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XVA in a multi-currency setting with stochastic foreign exchange rates

Roberta Simonella, Carlos Vázquez*

Department of Mathematics, University of A Coruña, 15071 A Coruña, Spain
CITIC, University of A Coruña, 15071 A Coruña, Spain

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Abstract

In the present article we address the modelling and the numerical computation of the total value adjustment for European options in a multi-currency setting when the foreign exchange rates between the different involved currencies are assumed to be stochastic. Thus, we extend to a more realistic approach a previous work where constant exchange rates have been considered. New models are formulated both in terms of linear and nonlinear PDEs and expectations, the hedging arguments requiring the additional consideration of the exposure to foreign exchange risk. For the nonlinear models, Picard iteration methods are applied to the formulation in terms of expectations and compared with multilevel Picard iteration methods. In this way, we avoid the curse of dimensionality associated to the use of deterministic numerical methods (such as finite differences or finite element methods) for solving high dimensional PDEs. Some examples of option pricing problems illustrate the performance of the proposed models and numerical methods.

Keywords: Total value adjustment; Multi-currency; Stochastic foreign exchange rates; European options; Picard iteration methods

1. Introduction

Since the financial crisis in 2007–08, it is clear that the pricing of different financial products must include the adjustments associated to the presence of counterparty risk of both parts involved in the contract. In particular, this has to be taken into account in the pricing of financial derivatives. The initial and more classical adjustments were motivated by counterparty risks related to credit (CVA), funding (FVA) and collateral (CollVA). Later on, adjustments related to capital (KVA) and margin (MVA) have been added. Among the classical and more general references on the topic, we address the readers to the books [5,10,25] and the references therein. In the single currency setting, three main approaches have been widely developed. The first one is based on partial differential equations (PDEs) with seminal Refs. [6,45], the second one is based on expectations and started with [4], and the third one is formulated in terms of backward stochastic differential equations (BSDEs) with the main seminal articles [8,9]. These three approaches can be also considered in the multi-currency setting that we analyse in the present article.

* Corresponding author at: Department of Mathematics, University of A Coruña, 15071 A Coruña, Spain.
E-mail address: carlosv@udc.es (C. Vázquez).

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In a global economy, financial institutions operate in different currencies. Therefore, in the context of XVA they can either fund or post collateral in different currencies. Some attention to the multi-currency setting has been recently addressed in the literature. In the present work we start considering a multi-currency setting, following the ideas in [17], where the joint consideration of CVA, FVA, CollVA and repo adjustments are taken into account. We will refer to the set of these adjustments as total value adjustment or XVA. For the additional inclusion of KVA or MVA in the XVA, the ideas in [23,24] in the single currency case could be considered.

In the recent article [2], the same problem has been addressed when considering constant foreign exchange rates between the different involved currencies. More precisely, in [2] we have addressed the computation of the XVA for different European vanilla options by means of appropriate proposed models that are formulated in terms of expectations. Moreover, the XVA pricing problem was also formulated in terms of linear and nonlinear PDEs, although their numerical solution was not addressed.

The main objective of this article is the extension of the methodology developed in [2] to the more realistic modelling approach that considers stochastic foreign exchange (FX) rates. First, this extension requires the introduction of appropriate stochastic dynamics for FX rates and the statement of the new models based on PDEs and expectations. As a first approach, we will consider that the stochastic dynamics of each FX rate follows a geometric Brownian motion process, although in the future other more general dynamics could be considered. Note that the consideration of stochastic FX rates implies that the number of stochastic factors increases significantly, so that the curse of dimensionality comes into place when deterministic numerical methods (as finite differences or finite element methods) are considered for the solution of the corresponding linear and nonlinear high dimensional PDEs models. Actually, these deterministic numerical methods involve and exponentially increasing computational cost in the dimension of the PDE. Therefore, probabilistic methods based on Feynman Kac formula to obtain an equivalent formulation in terms of expectations seem an appropriate alternative. In the nonlinear case, a general nonlinear Feynman–Kac formula has been proposed in the seminal work [43]. Among the possible probabilistic methods, the most *naive* comes from the consideration of the so called “Monte Carlo of Monte Carlo” (also know as nested Monte Carlo simulation or straight-forward Monte Carlo method), which would give rise to an at least exponentially growing computational cost of the approximation method in the inverse of the prescribed approximation accuracy. In the nonlinear case, among the recent advanced nonlinear Monte Carlo techniques to solve the semilinear PDEs formulations, we point out three of them: branching diffusion methods, deep learning based methods and multilevel Picard iterations. In next paragraphs we address a short review of these three alternatives.

As pointed out in [28], a first possible way to improve nonlinear Monte Carlo techniques could be the use of first order BSDEs, although this would require the use of appropriate regressors to compute the involved conditional expectations. Therefore, in [28] the author proposes a new method based on branching diffusions with a marked Galton–Watson random tree to solve the semilinear PDE arising in CVA computation when the mark to market is equal to the risky derivative. For this purpose, the author extends the initial nonlinearities considered by the seminal article of McKean [40] to the case of a polynomial nonlinearity. Next, a suitable approximation of the sign nonlinearity arising in the CVA pricing problem by a polynomial one is proposed. However, as indicated in [28], this marked branching method requires that the product of the maturity times the maximum norm of the payoff is small enough. Recently, in [30] the branching diffusion method has been extended and combined with Monte Carlo automatic differentiation techniques for the solution of high dimensional PDEs with polynomial nonlinearities, both in the solution and in the gradient of the solution. More recently, in [31] the branching diffusion approach is extended to more general nonlinear Cauchy problems, including hyperbolic and higher order PDEs.

The use of deep learning methods for XVA computations can be framed into the highly increasing use of deep learning techniques for the solution of high dimensional linear and nonlinear PDEs and BSDEs. These techniques are based in the intensive use of artificial neuronal networks (ANN), which can approximate with any arbitrary accuracy any continuous function according to the Universal Approximation Theorem (see the seminal articles [11,32,37], for example). Actually, deep learning techniques are based on the connection of several layers of ANNs (deep ANNs). Concerning the use of deep learning techniques for the solution of high dimensional PDEs, we point out the works [26,34], for example. Following the lines in [26] for high dimensional semilinear PDEs, the work [15] provides deep learning techniques for the solution of high dimensional BSDEs by using asymptotic expansions to reduce the value of the loss function and speed up the convergence. By means of these deep learning techniques, in [29] a primal–dual approach has been developed to compute CVA and initial margin (IM). Recently, in [21] the authors consider a discretization of the generic BSDE modelling the dynamics of the XVA for a derivatives portfolio and parameterize the high dimensional hedging process at every time point by a family of ANNs. Thus, BSDEs can be understood as model-based reinforcement learning problems. Moreover, they show that the method can be used to compute sensitivities. In [1], also deep learning regression techniques are used to obtain conditional risk measures in the framework of XVA by following a BSDEs approach. However, despite the large amount of research and literature improving the application of deep learning approximation methods to overcome the curse of dimensionality in XVA computations and its performance in practical and industrial problems, there is still a lack of rigorous theoretical analysis that proves the convergence of these numerical techniques to the exact solution of the problem.

Picard iteration techniques are approximation methods for solving a fixed-point equation. These methods can be applied to solve nonlinear models formulated in terms of expectations that have been obtained from the corresponding nonlinear PDEs by means of a nonlinear Feynman–Kac formula. Once the Picard iteration method has been posed, it must be discretized by means of quadrature formulas. In this article, we propose Picard iteration methods to solve the nonlinear formulations arising in XVA computation by using rectangular and trapezoidal quadrature formulas.

Recently, in [13,33], and [14], the authors propose a family of multilevel Picard iteration methods, which mainly combine the multilevel Monte Carlo techniques from [18,27] with Picard iteration methods. More precisely, in [14] the authors develop the theoretical analysis under suitable smoothness conditions while in [13] they address simulation studies including applications to financial pricing problems. As indicated in [13], the computational complexity increases at most linearly in the dimension of the PDE and quartically in the inverse of the prescribed accuracy. In the present article we also apply the numerical multilevel Picard iteration methods proposed in [13].

The article is organized as follows. In Section 2 we deduce mathematical models for XVA formulated in terms of linear or nonlinear PDEs problems, depending on the choice of the mark-to-market derivative value. In Section 3 we formulate the problem in terms of expectations, so that Picard iteration methods can be applied, and introduce the proposed numerical methods to approximate XVA price. In Section 4 we show and discuss numerical results that correspond to different examples of European options. Finally, in Section 5 we summarize the main conclusions.

2. Mathematical model

In this section, we infer a PDE formulation for the value of a derivative which is traded between a default-free hedger H and a defaultable investor I in a multi-currency setting. Therefore, we take into account the valuation adjustment due to the fact that the investor may default (counterparty risk). More precisely, we extend the work in [2] by assuming that the FX rates have stochastic dynamics.

We denote by D the domestic currency and by $C_0, \ldots, C_N$ the foreign currencies. For $j = 0, \ldots, N$, let X_t^{D,C_j} be the FX rate between currencies D and C_j at time t, i.e., the value in domestic currency D of one unit of the foreign currency C_j at time t. The dynamics of the stochastic FX rate X_t^{D,C_j} under the real world measure P is described by the SDEs:

$$dX_t^{D,C_j} = \mu^{X^j} X_t^{D,C_j} dt + \sigma^{X^j} X_t^{D,C_j} dW_t^{X^j,P}, \tag{1}$$

where μ^{X^j} and σ^{X^j} are respectively the real world drift and the volatility of X^{D,C_j}, while $W^{X^j,P}$ is a standard P-Wiener process. Obviously, if $C_j = D$ for a certain j, then $X_t^{D,C_j} = 1$ at any time t. We denote by $X_t = (X_t^{D,C_0}, \ldots, X_t^{D,C_N})$ the vector of the FX rates values at time t and by $\bar{X}_t = (X_t^{D,C_1}, \ldots, X_t^{D,C_N})$ the vector of the FX rates values, except the value of X^{D,C_0}, at time t. Indeed, this notation will be useful in the following, since we will consider derivatives written on N underlying assets denominated in currencies $C_1, \ldots, C_N$, respectively, while C_0 will be the currency of the collateral account.

Alternative more complex models to the geometric Brownian motion defined by (1) are proposed in the literature (see [22,38], for example). The additional consideration of local, stochastic or local/stochastic volatilities leads to increasing the complexity and the number of stochastic factors, although the same methodology could be applied.

For $i = 1, \ldots, N$, let S_t^i denote the price of a foreign asset in units of the foreign currency C_i at time t and let $S_t = (S_t^1, \ldots, S_t^N)$ be the vector of the assets prices at time t. Moreover, let h_t be the investor's credit spread at time t. We assume that under the real world measure P the evolution of the prices of the foreign assets and of the investor's credit spread are respectively governed by the SDEs:

$$dS_t^i = \mu^{S^i,P} S_t^i dt + \sigma^{S^i} S_t^i dW^{S^i,P}, \tag{2}$$

$$dh_t = \mu^{h,P} dt + \sigma^h dW_t^{h,P}, \tag{3}$$

where $\mu^{S^i,P}$ and μ^h are the real world drifts of the processes, while σ^{S_i} and σ^h are their volatilities. Moreover, $W^{S^i,P}$ and $W^{h,P}$ are Wiener processes under the real world measure P.

We assume that all the processes are correlated with constant correlations. The correlation matrix is given by:

$$\bar{P} = \begin{bmatrix} 1 & \rho^{S^1,S^2} & \dots & \rho^{S^1,S^N} & \rho^{S^1,X^0} & \rho^{S^1,X^1} & \dots & \rho^{S^1,X^N} & \rho^{S^1,h} \\ \rho^{S^1,S^2} & 1 & \dots & \rho^{S^2,S^N} & \rho^{S^2,X^0} & \rho^{S^2,X^1} & \dots & \rho^{S^2,X^N} & \rho^{S^2,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{S^1,S^N} & \rho^{S^2,S^N} & \dots & 1 & \rho^{S^N,X^0} & \rho^{S^N,X^1} & \dots & \rho^{S^N,X^N} & \rho^{S^N,h} \\ \rho^{S^1,X^0} & \rho^{S^2,X^0} & \dots & \rho^{S^N,X^0} & 1 & \rho^{X^0,X^1} & \dots & \rho^{X^0,X^N} & \rho^{X^0,h} \\ \rho^{S^1,X^1} & \rho^{S^2,X^1} & \dots & \rho^{S^N,X^1} & \rho^{X^0,X^1} & \rho^{X^0,X^1} & \dots & \rho^{X^1,X^N} & \rho^{X^1,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{S^1,X^N} & \rho^{S^2,X^N} & \dots & \rho^{S^N,X^N} & \rho^{X^0,X^N} & \rho^{X^1,X^N} & \dots & 1 & \rho^{X^N,h} \\ \rho^{S^1,h} & \rho^{S^2,h} & \dots & \rho^{S^N,h} & \rho^{X^0,h} & \rho^{X^1,h} & \dots & \rho^{X^N,h} & 1 \end{bmatrix} \tag{4}$$

Although for simplicity we assume constant correlations and volatilities, the methodology can be straightforwardly extended to time dependent volatilities and correlations.

We denote by J_t the investor's default state at time t, that is to say:

$$\begin{cases} J_t = 1 & \text{if I defaults before or at time } t, \\ J_t = 0 & \text{otherwise.} \end{cases} \tag{5}$$

From the investor's point of view, the derivative value in domestic currency D at time t is given by $V_t^D = V^D(t, S_t, X_t, h_t, J_t)$. The price in currency D of the same derivative traded between two non-defaultable counterparties is referred to as risk-free derivative price and is denoted by $W_t^D = W^D(t, S_t, X_t)$.

In order to infer the dynamics of X^{D,C_j}, for $j = 0, \dots, N$, under the risk neutral probability measure of the domestic market, denoted by Q^D, we consider two bonds with maturity T in the domestic market and in the jth foreign market, for $j = 0, \dots, N$, the prices of which at time t are denoted by B_t^D and $B_t^{C_j}$, respectively.

By using Eq. (1), the discounted price of the foreign bond in the domestic currency D is given by

$$\begin{aligned} \hat{B}_t^{j,D} &= (B_t^D)^{-1} B_t^{C_j} X_t^{D,C_j} \\ &= \hat{B}_0^{j,D} \exp\left(\left(r^j - r^D + \mu^{X^j} - \frac{(\sigma^{X^j})^2}{2}\right)t + \sigma^{X^j} W_t^{X^j,P}\right), \end{aligned} \tag{6}$$

where r^D, r^j, $j = 0, \dots, N$, are the risk-free rates in the domestic market and in the jth foreign market, respectively. Thus, we are assuming that interest rates r^D and r^j are constant. Note that the consideration of stochastic evolution of these interest rates would increase the number of stochastic factors in a significant way. The idea is to address this step in a future work.

Thus, the dynamics of $\hat{B}_t^{j,D}$ under the real measure P is given by

$$d\hat{B}_t^{j,D} = (\mu^{X^j} + r^j - r^D)\hat{B}_t^{j,D} dt + \sigma^{X^j} \hat{B}_t^{j,D} dW_t^{X^j,P}. \tag{7}$$

From Girsanov's Theorem [19], there exists an equivalent measure Q^D, such that we can build a Wiener process W^{X^j,Q^D} under the measure Q^D, which is defined as:

$$W_t^{X^j,Q^D} = W_t^{X^j,P} + \int_0^t m_s ds,$$

or equivalently, W^{X^j,Q^D} satisfies the relation $dW_t^{X^j,Q^D} = dW_t^{X^j,P} + m_t dt$, where m_t is the process associated to the change of measure.

Therefore, the dynamics of $\hat{B}_t^{j,D}$ under the measure Q^D is given by

$$d\hat{B}_t^{j,D} = (\mu^{X^j} + r^j - r^D - m_t \sigma^{X^j})\hat{B}_t^{j,D} dt + \sigma^{X^j} \hat{B}_t^{j,D} dW_t^{X^j,Q^D}. \tag{8}$$

Since $\hat{B}_t^{j,D}$ must be a martingale under Q^D, we get

$$r^D - r^j = \mu^{X^j} - m_t \sigma^{X^j}, \tag{9}$$

that leads to

$$X_t^{D,C_j} = X_0^{D,C_j} \exp\left(\left(r^D - r^j - \frac{(\sigma^{X^j})^2}{2}\right)t + \sigma^{X^j} W_t^{X^j,Q^D}\right), \tag{10}$$

or equivalently,

$$dX_t^{D,C_j} = (r^D - r^j)X_t^{D,C_j} dt + \sigma^{X^j} X_t^{D,C_j} dW_t^{X^j,Q^D}. \tag{11}$$

Also, we need to infer the dynamics of S_t^i under the risk neutral measure of the domestic market Q^D. For this purpose, we assume that the dynamics of S_t^i under the risk neutral measure of the domestic market and under the risk neutral measure of the foreign market, denoted by Q^{C_i}, are respectively given by the SDEs:

$$dS_t^i = \mu^{S^i,Q^D} S_t^i dt + \sigma^{S^i} S_t^i dW_t^{S^i,Q^D}, \tag{12}$$

$$dS_t^i = (r^i - q^i)S_t^i dt + \sigma^{S^i} S_t^i dW_t^{S^i,Q^{C_i}}, \tag{13}$$

where μ^{S^i,Q^D} is the drift of S_t^i under the measure Q^D, σ^{S^i} is its volatility, r^i is the risk-free rate of the ith foreign market and q^i is the continuous dividend yield paid by S_t^i. Moreover, $W_t^{S^i,Q^D}$ and $W_t^{S^i,Q^{C_i}}$ are a Q^D-Wiener process and a Q^{C_i}-Wiener process, respectively.

We denote by $S_t^{i,D}$ the price of the foreign underlying asset S^i in the domestic currency D, that is to say, $S_t^{i,D} = S_t^i X_t^{D,C_i}$. From (11) and (12), by applying the classical Itô formula we obtain that the dynamics of $S^{i,D}$ follows the SDE:

$$\begin{aligned} dS_t^{i,D} &= d(S_t^i X_t^{D,C_i}) \\ &= (r^D - r^i + \mu^{S^i,Q^D} + \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i}) S_t^i X_t^{D,C_i} dt \\ &\quad + (\sigma^{S^i} dW^{S^i,Q^D} + \sigma^{X^i} dW^{X^i,Q^D}) S_t^i X_t^{D,C_i}. \end{aligned} \tag{14}$$

Since the drift of $S^{i,D}$ under the risk neutral measure of the domestic market is given by $(r^D - q^i)$, we obtain

$$\mu^{S^i,Q^D} = r^i - q^i - \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i}.$$

Therefore, from (12) we get that the dynamics of S_t^i under the risk neutral measure of the domestic market follows the SDE:

$$dS_t^i = (r^i - q^i - \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i}) S_t^i dt + \sigma^{S^i} S_t^i dW_t^{S^i,Q^D}. \tag{15}$$

By changing the probability measure from P to Q^D in (3) and denoting by M^h the investor's market price of credit risk, the drift of the investor's credit spread is given by $\mu^{h,P} - M^h \sigma^h$, that can be written as

$$\mu^{h,P} - M^h \sigma^h = -\kappa \lambda. \tag{16}$$

In (16) λ represents the investor's intensity of default, defined as

$$\lambda = \frac{h}{1-R}, \tag{17}$$

where R is the constant investor's recovery rate.

Once we have deduced the dynamics of the processes involved in the model under the domestic risk neutral measure, in order to compute the derivative value, we implement a self-financing strategy by building a portfolio Π which hedges all the risk factors. More precisely, extending the work in [16] to the multi-currency framework, we assume that:

- H hedges the market risk due to changes in S^i, for $i = 1, \ldots, N$, by trading in fully collateralized derivatives on the same underlying assets. For $i = 1, \ldots, N$, the net present value in currency C_i of the derivative written on the underlying asset S^i is denoted by H^i, so that $H^{i,D} = H^i X^{D,C_i}$ represents the net present value of H^i in the currency D;

- the exposure to the FX risk due to changes in X^{D,C_j}, for $j = 0, \dots, N$, is hedged by trading in FX derivatives. For $j = 0, \dots, N$, E^j denotes the net present value in currency D of the derivative written on the FX rate X^{D,C_j};
- in order to hedge the spread risk due to changes in I's credit spread h and the I's default risk, H has to trade in two credit default swaps with different maturities written on the investor:
  - a short term credit default swap, $CDS^D(t, t + dt)$, that is an overnight credit default swap with unit notional. The protection buyer pays a premium at time t equal to $h_t dt$ and receives $(1 - R)$ at time $t + dt$ if the investor defaults between t and $t + dt$. We assume that $h_t dt$ is such that $CDS^D(t, t + dt) = 0$;
  - a long term credit default swap, $CDS^D(t, T)$, that is a cash collateralized credit default swap maturing on T. In general, $CDS^D(t, T)$ is not null.

Furthermore, we consider a collateral account C^{C_0}, denominated in currency C_0 and composed of a portfolio of bonds R^{C_0} and cash M^{C_0}, i.e.:

$$C^{C_0} = R^{C_0} + M^{C_0}. \tag{18}$$

According to the self-financing condition of the replicating strategy [16], the hedger trades in short term bonds maturing on $t + dt$ to match the spread duration of the uncollateralized part of the derivative. Hence, denoting by $B^D(t, t + dt)$ the value of the short term bond at time t and by Ω_t the number of units of $B^D(t, t + dt)$ at time t, we have:

$$\Omega_t B^D(t, t + dt) = V_t^D - C_t^{C_0} X^{D,C_0}. \tag{19}$$

Therefore, the replicating portfolio Π_t at time t is made as follows:

$$\begin{aligned}\Pi_t = &\sum_{i=1}^{N} \alpha_t^i H_t^{i,D} + \sum_{j=0}^{N} \eta_t^j E_t^j + \gamma_t CDS^D(t, T) + \epsilon_t CDS^D(t, t + dt)\\ &+ \Omega_t B^D(t, t + dt) + \beta_t^D,\end{aligned} \tag{20}$$

where

- α_t^i represents the weight of the derivative $H_t^{i,D}$, for $i = 1, \dots, N$, in the portfolio composition at time t;
- η_t^j denotes the weight of the derivative E_t^j, for $j = 0, \dots, N$, in the portfolio composition at time t;
- γ_t and ϵ_t are the units of the long term credit default swap and of the short term credit default swap, respectively, in the portfolio composition at time t;
- Ω_t represents the number of units of the short term bond in the portfolio composition at time t;
- β_t^D denotes the amount of cash in the portfolio bank account at time t, which is composed of

$$\beta_t^D = -\sum_{i=1}^{N} \alpha_t^i H_t^{i,D} - \sum_{j=0}^{N} \eta_t^j E_t^j - \gamma_t CDS^D(t, T) + C_t^{C_0} X_t^{D,C_0}. \tag{21}$$

The portfolio composition in (20) is an extension to the multi-currency framework with stochastic FX rates of the portfolio built in [2], where deterministic FX rates have been considered.

As a consequence of the no arbitrage condition, we have

$$V^D(t, S_t, X_t, h_t, J_t) = \Pi_t(t, S_t, X_t, h_t, J_t),$$

so that $dV_t^D = d\Pi_t$ and the self-financing condition leads to

$$\begin{aligned}dV_t^D = &\sum_{i=1}^{N} \alpha_t^i dH_t^{i,D} + \sum_{j=0}^{N} \eta_t^j dE_t^j + \gamma_t dCDS^D(t, T) + \epsilon_t dCDS^D(t, t + dt)\\ &+ \Omega_t dB(t, t + dt) + d\beta_t^D.\end{aligned} \tag{22}$$

Note that V^D depends on both diffusion and jump processes due to the presence of the investor's default state J. Therefore, Itô formula for jump–diffusion processes [44] is applied to obtain the variation of V^D from t to $t+dt$:

$$\begin{aligned} dV_t^D = {} & \frac{\partial V^D}{\partial t}dt + \sum_{i=1}^{N}\frac{\partial V^D}{\partial S^i}dS_t^i + \sum_{j=0}^{N}\frac{\partial V^D}{\partial X^j}dX_t^j + \frac{\partial V^D}{\partial h}dh_t + \Delta V^D dJ_t \\ & + \Bigg[\frac{1}{2}\sum_{i,k=1}^{N}\rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S_t^iS_t^k\frac{\partial^2 V^D}{\partial S^i S^k} + \frac{1}{2}\sum_{j,l=0}^{N}\rho^{X^jX^l}\sigma^{X^j}\sigma^{X^l}X_t^jX_t^l\frac{\partial^2 V^D}{\partial X^j X^l} \\ & + \frac{1}{2}(\sigma^h)^2\frac{\partial^2 V^D}{\partial h^2} + \sum_{i=1}^{N}\sum_{j=0}^{N}\rho^{S^iX^j}\sigma^{S^i}\sigma^{X^j}S_t^iX_t^j\frac{\partial^2 V^D}{\partial S^i\partial X^j} \\ & + \sum_{i=1}^{N}\rho^{S^ih}\sigma^{S^i}\sigma^hS_t^i\frac{\partial^2 V^D}{\partial S^i\partial h} + \sum_{j=0}^{N}\rho^{X^jh}\sigma^{X^j}\sigma^hX_t^j\frac{\partial^2 V^D}{\partial X^j\partial h}\Bigg]dt, \end{aligned}$$

where ΔV^D is the variation of V_t^D at default, defined as

$$\Delta V^D = V^D(t, S_t, X_t, h_t, 1) - V^D(t, S_t, X_t, h_t, 0). \tag{23}$$

Note that if the investor defaults at time t the value of the risky derivative at time t is given by

$$V^D(t, S_t, X_t, h, 1) = RM^+(t, S_t, X_t, h_t) + M^-(t, S_t, X_t, h_t), \tag{24}$$

where $M(t, S_t, X_t, h_t)$ denotes the mark-to-market price and we have used the notation $M^+ = \max(M, 0)$ and $M^- = \min(M, 0)$. Therefore, the variation of V^D at default can be written as

$$\Delta V^D = RM^+ + M^- - V^D. \tag{25}$$

In order to compute the variation of Π_t in the time interval $[t, t+dt]$, $d\Pi_t$, we first consider the dynamics of the short term credit default swap and the overnight bond, that are respectively given by:

$$\begin{aligned} dCDS^D(t, t+dt) &= h_t dt - (1-R)dJ_t, \\ dB^D(t, t+dt) &= f^{H,D}B^D(t, t+dt)dt, \end{aligned} \tag{26}$$

where $f^{H,D}$ is the hedger's domestic funding rate. Moreover, the weight of the short term bond in the portfolio Π_t at time t, Ω_t, is deduced from the self-financing condition (19) and given by:

$$\Omega_t = \frac{V_t^D - C_t^D}{B^D(t, t+dt)}. \tag{27}$$

In addition, from (21) we obtain that the variation of β_t^D in the time interval from t to $t+dt$ is given by

$$\begin{aligned} d\beta_t^D = {} & -\left[\sum_{i=1}^{N}\alpha_t^i(c^D + b^{D,C_j})H_t^{i,D} + \sum_{j=0}^{N}\eta_t^jc^DE_t^j + \gamma_tc^DCDS^D(t,T)\right]dt \\ & + \left[\left(r^R + b^{D,C_0}\right)R_t^{C_0} + \left(c^D + b^{D,C_0}\right)M_t^{C_0}\right]X_t^{D,C_0}dt, \end{aligned} \tag{28}$$

where r^R is the instantaneous repo rate associated to the bond R^{C_0}, b^{D,C_0} is the cross-currency basis and c^D is the OIS rate in the domestic market. Thus, the change in Π_t in the infinitesimal interval $[t, t+dt]$ is given by

$$\begin{aligned} d\Pi_t = \sum_{i=1}^{N}\alpha_t^i\Bigg(& \frac{\partial H^{i,D}}{\partial t} + (r^i - q^i - \rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i})S_t^i\frac{\partial H^{i,D}}{\partial S^i} + (r^D - r^i)X_t^i\frac{\partial H^{i,D}}{\partial X^i} \\ & + \frac{1}{2}(S_t^i\sigma^{S^i})^2\frac{\partial^2 H^{i,D}}{\partial (S_t^i)^2} + \frac{1}{2}(X_t^i\sigma^{X^i})^2\frac{\partial^2 H^{i,D}}{\partial (X^i)^2} \\ & + \rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i}S_t^iX_t^i\frac{\partial^2 H^{i,D}}{\partial S^i\partial X^i} - (c^D + b^{D,C_j})H^{i,D}\Bigg)dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{N}\eta^j\left[\frac{\partial E^j}{\partial t}+(r^D-r^j)X_t^j\frac{\partial E^j}{\partial X^j}+\frac{1}{2}(X_t^j\sigma^{X^j})^2\frac{\partial^2 E^j}{\partial (X^j)^2}-c^D E^j\right]dt\\
&+\sum_{i=1}^{N}\alpha_t^i\sigma^{S^i}S_t^i\frac{\partial H^{i,D}}{\partial S^i}dW_t^{S^i}\\
&+\sum_{i=1}^{N}\alpha_t^i\sigma^{X^i}X_t^i\frac{\partial H^{i,D}}{\partial X^i}dW_t^{X^i}+\sum_{j=0}^{N}\eta_t^j\sigma^{X^j}X_t^j\frac{\partial E^j}{\partial X^j}dW_t^{X^j}\\
&+\gamma_t\left[\frac{\partial CDS^D(t,T)}{\partial t}dt+\frac{\partial CDS^D(t,T)}{\partial h}dh+\frac{1}{2}(\sigma^h)^2\frac{\partial^2 CDS^D(t,T)}{\partial h^2}dt\right]\\
&-\gamma_t c^D CDS^D(t,T)dt+\gamma_t\Delta CDS^D(t,T)dJ_t+\epsilon_t\left[h_t dt-(1-R_C)dJ\right]\\
&+(V_t^D-C_t^{C_0}X^0)f^{H,D}dt+\left[(r^R+b^{D,C_0})R_t^{C_0}+(c^D+b^{D,C_0})M_t^{C_0}\right]X_t^{D,C_0}dt.
\end{aligned}
$$

Therefore, in order to hedge the risks in the portfolio Π, we choose:

$$
\begin{aligned}
\alpha_t^i&=\frac{\frac{\partial V^D}{\partial S^i}}{\frac{\partial H^{i,D}}{\partial S^i}},\quad i=1,\dots,N,\\
\eta_t^0&=\frac{\frac{\partial V^D}{\partial X^0}}{\frac{\partial E^0}{\partial X^0}},\quad \eta_t^i=\frac{\frac{\partial V^D}{\partial X^i}-\alpha_t^i\frac{\partial H^{i,D}}{\partial X^i}}{\frac{\partial E^i}{\partial X^i}},\quad i=1,\dots,N,\\
\gamma_t&=\frac{\frac{\partial V^D}{\partial h}}{\frac{\partial CDS^D(t,T)}{\partial h}},\\
\epsilon_t&=\frac{1}{1-R}\left(\gamma_t\Delta CDS^D(t,T)-\Delta V^D\right).
\end{aligned}
$$

Next, we take into account the equations satisfied by $H^{i,D}$, E^j and $CDS^D(t,T)$, which are respectively given by:

$$
\begin{aligned}
&\frac{\partial H^{i,D}}{\partial t}+\frac{(S^i\sigma^{S^i})^2}{2}\frac{\partial^2 H^{i,D}}{\partial (S^i)^2}+\frac{(X^i\sigma^{X^i})^2}{2}\frac{\partial^2 H^{i,D}}{\partial (X^i)^2}\\
&\qquad+\rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i}S^iX^i\frac{\partial^2 H^{i,D}}{\partial S^i\partial X^i}+(r^i-q^i-\rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i})S^i\frac{\partial H^{i,D}}{\partial S^i}\\
&\qquad+(r^D-r^i)X^i\frac{\partial H^{i,D}}{\partial X^i}=(c^D+b^{D,C_i})H^{i,D},\\
&\frac{\partial E^j}{\partial t}+\frac{1}{2}(X^j\sigma^{X^j})^2\frac{\partial^2 E^j}{\partial (X^j)^2}+(r^D-r^j)X^j\frac{\partial E^j}{\partial X^j}=c^D E^j,\\
&\frac{\partial CDS^D(t,T)}{\partial t}+\frac{1}{2}(\sigma^h)^2\frac{\partial^2 CDS^D(t,T)}{\partial h^2}+(\mu^h-M^h\sigma^h)\frac{\partial CDS^D(t,T)}{\partial h}\\
&\qquad+\frac{h}{1-R}\Delta CDS^D(t,T)=c^D CDS^D(t,T).
\end{aligned}
$$

Finally, from the previous arguments, Eq. (22) turns into

$$
\begin{aligned}
&\frac{\partial V^D}{\partial t}+\mathcal{L}_{SXh}V^D\\
&\quad=-\frac{h}{1-R}\Delta V^D+f^{H,D}V^D\\
&\quad+[(r^R+b^{D,C_0}-f^{H,D})R^{C_0}+(c^D+b^{D,C_0}-f^{H,D})M^{C_0}]X^{D,C_0},
\end{aligned}
\tag{29}
$$

where the differential operator $\mathcal{L}_{SXh}$ is given by

$$\begin{aligned}\mathcal{L}_{SXh} &= \frac{1}{2}\sum_{i,k=1}^{N}\rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k\frac{\partial^2}{\partial S^i\partial S^k} + \frac{1}{2}\sum_{j,l=0}^{N}\rho^{X^jX^l}\sigma^{X^j}\sigma^{X^l}X^jX^l\frac{\partial^2}{\partial X^j\partial X^l}\\ &+\sum_{i=1}^{N}\sum_{j=0}^{N}\rho^{S^iX^j}\sigma^{S^i}\sigma^{X^j}S^iX^j\frac{\partial^2}{\partial S^i\partial X^j} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2}{\partial h^2}\\ &+\sum_{i=1}^{N}\rho^{S^ih}\sigma^{S^i}\sigma^hS^i\frac{\partial^2}{\partial S^i\partial h} + \sum_{j=0}^{N}\rho^{X^jh}\sigma^{X^j}\sigma^hX^j\frac{\partial^2}{\partial X^j\partial h}\\ &+\sum_{i=1}^{N}(r^i - q^i - \rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i})S^i\frac{\partial}{\partial S^i}\\ &+\sum_{j=0}^{N}(r^D - r^j)X^j\frac{\partial}{\partial X^j} + (\mu^h - M^h\sigma^h)\frac{\partial}{\partial h}.\end{aligned} \tag{30}$$

By taking into account (16), the differential operator (30) turns into

$$\begin{aligned}\mathcal{L}_{SXh} &= \frac{1}{2}\sum_{i,k=1}^{N}\rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k\frac{\partial^2}{\partial S^i\partial S^k} + \frac{1}{2}\sum_{j,l=0}^{N}\rho^{X^jX^l}\sigma^{X^j}\sigma^{X^l}X^jX^l\frac{\partial^2}{\partial X^j\partial X^l}\\ &+\sum_{i=1}^{N}\sum_{j=0}^{N}\rho^{S^iX^j}\sigma^{S^i}\sigma^{X^j}S^iX^j\frac{\partial^2}{\partial S^i\partial X^j} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2}{\partial h^2}\\ &+\sum_{i=1}^{N}\rho^{S^ih}\sigma^{S^i}\sigma^hS^i\frac{\partial^2}{\partial S^i\partial h} + \sum_{j=0}^{N}\rho^{X^jh}\sigma^{X^j}\sigma^hX^j\frac{\partial^2}{\partial X^j\partial h}\\ &+\sum_{i=1}^{N}(r^i - q^i - \rho^{S^iX^i}\sigma^{S^i}\sigma^{X^i})S^i\frac{\partial}{\partial S^i}\\ &+\sum_{j=0}^{N}(r^D - r^j)X^j\frac{\partial}{\partial X^j} - \frac{\kappa}{1-R}h\frac{\partial}{\partial h}.\end{aligned} \tag{31}$$

Usually, there are two possible choices for the mark-to-market value M in (25), either equal to the risky derivative value or equal to the value of the risky-free derivative in terms of counterparty risk [6]. Therefore, from these two possibilities, we get two alternative PDE models:

- if $M = V^D$, the PDE (29) turns into:

$$\frac{\partial V^D}{\partial t} + \mathcal{L}_{SXh}V^D - fV^D = (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0} + h(V^D)^+, \tag{32}$$

- if $M = W^D$, the PDE (29) turns into:

$$\begin{aligned}&\frac{\partial V^D}{\partial t} + \mathcal{L}_{SXh}V^D - \left(\frac{h}{1-R} + f\right)V^D\\ &= (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0} + h(W^D)^+ - \frac{h}{1-R_C}W^D,\end{aligned} \tag{33}$$

where $\bar{r} = r^R + b^{D,C_0} - f^{H,D}$, $\bar{m} = c^D + b^{D,C_0} - f^{H,D}$ and $f = f^{H,D}$.

Note that the value of the risk-free derivative, W^D, satisfies the PDE:

$$\frac{\partial W^D}{\partial t} + \mathcal{L}_{S\bar{X}}W^D - fW^D = 0,$$

where

$$
\begin{aligned}
\mathcal{L}_{S\bar{X}} =& \frac{1}{2}\sum_{i,k=1}^{N} \rho^{S^i S^k}\sigma^{S^i}\sigma^{S^k} S^i S^k \frac{\partial^2}{\partial S^i \partial S^k} + \frac{1}{2}\sum_{j,l=1}^{N} \rho^{X^j X^l}\sigma^{X^j}\sigma^{X^l} X^j X^l \frac{\partial^2}{\partial X^j \partial X^l} \\
&+ \sum_{i,j=1}^{N} \rho^{S^i X^j}\sigma^{S^i}\sigma^{X^j} S^i X^j \frac{\partial^2}{\partial S^i \partial X^j} \\
&+ \sum_{i=1}^{N}(r^i - q^i - \rho^{S^i X^i}\sigma^{S^i}\sigma^{X^i})S^i \frac{\partial}{\partial S^i} + \sum_{j=1}^{N}(r^D - r^j)X^j \frac{\partial}{\partial X^j}.
\end{aligned} \tag{34}
$$

If we denote the total value adjustment by U, then we have $U = V^D - W^D$. Moreover, as both the risky derivative and the risk-free derivative at time T are equal to the payoff G, i.e.,

$$W^D(T, S, \bar{X}) = V^D(T, S, X, h) = G(S, X),$$

then the value of U at maturity T is zero.

Therefore, by taking into account the equations satisfied by the risky and risk-free derivatives, we obtain the following alternative PDE problems satisfied by the total value adjustment:

- if $M = V^D$ the XVA is the solution of the nonlinear PDE problem:

$$
\begin{cases}
\frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - fU = h(W^D + U)^+ + (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0}, \\
U(T, S, X, h) = 0;
\end{cases} \tag{35}
$$

- if $M = W^D$ the XVA is the solution of the linear PDE problem:

$$
\begin{cases}
\frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - \left(\frac{h}{1-R} + f\right) U = h(W^D)^+ + (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0}, \\
U(T, S, X, h) = 0.
\end{cases} \tag{36}
$$

In both cases, the PDEs are posed in the unbounded domain

$$D = \{(t, S, X, h) \in [0, T) \times (0, +\infty)^N \times (0, +\infty)^{(N+1)} \times (0, +\infty)\}.$$

Note that the PDE model inferred in this section has the same structure as the one in [2]. The difference between the two models lies in the fact that here we have $N+1$ extra stochastic factors, namely $X^{D,C_0}, X^{D,C_1}, \dots, X^{D,C_N}$, which are assumed to be constant in [2]. Thus, the new model that incorporates the more realistic approach by considering stochastic foreign exchange rates also involves much higher spatial dimension in the PDEs problems. Therefore, as indicated in the introduction, when trying to solve these PDEs formulations by means of finite differences or finite element methods, the curse of dimensionality comes into place in a more relevant way. Three possible alternative probabilistic numerical approaches are based on branching diffusion methods, deep learning techniques or multilevel Picard iterations. In the present article we mainly consider a particular Picard iteration scheme that we will compare with the application of a multilevel Picard iteration method.

3. Formulation in terms of expectations and numerical methods

In order to apply Picard iteration methods based on Monte Carlo simulation techniques, we follow the same approach as in [2] and we first apply appropriate Feynman–Kac formulas for the nonlinear [3] and linear [44] PDEs to obtain their equivalent formulations in terms of expectations. In order to shorten some notations, we define:

$$\bar{C}^{C_0} = \bar{r}R^{C_0} + \bar{m}M^{C_0}, \quad \bar{C}^D = \bar{C}^{C_0}X^{D,C_0}.$$

Thus, after applying the previously indicated Feynman–Kac formulas, we get two alternative integral equations.

- If $M = V^D$, the total value adjustment at time t is given by:

$$
\begin{aligned}
U(t, S, X, h) = \mathrm{E}_t^Q \Bigg[-\int_t^T e^{-f(u-t)} \Bigg(& h_u(W^D(u, S_u, \bar{X}_u) + U(u, S_u, X_u, h_u))^+ \\
&+ \bar{C}_u^{C_0} X_u^{D,C_0} \Bigg) du | S_t = S, X_t = X, h_t = h \Bigg],
\end{aligned} \tag{37}
$$

so that the XVA value at time $t = 0$ (also referred as XVA price), is:

$$U(0,S,X,h) = \mathrm{E}_0^Q\left[-\int_0^T e^{-fu}\left(h_u(W^D(u,S_u,\bar{X}_u) + U(u,S_u,X_u,h_u))^+ + \bar{C}_u^{C_0} X_u^{D,C_0}\right) du | S_0 = S, X_0 = X, h_0 = h\right]. \tag{38}$$

- If $M = W^D$, the total value adjustment at time t is given by:

$$U(t,S,X,h) = \mathrm{E}_t^Q\left[-\int_t^T e^{-\int_t^u (\frac{h_r}{1-R}+f)dr}\left(h_u(W^D(u,S_u,\bar{X}_u))^+ + \bar{C}_u^{C_0} X_u^{D,C_0}\right) du | S_t = S, X_t = X, h_t = h\right]. \tag{39}$$

Therefore, the XVA price is:

$$U(0,S,X,h) = \mathrm{E}_0^Q\left[-\int_0^T e^{-\int_0^u (\frac{h_r}{1-R}+f)dr}\left(h_u(W^D(u,S_u,\bar{X}_u))^+ + \bar{C}_u^{C_0} X_u^{D,C_0}\right) du | S_0 = S, X_0 = X, h_0 = h\right]. \tag{40}$$

In order to numerically approximate the XVA value in both the nonlinear case (38) and the linear case (40), we first introduce a time discretization based on a uniform mesh with Z time nodes $t_Z = z\Delta t,\ z = 0,\ldots,Z-1$, the constant $\Delta t = T/(Z-1)$ being the time step.

Taking into account the previous time mesh, we can discretize the dynamics of the underlying assets S^i (for $i = 1,\ldots,N$), the FX rates X^j (for $j = 0,\ldots,N$), the investor's credit spread h, and the two components of the collateral account, i.e., R^{C_0} and M^{C_0}, by using the Euler–Maruyama scheme [36]. Thus, for $z = 0,\ldots,Z-2$, we consider the iterative procedure:

$$\begin{aligned}
S_{t_{z+1}}^i &= S_{t_z}^i + (r^i - q^i - \rho^{S^i X^i}\sigma^{S^i}\sigma^{X^i}) S_{t_z}^i \Delta t + \sigma^{S^i} S_{t_z}^i \Delta W_{t_{z+1}}^{S^i},\\
X_{t_{z+1}}^j &= X_{t_z}^j + (r^D - r^j) X_{t_z}^j \Delta t + \sigma^{Xj} X_{t_z}^j \Delta W_{t_{z+1}}^{Xj},\\
h_{t_{z+1}} &= h_{t_z} - \frac{\kappa}{1-R_C} h_{t_z}\Delta t + \sigma^h \Delta W_{t_{z+1}}^h,\\
R_{t_{z+1}}^{C_0} &= R_{t_z}^{C_0} + (r^R + b^{D,C_0}) R_{t_z}^{C_0}\Delta t,\\
M_{t_{z+1}}^{C_0} &= M_{t_z}^{C_0} + (c^D + b^{D,C_0}) M_{t_z}^{C_0}\Delta t,
\end{aligned}$$

where $\Delta W_{t_{z+1}}^{S^i} = W_{t_{z+1}}^{S^i} - W_{t_z}^{S^i}$, for $i = 1,\ldots,N$, $\Delta W_{t_{z+1}}^{Xj} = W_{t_{z+1}}^{Xj} - W_{t_z}^{Xj}$, for $j = 0,\ldots,N$, and $\Delta W_{t_{z+1}}^h = W_{t_{z+1}}^h - W_{t_z}^h$ are increments of the corresponding Brownian motions, which are correlated according to the correlation matrix (4). Note that in practice the correlated Brownian motions can be built from independent Brownian motions by using the Cholesky factorization of the correlation matrix.

In both cases (38) and (40), the computation of XVA value requires integral approximation techniques by numerical quadrature formulas. In next paragraphs we describe the different methods we have used in the nonlinear and linear cases.

Nonlinear case ($M = V^D$). We denote by I^{NL} the integral on the right hand side of (38), i.e.,

$$I^{NL} = \int_0^T e^{-fu}\left(h_u(W^D(u,S_u,\bar{X}_u) + U(u,S_u,X_u,h_u))^+ + \bar{C}_u^{C_0} X_u^{D,C_0}\right) du. \tag{41}$$

We will approximate the integral by using the simple rectangular formula, as follows:

$$I^{NL} \simeq T\left(h_0(W^D(0,S_0,\bar{X}_0) + U(0,S_0,X_0,h_0))^+ + \bar{C}_0^{C_0} X_0^{D,C_0}\right), \tag{42}$$

or the simple trapezoidal formula:

$$I^{NL} \simeq \frac{T}{2}\left(e^{-fT}\left(h_T(W^D(T,S_T,\bar{X}_T))^+ + \bar{C}_T^{C_0}X_T^{D,C_0}\right) + h_0(W^D(0,S_0,\bar{X}_0)+U(0,S_0,X_0,h_0))^+ + \bar{C}_0^{C_0}X_0^{D,C_0}\right). \tag{43}$$

Since (38) is an integral equation, the implementation of a fixed-point method (Picard iteration method) is required to compute the XVA price. Thus, starting both from $U^0 = 0$, in the case of the simple rectangular formula or the simple trapezoidal formula we respectively consider the iteration procedure:

$$U^{l+1}(0,S_0,X_0,h_0) = T\,\mathrm{E}\Big[h_0(W^D(0,S_0,\bar{X}_0)+U^l(0,S_0,X_0,h_0))^+ + \bar{C}_0^{C_0}X_0^{D,C_0}\Big], \tag{44}$$

or

$$U^{l+1}(0,S_0,X_0,h_0) = \frac{T}{2}\,\mathrm{E}\Big[\, e^{-fT}\left(h_T(W^D(T,S_T,\bar{X}_T))^+ + \bar{C}_T^{C_0}X_T^{D,C_0}\right) + h_0(W^D(0,S_0,\bar{X}_0)+U^l(0,S_0,X_0,h_0))^+ + \bar{C}_0^{C_0}X_0^{D,C_0}\Big], \tag{45}$$

for $l = 0, 1, 2, \ldots$, until a convergence test with a prescribed tolerance is fulfilled.

Linear case ($M = W^D$). In this case, we denote by I^L the integral on the right hand side of (40), i.e.,

$$I^L = \int_0^T e^{-\int_0^u \left(\frac{h_r}{1-R}+f\right)dr}\left(h_u(W^D(u,S_u,\bar{X}_u))^+ + \bar{C}_u^{C_0}X_u^{D,C_0}\right)du, \tag{46}$$

and approximate the integral by using the composite rectangular formula:

$$I^L \simeq \Delta t \sum_{z_1=0}^{Z-2} \exp\left(-\Delta t \sum_{z_2=0}^{z_1-1}\left(\frac{h_{t_{z_2}}}{1-R}+f\right)\right) \cdot \left(h_{t_{z_1}}(W^D(t_{z_1},S_{t_{z_1}},\bar{X}_{t_{z_1}}))^+ + \bar{C}_{t_{z_1}}^{C_0}X_{t_{z_1}}^{D,C_0}\right), \tag{47}$$

or the composite trapezoidal formula:

$$\begin{aligned} I^L \simeq \frac{\Delta t}{2}\sum_{z_1=0}^{Z-2}\Bigg[&\exp\left(-\frac{\Delta t}{2}\sum_{z_2=0}^{z_1-1}\left(\frac{h_{t_{z_2}}+h_{t_{z_2+1}}}{1-R}+2f\right)\right) \cdot \left(h_{t_{z_1}}(W^D(t_{z_1},S_{t_{z_1}},\bar{X}_{t_{z_1}}))^+ + \bar{C}_{t_{z_1}}^{C_0}X_{t_{z_1}}^{D,C_0}\right) \\ &+ \exp\left(-\frac{\Delta t}{2}\sum_{z_2=0}^{z_1}\left(\frac{h_{t_{z_2}}+h_{t_{z_2+1}}}{1-R}+2f\right)\right) \cdot \left(h_{t_{z_1+1}}(W^D(t_{z_1+1},S_{t_{z_1+1}},\bar{X}_{t_{z_1+1}}))^+ + \bar{C}_{t_{z_1+1}}^{C_0}X_{t_{z_1+1}}^{D,C_0}\right)\Bigg]. \end{aligned} \tag{48}$$

Note that in the nonlinear case we consider only simple rectangular and simple trapezoidal formulas due to the fact that the use of composite formulas requires to know the values of U at intermediate time nodes, but we only know the final value of U, that is, U is null at the final node $t_{Z-1} = T$. Therefore, one could approximate the value of U at each node going backwards from the last node, although in this way a nested Monte Carlo problem arises.

Besides the previously described Picard iteration methods for the nonlinear model, we have also coded and applied to the nonlinear case the multilevel Picard iteration method proposed in [13]. Note that this method is also applied in [13] to a special case of the model in [6] to obtain the CVA in a single currency setting. In the next section we will include the comparison between the results of the previously described Picard iteration methods and the multilevel Picard iteration.

Table 1
Data.

$r = (0.07, 0.09, 0.12)$	$\sigma^S = (0.30, 0.20)$	$q = (0.07, 0.08)$	$r^D = 0.14$
$X_0 = (0.13, 0.89, 1.12)$	$\sigma^X = (0.38, 0.40, 0.35)$	$R_0^{C_0} = 25$	$M_0^{C_0} = 25$
$h_0 = 0.20$	$\kappa = 0.01$	$\sigma^h = 0.2$	$R = 0.3$
$c^D = 0.06$	$r^R = 0.05$	$b^{D,C_0} = 0.02$	$f = 0.06$

4. Numerical results

In this section we present some results obtained by using the techniques described in the previous one. More precisely, we consider the pricing of some multiasset derivatives written on underlying assets denominated in different currencies [46], taking into account counterparty risk.

In particular, we use different quadrature formulas to approximate integrals involved in the expressions of XVA price (38) and (40) and analyse how this affects the Monte Carlo confidence intervals for risk free price, risky price and XVA price, as well as the elapsed computational time. Moreover, we are interested in how different choices of the initial values of the underlying assets, of the mark-to-market value M, and of the FX rates volatilities, affect both the risky derivative and the XVA prices.

In all forthcoming numerical examples, for the linear case and the nonlinear case with Picard iteration method we have set the number of simulations equal to 10^4 and the number of time nodes in the discretized dynamics of the involved processes to $Z = 10^3$. Unless otherwise stated, we have used data listed in Table 1, where we denote by $r = (r^0, r^1, r^2)$ the vector of the risk-free rates in the foreign markets, $q = (q^1, q^2)$ the vector of the dividends paid by the corresponding underlying assets, $\sigma^S = (\sigma^{S^1}, \sigma^{S^2})$ the vector of the assets volatilities, and $\sigma^X = (\sigma^{X^0}, \sigma^{X^1}, \sigma^{X^2})$ the vector of the FX rates volatilities.

Moreover, for the linear case and the nonlinear case with the Picard iteration methods, we report Monte Carlo 99% confidence intervals in the corresponding tables while the average Monte Carlo value is used for each point in the figures.

As indicated at the end of the previous section, we have also applied to all forthcoming numerical examples the multilevel Picard iteration method proposed in [13] to solve the nonlinear model. More precisely, for the numerical solution with multilevel Picard iteration we have considered the Picard parameter $\rho = 5$ and the number of Picard iterations $k = \rho$. At each Picard iteration l, we consider a composite rectangular quadrature formula with ρ^{k-l} rectangles and a number of Monte Carlo paths $m_{k,l,\rho} = \rho^{k-l}$, for $l = 1, \dots, k$. Moreover, as also suggested in [13], we consider 10 runs of the multilevel Picard iteration method with the previous configuration to obtain the mean values we report in the corresponding tables for each numerical example. For further details on the method and the notation, we address the reader to [13].

All the described numerical methods for the examples have been implemented from scratch in Matlab codes on an Intel(R) Core(TM) i7-8550U, 1.99 GHz, 16 GB (RAM), x64-based processor.

4.1. Risk-free, risky and XVA prices as functions of the underlying assets

We assume that the default-free hedger H buys from the defaultable counterparty an *option on the maximum of two underlying assets*: the price of the first one, S_t^1, is denominated in the foreign currency C_1, while the price of the second one, S_t^2, is denominated in the foreign currency C_2. The payoff function is given by:

$$G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = (\max(S_t^1 X_t^{D,C_1}, S_t^2 X_t^{D,C_2}) - K)^+, \quad (49)$$

where K denotes the strike value in domestic currency D, which is set to $K = 15$ in the numerical tests.

Fig. 1 represents the risk-free price and the risky price in the nonlinear case. The simple trapezoidal formula has been used to approximate the integrals in the XVA pricing equation (38). As expected, the risky derivative is less expensive than the risk-free derivative, so that the XVA price is negative, because the counterparty may default and, therefore, owes the hedger H a reduction in the derivative price.

Obviously, the XVA price is negative regardless of the choice of the mark-to-market value, as shown in Fig. 2, where we plot the XVA price in the nonlinear case, when the mark-to-market value M is equal to the risky derivative price V^D, and in the linear case, when M is equal to the risk-free price W^D. Results have been obtained using both

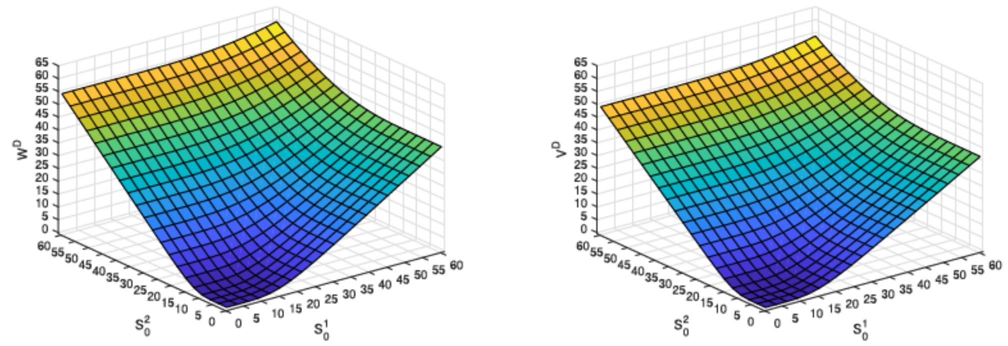
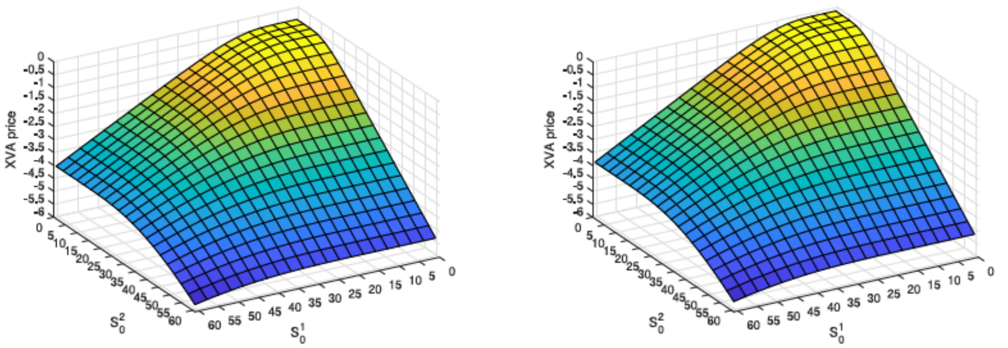
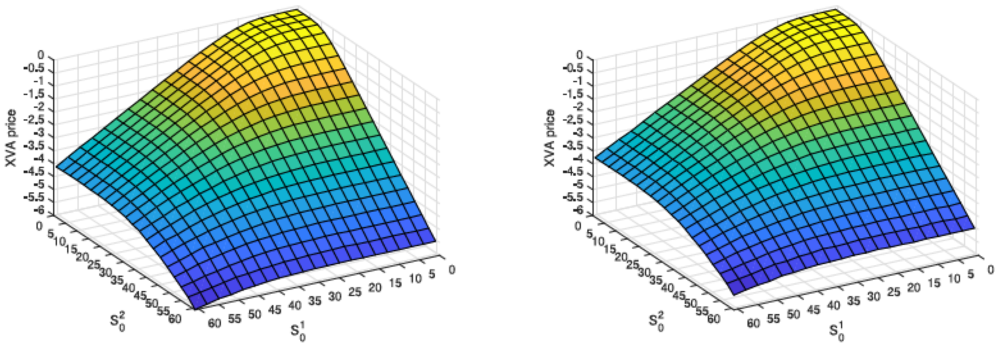


Fig. 1. Option on the maximum in the nonlinear case. Risk-free price (left) and risky price (right) as functions of the initial values of the underlying assets. Integrals are approximated by the simple trapezoidal rule.



(a) Nonlinear case. The integral is approximated by the simple rectangular formula

(b) Nonlinear case. The integral is approximated by the simple trapezoidal formula



(c) Linear case. The integral is approximated by the simple rectangular formula

(d) Linear case. The integral is approximated by the simple trapezoidal formula

Fig. 2. Option on the maximum. XVA price in both the nonlinear case (top) and in the linear case (bottom) as function of the initial values of the underlying assets. Simple rectangular (left) and trapezoidal (right) quadrature formulas have been used.

simple rectangular and simple trapezoidal quadrature formulas. In general, the simple rectangular rule gives greater estimate values, in absolute terms, than the simple trapezoidal rule, both in the nonlinear and in the linear cases.

In order to better illustrate the difference between the nonlinear case and the linear case, in Table 2 we report the Monte Carlo confidence intervals for the risky price and the XVA price for different values of the initial price of the underlying asset S^2, while the initial price of S^1 is fixed to 20.

Table 2
Option on the maximum. Risky price and XVA price confidence intervals in the nonlinear and the linear case for different initial values of the underlying assets with different numerical integration formulas.

Nonlinear case

		Simple rectangular formula		Simple trapezoidal formula	
S^1	S^2	V^D	XVA	V^D	XVA
20	10	[3.6869, 3.9608]	[−0.7830, −0.7561]	[3.9789, 4.252]	[−0.4935, −0.4626]
20	15	[5.1665, 5.4484]	[−0.9260, −0.8982]	[5.4525, 5.7341]	[−0.6423, −0.6101]
20	20	[8.2995, 8.6308]	[−1.2401, −1.2071]	[8.5750, 8.9072]	[−0.9670, −0.9282]
20	25	[12.4525, 12.8606]	[−1.6615, −1.6208]	[12.7147, 13.1249]	[−1.4024, −1.3533]
20	30	[17.0868, 17.5811]	[−2.1332, −2.0838]	[17.3342, 17.8318]	[−1.8900, −1.8290]

Linear case

		Simple rectangular formula		Simple trapezoidal formula	
S^1	S^2	V^D	XVA	V^D	XVA
20	10	[3.9500, 4.2207]	[−0.5231, −0.4931]	[3.9946, 4.2685]	[−0.47652, −0.4473]
20	15	[5.4095, 5.6882]	[−0.6862, −0.6552]	[5.4724, 5.7549]	[−0.62112, −0.5907]
20	20	[8.5072, 8.8350]	[−1.0358, −0.9994]	[8.6051, 8.9380]	[−0.93541, −0.8990]
20	25	[12.6172, 13.0212]	[−1.5010, −1.4561]	[12.7589, 13.1699]	[−1.3563, −1.3103]
20	30	[17.2048, 17.6942]	[−2.0202, −1.9658]	[17.3944, 17.8928]	[−1.8273, −1.7705]

Linear case

		Composite rectangular formula		Composite trapezoidal formula	
S^1	S^2	V^D	XVA	V^D	XVA
20	10	[3.9939, 4.2681]	[−0.4777, −0.4472]	[3.9940, 4.2681]	[−0.4776, −0.4472]
20	15	[5.4724, 5.7552]	[−0.6217, −0.5898]	[5.4725, 5.7553]	[−0.6216, −0.5897]
20	20	[8.6056, 8.9391]	[−0.9356, −0.8971]	[8.6057, 8.9393]	[−0.9355, −0.8970]
20	25	[12.7592, 13.1712]	[−1.3570, −1.3079]	[12.7594, 13.1714]	[−1.3569, −1.3077]
20	30	[17.3943, 17.8940]	[−1.8290, −1.7677]	[17.3945, 17.8942]	[−1.8287, −1.7675]

Table 3
Option on the maximum with the nonlinear model. XVA values computed with simple quadrature formulas (confidence intervals) and multilevel Picard iteration (mean over 10 runs).

S^1	S^2	Rectangular	Trapezoidal	Multilevel Picard Iteration
20	10	[−0.7830, −0.7561]	[−0.4935, −0.4626]	−0.4847
20	15	[−0.9260, −0.8982]	[−0.6423, −0.6101]	−0.6333
20	20	[−1.2401, −1.2071]	[−0.9670, −0.9282]	−0.9506
20	25	[−1.6615, −1.6208]	[−1.4024, −1.3533]	−1.3692
20	30	[−2.1332, −2.0838]	[−1.8900, −1.8290]	−1.8542

Table 2 includes the nonlinear case, the linear case with simple quadrature formulas, and the linear case with composite quadrature formulas. In any case, we observe the same trend: the total value adjustment becomes more and more negative by increasing the initial price of S^2 and, therefore, the derivative price increases. However, it is shown that the simple rectangular formula gives more negative XVA estimates than the simple trapezoidal formula in the nonlinear case and than all the other considered quadrature formulas in the linear case.

In Table 3, for the nonlinear case we show that the results obtained with the multilevel Picard iteration method belong to the confidence interval computed with the simple trapezoidal quadrature formula, while it is not the case when using the rectangle formula. Therefore, if we consider that the multilevel Picard iteration method provides a reference solution, then by using simple trapezoidal rule we get enough accuracy.

Table 4
Best of put/put option with nonlinear model. XVA price for different sets of FX rates volatilities values with simple quadrature formulas (confidence intervals) and with multilevel Picard iteration (mean over 10 runs). Results in the last row correspond to constant FX rates.

σ^X	Simple formulas		Multilevel Picard Iteration
	Rectangular	Trapezoidal	
(0.275, 0.05, 0.05)	[−0.4948, −0.4882]	[−0.2039, −0.1952]	−0.1984
(0.275, 0.05, 0.50)	[−0.5826, −0.5728]	[−0.3027, −0.2900]	−0.2940
(0.275, 0.50, 0.05)	[−0.5809, −0.5702]	[−0.2898, −0.2766]	−0.2844
(0.275, 0.50, 0.50)	[−0.6515, −0.6399]	[−0.3710, −0.3563]	−0.3640
(0.000, 0.00, 0.00)	[−0.4906, −0.4842]	[−0.2000, −0.1914]	−0.1943
$X^{D,C_j} \equiv X_0^{D,C_j}$	[−0.5045, −0.4979]	[−0.2153, −0.2064]	−0.2108

4.2. *XVA price for different values of FX rates volatilities*

In this example, we assume that the hedger buys from the counterparty a *best of put/put option*, the payoff of which is given by:

$$G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \max((K^1 - S_t^1 X_t^{D,C_1})^+, (K^2 - S_t^2 X_t^{D,C_2})^+), \tag{50}$$

where S_t^1 and S_t^2 are the prices of the two underlying assets denominated in their respective foreign currencies C_1 and C_2. Moreover, we denote by $K = (K^1, K^2)$ the vector of the strike values for the involved put options. In our numerical examples we have taken $S_0 = (12, 15)$ and $K = (12, 15)$.

We investigate how different values of the FX rates volatilities affect the XVA price. In particular, we keep fixed the volatility of X^{D,C_0}, say $\sigma^{X^0} = 0.275$, while for the volatilities of X^{D,C_1} and X^{D,C_2} we choose either a high volatility value, say 0.50, or a low volatility value, say 0.05.

Moreover, we also consider the case of null FX rates volatilities, so that FX rates are deterministic time-dependent functions. In fact, by considering $\sigma^{X^j} \equiv 0$ in (11), we obtain $dX_t^{D,C_j} = (r^D - r^j)X_t^{D,C_j} dt$, so that

$$X_t^{D,C_j} = X_0^{D,C_j} \exp((r^D - r^j)t).$$

Furthermore, we also consider the case of constant FX rates with $X_t^{D,C_j} = X_0^{D,C_j}$, that corresponds to the modelling approach in [2].

In Table 4 we show the XVA confidence intervals for the nonlinear case, while in Table 5 we report the results for the linear case. With reference to the cases with no null FX rates volatilities, as expected, regardless of the choice of the mark-to-market values and of the quadrature formula, XVA is more negative when both X^{D,C_1} and X^{D,C_2} have high volatility values, increases when only one of the two FX rates has a high volatility, and is greater when both the FX rates have low volatility values. In fact, higher volatilities correspond to higher levels of risk, therefore, in that case XVA becomes more negative, thus making the risky derivative price lower. Note that when all volatilities are null we are considering time dependent deterministic FX rates that decrease with time from their initial value X_0^{D,C_j}. In this case the XVA is less negative than any stochastic FX rates case, as shown in the corresponding rows of Tables 4 and 5. In order to compare with the case of constant FX rates developed in [2], we show in the last row of both tables the results for $X^{D,C_j} \equiv X_0^{D,C_j}$, which are a bit more negative than in the corresponding deterministic time dependent case (where the FX rates values decrease with time) and less negative than in the stochastic FX rates case when certain level of volatility is assumed.

Note that the last column of Table 4 shows the XVA obtained with multilevel Picard iteration. As in the previous case, for all sets of FX rates volatilities, the confidence interval obtained with the simple trapezoidal formula includes the values computed with multilevel Picard iteration method, which is not the case for the confidence intervals provided by the rectangle formula.

In Fig. 3 we plot the risk-free price, the risky price and the XVA price as functions of the initial prices of the underlying assets in the nonlinear case. The approximation of the integral in (38) has been obtained by the simple trapezoidal formula, the results of which are in agreement with the reference solution obtained with the multilevel Picard iteration method (see Table 4). As already discussed in the case of the option on the maximum between two

Table 5
Best of put/put option with the linear model. XVA price confidence intervals for different sets of FX rates volatilities values with simple and composite quadrature formulas. Results in the last row correspond to constant FX rates.

σ^X	Simple formulas		Composite formulas	
	Rectangular	Trapezoidal	Rectangular	Trapezoidal
(0.275, 0.05, 0.05)	[−0.2060, −0.1982]	[−0.1959, −0.1877]	[−0.1973, −0.1887]	[−0.1973, −0.1887]
(0.275, 0.05, 0.50)	[−0.3069, −0.2956]	[−0.2911, −0.2792]	[−0.2935, −0.2809]	[−0.2935, −0.2809]
(0.275, 0.50, 0.05)	[−0.3009, −0.2885]	[−0.2793, −0.2668]	[−0.2800, −0.2668]	[−0.2799, −0.2668]
(0.275, 0.50, 0.50)	[−0.3827, −0.3695]	[−0.3574, −0.3436]	[−0.3591, −0.3445]	[−0.3591, −0.3444]
(0.000, 0.00, 0.00)	[−0.2013, −0.1936]	[−0.1921, −0.1841]	[−0.1937, −0.1852]	[−0.1936, −0.1852]
$X^{D,C_j} \equiv X_0^{D,C_j}$	[−0.2180, −0.2101]	[−0.2069, −0.1986]	[−0.2086, −0.1998]	[−0.2086, −0.1998]

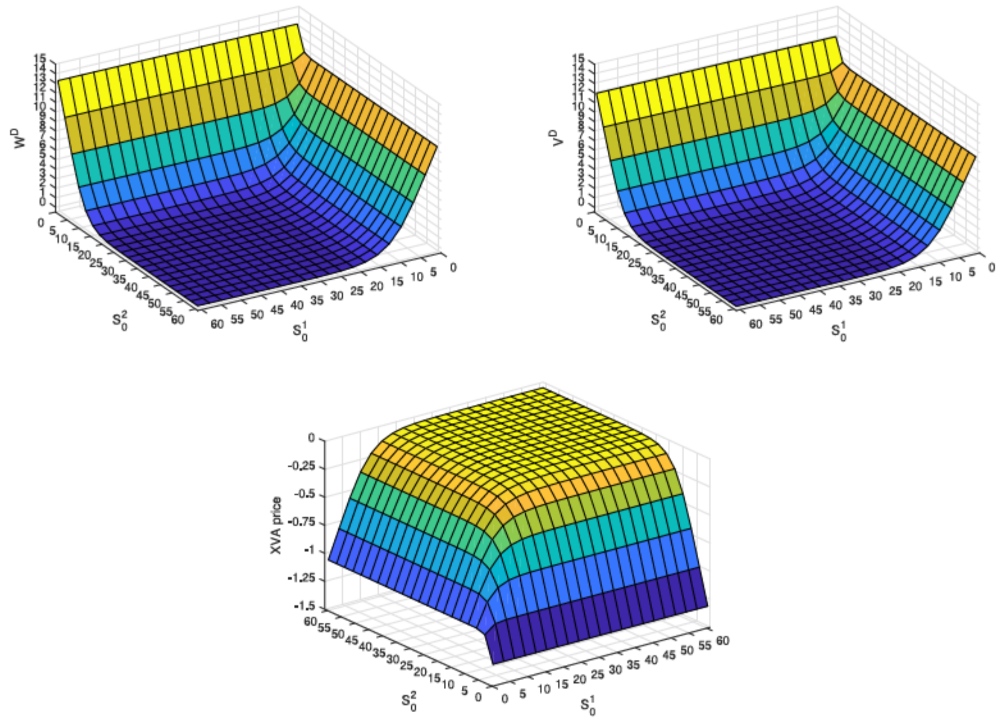


Fig. 3. Best of put/put option in the nonlinear case. Risk-free price, risky price and XVA price as functions of the initial values of the underlying assets. Integrals are approximated by the simple trapezoidal rule.

underlying assets, the total value adjustment is negative and reduces the price of the risky derivative with respect to the corresponding risk-free derivative. Moreover, it is shown that the derivative value increases by decreasing one or both the initial prices of the underlying assets and, consequently, XVA becomes more negative, because when the derivative is more valuable the counterparty default would lead the hedger to a worse loss.

4.3. Elapsed time

Finally, in order to investigate how the elapsed computational time changes for the different methods when increasing the number of the underlying assets, in the following example we assume that the hedger buys from the counterparty a *basket call option* written on N underlying assets $S^1, \ldots, S^N$ with weights $\alpha^1, \ldots, \alpha^N$. Therefore, the payoff function is given by:

$$G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \left(\sum_{i=1}^{N} \alpha^i S_t^i X_t^{D,C^i} - K\right)^+, \tag{51}$$

Table 6
Data for the basket option. For $N = 2, 4, 8, 16$ we respectively consider the first 3, 5, 9, 17 rows of the table on the left and the corresponding column of the table on the right for the vector of weights α.

i	S_0^i	r^i	q^i	σ^{S^i}	X_0^{D,C_i}	σ^{X^i}
0	–	0.30	–	–	0.47	0.27
1	10	0.30	0.24	0.35	0.89	0.34
2	14	0.24	0.15	0.28	0.54	0.35
3	15	0.25	0.20	0.26	0.15	0.34
4	12	0.31	0.26	0.24	1.21	0.30
5	10	0.28	0.22	0.35	0.33	0.26
6	10	0.29	0.23	0.25	0.16	0.24
7	13	0.32	0.30	0.27	0.61	0.29
8	14	0.28	0.22	0.24	0.14	0.28
9	15	0.34	0.29	0.26	1.20	0.30
10	12	0.33	0.26	0.28	0.12	0.29
11	11	0.25	0.18	0.34	1.13	0.30
12	15	0.23	0.16	0.28	0.19	0.30
13	16	0.22	0.17	0.35	0.62	0.33
14	12	0.26	0.19	0.24	1.11	0.32
15	17	0.32	0.29	0.28	0.17	0.31
16	18	0.26	0.21	0.28	0.28	0.29

α			
$N = 2$	$N = 4$	$N = 8$	$N = 16$
0.4982	0.1695	0.1490	0.0531
0.5018	0.3501	0.1197	0.0632
	0.1130	0.1959	0.0666
	0.3674	0.1004	0.0625
		0.1706	0.0613
		0.1169	0.0722
		0.0815	0.0719
		0.0659	0.0597
			0.0465
			0.0556
			0.0652
			0.0711
			0.0702
			0.0511
			0.0641
			0.0598

In the numerical tests we have chosen $K = 5$.

In particular, we choose $N = 2, 4, 8, 16$ and for any chosen value of N we consider data in the first $N + 1$ rows of the left part in Table 6 and the corresponding column of the right part in Table 6, where we have denoted by α the vector of the underlying assets weights, that is to say, $\alpha = (\alpha^i, \dots, \alpha^N)$.

In Table 7 we show Monte Carlo 99% confidence intervals for the prices of the risky derivative and the XVA, jointly with the elapsed computational time for both the linear and the nonlinear cases with different numbers of underlying assets. Rectangular and trapezoidal quadrature formulas are used to approximate integrals involved in the XVA price formulas. In addition, for the nonlinear case also the number of Picard iterations is reported. Concerning the prices of the risky derivative and XVA, in the linear case the confidence intervals coincide in the three or four decimal figures when using both composite formulas (they could be taken as reference values), the result of the simple trapezoidal rule being the closest to these values while the one from single rectangular formula is usually a bit more far than those ones. In the nonlinear case, the confidence intervals provided by the trapezoidal rule are closer to the corresponding reference values obtained in the linear case. With respect to elapsed times, as expected, the simple rectangular and the simple trapezoidal formulas are faster in the linear case, where the fixed point method is not employed. The elapsed computational time in the nonlinear case depends not only on the quadrature formula itself, but also on the number of iterations needed by the fixed point method: the rectangular formula always needs more iterations than the trapezoidal formula and turns out to be slower if $N = 4$ or $N = 16$. In the linear case the simple and the composite rectangular formulas are faster than the simple and the composite trapezoidal formula, respectively. Moreover, the composite formulas are obviously slower than the corresponding simple formulas.

Finally, in Table 8 we compare the solution of the nonlinear case with simple rectangular, simple trapezoidal and multilevel Picard iteration method. As in the previous numerical examples, the computed confidence intervals with simple trapezoidal rule include the values provided by the multilevel Picard iteration method for all the chosen values of N. Moreover, the reported computational times for both methods make the simple trapezoidal quadrature formula more competitive. Note that the computational time reported for the multilevel Picard iteration method corresponds to the 10 runs to compute the mean value reported in Table 8.

5. Conclusions

In the previous work [2], the models and computations of the XVA for a multi-currency setting have been developed when constant exchange rates between the different currencies are considered. In order to pose a more

Table 7
Basket option. Risky price and XVA price confidence intervals in the nonlinear case (NL) and in the linear case (L) for increasing number N of underlying assets. Different quadrature formulas are used to approximate integrals. The elapsed computational time in seconds and the number of fixed point iterations (FPI) for the nonlinear case are reported.

N			V^D	XVA	Time	FPI
2	NL	simple rect	[2.2616, 2.3476]	[−0.6135, −0.6042]	1.8803	13
		simple trap	[2.5433, 2.6296]	[−0.3325, −0.3215]	2.0698	11
	L	simple rect	[2.5919, 2.6804]	[−0.2833, −0.2714]	1.5015	
		simple trap	[2.5538, 2.6408]	[−0.3211, −0.3112]	2.4905	
		comp rect	[2.5528, 2.6397]	[−0.3225, −0.3119]	8.2648	
		comp trap	[2.5528, 2.6398]	[−0.3225, −0.3119]	8.3771	
4	NL	simple rect	[3.4511, 3.5332]	[−0.7290, −0.7199]	3.8816	14
		simple trap	[3.6617, 3.7444]	[−0.5197, −0.5074]	3.3307	11
	L	simple rect	[3.7352, 3.8202]	[−0.4460, −0.4317]	2.4135	
		simple trap	[3.6773, 3.7606]	[−0.5030, −0.4924]	2.9364	
		comp rect	[3.6766, 3.7600]	[−0.5042, −0.4925]	8.7305	
		comp trap	[3.6766, 3.7601]	[−0.5041, −0.4924]	9.8473	
8	NL	simple rect	[0.0987, 0.1271]	[−0.4048, −0.4024]	6.8170	11
		simple trap	[0.0323, 0.0605]	[−0.4720, −0.4682]	7.1890	9
	L	simple rect	[0.1014, 0.1301]	[−0.4031, −0.3983]	5.7553	
		simple trap	[0.0511, 0.0791]	[−0.4532, −0.4497]	6.6416	
		comp rect	[0.0525, 0.0806]	[−0.4519, −0.4480]	13.2813	
		comp trap	[0.0526, 0.0807]	[−0.4519, −0.4479]	14.3597	
16	NL	simple rect	[1.0071, 1.0308]	[−0.4784, −0.4758]	14.0100	13
		simple trap	[1.1801, 1.2040]	[−0.3061, −0.3020]	13.3564	11
	L	simple rect	[1.2212, 1.2457]	[−0.2651, −0.2603]	12.3217	
		simple trap	[1.1879, 1.2119]	[−0.2979, −0.2944]	12.5906	
		comp rect	[1.1881, 1.2121]	[−0.2979, −0.2940]	19.7964	
		comp trap	[1.1881, 1.2122]	[−0.2979, −0.2940]	20.4875	

Table 8
Basket option for increasing number N of underlying assets for the nonlinear case. XVA confidence intervals computed with simple rectangular and trapezoidal formulas and XVA mean values over 10 runs of the multilevel Picard iteration (MPI) method. The elapsed computational times in seconds for all methods and the number of fixed point iterations (FPI) for the simple formulas with Picard iteration methods are reported.

N	Method	XVA	Time	FPI
2	simple rect	[−0.6135, −0.6042]	1.88	13
	simple trap	[−0.3325, −0.3215]	2.07	11
	MPI	−0.3279	4543.18	
4	simple rect	[−0.7290, −0.7199]	3.8816	14
	simple trap	[−0.5197, −0.5074]	3.3307	11
	MPI	−0.5097	5935.35	
8	simple rect	[−0.4048, −0.4024]	6.8170	11
	simple trap	[−0.4720, −0.4682]	7.1890	9
	MPI	−0.4686	8697.76	
16	simple rect	[−0.4784, −0.4758]	14.0100	13
	simple trap	[−0.3061, −0.3020]	13.3564	11
	MPI	−0.3034	14350.53	

realistic modelling approach, the main objective of the present work has been to extend the previous work to the consideration of stochastic models for the evolution of foreign exchange rates. Note that the consideration of stochastic exchange rates significantly increases the number of underlying risk factors. Thus, after proposing suitable dynamics for exchange rates evolution, the portfolio replication and the dynamic hedging methodologies have provided the formulation of the XVA pricing problem in terms of linear and nonlinear PDEs with larger dimensions than in the case of constant exchange rates. Moreover, the use of Feynman–Kac formulas allows to obtain the equivalent formulations in terms of expectations, so that Monte Carlo simulation techniques can be applied to the corresponding linear and nonlinear models. In the nonlinear case, Picard iteration methods based on simple rectangular and trapezoidal quadrature formulas can be compared with the recently introduced multilevel Picard iteration methods [13,14].

Concerning the comparison of numerical methods when applied to all the considered examples, it seems that Picard iteration method with simple trapezoidal quadrature formula exhibits the best balance of accuracy and computational cost. Thus, the 99% Monte Carlo confidence interval provided by this method always contains the reference value obtained with multilevel Picard iteration method, with a significantly shorter computational time. With the same order of computational cost, the confidence interval obtained with the simple rectangular formula does not contain the XVA price provided by the multilevel Picard iteration method.

From the modelling point of view, numerical examples illustrate the consequences of using the more realistic stochastic models for the evolution of exchange rates. As illustrated in Tables 4 and 5, when considering time dependent deterministic FX rates, the price of the XVA is underestimated with respect to the case of more realistic stochastic rates evolution, namely the confidence interval of the XVA is located in a less negative region than in the stochastic case. In the case of constant FX rates, this underestimation also appears when compared with stochastic FX rates with a large enough level of volatility.

As possible future extensions of the present work, a first step could be the consideration of more sophisticated models for the stochastic evolution of FX rates, by incorporating local [12], stochastic [7] or local/stochastic volatility [39], see also the books [22,38] and the references therein. Another extension could be the consideration of stochastic dynamics for the risk free rates operating in each market, as proposed in [35] in a multi-currency setting, thus additionally increasing the number of stochastic factors. Note that the joint consideration of stochastic volatility models for FX combined with stochastic interest rates for cross-currency markets has been widely addressed for two currencies, see [42] and the references therein. For example, a Heston model for the FX rate evolution can be combined with a one factor model for each stochastic rate [41] or the one proposed in [20].

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