MARIA J. WONENBURGER

THE SPIN REPRESENTATION OF THE UNITARY GROUP

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obteniendo la calificación de SOBRESALIENTE «cum laude».

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THE SPIN REPRESENTATION OF THE UNITARY GROUP

por

MARIA J. WONENBURGER (1)

RESUMEN

Dada una forma hermitiana *f* no degenerada, definida sobre el cuerpo conmutativo F, y siendo el automorfismo involutivo asociado a *f* distinto de la identidad, se define una forma cuadrática Q asociada a *f*. Entonces las transformaciones unitarias y semejanzas unitarias respecto a *f* son transformaciones ortogonales y semejanzas respecto a Q. La representación espinorial del grupo ortogonal definido por Q induce una representación del grupo unitario definido por *f*. Llamamos a esta representación del grupo unitario su representación espinorial.

En el caso de que la característica de F sea cero o mayor que la dimensión de M sobre F, se demuestra que la representación espinorial del grupo unitario es completamente reducible y se hallan sus componentes irreducibles de las que se determinan algunas propiedades. La representación espinorial del grupo unitario puede extenderse a una representación del grupo de semejanzas unitarias, obteniéndose para este caso la misma descomposición en componentes irreducibles.

Finalmente, el método usado para el estudio de la representación espinorial del grupo unitario permite definir representaciones del grupo proyectivo de semejanzas unitarias en grupos ortogonales. Es posible que estas representaciones puedan ser obtenidas también usando las componentes homogéneas del álgebra exte-

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rior del espacio M, pero nuestra definición nos proporciona mejores medios para efectuar su estudio, que será llevado a cabo en un próximo artículo.

INTRODUCTION

The Clifford algebra [4] is an algebra defined by a quadratic form. If Q is a quadratic form on the vector space M over the field K, we will denote the Clifford algebra of the form Q by C (Q); C (Q) is an algebra over K. The definition of this algebra as well as the subalgebra C^+ (Q) and the study of their properties can be found in [1], [5], [6], [11] and [15], being [5] the most complete study.

We will consider only the case of a space M even dimensional over K, i. e. (M : K) = $2n$. Then C (Q) is a central simple algebra over K of dimensión 2^{2n} and, therefore, it is isomorphic to a total matric algebra over a sfield R whose center is K. The algebra C (Q) contains a subspace over K which is identified with the vector space M. Any basis of this subspace is a set of generators over K.

Let c be an invertible element of C (Q). Then c defines an inner automorphism taking the element b into $c^{-1} b c$. The set of invertible elements of C (Q) defining inner automorphisms which leave invariant the subspace M form a group which is called the Clifford group. The transformations induced in M by these automorphisms are orthogonal transformations with respect to Q. The mapping χ which takes an element c of the Clifford group into the orthogonal transformation induced in M by the inner automorphism defined by c is a homomorphism of the Clifford group on the orthogonal group O (Q). The kernel of this homomorphism consists of the multiplicative group of non-zero elements of K.

Since C (Q) is a simple algebra all its irredudible representations are equivalent. Any one of these representations, which are called spin representations, induces a representation of the Clifford group.

C (Q) has an involutive anti-automorphism leaving invariant the elements of M. This anti-automorphism will be denoted

by *. If c is an element of the Clifford group, $c c^*$ is an element of K called the norm of c.

Let f be a hermitian form on the vector space M of dimension n over the field F. Let K the subfield of F consisting of the elements invariant under the involutive automorphism J of F associated to the hermitian form. Then there exists a quadratic form on M considered as a vector space over K such that the unitarian transformations of M with respect to f are orthogonal transformations with respect to Q (cf. [12] and [17]). Of course the converse is not true.

We take the subgroup U $(f) \subset$ O (Q) of orthogonal transformations of Q which are unitarian transformations of f. The spin representation of the Clifford group induces a representation of the subgroup Δ consisting of the elements of the Clifford group which are mapped by χ into elements of U (f). We called this induced representation the spin representation of the unitary group and its study forms the main subject of this paper.

In order to find the irreducible components of the spin representation of U (f) it is sufficient to know the structure of the algebra G over K generated by the elements of Δ. It does not seem easy to find directly the structure of G, so that we start defining a subalgebra D (f) of C^+ (Q). Then we show that, when the characteristic of K is zero or greater than (M : F), D (f) is semisimple and we determine its simple components.

In chapter II we make a further study of D (f) in order to prove that it coincides with G. Therefore we can conclude that, when the characteristic of K fulfills the conditions mentioned above, the spin representation of U (f) is a direct sum of inequivalent irreducible representations.

In [16] we have defined in C (Q), considered as a vector espace over K, a gradation with indices 0, 1, ..., (M : K). Then C^+ (Q) is the sum of the subspaces of even degree and the Clifford group is the set of invertible elements which define inner automorphisms homogeneous of degree zero. Moreover the inner automorphisms of C (Q) which induce in C^+ (Q) homogeneous automorphisms of degree zero are the automorphims of C^+ (Q) associated to a similitude of Q. With respect to this gradation D (f) is a homogeneous subspace of C^+ (Q) and if a similitude of Q is a unitarian similitude with respect to f, the automorphism of $C^+(Q)$ associated to this similitude induces in $D(F)$ an inner automorphism.

Using these results, in chapter III we obtain faithful representation of the projective group of unitarian similitudes of Q into orthogonal groups. To do this we define a non degenerate quadratic form on the subspaces of $D(f)$ of degree

$$2i, \quad i = 1, 2, \ldots, (M:F),$$

and consider the transformations induced in those subspaces by the automorphisms of $C^+(Q)$ associated to the similitudes of Q which are unitarian similitudes with respect to f.

Chapter I

We start this chapter recalling the definitions of hermitian forms and unitarian similitudes and, at the same time, we set down our notation. These definitions will be given with the generality needed for our purpose; in [9] chap. I, §§ 5,6,9 the reader can find more general definitions.

The subalgebra $D(f)$ of $C(Q)$ is defined and studied, as well as the involutive anti-automorphism induced in it by the anti-automorphism* of $C(Q)$.

§ 1

Let F be a field of characteristic different from 2 and J an involutive automorphism of F different from the identity. The elements of F will denoted by small Greek letters. Let

$$K = \{\alpha \mid \alpha^J = \alpha, \quad \alpha \in F\}$$

be the subfield of elements of F invariant under J. Then F is a quadratic extension of K obtained adjoining any element θ such that $\theta^J = -\theta$ and therefore $\theta^2 = \rho \in K$.

Let M be a left vector space over F whose elements will be denoted by small Latin letters. It is said that $f(x, y)$ is a hermitian form on M relative to the automorphism J if it is a function with values in F satisfying the following conditions,

I) it is biadditive, i. e.,

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y); \qquad f(x, y_1 + y_2) = \\ = f(x, y_1) + f(x, y_2);$$

II) sesquilinear,

$$f(\lambda x, y) = \lambda f(x, y) \quad \text{and} \quad f(x, \lambda y) = f(x, y)\lambda^J; \quad \text{and}$$

III) reflexive,

$$f(x, y) = (f(y, x))^J.$$

If S is a linear transformation of M, uS will be the image of $u \in M$ under S. It is said that the linear transformation S is a unitarian similitude of ratio μ with respect to f (or a similitude of f) if

$$f(xS, yS) = \mu f(x, y).$$

When $\mu = 1$, the unitarian similitude S is called a unitarian transformation. We denote by $T_{\alpha+\beta\theta}$ the untarian similitudes defined by

$$xT_{\alpha+\beta\theta} = (\alpha + \beta\theta)x,$$

which are called unitarian homoteties.

When $f(x, y) = 0$ for every $y \in M$ implies $x = 0$, it is said that the form f is non-degenerate. In what follows M will always be a finite dimensional vector space over F and f a non-degenerate hermitian form on M.

Since M is a vector space over F, it has an underlying structure of vector space over $K \subset F$, and $(M:K) = 2(M:F)$. Taking

$$(x, y) = f(x, y) + f(y, x),$$

(x, y) is a non-degenerate symmetric bilinear form on M, considered as a vector space over K, associated to the quadratic form $Q(x) = \frac{1}{2}(x, x)$.

Any unitarian similitude with respect to f is a similitude of the same ratio with respect to Q. The unitarian similitudes of the form f are the similitudes of Q commuting with the similitude T defined by the unitarian homotecy T_θ (cf. [17]).

Let C(Q) be the Clifford algebra of the quadratic form Q and $x_1, x_2, \ldots, x_{2n}$ an orthogonal basis of M with respect to Q. If we consider C(Q) as a graded vector space, the elements

$$x_{i_1} x_{i_2} \ldots x_{i_h}, \quad i_1 < i_2 < \ldots < i_h, \quad 0 \leq h \leq 2n,$$

form a basis of the subspace of degree h. As usual, we have identified the subspace of degree 1 with the elements of M. $C^+(Q)$ as vector space over K is the sum of the subspaces of even degree,

It is known that we can associate to any similitude of Q an automorphism of $C^+(Q)$ (cf. [9], pag. 72, [10], [11]). A complete definition of these automorphisms given in an unpublished paper by N. Jacobson is reproduced in [16]. It follows from the definition that such automorphisms are homogeneous of degree zero with respect to the gradation of $C^+(Q)$. The automorphisms associated to the similitudes of Q, S and S′ coincide if and only if $S' = S T_\alpha$, $\alpha \in K$. Given any similitude S, there exist invertible elements of C(Q) which define inner automorphisms of this algebra inducing in $C^+(Q)$ the automorphism associated to S (cf. [16]); in particular, if S is an orthogonal transformation the inner automorphism defined by any element of the Clifford group mapped by χ into S induces in $C^+(Q)$ the automorphism associated to S. The mapping which takes a similitude of Q into the automorphism of $C^+(Q)$ associated to it is a homomorphism.

Since any unitarian similitude U with respect to f is a similitude of Q, we can associate to U an automorphism of $C^+(Q)$; in particular, if U is a unitarian transformation by its associated automorphism we will mean the inner automorphism of C(Q) defined by any element of the Clifford group mapped by χ into U.

Definition.—D(f) is the subalgebra of $C^+(Q)$ consisting of the elements invariant under the automorphisms of $C^+(Q)$ associated to the unitarian homotecies.

D(f) is an algebra over K and it follows from its definition that it is a homogeneous subspace of $C^+(Q)$ considered as a graded vector space.

If $x_1, x_2, \ldots, x_n$ is an orthogonal basis of M with respect to f, the elements

$$x_1, x_2, \ldots, x_n, y_1 = \theta x_1, y_2 = \theta x_2, \ldots, y_n = \theta x_n$$

form an orthogonal basis with respect to Q. When $\alpha + \beta\theta$, $\beta \neq 0$, is an element of F of norm 1, i. e.,

$$N(\alpha + \beta\theta) = (\alpha + \beta\theta)(\alpha - \beta\theta) = \alpha^2 - \rho\beta^2 = 1,$$

let U_i be the quasi-symmetry defined as follows,

$$x_i U_i = (\alpha + \beta\theta) x_i = \alpha x_i + \beta y_i; \quad x_j U_i = x_j \quad \text{for} \quad j \neq i;$$

and therefore

$$y_i U_i = (\theta x_i) U_i = \theta(\alpha + \beta\theta) x_i = \alpha\theta x_i + \beta\rho x_i =$$
$$= \alpha y_i + \beta\rho x_i; \quad y_j U_i = \theta x_j U_i = y_j.$$

Lemma 1. The automorphism of C(Q) associated to the unitarian transformation U is the inner automorphism defined by the element

$$u_i = \frac{1+\alpha}{\beta} + Q(x_i)^{-1} x_i y_i.$$

Proof. Since C(Q) is generated by its elements of degree 1, it is sufficient to prove that on these elements the automorphism associated to U_i coincides with the inner automorphism defined by u_i.

The inverse of u_i is

$$u_i^{-1} = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \left(\frac{1+\alpha}{\beta} - Q(x_i)^{-1} x_i y_i\right) \quad \text{for} \quad u_i^{-1} u_i =$$

$$= \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \left(\frac{(1+\alpha)^2}{\beta^2} - \rho\right) = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \cdot$$

$$\cdot \frac{1 + 2\alpha + \alpha^2 - \rho\beta^2}{\beta^2} = 1 \quad \text{since} \quad \alpha^2 - \rho\beta^2 = 1.$$

Since u_i commutes with x_j, y_j for $j \neq i$

$$u_i^{-1} x_j u_i = x_j = x_j U_i; \qquad u_i^{-1} y_j u_i = y_j = z_j U_i.$$

As to x_i and y_i,

$$u_i^{-1} x_i u_i = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \left(\frac{1+\alpha}{\beta} - Q(x_i)^{-1} x_i y_i\right) \cdot$$

$$\cdot x_i \left(\frac{1+\alpha}{\beta} + Q(x_i)^{-1} x_i y_i\right) = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \cdot$$

$$\cdot \left(\frac{1+\alpha}{\beta} - Q(x_i)^{-1} x_i y_i\right)^2 x_i = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \cdot$$

$$\cdot \left(\frac{2\alpha(1+\alpha)}{\beta^2} x_i + \frac{2(1+\alpha)}{\beta} y_i\right) = \alpha x_i + \beta y_i = x_i U_i;$$

$$u_i^{-1} y_i u_i = \left(\frac{2(1+\alpha)}{\beta^2}\right)^{-1} \left(\frac{2\alpha(1+\alpha)}{\beta^2} y_i + \frac{2(1+\alpha)}{\beta} \rho x_i\right) =$$

$$= \alpha y_i + \beta \rho x_i = y_i U_i,$$

which proves the lemma.

The unitarian homotecy defined by $\alpha + \beta\theta$, if $N(\alpha + \beta\theta) = 1$, is equal to the transformation $U = U_1 U_2 \dots U_n$ and the automorphism of $C(Q)$ associated to U coincides with the inner automorphism defined by $u = u_1 u_2 \dots u_n$.

First of all we are going to study the unitarian homotecies defined by elements of norm 1. We take any element of the form

$\mu + \theta$ and divide its square $\mu^2 + \rho + 2\mu\theta$ by its norm $\mu^2 - \rho$: so we get the element of norm 1,

$$\frac{\mu^2 + \rho}{\mu^2 - \rho} + \frac{2\mu}{\mu^2 - \rho}\theta = \alpha + \beta\theta.$$

Then

$$\frac{1+\alpha}{\beta} = \frac{2\mu^2}{2\mu} = \mu$$

and the automorphism of $C(Q)$ associated to the unitarian homotecy U coincides with the inner automorphism defined by

$$u = \prod_{i=1}^{n} (\mu + Q(x_i)^{-1} x_i y_i) = \mu^n + \mu^{n-1} r_1 + \dots + \mu^{n-i} r_i + \dots + r_n$$

where

$$r_h = \sum_{i_1 < i_2 < \dots < i_h} Q(x_{i_1})^{-1} Q(x_{i_2})^{-1} \dots Q(x_{i_h})^{-1} x_{i_1} y_{i_1} \dots x_{i_h} y_{i_h},$$

and the sum extends over all combinations of h indices.

Lemma 2. When K has at least n elements, the necessary condition for an element $c \in C^+(Q)$ to belong to $D(f)$ is that it commutes with r_i, $i = 1, 2, \dots, n$.

Proof. By definition $D(f)$ is elementwise invariant under the automorphisms of $C^+(Q)$ associated to the homotecies of f, and therefore, in particular, $D(f)$ is elementwise invariant under the automorphisms associated to the homotecies of norm 1. This means that the elements of $D(f)$ must commute with u for any value of $\mu \in K$. Since μ^n and r_n belong to the center of $C^+(Q)$, the elements of $D(f)$ commute with

$$\mu^{n-1} r_1 + \dots \mu r_{n-1} \quad \text{for evary} \quad \mu \in K. \tag{1}$$

When K has at least n elements if we give to μ $n-1$ different values and different from zero, the expression (1) will give us

$n-1$ elements belonging to the centralizer of $D(f)$ in $C^+(Q)$. These elements belong to the vector space over K generated by $r_1, r_2, \dots, r_{n-1}$ and are linearly independent since the determinant of the matrix formed by the coefficients is a determinant of Vandeermonde different from zero. Therefore the r_i are linear combinations of these elements and commute with the elements of $D(f)$.

Given a homotecy $T_{\alpha+\beta\theta}$, where $\alpha+\beta\theta$ has any norm and $\beta \neq 0$, since the automorphism of $C^+(Q)$ associated to this homotecy is the same that the one associated to

$$T_{\alpha+\beta\theta} T_{\beta^{-1}} = T_{\alpha\beta^{-1}+\theta}$$

we can suppose that $\beta = 1$. Let

$$N(\alpha+\theta) = \alpha^2 - \rho = \delta, \quad \text{and} \quad P = K(\sqrt{\delta}).$$

We consider M as a vector space over K and make the extension $M_P = P \otimes_K M$, so that M_P is a vector space over P. If we call Q_P the extension of Q to M_P, it is well known that

$$C(Q) \otimes_K P \cong C(Q_P)$$

(cf. [5] II.1.5); when $\sqrt{\delta} \in K$, $P = K$ and $C(Q) = C(Q_P)$. Every similitude S of Q can be extended in only one way to a similitude S of Q_P.

By lemma 1 we know that the automorphism of $C^+(Q_P)$ associated to the orthogonal transformation

$$x_i U_i = \frac{\alpha}{\sqrt{\delta}} x_i + \frac{1}{\sqrt{\delta}} y_i; \quad y_i U_i = \frac{\alpha}{\sqrt{\delta}} y_i + \frac{\rho}{\sqrt{\delta}} x_i;$$

$$x_j U_i = x_j; \quad y_j U_i = y_j$$

coincides with the inner automorphism defined by

$$u_i = \frac{1+\alpha/\sqrt{\delta}}{1/\sqrt{\delta}} + Q(x_i)^{-1} x_i y_i = \sqrt{\delta} + \alpha + Q(x_i)^{-1} x_i y_i.$$

Therefore the automorphism associated to $U = U_1 U_2 \dots U_n$ is the inner automorphism defined by

$$u = u_1 u_2 \dots u_n = (\sqrt{\delta}+\alpha)^n + (\sqrt{\delta}+\alpha)^{n-1} r_1 + \dots$$
$$+ (\sqrt{\delta}+\alpha)^{n-i} r_i + \dots + r_n.$$

On the other hand the automorphism of $C^+(Q_P)$ associated to U is the same that the automorphism associated to $U' = U\, T_{\sqrt{\delta}}$, that is,

$$x_i U' = \alpha x_i + y_i; \quad y_i U' = \alpha y_i + \rho x_i; \quad i = 1, 2, \dots n.$$

This means that the element $u \in C^+(Q_P)$ defines an inner automorphism of $C^+(Q_P)$ which induces in $C^+(Q)$ the automorphism associated to the homotecy $T_{\alpha+\theta}$. Therefore the elements of $C^+(Q)$ which commute with the r_i are left invariant by the automorphisms associated to the homotecies $T_{\alpha+\theta}$, hence they belong to $D(f)$. We have proved then.

Lemma 3. The condition of lemma 2 is also sufficient.

When $\sqrt{\delta} \notin K$ it is easy to find the element of $C^+(Q)$ which defines the same inner automorphism that the one defined by u. For, since r_n is in the center of $C^+(Q)$, u defines the same inner automorphism that

$$v = u\left(1 + \left(\frac{\alpha-\sqrt{\delta}}{\rho}\right)^n r_n\right).$$

Taking in account that

$$r_h r_n = \left(\sum Q(x_{i_1})^{-1} \dots Q(x_{i_h})^{-1} x_{i_1} y_{i_1} \dots x_{i_h} y_{i_h}\right)\left(Q(x_1)^{-1} \dots\right.$$

$$\left.\dots Q(x_n)^{-1} x_1 y_1 \dots x_n y_n\right) = \rho^h r_{n-h}, \quad \text{and} \quad (\alpha+\sqrt{\delta})\frac{\alpha-\sqrt{\delta}}{\rho} = 1,$$

we have

$$v = u\left(1 + \left(\frac{\alpha-\sqrt{\delta}}{\rho}\right)^n r_n\right) = (\alpha+\sqrt{\delta})^n + (\alpha-\sqrt{\delta})^n + \dots$$

$$+ \left[(\alpha+\sqrt{\delta})^{n-i} + (\alpha-\sqrt{\delta})^{n-i}\right] r_i + \dots + 2 r_n,$$

therefore $v \in C^+(Q)$.

Since every element of $C(Q)$ which commutes with r_n belongs to $C^+(Q)$ we can define $D(f)$ as the centralizer of the elements $r_1, r_2, \dots, r_n$ in $C(Q)$.

Let us suppose now that K has characteristic $p > n$ or zero. If we take $s = r_1^h, h \leqslant n$, s is a linear combination of the $r_i, i = 1, \dots, h$. Moreover the coefficient of r_h in s is different from zero, because $h! \neq 0$ in a field of characteristic zero or $p > n \geqq h$. Therefore if an element commutes with r_1 it commutes also with r_i, since the r_i, $i = 1, 2, \dots, n$, are linear combination of powers of r_1. When the characteristic of K is zero or $p > n$, K has more than elements, hence we can stablish

LEMMA 4. If the characteristic of K is zero or greater than $(M : F)$, the algebra $D(f)$ is the centralizer of r_1 in $C(Q)$.

§ 2

Now our problem is to find a suitable representation of $C(Q)$ so that we can determine the centralizer of r_1 in $C(Q)$, i. e., the algebra $D(f)$. We will make use of tensor products whose properties can be studied in [3].

As before we suppose that Q is the quadratic form associated to the non degenerate hermitian for f defined on the vector space M over the field $F = K(\theta)$ and that $x_1, x_2, \dots, x_n$ is an orthogonal basis of M with respect to f. Then we know that

$$x_i, \quad y_i = \theta x_i, \quad i = 1, 2, \dots, n \tag{2}$$

is an orthogonal basis with respect to Q.

We consider M as a vector space over K, make the extension $M_F = F \otimes_K M$ and identify $1 \otimes x$ with x. Then the elements (2) form an orthogonal basis of M_F with respect to Q_F.

The algebra $C(Q)$ can be expressed as a tensor product of quaternions over K. We define each one of these quaternions by a basis of the type $1, i, j, k$. We have then

$$C(Q) \cong (1, x_1, y_1, x_1 y_1)_1 \otimes_K \dots \otimes_K (1, u_i, v_i, \rho^{i-1} x_i y_i)_i \cdot \\ \cdot \otimes_K \dots \otimes_K (1, u_n, v_n, \rho^{n-1} x_n y_n)_n$$

where

$$u_i = Q(x_1)^{-1} \dots Q(x_{i-1})^{-1} x_1 y_1 x_2 y_2 \dots x_{i-1} y_{i-1} x_i = \\ = \left(\prod_{h=1}^{i-1} Q(x_h)^{-1} x_h y_h\right) x_i, \quad v_i = \left(\prod_{h=1}^{i-1} Q(x_h)^{-1} x_h y_h\right) y_i.$$

Therefore

$$C(Q_F) \cong C(Q) \otimes_K F$$

is also a tensor product of quaternions, but now the quaternions are taken over F, that is,

$$C(Q_F) \cong \left((1, x_1, y_1, x_1 y_1)_F\right)_1 \otimes_F \dots \otimes_F \left((1, u_n, v_n, \rho^{n-1} x_n y_n)_F\right)_n.$$

Since

$$(\rho^{i-1} x_i y_i)^2 = \rho^{2i-1} Q(x_i)^2$$

is a square in F, there exists an isomorphism of each one of these quaternions onto the algebra F_2, the total algebra of 2×2 matrices with entries in F.

Taking a suitable isomorphism, the element

$$Q(x_i)^{-1} \theta^{-1} x_i y_i \in C(Q_F)$$

whose square is equal to 1 is mapped into the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

If we denote by $e_1^1, e_1^2, e_2^1, e_2^2$ the matric units

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we can write

$$\theta^{-1} x_i y_i \cong (1)_1 \otimes_F \dots \otimes_F (e_1^1 - e_2^2)_i \otimes_F \dots \otimes_F (1)_n.$$

In the algebra

$$C(Q_F) \cong (F_2)_1 \otimes_F \ldots \otimes_F (F_2)_n \cong F_{2^n}$$

we will denote by

$$u^{h_1, h_2, \ldots, h_n}_{m_1, m_2, \ldots, m_n}$$

the elements of F_{2^n} defined by

$$\left(e^{h_1}_{m_1}\right)_1 \otimes_F \left(e^{h_2}_{m_2}\right)_2 \otimes_F \cdots \otimes_F \left(e^{h_n}_{m_n}\right)_n; \qquad h_j, m_j = 1, 2; \quad j = 1, 2, \ldots, n.$$

It is readily seen that the 2^{2^n} elements $u^{h_1 \cdots h_n}_{m_1 \cdots m_n}$ form a set of matric units for F_{2^n}. Let us order the 2^n sets

$$P_i = (m_1, m_2, \ldots, m_n),$$

$m = 1, 2$, in such a way that the set with s_1 elements equal 2 preceeds the set with s_2 elements 2 if $s_1 < s_2$, and among the sets with the same number of 2 we take any order. We make

$$u^{h_1 \ldots h_n}_{m_1 \ldots m_n} = u^s_r$$

if the sets $P_r = (m_1, m_2, \ldots, m_n)$ and $P_s = (h_1, h_2, \ldots, h_n)$ are in the r-th and s-th places, respectively, in the given order.

With the chosen bases for

$$(F_2)_1 \otimes \ldots \otimes (F_2)_n$$

and F_{2^n} the element $\theta^{-1} Q(x_i)^{-1} x_i y_i$ has the form

$$(e^1_1 + e^2_2)_1 \otimes \ldots \otimes (e^1_1 - e^2_2)_i \otimes \ldots \otimes (e^1_1 + e^2_2)_n \cong \sum_r \varepsilon_{ir} u^r_r$$

where ε_{ir} is 1 if the set P_r has a 1 in the i-th place and ε_{ir} is -1 if it has a 2.

Then the element

$$\sum_{i=1}^{r} \theta^{-1} Q(x_i)^{-1} x_i y_i \cong \sum_{r=1}^{2^n} \left(\sum_{i=1}^{n} \varepsilon_{ir}\right) u^r_r.$$

The coefficient $\sum_{i=1}^{n} \varepsilon_{is}$ of u_s^s is a sum of elements 1 and -1 and the number of -1 is the number b of elements 2 in the set P_s. Therefore the coefficient of u^s_s is $n - 2b$ and

$$\sum \theta^{-1} Q(x_i)^{-1} x_i y_i$$

is represented by the diagonal matrix

$$B = \operatorname{diag}(n, n-2, n-2, \ldots, n-2j, \ldots, -n)$$

where there are $\binom{n}{j}$ elements equal to $n - 2j$, $j = 0, 1, \ldots, n$.

Then the element

$$r_1 = \sum Q(x_i)^{-1} x_i y_i$$

is represented by the matrix

$$B' = \operatorname{diag}\big(n\theta, (n-2)\theta, \ldots, -n\theta\big)$$

whose characteristic polynomial is

$$\prod_{i=0}^{\left[\frac{n}{2}\right]} \big(x - (n-2i)\theta\big)^{\binom{n}{i}} \big(x + (n-2i)\theta\big)^{\binom{n}{i}} = \prod_{i=0}^{\left[\frac{n}{2}\right]} \big(x^2 - (n-2i)^2 \rho\big)^{\binom{n}{i}}$$

if n is odd, where $\left[\dfrac{n}{2}\right]$ denotes the greatest integer in $\dfrac{n}{2}$, and

$$\left[\prod_{i=0}^{\frac{n}{2}-1} \big(x - (n-2i)\theta\big)^{\binom{n}{i}} \big(x + (n-2i)\theta\big)^{\binom{n}{i}}\right] x^{\binom{n}{n/2}} =$$

$$= x^{\binom{n}{n/2}} \prod_{i=0}^{\frac{n}{2}-1} \big(x^2 - (n-2i)^2 \rho\big)^{\binom{n}{i}}$$

if n is even.

The matrix

$$B'' = \operatorname{diag}\left(\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{\left[\frac{n}{2}\right]}\right)$$

where

$$\alpha_i = \begin{pmatrix} 0 & 1 \\ (n-2i)^2 \rho & 0 \end{pmatrix}$$

and α_i appears $\binom{n}{i}$ times if $i \neq \frac{n}{2}$ and $\frac{1}{2}\binom{n}{n/2}$ times if $i = \frac{n}{2}$, is similar to B′ since it has the same elementary divisors. Moreover B″ is a matrix with entries in K.

Let us study now the simple algebra C (Q) whose center is K, since (M : K) has even dimension. We have then

$$C(Q) \cong K_r \otimes_K R,$$

where R is a division algebra of center K.

On the other hand we have seen that

$$C(Q) \otimes_K F \cong C(Q_F) \cong F_{2^n}$$

which shows that F is a splitting field for R. Therefore (F : K) = 2 is a multiple of $\sqrt{(R : K)}$ (see [2] cor. 8. 3. C). This shows that either (R : K) = 1, R = K or (R : K) = 4 and R is a quaternion division algebra over K.

It is immediate to see that both cases are possible. We are going to consider then separately.

Case 1: $R = K$, $C(Q) \cong K_{2^n}$.

We have seen that in $C(Q_F) \cong F_{2}$ the element

$$r_1 = \sum Q(x_i)^{-1} x_i y_i$$

can be represented by the matrix

$$B'' = \operatorname{diag}\left(\alpha_0, \ldots, \alpha_{\left[\frac{n}{2}\right]}\right)$$

Let B‴ be the image of r_1 in a representation of C (Q) onto K_{2^n}; B‴ will be also the image of r_1 in a representation of

$$C(Q_F) \cong F_{2^n} \cong K_{2^n} \otimes_K F.$$

Since there exists a representation of $C(Q_F)$ onto F_{2^n} which maps r_1 into $B'' \in K_{2^n}$, B″ is similar to B‴ in F_{2^n} and therefore it is also similar to B″ in K_{2^n}. Hence there exists an isomorphism of C (Q) onto K_{2^n} which maps r_1 into B″.

Now we have to find the centralizer of B″ in K_{2^n}. We consider K_{2^n} as the algebra of linear transformations of a vector space N over K of dimension 2^n. As before we suppose that the characteristic p of K is 0 or greater than n.

The transformation B″ is completely reducible. Its irreducible components belong to $1 + \left[\frac{n}{2}\right]$ classes of non equivalent irreducible transformations defined by the matrices $\alpha_0, \alpha_1, \ldots, \alpha_{\left[\frac{n}{2}\right]}$.

Let us consider N as a module over the ring A generated by the transformation B″ and the scalar multiplications and express the A-module N as a direct sum

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus N_i$$

of its $1 + \left[\frac{n}{2}\right]$ homogeneous components. These components N_i as A-modules are isomorphic to vector spaces over $F = K(\theta)$ of dimension $\binom{n}{i}$ for $i = 0, 1, \ldots, \left[\frac{n}{2}\right]$ if n is odd, and $i = 0, 1, \ldots, \frac{n}{2} - 1$, if n is even, for in this case the homogeneous component $N_{n/2}$, direct sum of the irreducible submodules corresponding to $\alpha_{n/2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, is isomorphic to a vector space over K of dimension $\binom{n}{n/2}$.

The centralizer D of A in the ring of endomorphisms of N considered as an additive group coincides with the centralizer in the algebra of linear transformations K_{2^n} since A contains the scalar multiplications by elements $\alpha \in K$. This means that $D \cong D(f)$. Moreover D as an algebra of linear transformations is completely reducible and has the same homogeneous components that A (see [14] theorem 6.1.1). Therefore

$$D \cong D(f) \cong \sum_{i=0}^{r} \oplus F_{\binom{n}{i}} \quad \text{if} \quad n = 2r+1, \quad \text{and}$$

$$D \cong D(f) \cong \sum_{i=0}^{r-1} \oplus F_{\binom{n}{i}} \oplus K_{\binom{n}{r}} \quad \text{if} \quad n = 2r.$$

In both cases the dimension of $D(f)$ over K is equal to

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}.$$

Case 2: $C(Q) \cong K_{2^{n-1}} \otimes_K R$, where R is a quaternion division algebra over K. By Wedderburn theorem for finite fields this case can occur only when K has an infinite number of elements.

Since F is a splitting field for R and $(F:K) = \sqrt{(R:K)}$, R contais a field isomorphic to F (cf. [2] th. 8.3.A (3) and th. 7.3.C (4). We denote by i_1 the element of R such that $i_1^2 = \rho$.

There exists an isomorphism of R considered as an algebra over K onto the subalgebra over K of F_2 with basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{i}_1 = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \theta \\ -\alpha\theta & 0 \end{pmatrix}$$

where $\alpha \in K$ is such that there exists an element $i_2 \in R$ which satisfies $i_2^2 = \alpha$, $i_1 i_2 = -i_2 i_1$.

Using this representation of R and taking

$$u^{2(j-1)+s}_{2(i-1)+r} \cong e^j_i \otimes e^s_r, \; i, j = 1, 2, \ldots, 2^{n-1}; \; r, s = 1, 2;$$

as matric units we get a representation of

$$C(Q) \cong K_{2^{n-1}} \otimes_K R$$

into a subalgebra S of F_{2^n}.

If we adjoint to this subalgebra the element $\theta I_{2^n} \in F_{2^n}$, where I_{2^n} is the unit matrix, we get the algebra

$$F_{2^n} \cong K_{2^{n-1}} \otimes_K R \otimes_K F \cong C(Q) \otimes_K F \cong C(Q_F).$$

It has been proved before that there exists a representation of $C(Q_F)$ onto F_{2^n} which maps r_1 into

$$B' = \operatorname{diag}(n\theta, \ldots, (n-2i)\theta, \ldots, -n\theta)$$

and therefore r_1 can also be represented by the matrix similar to B'

$$B'' = \operatorname{diag}\big(n\theta, -n\theta, \ldots, (n-2i)\theta, -(n-2i)\theta, \ldots\big) =$$
$$= \operatorname{diag}\big(n\bar{i}_1, (n-2)\bar{i}_1, \ldots, (n-2i)\bar{i}_1, \ldots\big)$$

where there are $\binom{n}{i}$ blocks of the form

$$(n-2i)\bar{i}_1 = \begin{pmatrix} (n-2i)\theta & 0 \\ 0 & -(n-2i)\theta \end{pmatrix} \quad \text{if} \quad i \neq \frac{n}{2}$$

and

$$\frac{1}{2}\binom{n}{n/2} \quad \text{if} \quad i = \frac{n}{2}.$$

Let E be the image of r_1 in a representation of $C(Q)$ onto $S \subset F_{2^n}$. Then E will be also the image of r_1 in a representation of

$$C(Q_F) \cong F_{2^n} \cong S \otimes_K F.$$

Therefore E and B'' are similar and there exists an invertible matrix M such that

$$B'' = M E M^{-1}; \quad M \in F_{2^n}. \tag{4}$$

Let us write M in the form $M = M_1 + \theta M_2$, where

$$M_1, M_2 \in S \cong K_{2^{n-1}} \otimes_K R.$$

Substituting this expression in (4) we get

$$B'' M_1 + \theta B'' M_2 = M_1 E + \theta M_2 E,$$

where

$$B'' M_1, M E \in S$$

and

$$\theta B'' M_2, \theta M_2 E \in \theta S.$$

Therefore

$$B'' M_1 = M_1 E \quad \text{and} \quad B'' M_2 = M_2 E$$

and in general

$$B'' (a_1 M_1 + a_2 M_2) = (a_1 M_1 + a_2 M_2) E,$$

where a_1 and a_2 are indeterminates.

The determinant of $a_1 M_1 + a_2 M_2$ is a homogeneous polynomial in a_1 and a_2 of degree 2^n and not identically zero since for $a_1 = 1$ and $a_2 = 0$ is different from zero. Since K has infinite elements, there exists value $\lambda_1, \lambda_2 \in K$ for a_1, a_2 such that

$$\det(\lambda_1 M_1 + \lambda_2 M_2) \neq 0$$

therefore

$$B'' = (\lambda_1 M_1 + \lambda_2 M_2) E (\lambda_1 M_1 + \lambda_2 M_2)^{-1}.$$

Hence r_1 is mapped into B″ in a suitable representation of C (Q) onto $S \subset F_{2^n}$.

In the representation of

$$C(Q) \cong K_{2^{n-1}} \otimes_K R$$

onto S that we have defined first, B″ is the image of $P \otimes_K \bar{i}_1$ where

$$P = \operatorname{diag}(n, n-2, \ldots, n-2i, \ldots) \in K_{2^{n-1}}$$

and the number of elements $n - 2i$ is $\binom{n}{i}$ for any i, i. e.,

$$i = 0, 1, \ldots, \left[\frac{n}{2}\right],$$

if n is odd and for any $i \neq \frac{n}{2}$ if n is even, since for $i = \frac{n}{2}$ the number of element equal to $n - 2i = 0$ is $\frac{1}{2}\binom{n}{n/2}$.

Therefore in this case the algebra D (f) which is the centralizer of $P \otimes_K \bar{i}_1$ in $S \cong K_{2^{n-1}} \otimes R$ has the following structure,

$$D(f) \cong \sum_{i=0}^{r} \oplus F_{\binom{n}{i}}, \quad \text{if} \quad n = 2r + 1. \quad \text{and}$$

$$D(f) \cong \sum_{i=0}^{r-1} \oplus F_{\binom{n}{i}} \oplus R_{\frac{1}{2}\binom{n}{r}}, \quad \text{if} \quad n = 2r.$$

As in case 1 the dimension of D (f) over K is

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}.$$

We sum up these results in:

THEOREM 1.—Let f be a non degenerate hermitian form on the vector space M of dimension n over the field F, J the involution associated to f and

$$K = \{ \alpha \mid \alpha^J = \alpha, \ \alpha \in F \} \neq F.$$

Then if the charateristic of F is zero or greater than n, the algebra D (f) has dimension $\binom{2n}{n}$ over K, and

$$\mathrm{D}(f) \cong \sum_{i=0}^{r} \oplus \mathrm{F}_{\binom{n}{i}}, \quad \text{if} \quad n = 2r + 1,$$

$$\mathrm{D}(f) \cong \sum_{i=0}^{r-1} \oplus \mathrm{F}_{\binom{n}{i}} \oplus \mathrm{T}. \quad \text{if} \quad n = 2r,$$

where T can be either $\mathrm{K}_{\binom{n}{r}}$ or $\mathrm{R}_{\frac{1}{2}\binom{n}{r}}$, R a quaternion division algebra over K.

§ 3

Now that we know the structure of D (f) when the characteristic of F is O or greater than (M : F) we see that in any of the possible cases the dimension over K of the center of D (f) is $n + 1$. Therefore the center is the vector space over K with basis $1, r_1, r_2, \ldots, r_n$ which coincides with the algebra over K generated by 1 and r_1.

The involutive antiautomorphism of C (Q) * leaves invariant the homogeneous elements of degree $4m$ or $4m + 1$ and takes the elements of degree $4m + 2$ and $4m + 3$ into their opposites. Since D (f) is a homogeneous subspace of C (Q) such antiautomorphism induces an antiautomorphism in D (f) which we are going to denote also by *.

Let us take an isomorphism σ of D (f) onto $\sum_{i=0}^{r-1} \oplus \mathrm{F}_{\binom{n}{i}}$ if $n = 2r - 1$, and onto

$$\sum_{i=0}^{r-1} \oplus \mathrm{F}_{\binom{n}{i}} \oplus \mathrm{T}$$

if $n = 2r$. If $c \in \mathrm{D}(f)$ we called c^σ the spin representation of c; the element r^σ_1 must be equal to

$$\sum_{i=0}^{r-1} \oplus (n - 2i)\, \theta\, 1_{\binom{n}{i}}$$

if $n = 2r - 1$ or $n = 2r$, or to a sum obtained from this one by substituting $-\theta$ for some θ. The antiautomorphism * of D (f) defines an antiautomorphism in the spin representation which we still denote by * and it is defined by $(c^\sigma)^* = (c^*)^\sigma$.

Let γ be the antiautomorphism of the spin representation of D (f) which takes any matrix belonging to $\mathrm{F}_{\binom{n}{i}}$ into its conjugate transpose with respect to the automorphism J of F and a matrix belonging to T into its transpose if $\mathrm{T} = \mathrm{K}_{\binom{n}{r}}$ and into its conjugate transpose with respect to any involutive antiautomorphism i' of R leaving invariant the elements of K if $\mathrm{T} = \mathrm{R}_{\frac{1}{2}\binom{n}{r}}$.

The product of the antiautomorphisms * and γ is an automorphism of the spin representation of D (f) which leaves invariant the elements 1 and $r_1{}^\sigma$, for, since r_1 has degree 2,

$$(r_1^\sigma)^{*\gamma} = \left((r_1^*)^\sigma\right)^\gamma = \left((-r_1)^\sigma\right)^\gamma = -(r_1^\sigma)^\gamma = r_1.$$

Hence the center of the spin representation of D (f) generated by 1 and r_1 is left invariant elementwise by this automorphism. Therefore, since D (f) is semi-simple, this automorphism is inner.

Let

$$\mathrm{P} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (\mathrm{P})_i,$$

where $(\mathrm{P})_i \in \mathrm{F}_{\binom{n}{i}}$ for $i = 0, 1, \ldots, \left[\frac{n}{2}\right] - 1$, $(\mathrm{P})_{\left[\frac{n}{2}\right]} \in \mathrm{F}_{\binom{n}{r}}$

if $n = 2r + 1$ and

$$(P)_{\left[\frac{n}{2}\right]} \in T \quad \text{if} \quad n = 2r,$$

be an element which defines the inner automorphism $*\gamma$ of the spin representation of $D(f)$. Then, for every

$$a^{\sigma} = A = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (A)_i,$$

$$(a^{\sigma})^{*\gamma} = A^{*\gamma} = P^{-1} A P = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (P)_i^{-1} (A)_i (P)_i.$$

If we denote by $Q = \Sigma \oplus (Q)_i$ the element $P^{\gamma} = \Sigma \oplus (P)_i^{\gamma}$,

$$A^* = Q A^{\gamma} Q^{-1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (Q)_i (A)_i^{\gamma} (Q)_i^{-1}.$$

It is a well known question to show that P can be chosen in such a way that the $(Q)_i$ are either hermitian or skew-hermitian matrices with respect to γ. For, since $*$ is an involutive antiautomorphism

$$A = (A^*)^* = (Q A^{\gamma} Q^{-1})^* = Q Q^{-\gamma} A Q^{\gamma} Q^{-1} =$$
$$= \sum_i \oplus (Q)_i (Q)_i^{-\gamma} (A)_i (Q)_i^{\gamma} (Q)_i^{-1} = \sum_i \oplus (A)_i,$$

that is, $(Q)_i^{\gamma} (Q)_i^{-1}$ is a central element of the simple algebra to which it belongs. This implies $(Q)_i^{\gamma} = \varepsilon (Q)_i$, where $\varepsilon \in F$ if $(Q)_i \in F_{\binom{n}{i}}$, and $\varepsilon \in K$ if $(Q)_i \in T$.

If the matrix $(Q)_i$ is not skew-hermitian with respect to γ, $\varepsilon \neq -1$ and therefore

$$(Q)_i + (Q)_i^{\gamma} = (1 + \varepsilon)(Q)_i$$

is a hermitian matrix with respect to γ and has an inverse. Moreover $(Q)_i$ and $(1 + \varepsilon)(Q)_i$ define the same inner automorphism.

If $(Q)_i$ is a skew-hermitian matrix, let us suppose

1) $(Q)_i \in F$. Then $\theta (Q)_i$ is a hermitian matrix and defines the same inner automorphism that $(Q)_i$.

2) $(Q)_r \in R_{\frac{1}{2}\binom{n}{r}}$. Then instead of defining the anti-automorphism induced by γ in $R_{\frac{1}{2}\binom{n}{r}}$ by the conjugate transpose with respect to the involutive antiautomorphism J' we define it as the antiautomorphism taking any element of $R_{\frac{1}{2}\binom{n}{r}}$ into its conjugate transpose with respect to the involutive antiautomorphism J'' of R defined as follows, $b^{J''} = a^{-1} b^{J'} a$ for any $b \in R$, where $a \in R$ is such that $a^{J'} = -a$. Hence we have now

$$a^{J''} = -a, \qquad b^{J'} = a\, b^{J''} a^{-1}$$

and

$$\left.\begin{aligned} (A)_r^* &= (Q)_r (A^{J'})_r' (Q)_r^{-1} = (Q)_r\, a\, (A^{J''})_r'\, a^{-1} (Q)_r^{-1} = \\ &= (Q)_r\, a\, (A)_r^{\gamma}\, a^{-1} (Q)_r^{-1}, \end{aligned}\right\} \tag{7}$$

where $(B)'$ stands for the transpose of (B). The matrix $(Q)_r a$ is hermitian with respect to the new γ, for

$$\left((Q)_r\, a\right)^{\gamma} = \left((Q)_r'\, a\right)^{J''} = -a\, a^{-1} (Q^{J'})_r'\, a = (Q)_r\, a$$

taking into account that $(Q^{J'})_r' = -(Q)_r$. But (7) shows that for this γ $(Q)_r a$ is the matrix which replaces $(Q)_r$.

3) $(Q)_r \in K_{\binom{n}{r}}$. Then there does not exist a symmetric matrix which can replace $(Q)_r$.

We have prove then.

Theorem 2. Let us assume that f and F fulfill the conditions of theorem 1. Then the antiautomorphism $*$ of the spin representation of $D(f)$ has the following form, if the matrix

$$A = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (A)_i$$

belongs to the spin representation,

$$A^* = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (Q)_i (A)_i^\gamma (Q)_i^{-1} = Q A^\gamma Q$$

where $(A)_i^\gamma$ is the conjugate transpose of $(A)_i$ with respect to the automorphism identity, J or an involutive antiautomorphism of R leaving K invariant elementwise if $(A)_i$ belongs to $K_{\binom{n}{r}}$, $F_{\binom{n}{i}}$ or $R_{\frac{1}{2}\binom{n}{r}}$, respectively. Moreover the matrices $(Q)_i \in F_{\binom{n}{i}}$ are hermitian with respect to γ, as well as $(Q)_r \in R_{\frac{1}{2}\binom{n}{r}}$ under a suitable choice of the antiautomorphism of R. On the contrary if $(Q)_r \in K_{\binom{n}{r}}$, $(Q)_r$ can be either symmétric or skew-symmetric.

In chapter II § 2 we will see when $(Q)_r \in K_{\binom{n}{r}}$ is symmetric and when skew-symmetric.

If all the $(Q)_i$ are hermitian with the possible exception of $(Q)_r \in K_{\binom{n}{r}}$ which might be skew symmetric we call

$$Q = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (Q)_i$$

matrix associated to the antiautomorphism * in the representation σ.

Let us take a different isomorphism σ' of D (f) onto

$$\sum_{i=0}^{r-1} \otimes F_{\binom{n}{i}}$$

if $n = 2r - 1$ and onto

$$\sum_{i=0}^{r-1} \oplus F_{\binom{n}{i}} \oplus T$$

if $n = 2r$. Then each component of the matrix $a^{\sigma'} = \Sigma \oplus (\bar{A})_i$ is similar to the corresponding component of $a^\sigma = \Sigma \oplus (A)_i$ or, if

$(\bar{A})_i \in F_{\binom{n}{i}}$ might be similar to the conjugate of $(A)_i$ with respect to J. That is,

$$(\bar{A})_r = (N)_r (A)_r (N)_r^{-1} \quad \text{if} \quad 2r \neq n \quad \text{and} \quad (N)_r \in R_{\frac{1}{2}\binom{n}{r}} \text{ or } K_{\binom{n}{r}}$$

and for any other i either

$$(\bar{A})_i = (N)_i (A)_i (N)_i^{-1}$$

or

$$(\bar{A})_i = (N)_i (A^J)_i (N)_i^{-1}$$

where $(N)_i \in F_{\binom{n}{i}}$.

We denote by $(A)_i^{\varepsilon_i}$ the matrix $(A)_i$ if $(\bar{A})_i$ is similar to $(A)_i$, $(A)_i^{\varepsilon_i} = (A^J)_i$ if $(\bar{A})_i$ is similar to $(A^J)_i$ and make $A^\varepsilon = \Sigma \oplus (A)_i^{\varepsilon_i}$, where $(A)_r^{\varepsilon_r} = (A)_r$ if $2r = n$. Then

$$a^{\sigma'} = \bar{A} = N A^\varepsilon N^{-1} = \sum \oplus (N)_i (A)_i^{\varepsilon_i} (N)_i^{-1}; \quad \bar{A}^\gamma = N^{-\gamma} A^{\varepsilon\gamma} N^\gamma \quad \text{and}$$

$$(a^*)^{\sigma'} = \bar{A}^* = N (A^*)^\varepsilon N^{-1} = N Q^\varepsilon A^{\gamma\varepsilon} Q^{-\varepsilon} N^{-1} = N Q^\varepsilon N^\gamma N^{-\gamma} \cdot$$

$$\cdot A^{\varepsilon\gamma} N^\gamma N^{-\gamma} Q^{-\varepsilon} N^{-1} = (N Q^\varepsilon N^\gamma)(N^{-\gamma} A^{\varepsilon\gamma} N^\gamma)(N Q^\varepsilon N^\gamma)^{-1} =$$

$$= (N Q^\varepsilon N^\gamma) \bar{A}^\gamma (N Q^\varepsilon N^\gamma)^{-1}.$$

If $(Q)_i$ is hermitian with respect to γ, $(Q)_i^{\varepsilon_i}$ is also hermitian. Therefore $N Q^\varepsilon N^\gamma$ is a matrix associated to the antiautomorphism * in the representation σ'. By choosing a suitable σ' the matrices $(N)_i (Q)_i^{\varepsilon_i} (N)_i^\gamma$ will be diagonal matrices if $2i \neq n$; for $2r = n$, ε_r is the identity and therefore $(N)_r (Q)_r^{\varepsilon_r} (N)_r^\gamma$ and $(Q)_r$ are cogredient relative to γ (see [13], p. 149).

Let us suppose that Q is a matrix associated to * in such a representation. Then we see that a matrix associated to * in any other spin representation is a direct sum of matrices cogredient to the components of Q relative to γ since $(Q)_i^{\varepsilon_i} = (Q)_i$ if $(Q)_i$ is a diagonal hermitian matrix.

CHAPTER II

In this chapter it will be proved that D (f) is the envelopping algebra over K of the elements of the Clifford group mapped by χ into the unitarian transformations of f. Then the simple components of the spin representation of the unitary group will be known. As before we assume that the characteristic of F is zero or greater than (M : F) and that f is non degenerate.

First of all we compute the dimension of the subspace of D (f) of degree $2h$, $h = 1, 2, \dots, n$. It will be always supposed that the elements $x_1, x_2, \dots, x_n$ form an orthogonal basis for f and that $y_i = \theta x_i$.

§ 1

The elements

$$x_1^{\varepsilon_1} y_1^{\delta_1} x_2^{\varepsilon_2} y_2^{\delta_2} \dots x_n^{\varepsilon_n} y_n^{\delta_n}, \quad \text{where} \quad \varepsilon_i, \delta_i = 0, 1 \qquad (1)$$

$$\text{and} \quad \Sigma\, \varepsilon_i + \Sigma\, \delta_i = 2h,$$

form a basis for the subspace of C (Q) of degree $2h$. Let us write in the order in which they appear in the expression (1) the subindices of the elements with exponent 1. We get then for each element of the basis a set of $2h$ numbers between 0 and $n + 1$ in no decreasing order and where each number appear at most twice. We will call the set of $2h$ numbers deduced from an element of the form (1) the index system of such element and will say that the system has degree $2h$.

Let us divide the set of elements (1) of degree $2h$ into subsets with the same index system. We consider the vector spaces over K generated by each of these subsets and get in this form a decomposition of the subspace of C (Q) of degree $2h$ in a direct sum of subspaces which will be called the subspaces of the index system or index subspaces. Of course this decomposition depends on the chosen orthogonal basis.

When $h = 1$ the elements

$$x_i y_i;\ x_i y_j;\ x_j y_i;\ x_i x_j;\ y_i y_j; \qquad i, j = 1, 2, \dots, n, \quad j > i$$

form a basis of the space of degree 2 of C (Q).

The subspace of the index system $i\,i$, $i = 1, 2, \dots, n$, i. e., the subspace generated by $x_i\, y_i$ belongs to D (f), for it is obvious that such element commutes with

$$r_1 = \sum_i Q\,(x_i)^{-1}\, x_i\, y_i$$

and then lemma 3 of chapter I asserts that it belongs to D (f). As to the elements of the index space $i\,j$, $i < j$, we are going to find their images under the automorphism of C (Q) of order 2. τ_0, associated to the homotecy T of ratio $-\rho$. We have

$$(x_i y_j)^{\tau_0} = -\rho^{-1}(x_i \mathrm{T})(y_j \mathrm{T}) = -y_i x_j;$$

$$(x_i y_j)^{\tau_0} = -\rho^{-1}(x_i \mathrm{T})(x_j \mathrm{T}) = -\rho^{-1} y_i y_j$$

and since τ_0 has order 2,

$$(-y_i x_j)^{\tau_0} = x_i y_j \quad \text{and} \quad (-\rho^{-1} y_i y_j)^{\tau_0} = x_i x_j.$$

Therefore the elements

$$u_{ij} = x_i y_j - y_i x_j; \quad v_{ij} = x_i x_j - \rho^{-1} y_i y_j$$

are left invariant by the automorphism τ_0 and the elements

$$r_{ij} = x_i y_j + y_i x_j; \quad s_{ij} = x_i x_j + \rho^{-1} y_i y_j$$

are taken by τ_0 into their opposites and hence they do not belong to D (f).

Let us see now that u_{ij}, v_{ij} are invariant under the automorphism $\tau_{\alpha+\beta\theta}$ of C (Q) associated to any homotecy $T_{\alpha+\beta\theta}$. We have

$$\begin{aligned}
u_{ij}^{\tau_{\alpha+\beta\theta}} &= (x_i y_j - y_i x_j)^{\tau_{\alpha+\beta\theta}} = (\alpha^2 - \beta^2 \rho)^{-1}\big((\alpha x_i + \beta y_i)(\alpha y_j + \rho \beta x_j) - \\
&- (\alpha y_i + \rho \beta x_i)(\alpha x_j + \beta y_j)\big) = (\alpha^2 - \rho \beta^2)^{-1}\big((\alpha^2 - \rho \beta^2) \cdot \\
&\cdot (x_i y_j - y_i x_j) + (\alpha \beta - \beta \alpha)(\rho x_i x_j + y_i y_j)\big) = \\
&= x_i y_j - y_i x_j = u_{ij}; \\
v_{ij}^{\tau_{\alpha+\beta\theta}} &= (x_i x_j - \rho^{-1} y_i y_j)^{\tau_{\alpha+\beta\theta}} = (\alpha^2 - \beta^2 \rho)^{-1}\big((\alpha x_i + \beta y_i) \cdot \\
&\cdot (\alpha x_j + \beta y_j) - \rho^{-1}(\alpha y_i + \rho \beta x_i)(\alpha y_j + \rho \beta x_j)\big) = \\
&= x_i x_j - \rho^{-1} y_i y_j = v_{ij}.
\end{aligned}$$

Therefore any element of C (Q) of degree 2 invariant under the automorphism τ_θ belongs to D (f).

The computation that we have carried out to check that u_{ij} and v_{ij} are invariant under the automorphism $\tau_{\alpha+\beta\theta}$ is independent of the value of the indices $i\,j$. It is immediate to see that this is also true for the elements of any index space with indices all different.

LEMMA 1. Let g be an element of degree $2h$ of C (Q). Then $g \in$ D (f) if and only if its projections on each index subspace belongs to D (f).

PROOF. Let

$$x_1^{\varepsilon_1} y_1^{\delta_1} x_2^{\varepsilon_2} y_2^{\delta_2} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}$$

be any element of degree $2h$. The automorphism of C^+ (Q) associated to the homotecy $T_{\alpha+\beta\theta}$ of f takes this element into another element of the same degree given by the expression

$$(\alpha^2 - \rho \beta^2)^{-h} (\alpha x_1 + \beta y_1)^{\varepsilon_1} (\alpha y_1 + \beta \rho x_1)^{\delta_1} \ldots (\alpha y_n + \beta \rho x_n)^{\delta_n}.$$

Taking into account that in the result only appear terms of degree $2h$ it is easily seen that we get a linear combination of elements of the set (1) all of them with the same index system that the taken element.

If $g \in$ D (f), g is left invariant by the automorphism associated to any homotecy. For what we have just seen it is clear that this is possible only if its projection on any index subspace is left invariant by such automorphism. This implies that these projections belong to D (f). On the other hand it is obvious that if each projection belongs to D (f), g also belongs to this algebra.

We have, then, that the decomposition of the space of C (Q) of degree $2h$ in a direct sum of index subspaces induces a decomposition of the space of degree $2h$ of D (f) in a direct sum of its index subspaces. In other words we could say that D (f) is a homogeneous subspace with respect to the decomposition of C (Q) in index subspaces. The space of degree O is the space of the vacuous index system.

The dimension of the space of degree $2h$ of D (f), $h = 1, 2, \ldots, n$, can be computed when we know the dimension of the index subspaces. First of all we remark that the space of degree $2n - 2h$ has the same dimension that the space of degree $2h$. For, if we multiply each element of degree $2h$ by r_n we have a $1 - 1$ linear transformation of the space of degree $2h$ onto the space of degree $2n - 2h$. Therefore we need to compute only the dimension of the spaces of degree $2h$ when $2h \leqslant n$.

We classify the index systems of degree $2h$ into $h + 1$ families

$$G_{2h}^0, G_{2h}^1, \ldots, G_{2h}^h;$$

G^i_{2h} being the set of index systems in which there are i and only i indices which appear twice.

LEMMA 2. All the index subspaces of D (f) which belong to the same family of index systems have the same dimension. Moreover the dimension of an index subspace whose index system belongs to the family G^r_{2h} equals the dimension of an index subspace whose index system belongs to $G^o_{2(h-r)}$.

PROOF. Let us consider first the family G^o_{2h}, that is, that family whose index systems consist of $2h$ different indices. This is only possible if $n \geqslant 2h$.

Let $i_1, \ldots, i_{2h}$; $i'_1, \ldots, i'_{2h}$ be two different index systems of

G^0_{2h} and let us take a basis for the index space corresponding to the first index system and suppose that each element of this basis is expressed as a linear combination of elements of the form (1) belonging to that index system. If in this expression we substitute i'_j for i_j, we get linearly independent elements of the index subspace corresponding to the second index system. Therefore the dimension d of the subspace defined by the first index system is less than, or equal to, the dimension d' of the index subspace defined by the second system. By symmetry $d' \leqslant d$ and hence $d' = d$.

Let us take now an index system of the family G^r_{2h} and suppose that $j_1, j_2, \ldots, j_r$ are the indices which appear twice. Then any element of the corresponding index subspace can be expressed as a product of

$$\prod_{s=1}^{r} x_{j_s} \, y_{j_s}$$

by an element of degree $2(h-r)$ of the index subpace defined by the system obtained from the index system we started with by leaving out

$$j_1, j_1, j_2, j_2, \ldots, j_r, j_r.$$

Therefore the dimension of an index subspace defined by a system of the family G^r_{2h} equals the dimension of a subspace defined by an index system of $G^0_{2(h-r)}$.

Lemma 3. The dimension of the subspace of an index system of G^0_{2h} is $\binom{2h}{h}$.

Proof. We are going to use induction on h, starting with $h = 1$, even though we take $\binom{0}{0} = 1$.

If we know the dimension of the subspace of an index system of G^0_{2r} for $r < h$ we can compute the dimension of the space of degree $2r$, for it will be equal to the sum of the dimensions of the subspaces of all the index systems of degree $2r$.

The dimension of any of these subspaces equals the dimension of a subspace of an index system of $G^0_{2(r-i)}$.

The number of different index systems of the family G^0_{2r} is $\binom{n}{2r}$, where $n = (M : F)$ is the number of different indices. In general the number of different index systems of the family G^s_{2r} is

$$\binom{n}{s}\binom{n-s}{2(r-s)}.$$

Let d_{2r} be the dimension of the subspace of an index system of G^0_{2r}. Then the dimension of the space of degree $2r$ is

$$e_{2r} = \sum_{i=0}^{r} \binom{n}{i}\binom{n-i}{2(r-i)} d_{2(r-i)}$$

If $r = 1$ it is true that the dimension d_2 is

$$\binom{2r}{r} = \binom{2}{1} = 2,$$

because the elements u_{ij}, v_{ij} form a basis for the space of indices ij. If we suppose that for $r < h$ $d_{2r} = \binom{2r}{r}$, the dimension of the space of degree $2r$ must be

$$\left.\begin{aligned} e_{2r} &= \sum_{i=0}^{r} \binom{n}{i}\binom{n-i}{2(r-i)}\binom{2(r-i)}{(r-i)} = \\ &= \sum_{i=0}^{r} \frac{n!}{i!(n-i)!} \cdot \frac{(n-i)!}{(2(r-i))!(n-2r+i)!} \cdot \\ &\cdot \frac{(2(r-i))!}{(r-i)!(r-i)} = \sum_{i=0}^{r} \frac{n!}{i!(n-2r+i)!((r-i)!)} = \\ &= \sum_{i=0}^{r} \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(n-2r+i)!(r-i)} \cdot \frac{r!}{(r-i)!\,i!} = \\ &= \binom{n}{r}\sum_{i}\binom{n-r}{r-i}\binom{r}{i} = \binom{n}{r}^2. \end{aligned}\right\} \quad (3)$$

Now let us prove that if $d_{2r} = \binom{2r}{r}$ for $r < h$ the formula is also true for $r = h$. If $(M : F) = 2h$ by theorem 1 of chapter I we know that the dimension of $D(f)$ is

$$\binom{4h}{2h} = \sum_{i=0}^{2h} \binom{2h}{i}^2 = 2\sum_{i=0}^{h-1} \binom{2h}{i}^2 + \binom{2h}{h}^2. \tag{4}$$

On the other hand, since in this case $e_{2i} = e_{2(2h-i)}$, the dimension of $D(f)$ taking into account (3) is

$$\sum_{i=0}^{2h} e_{2i} = 2\sum_{i=0}^{h-1} e_{2i} + e_{2h} = 2\sum_{i=0}^{h-1} \binom{2h}{i}^2 + e_{2h}. \tag{5}$$

Equating the expresions (4) and (5), which give the dimension of $D(f)$, we have $e_{2h} = \binom{2h}{h}^2$.

We know also that

$$\binom{2h}{h}^2 = e_{2h} = \sum_{i=1}^{h} \binom{2h}{i}\binom{2h-i}{2(h-i)}\binom{2(h-i)}{h-i} + d_{2h}. \tag{6}$$

If in (3) we make $n = 2h$ and $r = h$, comparing the first sum with (6) we get $d_{2h} = \binom{2h}{h}$ which proves the lemma.

Now that the lemma is proved, expression (3) proves the following

THEOREM 1. The dimension of the space of $D(f)$ of degree $2h$ is $\binom{n}{h}^2$ where $n = (M : F)$.

The dimension of the space of $D(f)$ of degree $2h$ has been computed taking into account that this space is the direct sum of the index subspaces of degree $2h$. If we sum the dimensions of all the index subspaces we get the dimension of $D(f)$. In this way we are going to get a formula which will be used later on.

In lemma 2 it has been seen that all the index subspaces corresponding to any index system where there are precisely $2i$ indices which appear only once have the same dimension. In lemma 3 we have proved that this dimension is $\binom{2i}{i}$. Let us denote by E_i the dimension of the subspace of $D(f)$ direct sum of all the index subspaces whose index systems contains precisely $2i$ indices appearing only once.

Since there are n different indices the index systems can be divided into sets of index systems where each set consists of all the systems with precisely $2i$ indices appearing only once,

$$i = 0, 1, \ldots, \left[\frac{n}{2}\right].$$

Hence

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} E_i = \binom{2n}{n}.$$

If we choose an index system of degree $2i$ with $2i$ different indices we can get index systems where these $2i$ indices are the only ones which appear only once by adding to the chosen system $0, 1, \ldots, n-2i$ pairs of indices picked up among the $n-2i$ indices different from the given ones. In general we can add r pairs of indices in $\binom{n-2i}{r}$ different ways; therefore from an index system of degree $2i$ with $2i$ different indices we get

$$\sum_{r=0}^{n-2i} \binom{n-2i}{r} = 2^{n-2i}$$

index systems in which the indices appearing only once are the the chosen $2i$ indices. Since these $2i$ indices can be chosen in

$\binom{n}{2i}$ different ways we will get $\binom{n}{2i} 2^{n-2i}$ different index systems in which there are $2i$ indices which appear only once. Hence

$$E_i = \binom{n}{2i} 2^{n-2i} \binom{2i}{i}.$$

Summing up the E_i we get

LEMMA 4.

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n}{2i} \binom{2i}{i} 2^{n-2i} = \binom{2n}{n}.$$

§ 2

It will be proved now that the elements of **D** (f) of degree 2 form a set of generators. In order to do so we will start defining by induction canonical bases for the different index subspaces. To simplify the notation when we refer to a subspace of the index system of degree $2r$ with $2r$ different indices we assume that these indices are $1, 2, \ldots, 2r$. In doing so there is not loss of generality since we get the subspace of indices $j_1, j_2, \ldots, j_{2r}$ by substituting j_i for i and the computations that we carry out do not depend upon the particular value of the indices.

Chosen a basis of $C^+(Q)$ of the form (1), the index subspace of indices $1, 2, \ldots, 2m$ has as basis the set of 2^{2m} elements obtained from

$$z_1 z_2, \ldots, z_{2m} \tag{7}$$

writing x_j or y_j instead of z_j, $j = 1, 2, \ldots, 2m$.

Since the number of factors is even we can write the product in the form

$$(z_1 z_2) \ldots (z_{2i-1} z_{2i}) \ldots (z_{2m-1} z_{2m}) \tag{7'}$$

where $z_{2i-1} z_{2i}$ can take one of the 4 different forms,

$$x_{2i-1} x_{2i}; \quad x_{2i-1} y_{2i}; \quad y_{2i-1} x_{2i}; \quad y_{2i-1} y_{2i}.$$

We adopt the following notation,

$$u_i = u_{2i-1,2i} = x_{2i-1} y_{2i} - y_{2i-1} x_{2i};$$

$$v_i = v_{2i-1,2i} = x_{2i-1} x_{2i} - \rho^{-1} y_{2i-1} y_{2i};$$

$$r_i = r_{2i-1,2i} = x_{2i-1} y_{2i} + y_{2i-1} x_{2i};$$

$$s_i = s_{2i-1,2i} = x_{2i-1} x_{2i} + \rho^{-1} y_{2i-1} y_{2i},$$

and we get

$$x_{2i-1} x_{2i} = \frac{1}{2}(v_i + s_i); \quad x_{2i-1} y_{2i} = \frac{1}{2}(u_i + r_i);$$

$$y_{2i-1} y_{2i} = \frac{1}{2}(r_i - u_i); \quad y_{2i-1} y_{2i} = \frac{\rho}{2}(s_i - v_i).$$

Substituting these values in (7') we see that any element of (7) is a linear combination of elements

$$t_1 t_2, \ldots, t_m, \tag{8}$$

where t_i can be any of the four terms u_i, v_i, r_i or s_i. Conversely any element obtained form (8) writing instead of t_i any one of the terms u_i, v_i, r_i or s_i, $i = 1, 2, \ldots, m$, is a linear combination of elements of the form (7). Therefore the elements obtained from (8) by different substitutions of t_i generate the subspace of indices $1, 2, \ldots, 2m$.

The number of such elements is $4^m = 2^{2m}$. This number being equal to the dimension of the subspace of $C^+(Q)$ of indices $1, 2, \ldots, 2m$, these elements must be linearly independent.

Let us denote by

$$w_{i_1 i_2} w_{i_3 i_4}, \ldots, w_{i_{2m-1} i_{2m}} \tag{9}$$

the set of 2^m different elements obtained by substituting $u_{i_{2j-1}i_{2j}}$ or $v_{i_{2j-1}i_{2j}}$ for $w_{i_{2j-1}i_{2j}}$. As to the indices we assume that $i_1, i_2, \ldots, i_{2m}$ is a reordenation of $1, 2, \ldots, 2m$ such that

$$i_{2j-1} < i_{2j}, \quad j = 1, \ldots, m.$$

Since for different values of j the $w_{i_{2j-1}i_{2j}}$'s commute with each other we will not take into account the order in which they are written.

We will say that an element of $D(f)$ belonging to the subspace of indices $1, 2, \ldots, 2m$ is canonical if it can be written as a linear combination of elements of the form (9). It will be said that such a linear combination is a canonical expression of the element. It follows from its definition that a canonical element of $D(f)$ belongs to the algebra generated by u_{ij}, v_{ij},

$$i, j = 1, 2, \ldots, n, \quad i < h.$$

Given a canonical element $c \in D(f)$ of the subspace of indices $1, 2, \ldots, 2m$ writing instead of

$$w_{i_{2j-1}i_{2j}} = \begin{cases} u_{i_{2j-1}i_{2j}} = x_{i_{2j-1}} y_{i_{2j}} - y_{i_{2j-1}} x_{i_{2j}} \\ v_{i_{2j-1}i_{2j}} = x_{i_{2j-1}} x_{i_{2j}} - \rho^{-1} y_{i_{2j-1}} y_{i_{2j}} \end{cases}$$

its value in terms of the x_i's and y_i's and taking into account only that $C^+(Q)$ is an associative linear algebra we get an expression of c as linear combination of elements

$$z_{i_1} z_{i_2}, \ldots, z_{i_{2m}} \tag{10}$$

which differs from the expression in terms of the elements

$$z_1 z_2, \ldots, z_{2m}$$

only in the order of the factors what can give place to a change in the sign.

In the canonical expression of c let us write instead of $w_{i_{2j-1}i_{2j}}$

$$W_{i_{2j-1}i_{2j}} = \begin{cases} U_{i_{2j-1}i_{2j}} = s_{i_{2j-1}} r_{i_{2j}} - r_{i_{2j-1}} s_{i_{2j}} \\ V_{i_{2j-1}i_{2j}} = s_{i_{2j-1}} s_{i_{2j}} - \rho^{-1} r_{i_{2j-1}} r_{i_{2j}} \end{cases}$$

and express this sum of products of W_{hf}'s as a sum of products of elements s_{hf}, r_{hf} using only the fact that we are operating in a linear associative algebra. Then the element that we obtain is the element derived from the expression of c as linear combination of elements (10) substituting s_i for x_i and r_i by y_i.

But since

$$\begin{aligned} U_{ji} = s_j r_i - r_j s_i &= (x_{2j-1} x_{2j} + \rho^{-1} y_{2j-1} y_{2j})(x_{2i-1} y_{2i} + y_{2i-1} x_{2i}) - \\ &- (x_{2j-1} y_{2j} + y_{2j-1} x_{2j})(x_{2i-1} x_{2i} + \rho^{-1} y_{2i-1} y_{2i}) = \\ &= v_{2j-1, 2i-1} u_{2j, 2i} + u_{2j-1, 2i-1} v_{2j, 2i} \in D(f) \end{aligned}$$

and

$$\begin{aligned} V_{ji} &= s_j s_i - \rho^{-1} r_j r_i = \\ &= -(v_{2j-1, 2i-1} v_{2j, 2i} + \rho^{-1} u_{2j-1, 2i-1} u_{2j, 2i}) \in D(f) \end{aligned}$$

the element of $C^+(Q)$ obtained by putting W_{ji} instead of w_{ji} belongs to the subspace of $D(f)$ of indices $1, 2, \ldots, 4m$. Moreover, since r_i (or s_i) commutes with any r_j and s_j if $i \neq j$, we get the same element if we substitute s_i and r_i for x_i and y_i, respectively, in the expression of c as linear combination of elements (10) or as combination of elements (7).

Lemma 5. From linearly independent canonical elements of degree $2m$ belonging to a subspace of $2m$ different indices we can derive canonical elements of a subspace of $4m$ different indices, which are linearly independent.

Proof. As before we suppose that the canonical elements of degree $2m$ belong to the subspace of indices $1, 2, \ldots, 2m$. We have just seen that if in the expression of these elements as linear combinations of elements (7) we substitute s_i for x_i and r_i for y_i we get canonical elements of the subspace of indices $1, 2, \ldots, 4m$. Therefore we only need to prove that if the canonical elements of degree $2m$ are linearly independent the elements of degree $4m$ obtained from these elements are also linearly independent. The linear independence of the elements of degree $4m$ so obtained follows from the fact that for any m the 4^m different elements obtained from (8) by substituting u_i, v_i, r_i or s_i for t_i are linearly independent. For, if the linear combinations of elements (7) which express the given canonical elements of degree $2m$ are linearly independent substituting in these elements s_i for x_i and r_i for y_i we get linearly independent elements.

LEMMA 6. Every subspace of an index system of degree $2r$ with $2r$ different indices has a basis consisting of canonical elements. Therefore such index subspaces belong to the algebra generated by u_{ij}, v_{ij}, $i<j$.

PROOF. Let $1, 2, \ldots, 2r$ be the index system. Since for $r=1$, u_{12}, v_{12} form a basis for the subspace of indices 1, 2, we are going to use induction on r. Therefore we assume that the lemma is true for $2r < 2h$. Then it will be seen that it is true for $2r = 2h$ or to be precise, we will see that the lemma is true for $2r = 2h$, if it is true for any r such that $2r \leqslant h$.

We take the 4^m elements of the form (8) as a basis for the subspace of $C^+(Q)$ of indices $1, 2, \ldots, 2h$ for $m = h$. Among these the 2^h elements containing only u_i's and v_i's belong to $D(f)$ since $u_i, v_i \in D(f)$. Moreover these elements are linearly independent.

Let us choose $2j$ indices, $2j \leqslant h$, $i_1, i_2, \ldots, i_{2j}$ among $1, 2, \ldots, h$. In the $\binom{2j}{j}$ canonical elements that by the induction assumption form a basis for the subspace of indices $1, 2, \ldots, 2j$ we write s_{i_m} instead of x_m and r_{i_p} instead of y_p, $m, p = 1, 2, \ldots, 2j$. Lemma 5 asserts that in this form we get $\binom{2j}{j}$ linearly independent canonical elements of degree $4j$. Now let $i'_1, i'_2, \ldots, i'_{h-2j}$ be the complementary set of $i_1, \ldots, i_{2j}$ with respect to $1, 2, \ldots, h$. If we multiply each one of the 2^{h-2j} different elements obtained from

$$t_{i'_1} t_{i'_2} \ldots t_{i'_{h-2j}} \tag{11}$$

substituting $u_{i'_m}$ or $v_{i'_m}$ for $t_{i'_m}$, $m = 1, 2, \ldots, h-2j$, by each one of the elements obtained before we get $\binom{2j}{j} 2^{h-2j}$ canonical elements of the subspace of indices $1, 2, \ldots, 2h$. We say that such elements belong to the index family $i_1, i_2, \ldots, i_{2j}$. These elements are linearly independent, for if there exists a linear combination which equals zero the partial sums extended over the elements with the same factor of the form (11) must be zero, since the elements containing $u_{i'_g}$ can not be cancelled with the elements containing $v_{i'_g}$. In each one of these partial sums the factor of the form (11) is multiplied by a linear combination of the $\binom{2j}{j}$ canonical elements mentioned above and it has been seen that such elements are linearly independent. Therefore all the coefficients of the linear combination which equals zero must be zero. In other words, the $\binom{2j}{j}$ 2^{h-2j} canonical elements of the index family $i_1, \ldots, i_{2j}$ are linearly independent.

The $2j$ indices can be chosen in $\binom{h}{2j}$ different ways, hence for each value of j we get $\binom{h}{2j}\binom{2j}{j} 2^{h-2j}$ elements. If we take all possible values of j we have

$$\sum_{j=0}^{\left[\frac{h}{2}\right]} \binom{h}{2j}\binom{2j}{j} 2^{h-2j} = \binom{2h}{h}$$

elements according to lemma 4.

Moreover these $\binom{2h}{h}$ canonical elements are linearly independent because if a linear combination of such elements is zero each one of the partial sums extended over all the elements of an index family should be zero. Since we have just proved that the elements of an index family are linearly independent, the $\binom{2h}{h}$ canonical elements are linearly independent and form a basis of the subspace of indices $1, 2, \ldots, 2h$ which has dimension $\binom{2h}{h}$.

THEOREM 2. The algebra $D(f)$ is generated by its elements of degree 2.

PROOF. By lemma 6 we know that the subspace of any index system of degree $2r$ with $2r$ different indices belongs to the algebra generated by the elements u_{ij}, v_{ij}, $i<j$, of degree 2.

If d is an element of a subspace of an index system in which there are $2r'$ indices appearing only once and r'' indices appearing twice, d is the product of an element of the subspace whose index system consists of the $2r'$ indices appearing only once by the element

$$d' = \prod_{s=1}^{r''} x_{i_s} y_{i_s},$$

where $i_1, \ldots, i_{r''}$ are the indices appearing twice. Since d' belongs to the algebra G generated by the elements of $D(f)$ of degree 2, d belongs to G.

Since $(x_i y_i)^2 \neq 0$ belongs to the subspace of the vacuous index system, the elements of degree zero also belong to G. Therefore G contains all the index subspaces and coincides with the algebra $D(f)$ which is direct sum of such subspaces.

THEOREM 3. The algebra $D(f)$ is generated by the elements of the Clifford group of $C(Q)$ mapped by χ into the symmetries of the hermitian form f. Moreover $D(f)$ contains also the elements of the Clifford group mapped by χ into the unitary transformations as well as the elements of $C(Q)$ which define inner automorphisms which induce in $C^+(Q)$ automorphisms associated to the unitarian similitudes.

PROOF. To prove the first part, by theorem 2, it suffices to show that the algebra over K generated by the elements of the Clifford group mapped by χ into the symmeries of f contains the space of degree 2 of $D(f)$ and that, conversely, this space contains all such elements.

Let H be the hyperplane orthogonal to the non-isotropic vector x with respect to f, and let $\bar{x}$ be the symmetry with respect to H. Then the symmetry $\bar{x}$ as a transformation of M over K is the involutive orthogonal transformation which takes the vectors

of the non-isotropic plane P generated by x and $y = \theta x$ in their opposites and leaves invariant the vectors of the subspace $P^{\perp}$ orthogonal to P with respect to Q.

The elements of the Clifford group mapped by χ into this orthogonal transformation are of the form $\alpha x y$, $0 \neq \alpha \in K$. Since $\alpha x y$ is of degree 2 and is invariant under τ_θ it belongs to $D(f)$.

On the other hand, if K has more than 3 elements it is possible to find $\alpha_{ij} \neq 0$ such that $z_{ij} = x_i + \alpha_{ij} x_j$ is a non-isotropic vector. Then the elements of the Clifford group mapped by χ into the symmetries of f with respect to the hyperplanes orthogonal to the vectors $x_i, z_{ij}, i, j = 1, 2, \ldots, n$, $i<j$, are

$$x_i y_j;\ (x_i + \alpha_{ij} x_j)(y_i + \alpha_{ij} y_i) = x_i y_i + \alpha_{ij}^2 x_j y_j + \alpha_{ij}(x_i y_j - y_i x_j);$$

where $y_i = \theta x_i$. The algebra generated by such elements contains a basis of the space of degree 2 of $D(f)$, for

$$(x_i y_j - y_i x_j) x_j y_j = u_{ij} x_j y_j = \rho\, Q(x_j)(x_i x_j - \rho^{-1} y_i y_j) = \rho\, Q(x_j)\, v_{ij}.$$

Therefore if K has more than 3 elements, the elements of the Clifford group mapped by χ into the symmetries of f generate $D(f)$.

If K has only 3 elements, F is the field with 9 elements and there exists then an orthogonal basis of M with respect to f such that $f(x_i, x_i) = 1$ for any i; since we suppose that $(M:F)$ is less than the characteristic of F, $n<3$. If $(M:F)=1$ the theorem is obvious and if $(M:F)=2$ we can take $\alpha_{12}=1$ and apply the preceding argument.

Now let us see that $D(f)$ contains the elements mapped by χ into the quasi-symmetries of f. The quasi-symmetry which leaves invariant elementwise the hyperplane H orthogonal to the non-isotropic vector x and takes x into $(\alpha + \beta\theta)x$, where $N(\alpha + \beta\theta) = 1$, is the image under χ of the elements of the form

$$\nu\left(\frac{1+\alpha}{\beta} + Q(x)^{-1} x y\right),$$

where

$$y = \theta x, \quad 0 \neq \nu \in K.$$

Since these elements belong to $D(f)$ the second assertion is proved because the unitary group is generated by the quasi-symmetries (cf. [8] or [9], p. 41).

It only remains to see that $D(f)$ contains the invertible elements of $C(Q)$ defining inner automorphisms which induce in $C^+(Q)$ the automorphisms associated to the unitarian similitudes of f. Such elements are defined up to an invertible factor of the form $\alpha + \beta r_n$, since the algebra $K + K r_n$ is the center of $C^+(Q)$.

Since the automorphism σ of $C^+(Q)$ associated to a unitarian similitude of f is homogeneous of degree zero, it must take the element r_1, which is a basis of the space of central elements of degree 2, into αr_1. But the component of degree zero of $(r_1^2)^\sigma$, $n\rho \neq 0$, must be equal to the component of degree zero of

$$(r_1^\sigma)^2 = (\alpha r_1)^2, \; \alpha^2 n \rho;$$

therefore $\alpha^2 = 1$, $\alpha = \pm 1$.

If $\alpha = -1$, since r_h is the component of degree $2h$ of $\frac{r_1^h}{h!}$, $(r_{2t+1})^\sigma = -r_{2t+1}$ and $r^\sigma_{2t} = r_{2t}$. Let $c \in C^+(Q)$ and let us apply to c the automorphism associated to the homotecy defined by the element

$$\frac{\mu^2 + \rho}{\mu^2 - \rho} - \frac{2\mu\theta}{\mu^2 - \rho} = \alpha + \beta\theta, \quad \mu \neq 0$$

of norm 1, and then the automorphism σ. We get (cf., Ch. I, p. 9):

$$\begin{aligned} c^{\tau_{\alpha+\beta\theta}\sigma} &= \left((\mu^n + \mu^{n-1} r_1 + \ldots + \mu^{n-i} r_i + \ldots + r_n)^{-1} \cdot \right. \\ &\left. \cdot\, c\,(\mu^n + \mu^{n-1} r_1 + \ldots + r_n)\right)^\sigma = (\mu^n - \mu^{n-1} r_1 + \ldots \\ &\ldots + (-1)^i \mu^{n-i} r_i + \ldots + (-1)^n r_n)^{-1} \cdot c^\sigma (\mu^n + \ldots + (-1)^n r_n) = \\ &= \left((-\mu)^n + \ldots + (-\mu)^{n-i} r_i + \ldots + r_n\right)^{-1} \cdot \\ &\cdot\, c^\sigma \left((-\mu)^n + \ldots + r_n\right) = c^{\sigma \tau_{\alpha - \beta\theta}}, \end{aligned}$$

that is,

$$c^{\tau_{\alpha+\beta\theta}\sigma} = c^{\sigma\tau_{\alpha-\beta\theta}}. \tag{12}$$

On the other hand if S is a unitarian similitude of f and $T_{\alpha+\beta\theta}$ a homotecy,

$$S\, T_{\alpha+\beta\theta} = T_{\alpha+\beta\theta}\, S.$$

Let σ and $\tau_{\alpha+\beta\theta}$ be the automorphisms of $C^+(Q)$ associated to S and $T_{\alpha+\beta\theta}$, respectively. Then since $S\,T_{\alpha+\beta\theta} = T_{\alpha+\beta\theta}\,S$ we must have $\sigma\,\tau_{\alpha+\beta\theta} = \tau_{\alpha+\beta\theta}\,\sigma$ and therefore

$$c^{\tau_{\alpha+\beta\theta}\sigma} = c^{\sigma\tau_{\alpha+\beta\theta}} \tag{13}$$

From (12) and (13) we get

$$(c^\sigma)^{\tau_{\alpha+\beta\theta}} = (c^\sigma)^{\tau_{\alpha-\beta\theta}}$$

and this can not be true for any $c^\sigma \in C^+(Q)$ since $\alpha + \beta\theta$ and $\alpha - \beta\theta$ do not differ by a factor $\delta \in K$. Hence the assumption $\alpha = -1$ leads to contradiction. Therefore $\alpha = 1$ and any element of $C(Q)$ which defines an inner automorphism inducing in $C^+(Q)$ the automorphism σ associated to a unitarian similitude commutes with r_1. Then the lemma 4 of Ch. I shows that such element belongs to $D(f)$.

In Ch. I, theorem 1, we have seen that $D(f)$ is a semisimple subalgebra of $C(Q)$ direct sum of $1 + \left[\frac{n}{2}\right]$ simple algebras. The theorem that we have just proved shows now that the spin representation of the elements of the Clifford group mapped by χ into unitarian transformation of f decomposes in a direct sum of $1 + \left[\frac{n}{2}\right]$ irreducible representations.

We get the same decomposition in simple representations if we consider only the spin representation of the elements of the Clifford group mapped by χ into elements of the group $U^\pm(f)$ generated by the symmetries of f or if we consider the spin representation of the group of invertible elements of $C(Q)$ which define inner automorphisms inducing in $C^+(Q)$ the automorphisms associated to the unitarian similitudes. Each unitarian similitude defines one of this invertible elements up to an invertible factor of the form $\alpha + \beta r_n$.

§ 3

Let us take a spin representation σ of $D(f)$ and let Q be a matrix of the antiautomorphism $*$ in the representation σ. Then Q is a direct sum of hermitian matrices, with the exception of the component $(Q)_r$ which when $n = 2r$ and $T = K_{\binom{n}{r}}$ can be either symmetric or skew-symmetric (Ch. I, theorem 2).

If $c \in D(f) \subset C^+(Q)$ defines the inner automorphism associated to the unitarian transformation U, let $(C)_i$, $i = 0, 1, \ldots, \left[\frac{n}{2}\right]$ denote the i-th component of the matrix $C = c^\sigma$. In particular, if $\alpha \in K$, $\alpha^\sigma = \Sigma \oplus \alpha (I)_i$, where $(I)_i$ is the i-th component of the unit matrix.

If an element c belongs to the Clifford group its norm $c\,c^* = \alpha \in K$. When $c \in D(f)$ in the spin representation σ

$$(c\,c^*)^\sigma = c^\sigma (c^*)^\sigma = C\,Q\,C^T Q^{-1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus (C)_i (Q)_i (C)_i^T (Q)_i^{-1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \oplus \alpha (I)_i$$

which implies

$$(C)_i (Q)_i (C)_i^T = \alpha (Q)_i. \tag{14}$$

If $n = 2r + 1$, then $(Q)_i \in F_{\binom{n}{i}}$, $i = 0, 1, \ldots, r$, and the $(Q)_i$ are hermitian matrices with respect to γ. Each one of these matrices can be considered as the matrix of a hermitian form relative to a basis (cf. [13], pp. 149-50) where J is the involutive automorphism associated to the hermitian form. Then the $(C)_i$ represent linear transformations of the vector spaces on which the hermitian forms $(Q)_i$ are defined relative to the given bases.

Relation (14) shows that these transformations are unitarian similitudes. Given σ, the $(Q)_i$ are defined up to a factor $\delta \in K$. Therefore we can say that the hermitian forms $(Q)_i$ are defined by $D(f)$ up to a factor δ, because if we take another spin representation we have seen in Ch. I, § 3 that the new matrices $(\bar{Q})_i$ are cogredient with the $(Q)_i$ and therefore define the same hermitian forms with respect to new bases.

When $n = 2r$, what we have just said is still true for the $(Q)_i$ if $i \neq r$. As to $(Q)_r$ we know that there are two possible cases:

1) $(Q)_r \in R_{\frac{1}{2}\binom{n}{r}}$. Then we can applied what has been said for the $(Q)_i \in F_{\binom{n}{i}}$ with the only difference that now we have a hermitian form on a vector space over a sfield of quaternions (see [7], pp. 74-75) and J′ is the involutive anti-automorphism associated to the hermitian form.

2) $(Q)_r \in K_{\binom{n}{r}}$. In this case we are going to see that $(Q)_r$ is symmetric if $r = 2s$ and skew-symmetric if $r = 2s + 1$. Then

a) if $(Q)_r$ is symmetric we can applied what has been said and the $(C)_r$ define similitudes with respect to the symmetric bilinear form $(Q)_r$,

b) if $(Q)_r$ is skew-symmetric we have an alternate bilinear form. The $(C)_r$ define sympletic similitudes.

To prove that $(Q)_r$ is symmetric (skew-symmetric) if $r = 2s$ $(r = 2s + 1)$ we compute the dimension over K of the space of elements of $D(f)$ invariant under the anti-automorphism $*$.

Let d_0 be the dimension of the space of elements of $D(f)$ invariant under $*$ when $n = 2r$. Since the elements of degree $4i$ are invariant under the anti-automorphism $*$ and the elements of degree $4i + 2$ are changed by this anti-automorphism in their opposites, by theorem 1, we get

$$d_0 = \sum_{i=0}^{r} \binom{n}{2i}^2.$$

To compute this sum we compute first

$$B = \sum_{i=0}^{n} (-1)^i \binom{n}{i}^2 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n}{n-i}.$$

Hence B is the coefficient of x^n in

$$\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}x^i\right)\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)=(1-x)^n(1+x)^n=(1-x^2)^n,$$

and therefore $B=(-1)^r\binom{n}{r}$

Since $A=\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$,

$$d_0=\sum_{i=0}^{r}\binom{n}{2i}^2=(A+B)/2=\frac{1}{2}\left(\binom{2n}{n}+(-1)^r\binom{n}{r}\right).$$

Let us determine now the space of invariant elements of $F_{\binom{n}{i}}$ with respect to the anti-automorphism $(A)_i^*=(Q)_i(A)_i^T(Q)_i^{-1}$, where $(Q)_i=\operatorname{diag}\left(\alpha_1,\alpha_2,\ldots,\alpha_{\binom{n}{i}}\right)$. The $\binom{n}{i}^2$ elements

$$e_{hh},\ e_{hj}+\alpha_j e_{jh}\alpha_h^{-1},\ \theta e_{hj}-\theta\alpha_j e_{jh}\alpha_h^{-1};\ h,j=1,2,\ldots,\binom{n}{i};\ h<j,$$

where e_{hj} is the matrix with 1 in the intersection of the h-th row and the j-th column and 0 elsewhere, form a basis over K of the space of elements invariant under the anti-automorphism *.

If $(Q)_r\in K_{\binom{n}{r}}$, $n=2r$ and $(Q)_r=\operatorname{diag}\left(\alpha_1,\alpha_2,\ldots,\alpha_{\binom{n}{r}}\right)$ the space of elements of $K_{\binom{n}{r}}$ invariant under the anti-automorphism $(A)_r^*=(Q)_r(A)_r'(Q)_r^{-1}$, has as basis the $\binom{\binom{n}{r}+1}{2}$ elements e_{hh}, $e_{hj}+\alpha_j e_{jh}\alpha_h^{-1}$. Therefore when $(Q)_r=\operatorname{diag}\left(\alpha_1,\ldots,\alpha_{\binom{n}{r}}\right)$, the space of elements of the spin representation of D (f) invariant under the anti-automorphism * has dimension

$$\sum_{i=0}^{r-1}\binom{n}{i}^2+\binom{\binom{n}{r}+1}{2}=\sum_{i=0}^{r-1}\binom{n}{i}^2+\frac{1}{2}\binom{n}{r}^2+$$

$$+\frac{1}{2}\binom{n}{r}=\frac{1}{2}\left(\binom{2n}{n}+\binom{n}{r}\right).$$

This value coincides with d_0 only when r is even. Therefore if $r=2s+1$, the matrix $(Q)_r$ is skew-symmetric. When $r=2s$, the matrix $(Q)_r$ must be symmetric, otherwise a straighforward computation will show that the subspace of D (f) ef elements invariant under * should have dimension

$$\frac{1}{2}\left(\binom{2n}{n}-\binom{n}{r}\right).$$

Let us remark that when an element $c\in D(f)$ belongs to the group of invertible elements defining inner automorphisms of C (Q) which induce on C^+ (Q) the automorphisms associated to unitarian similitudes, $c\,c^*$ is an element of K if $n=2r+1$ and it belongs to the space over K generated by 1 and r_n if $n=2r$ (cf. [9], p. 72 or [16], taking into account that $(M:K)=2n$). Since r_n belongs to the center of D (f) and for $n=2r$, $r_n^2=\rho^n$, $\rho^{-r}r_n$ is a direct sum of matrices each one equal to the unit matrix or to its opposite. Therefore for any n,

$$(C)_i(Q)_i(C)_i^T(Q)_i^{-1}=\alpha_i(I)_i,$$

that is,

$$(C)_i(Q)_i(C)_i^T=\alpha_i(Q)_i.$$

What has been said for the components of the matrices images by σ of the elements of the Clifford group mapped by χ into unitarian transformations is also true for the components $(C)_i$ of a matrix C image by σ of an element of the group mentioned above. That is to say, the $(C)_i$ are unitarian similitudes of the hermitian form $(Q)_i$ defined on a vector space over F if $2i \neq n$; for $2i = n$ $(C)_i$ is a unitarian similitude of the hermitian form $(Q)_i$ defined on a vector space over a sfield of quaternions if $(C)_i \in R_{\frac{1}{2}\binom{n}{r}}$, and, if $(C)_i \in K_{\binom{n}{r}}$, $(C)_i$ is a similitude with respect to the quadratic form $(Q)_r$ if $r = 2s$ or a similitude with respect to an alternate bilinear form if $r = 2s + 1$.

CHAPTER III

In this chapter the algebra $D(f)$ will be used to obtain representations of the projective group of unitarian similitudes of f into orthogonal groups. The spaces of the representations are the subspaces of $D(f)$ of degree $2i$, $i = 1, 2, \ldots, n-1$, which are vector spaces over K. We map the unitarian similitude S into the linear transformation induced in the subspace of $D(f)$ of degree $2i$ by the automorphism of $C^+(Q)$ associated to S. It will be seen that in this form we get a representation of the projective group of unitarian similitudes.

In § 1 we define a symmetric bilinear form on the subspaces of degree $2i$ of $D(f)$, $i = 1, 2, \ldots, n-1$. Then it will be shown that for any i this form is non-degenerate and the linear transformation induced in one of these subspaces by the automorphism associated to a unitarian similitude is an orthogonal transformation with respect to this symmetric bilinear form.

As before it will be assumed that the hermitian form f is non-degenerate and the characteristic of F is zero or greater than (M : F).

§ 1

We define a symmetric bilinear form (x, y) on the subspace of degree r of $C(Q)$ in the following way:

Let a and b be any two elements of the subpace of degree r of $C(Q)$. Then we take as value of (a, b) the component of degree zero of $a\,b^*$. It is obvious that the form so defined is bilinear.

Since a and b are homogeneous elements, the anti-automorphism * leaves them invariant if $r = 4m$ or $r = 4m + 1$ and takes them into their opposites if $r = 4m + 2$ or $r = 4m + 3$. Therefore, since the component of degree zero is left invariant by the homogeneous anti-automorphism *, $a\,b^*$ has the same component of degree zero that $(a\,b^*)^* = b\,a^*$. Then $(a, b) = (b, a)$, which shows that (x, y) is a symmetric form.

Lemma 1. The symmetric bilinear form defined on the spaces of degree r of $C(Q)$ is non-degenerate if and only if Q is non-degenerate.

Proof. Let $z_1, z_2, \ldots, z_N$ be an orthogonal basis with respect to Q, of the vector space on which Q is defined. Then the elements

$$z_{i_1} z_{i_2} \ldots z_{i_r}, \qquad i_1 < i_2 < \ldots < i_r,$$

form an orthogonal basis of the space of degree r with respect to the form (x, y). If all the vectors z_i, $i = 1, 2, \ldots, N$ are non-isotropic and

$$a = z_{i_1} z_{i_2} \ldots z_{i_r}, \qquad (a, a) = Q(z_{i_1})\,Q(z_{i_2}) \ldots Q(z_{i_r}) \neq 0$$

and the form (x, y) is non-degenerate. But, if one of the vectors z_i is isotropic, there are isotropic vectors in the chosen orthogonal basis of the space of degree r.

The form (x, y) induces a symmeric bilinear form on the subspaces of degree $2i$ of $D(f)$. It will be proved that this induced bilinear form is non degenerate.

We are going to use the notation of chapter II; in particular u_i, v_i, r_i, s_i have the same meaning that in chapter II, § 1, where x_i, $y_i = \theta x_i$, $i = 1, 2, \ldots, n$ is an orthogonal basis with respect to Q.

Lemma 2. Let

$$a = w_{i_1} w_{i_2} \ldots w_{i_m} \quad \text{and} \quad b = w'_{i_1} w'_{i_2} \ldots w'_{i_m},$$

where w_{i_k}, w'_{i_k} stand for u_{i_k} or v_{i_k}. Then

$$(a, b) = \left(w_{i_1}, w'_{i_1}\right)\left(w_{i_2}, w'_{i_2}\right) \ldots \left(w_{i_m}, w'_{i_m}\right)$$

and is different from zero if and only if $a = b$, that is,

$$w_{i_k} = w'_{i_k}, \qquad k = 1, 2, \ldots, m.$$

Proof. The elements w_h and w_f, $h \neq f$, commute with each other and $w_h^* = -w_h$ for any h. Therefore

$$ab^* = w_{i_1} w'^*_{i_1} w_{i_2} w'^*_{i_2} \ldots w_{i_m} w'^*_{i_m}. \tag{1}$$

The product $w_j w_j'^*$ has one of the following forms,

$$\left.\begin{aligned}
v_j v_j^* &= -v_j^2 = 2\,Q(x_{2j-1})\,Q(x_{2j}) + 2\,\rho^{-1} x_{2j-1} x_{2j} y_{2j-1} y_{2j},\\
u_j u_j^* &= -u_j^2 = -2\,\rho\,Q(x_{2j-1})\,Q(x_{2j}) -\\
&\quad - 2\,x_{2j-1} y_{2j-1} x_{2j} y_{2j} = -\rho\, v_j v_j^*\\
u_j v_j^* &= -2\,Q(x_{2j-1})\, x_{2j} y_{2j} + 2\,Q(x_{2j})\, x_{2j-1} y_{2j-1}\\
v_j u_j^* &= -u_j v_j^*.
\end{aligned}\right\} \tag{2}$$

The subindices $2\,i_h - 1$, $2\,i_h$ of x and y in the product $w_{i_h} w^*_{i_h}$ are different of the subindices $2\,i_k - 1$, $2\,i_k$ of x and y in $w_{i_k}^* w'^*_{i_k}$, if $h \neq k$. If the index systems of two elements of the form

$$x_1^{\varepsilon_1} y_1^{\delta_1} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}$$

have no common indices the degree of their product is the sum of the degrees of these elements. Therefore the zero component of

$$\prod_{j=1}^{m} w_{i_j} w'^*_{i_j}$$

is the product of the zero components of each factor $w_{i_j} w'^*_{i_j}$. Equating the zero components of (1) we have

$$(a, b) = (w_{i_1}, w'_{i_1})\,(w_{i_2}, w'_{i_2}) \ldots (w_{i_m}, w'_{i_m}).$$

On the other hand the equalities (2) show that

$$(w_{i_j}, w'_{i_j}) \begin{cases} = 0 & \text{if} \quad w_{i_j} \neq w'_{i_j} \\ \neq 0 & \text{if} \quad w_{i_j} = w'_{i_j} \end{cases}$$

and therefore

$$(a, b) \begin{cases} = 0 & \text{if} \quad a \neq b \\ \neq 0 & \text{if} \quad a = b \end{cases}.$$

Lemma 2'. If in lemma 2 we assume that w_{i_k} and w'_{i_k} stand for s_{i_k} or r_{i_k} the lemma is also true.

The proof is the same as before, but instead of (2) we have

$$\left.\begin{aligned}
s_j s_j^* &= -s_j^2 = 2\,Q(x_{2j-1})\,Q(x_{2j}) + 2\,\rho^{-1} x_{2j-1} y_{2j-1} x_{2j} y_{2j}\\
r_j r_j^* &= -r_j^2 = -2\,\rho\,Q(x_{2j-1})\,Q(x_{2j}) -\\
&\quad - 2\,x_{2j-1} y_{2j-1} x_{2j} y_{2j} = -\rho\, s_j s_j^*\\
s_j r_j^* &= 2\,Q(x_{2j-1})\, x_{2j} y_{2j} + 2\,Q(x_{2j})\, x_{2j-1} y_{2j-1}\\
r_j s_j^* &= -s_j r_j^*
\end{aligned}\right\} \tag{2'}$$

In the proof of lemma 6 of chapter II we have defined by induction on h a basis of the space of $2\,h$ different indices. To simplify the notation we had supposed there that the indices were $1, 2, \ldots, 2\,h$ and had pointed out how to deduce a basis for any space of $2\,h$ different indices from the basis of the space of indices $1, 2, \ldots, 2\,h$. The bases so defined for the index spaces of any system of $2\,h$ different indices will be called the canonical bases.

We are going to recall the form of these bases and, at the same time, we introduce a new notation which will be used later on. Let us assume that we know already the form of the elements of the canonical basis of a space of an index system formed by $2\,j$ different indices, $2\,j \leqslant h$. Let $i_1 < i_2 < \ldots < i_{2j}$ be $2\,j$ numbers chosen among the numbers $1, 2, \ldots, h$. The $\binom{2j}{j}$ elements

$c_1, c_2, \ldots, c_{\binom{2j}{j}}$ of the canonical basis of the space of indices $i_1, i_2, \ldots, i_{2j}$ can be written as follows,

$$c_f = \sum_{\nu_i=0}^{\nu_i=1} \alpha^f_{\nu_1 \ldots \nu_{2j}} p^{\nu_1}_{i_1} p^{\nu_2}_{i_2} \ldots p^{\nu_{2j}}_{i_{2j}} \quad \text{where} \quad \alpha^f_{\nu_1 \ldots \nu_{2j}} \in \mathrm{K}, \tag{3}$$

$$p^{\nu_k}_{i_k} = \begin{cases} x_{i_k} & \text{if} \quad \nu_k = 0 \\ y_{i_k} & \text{if} \quad \nu_k = 1 \end{cases}$$

and the sum extends over the 2^{2j} different j-tuples deduce from $(\nu_1, \nu_2, \ldots, \nu_{2j})$ letting $\nu_i = 0, 1$.

If we change the meaning of $p^{\nu_k}_{i_k}$ taking

$$p^{\nu_k}_{i_k} = \begin{cases} s_{i_k} & \text{if} \quad \nu_k = 0 \\ r_{i_k} & \text{if} \quad \nu_k = 1 \end{cases} \tag{4}$$

we get a canonical element of the space of indices $2\,i_1 - 1,\ 2\,i_1, \ldots, 2\,i_{2j} - 1,\ 2\,i_{2j}$. This element will be denoted by C_f.

Let $i'_1 < i'_2 < \ldots < i'_{h-2j}$ be the complementary set of $i_1, i_2, \ldots, i_{2j}$ with respect to $1, 2, \ldots, h$. Let us mulpiply each C_f by each one of the elements D_g, $g = 1, 2, \ldots, 2^{h-2j}$ deduce from $w_{i'_1} w_{i'_2} \ldots w_{i'_{h-2j}}$ sustituting $u_{i'_k}$ or $v_{i'_k}$ for each $w_{i'_k}$. The elements $C_f D_g$ form the canonical basis for the subspace of the index family $i_1, i_2, \ldots, i_{2j}$, which is a subspace of the space of indices $1, 2, \ldots, 2\,h$. The union of the canonical bases of all the subspaces belonging to the different index families of the space of indices $1, 2, \ldots, 2\,h$ is the canonical basis for this index space.

It will be proved first that if the spaces of degree less than $2\,h$ have an orthogonal basis of non-isotropic vectors, the space of degree $2\,h$ has an orthogonal basis with the same property. This conclusion will be reached through a sequence of lemmas.

Lemma 3. The subspaces of the same degree defined by two different index systems are orthogonal to each other.

Proof. The elements a and b of degree $2\,h$ are linear combinations of elements

$$x_1^{\varepsilon_1} y_1^{\delta_1} x_2^{\varepsilon_2} y_2^{\delta_2} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}, \quad \varepsilon_i, \delta_i = 0,1; \quad \text{and} \quad \sum_i \varepsilon_i + \sum_i \delta_i = 2h. \tag{5}$$

The product of two elements of the form (5) is a homogeneous element and it has degree zero if and only if both elements are equal. If a and b belong to two index subspaces with different index systems the elements (5) which appear in the expression of a do not appear in the expression of b. Therefore the component of degree zero of the product $a\,b$ is zero. Since

$$a\,b^* = (-1)^{\binom{2h}{2}} a\,b, \qquad (a, b) = 0.$$

It follows from this lemma that if we have an orthogonal basis of non-isotropic vectors for each one of the index subspaces of degree $2\,h$, the union of these bases is an orthogonal basis of non-isotropic vectors for the space of degree $2\,h$.

Lemma 4. If the index spaces of degree $2\,(h-1)$ have orthogonal bases of non-isotropic vectors the same is true for the subspaces of an index system of degree $2\,h$ in which at least one index appears twice.

Proof. Let k be an index which appears twice in a given index system of degree $2\,h$. We consider the index system of degree $2\,(h-1)$ deduced from the given system of degree $2\,h$ by leaving out the pair of indices $k\,k$. If we multiply each element of any basis of the subspace of the index system of degree $2\,(h-1)$ by $x_k\,y_k$ we get a basis for the subspace of the given index system. Let us denote by m_j, $j = 1, 2, \ldots, \mathrm{N}$ the elements of an orthogonal basis of non-isotropic vectors of the index subspace of degre $2\,(h-1)$, i. e., $(m_i, m_j) = 0$ if $i \neq j$, $(m_i, m_i) \neq 0$. Since

$$m_i\,x_k\,y_k\,(m_j\,x_k\,y_k)^* = m_i\,x_k\,y_k\,y_k\,x_k\,m_j^* = -\rho\,\mathrm{Q}\,(x_k)^2\,m_i\,m_j^*,$$

equating the components of degree zero of the right hand and left hand expressions we get

$$(m_i x_k y_k,\ m_j x_k y_k) = -\rho\, Q(x_k)^2 (m_i,\ m_j) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

which proves the lemma.

Now we need to find orthogonal bases of non-isotropic vectors for the index subspaces of any system of $2h$ different indices. Without loss of generality we can assume that we are dealing with the subspace of indices $1, 2, \ldots, 2h$. We are going to consider this subspace as the direct sum of the subspaces of the different index families (see chapter II, proof of lemma 6).

Lemma 5. The subspaces of the different index families of the space of index $1, 2, \ldots, 2h$ are orthogonal to each other.

Proof. Let a and b be two elements of the canonical basis of the space of indices $1, 2, \ldots, 2h$, belonging to two different subspaces of the index families. Let k be an index which belongs to the family of indices of the subspace containing b and does not belong to the family of indices of the subspace containing a.

Let us express a and b as linear combination of elements of the form

$$t_1\, t_2 \ldots t_h \tag{6}$$

where t_j stands for u_j, v_j, r_j or s_j, $j = 1, \ldots, h$. Then, in the terms which appear in the expression of b, t_k stands for s_k or r_k, whereas in the terms which appear in the expression of a, t_k stands for u_k or v_k.

On the other hand the antiautomorphism * takes the element $\alpha\, t_1\, t_2 \ldots t_h$ into $(-1)^h \alpha\, t_1\, t_2 \ldots t_h$ for t_i and t_j commute if $i \neq j$ and $t_j^* = -t_j$, since t_j has degree 2.

Then $a\, b^* = (-1)^h a\, b$ is a sum of products and each one of these products contains one of the following pairs of factors, u_k, r_k ; u_k, s_k ; v_k, r_k or v_k, s_k among other factors wich commute with these ones. Since

$$u_k r_k = u_k s_k = v_k r_k = v_k s_k = 0 \qquad a\, b^* = (-1)^h a\, b = 0$$

and therefore $(a, b) = 0$.

Lemma 6. If the canonical bases of the spaces of $2j$ different indices, $2j \leqslant h$, are orthogonal bases of non-isotropic vectors, the canonical basis of a subspace of any index family belonging to the index system $1, \ldots, 2h$ is an orthogonal basis of non-isotropic vectors.

Proof. It has been seen that the elements of the canonical basis of the subspace of the index family $i_1, i_2, \ldots, i_{2j}$ have the form $C_f\, D_g$.

It will be first proved that any two elements of the canonical basis are orthogonal to each other. We consider two different cases

Case I. Let us suppose that the two elements are $E_1 = C_{f_1} D_{g_1}$ and $E_2 = C_{f_2} D_{g_2}$ where $g_1 \neq g_2$. This means that if $i'_1, \ldots, i'_{h-2j}$ is the complementary set of $i_1, i_2, \ldots, i_{2j}$ the factor D_g which has the form $w_{i'_1} w_{i'_2} \cdots w_{i'_{h-2j}}$ is different for the two given elements.

In the expression of the elements of the subspace of an index family the r_k or s_k which appear in the factor C_f have different indices that the u_m or v_m which appear in D_g. Therefore any C_f commutes with any D_g. Then,

$$E_1\, E_2^* = C_{f_1} D_{g_1} (C_{f_2} D_{g_2})^* = C_{f_1} D_{g_1} D_{g_2}^* C_{f_2}^* = C_{f_1} C_{f_2}^* D_{g_1} D_{g_2}^*.$$

The product $C_{f_1} C_{f_2}$ is a linear combination of elements (5) where only the indices $2i_1 - 1, 2i_1, \ldots, 2i_{2j} - 1, 2i_{2j}$ may appear. None of these indices appears in the product of $D_{g_1} D_{g_2}^*$; therefore the component of degree zero of $E_1 E_2^*$ is the product of the components of degree zero of $C_{f_1} C_{f_2}^*$ and $D_{g_1} D_{g_2}^*$, that is,

$$(E_1,\ E_2) = (C_{f_1},\ C_{f_2})\,(D_{g_1},\ D_{g_2}).$$

Now lemma 2 asserts that

$$(D_{g_1},\ D_{g_2}) \begin{cases} = 0 & \text{if } g_1 \neq g_2 \\ \neq 0 & \text{if } g_1 = g_2 \end{cases}$$

Therefore $(E_1, E_2) = 0$ if $g_1 \neq g_2$.

CASE II. $E_1 = C_{f_1} D_g$, $E_2 = C_{f_2} D_g$. Since we suppose that $E_1 \neq E_2$ we must have $f_1 \neq f_2$.

The C_f are defined by (3) where the $p_{i_k}^{\nu_k}$ have the meaning given by (4) and if we take

$$p_{i_k}^{\nu_k} = \begin{cases} x_{i_k} & \text{if} \quad \nu_k = 0 \\ y_{i_k} & \text{if} \quad \nu_k = 1 \end{cases}$$

we get the element c_f of the canonical basis of the space of indices $i_1, i_2, \ldots, i_{2j}$. Since we assume that the canonical basis of such space is an orthogonal basis of non-isotropic vectors, the component of degree zero of $c_{f_1} c_{f_2}^*$ must be zero if $f_1 \neq f_2$ and different from zero if $f_1 = f_2$. In the product

$$c_{f_1} c_{f_2}^* = \left(\sum_{\nu_i = 0}^{1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} p_{i_1}^{\nu_1} \ldots p_{i_{2j}}^{\nu_{2j}} \right) \left(\sum_{\nu_i = 0}^{1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} p_{i_{2j}}^{\nu_{2j}} \ldots p_{i_1}^{\nu_1} \right)$$

we only get elements of degree zero when we multiply two terms with the same set of values for $\nu_1, \nu_2, \ldots, \nu_{2j}$. Therefore

$$\left. \begin{aligned} (c_{f_1}, c_{f_2}) = \left(\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \right) \times \\ \times Q(x_{i_1}) \ldots Q(x_{i_{2j}}) \begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases} \end{aligned} \right\} \tag{7}$$

If we compute the component of degree zero of the product $C_{f_1} C_{f_2}^*$ taking into account lemma 2′ and the equalities (2′) we get

$$\left. \begin{aligned} (C_{f_1}, C_{f_2}) = \sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} \left(p_{i_1}^{\nu_1}, p_{i_1}^{\nu_1}\right) \ldots \left(p_{i_{2j}}^{\nu_{2j}}, p_{i_{2j}}^{\nu_{2j}}\right) = \\ = \left(\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \right) 2^{2j} Q(x_{2i_1 - 1}) \\ Q(x_{2i_1}) \ldots Q(x_{2i_{2j}-1}) Q(x_{2i_{2j}}) \begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases} \end{aligned} \right\} \tag{8}$$

for $\prod_{k=1}^{2j} Q(x_{2i_k - 1}) Q(x_{2i_k}) \neq 0$, and (7) shows that

$$\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases}.$$

Therefore it has been proved that the vectors $C_f D_g$ are orthogonal to each other. It remains to be proved that they are non-isotropic. This is immediate, for if

$$E = C_f D_g, \qquad (E, E) = (C_f, C_f)(D_g, D_g);$$

since (8) proves that $(C_f, C_f) \neq 0$ and lemma 2 asserts that

$$(D_g, D_g) \neq 0, \qquad (E, E) \neq 0.$$

Lemmas 5 and 6 prove that if the canonical basis of any space of a system of $2j$ different indices, $2j \leqslant h$, is an orthogonal basis of non-isotropic vectors, the same is true for the canonical basis of a space of a system of $2h$ different indices. If $j = 1$, u_i and v_i form the canonical basis for the space of indices 1, 2 and from this basis we get by induction and substitution of indices the canonical bases of all the spaces of $2h$ different indices. The equalities (2) show that u_1 and v_1 form a canonical basis of non-isotropic vectors. Therefore the canonical basis of any index space of a system of $2h$ different indices is an orthogonal basis of non-isotropic elements.

If we determine now orthogonal bases of non-isotropic vectors for each space of an index system of degree $2h$ in which at least an index appears twice, lemma 3 asserts that the union of these bases and the canonical bases of the spaces of index systems of $2h$ different indices form an orthogonal basis of non-isotropic vectors for the space of $D(f)$ of degree $2h$. Using now lemma 4, it suffices to prove that the space of degree 2 has a basis with this property. Since the element $x_i y_i$ is non-isotropic, for

$$(x_i y_i, x_i y_i) = -\rho Q(x_i) \neq 0,$$

the elements

$$x_i y_i,\quad u_{ij},\quad v_{ij},\quad i, j = 1, 2, \ldots, n,\quad i < j,$$

form an orthogonal basis of non-isotropic elements for the space of degree 2. Therefore we have established

THEOREM 1. Let Q be the quadratic form associated to the non-degenerate hermitian form f. Then the symmetric bilinear forms (a, b) that have been defined on the spaces of degree r of $C(Q)$ induce non-degenerate symmetric bilinear forms on the spaces of degree $2i$ of $D(f)$, $i = 1, \ldots, n$.

§ 2

In chapter I we have associated to any unitarian similitude of f an automorphism of $C^+(Q)$, where Q is the quadratic form associated to f. If the invertible element $c \in C(Q)$ defines an inner automorphism inducing in $C^+(Q)$ the automorphism associated to a unitarian similitude S, then theorem 3 of chapter II asserts that $c \in D(f)$. Therefore the automorphism of $C^+(Q)$ associated to a unitarian similitude induces in $D(f)$ an inner automorphism. Moreover it is known that such automorphism is homogeneous of degree 0 and therefore induces linear transformations in the spaces of $D(f)$ of degree $2i$, $i = 0, 1, \ldots, n$.

THEOREM 2. The linear transformation of the space of degree $2h$ of $D(f)$, $h = 0, 1, \ldots, n$, induced by an automorphism of $C^+(Q)$ associated to a unitarian similitude is an orthogonal transformation with respect to the symmetric bilinear form (a, b).

PROOF. Let a and b be two elements of degree $2h$ of $D(f)$ By definition (a, b) is the component of degree zero of $a\, b^*$.

Let σ be the automorphism associated to the unitarian similitude S. Since σ is homogeneous of degree o, it commutes with the antiautomorphism $*$. Moreover σ leaves K invariant elementwise; therefore the zero component of $(a\, b^*)^\sigma = a^\sigma (b^\sigma)^*$ coincides with the zero component of $a\, b^*$, which implies $(a, b) = (a^\sigma, b^\sigma)$. Hence the linear transformation induced by σ in the space of degree $2h$ of $D(f)$ is an orthogonal transformation.

The quadratic form associated to the bilinear form (a, b) defined on the space D_{2h} of $D(f)$ of degree $2h$ will be denoted by Q_{2h}.

The mapping φ_{2h} which takes the element U of the group of unitarian similitudes of f, $S(f)$, into the orthogonal transformation induced in D_{2h} by the automorphism of $C^+(Q)$ associated to U is a homomorphism of the group $S(f)$ into the orthogonal group $O(Q_{2h})$. Moreover, since $D(f)$ has been defined as the subalgebra of $C^+(Q)$ consisting of the elements invariant under the automorphisms of $C^+(Q)$ associated to the homotecies of f, φ_{2h} maps these homotecies into the identity of $O(Q_{2h})$. Therefore by means of φ_{2h} we get a homomorphism ψ_{2h} of the factor group of $S(f)$ by the group of homotecies into the orthogonal group $O(Q_{2h})$. Since the factor group of $S(f)$ by the group of homotecies, which is its center, is the projective group of unitarian similitudes $PS(f)$, we get.

THEOREM 3. For $h = 1, 2, \ldots, n-1$, ψ_{2h} defines a representation of $PS(f)$ into the orthogonal group of the space D_{2h} with respect to the form (a, b).

It is not difficult to prove now that when K has more than 5 elements each one of these representations is faithful. We omit here this proof since we will publish a more refined result in a paper where the irreducible components of each one of these representations will be determined.

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