Engineering Structures 273 (2022) 115071

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Topology optimization of structures considering minimum weight and stress constraints by using the Overweight Approach

Diego Villalba *, José París, Iván Couceiro, Ignasi Colominas, Fermín Navarrina

Group of Numerical Methods in Engineering, Universidade da Coruña, Campus de Elviña, 15071, A Coruña, Spain

ARTICLE INFO

Keywords:
Structural topology optimization
Stress constraint
Overweight Approach
Finite element method
Aggregation techniques

ABSTRACT

In this paper, a new method for structural topology optimization considering minimum weight and local stress constraints is proposed. For this purpose, the Overweight Approach, an improvement of the so-called Damage Approach, is used. In this method, a virtual relative density is defined as a function of the violation of the local stress constraints. The virtual relative density is increased as stresses exceed the maximum allowable value. The optimization algorithm will provide a design with a minimal variation of the relative density. The structural analysis is performed by means of the Finite Element Method (FEM) and the distribution of material is modelled in terms of a uniform relative density within each element. Moreover, the optimization is addressed by means of the Sequential Linear Programming algorithm (SLP). Finally, the proposed methodology is tested by means of some benchmark problems, and the results show that the Overweight Approach is a feasible alternative for the Damage Approach and the stress constraint aggregation techniques.

1. Introduction

The first works about structural topology optimization were conducted by Bendsøe and Kikuchi in 1988 [1]. These works meant the establishment of the basis of what was then a new field. Since then, an important number of contributions has been made covering a wide range of different approaches and applications. Although the more extended in the literature is the Maximum Stiffness Problem [1–8], the attention in this publication will be only focused in the Minimum Weight with Stress Constraints Problem [9–14]. This circumstance is due to its high interest from the engineering point of view where the main objective is to design structures which are able to support different loads with the requirement that structural stresses have to be lower than a maximum value. On the other hand, the variety of methods developed to solve these problems is extremely high due to the drawbacks emerged during the solution process.

The different alternatives used to solve the Minimum Weight with Stress Constraints problem are related with the different ways to formulate the stress constraints in the topology optimization problem. Several strategies have been used to impose the stress constraints in the problem until now. The Local Stress Constraint approach [9,10,12,15–20] establishes one constraint for each local stress considered. In this case the number of constraints is equal to the number of local stresses analysed. This number is related with the number of design variables used in the definition of the topology optimization problem. The Global Stress Constraint approach [12,18,21–24] formulates only one constraint to consider all the local stresses regardless the number of design variables used in the definition of the problem. The Block Aggregation of Stress Constraints approach [11,13,18,25] formulates a certain number of constraints to consider all the local stresses. This approach is an intermediate situation between the previous approaches, since the number of constraints is higher than 1 but considerably lower than the number of local stresses considered in the formulation of the problem. Finally, the Damage Approach [26] creates an alternative model to the original one. One of the characteristics of the original model is modified if the local stresses are greater than its maximum allowable value in the alternative one.

The Global Stress Constraint approach and the Block Aggregation of Stress Constraints approach require the use of functions that let to estimate the maximum value for a certain set of values: the maximum stress value of a certain number of local stresses. Until now, several functions have been used: the Kreisselmeier–Steinhauser (KS) function [22,25,27–30], the p-norm approach [23,30–34], and the induced constraint aggregation [35,36]. On the other hand, in this research an alternative formulation of the Damage Approach previously developed [26] will be established to impose the stress constraints in the Topology Optimization problem, this formulation will be known as Overweight Approach. The proposed formulation creates the overweight model by using an alternative characteristic to the global structural stiffness with the objective of not requiring additional calculations, in this case

* Corresponding author.
E-mail address: diego.villalba.rama@udc.es (D. Villalba).

https://doi.org/10.1016/j.engstruct.2022.115071
Received 25 April 2022; Received in revised form 29 September 2022; Accepted 1 October 2022
Available online 17 October 2022

the structural weight. Although the Method of Moving Asymptotes (MMA) could be used to solve the problem, the Sequential Linear Programming (SLP) will be used in the optimization process since it has been previously used for the authors of this paper to solve the topology optimization problem successfully.

The material layout in the domain will be defined by using the Solid Isotropic Material with Penalty (SIMP) [3,37–43], considering that the relative density will be uniform per element. The structural analysis used in the topology optimization problem proposed in this paper is stated in terms of a Finite Element formulation, considering several hypotheses: elastic and linear material, small displacements and small gradients of displacements. Since the design variables used in the topology optimization problem define the material layout, the formulation has to include the effect of the design variables in the structural analysis [10–12]. The remainder of this paper is structured as follows. Section 2 describes the topology optimization problem with all its components and discusses numerical implementation aspects. Section 3 describes the proposed optimization algorithms. In Section 4 the sensitivity analysis required by the optimization algorithms is formulated. In Section 5 the most important aspects of the numerical implementation are commented. In Section 6 the optimization method is applied on different numerical examples. Conclusions are commented in Section 7. Finally, an appendix with a comparative analysis between the methodology developed in this manuscript and the previous formulations of the same problem (local stress constraints and global stress constraint with an aggregation function) is included.

2. Optimization problem

The optimization problem solved in this paper can be stated as:

$$\begin{aligned} &\text{Calculate} && \boldsymbol{\rho} = \{\rho_i\} && i = 1, \dots, n \\ &\text{which minimize} && F(\boldsymbol{\rho}) && \\ &\text{subject to} && g\left(\mathbf{r}^0, \boldsymbol{\rho}\right) \le 0 && \\ & && \rho_{min} \le \rho_i \le \rho_{max} && i = 1, \dots, n \end{aligned} \tag{1}$$

where $\boldsymbol{\rho} = \{\rho_i\}$ is the design variables vector, $\mathbf{r}^0$ is the coordinates vector of the points considered, $F(\boldsymbol{\rho})$ is the objective function, $g\left(\mathbf{r}^0, \boldsymbol{\rho}\right)$ is the Overweight Constraint. Furthermore, n is the number of design variables, and ρ_{min} and ρ_{max} are the side constraints of the design variables. The material model used to solve the problem will be: uniform density per element. For this reason, the design variables, whose number is equal to the elements of the mesh, represent the value of the relative density at each element of the mesh. The rest of the terms of the optimization problem will be described in the next subsections. Finally, the optimization algorithms used to solve it will be described in Section 3.

2.1. Objective function

Minimum weight formulations with stress constraints [9–14] have been stated with the purpose of reaching structures with the lowest cost. Since the cost of the structure is related with the amount of material required in its manufacture, the objective function of the problem will be the structural weight, whose formulation will be:

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega_e} \rho_e \, d\Omega_e, \tag{2}$$

since the value of the relative density will take a constant value in each element of the mesh, Eq. (2) can be reduced to:

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \rho_e V_e \quad \text{and} \quad V_e = \int_{\Omega_e} d\Omega_e, \tag{3}$$

where ρ_e is the value of the relative density of element e and V_e is the volume occupied by element e. Finally, and due to the objective of attaining solid-void solutions, a factor p which penalizes intermediate relative density values is introduced [10–13,17,25]. Thus, the objective function will be:

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \left(\rho_e\right)^{\frac{1}{p}} V_e. \tag{4}$$

The penalty coefficient p will be initially 1 to reduce the non-linearity of the problem. However, it will be increased to reduce the appearance of areas with intermediate values of relative density. In the same way that in [26], this penalty coefficient p is typically introduced in the calculation of the effective Young's modulus used to compute the global stiffness matrix in the literature. If stress constraints are considered, this circumstance can suppose an important handicap in case of using different values of the coefficient p during the solution of the problem, since the value of the structural stresses will be also modified. This circumstance does not happen when the penalty coefficient p is introduced in the objective function. The effectiveness of the penalty coefficient p is tested with the measure of discreteness [44]. Once the objective function of the optimization problem has been stated, the next step is to formulate the overweight constraint.

2.2. Overweight constraint

In the same way that the Damage Approach [26], the Overweight Approach is an alternative method to the classical stress constraints aggregation functions used to combine the effect of a certain number of local stress constraints. The Overweight Approach penalizes the violation of the local stress constraints through a material overweight. This overweight consists in the increase of the relative density in the areas where the local stresses are violated and is related with the magnitude of this violation. For this purpose, an alternative model of the original structure is created by considering this structural overweight. Therefore, two different models that describe the same structure are defined, hereinafter: the original model and the overweight model. Consequently, the overweight model is always heavier or has the same structural weight than the original model. In other words, if both models have the same weight, all the constraints imposed are satisfied, otherwise, there are violated constraints. As a consequence, the Overweight Approach will be used to consider the local stress constraints in the problem.

The overweight model will be defined from the original one following a sequence of steps. First, the structural analysis of the original model is computed and the displacement field is obtained. Then, the stresses in the central point of each element of the mesh are computed. The last step is to increase the structural weight in the points where the stresses exceed their maximum allowable value. This circumstance supposes an important difference with the Damage Approach [26] where the structural stiffness was reduced. The use of the structural weight in the formulation of the overweight constraint is more appropriate than the use of the global stiffness in the formulation of the damage constraint if a minimum weight problem is stated. In this case, only the calculation of the structural weight of the overweight model is required since the structural weight of the original one is computed with the objective function. However, the Damage Approach developed in [26] would require to compute the structural stiffness of both models. The Overweight Constraint is a comparison between the structural weight of both models. Furthermore, the relative density in the overweight model will have to satisfy these conditions:

$$\begin{cases} \tilde{\rho}(x) > \rho(x) & \forall x \in \Omega_\sigma \\ \tilde{\rho}(x) = \rho(x) & \forall x \in \Omega \backslash \Omega_\sigma \end{cases} \tag{5}$$

where x is the point where the stresses σ are computed, σ_{max} is the value of the maximum allowable stress, Ω_σ is the part of the domain where the stresses are higher than their maximum allowable value, ρ is the relative density in the original model and $\tilde{\rho}$ in the overweight one.

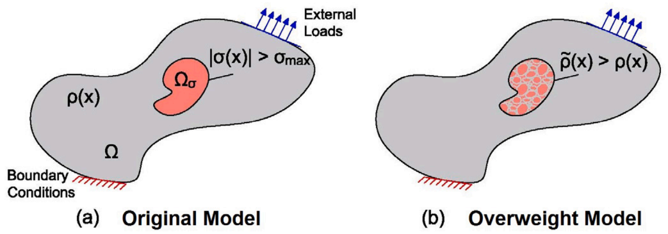

Fig. 1. Schematic models: **(a)** the original model, where the stress exceeds the allowable maximum stress in the red subregion Ω_σ, and **(b)** the overweight model with degraded material properties in Ω_σ. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

A graphical representation of the Overweight Approach can be seen in Fig. 1. Therefore, the overweight constraint can be defined as:

$$g(\rho) = \frac{\tilde{W}}{W} - 1 = 0, \tag{6}$$

where W is the structural weight of the original model and $\tilde{W}$ is the structural weight of the overweight model. This means that the structural weight of both models will be always equal unless there are some violated stresses. In other words, all the local stresses are checked by comparing only one global property of the structure. The structural weight of both models can be calculated as follows:

$$W = \int_\Omega \rho \, d\Omega \qquad \tilde{W} = \int_\Omega \tilde{\rho} \, d\Omega. \tag{7}$$

The definition of the Overweight constraint in (6) means that it will be always active even when all the structural stresses will be lower than their maximum value, since the structural weight of both models will coincide. This can be explained from the definition of the relative density in the overweight model established in (5), where the relative density of the overweight model will be higher or equal to the original one. Consequently, the Overweight constraint will be always higher or equal to zero, since there is no possibility that the structural weight of the overweight model will be lower than the original one to obtain negative values of the overweight constraint. The permanent activation of the Overweight constraint is not desirable since its derivatives have to be computed in any case even when all the stresses are considerably lower than their maximum allowable value. For this reason, a small positive parameter ζ is introduced in the overweight constraint to relax the equality constraint. Thus, Eq. (6) becomes:

$$g[\rho] = \frac{\tilde{W}}{W} - 1 \leq \zeta. \tag{8}$$

Theoretically, this circumstance will mean that stress constraints may be slightly violated. Consequently, the value of ζ should not be extremely high to avoid the appearance of areas with an important failure of local stress constraints. However, if the parameter ζ takes an extremely small value the number of iterations required is increased considerably. The value of the relaxation parameter of the overweight constraint ζ will be in this paper 10^{-3}.

2.2.1. Overweight model

The formulation of the relative density in the overweight model can be defined by different approaches. However, a similar scheme of that in [26] is used for this purpose. Furthermore, the relative density of the overweight model can be calculated as:

$$\tilde{\rho} = \rho_{min} + \beta\left(\rho - \rho_{min}\right), \tag{9}$$

where β is the overweight function and $\beta\left(\sigma; \sigma_{max}\right) \geq 1$. Although the relative density of the overweight model $\tilde{\rho}$ can be obtained directly by multiplying the overweight function β times the relative density of the original model ρ, the term ρ_{min} is included in the formulation to avoid penalizing the areas with this relative density when the stresses exceed their maximum. This intends to simulate the effect of the areas with relative density equal to zero, where the stresses does not exist properly, since the main objective is the attainment of full-void solutions and ρ_{min} has been introduced for numerical issues related with the singularity of the stiffness matrix. This is the main advantage of this formulation with respect to the aggregation functions used to formulate the global stress constraint.

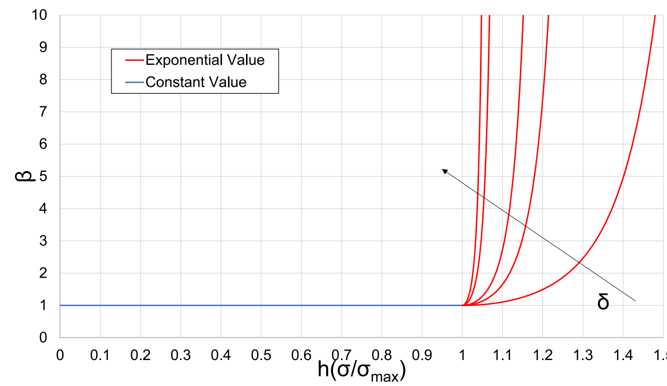

Fig. 2. Overweight function for increasing values of $\delta > 0$.

The overweight function β needs to satisfy two conditions: to be at least first order differentiable and to be monotonically increasing when stresses exceed its allowable limit. These conditions are required to solve the problem by using gradient-based optimization methods. Although there are many functions which satisfy both criteria, an exponential function will be used. The main reason to use this kind of function is that important penalizations are obtained for a small violation of the stress constraints. The function used in this work is:

$$\beta(\sigma;\delta) = \begin{cases} 1, & \text{if} \frac{\sigma}{\sigma_{max}} < 1 \\ e^{\delta\left(h(\sigma,\sigma_{max})\right)^2}, & \text{if} \frac{\sigma}{\sigma_{max}} \geq 1 \end{cases} \tag{10}$$

where h is the local stress constraint, $\delta > 0$ is the exponential degradation parameter which controls the gradient of the overweight function and the amount of overweight for each stress level. Extremely high values for the parameter δ means high overweight for a little failure of the stress constraints and an increase in the non-linearity of the overweight constraint. For this reason, it is advisable that this parameter takes intermediate values (10–100). A graphical representation of the overweight function for different values of δ can be observed in Fig. 2. At this point, the formulation developed in [26] introduces an inherent problem when high values of the parameter α are used, since the value of the derivative of β will be equal to zero when stresses are higher than their maximum allowable value. This means that the damage constraint does not work properly. This problem has been solved in this paper, since the second derivative of β is always greater than 0. Finally, it is important to remark that this function does not intend to model accurately the structural overweight, since the main objective is to obtain an optimal design without overweight.

The main disadvantage of the overweight function proposed in (10) is that its derivative is null when stresses are lower than its maximum allowable value. This constitutes an important drawback when stresses are near to their maximum allowable value. As a result of this, some changes are introduced in the overweight function to avoid this issue and to facilitate the numerical resolution of the topology optimization problem. The modified overweight function will have to provide information about the structural stresses when they are near to their maximum allowable value. First, the overweight function is moved to the left. Thus, it will take a value slightly superior to 1 when stresses take their maximum allowable value. This does not have an

Table 1
Overweight constraint parameters.

Parameter	Value
δ	50
τ	0.01
ψ	0.1

impact in the operation of the algorithm because it mitigates the effect of the relaxation of the overweight constraint proposed in (8). The translated overweight function is:

$$\beta(\sigma;\delta) = \begin{cases} 1, & \text{if}\frac{\sigma}{\sigma_{max}} < (1-\tau) \\ e^{\delta\left(h(\sigma,\sigma_{max})+\tau\right)^2}, & \text{if}\frac{\sigma}{\sigma_{max}} \geq (1-\tau) \end{cases} \tag{11}$$

where τ is the magnitude of the overweight function movement to the left. This traslation guarantee that the overweight function derivatives are not equal to zero when the stresses are slightly lower than their maximum. Moreover, τ should not take an extremely higher value to avoid an excessive increase of the value of the overweight function when the stresses are equal to their maximum. Then, a transition function between the exponential and constant parts of the translated overweight function is introduced. This transition function replaces the exponential part when stresses are under their maximum and includes also a part of the range where the overweight function takes a constant value. This transition function extends the range where the overweight function derivatives do not take a null value. The transition function has to satisfy the same conditions than the original overweight function since the continuity of the overweight function and its first derivative has to be maintained to avoid numerical problems. The general structure of the overweight function is:

$$\beta(\sigma;\delta) = \begin{cases} 1, & \text{if } \frac{\sigma}{\sigma_{max}} \leq (1-\psi) \\ t(\sigma,\delta,\tau,\psi), & \text{if } (1-\psi) < \frac{\sigma}{\sigma_{max}} < 1 \\ e^{\delta\left(h(\sigma,\sigma_{max})+\tau\right)^2}, & \text{if } \frac{\sigma}{\sigma_{max}} \geq 1 \end{cases} \tag{12}$$

and ψ is the size of the definition range of the transition function. The introduction of the transition function makes sense if ψ is greater than τ. The transition function will be:

$$t(\sigma,\delta,\tau,\psi) = e^{\frac{\delta\tau^2}{\psi^{(2\psi/\tau)}}\left(h(\sigma,\sigma_{max})+\psi\right)^{(2\psi/\tau)}} \tag{13}$$

Fig. 3 shows a graphical representation of the overweight function used in this paper. This transition function allows the use of gradient-based optimization methods since the derivative of the overweight function will not be equal to zero in the proximities of the border between the feasible and non-feasible region. Moreover, the use of a comparison between both models to formulate the constraint allows to determine if all the local stresses are lower than their maximum. Table 1 shows the value of parameters introduced in this section used to solve the examples of this paper. Once the overweight constraint of the optimization problem has been stated, the next step is to establish the most appropriate stress criterion h.

2.2.2. Stress criterion

The definition of the overweight constraint will be completed with the choice of the stress criterion h. The stress criterion incorporates the information of the local stress constraints in the overweight constraint. Steel is the material considered in this paper to solve the problems, and therefore, the most appropriate failure criterion is Von Mises. Other failure criteria can be used instead of the Von Mises criterion if other materials are considered in the definition of the stress criterion h. Moreover, several local stress constraints for the same point can be used to define the overweight constraint in case of materials without an appropriate failure criterion. As it was established in (10), the stress criterion h has to be normalized with respect to the maximum allowable stress. Therefore, the normalized local stress constraint with the equivalent Von Mises criterion can be expressed as:

$$h\left(\boldsymbol{\sigma},\sigma_{max}\right) = \frac{\hat{\sigma}_{VM}(\boldsymbol{\sigma})}{\hat{\sigma}_{max}} - 1 \leq 0 \tag{14}$$

where $\hat{\sigma}_{VM}$ is the Von Mises stress and $\hat{\sigma}_{max}$ is the maximum allowable value of stresses. Finally, a stress relaxation coefficient φ is included in the formulation in order to avoid singularity phenomena of the stress criterion h when the value of the relative density tends to zero. In this paper, φ will depend on the value of the relative density according to [9–12,45] as:

$$\varphi = 1 - \varepsilon + \frac{\varepsilon}{\rho}, \tag{15}$$

where ε is the stress relaxation parameter whose value is increased progressively to make possible the appearance of areas with relative density equal to its lower limit. However, an initial stress relaxation is not advisable to avoid removing parts that might appear again later if necessary. A graphical representation of the stress relaxation coefficient φ can be seen in Fig. 4. The introduction of the stress relaxation coefficient is essential from a theoretical and a practical point of view. Thus, the formula of the stress criterion h to introduce in (12) will be:

$$h\left(\boldsymbol{\sigma},\sigma_{max}\right) = \frac{\hat{\sigma}_{VM}(\boldsymbol{\sigma})}{\hat{\sigma}_{max}\ \varphi} - 1 \leq 0 \tag{16}$$

2.3. Side constraints

The side constraints establishes the upper and lower limit of the design variables. The upper limit is equal to 1, what represents full of material. Although the theoretical lower limit should be zero to represent the lack of material, this produces the singularity of the global stiffness matrix. This singularity can be solved by considering a small value of the Young Modulus when the relative density is equal to zero. However, this fictional stiffness in some parts of the domain can produce a little perturbation of the structural analysis. For this reason, the lower limit takes a slightly superior to zero value. In this paper the minimum value is $\rho_{min} = 0.001$ according to other references in the literature [1,3,11,14,23].

3. Optimization algorithms

The solution of the topology optimization problem (1) requires the use of an iterative algorithm that involves different approaches. For this reason, the design variables $\boldsymbol{\rho}$ are updated in each iteration as:

$$\boldsymbol{\rho}_{k+1} = \boldsymbol{\rho}_k + \theta_k \boldsymbol{s}_k \tag{17}$$

where θ_k is the improvement factor which establishes the magnitude of modification between two consecutive iterations and $\boldsymbol{s}_k$ is the improvement design direction. Excessive modifications of the design between two consecutive iterations are avoided by considering a move limits approach of the design variables. This limitation in the modification of the design variables $\left(\Delta\rho_{i,j}\right)_{max}$ is introduced due to the high non-linearity of the problem to facilitate the convergence to the optimal solution. This move limits are reduced by multipling them for a reduction factor (F_{Red}) each certain number of iterations $(N_{It,Red})$ to avoid the oscillation of the solution around its optimal. Therefore, it will be necessary to calculate, first, the best improvement direction and then to establish the most appropriate value of the improvement factor. The algorithm used to calculate the improvement direction in this publication is the Sequential Linear Programming algorithm based on the Simplex Algorithm. A linear approximation is considered to compute the improvement factor θ_k, since the Simplex Algorithm uses the first order Taylor Series to compute the improvement design direction $\boldsymbol{s}_k$, both parameters are calculated simultaneously. Finally, once both parameters are known, the solution of the topology optimization problem can be updated, and the optimization algorithm can be repeated until convergence. Once the optimization algorithms have been introduced, the next step is to compute the first order sensitivity analysis.

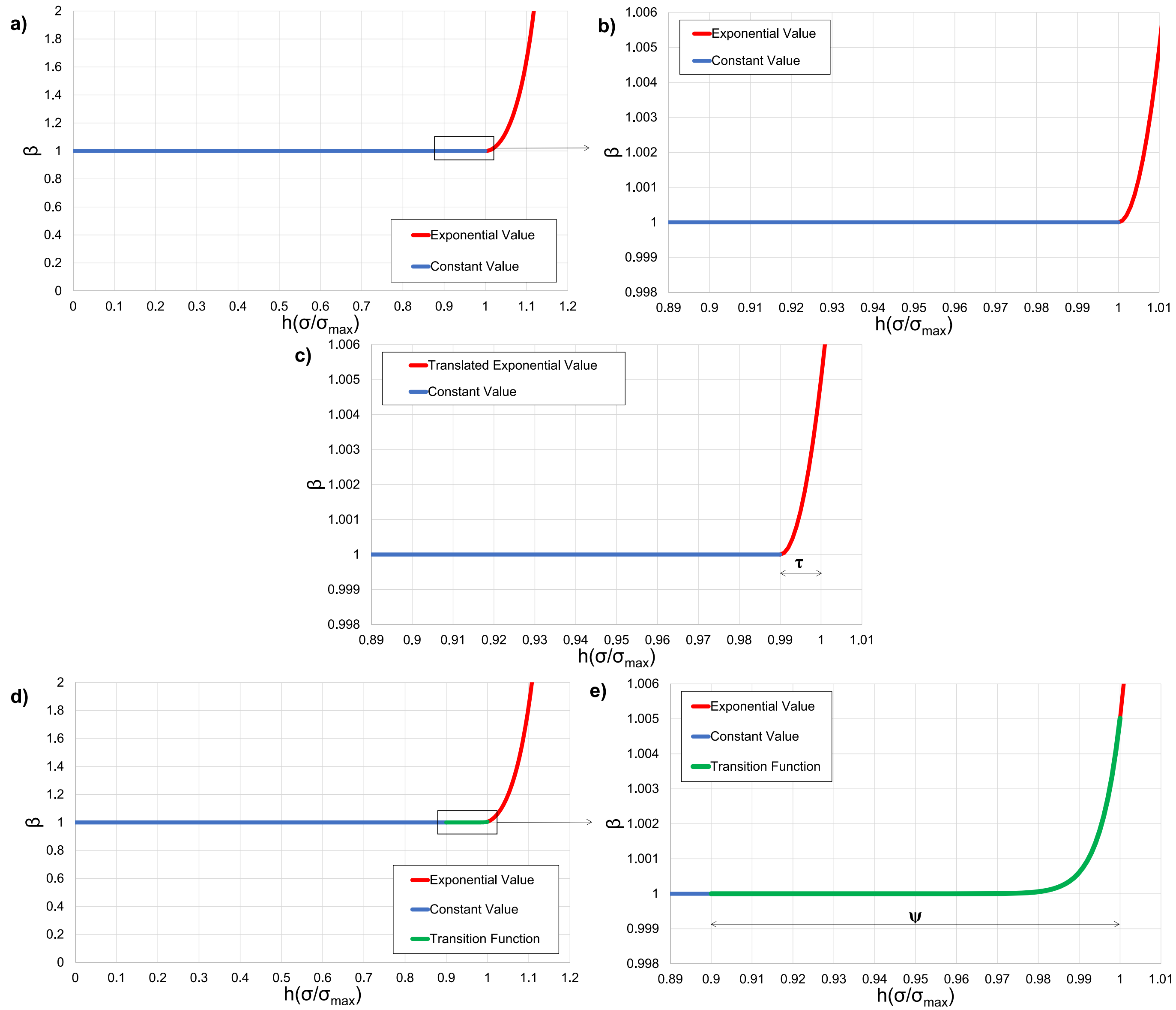

Fig. 3. Overweight function. (a) Original Overweight function (10). (b) Detail of Original Overweight function (10). (c) Detail of translated Overweight function (11). (d) Modified Overweight function (12). (e) Detail of modified Overweight function (12).

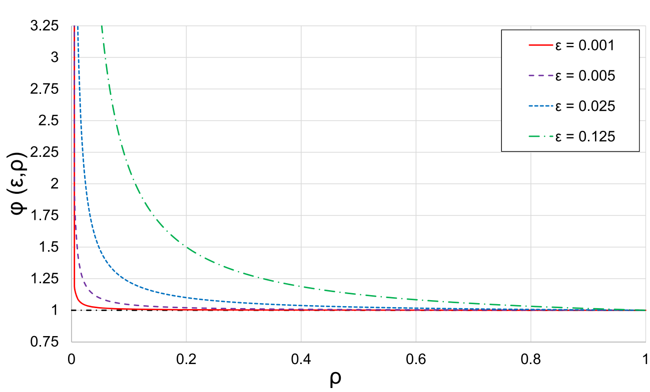

Fig. 4. Stress relaxation coefficient.

4. Sensitivity analysis

The optimization algorithms require the calculation of the first order derivatives of the objective function and all the constraints. However, it is necessary to use two different approaches for computing the first order sensitivity analysis of the objective function and the overweight constraint.

4.1. Sensitivity analysis of the objective function

First, the sensitivity analysis of the objective function can be computed by using direct differentiation, since the structural weight depends directly on the design variables. The sensitivity analysis of the objective function (4) is obtained considering the penalization factor of the intermediate values of relative density $p \geq 1$. The analytical expression of the first order derivative of the objective function is:

$$\frac{\partial F}{\partial \rho_e} = \frac{1}{p}\rho_e^{\frac{1-p}{p}} V_e \tag{18}$$

4.2. Sensitivity analysis of the overweight constraint

The sensitivity analysis of the overweight constraint can be also computed by using direct differentiation. However, the relationship between the overweight constraint and the design variables is not as straightforward as in the objective function, since it depends on the design variables directly and indirectly through the structural stresses. This dependence would mean to solve as many structural analyses as design variables to obtain the sensitivity analysis of the overweight constraint. If the number of constraints is considerably lower than the number of design variables, the Adjoint Variable Approach is the most efficient method to compute the sensitivities of constraints [13,22,45–48]. In this problem, only one constraint is defined and the Adjoint Variable Approach is used. For this reason, the analytical expression of the first order derivatives of the overweight constraint can be stated as:

$$\frac{dg}{d\rho_i} = \frac{\partial g}{\partial \sigma_{VM,s}}\bigg|_{\substack{\sigma_{VM,s}(\sigma_s)\\ \sigma_s(\alpha)\\ \alpha(\rho)}} \frac{d\sigma_{VM,s}}{d\boldsymbol{\sigma_s}}\bigg|_{\substack{\sigma_s(\alpha)\\ \alpha(\rho)}} \frac{d\boldsymbol{\sigma_s}}{d\boldsymbol{\alpha}}\bigg|_{\alpha(\rho)} \frac{d\boldsymbol{\alpha}}{d\rho_i} + \frac{\partial g}{\partial \rho_i}\bigg|_{\substack{\sigma_{VM,s}(\sigma_s)\\ \sigma_s(\alpha)\\ \alpha(\rho)}} \tag{19}$$

where σ_s is the local stresses vector used to compute the Von Mises stress $\sigma_{VM,s}$ and α is the nodal displacement vector. These derivatives can be computed by applying the Adjoint Variable Approach described in [13,22,45–48] as:

$$\frac{dg}{d\rho_i} = \lambda^T \left(\frac{d\mathbf{f}}{d\rho_i} - \frac{d\mathbf{K}}{d\rho_i}\boldsymbol{\alpha} \right) + \frac{\partial \hat{g}}{\partial \rho_i}\bigg|_{\substack{\sigma_{VM,s}(\sigma_s)\\ \sigma_s(\alpha)\\ \alpha(\rho)}} \tag{20}$$

where λ is the adjoint variable, $\mathbf{K}$ is the stiffness matrix of the structure and $\mathbf{f}$ is the vector of applied loads. The Adjoint Variable λ is computed by solving the next system of linear equations:

$$\mathbf{K}^T \lambda = \left(\frac{\partial \hat{g}}{\partial \sigma_{VM,s}}\bigg|_{\substack{\sigma_{VM,s}(\sigma_s)\\ \sigma_s(\alpha)\\ \alpha(\rho)}} \frac{d\sigma_{VM,s}}{d\boldsymbol{\sigma}_s}\bigg|_{\substack{\sigma_s(\alpha)\\ \alpha(\rho)}} \frac{d\boldsymbol{\sigma}_s}{d\boldsymbol{\alpha}}\bigg|_{\alpha(\rho)} \right)^T \tag{21}$$

The terms $\frac{\partial g}{\partial \sigma_{VM,s}}$ and $\frac{\partial g}{\partial \rho_i}$ can be computed by using the chain rule as:

$$\frac{\partial g}{\partial \sigma_{VM,s}} = \frac{\partial g}{\partial \tilde{W}} \frac{\partial \tilde{W}}{\partial \tilde{\rho}_i} \frac{\partial \tilde{\rho}_i}{\partial \beta_s} \frac{\partial \beta_s}{\partial \sigma_{VM,s}} \tag{22}$$

and

$$\frac{\partial g}{\partial \rho_i} = \frac{\partial g}{\partial W} \frac{\partial W}{\partial \rho_i} + \frac{\partial g}{\partial \tilde{W}} \frac{\partial \tilde{W}}{\partial \tilde{\rho}_i} \frac{\partial \tilde{\rho}_i}{\partial \rho_i} \tag{23}$$

Finally, the formulation of the rest of the terms can be seen in [45].

5. Numerical implementation

The most relevant numerical aspects of the algorithm developed in this paper will be commented. First, Fig. 5 shows the flowchart of the algorithm developed. This procedure is repeated until convergence. The numerical implementation has been made in Fortran. The non-linearity of the problem solved in this paper produces an important oscillation of the solutions obtained around its optimum. For this reason, some of the parameters of the problem are modified to make possible the convergence of the solution to its optimum and to obtain full-void solution: the move limits of the design variables $(\Delta\rho_{i,j})_{max}$, the penalty coefficient of intermediate values of relative density p and the stress relaxation coefficient ε. The value of these parameters is changed each certain number of iterations. And finally, when all the values of the parameters have been used and the solution does not experience important changes, the optimum has been attained. This is the convergence criterium considered.

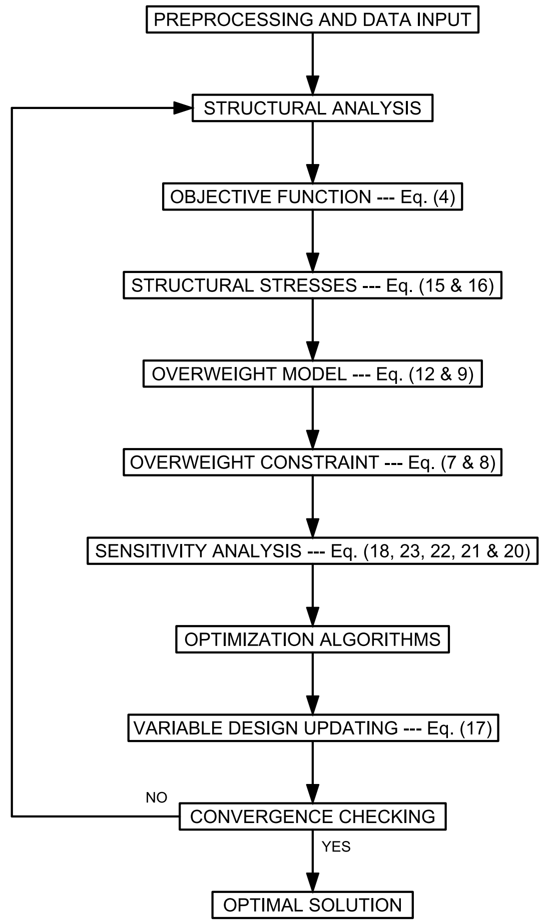

Fig. 5. Algorithm's flowchart.

6. Application examples

The formulation proposed is validated by studying three structural problems frequently analysed in the topology optimization field to test the performance of the overweight approach technique as an indirect way to impose stress constraints in the problem. These examples are two-dimensional structures in plane stress and the material considered is steel with density $\rho_{mat} = 7850$ kg/m^3, Young's modulus $E = 210$ GPa, Poisson's ratio $\nu = 0.3$ and yield stress $\hat{\sigma}_{max} = 230$ MPa. Structural self-weight is included as a structural load and the initial design of all the examples consists on a domain full of material. All the examples have been computed on an Intel(R) Xeon(R) CPU E5-2697 A v4 processor of 2.60 GHz with 64GB of RAM.

6.1. Cantilever beam

The first example corresponds to a cantilever beam with null displacements in the left edge and a vertical force applied in the middle of the right edge. Fig. 6 shows the dimensions of the domain and the position of the vertical forces applied. The domain of the structure is discretized with a homogeneous mesh of 200 x 100 eight-node quadrilateral elements. The structural thickness is 0.30 m and the punctual load (1000 kN) is distributed on twelve elements to avoid stress accumulation phenomena. This number of elements only has influence in the vicinity of the loading point and it diminishes as the distance with this area increases.

Fig. 7 shows the optimal solution of the problem that consists in a set of bars which coincides with isostatic lines. The most important characteristic of the optimal solution is the existence of a certain

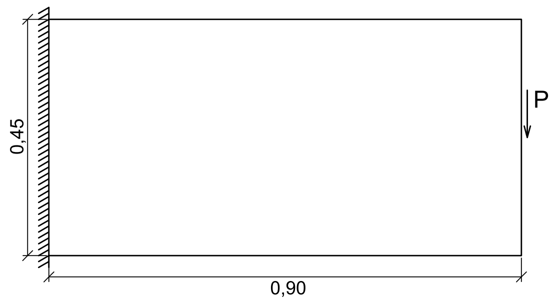

Fig. 6. Cantilever beam: Domain dimensions. (Units - m).

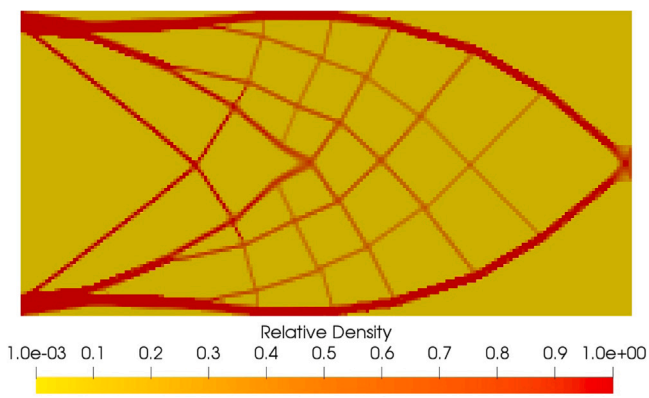

Fig. 7. Cantilever beam: Optimal solution.

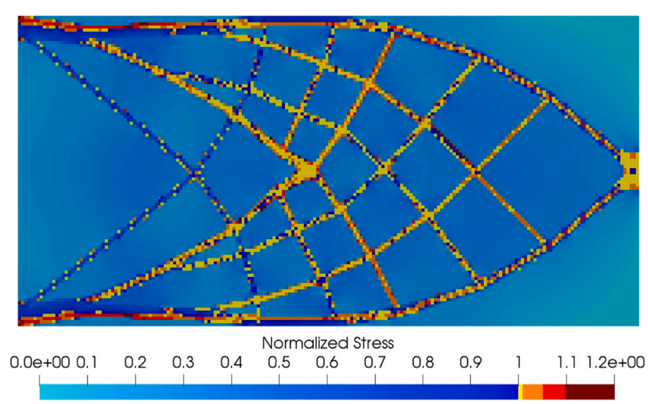

Fig. 8. Cantilever beam: Normalized stress.

symmetry between the upper and the lower part of the domain. This can be explained since the structural model is linear and the material is steel whose behaviour to traction and compression forces is similar. However, the solution of this example requires more iterations than the other examples of this paper, since its theoretical solution consists in an infinite number of bars that coincides with the isostatic lines. This means that the topology optimization algorithm has to select and remove the majority of these bars, what supposes an increase in the difficulty of the problem. Fig. 8 shows the normalized stress state obtained through the quotient between the stress and the stress relaxation coefficient times the maximum allowable stress in each point of the domain.

Fig. 9 shows the evolution of the structural weight and the objective function. The value of the structural weight is constant during the most of the process due to the redistribution of the material in the domain. This phenomena can be observed in Fig. 10. The objective function is always reduced except in the iterations in which the penalty coefficient

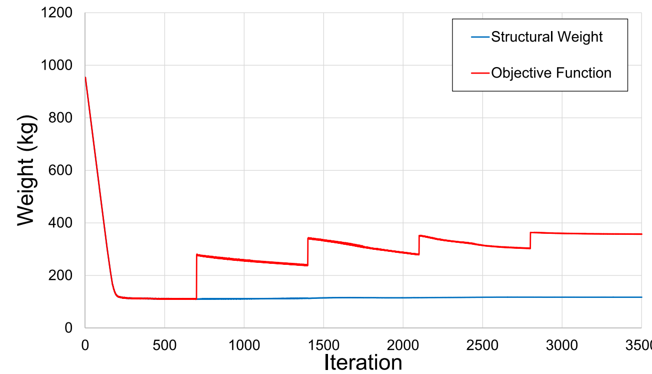

Fig. 9. Cantilever beam: Structural weight.

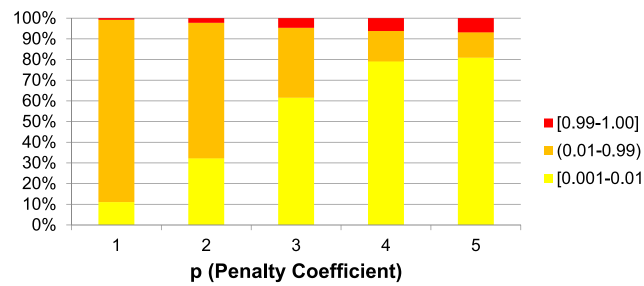

Fig. 10. Cantilever beam: Design variables distribution.

Table 2
Cantilever Beam: General parameters of the problem.

Data input	Value
n	20000
$(\Delta\rho_{i,0})_{max}$	0.005
F_{Red}	0.80
$N_{It,Red}$	350

Table 3
Cantilever Beam: Evolution of the optimization parameters.

Iterations	p	ε
0–699	1	0
700–1399	2	0.001
1400–2099	3	0.002
2100–2799	4	0.003
2800–3500	5	0.005

Table 4
Cantilever Beam: Results.

Data output	Value	Units
Number of iterations	3500	
Final weight	117.3	kg
Initial weight percentage	12.30%	
Measure of discreteness	8.95	
Total CPU time	22.25	h

Table 5
Cantilever Beam: Distribution of CPU time per iteration.

Algorithms	Time
Structural analysis	74.27%
Sensitivity analysis	23.23%
Optimization algorithms	1.81%
Rest of the process	0.69%

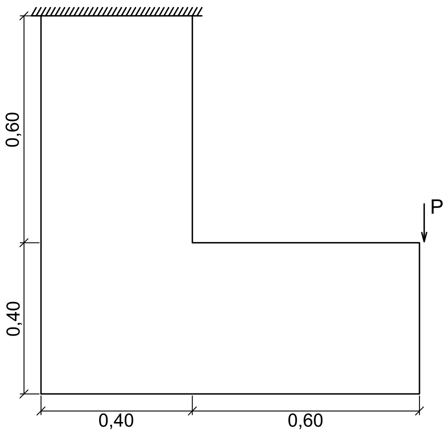

Fig. 11. L-shaped beam: Domain dimensions. (Units - m).

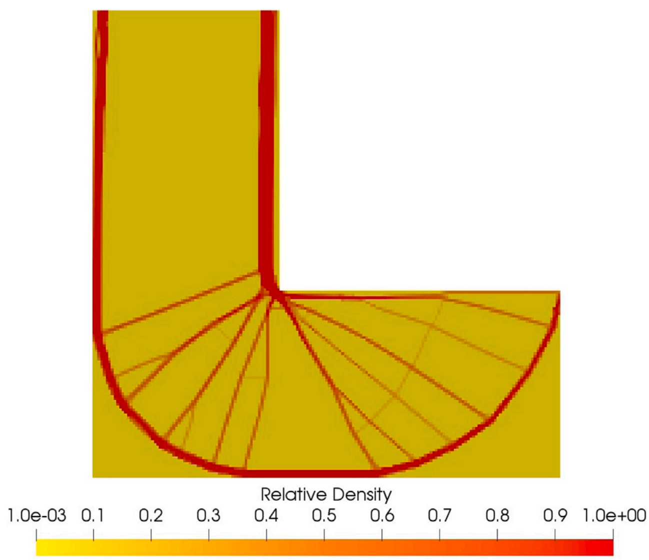

Fig. 12. L-shaped beam: Optimal solution.

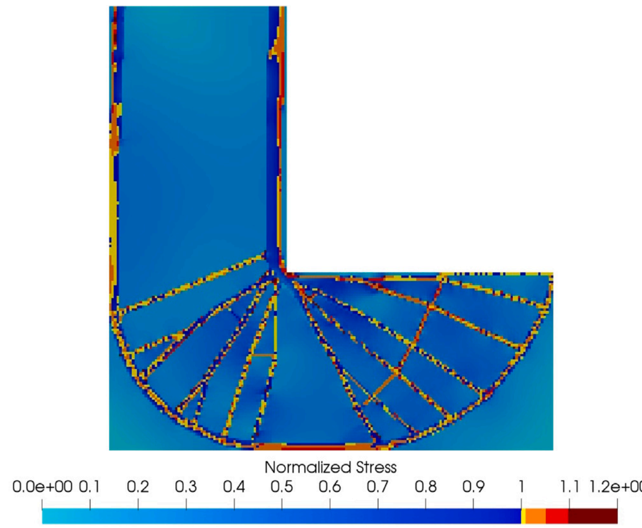

Fig. 13. L-shaped beam: Normalized stress.

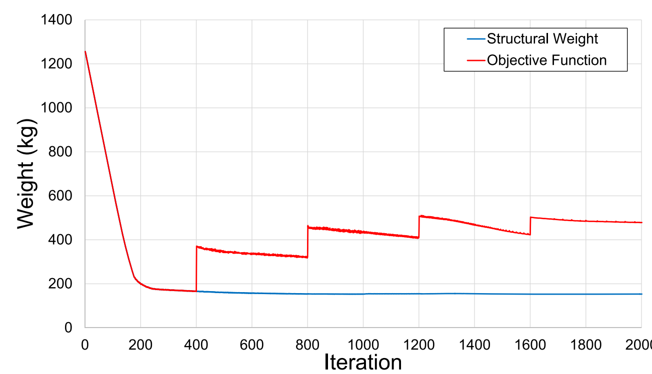

Fig. 14. L-shaped beam: Structural weight.

of intermediate values of relative density is modified. Fig. 10 shows the design variables distribution in the last iteration of each value of the penalty coefficient. The presence of intermediate values of the design variables is reduced as the penalty coefficient is increased. Tables 2 and 3 show the value of the most important parameters of the problem. Table 4 shows the results obtained with the solution of the problem, and Table 5 shows the distribution of the computing time per algorithm of an average iteration.

6.2. L-shaped beam

The second example corresponds to a L-shaped beam with null displacements in the upper edge and a vertical force applied in the upper part of the right edge. Fig. 11 shows the dimensions of the domain and the position of the vertical forces applied. The domain of the structure is discretized with a mesh of 19600 eight-node quadrilateral elements. The structural thickness is 0.25 m and the external force applied (650 kN) has been distributed on five elements.

Fig. 12 shows the optimal solution of the problem. Fig. 13 shows the normalized stress state. There are regions where stresses are slightly higher to their maximum. This is common to all the examples and is related with the relaxation of the overweight constraint needed to have an inequality constraint in the problem. This constraint violation can be reduced, by decreasing the coefficient used to relax the overweight constraint or by increasing the exponential degradation parameter of the overweight function. Moreover, this is also related with the definition of the relative density in the overweight model. The minimum value is introduced in (9) to avoid adding a penalization over the areas with minimum relative density when stresses are higher than their maximum. That is to facilitate removing material in low density areas. This intends to simulate the non-existence of material in this areas what means the non-existence of stresses, the singularity phenomena. Lastly, the stresses slightly higher to their maximum are related with the stress concentration phenomena in the inside corners of the domain. This can be avoided by doing a corner smoothing, however, the geometry considered in the solution of the L-shaped beam problem, is the geometry traditionally used in the literature. On the other hand, the formulation with stress constraints can avoid the stress concentration phenomena by rounding the geometry of the optimal solution in the corners where the stress concentration phenomena appears as it can be see in Fig. 12.

Fig. 14 shows the evolution of the structural weight and the objective function. Fig. 15 shows the design variables in the last iteration of each value of the penalty coefficient of intermediate values of density. Both figures show the same behaviour than in the previous example. Table 6 shows the value of the parameters related with the definition of the problem and with the optimization process. Table 7 shows the different values of the penalty coefficient and the stress relaxation coefficient during the optimization process and the application range of iterations. Table 8 shows the most relevant parameters of the optimization process. Finally, Table 9 shows the distribution of the average computing time per algorithm of one iteration.

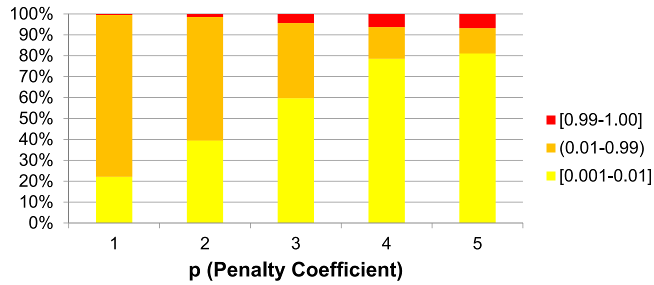

Fig. 15. L-shaped beam: Design variables distribution.

Table 6
L-shaped Beam: General parameters of the problem.

Data input	Value
n	19600
$(\Delta\rho_{i,0})_{max}$	0.005
F_{Red}	0.80
$N_{It,Red}$	200

Table 7
L-shaped Beam: Evolution of the optimization parameters.

Iterations	p	ε
0–399	1	0
400–799	2	0.001
800–1199	3	0.002
1200–1599	4	0.003
1600–2000	5	0.005

Table 8
L-shaped Beam: Results.

Data output	Value	Units
Number of iterations	2000	
Final weight	153.4	kg
Initial weight percentage	12.21%	
Measure of discreteness	8.37	
Total CPU time	9.79	h

Table 9
L-shaped Beam: Distribution of CPU time per iteration.

Algorithms	Time
Structural analysis	69.08%
Sensitivity analysis	28.02%
Optimization algorithms	2.02%
Rest of the process	0.88%

6.3. MBB beam

The last example corresponds to a MBB beam with null vertical displacements in the supports and a vertical distributed force applied in the central part of the structure over the upper edge. Fig. 16 shows the dimensions of the domain and the position of the vertical forces applied of the entire structure and the part of the structure considered in the solution of the problem. The structural domain has been discretized with a homogeneous mesh of 240 times 80 eight-node quadrilateral elements. The structural thickness is 0.30 m and the distributed load of 3200 kN/m is applied over 32 adjacent elements in the upper part of the domain in the proximity of the symmetry axis.

Fig. 17 shows the optimal solution of the problem with the symmetrical replication. Fig. 18 shows the normalized stress state and it can be observed that in the same way that the previous examples there are regions where stresses are slightly higher to their maximum. Fig. 19 shows the evolution of the structural weight and the objective function and Fig. 20 shows the design variables distribution in the last iteration of each value of the penalty coefficient of intermediate values of density. While the structural weight is approximately constant during the most of the process due to the redistribution of material, the objective function is always reduced except in the iterations in which the penalty coefficient of intermediate values of density is modified, as it can be seen in Fig. 19. On the other hand, the presence of intermediate values of relative density is reduced as the penalty coefficient is increased, as it can be observed in Fig. 20. Therefore, both figures represent the same behaviour than previous examples. The self-weight does not have influence in the results since its magnitude only supposes at most 5% of the loads applied over the structure.

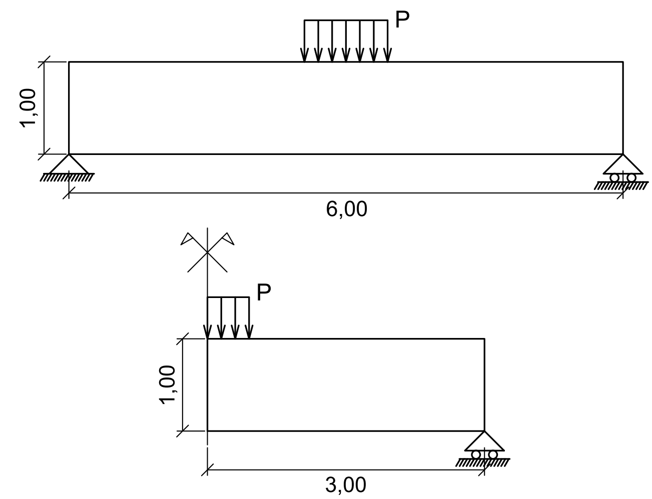

Fig. 16. MBB beam: Domain dimensions. (Units - m).

Table 10
MBB Beam: General parameters of the problem.

Data input	Value
n	19200
$(\Delta\rho_{i,0})_{max}$	0.005
F_{Red}	0.80
$N_{It,Red}$	150

Table 11
MBB Beam: Evolution of the optimization parameters.

Iterations	p	ε
0–299	1	0
300–599	2	0.001
600–899	3	0.002
900–1199	4	0.003
1200–1500	5	0.005

Tables 10 and 11 show the value of the most important parameters of the problem. Table 12 shows the results obtained with the solution of the problem and Table 13 shows the distribution of the average computing time per algorithm at each iteration. Table 12 shows that 1500 iterations are required to guarantee the convergence to the optimal solution. It is important to remark that the complexity of the problem is related with the number of iterations required to solve it. Moreover, the number of iterations required to solve the problem can be reduced considerably if a quadratic approximation of the objective function and the overweight constraint is used to compute the improvement factor since the oscillations around the optimal solution would be mitigated. Table 13 shows, in the same way that previous examples, that the structural analysis and the sensitivity analysis mean 98% of the total CPU time required to solve the problem. The CPU time required for the rest of the algorithms including the data initialization can be considered negligible.

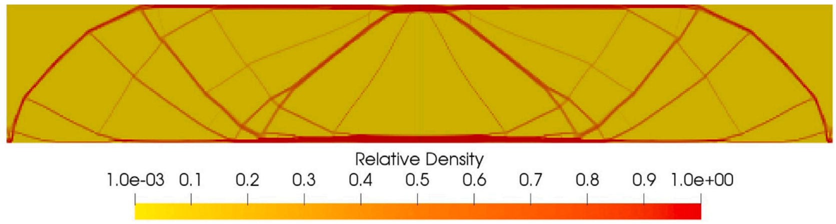

Fig. 17. MBB beam: Optimal solution.

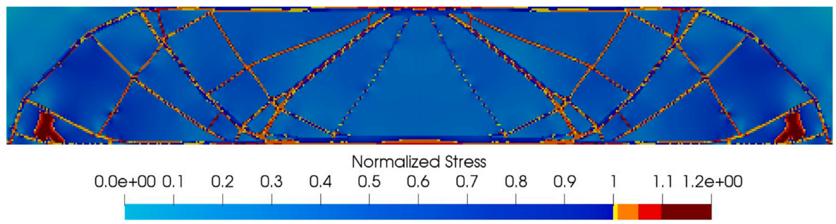

Fig. 18. MBB beam: Normalized stress.

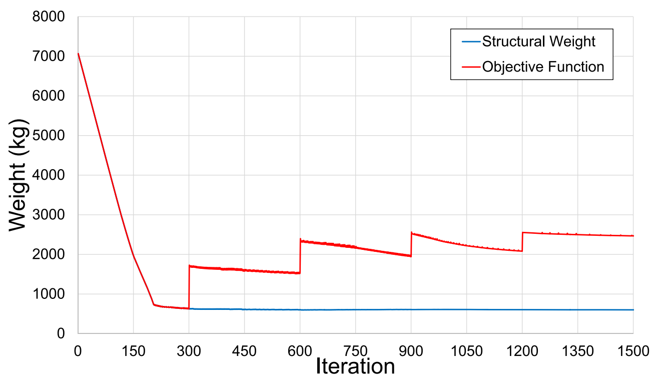

Fig. 19. MBB beam: Structural weight.

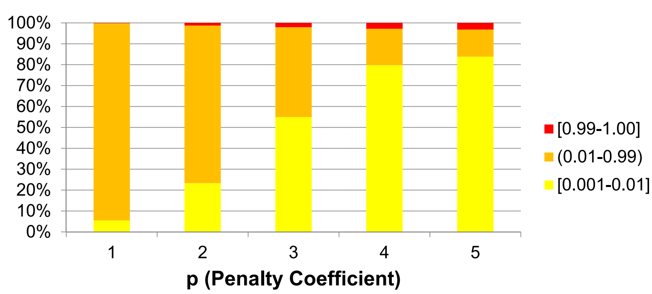

Fig. 20. MBB beam: Design variables distribution.

Table 12
MBB Beam: Results.

Data output	Value	Units
Number of iterations	1500	
Final weight	604.2	kg
Initial weight percentage	8.56%	
Measure of discreteness	9.13	
Total CPU time	6.75	h

7. Conclusions

This paper introduces an alternative method to the original damage approach to solve the structural topology optimization problem with minimum weight and stress constraints. This method requires only one overweight constraint to consider the effect of all the local stress constraints. This method provides solutions with high spatial definition since problems with a large number of design variables are solved in an affordable amount of time. On the other hand, the use of the structural weight to define the overweight constraint requires to calculate it only for the overweight model since the objective function requires to compute it for the original model. By contrast, the Damage Approach requires to compute the structural stiffness of both models, the original and the alternative. First, the computation of the structural stiffness of the original model is not considerably expensive, since the structural displacements have been computed to obtain the structural stresses. However, the computation of the structural stiffness in the alternative model, requires to do an additional structural analysis, since the stiffness matrix in this case will be different due to the modification of the Young Modulus in the alternative model. Since the structural analysis in this manuscript is the critical part of the algorithm in terms of CPU time, the Overweight approach introduces a reduction of CPU time required to solve the topology optimization problem with respect to the previously developed Damage Approach. The solution of the topology optimization problem with local stress constraints requires to compute all the active local stress constraint derivatives, whose number increases as the solution converges to its optimum. Conversely, the overweight approach only defines just one constraint: the overweight constraint. This means that the CPU time required for the search of the improvement direction and the sensitivity analysis will be considerably reduced in the optimization process. Consequently, the structural analysis becomes the critical step in terms of the CPU time to solve the problem with the method developed, since the sensitivity analysis is computed in an efficient way by using the adjoint variable approach. Even though the use of the aggregation techniques of constraints (as the overweight constraint) requires more iterations than the use of local stress constraints, the CPU time required for each iteration is much smaller and consequently the formulation proposed require less CPU time. Thus, cases or examples with a large number of design variables can be solved in an affordable amount of CPU time. Furthermore, the

Table 13
MBB Beam: Distribution of CPU time per iteration.

Algorithms	Time
Structural analysis	66.79%
Sensitivity analysis	30.16%
Optimization algorithms	2.10%
Rest of the process	0.95%

approach developed could improve its efficiency with respect to the CPU time if parallel computing techniques were implemented.

Finally, the Overweight approach developed in this paper has some advantages with respect to the aggregation functions used to state the global stress constraints. While the aggregation functions consider all the local stress constraints during the solution of the optimization problem, the Overweight Approach does not take into account the local stress constraints placed in areas with $\rho = \rho_{min}$, since the relative density of both models will coincide even when the stresses are higher than their maximum value. This advantage of the Overweight Approach is also common with respect to the local stress constraints approach. In other words, the Overweight Constraint model proposed does not penalize the weight of areas with minimum relative density even though the stresses exceed their maximum value. This characteristic allows the existence of these areas without using a high stress relaxation coefficient. That is, it helps to avoid the singularity phenomena. As it was mentioned, this intends to simulate the effect of the areas with relative density equal to zero, where the stresses does not exist properly.

The very definition of the overweight constraint allows to check whether all the stresses are below their maximum value by comparing two different models: the original and the overweight. However, a small violation of the local stress constraints is allowed due to numerical reasons. That is, the overweight constraint introduced to have a inequality constraint and the use of stress relaxation coefficients. This effect can be mitigated by reducing the value of the stress relaxation coefficient and/or by increasing the value of the exponential degradation parameter. The latter means also an increase in the number of iterations required to solve the problem due to its high non-linearity. The quality of the results obtained with the overweight approach is very good, since problems with a considerable number of elements can be solved in a reasonable amount of CPU time. Moreover, the use of a penalty coefficient over the intermediate values of the relative density provides material distributions with only 10%–15% of the domain with intermediate values of relative density. This is very important from a practical point of view since the solutions obtained could be easily manufactured. Moreover, the use of a penalty coefficient over the intermediate values of the relative density provides solutions whose topology is clearly defined and manufactured after submitting the solutions to a postprocessing process. In conclusion, the use of the Overweight Approach proposed reports important benefits from a computational point of view in topology optimization problems with stress constraints. The CPU time required to obtain the solution of problems with a large number of design variables is affordable, what makes manageable in practice to attain solutions with high spatial definition.

Replication of results

The code and data of this study are available on request from the corresponding author.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

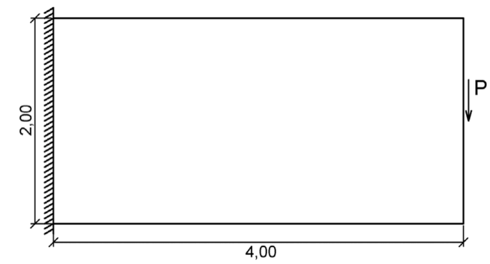

Fig. 21. Cantilever beam: Domain dimensions (Units - m).

Acknowledgements

This work has been partially supported by the "Xunta de Galicia, Spain (Secretaría Xeral de Universidades)" and the European Union (European Social Fund - ESF) through "Grants for supporting the predoctoral stage in universities of the Galician University System, in research public organisms of Galicia and in other entities of the "R & D system of the Galician Government - 2016" ED481A-2016/387, by Feder funds of the European Union, by the "Ministerio de Economía y Competitividad, Spain" of the Spanish Government through grants DPI2015-68431-R and RTI2018-093366-B-I00, by the "Consellería de Educación, Universidade e Formación Profesional, Spain" of the "Xunta de Galicia" through "grants for the consolidation and structuring of competitive research units of the Galician University System: Competitive reference group" GRC2014/039 and GRC2018/41, and by research fellowships of the University of A Coruña, Spain and the "Fundación de la Ingeniería Civil de Galicia, Spain".

Appendix. Comparison of the Overweight Approach with the previous formulations of the same problem

A limited comparison between performance of the local stress constraint approach and the global stress constraint with a classical aggregation function and a Overweight aggregation function is presented here, based on a benchmark problem previously solved by the authors. A more comprehensive investigation of the comparative performance between local and global stress constraint approaches is presented in [49]. The analysis herein aims to demonstrate how the proposed method in this manuscript compares with the classical methods in the literature. The results of this benchmark problem with the three formulations: Local strategy, Global strategy (Aggregation Function and Overweight Approach) computed in the same computer are shown below. The benchmark problem considered to compare all the formulations corresponds to a cantilever beam (similar to the problem solved in Section 6.1) with null displacements in the left edge and vertical force applied in the middle of the right edge. Fig. 21 shows the dimensions of the domain and the position of the vertical forces applied. The domain of the structure is discretized with a homogeneous mesh of 120 × 60 eight-node quadrilateral elements. The structural thickness is 0.2 m and the punctual load (4000 kN) is distributed on eight elements to avoid stress concentration phenomena. Figs. 22–24 show the solution of the problem with the Local Stress Constraints, with a Global Stress Constrain by using an aggregation function and the Overweight Constraint respectively. In all the cases, the optimal solution consists in a set of bars which coincides with isostatic lines. Figs. 25–27 show the normalized stress state obtained through the quotient between the stress and the stress relaxation coefficient times the maximum allowable stress in each point of the domain for the three formulations. Table 14 show the value of the most important parameters of the problem that will be used to do the comparison between the three formulations. The comparative analysis is shown below.

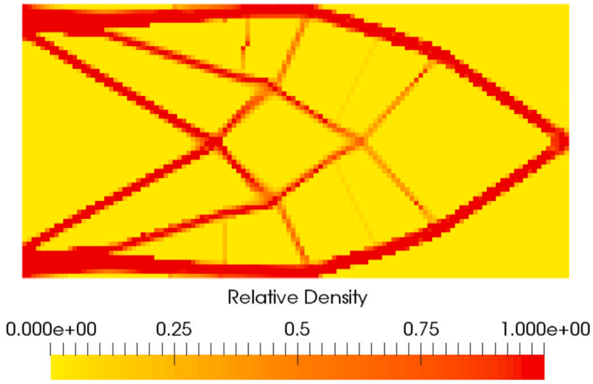

Fig. 22. Cantilever beam: Optimal solution - Local stress constraints.

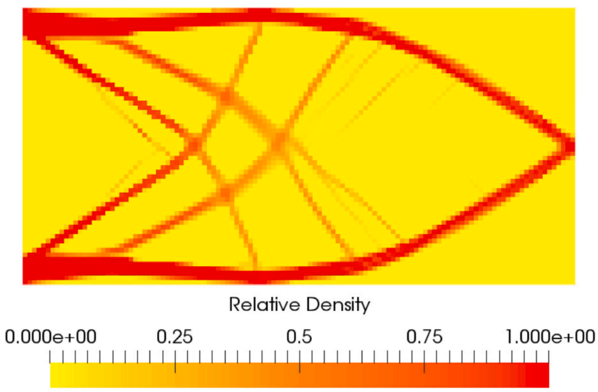

Fig. 23. Cantilever beam: Optimal solution - Global stress constraint.

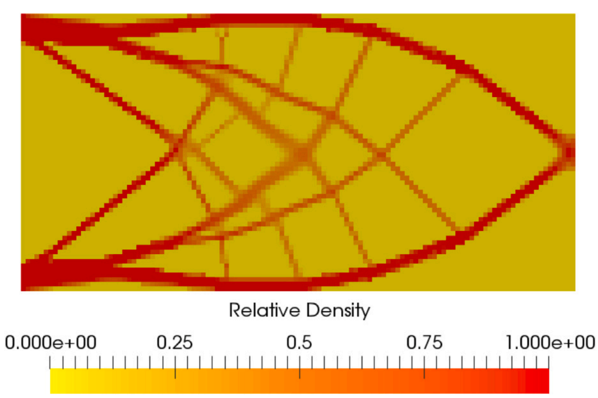

Fig. 24. Cantilever beam: Optimal solution - Overweight constraint.

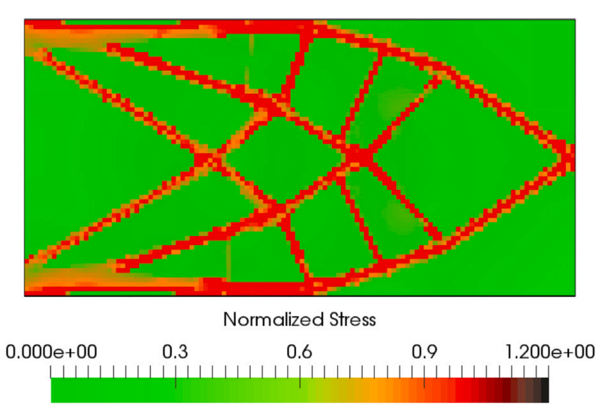

Fig. 25. Cantilever beam: Normalized stress - Local stress constraint.

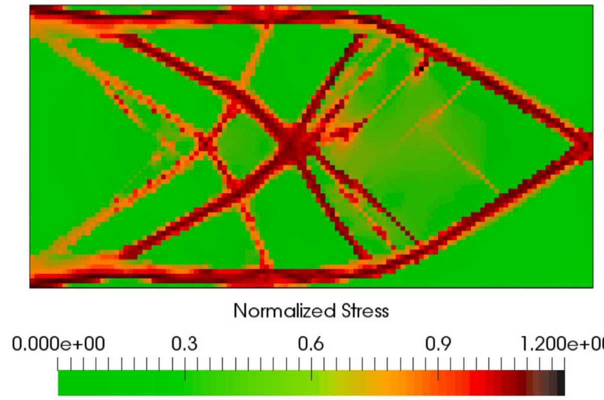

Fig. 26. Cantilever beam: Normalized stress - Global stress constraint.

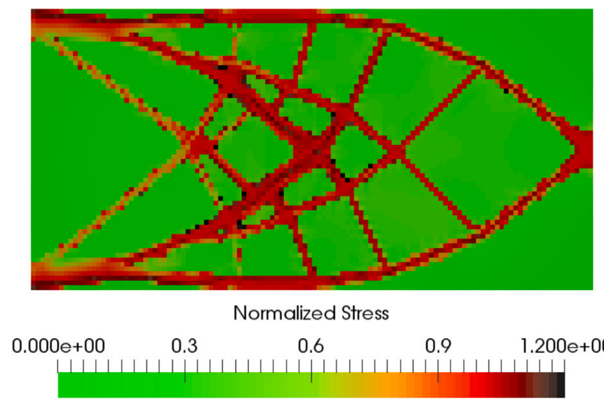

Fig. 27. Cantilever beam: Normalized stress - Overweight constraint.

Table 14
Cantilever Beam: Comparative analysis.

Data output	Local stress constraint	Global stress constraint	Overweight constraint
Number of iterations	157	971	1400
Initial Volume percentage	18.27%	16.05%	17.08%
Total CPU time	729.4 h	10.4 h	1.3 h

- Quality of the solutions: The topology is perfectly defined in the solutions obtained with the local stress constraints and the overweight constraint, by contrast, the solution obtained with the global stress constraints by using an aggregation function shows parts of the structure that are non-connected with the rest of the structure.
- Structural Stresses: The structural stresses with the local stress constraints are in all the domain lower than their maximum, on the contrary, in the solutions obtained with the global stress constraint by using an aggregation function and the Overweight constraint the structural stresses can be slightly higher than their maximum in some parts of the domain.
- Number of Iteration. The local stress constraints approach requires less iterations than the global stress constraint approach with an aggregation function and the overweight approach. Moreover, the overweight approach requires more iterations than the global stress constraint due to the high non-linearity of the overweight constraint.
- Final Weight: The solutions obtained with the three formulations have approximately the same structural weight, since the differences in the final volume are lower than a 3% of the volume

of the domain, consequently, these differences can be considered negligible.

- CPU time: The CPU time required for the Overweight Approach is considerable lower than the CPU time required for the previous formulations, Local stress constraints and global stress constraint with an aggregation function. The use of Local stress constraints requires more time than the other formulations to solve the problem since the number of active constraints increases as the solution converges to its optimum, by contrast, the other two formulations only defines one constraint to control the same structural stresses. On the other hand, the Overweight approach requires less CPU time than the global stress constraint approach with an aggregation function, since only the derivative of the stresses higher than a certain percentage of the maximum value has to be considered to compute the Overweight Constraint derivative, on the contrary, the global stress constraint derivative with an aggregation function requires to compute the derivatives of all the structural stresses.

Finally, it is possible to conclude that the Overweight Approach is a valid alternative to the previous formulations used to consider the stress constraints in the topology optimization problem of minimum weight.

References

[1] Bendsøe M, Kikuchi N. Generating optimal topologies in structural design using a homogenization method. Comput Methods Appl Mech Engrg 1988;71:197–224. http://dx.doi.org/10.1016/0045-7825(88)90086-2.

[2] Bendsøe M. Optimal shape design as a material distribution problem. Struct Multidiscip Optim 1989;1:193–202. http://dx.doi.org/10.1007/BF01650949.

[3] Bendsøe M. Optimization of structural topology. Springer International Publishing; 1995, http://dx.doi.org/10.1007/978-3-662-03115-5,

[4] Stolpe M, Svanberg K. An alternative interpolation scheme for minimum compliance topology optimization. Struct Multidiscip Optim 2001;22:116–24. http://dx.doi.org/10.1007/s001580100129.

[5] Mahdavi A, Balaji R, Frecker M, Mockensturm E. Topology optimization of 2D continua for minimum compliance using parallel computing. Struct Multidiscip Optim 2006;32:121–32. http://dx.doi.org/10.1007/s00158-006-0006-1.

[6] Niu F, Xu S, Cheng G. A general formulation of structural topology optimization for maximizing structural stiffness. Struct Multidiscip Optim 2011;43:561–72. http://dx.doi.org/10.1007/s00158-010-0585-8.

[7] Bochenek B, Mazur M. A novel heuristic algorithm for minimum compliance topology optimization. Eng Trans 2016;64:541–6, URL http://entra.put.poznan.pl/index.php/et/article/viewFile/692/594.

[8] Wang H, Cheng W, Du R, Wang S, Yang Y. Improved proportional topology optimization algorithm for solving minimum compliance problem. Struct Multidiscip Optim 2020;62:475–93. http://dx.doi.org/10.1007/s00158-020-02504-8.

[9] Duysinx P, Bendsøe M. Topology optimization of continuum structures with local stress constraints. Int J Numer Methods Eng 1998;43:1453–78. http://dx.doi.org/10.1002/(SICI)1097-0207(19981230)43:8.

[10] Navarrina F, Muinos I, Colominas I, Casteleiro M. Topology optimization of structures: A minimum weight approach with stress constraints. Adv Eng Softw 2005;36:599–606. http://dx.doi.org/10.1016/j.advengsoft.2005.03.005.

[11] París J, Navarrina F, Colominas I, Casteleiro M. Block aggregation of stress constraints in topology optimization of structures. Comput Aided Optim Des Eng X 2007;91:25–34. http://dx.doi.org/10.2495/OP070031.

[12] París J, Navarrina F, Colominas I, Casteleiro M. Topology optimization of continuum structures with local and global stress constraints. Struct Multidiscip Optim 2009;39:419–37. http://dx.doi.org/10.1007/s00158-008-0336-2.

[13] París J, Navarrina F, Colominas I, Casteleiro M. Improvements in the treatment of stress constraints in structural topology optimization problems. J Comput Appl Math 2010;234:2231–8. http://dx.doi.org/10.1016/j.cam.2009.08.080.

[14] Bruggi M, Duysinx P. Topology optimization for minimum weight with compliance and stress constraints. Struct Multidiscip Optim 2012;46:369–84. http://dx.doi.org/10.1007/s00158-012-0759-7.

[15] Cheng G, Jiang Z. Study on topology optimization with stress constraints. Eng Optim 1992;20:129–48. http://dx.doi.org/10.1080/03052159208941276.

[16] Yang R, Chen C. Stress-base topology optimization. Struct Multidiscip Optim 1996;12:98–105. http://dx.doi.org/10.1007/BF01196941.

[17] Navarrina F, Muinos I, Colominas I, Casteleiro M. Minimum weight with stress constraints topology optimization. In: 8th International conference on computer aided optimum design of structures. 2003, URL https://ruc.udc.es/dspace/handle/2183/402.

[18] Le C, Norato J, Bruns T, Ha C, Tortorelli D. Stress-based topology optimization for continua. Struct Multidiscip Optim 2010;41:605–20. http://dx.doi.org/10.1007/s00150-009-0440-y.

[19] da Silva GA, Aage N, Beck AT, Sigmund O. Three-dimensional manufacturing tolerant topology optimization with hundreds of millions of local stress constraints. Internat J Numer Methods Engrg 2020;122(2):548–78. http://dx.doi.org/10.1002/nme.6548.

[20] Giraldo-Londono O, Paulino G. A unified approach for topology optimization with local stress constraints considering various failure criteria: von mises, drucker-prager, tresca, mohr-Coulomb, bresler-pister and willam-warnke. Proc R Soc A 2020;476. http://dx.doi.org/10.1098/rspa.2019.0861.

[21] Martins J, Poon N. On structural optimization using constraint aggregation. In: 6th World congress on structural multidisciplinary optimization. 2005, URL https://citeseerx.ist.psu.edu/viewdoc/download?-doi=10.1.1.140.3612&rep=rep1&type=pdf.

[22] Poon N, Martins J. An adaptive approach to constraint aggregation using adjoint sensitivity analysis. Struct Multidiscip Optim 2007;34:61–73. http://dx.doi.org/10.1007/s00158-006-0061-7.

[23] Liu H, Yang D, Hao P, Zhu X. Isogeometric analysis based topology optimization design with global stress constraint. Comput Methods Appl Mech Engrg 2018;342:625–52. http://dx.doi.org/10.1016/j.cma.2018.08.013.

[24] Fan Z, Xia L, Lai W, Xia Q, Shi T. Evolutionary topology optimization of continuum structures with stress constraints. Struct Multidiscip Optim 2019;59:647–58. http://dx.doi.org/10.1007/s00158-018-2090-4.

[25] París J, Navarrina F, Colominas I, Casteleiro M. Block aggregation of stress constraints in topology optimization of structures. Adv Eng Softw 2010;41:433–41. http://dx.doi.org/10.1016/j.advengsoft.2009.03.006.

[26] Verbart A, Langelaar M, van Keulen F. Damage approach: A new method for topology optimization with local stress constraints. Struct Multidiscip Optim 2016;53:1081–98. http://dx.doi.org/10.1007/s00158-015-1318-9.

[27] Kreisselmeier G, Steinhauser R. Systematic control design by optimizing a vector performance indicator. Symp Comput-Aided Des Control Syst 1979;12:113–7. http://dx.doi.org/10.1016/S1474-6670(17)65584-8.

[28] Kreisselmeier G, Steinhauser R. Application of vector performance optimization to a robust control loop design for a fighter aircraft. Int J Control 1983;37:251–84. http://dx.doi.org/10.1080/00207179.1983.9753066.

[29] Zhu J, Zhang W, Xia L. Topology optimization in aircraft and aerospace structures design. Archi Comput Methods Eng 2016;23:595–622. http://dx.doi.org/10.1007/s11831-015-9151-2.

[30] Verbart A, Langelaar M, van Keulen F. A unified aggregation and relaxation approach for stress-constrained topology optimization. Struct Multidiscip Optim 2017;55:663–79. http://dx.doi.org/10.1007/s00158-016-1524-0.

[31] Duysinx P, Sigmund O. New development in handling stress constraints in optimal material distribution. In: 7th AIAA/USAF/NASA/ISSMO Symposium on multidisciplinary design optimization. 1998, http://dx.doi.org/10.2514/6.1998-4906.

[32] Qiu G, Liu X. A note on the derivation of global stress constraints. Struct Multidiscip Optim 2010;40:625–8. http://dx.doi.org/10.1007/s00158-009-0397-x.

[33] Lee K, Ahn K, Yoo J. A novel P-norm correction method for lightweight topology optimization under maximum stress constraints. Comput Struct 2016;171:18–30. http://dx.doi.org/10.1016/j.compstruct.2016.04.005.

[34] Conlan-Smith C, James K. A stress-based topology optimization method for heterogeneous structures. Struct Multidiscip Optim 2019;60:167–83. http://dx.doi.org/10.1007/s00158-019-02207-9.

[35] Kennedy G, Hicken J. Improved constraint-aggregation methods. Comput Methods Appl Mech Engrg 2015;289:332–54. http://dx.doi.org/10.1016/j.cma.2015.02.017.

[36] Lambe A, Kennedy G, Martins J. An evaluation of constraint aggregation strategies for wing bos mass minimization. Struct Multidiscip Optim 2016;55:257–77. http://dx.doi.org/10.1007/s00158-016-1495-1.

[37] Rossow H, Taylor J. A finite element method for the optimal design of variable thickness sheets. AAIA J 1973;11:1566–9. http://dx.doi.org/10.2514/3.50631.

[38] Cheng G, Guo X. ϵ-Relaxed approach in structural topology optimization. Struct Multidiscip Optim 1997;13:258–66. http://dx.doi.org/10.1007/BF01197454.

[39] Bendsøe M, Sigmund O. Material interpolation schemes in topology optimization. Arch Appl Mech 1999;69:635–54. http://dx.doi.org/10.1007/s004190050248.

[40] Eschenauer H, Olhoff N. Topology optimization of continuum structures: A review. Appl Mech Rev 2001;54:331–90. http://dx.doi.org/10.1115/1.1388075.

[41] Bruggi M. On an alternative approach to stress constraints relaxation in topology optimization. Struct Multidiscip Optim 2008;36:125–41. http://dx.doi.org/10.1007/s00158-007-0203-6.

[42] Zuo W, Saitou K. Multi-material topology optimization using ordered SIMP interpolation. Struct Multidiscip Optim 2017;55:477–91. http://dx.doi.org/10.1007/s00158-016-1513-3.

[43] Gebremedhen HS, Woldemicahel DE, Hashim F. Three-dimensional stress-based topology optimization using simp method. Int J Simul Multidiscip Des Optim 2019;10. http://dx.doi.org/10.1051/smdo/2019005.

[44] Sigmund O. Morphology-based black and white filters for topology optimization. Struct Multidiscip Optim 2007;33:401–24. http://dx.doi.org/10.1007/s00158-006-0087-x.

[45] París J, Navarrina F, Colominas I, Casteleiro M. Stress constraints sensitivity analysis in structural topology optimization. Comput Methods Appl Mech Engrg 2010;199:2110–22. http://dx.doi.org/10.1016/j.cma.2010.03.010.

[46] Luo Y, Kang Z. Topology optimization of continuum structures with Drucker-Prager yield stress constraints. Comput Struct 2012;90–91:65–75. http://dx.doi.org/10.1016/j.compstruc.2011.10.008.

[47] Holmberg E, Torstenfelt B, Klarbring A. Stress constrainted topology optimization. Struct Multidiscip Optim 2013;48:33–47. http://dx.doi.org/10.1007/s00158-012-0880-7.

[48] Luo Y, Xing J, Niu Y, Li M, Kang Z. Wrinkle-free design of thin membrane structures using stress-based topology optimization. J Mech Phys Solids 2017;102:277–93. http://dx.doi.org/10.1016/j.jmps.2017.02.003.

[49] da Silva GA, Aage N, Beck AT, Sigmund O. Local versus global stress constraint strategies in topology optimization: A comparative study. Internat J Numer Methods Engrg 2021;122:6003–36. http://dx.doi.org/10.1002/nme.6781.