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ANALYTICAL INTEGRATION TECHNIQUES FOR EARTHING GRID COMPUTATION BY BOUNDARY ELEMENT METHODS

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SUMMARY

Analysis and design of substation earthing involves computing the equivalent resistance of grounding systems, but also distribution of potentials on the earth surface due to fault currents [1]. While very crude approximations were available in the sixties, several methods have been proposed in the last two decades, must of them on the basis of intuitive ideas such as superposition of punctual current sources and error averaging [2,3]. Although these techniques represented a significant improvement in the area of earthing analysis, a number of problems have been reported. Namely: large computational requirements, unrealistic results when segmentation of conductors is increased, and uncertainty in the margin of error [3].

In this paper, a 1D Boundary Element formulation is presented. Several widespread intuitive methods (such as APM) are identified as particular cases of this general approach. Thus, former intuitive ideas can now be explained as suitable assumptions introduced in the BEM formulation to reduce computational cost. The anomalous asymptotic behaviour of this kind of methods is mathematically explained, and sources of error are pointed out. While linear and parabolic leakage current elements allow to increase accuracy, computing time is drastically reduced by means of new analytical integration techniques. Finally, an application example to a real problem is presented.

1. INTRODUCTION

Physical phenomena underlying to fault currents dissipation into the earth can be modelled by means of Maxwell's Electromagnetic Theory [4]. Constraining the analysis to the obtention of the electrokinetic steady-state response, and neglecting the resistivity of the earthing electrode (system of interconnected buried conductors), the 3D problem associated to an electrical current derivation to earth can be written as

$$\begin{gathered}\boldsymbol{\sigma} = -\boldsymbol{\gamma}\ \mathbf{grad}\, V, \qquad \operatorname{div}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad E, \\ \boldsymbol{\sigma}^t \boldsymbol{n}_E = 0 \quad \text{in} \quad \Gamma_E, \qquad V = V_\Gamma \quad \text{in} \quad \Gamma, \qquad V \longrightarrow 0 \quad \text{if} \quad |\boldsymbol{x}| \to \infty,\end{gathered} \tag{1}$$

where E is the earth and $\boldsymbol{\gamma}$ its conductivity tensor, Γ_E is the earth surface and $\boldsymbol{n}_E$ its normal exterior unit field, and Γ is the earthing electrode surface [5,6]. The solution to this problem gives the potential V and the current density $\boldsymbol{\sigma}$

at an arbitrary point $\boldsymbol{x}$ when the earthing electrode is energized to potential V_Γ (Ground Potential Rise or GPR) with respect to remote earth. Since V and $\boldsymbol{\sigma}$ are proportional to the GPR, the assumption $V_\Gamma = 1$ is not restrictive at all.

In this terms, being $\boldsymbol{n}$ the normal exterior unit field to Γ, the leakage current density σ at an arbitrary point of the earthing electrode surface, the ground current I_Γ (total surge current being leaked into the earth) and the equivalent resistance of the earthing system R_{eq} (apparent resistance of the electrode-earth circuit) can be written as

$$\sigma = \boldsymbol{\sigma}^t \boldsymbol{n}, \qquad I_\Gamma = \int\int_\Gamma \sigma \ d\Gamma, \qquad R_{eq} = \frac{V_\Gamma}{I_\Gamma}. \tag{2}$$

For most practical purposes, the assumption of homogeneous and isotropic soil can be considered acceptable [7], and the tensor $\boldsymbol{\gamma}$ can be substituted by a meassured apparent scalar conductivity γ. Otherwise, a multi-layer model can be accepted without risking a serious calculation error [7]. Since the kind of techniques described in this paper can be extended to multi-layer soil models [8], further discussion and examples are restricted to uniform soils. Hence, problem (1) reduces to the Laplace equation with mixed boundary conditions [4]. If one further assumes that the earth surface is horizontal, symmetry allows to rewrite (1) in terms of a Dirichlet Exterior Problem [6].

Although this classical problem has been rigorously studied [9], and its solution can be efficiently obtained in many other technical applications by standard numerical techniques, additional difficulties appear in our case due to the complexity of the boundary Γ. In most practical cases, the earthing electrode (grounding grid) consist of a number of interconnected bare cylindrical conductors, horizontally buried and supplemented by a number of vertical rods, which ratio diameter/lenght uses to be relatively small ($\approx 10^{-3}$). Therefore, discretization of domain E is extremely difficult, and the obtention of sufficiently accurate results should imply unapproachable computing requirements.

On the other hand, two basic goals must be achieved in a grounding system design: human safety must be preserved (by limiting step and touch voltages), and integrity of equipment and continuity of service must be granted (by ensuring fault currents dissipation into the earth) when a fault condition occurs [1,5,6]. Since computation of potential is only required on the earth surface Γ_E, and the equivalent resistance can be easily obtained in terms of the leakage current (2), a Boundary Element approach seems to be the right choice.

2. VARIATIONAL STATEMENT OF THE PROBLEM

Applying Green's Identity [10] to (1), one gets the following expression for the potential V in E, in terms of the unknown leakage current σ:

$$V(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int\int_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{x}, \boldsymbol{\xi})\, \sigma(\boldsymbol{\xi}) \ d\Gamma, \tag{3}$$

with the weakly singular kernel

$$k(\boldsymbol{x}, \boldsymbol{\xi}) = \left(\frac{1}{r(\boldsymbol{x}, \boldsymbol{\xi})} + \frac{1}{r(\boldsymbol{x}, \boldsymbol{\xi}')} \right), \qquad r(\boldsymbol{x}, \boldsymbol{\xi}) = |\boldsymbol{x} - \boldsymbol{\xi}|, \tag{4}$$

where $\boldsymbol{\xi}'$ is the symmetric of $\boldsymbol{\xi}$ with respect to the earth surface [5,6].

Since (3) holds on the earthing electrode surface [5,6], the boundary condition $V_\Gamma = 1$ leads to the Fredholm integral equation of the first kind on Γ

$$1 = \frac{1}{4\pi\gamma} \int\!\!\int_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{\chi}, \boldsymbol{\xi})\, \sigma(\boldsymbol{\xi})\; d\Gamma \qquad \forall \boldsymbol{\chi} \in \Gamma, \tag{5}$$

which solution is the unknown leakage current density σ.

Equation (5) can now be written in a weaker variational form as:

$$\int\!\!\int_{\boldsymbol{\chi}\in\Gamma} w(\boldsymbol{\chi}) \left[1 - \frac{1}{4\pi\gamma} \int\!\!\int_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{\chi}, \boldsymbol{\xi})\, \sigma(\boldsymbol{\xi})\, d\Gamma \right] d\Gamma = 0, \tag{6}$$

for all members $w(\boldsymbol{\chi})$ of a suitable class of test functions on Γ.

2.1. Boundary Element Formulation

For a given set of $\mathcal{N}$ trial functions $\{N_i(\boldsymbol{\xi})\}$ defined on Γ, and for a given set of $\mathcal{M}$ 2D boundary elements $\{\Gamma^\alpha\}$, the unknown leakage current density σ and the earthing electrode surface Γ can be discretized in the form

$$\sigma(\boldsymbol{\xi}) = \sum_{i=1}^{\mathcal{N}} \sigma_i\, N_i(\boldsymbol{\xi}), \qquad \Gamma = \bigcup_{\alpha=1}^{\mathcal{M}} \Gamma^\alpha, \tag{7}$$

and a discretized form of (3) can be written as

$$V(\boldsymbol{x}) = \sum_{i=1}^{\mathcal{N}} \sigma_i\, V_i(\boldsymbol{x}), \qquad V_i(\boldsymbol{x}) = \sum_{\alpha=1}^{\mathcal{M}} V_i^\alpha(\boldsymbol{x}), \tag{8}$$

$$V_i^\alpha(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int\!\!\int_{\boldsymbol{\xi}\in\Gamma^\alpha} k(\boldsymbol{x}, \boldsymbol{\xi})\, N_i(\boldsymbol{\xi})\; d\Gamma. \tag{9}$$

Finally, for a given set of $\mathcal{N}$ test functions $\{w_j(\boldsymbol{\chi})\}$ defined on Γ, the variational statement (6) is reduced to the system of linear equations

$$\sum_{i=1}^{\mathcal{N}} R_{ji}\sigma_i = \nu_j, \qquad j = 1, \ldots, \mathcal{N}; \tag{10}$$

$$R_{ji} = \sum_{\beta=1}^{\mathcal{M}} \sum_{\alpha=1}^{\mathcal{M}} R_{ji}^{\beta\alpha}, \qquad \nu_j = \sum_{\beta=1}^{\mathcal{M}} \nu_j^\beta, \qquad i = 1, \ldots, \mathcal{N};\; j = 1, \ldots, \mathcal{N}; \tag{11}$$

$$R_{ji}^{\beta\alpha} = \frac{1}{4\pi\gamma} \int\!\!\int_{\boldsymbol{\chi}\in\Gamma^\beta} w_j(\boldsymbol{\chi}) \left[\int\!\!\int_{\boldsymbol{\xi}\in\Gamma^\alpha} k(\boldsymbol{\chi}, \boldsymbol{\xi})\, N_i(\boldsymbol{\xi})\, d\Gamma \right] d\Gamma, \tag{12}$$

$$\nu_j^\beta = \int\!\!\int_{\boldsymbol{\chi}\in\Gamma^\beta} w_j(\boldsymbol{\chi})\; d\Gamma. \tag{13}$$

It can be easily understood that 2D discretizations required to solve the above stated equations in real cases (grounding grids) imply an extremely large number of degrees of freedom. Taking into account that the coefficients matrix in (10) is not sparse, and that 2D integration in (12) must be performed twice over the electrode surface, some reasonable additional assumptions must be introduced to overcome the problem complexity.

3. APPROXIMATED 1D VARIATIONAL STATEMENT OF THE PROBLEM

For a given generic point $\boldsymbol{\xi}$ at the boundary of a cylindrical bar, let $\widehat{\boldsymbol{\xi}}$ be its orthogonal projection over the bar axis, and let $\phi(\widehat{\boldsymbol{\xi}})$ be the diameter (assumed much smaller than the bar length) and $C(\widehat{\boldsymbol{\xi}})$ be the circumferential perimeter of the cross section at this point. Let L be the whole set of axial lines of the buried conductors.

If the leakage current is assumed uniform around the perimeter of every cross section, that is $\sigma(\boldsymbol{\xi}) = \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\ \forall \boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})$, (3) can be written in the form

$$\widehat{V}(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x}, \boldsymbol{\xi})\, dC \right] \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\, dL. \tag{14}$$

The assumption of circumferential uniformity seems to be quite adecquate and not too restrictive, if we take into account the real geometry of grounding grids. Nevertheless, boundary condition $V = 1$ can not be exactly satisfied now at every point on the electrode surface, and (6) does not hold (except in particular cases, where the leakage current is really uniform around the perimeter). However, (6) can hold if we restrict the class of trial functions to those with circumferential uniformity, that is $w(\boldsymbol{\chi}) = \widehat{w}(\widehat{\boldsymbol{\chi}})\ \forall \boldsymbol{\chi} \in C(\widehat{\boldsymbol{\chi}})$, resulting in:

$$\int_{\widehat{\boldsymbol{\chi}} \in L} \widehat{w}(\widehat{\boldsymbol{\chi}}) \left[\pi\phi(\widehat{\boldsymbol{\chi}}) - \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}})\, \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\, dL \right] dL = 0 \tag{15}$$

for all members $\widehat{w}(\widehat{\boldsymbol{\chi}})$ of a suitable class of test functions on L, where

$$K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) = \int_{\boldsymbol{\chi} \in C(\widehat{\boldsymbol{\chi}})} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{\chi}, \boldsymbol{\xi})\, dC \right] dC. \tag{16}$$

In this way, boundary condition $V = 1$ is forced to be satisfied on the average at every cross section. In fact, (15) can be considered as a weaker variational statement of the Fredholm integral equation of the first kind on L

$$\pi\phi(\widehat{\boldsymbol{\chi}}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}})\, \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\, dL \qquad \forall \widehat{\boldsymbol{\chi}} \in L. \tag{17}$$

Since ends and junctions of conductors are not taken into account in this formulation, slightly anomalous local effects are expected at these points, but global results should not be noticeably affected.

3.1. Boundary Element Formulation

For a given set of n trial functions $\{\widehat{N}_i(\widehat{\boldsymbol{\xi}})\}$ defined on L, and for a given set of m 1D boundary elements $\{L^\alpha\}$, the unknown leakage current $\widehat{\sigma}$, and the whole set of axial lines of the buried conductors L, can be discretized in the form

$$\widehat{\sigma}(\widehat{\boldsymbol{\xi}}) = \sum_{i=1}^{n} \widehat{\sigma}_i\, \widehat{N}_i(\widehat{\boldsymbol{\xi}}), \qquad L = \bigcup_{\alpha=1}^{m} L^\alpha, \tag{18}$$

and a discretized version of (14) can be written as

$$\widehat{V}(\boldsymbol{x}) = \sum_{i=1}^{n} \widehat{\sigma}_i \, \widehat{V}_i(\boldsymbol{x}), \qquad \widehat{V}_i(\boldsymbol{x}) = \sum_{\alpha=1}^{m} \widehat{V}_i^{\alpha}(\boldsymbol{x}), \tag{19}$$

$$\widehat{V}_i^{\alpha}(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L^{\alpha}} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x}, \boldsymbol{\xi}) \, dC \right] \widehat{N}_i(\widehat{\boldsymbol{\xi}}) \; dL. \tag{20}$$

Finally, for a given set of n test functions $\{\widehat{w}_j(\widehat{\boldsymbol{\chi}})\}$ defined on L, (15) is reduced to the system of linear equations

$$\sum_{i=1}^{n} \widehat{R}_{ji} \widehat{\sigma}_i = \widehat{\nu}_j, \qquad j = 1, \ldots, n; \tag{21}$$

$$\widehat{R}_{ji} = \sum_{\beta=1}^{m} \sum_{\alpha=1}^{m} \widehat{R}_{ji}^{\beta\alpha}, \qquad \widehat{\nu}_j = \sum_{\beta=1}^{m} \widehat{\nu}_j^{\,\beta}, \qquad i = 1, \ldots, n; \; j = 1, \ldots, n; \tag{22}$$

$$\widehat{R}_{ji}^{\beta\alpha} = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\chi}} \in L^{\beta}} \widehat{w}_j(\widehat{\boldsymbol{\chi}}) \left[\int_{\widehat{\boldsymbol{\xi}} \in L^{\alpha}} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) \, \widehat{N}_i(\widehat{\boldsymbol{\xi}}) \, dL \right] dL, \tag{23}$$

$$\widehat{\nu}_j^{\beta} = \int_{\widehat{\boldsymbol{\chi}} \in L^{\beta}} \pi \, \phi(\widehat{\boldsymbol{\chi}}) \, \widehat{w}_j(\widehat{\boldsymbol{\chi}}) \; dL. \tag{24}$$

The computational work required to solve a real problem is drastically reduced by means of this 1D formulation with respect to the one given in **2.1**. However, extensive computing is still required, mainly because of circumferential integration in (20) and (23), and further simplifications are necessary to reduce computing time under acceptable levels.

3.2. Simplified 1D Boundary Element Formulation

The inner integral in (20) can be approximated as

$$\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x}, \boldsymbol{\xi}) \, dC \approx \pi \, \phi(\widehat{\boldsymbol{\xi}}) \, \widehat{k}(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}), \tag{25}$$

where

$$\widehat{k}(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}) = \left(\frac{1}{\widehat{r}(\boldsymbol{x}, \widehat{\boldsymbol{\xi}})} + \frac{1}{\widehat{r}(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}')} \right), \qquad \widehat{r}(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}) = \sqrt{|\boldsymbol{x} - \widehat{\boldsymbol{\xi}}|^2 + \frac{\phi^2(\widehat{\boldsymbol{\xi}})}{4}}, \tag{26}$$

and $\widehat{\boldsymbol{\xi}}'$ is the symmetric of $\widehat{\boldsymbol{\xi}}$ with respect to the earth surface. This approximation is quite accurate, unless distance between points $\boldsymbol{x}$ and $\widehat{\boldsymbol{\xi}}$ was in the order of magnitude of the diameter $\phi(\widehat{\boldsymbol{\xi}})$. Then, (16) can be approximated as

$$K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) \approx \pi \, \phi(\widehat{\boldsymbol{\chi}}) \, \pi \, \phi(\widehat{\boldsymbol{\xi}}) \, \frac{1}{2} \left(\widehat{k}(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) + \widehat{k}(\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\chi}}) \right), \tag{27}$$

where the arithmetic mean avoids the lack of symmetry in the system of equations (21) when the conductor diameter is different at points $\widehat{\boldsymbol{\chi}}$ and $\widehat{\boldsymbol{\xi}}$.

Now, for different selections of the sets of trial and test functions, specific formulations are achieved. Thus, for constant leakage current elements (one centered node per element), Point Collocation (Dirac deltas as trial functions) leads to the very early intuitive methods, based on the idea that each segment of conductor is substituted by an "imaginary sphere". On the other hand, Galerkin (test functions identical to trial functions) leads to a kind of more recent methods (such as APM), based on the idea that each segment of conductor is substituted by a "line of point sources over the lenght of the conductor" [3]. Coefficients (23) correspond to "mutual and self resistances" between "segments of conductor" [3]. For higher order elements more advanced formulations can be derived [5,6],

The problems encountered with the application of these methods [3,6,11] can now be explained from a mathematically rigorous point of view. The fact is that approximation (25) is not valid for short distances. Hence, when discretization is increased, and the conductor diameter becomes comparable to the size of the elements, approximation (27) introduces significant errors in the coefficients of the linear system (21) corresponding to adjacent nodes (including diagonal terms). From another point of view, since the approximation error increases as discretization does, numerical results for dense discretizations do not trend to the solution of the integral equation (17) with kernel (16), but to the solution of a different ill-conditioned integral equation with kernel (27). For the test problem presented in [11] (single bar in an infinite domain), the circumferential uniformity hypothesis is strictly satisfied. In this case, it can be easily verified that solving (17) with the simplified kernel (27) is absolutely equivalent to solve (5), but both free ends are disregarded and the boundary condition $V = 1$ is imposed on the axis, not on the boundary of the bar.

This explains why unrealistic results are obtained when discretization increases [3], and convergence is precluded [6]. However, results obtained for low and medium levels of discretization have been proved to be sufficiently accurate for practical purposes [11].

Further discussion and examples are restricted to Galerkin type formulations, where the matrix of coefficients in (10) is symmetric and positive definite [12]. Diameter of conductors is assumed constant within each element. Therefore, (20) and (23) can be rewritten as

$$\widehat{V}_i^{\alpha}(\boldsymbol{x}) = \frac{1}{4\pi\gamma}\pi\,\phi^{\alpha}\int_{\widehat{\boldsymbol{\xi}}\in L^{\alpha}} \widehat{k}(\boldsymbol{x},\widehat{\boldsymbol{\xi}})\,\widehat{N}_i(\widehat{\boldsymbol{\xi}})\;dL, \tag{28}$$

$$\widehat{R}_{ji}^{\beta\alpha} = \frac{\pi\phi^{\beta}\,\pi\phi^{\alpha}}{4\pi\gamma}\int_{\widehat{\boldsymbol{\chi}}\in L^{\beta}} \widehat{N}_j(\widehat{\boldsymbol{\chi}})\left[\int_{\widehat{\boldsymbol{\xi}}\in L^{\alpha}} \frac{\widehat{k}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}})+\widehat{k}(\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\chi}})}{2}\,\widehat{N}_i(\widehat{\boldsymbol{\xi}})\,dL\right]dL, \tag{29}$$

where ϕ^{α} and ϕ^{β} represent the constant conductor diameter within elements L^{α} and L^{β}. Obviously, (29) leads to a symmetric matrix, and can also be written as

$$\widehat{R}_{ji}^{\beta\alpha} = \frac{1}{2}\left(\pi\,\phi^{\beta}\int_{\widehat{\boldsymbol{\chi}}\in L^{\beta}} \widehat{N}_j(\widehat{\boldsymbol{\chi}})\,\widehat{V}_i^{\alpha}(\widehat{\boldsymbol{\chi}})\;dL + \pi\,\phi^{\alpha}\int_{\widehat{\boldsymbol{\xi}}\in L^{\alpha}} \widehat{N}_i(\widehat{\boldsymbol{\xi}})\,\widehat{V}_j^{\beta}(\widehat{\boldsymbol{\xi}})\;dL\right). \tag{30}$$

4. ANALYTICAL INTEGRATION AND OVERALL EFFICIENCY

Computation of remaining integrals in (28) and (30) is not obvious. Gauss quadratures can not be used due to the undesirable behaviour of the integrands. Although very costly, a compound adaptative Simpson quadrature (with Richardson extrapolation error estimates) seems to be the best numerical choice [5]. Therefore, we turn our attention to analytical integration techniques.

Explicit formulae have been recently derived to compute (28) in the case of constant (1 functional node), linear (2 functional nodes) and parabolic (three functional nodes) leakage current elements. Explicit expressions have also been derived for contributions (30) related to parallel elements, including the case in which both elements coincide. These formulae generalize those obtained by other authors for the most simple cases (i.e. constant leakage current elements in APM [2]). Derivation of these formulae requires quite a lot of analytical work, being too cumbersome to be made explicit in this paper. Numerical integration is unexpensive in the remaining cases, since analytical expressions for (28) can be substituted into (30) resulting in quite smooth integrands, and accurate results can be obtained by means of a suitable adaptive quadrature. Anyhow, further research is under development to extend completely this analytical approach.

With regard to overall computational cost, for a given discretization (m elements of p nodes each, and a total number of n degrees of freedom) a linear system (21) of order n must be generated and solved. Since the matrix is symmetric, but not sparse, resolution requires $O(n^3/3)$ operations. Matrix generation requires $O(m^2p^2/2)$ operations, since p^2 contributions of type (30) have to be computed for every pair of elements, and approximately half of them are discarded because of symmetry. Hence, most of computing effort is devoted to matrix generation in small/medium problems, while linear system resolution prevails in medium/large ones. At present, the size of the largest problem that can be solved is limited by memory storage. Thus, for a problem with 2000 degrees of freedom, at least 16Mb would be needed, while computing times for matrix generation and system resolution would be acceptable, but noticeable, and in the same order of magnitude (a couple of hours on a nice workstation). On the other hand, once the leakage current has been obtained, the cost of computing the equivalent resistance (2) is negligible. The additional cost of computing potential at any given point (normally at the earth surface) by means of (19) and (28) requires only $O(mp)$ operations, since analytical formulae for (28) are available. However, if it is necessary to compute potentials at a large number of points (i.e. to draw contours), computing time may also be important.

5. CONCLUSIONS

A Boundary Element approach for the analysis of substation earthing systems has been presented. For 3D problems, some reasonable assumptions allow to reduce a general 2D BEM formulation to an approximated less expensive 1D version. Further simplifications reduce computing requirements under acceptable levels. Several widespread intuitive methods are identified as particular cases of this approach. Problems encountered with the application of these methods can be finally explained from a mathematically rigorous point of view, while more efficient and accurate formulations can be derived. By means of analytical integration techniques for the discretized equations, accurate results can be obtained in practical cases with acceptable computing requirements.

This approach has been applied to a real case: the E. R. Barberá substation grounding (close to Barcelona, Spain, and under construction at present). The plan and characteristics are presented in Figure 1. Results are given in Figure 2. Each bar was discretized in one single linear element. The model (408 elements and 238 degrees of freedom) required only 6 minutes of cpu time on a Vax-4300/32Mb computer. At the scale of the whole grid, results are not noticeably improved by increasing discretization. In cases like this, higher order elements are advantageous in comparison with constant elements, since accuracy is much higher for a remarkably smaller total number of degrees of freedom. Results were obtained by a Computer Aided Design System based upon the suggested approach, that has been under development during the last few years.

6. ACKNOWLEDGEMENTS

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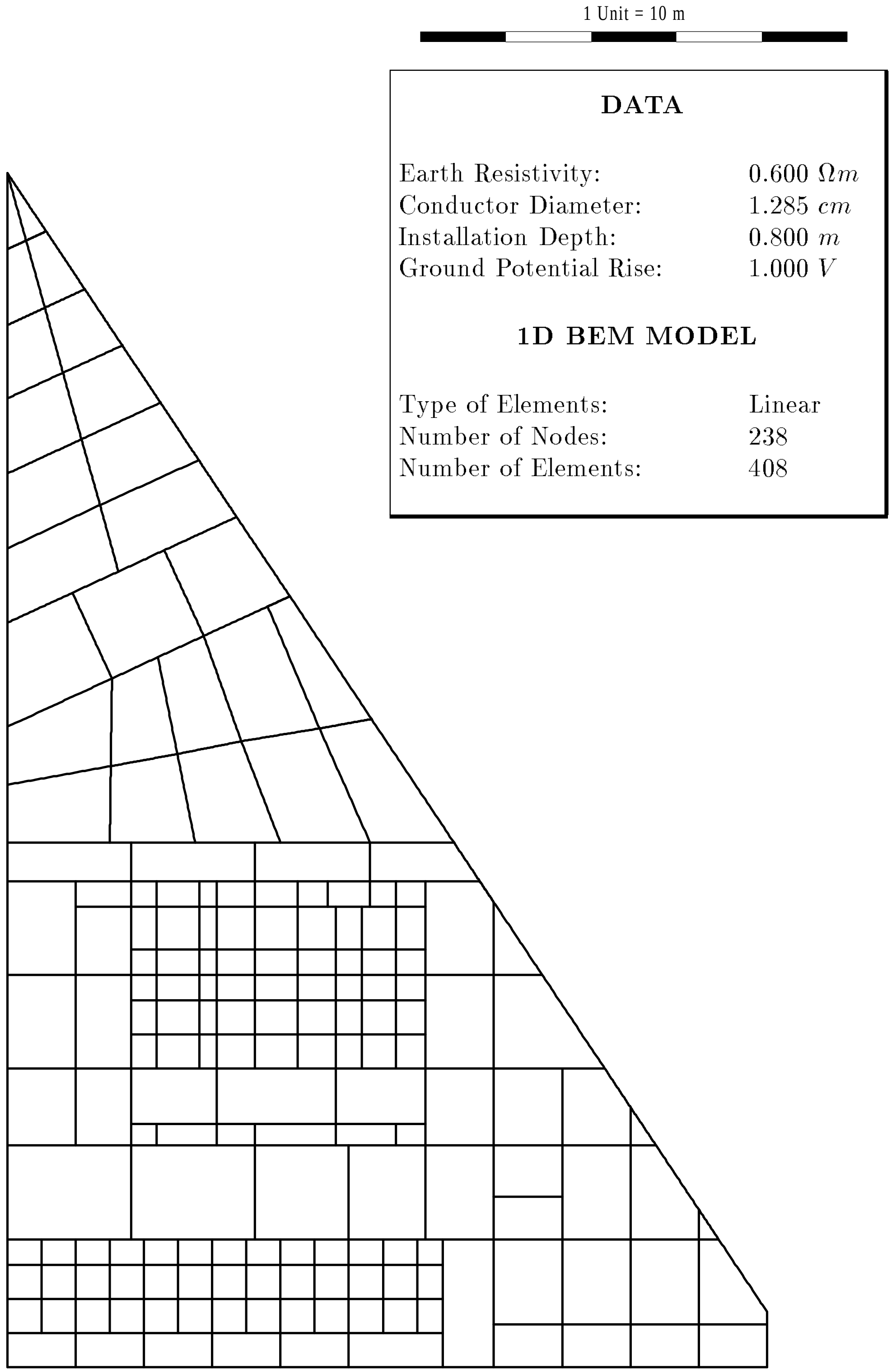

Figure 1.—E. R. Barberá Grid: Plan, Problem Characteristics and Numerical Model (1 Linear Element per bar).

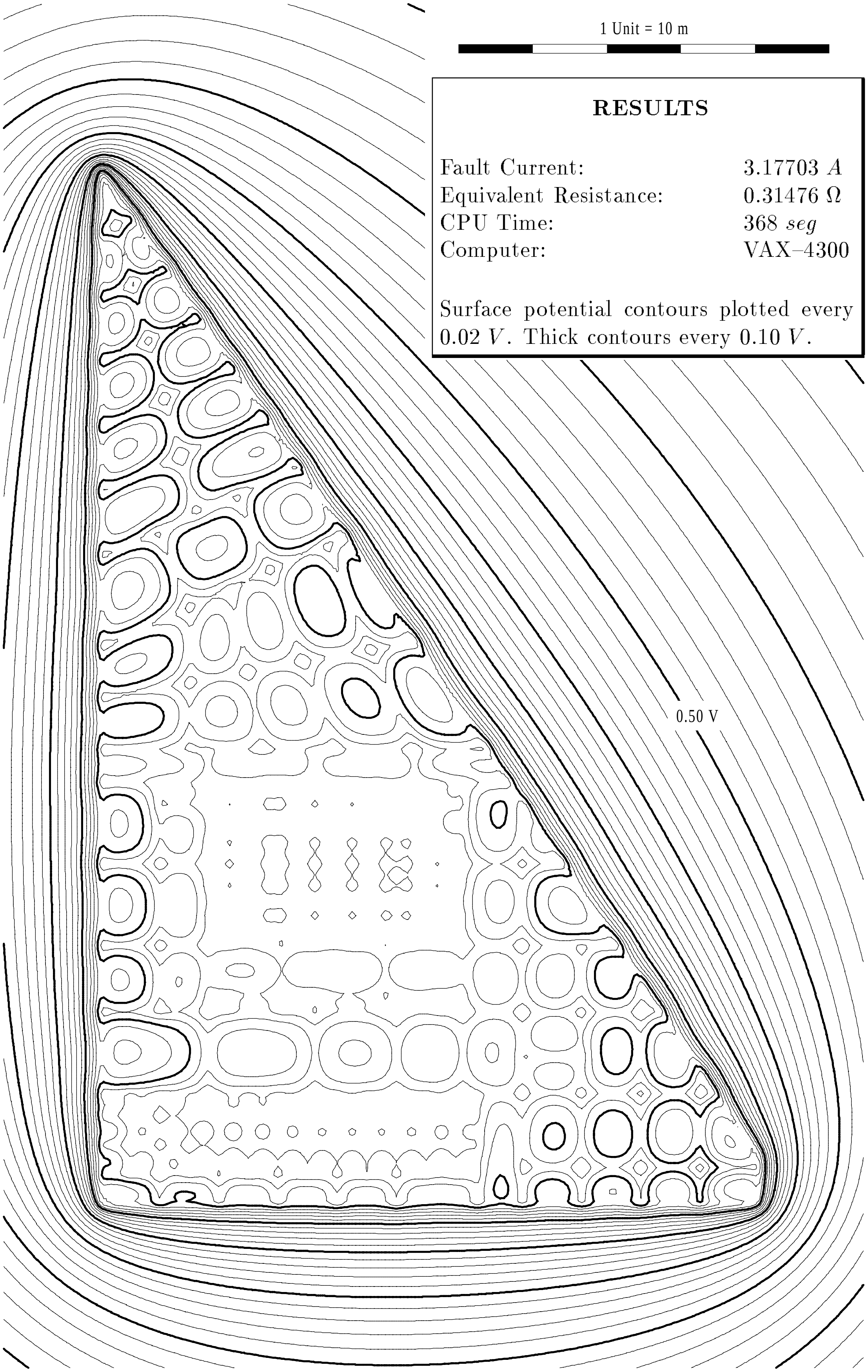

Figure 2.—E. R. Barberá Grid: Results obtained by BEM (1 linear element per bar). Ground surface potential distribution.