1}}{(\Delta\hat{G})^2}\right)\\ &-\frac{(1-\hat{\theta})\hat{g}^2\sigma_g^2}{2}\left(\frac{P_{\chi^n,j+1}^n-2P_{\chi^n,j}^n+P_{\chi^n,j-1}^n}{(\Delta\hat{G})^2}\right)\\ &-\hat{\theta}\hat{g}\alpha_g\left(f(T-\tau^{n+1})-\left(\hat{G}_j\hat{g}-\bar{g}\right)+\frac{\beta_g\omega}{2\alpha_g}\right)\left(\frac{P_{i,j+1}^{n+1,k+1}-P_{i,j-1}^{n+1,k+1}}{2\Delta\hat{G}}\right)\\ &-(1-\hat{\theta})\hat{g}\alpha_g\left(f(T-\tau^n)-\left(\hat{G}_j\hat{g}-\bar{g}\right)+\frac{\beta_g\omega}{2\alpha_g}\right)\left(\frac{P_{\chi^n,j+1}^n-P_{\chi^n,j-1}^n}{2\Delta\hat{G}}\right)\\ &-\frac{\hat{\theta}\hat{g}\beta_g}{2}\left(\frac{\theta_{i,j+1}^{n+1,k}-\theta_{i,j-1}^{n+1,k}}{2\Delta\hat{G}}\right)-\frac{(1-\hat{\theta})\hat{g}\beta_g}{2}\left(\frac{\theta_{i,j+1}^n-\theta_{i,j-1}^n}{2\Delta\hat{G}}\right)\\ &+r\hat{\theta}P_{i,j}^{n+1}+r(1-\hat{\theta})P_{\chi^n,j}^n=0.\end{aligned}$$

where $\hat{\theta}=0.5$, $P_{r,s}^{l,m}=P^m(\tau^l,\hat{B}_r,\hat{G}_s)$, $P_{\chi^l,s}^{l,m}=P^m(\tau^l,\chi^l,\hat{G}_s)$ and $\theta_{r,s}^{l,m}=\theta^m(\tau^l,\hat{B}_r,\hat{G}_s)$.

5 Numerical examples

5.1 Academic test

As a sanity check of the code and numerical methods, in the first example we show an academic test with known analytical solution. For this purpose, we consider the following non homogeneous non-linear PDE:

$$\mathcal{L}[P] = h,$$

where the differential operator $\mathcal{L}$ is defined by (32.2) and h is given by

$$h(t, B, \tilde{G}) = \Big[-B\tilde{G} + \frac{1}{2}\sigma_g^2 t^2 B^2 - tB\alpha_g \Big(f(t) + \frac{\beta_g}{\alpha_g} \exp\big((T-t)B\tilde{G}\big) - \tilde{G} \Big) - \exp(\tilde{G})t\tilde{G} - r \Big] P(t, B, \tilde{G}), \qquad (32.12)$$

with $P(t, B, \tilde{G}) = \exp\big((T-t)B\tilde{G}\big)$ is the analytical solution of the PDE (32.12).

By choosing $\hat{b} = 1$ and $\tilde{g} = 0.5$ for the change of variables, we pose the PDE problem in the bounded domain $\tilde{\Omega} = [0, 1] \times [0, 1] \times [0, 1]$ with Dirichlet boundary conditions on Γ_1^+, Γ_2^- and Γ_2^+ that are given by the evaluation of the solution at the corresponding boundaries.

Table 1: Parameters in the PDE model for the academic test.

Parameter	T	γ	α_g	β_g	σ_g	r	ω	ϵ
Value	1	1	2	1.27×10^{-3}	0.1863	0.02	2	10^{-5}

In this academic test we do not include the seasonality effect, so that we take $f = 0$. Parameters in the PDE are collected in Table 1 and mostly taken from M. Coulon, J. Khazaei, W. B. Powell (2015).

Table 2: Relative errors and empirical convergence order in academic test.

Time steps	**Space steps**	**Error**	**R**	**Order**
40	32	0.0108651	-	-
80	64	0.0055962	1.9415	0.9572
160	128	0.0028748	1.9466	0.9610
320	256	0.0014710	1.9543	0.9667
640	512	0.0007474	1.9681	0.9768
1280	1024	0.0003780	1.9771	0.9834
2560	2048	0.0001906	1.9832	0.9879

Table 2 shows the errors, convergence ratio and empirical order of convergence with different time and spatial discretizations computed as in Y. d'Halluin, P. A. Forsyth, G. Labahn (2005). Thus, we can conclude that a first order convergence is achieved.

5.2 Real case

In this example, we have used the real New Jersey market data presented for SREC markets, i.e., markets for solar renewable energy certificates, in M. Coulon, J. Khazaei, W. B. Powell (2015). In this market, the energy year refers to the 12-month period ending on May 31. We assume that the maturity is $T = 13$, i.e., May 31, 2013. For convenience, the energy year 2013 is defined as the time interval $(12, 13]$. Thus, we consider the requirement schedule 2010-2013, i.e., $\gamma = 3$, with initial year $t = t_0 = T - \gamma = 10$ and the first compliance date at $t = 11$ (at the end of the year 2011) which corresponds to $\tau = 2$. We consider the rest of the PDE parameters as those in the Table 2. Moreover, the seasonality function is chosen as $f(s) = -0.1209\sin(4\pi s) + 0.0900\cos(4\pi s) + 0.2151\sin(2\pi s) + 0.3859\cos(2\pi s)$ and represents the influence of weather conditions.

The requirement, R_i, and penalty, π_{Ti}, values at the end of each year for $i = 1, 2, 3$ are indicated in M. A. Baamonde-Seoane, M. C. Calvo-Garrido, M. Coulon, C. Vázquez (2021), M.A. Baamonde-Seoane, M.C. Calvo-Garrido, C. Vázquez (2023) and M. A. Baamonde-Seoane, M. C. Calvo-Garrido, C. Vázquez (2023). For the numerical methods, we start by choosing $\hat{b} = 7 \times 10^5$ and $\bar{g} = \ln(7 \times 10^5)$. Concerning the discretization parameters, we consider $\Delta\tau = \frac{1}{1200}$, and a uniform mesh with $\Delta\hat{B} = \Delta\hat{G} = 1/32$.

On the one hand, in Figure 1 we can observe that when the accumulated supply (B) and renewable energy (G) are low, the price of the certificate tends to the penalty value. On the other hand, when the value of both state variables increases, the price of the certificate decreases.

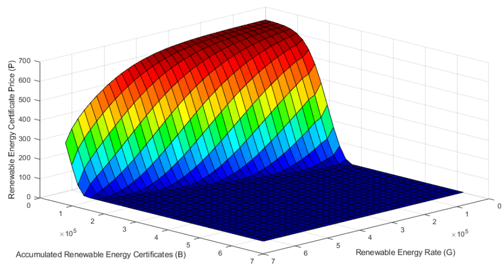

Figure 1: Renewable energy certificate price at time $t = T - 2/3$ in the real test.

In Figure 2 we can observe that, for values of accumulated certificates nearer to requirement, low values of generation rate are linked to certificate prices equal to the penalty amount. Additionally, for lower values of banked certificates, high enough values of generate rate are associated with prices equal to the penalty.

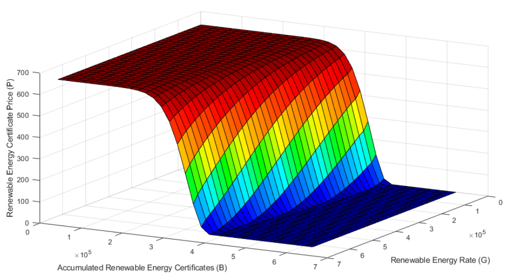

Figure 2: Renewable energy certificate price at time $t = T - 1/3$ in the real test.

Finally, in Figure 3 we represent the price of the certificate versus the number of accumulated renewable energy certificates for different times, obtaining some cross-sectional plots. At $t = T$, the REC price matches the penalty if the requirement is not met, otherwise the price is zero. Then, as we move backwards in time, the curves take lower values and move to the left, due to the diffusion of the final value. This behaviour can be also observed in M.A. Baamonde-Seoane, M.C. Calvo-Garrido, C. Vázquez (2023) and M. Coulon, J. Khazaei, W. B. Powell (2015).

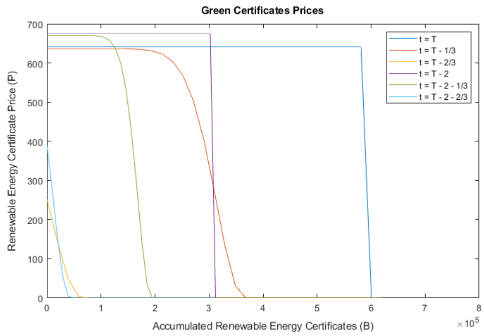

Figure 3: Price curves for different times in the real test.

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