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Resolving sets tolerant to failures in three-dimensional grids

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Abstract. An ordered set S of vertices of a graph G is a *resolving set* for G if every vertex is uniquely determined by its vector of distances to the vertices in S. The *metric dimension* of G is the minimum cardinality of a resolving set. In this paper we study resolving sets tolerant to several failures in three-dimensional grids. Concretely, we seek for minimum cardinality sets that are resolving after removing any k vertices from the set. This is equivalent to finding $(k+1)$-*resolving sets*, a generalization of resolving sets, where, for every pair of vertices, the vector of distances to the vertices of the set differs in at least $k+1$ coordinates. This problem is also related with the study of the $(k+1)$-*metric dimension* of a graph, defined as the minimum cardinality of a $(k+1)$-resolving set. In this work, we first prove that the metric dimension of a three-dimensional grid is 3 and establish some properties involving resolving sets in these graphs. Secondly, we determine the values of $k \geq 1$ for which there exists a $(k+1)$-resolving set and construct such a resolving set of minimum cardinality in almost all cases.

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1. Introduction

Resolving sets can be used to distinguish the vertices of a graph G comparing distances to fixed vertices. Fault-tolerant resolving sets were defined to distinguish the vertices of G even though one of the vertices of the set fails. Here we consider a more general case, concretely, resolving sets that distinguish

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the vertices of a graph when any k vertices of the set fail, where k is a fixed integer.

The notion of resolving sets in graphs was defined independently by Harary and Melter [13] and Slater [25] and have since received a lot of attention due to their applications in several areas, such as network discovery and verification [3], robot navigation [17], chemistry [5] or games [7]. Fault-tolerant resolving sets were introduced in [14] and have been studied in [2,6,15,19,21,23,24,27]. For more applications and properties on metric dimension and its variants, the reader is addressed to the surveys [20,26] and the references herein.

Let G be a simple finite connected graph. For two vertices $u, v \in V(G)$, let $d(u, v)$ denote the length of a shortest path from u to v. A vertex u of G *resolves* two vertices x and y if $d(u, x) \neq d(u, y)$. A set of vertices $S \subseteq V(G)$ is a *resolving set* for G if for every pair of different vertices x and y of G there is a vertex u in S that resolves x and y. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality of a resolving set, and a *metric basis* is a resolving set of cardinality $\dim(G)$. If $S = \{u_1, \ldots, u_k\} \subseteq V(G)$, we denote by $r(x|S)$ the vector of distances from x to the vertices of S, that is, $r(x|S) = (d(x, u_1), \ldots, d(x, u_k))$. Thus, S is a resolving set if and only if $r(x|S) \neq r(y|S)$ for every pair of distinct vertices $x, y \in V(G)$. The elements of $r(x|S)$ are the *metric coordinates* of x with respect to S. A resolving set $S \subseteq V(G)$ is *fault-tolerant* if $S - \{u\}$ is a resolving set, for every $u \in S$.

Resolving sets can be used to locate nodes in a network modeled as a graph, and fault-tolerant resolving sets can be used to locate nodes even though one of the nodes of the resolving set fails. Here we consider the possibility of more than one failure. Note that the set obtained after the removal of any k vertices of a resolving set S remains resolving if and only if every pair of vertices of the graph is resolved by at least $k + 1$ distinct vertices of S. This last concept was introduced in [9], concretely, a set S of vertices of a graph is a k-*resolving set* if for every pair of vertices x and y, the vectors $r(x|S)$ and $r(y|S)$ have at least k different coordinates. Hence, a set S remains resolving even though k vertices fail if and only if S is a $(k+1)$-resolving set. The k-metric dimension and k-resolving sets of a graph and, concretely, of some product graphs have been studied in [2,9–12,18,22,28]. In the survey [20], an extensive summary of known results and applications of the k-metric dimension is given.

It is worth mentioning that, in contrast to resolving sets or fault-tolerant resolving sets, k-resolving sets do not always exist for $k \geq 3$ [9]. Whenever a graph G has at least one k-resolving set, a k-resolving set of minimum cardinality is a k-*metric basis*, and its cardinality is the k-*metric dimension* of G, denoted by $\dim_k(G)$ [9]. With this terminology, 1-resolving sets correspond to resolving sets and 2-resolving sets correspond to fault-tolerant resolving sets.

Here we are interested in finding resolving sets tolerant to k failures, that is, $(k + 1)$-resolving sets and, more precisely, $(k + 1)$-metric bases and the $(k + 1)$-metric dimension of three-dimensional grids. Grid graphs have been proven to be very useful in diverse areas such as robotics, video games

and telecommunications [1,8,19]. The value of the k-metric dimension of two-dimensional grids is determined in [2].

Since a superset of a resolving set is also a resolving set, the following result is obvious and provides a necessary and sufficient condition for the existence of a k-resolving set.

Remark 1. A graph G has a k-resolving set if and only if $V(G)$ is a k-resolving set.

Next result gives a lower bound on the k-metric dimension of a graph, whenever it is defined.

Proposition 2 [9]. *If a graph G has a k-resolving set, then* $\dim_k(G) \geq \dim(G) + k - 1$.

The paper is organised as follows. In Sect. 2, some properties concerning resolving sets for three-dimensional grids are given. Concretely, we prove that the metric dimension of a three-dimensional grid is exactly 3 and establish lower bounds on the number of vertices resolving a fixed pair of vertices. These bounds will be very useful in Sect. 3, devoted to the study of resolving sets tolerant to k failures in three-dimensional grids. Concretely, we determine the $(k+1)$-metric dimension and describe $(k+1)$-metric bases of these grids in almost all cases.

2. Resolving Sets in Three-Dimensional Grids

In this section we calculate the metric dimension of three-dimensional grids and prove some new results concerning resolving sets in these grids. We also point out sets of vertices resolving a fixed pair of vertices, that will be very useful in Sect. 3.

Formally, an *r-dimensional grid*, or *rD grid* for short, is any graph obtained as the cartesian product of r non-trivial paths, that is, $P_{n_1} \Box \cdots \Box P_{n_r}$, with $n_1, \ldots, n_r \geq 2$. We can assume that the set of vertices of $P_{n_1} \Box \cdots \Box P_{n_r}$ is:

$$V(P_{n_1} \Box \cdots \Box P_{n_r}) = \{(x_1, \ldots, x_r) : 0 \leq x_i \leq n_i - 1 \text{ for every } i \in \{1, \ldots, r\}\}$$

and two vertices $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ of $V(P_{n_1} \Box \cdots \Box P_{n_r})$ are adjacent if and only if $y_i \in \{x_i - 1, x_i + 1\}$, for some $i \in \{1, \ldots, r\}$, and $x_j = y_j$, whenever $j \neq i$. Hence, the degrees of the vertices of an rD grid are $r, r+1, \ldots, 2r$, and the distance between two vertices $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ in $P_{n_1} \Box \cdots \Box P_{n_r}$ is:

$$d((x_1, \ldots, x_r), (y_1, \ldots, y_r)) = \sum_{i=1}^{r} |y_i - x_i|.$$

There are some vertices that play an important role in resolving pairs of vertices of a 3D grid, concretely, vertices with degree different from the maximum degree, 6. The vertices of degree 3 are called *corners*. A vertex of degree 6 is an *interior* vertex. With the specified labeling of the vertices, the set of corners of a 3D grid is $\{(x_1, x_2, x_3) : x_i \in \{0, n_i - 1\} \text{ for } i \in \{1, 2, 3\}\}$.

A *face* of a 3D grid is a set of vertices with a constant coordinate x_{i_0} equal to either 0 or to $n_{i_0} - 1$, for some $i_0 \in \{1, 2, 3\}$. Let $F(n_1, n_2, n_3)$ denote the set of vertices belonging to any face, that is, $F(n_1, n_2, n_3) = \{(x_1, x_2, x_3) : x_i \in \{0, n_i - 1\} \text{ for some } i \in \{1, 2, 3\}\}$. We write simply F if the values n_1, n_2 and n_3 are clear from context. Note that F consists of all non-interior vertices and $|F(n_1, n_2, n_3)| = n_1 n_2 n_3 - (n_1 - 2)(n_2 - 2)(n_3 - 2) = 2(n_1 n_2 + n_2 n_3 + n_1 n_3) - 4(n_1 + n_2 + n_3) + 8$.

It is known that the metric dimension of an rD grid is at most r [17]. Also in this paper, it is said that it is exactly r and the proof is left to the reader. However, in general this is not in true. For example, it is known that the metric dimension of the hypercube, that can be viewed as a rD grid with $n_1 = \cdots = n_r = 2$, is less than r for $r \geq 5$ [4]. It remains an open problem to determine the exact value of the metric dimension for general grids. A discussion on asymptotic values of the metric dimension of a grid is included in [16]. Next, we prove that the result stated in [17] holds for 3D grids.

Theorem 3. *If $n_1, n_2, n_3 \geq 2$, then $\dim(P_{n_1} \square P_{n_2} \square P_{n_3}) = 3$. Moreover, a set formed by three corners of a face is a metric basis.*

Proof. We begin by proving that $\dim(P_{n_1} \square P_{n_2} \square P_{n_3}) \geq 3$. Paths are the only graphs with metric dimension 1 [5], thus, it is enough to prove that the metric dimension of $P_{n_1} \square P_{n_2} \square P_{n_3}$ is different from 2.

Suppose to the contrary that $\dim(P_{n_1} \square P_{n_2} \square P_{n_3}) = 2$ and $S = \{u, v\}$ is a resolving set for G. Then, the degree of u and v is at most 3 [17]. Hence, u and v must be corners in $P_{n_1} \square P_{n_2} \square P_{n_3}$. We can assume without loss of generality that $u = (0, 0, 0)$ and $v \in \{(n_1 - 1, 0, 0), (n_1 - 1, n_2 - 1, 0), (n_1 - 1, n_2 - 1, n_3 - 1)\}$.

But, if $v = (n_1 - 1, 0, 0)$, then $r((0, 1, 0)|S) = r((0, 0, 1)|S) = (1, n_1)$;

if $v = (n_1 - 1, n_2 - 1, 0)$, then $r((0, 1, 0)|S) = r((1, 0, 0)|S) = (1, n_1 + n_2 - 3)$; and

if $v = (n_1 - 1, n_2 - 1, n_3 - 1)$, then $r((0, 1, 0)|S) = r((1, 0, 0)|S) = (1, n_1 + n_2 + n_3 - 4)$, which is a contradiction.

In [17], the authors prove that the set $\{(0, 0, 0), (n_1 - 1, 0, 0), (0, n_2 - 1, 0)\}$ is a resolving set for $P_{n_1} \square P_{n_2} \square P_{n_3}$. Hence, $\dim(P_{n_1} \square P_{n_2} \square P_{n_3}) = 3$. By symmetry, we have that every set formed by three corners of a face is a metric basis. $\qquad\square$

The following lemma describes some resolving sets of cardinality 4 for a 3D grid. Notice that similar resolving sets can be given by interchanging the role of the three coordinates.

Lemma 4. *If $h, h' \in \{0, \dots, n_3 - 1\}$, with $h \neq h'$, $i \in \{0, \dots, n_1 - 1\}$ and $j \in \{0, \dots, n_2 - 1\}$, then $S = \{(0, 0, h), (n_1 - 1, 0, h), (0, n_2 - 1, h), (i, j, h')\}$ is a resolving set for $P_{n_1} \square P_{n_2} \square P_{n_3}$.*

Proof. Let $G = P_{n_1} \square P_{n_2} \square P_{n_3}$. By Theorem 3, the set $S' = \{(0, 0, h), (n_1 - 1, 0, h), (0, n_2 - 1, h)\}$ is a metric basis for the 3D grid formed by the vertices $\{(x_1, x_2, x_3) \in V(G) : x_3 \leq h\}$ and for the 3D grid formed by the vertices $\{(x_1, x_2, x_3) \in V(G) : x_3 \geq h\}$. Hence, if $h \in \{0, n_3 - 1\}$, then S is a resolving
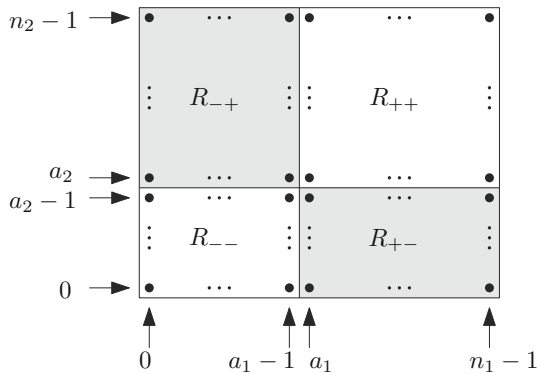
Figure 1. The sets $R_{--}(a_1, a_2)$, $R_{++}(a_1, a_2)$, $R_{-+}(a_1, a_2)$ and $R_{+-}(a_1, a_2)$

set for G. Otherwise, by symmetry, two different vertices u and v of G have the same coordinates with respect to S if and only if $u = (a, b, c_1)$ and $v = (a, b, c_2)$, with $c_1 \neq c_2$ and $c_1 + c_2 = 2h$, for some integers a, b, c_1, c_2. We claim that u and v are resolved by (i, j, h'). Indeed, suppose to the contrary that (i, j, h') does not resolve u and v. In such a case, since $d(u, (i, j, h')) = |i - a| + |j - b| + |h' - c_1|$ and $d(v, (i, j, h')) = |i - a| + |j - b| + |h' - c_2|$, we have $|h' - c_1| = |h' - c_2|$. But this equality is true if and only if $c_1 = c_2$ or $c_1 + c_2 = 2h'$. Hence, $h = h'$, a contradiction. □

Now, we give a sufficient condition for a set not to be resolving in a 3D grid. First we need some terminology. For every $a_1 \in \{1, \ldots, n_1 - 1\}$ and $a_2 \in \{1, \ldots, n_2 - 1\}$, we define the sets of vertices in $P_{n_1} \square P_{n_2}$:

$$R_{--}(a_1, a_2) = \{(x_1, x_2) : 0 \leq x_1 < a_1, 0 \leq x_2 < a_2\},$$
$$R_{++}(a_1, a_2) = \{(x_1, x_2) : a_1 \leq x_1 < n_1, a_2 \leq x_2 < n_2\},$$
$$R_{-+}(a_1, a_2) = \{(x_1, x_2) : 0 \leq x_1 < a_1, a_2 \leq x_2 < n_2\},$$
$$R_{+-}(a_1, a_2) = \{(x_1, x_2) : a_1 \leq x_1 < n_1, 0 \leq x_2 < a_2\}.$$

Notice that these sets form a partition of the set of vertices of the 2D grid $P_{n_1} \square P_{n_2}$ (see Fig. 1).

Let $\mathrm{pr}_3(S)$ be the projection of S onto $P_{n_1} \square P_{n_2}$, that is, $\mathrm{pr}_3(S)$ contains all the vertices of the 2D grid $P_{n_1} \square P_{n_2}$ obtained by deleting the third coordinate from the vertices of S. Analogously, the projection $\mathrm{pr}_i(S)$, $i \in \{1, 2\}$, contains all the vertices of the 2D grid obtained by deleting the i-th coordinate from the vertices of S.

Lemma 5. *If there exist $a_1 \in \{1, \ldots, n_1 - 1\}$ and $a_2 \in \{1, \ldots, n_2 - 1\}$ such that $\mathrm{pr}_3(S) \subseteq R_{--}(a_1, a_2) \cup R_{++}(a_1, a_2)$ or $\mathrm{pr}_3(S) \subseteq R_{-+}(a_1, a_2) \cup R_{+-}(a_1, a_2)$, then S is not a resolving set for $P_{n_1} \square P_{n_2} \square P_{n_3}$.*

Proof. If $\mathrm{pr}_3(S) \subseteq R_{--}(a_1, a_2) \cup R_{++}(a_1, a_2)$, then the vertices $(a_1 - 1, a_2, 0)$ and $(a_1, a_2 - 1, 0)$ are not resolved by S. Analogously, if $\mathrm{pr}_3(S) \subseteq R_{-+}(a_1, a_2) \cup$
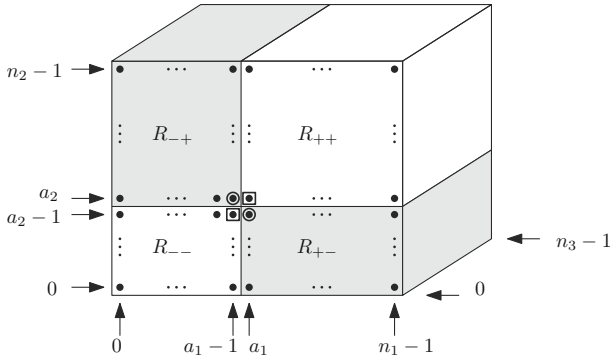
Figure 2. If $\mathrm{pr}_3(S) \subseteq R_{--}(a_1, a_2) \cup R_{++}(a_1, a_2)$, then S is included in the white region and circled vertices are not resolved by any vertex of S. Analogously, if $\mathrm{pr}_3(S) \subseteq R_{-+}(a_1, a_2) \cup R_{+-}(a_1, a_2)$, then S is included in the gray region and squared vertices are not resolved by any vertex of S

$R_{+-}(a_1, a_2)$, then the vertices $(a_1-1, a_2-1, 0)$ and $(a_1, a_2, 0)$ are not resolved by S (see Fig. 2).   □

Note that, by symmetry, the preceding result can also be stated for $\mathrm{pr}_1(S)$ and $\mathrm{pr}_2(S)$.

Now, we prove some properties for general graphs that provide vertices resolving a fixed pair of vertices under certain conditions.

Lemma 6. *Let u and v be vertices of a graph G. If $d(u, v)$ is odd, then every vertex of a shortest path between u and v resolves u and v. If $d(u, v)$ is even, then all but one of the vertices of a shortest path between u and v resolve u and v.*

Proof. For every vertex x belonging to a shortest path from u to v, the equality $d(u, x) + d(x, v) = d(u, v)$ holds. If $d(u, v)$ is odd, then there is no vertex x in the shortest path satisfying $d(u, x) = d(v, x)$. If $d(u, v)$ is even, then there is exactly one vertex x in the shortest path satisfying $d(u, x) = d(v, x) = d(u, v)/2$.   □

Lemma 7. *Let u and v be vertices of a bipartite graph G.*
    If $d(u, v)$ is odd, then every vertex of G resolves u and v.

Proof. The distance between two vertices of the same partite set of G is even, and the distance between vertices of different partite sets is odd. If $d(u, v)$ is odd, then u and v belong to different partite sets of G, and for every vertex $x \in V(G)$, the distances $d(x, u)$ and $d(x, v)$ have different parity. Hence, x resolves u and v.   □

Next results establish a lower bound on the number of vertices in F resolving a pair of fixed vertices u and v, taking into account the number of different coordinates of u and v.

Since grids are bipartite graphs, by Lemma 7, all the vertices of a grid resolve a pair of vertices u and v, if $d(u, v)$ is odd. Thus, we focus on the case $d(u, v)$ even. For $n_1, n_2, n_3 \geq 2$, we define

$$\alpha_M(n_1, n_2, n_3) = \min\{n_1(n_2 + n_3 - 2), n_2(n_1 + n_3 - 2), n_3(n_1 + n_2 - 2)\}.$$

It is easy to check that $\alpha_M(n_1, n_2, n_3) = n_i(n_1 + n_2 + n_3 - n_i - 2)$, for $n_i = \min\{n_1, n_2, n_3\}$.

Lemma 8. *Let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ be two different vertices of $P_{n_1} \square P_{n_2} \square P_{n_3}$ with $x_i \neq y_i$ for exactly one value $i \in \{1, 2, 3\}$. Then, there are at least $\alpha_M(n_1, n_2, n_3)$ vertices in $F(n_1, n_2, n_3)$ resolving u and v.*

Proof. Assume without loss of generality that $x_1 = y_1$, $x_2 = y_2$ and $x_3 \neq y_3$. A vertex $z = (z_1, z_2, z_3)$ does not resolve u and v if and only $d(z, u) = |x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3| = |x_1 - z_1| + |x_2 - z_2| + |y_3 - z_3| = d(z, v)$, that is, if and only if $|x_3 - z_3| = |y_3 - z_3|$. Hence, the set of vertices not resolving u and v is $\{(a, b, (x_3 + y_3)/2) : 0 \leq a \leq n_1 - 1, 0 \leq b \leq n_2 - 1\}$ (note that this set is empty when x_3 and y_3 have different parity). In any case, there are at least $|F| - (2n_1 + 2n_2 - 4)$ vertices in F resolving u and v. But

$$|F| - (2n_1 + 2n_2 - 4) - n_3(n_1 + n_2 - 2)$$
$$= 2(n_1 - 2)(n_2 - 2) + (n_2 - 2)(n_3 - 2) + (n_1 - 2)(n_3 - 2) + 2(n_3 - 2) \geq 0.$$

Hence, $|F| - (2n_1 + 2n_2 - 4) \geq n_3(n_1 + n_2 - 2) \geq \alpha_M$. $\square$

Lemma 6 shows when a vertex of a shortest path between two fixed vertices u and v resolves the pair u and v. Next, we analyze which vertices not belonging to the shortest path resolve u and v. We denote by $S(u, v)$ the set of vertices belonging to a shortest path between u and v in a 3D grid. Concretely, for every pair $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ of distinct vertices of the grid $P_{n_1} \square P_{n_2} \square P_{n_3}$, $S(u, v)$ is the set of vertices:

$$[\min\{x_1, y_1\}, \max\{x_1, y_1\}] \times [\min\{x_2, y_2\}, \max\{x_2, y_2\}] \times [\min\{x_3, y_3\}, \max\{x_3, y_3\}].$$

Hence, $S(u, v)$ induces a 3D grid such that u and v are corners, if the three coordinates of u and v are different; a 2D grid, if u and v have exactly 2 different coordinates; and a path, if u and v differ by exactly one coordinate.

Next, we associate a vertex not in $S(u, v)$ with a vertex of the subgrid $S(u, v)$ (belonging to one of its faces, whenever $S(u, v)$ is a 3D grid) so that either both vertices resolve u and v or neither of them resolves u and v. The result is stated for the first coordinate, but it holds similarly for the second and third coordinates.

Lemma 9. *Let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ be two distinct vertices of $P_{n_1} \square P_{n_2} \square P_{n_3}$. Let $z = (z_1, z_2, z_3)$.*

(i) *If $z_1 \leq \min\{x_1, y_1\}$, then z resolves u and v if and only if vertex $(\min\{x_1, y_1\}, z_2, z_3)$ resolves u and v.*

(ii) *If $z_1 \geq \max\{x_1, y_1\}$, then z resolves u and v if and only if vertex $(\max\{x_1, y_1\}, z_2, z_3)$ resolves u and v.*
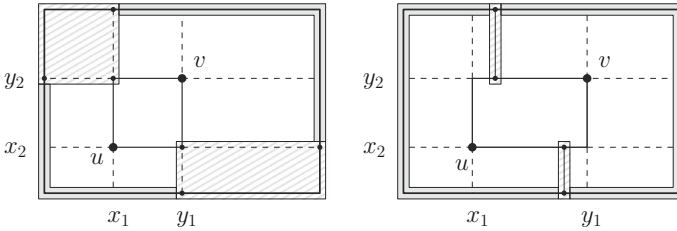
Figure 3. Vertices (a, b, c), with $c = x_3$, of the 3D grid are represented. Left, case $y_1 - x_1 = y_2 - x_2$ and right, $y_1 - x_1 \neq y_2 - x_2$. Vertices in the striped region do not resolve u and v. In both cases, there are at least $n_1 + n_2 - 2$ vertices in the boundary resolving u and v (gray region)

Proof. We prove only the first item, because the second one is derived analogously. Assume without loss of generality that $x_1 \leq y_1$. Let $z = (z_1, z_2, z_3)$ be such that $z_1 \leq x_1$ and let $z' = (x_1, z_2, z_3)$. Then, $d(z, u) = |x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3| = d(z, z') + d(z', u)$. Similarly, we have that $d(z, v) = |y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| = |y_1 - x_1| + |x_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| = |x_1 - z_1| + (|y_1 - x_1| + |y_2 - z_2| + |y_3 - z_3|) = d(z, z') + d(z', v)$. Hence, z resolves u and v if and only if z' resolves u and v. $\square$

Note that the preceding lemma implies that a vertex z not in $S(u, v)$ resolves u and v if and only if the vertex in $S(u, v)$ closest to z resolves u and v.

Lemma 10. *Let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ be two different vertices of $P_{n_1} \square P_{n_2} \square P_{n_3}$ with $x_i = y_i$ for exactly one value of $\{1, 2, 3\}$. Then, there are at least $\alpha_M(n_1, n_2, n_3)$ vertices of $F(n_1, n_2, n_3)$ resolving u and v.*

Proof. We may assume without loss of generality that $x_1 < y_1$, $x_2 < y_2$ and $x_3 = y_3$. Then, $d(u, v) = y_1 - x_1 + y_2 - x_2$. If $d(u, v)$ is odd, then all the vertices in F resolve u and v by Lemma 7. Now suppose that $d(u, v) = y_1 - x_1 + y_2 - x_2$ is even. If $y_1 - x_1 = y_2 - x_2$, then, using Lemma 9 we derive that the vertices in F belonging to the set

$$\{(0, b, c) : 0 \leq b < y_2, 0 \leq c \leq n_3 - 1\}$$
$$\cup \{(n_1 - 1, b, c) : x_2 < b < n_2 - 1, 0 \leq c \leq n_3 - 1\}$$
$$\cup \{(a, 0, c) : 0 \leq a < y_1, 0 \leq c \leq n_3 - 1\}$$
$$\cup \{(a, n_2 - 1, c) : x_1 < a < n_1 - 1, 0 \leq c \leq n_3 - 1\}$$

resolve u and v, and the number of vertices of this set is $(y_2 + (n_2 - 1 - x_2) + y_1 + (n_1 - 1 - x_1) - 2)n_3 = (y_2 - x_2 + y_1 - x_1 + n_1 + n_2 - 4)n_3 \geq (n_1 + n_2 - 2)n_3$ (see Fig. 3, left).

If $y_1 - x_1 \neq y_2 - x_2$, we may assume $y_1 - x_1 < y_2 - x_2$. Let $r = (y_1 - x_1 + y_2 - x_2)/2$. Then, the following vertices in F resolve u and v:

$$\{(0, b, c) : b \neq x_2 + r, 0 \leq c \leq n_3 - 1\}$$
$$\cup \{(n_1 - 1, b, c) : b \neq y_2 - r, 0 \leq c \leq n_3 - 1\}$$

$$\cup \{(a, 0, c) : 0 \le a \le n_1 - 1, 0 \le c \le n_3 - 1\}$$
$$\cup \{(a, n_2 - 1, c) : 0 \le a \le n_1 - 1, 0 \le c \le n_3 - 1\}.$$

and this set has at least $(2n_1 + 2n_2 - 4)n_3$ vertices. Hence, there are at least $(n_1 + n_2 - 2)n_3 \ge \alpha_M$ vertices in F resolving u and v (see Fig. 3, right). □

Lemma 11. *Let* $u = (x_1, x_2, 0)$ *and* $v = (y_1, y_2, n_3 - 1)$ *be two vertices of* $P_{n_1} \square P_{n_2} \square P_{n_3}$. *Then, there are at least* $n_3(n_1 + n_2 - 2)$ *vertices in* $F(n_1, n_2, n_3)$ *resolving* u *and* v.

Proof. We claim that for every pair a and b, such that $a \in \{0, n_1 - 1\}$ or $b \in \{0, n_2 - 1\}$, there is at most one vertex in the set $T_{a,b} = \{(a, b, c) : 0 \le c \le n_3 - 1\}$ not resolving u and v. Indeed, if there were at most two vertices not resolving them, then by Lemma 9 there would be two vertices z and z' belonging to some set $T'_{a',b'} = \{(a', b', c) : 0 \le c \le n_3 - 1\}$ not resolving u and v, where $x_1 \le a' \le y_1$, $x_2 \le b' \le y_2$ and either $a' \in \{x_1, y_1\}$ or $b' \in \{x_2, y_2\}$. But this is a contradiction by Lemma 6, because there is a shortest path between u and v that goes through both z and z'. Notice that there are $2n_1 + 2n_2 - 4$ sets of type $T_{a,b}$, with $a \in \{0, n_1 - 1\}$ or $b \in \{0, n_2 - 1\}$. Hence, there are at least $(2n_1 + 2n_2 - 4)(n_3 - 1)$ vertices in $F(n_1, n_2, n_3)$ resolving u and v, and $(2n_1 + 2n_2 - 4)(n_3 - 1) \ge (n_1 + n_2 - 2)(2n_3 - 2) \ge (n_1 + n_2 - 2)n_3$, where the last inequality holds because $n_3 \ge 2$. □

Lemma 12. *Let* $u = (x_1, x_2, x_3)$ *and* $v = (y_1, y_2, y_3)$ *be two different vertices of* $P_{n_1} \square P_{n_2} \square P_{n_3}$ *with* $x_i \ne y_i$, *for every* $i \in \{1, 2, 3\}$. *Then, there are at least* $n_1 + n_2 - 2$ *vertices* (z_1, z_2, z_3) *resolving* u *and* v *with* $z_3 = n_3 - 1$, *and either* $z_1 \in \{0, n_1 - 1\}$ *or* $z_2 \in \{0, n_2 - 1\}$.

Proof. Assume without loss of generality that $x_i < y_i$ for all $i \in \{1, 2, 3\}$. Let $C = \{(z_1, z_2, n_3 - 1) : z_1 \in \{0, n_1 - 1\}$ or $z_2 \in \{0, n_2 - 1\}\}$ and let $C' = \{(z_1, z_2, y_3) : z_1 \in \{0, n_1 - 1\}$ or $z_2 \in \{0, n_2 - 1\}\}$. Note that $|C| = |C'| = 2n_1 + 2n_2 - 4$. By Lemma 9, a vertex $z = (z_1, z_2, n_3 - 1) \in C$ resolves u and v if and only if $(z_1, z_2, y_3) \in C'$ resolves u and v. Therefore, it is enough to show that there are at least $n_1 + n_2 - 2$ vertices in C' resolving u and v.

Consider the sets $S_1 = \{(x_1, b, y_3) : x_2 \le b \le y_2\} \cup \{(a, y_2, y_3) : x_1 < a \le y_1\}$, $S_2 = \{(a, x_2, y_3) : x_1 \le a \le y_1\} \cup \{(y_1, b, y_3) : x_2 < b \le y_2\}$ and $S_3 = \{(x_1, x_2, c) : x_3 \le c < y_3\}$. The vertices of $S_1 \cup S_3$ form a shortest path between u and v as well as the vertices of $S_2 \cup S_3$. Hence, there is at most one vertex in S_1 and at most one vertex in S_2 not resolving u and v. Let S be the set of vertices in $S_1 \cup S_2$ not resolving u and v. Thus, $|S| \in \{0, 1, 2\}$.

Note that for $w = (z_1, z_2, y_3) \in S$, we have $d(w, v) = d(w, u) = d(w, (x_1, x_2, y_3)) + (y_3 - x_3) > d(w, (x_1, x_2, y_3))$

If $|S| = 0$, then all the vertices of C' resolve u and v by Lemma 9, and $|C'| = 2n_1 + 2n_2 - 4 > n_1 + n_2 - 2$.

If $|S| = 1$, then, since $|S_1| = |S_2|$, the vertex in S must be (x_1, x_2, y_3). By Lemma 9, there are $2n_1 + 2n_2 - 4 - (x_1 + x_2 + 1)$ vertices in C' resolving u and v. But $2n_1 + 2n_2 - 4 - (x_1 + x_2 + 1) \ge n_1 + n_2 - 2$, because $x_i < y_i \le n_i - 1$, for $i = 1, 2$ (see Fig. 4a).
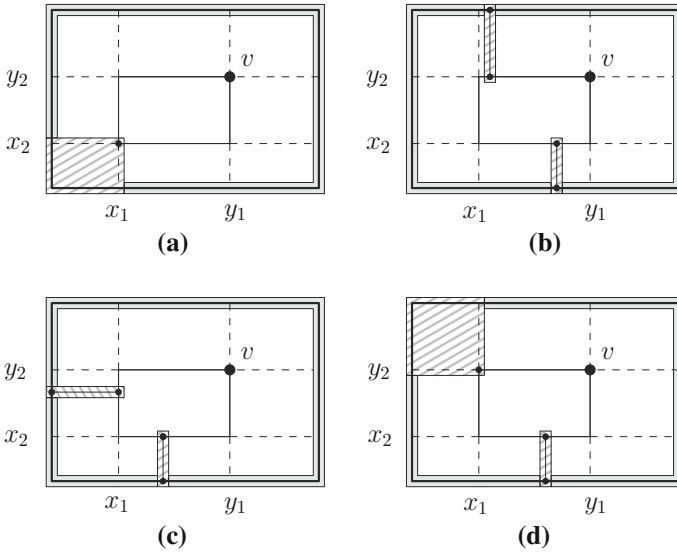
Figure 4. Vertices (a, b, c), with $c = y_3$, of the 3D grid are depicted. Vertices in the striped region do not resolve u and v. The vertices of C' resolving u and v are in the gray region. **a** $|S| = 1$. **b–d** $|S| = 2$

If $|S| = 2$, then $(x_1, x_2, y_3) \notin S$, one vertex of S is in $S_1 \backslash S_2$, the other one belongs to $S_2 \backslash S_1$, and both are at the same distance from (x_1, x_2, y_3).

Moreover, there is at most one vertex belonging to S in $\{(x_1, y_2, y_3), (y_1, x_2, y_3)\}$. Indeed, if $S = \{(x_1, y_2, y_3), (y_1, x_2, y_3)\}$, then $y_1 - x_1 = y_2 - x_2 + y_3 - x_3$ and $y_2 - x_2 = y_1 - x_1 + y_3 - x_3$, because the vertices of S are at the same distance from u and v, and this is not possible because $y_3 - x_3 > 0$. If $S \cap \{(x_1, y_2, y_3), (y_1, x_2, y_3)\} = \emptyset$, then by Lemma 9 there are at least $2n_1 + 2n_2 - 4 - 2$ vertices in C' resolving u and v, and $2n_1 + 2n_2 - 4 - 2 \geq n_1 + n_2 - 2$ because $n_1, n_2 \geq 2$ (see Fig. 4b, c).

If $|S \cap \{(x_1, y_2, y_3), (y_1, x_2, y_3)\}| = 1$, we may assume without loss of generality that $y_2 - x_2 < y_1 - x_1$ so that $S \cap \{(x_1, y_2, y_3), (y_1, x_2, y_3)\} = (x_1, y_2, y_3)$. By Lemma 9 there are at least $2n_1 + 2n_2 - 4 - 1 - (x_1 + n_2 - y_2)$ vertices in C' resolving u and v, $2n_1 + 2n_2 - 4 - 1 - (x_1 + n_2 - y_2) \geq n_1 + n_2 - 2$ because $x_1 < y_1 \leq n_1 - 1$ and $0 \leq x_2 < y_2$ (see Fig. 4d).

In any case, there are at least $n_1 + n_2 - 2$ vertices in C' resolving u and v, and so in C, as we wanted to prove. $\qquad \square$

We finish this section with a lower bound on the number of vertices belonging to the faces of a 3D grid resolving any fixed pair of vertices.

Proposition 13. *Let* $u, v \in V(P_{n_1} \square P_{n_2} \square P_{n_3})$. *Then, there are at least* $\alpha_M(n_1, n_2, n_3)$ *vertices in* $F(n_1, n_2, n_3)$ *resolving* u *and* v.

Proof. We proceed by induction on $h = n_1 + n_2 + n_3 \geq 6$.

If $h = 6$, then $n_1 = n_2 = n_3 = 2$ and $\alpha_M(n_1, n_2, n_3) = 4$. Besides, all the vertices of the grid are in $F(2, 2, 2)$ and it is easy to check that, for every pair of vertices u and v, there are at least 4 vertices resolving them.

Now we prove that the statement of the lemma holds whenever $n_1 + n_2 + n_3 = h$, assuming that it is true for grids $P_{n_1'} \square P_{n_2'} \square P_{n_3'}$ such that $n_1' + n_2' + n_3' \leq h - 1$. Let u and v be two different vertices of $P_{n_1} \square P_{n_2} \square P_{n_3}$ and let $R_{n_1, n_2, n_3}(u, v)$ denote the set of vertices of $F(n_1, n_2, n_3)$ resolving u and v. If u and v have at least one equal coordinate, then $|R_{n_1, n_2, n_3}(u, v)| \geq \alpha_M(n_1, n_2, n_3)$ by Lemmas 8 and 10. Now suppose that the three coordinates of u and v are distinct. We may assume $n_1 \leq n_2 \leq n_3$. Suppose first that $n_1 < n_3$. In such a case, $n_1 + n_2 + n_3 - 1 = h - 1$. If u, v are in $P_{n_1} \square P_{n_2} \square P_{n_3-1}$, then by induction hypothesis and using Lemmas 9 and 12,

$$
\begin{aligned}
|R_{n_1, n_2, n_3}(u, v)| &\geq |R_{n_1, n_2, n_3-1}(u, v)| + (n_1 + n_2 - 2) \\
&\geq \alpha_M(n_1, n_2, n_3 - 1) + (n_1 + n_2 - 2) \\
&= (n_2 + (n_3 - 1) - 2)n_1 + (n_1 + n_2 - 2) \\
&= (n_2 + n_3 - 2)n_1 + n_2 - 2 \\
&\geq (n_2 + n_3 - 2)n_1 = \alpha_M(n_1, n_2, n_3).
\end{aligned}
$$

If no both vertices u and v are included in a grid $P_{n_1} \square P_{n_2} \square P_{n_3-1}$, by symmetry it is enough to consider the case $u = (x_1, x_2, 0)$ and $v = (y_1, y_2, n_3 - 1)$. By Lemma 11, $|R_{n_1, n_2, n_3}(u, v)| \geq n_3(n_1 + n_2 - 2) \geq \alpha_M(n_1, n_2, n_3)$.

Now suppose $n_1 = n_3$, that is, $n_1 = n_2 = n_3 = n$. Arguing as in the preceding case, if u, v belong to $P_n \square P_n \square P_{n-1}$, then

$$
\begin{aligned}
|R_{n,n,n}(u, v)| &\geq |R_{n,n,n-1}(u, v)| + (2n - 2) \\
&\geq \alpha_M(n, n, n - 1) + (2n - 2) \\
&= (2n - 2)(n - 1) + (2n - 2) \\
&= (2n - 2)n = \alpha_M(n, n, n)
\end{aligned}
$$

Otherwise, assume that $u = (x_1, x_2, 0)$ and $v = (y_1, y_2, n - 1)$, and by Lemma 11, $|R_{n,n,n}(u, v)| \geq n(2n - 2) = \alpha_M(n, n, n)$. $\qquad\square$

3. Resolving sets tolerant to k failures in 3D Grids

Recall that a resolving set S of a graph remains resolving after the deletion of any k vertices if and only if S is a $(k+1)$-resolving set of G, and the minimum cardinality of such a set is the $(k+1)$-metric dimension of G. In this section we provide exact values and bounds on the $(k+1)$-metric dimension of 3D grids. Moreover, in almost all cases, a $(k+1)$-metric basis, that is, a $(k+1)$-resolving set of minimum cardinality, is constructed.

Let $\alpha_m(n_1, n_2, n_3)$ be defined as follows:

$$
\alpha_m(n_1, n_2, n_3) = 2(n_1 + n_2 + n_3) - 8,
$$

and recall the definition of $\alpha_M(n_1, n_2, n_3)$ given in the preceding section:

$$
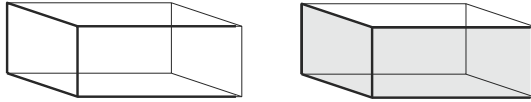\alpha_M(n_1, n_2, n_3) = \min\{n_1(n_2 + n_3 - 2), n_2(n_1 + n_3 - 2), n_3(n_1 + n_2 - 2)\}.
$$

Figure 5. Left, α_m is the number of vertices lying on the thicker edges of the grid. Right, α_M is the number of vertices lying on the gray faces of the grid

From now on, we assume $n_1, n_2, n_3 \geq 2$ and $k \geq 2$. Note that α_m and α_M are half the number of vertices lying on an "edge" of the grid and half the number of vertices lying on the four smallest faces of the grid, respectively (see Fig. 5).

Proposition 14. *If $k \geq \alpha_M(n_1, n_2, n_3)$, then $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ has no $(k+1)$-resolving set.*

Proof. Assume without loss of generality that $\alpha_M = (n_1 + n_2 - 2)n_3$. Suppose to the contrary that $S \subseteq V = V(P_{n_1} \Box P_{n_2} \Box P_{n_3})$ is a $(k+1)$-resolving set. Consider the partition $\{V_1, V_2\}$ of V, where $V_1 = \{u \in V : \mathrm{pr}_3(u) \in R_{--}(1,1) \cup R_{++}(1,1)\}$ and $V_2 = \{u \in V : \mathrm{pr}_3(u) \in R_{-+}(1,1) \cup R_{+-}(1,1)\}$. Then, $|V_2| = (n_1 + n_2 - 2)n_3 = \alpha_M(n_1, n_2, n_3) \leq k$. Hence, by removing from S the at most k vertices belonging to V_2 we have a set of vertices of $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ that is not resolving by Lemma 5, which is a contradiction. $\Box$

Proposition 15. *If $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ has a $(k+1)$-resolving set, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) \geq 2k + 2.$$

Proof. We prove that there is no $(k+1)$-resolving set of cardinality at most $2k+1$. Let $a_1 = \lfloor n_1/2 \rfloor$, $a_2 = \lfloor n_2/2 \rfloor$ and suppose that $S \subseteq V(P_{n_1} \Box P_{n_2} \Box P_{n_3})$ is a $(k+1)$-resolving set of cardinality at most $2k+1$. Consider the partition $\{V_1, V_2\}$ of the set $V = V(P_{n_1} \Box P_{n_2} \Box P_{n_3})$ such that $V_1 = \{(i,j,h) \in V : (i,j) \in R_{--}(a_1, a_2) \cup R_{++}(a_1, a_2)\}$ and $V_2 = \{(i,j,h) \in V : (i,j) \in R_{-+}(a_1, a_2) \cup R_{+-}(a_1, a_2)\}$. By the Pigeonhole Principle, at least one of the sets V_1 or V_2 has at most k vertices of S. Hence, by removing these vertices from S we have a set of vertices of $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ that is not resolving by Lemma 5, which leads to a contradiction. $\Box$

Proposition 16. *If $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ has a $(k+1)$-resolving set and k is even, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) \geq 2k + 3.$$

Proof. Let $V = V(P_{n_1} \Box P_{n_2} \Box P_{n_3})$ and let $a_1 = \lfloor n_1/2 \rfloor$, $a_2 = \lfloor n_2/2 \rfloor$, and $a_3 = \lfloor n_3/2 \rfloor$. Suppose that $S \subseteq V$ is a $(k+1)$-resolving set of cardinality at most $2k + 2$. Consider the sets of V:

$R_{---}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : 0 \leq x_1 < a_1, 0 \leq x_2 < a_2, 0 \leq x_3 < a_3\}$;

$R_{--+}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : 0 \leq x_1 < a_1, 0 \leq x_2 < a_2, a_3 \leq x_3 < n_3\}$;

$R_{+--}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : a_1 \leq x_1 < n_1, 0 \leq x_2 < a_2, 0 \leq x_3 < a_3\}$;
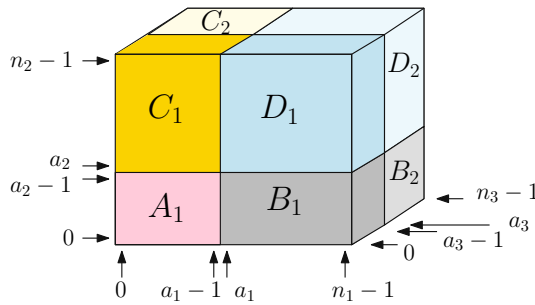
Figure 6. The sets $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are the vertices of S included in the regions obtained when partitioning the vertices of V into the eight octants determined by the chosen values of a_1, a_2 and a_3

$$R_{+-+}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : a_1 \leq x_1 < n_1, 0 \leq x_2 < a_2, a_3 \leq x_3 < n_3\};$$
$$R_{-+-}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : 0 \leq x_1 < a_1, a_2 \leq x_2 < n_2, 0 \leq x_3 < a_3\};$$
$$R_{-++}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : 0 \leq x_1 < a_1, a_2 \leq x_2 < n_2, a_3 \leq x_3 < n_3\};$$
$$R_{++-}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : a_1 \leq x_1 < n_1, a_2 \leq x_2 < n_2, 0 \leq x_3 < a_3\};$$
$$R_{+++}(a_1, a_2, a_3) = \{(x_1, x_2, x_3) : a_1 \leq x_1 < n_1, a_2 \leq x_2 < n_2, a_3 \leq x_3 < n_3\}.$$

Notice that, by definition, these sets form a partition of V. Moreover, if we consider the vertices of the grid as points of the 3D space, these sets are included in the eight regions defined by the semispaces $x_1 < a_1$, $x_1 \geq a_1$; $x_2 < a_2$, $x_2 \geq a_2$; and $x_3 < a_3$, $x_3 \geq a_3$. Consider the following partition of S (see Fig. 6):

$$A_1 = S \cap R_{---}(a_1, a_2, a_3), \qquad A_2 = S \cap R_{--+}(a_1, a_2, a_3),$$
$$B_1 = S \cap R_{+--}(a_1, a_2, a_3), \qquad B_2 = S \cap R_{+-+}(a_1, a_2, a_3),$$
$$C_1 = S \cap R_{-+-}(a_1, a_2, a_3), \qquad C_2 = S \cap R_{-++}(a_1, a_2, a_3),$$
$$D_1 = S \cap R_{++-}(a_1, a_2, a_3), \qquad D_2 = S \cap R_{+++}(a_1, a_2, a_3).$$

If $|A_1| + |A_2| + |D_1| + |D_2| \leq k$ or $|B_1| + |B_2| + |C_1| + |C_2| \leq k$, then S is not a $(k+1)$-resolving set because, by Lemma 5, the removal of the at most k vertices of S in $A_1 \cup A_2 \cup D_1 \cup D_2$ or in $B_1 \cup B_2 \cup C_1 \cup C_2$ produces a non-resolving set, which leads to a contradiction. Hence,

$$|A_1| + |A_2| + |D_1| + |D_2| \geq k + 1 \text{ and } |B_1| + |B_2| + |C_1| + |C_2| \geq k + 1.$$

But, since

$$|A_1| + |A_2| + |D_1| + |D_2| + |B_1| + |B_2| + |C_1| + |C_2| = |S| \leq 2k + 2,$$

we derive $|S| = 2k + 2$, and

$$|A_1| + |A_2| + |D_1| + |D_2| = |B_1| + |B_2| + |C_1| + |C_2| = k + 1.$$

By symmetry, we derive

$$|A_1| + |B_1| + |C_2| + |D_2| = |A_2| + |B_2| + |C_1| + |D_1| = k + 1,$$
$$|A_1| + |C_1| + |B_2| + |D_2| = |A_2| + |C_2| + |B_1| + |D_1| = k + 1.$$

Hence,

$$|A_1| + |A_2| + |D_1| + |D_2| = k + 1 \tag{1}$$
$$|A_1| + |B_1| + |C_2| + |D_2| = k + 1 \tag{2}$$
$$|A_1| + |C_1| + |B_2| + |D_2| = k + 1 \tag{3}$$
$$|A_2| + |B_2| + |C_1| + |D_1| = k + 1 \tag{4}$$

By subtracting Eq. (2) from Eq. (1), Eq. (3) from Eq. (1) and Eq. (4) from Eq. (3), we get

$$|D_1| + |A_2| = |B_1| + |C_2| = |C_1| + |B_2| = |A_1| + |D_2| = q,$$

for some $q \in \mathbb{Z}$ and, consequently,

$$|S| = |A_1| + |A_2| + |B_1| + |B_2| + |C_1| + |C_2| + |D_1| + |D_2| = 4q.$$

Hence, $|S| = 2k + 2 = 4q$, which implies that $k + 1$ is even, a contradiction. □

Note that the preceding lemma provides an alternative way of proving that the metric dimension of a 3D grid is at least 3 (see Theorem 3).

Next, we describe a $(k + 1)$-resolving set of minimum cardinality for some 3D grids. Bearing this in mind, we define the following $n_1 + n_2 + n_3 - 4$ disjoint sets of four vertices:

$S_{1,i} = \{(i, 0, 0), (i, n_2 - 1, 0), (i, 0, n_3 - 1), (i, n_2 - 1, n_3 - 1)\}$, for $0 \leq i \leq n_1 - 1$,
$S_{2,j} = \{(0, j, 0), (n_1 - 1, j, 0), (0, j, n_3 - 1), (n_1 - 1, j, n_3 - 1)\}$, for $1 \leq j \leq n_2 - 2$,
$S_{3,h} = \{(0, 0, h), (n_1 - 1, 0, h), (0, n_2 - 1, h), (n_1 - 1, n_2 - 1, h)\}$, for $1 \leq h \leq n_3 - 2$.

Proposition 17. *If k is an odd integer and $k < \alpha_m(n_1, n_2, n_3)$, then*

$$\dim_{k+1}(P_{n_1} \square P_{n_2} \square P_{n_3}) = 2k + 2.$$

Proof. By Proposition 15, it is enough to construct a $(k + 1)$-resolving set of cardinality $2k + 2$. Let $\alpha = \frac{k+1}{2}$. Note that by hypothesis $\alpha = \frac{k+1}{2} \leq n_1 + n_2 + n_3 - 4$, so we can consider the following set S formed by α of the previously defined sets of vertices (see an example in Fig. 7, left):

$$S = \bigcup_{i=0}^{\alpha-1} S_{1,i}, \text{ if } \alpha \leq n_1,$$

$$S = \left(\bigcup_{i=0}^{n_1-1} S_{1,i} \right) \cup \left(\bigcup_{j=1}^{\alpha-n_1} S_{2,j} \right), \text{ if } n_1 < \alpha \leq n_1 + n_2 - 2,$$

$$S = \left(\bigcup_{i=0}^{n_1-1} S_{1,i} \right) \cup \left(\bigcup_{j=1}^{n_2-2} S_{2,j} \right) \cup \left(\bigcup_{h=1}^{\alpha-n_1-n_2+2} S_{3,h} \right), \text{ if } n_1 + n_2 - 2 < \alpha.$$
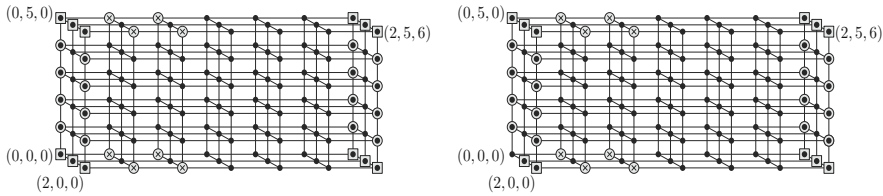
Figure 7. A 18-resolving set (left) and a 17-resolving set (right) of the grid $P_3 \Box P_6 \Box P_7$ consists of squared, circled and crossed vertices. Squared vertices belong to some set $S_{1,i}$, circled vertices to some $S_{2,j}$, and crossed vertices to some $S_{3,h}$

Obviously, $|S| = 4\alpha = 2k + 2$. We claim that S is a $(k+1)$-resolving set of $P_{n_1} \Box P_{n_2} \Box P_{n_3}$. Indeed, any subset S' obtained after the removal of k vertices from S, contains at least three vertices of either one of the sets S_{1,i_0} for some $0 \le i_0 \le n_1 - 1$, or S_{2,j_0} for some $1 \le j_0 \le n_2 - 2$, or S_{3,h_0}, for some $1 \le h_0 \le n_3 - 2$, since otherwise we have to remove at least $2\alpha = k + 1$ vertices. By Lemma 4, there exists a vertex such that together with these three vertices form a resolving set for $P_{n_1} \Box P_{n_2} \Box P_{n_3}$. Hence, S is a $(k+1)$-resolving set. $\qquad\square$

Proposition 18. *If k is an even integer and $k < \alpha_m(n_1, n_2, n_3)$, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) = 2k + 3.$$

Proof. By Proposition 16, it is enough to construct a $(k+1)$-resolving set of cardinality $2k + 3$. Let $\alpha = \frac{k}{2} + 1$. Observe that, by hypothesis, $\alpha \le n_1 + n_2 + n_3 - 4$. Thus, we can consider the following set $S = S^* - \{(0,0,0)\}$, where S^* is formed by the union of α of the previously defined sets of vertices (see an example in Fig. 7, right):

$$S^* = \bigcup_{i=0}^{\alpha-1} S_{1,i}, \text{ if } \alpha \le n_1,$$

$$S^* = \left(\bigcup_{i=0}^{n_1-1} S_{1,i} \right) \cup \left(\bigcup_{j=1}^{\alpha-n_1} S_{2,j} \right), \text{ if } n_1 < \alpha \le n_1 + n_2 - 2,$$

$$S^* = \left(\bigcup_{i=0}^{n_1-1} S_{1,i} \right) \cup \left(\bigcup_{j=1}^{n_2-2} S_{2,j} \right) \cup \left(\bigcup_{h=1}^{\alpha-n_1-n_2+2} S_{3,h} \right), \text{ if } n_1 + n_2 - 2 < \alpha.$$

Obviously, $|S| = 4\alpha - 1 = 2k + 3$. We claim that S is a $(k+1)$-resolving set of $P_{n_1} \Box P_{n_2} \Box P_{n_3}$. Indeed, any subset S' obtained after the removal of k

vertices from S, contains at least three vertices of either one of the sets S_{1,i_0} for some $0 \leq i_0 \leq n_1 - 1$, or S_{2,j_0} for some $1 \leq j_0 \leq n_2 - 2$, or S_{3,h_0}, for some $1 \leq h_0 \leq n_3 - 2$, since otherwise we have to remove at least $2(\alpha - 1) + 1 = k + 1$ vertices. By Lemma 4, there exists a vertex such that together with these three vertices form a resolving set for $P_{n_1} \Box P_{n_2} \Box P_{n_3}$. Hence, S is a $(k+1)$-resolving set.   □

Hence, the exact value of the $(k+1)$-metric dimension has been determined whenever $k < \alpha_m(n_1, n_2, n_3)$. In the remaining cases this parameter is defined, Proposition 13 provides us an upper bound on the $(k+1)$-metric dimension.

Corollary 19. *If $k < \alpha_M(n_1, n_2, n_3)$, then $F(n_1, n_2, n_3)$ is a $(k+1)$-resolving set of $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ and, hence,*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) \leq n_1 n_2 n_3 - (n_1 - 2)(n_2 - 2)(n_3 - 2).$$

We can summarize the previous results as follows.

Theorem 20. *Let $k, n_1, n_2, n_3 \geq 2$.*

1. *If $2 \leq k < \alpha_m(n_1, n_2, n_3)$, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) = \begin{cases} 2k + 2, & \text{if } k \text{ is odd;} \\ 2k + 3, & \text{if } k \text{ is even.} \end{cases}$$

2. *If $\alpha_m(n_1, n_2, n_3) \leq k < \alpha_M(n_1, n_2, n_3)$, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3}) \leq n_1 n_2 n_3 - (n_1 - 2)(n_2 - 2)(n_3 - 2)$$

3. *If $\alpha_M(n_1, n_2, n_3) \leq k$, then $P_{n_1} \Box P_{n_2} \Box P_{n_3}$ has no $(k+1)$-resolving set.*

Observe that $\alpha_m = \alpha_M$, whenever $\min\{n_1, n_2, n_3\} = 2$. Hence, the value of the $(k+1)$-metric dimension is completely determined for these cases.

We finish by posing a conjecture about the exact value of the $(k+1)$-metric dimension of 3D grids whenever $\alpha_m \leq k < \alpha_M$, based on the ideas used to construct $(k+1)$-resolving sets for $k < \alpha_m$.

Conjecture 21. *If $\alpha_m(n_1, n_2, n_3) \leq k < \alpha_M(n_1, n_2, n_3)$, then*

$$\dim_{k+1}(P_{n_1} \Box P_{n_2} \Box P_{n_3})$$
$$= \min\{4k - 2\alpha_m(n_1, n_2, n_3) + 4, n_1 n_2 n_3 - (n_1 - 2)(n_2 - 2)(n_3 - 2)\}.$$

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