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Variable elasticity of substitution and economic growth in the neoclassical model

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Abstract: We study the effect of factor substitutability in the neoclassical growth model with variable elasticity of substitution. We consider two otherwise identical economies differing uniquely in their initial factor substitutability with Variable-Elasticity-of-Substitution (VES), Sobelow or Sigmoidal technologies. If the initial capital per capita is below its steady-state value, the economy with the higher initial elasticity of substitution will feature a higher steady-state income and capital per capita irrespective of whether the production technology is VES, Sobelow or Sigmoidal. Numerical results are provided to compare the effect of a higher elasticity of substitution in the Constant-Elasticity-of-Substitution (CES) model versus the models with variable-elasticity-of-substitution technology.

Keywords: economic growth; elasticity of substitution; neoclassical growth.

JEL classification: O41; E21.

1 Introduction

The seminal papers of de La Grandville (1989) and Klump and de La Grandville (2000) uncovered the positive effect of the elasticity of factor substitution on economic growth in the Solow model. Since then, several papers have analyzed the relationship between factor substitution and economic growth in other frameworks (e.g., Gómez 2015, 2016, 2017; Irmen 2011; Irmen and Klump 2009; Klump and Preissler 2000; Miyagiwa and Papageorgiou 2003; Xue and Yip 2012). In particular, Xue and Yip (2012) present a comprehensive characterization of the link between the elasticity of substitution and the steady-state per capita capital and output in the Solow, Ramsey–Cass–Koopmans (RCK) and Diamond models. If initial per capita capital is below its steady-state value, a higher elasticity of substitution generates a higher steady-state income per capita in the Solow and the RCK models, whereas the effect is ambiguous in the Diamond model.

All these works rely on a normalized Constant-Elasticity-of-Substitution (CES) production function (Arrow et al. 1961; Solow 1956) to study the factor substitutability-growth nexus. However, there is some compelling evidence supporting that the elasticity of substitution depends on the stage of development. Actually, Arrow et al. (1961, p. 247) already speculated that "the process of economic development itself might shift the over-all elasticity of substitution". Piketty and Saez (2014, p. 841) also argue that "it makes sense to assume that the elasticity of factor substitution tends to rise over the development process, as there are more diverse uses and forms for capital and more possibilities to substitute capital for labor" (see also Piketty 2014; Piketty and Zucman 2014). Duffy and Papageorgiou (2000) and Karagiannis, Palivos, and Papageorgiou (2005) provide empirical evidence that the elasticity of substitution may be increasing as the economy develops. Duffy and Papageorgiou (2000) estimate a CES production function for a panel of 82 countries. When they divide the entire sample up into several subsamples according to the initial capital per worker, they find that the elasticity of substitution is above unity in well developed countries and below unity in less developed countries. Using the same data, Karagiannis, Palivos, and Papageorgiou (2005) report empirical results supporting a Variable-

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Elasticity-of-Substitution (VES) production function with an elasticity of substitution greater than one. Thus, it would be interesting to consider technologies with variable elasticity of substitution.

This paper studies the effect of the elasticity of substitution on steady-state magnitudes in the neoclassical growth model. The model with CES production function has been studied, e.g., by Klump and Preissler (2000), Klump (2001), and Xue and Yip (2012). Hence, we consider technologies that feature a variable elasticity of substitution between capital and labor. In particular, we consider two 'classic' concave variants: the VES production function proposed by Revankar (1971) and the Sobelow production function proposed by Jones and Manuelli (1990). Furthermore, we consider the Sigmoidal production function, a non-concave technology recently introduced by Capasso, Engbers, and Torre (2010). The previous literature has emphasized the importance of normalizing the CES production function.[1] Normalization consists on considering a specific family of CES functions that are tangent at the same baseline point, which differ uniquely in the elasticity of substitution. In this way, the effect of a change in the elasticity of substitution is determined by comparing different members of the same family rather than members from different families (Klump and Preissler 2000). Thus, to study the economic growth-substitutability link we will also normalize the underlying production functions.

We find that if initial per capita capital is below its steady-state value, a higher elasticity of substitution generates a higher steady-state income per capita in the three models. Furthermore, an increase in the elasticity of substitution has a positive or negative effect on the steady-state capital income share depending on whether the initial per capita capital is lower or greater than its steady-state value, respectively. We show that this is consequence of the distribution effect (Irmen and Klump 2009). Therefore, our results reinforce the conclusion of Klump and de La Grandville (2000) that factor substitution is a powerful engine of growth.

The rest of the paper is organized as follows. Section 2 outlines the model. Section 3 analyzes the link between factor substitution and steady-state magnitudes in the models with VES, Sobelow and Sigmoidal technologies. Section 4 presents some numerical results. Section 5 concludes.

2 The model

We consider a closed economy populated by identical and infinitely lived agents. Population grows at the exponential rate $n > 0$. Output Y is produced using capital K and labor L, which coincides with population so per capita and per worker quantities coincide. The technology is described by a production function $Y = F(K, L)$ that exhibits constant returns to scale with respect to K and L. Hence, we can define the production function in intensive form as $y = f(k) = F(k, 1)$, where $y = Y/L$ is the output-labor ratio and $k = K/L$ is the capital-labor ratio. The elasticity of substitution between capital and labor, σ, is given by

$$\sigma(k) = -\frac{f^{'}(k)}{kf(k)}\,\frac{f(k) - kf^{'}(k)}{f^{''}(k)}.$$

We shall make the following assumption regarding technology:

Assumption 1: *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is at least twice continuously differentiable,* f(0) ≥ *0, and* $f^{'} > 0$ *(strictly increasing).*

The representative agent maximizes the intertemporal utility

$$U = \int_0^\infty \frac{c^{1-\theta} - 1}{1 - \theta} e^{-\rho t} dt, \quad \theta > 0, \rho > 0,$$

1 Actually Klump, McAdam, and Willman (2012, p. 792) state that "In situations where the researchers wish to gauge the sensitivity of results steady-state or dynamic to variations in the substitution elasticity, normalization is imperative."

where c is agent's consumption, ρ is the rate of time preference and $1/\theta$ is the elasticity of intertemporal substitution, subject to her budget constraint

$$\dot{k} = rk + w - c - (n + \delta)\,k, \tag{1}$$

where r and w are the rental prices of capital and labor, respectively, and δ is the depreciation rate of capital. The first-order conditions for this problem are

$$\lambda = c^{-\theta}, \tag{2}$$

$$\dot{\lambda} = (\rho + \delta + n - r)\lambda, \tag{3}$$

where λ is the shadow value of capital, together with the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda k = 0$.

Profit maximization entails that the rental prices of capital and labor are their respective marginal products:

$$r = \frac{\partial F}{\partial K}(K, L) = f'(k), \tag{4}$$

$$w = \frac{\partial F}{\partial L}(K, L) = f(k) - kf'(k). \tag{5}$$

Using Eqs. (4) and (5) to substitute for r and w in Eq. (1), we get the resources' constraint

$$\dot{k} = f(k) - c - (n + \delta)k. \tag{6}$$

Log-differentiating Eq. (2), and using Eq. (3), the growth rate of per capita consumption is

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'(k) - n - \delta - \rho\right]. \tag{7}$$

The dynamic system Eqs. (6) and (7) drives the dynamics of the economy in the variables k and c. To ensure the existence of a (not necessarily unique) non-trivial steady state we shall make the following additional assumption:[2]

Assumption 2: $\lim_{k\to\infty} f'(k) < n + \delta + \rho < \lim_{k\to 0} f'(k)$.

In this case, the steady state(s) are given by the solution(s) to the system

$$f'\left(\bar{k}\right) = n + \delta + \rho, \tag{8}$$

$$\bar{c} = f\left(\bar{k}\right) - (n + \delta)\bar{k}. \tag{9}$$

If $f'' < 0$; i.e., f is strictly concave, there is a unique steady state solution. However, if the production function is not concave there could be none or multiple stationary solutions. As will be shown later, this may happen in the case of the Sigmoidal technology. The Jacobian matrix of system Eqs. (6) and (7) evaluated at the steady state is

$$\bar{J} = \begin{pmatrix} \rho & -1 \\ \dfrac{f''\left(\bar{k}\right)}{\theta}\bar{c} & 0 \end{pmatrix}.$$

The determinant of the Jacobian matrix is $\det(\bar{J}) = \bar{c}f''(\bar{k})/\theta$, and the trace is positive, $\mathrm{tr}(\bar{J}) = \rho > 0$. Hence, if $f''(\bar{k}) > 0$ there are two unstable roots and, therefore, the steady state is locally unstable. If $f''(\bar{k}) < 0$ there is one

2 If $\lim_{k\to\infty} f'(k) > n + \delta + \rho$, so Assumption 2 is not satisfied, the economy features endogenous growth. This case has been analyzed in Gómez (2020).

stable and one unstable eigenvalue and, therefore, the steady state is locally saddle-path stable. We will focus on saddle-path stable steady state(s). Let π denote the capital income share

$$\pi = \pi(k, \sigma_0) = \frac{k}{f(k, \sigma_0)} \frac{\partial f}{\partial k}(k, \sigma_0). \tag{10}$$

3 Elasticity of substitution and economic growth

This section analyzes the link between capital-labor substitutability and economic growth with technologies that feature a variable elasticity of substitution.

3.1 The model with VES technology

Let us first consider the VES production function proposed by Revankar (1971),

$$Y = F(K, L) = AK^{\alpha}(BK + L)^{1-\alpha}, \quad A > 0, 0 < \alpha < 1,$$

which converges to the Cobb–Douglas function if $B = 0$. The VES production function can be written in intensive form as

$$y = f(k) = F(k, 1) = Ak^{\alpha}(Bk + 1)^{1-\alpha}. \tag{11}$$

The marginal product of capital is $f'(k) = Ak^{\alpha-1}(Bk+1)^{-\alpha}(\alpha + Bk) > 0$. Thus a positive marginal product is guaranteed for $k \in [0, \infty)$ if $B \geq 0$, and for $k \in [0, -\alpha/B)$ if $B < 0$. If this condition is satisfied, then the production function is strictly concave in the relevant range as $f''(k) = -(1-\alpha)\alpha A k^{\alpha-2}(Bk+1)^{-\alpha-1} < 0$. We have that $\lim_{k\to 0} f'(k) = +\infty$. Furthermore, if $B < 0$ then $\lim_{k\to -\alpha/B} f'(k) = 0$, if $B = 0$ then $\lim_{k\to\infty} f'(k) = 0$, and if $B > 0$ then $\lim_{k\to\infty} f'(k) = AB^{1-\alpha}$. Hence, Assumption 2 amounts to i) $B \leq 0$, or ii) $B > 0$ and $AB^{1-\alpha} < n + \delta + \rho$. Given that $f'' < 0$ in the relevant range for k the production function is strictly concave and there exists a unique and saddle-path stable steady state.

The (variable) elasticity of substitution is

$$\sigma(k) = 1 + \frac{Bk}{\alpha},$$

which is above unity if $B > 0$ and below unity if $B < 0$. Furthermore, $\sigma'(k) = B/\alpha$, and so, as k increases the elasticity of substitution is increasing if $B > 0$ and decreasing if $B < 0$.

The production function Eq. (11) has three parameters, A, B and α. Thus, normalization requires three initial data for the baseline value of the capital-labor ratio, k_0. The first two are the baseline income per capita, $y_0 = f(k_0, \sigma_0)$, and the capital income share, $\pi_0 = k_0 \frac{\partial f}{\partial k}(k_0, \sigma_0)/f(k_0, \sigma_0)$. The third one is the initial value of the elasticity of substitution, $\sigma_0 = \sigma(k_0) = 1 + Bk_0/\alpha$. With these baseline conditions, we get the normalized VES production function in intensive form,

$$y = f(k, \sigma_0) = A(\sigma_0)[B(\sigma_0)k + 1]^{\alpha(\sigma_0)}, \tag{12}$$

where the productivity and distribution parameters can be computed as

$$A(\sigma_0) = y_0\left[\frac{\pi_0 + \sigma_0(1-\pi_0)}{\sigma_0}\right]^{\frac{(1-\pi_0)\sigma_0}{\pi_0+\sigma_0(1-\pi_0)}} k_0^{-\frac{\pi_0}{\pi_0+\sigma_0(1-\pi_0)}}, \tag{13}$$

$$\alpha(\sigma_0) = \frac{\pi_0}{\pi_0 + \sigma_0(1-\pi_0)}, \tag{14}$$

$$B(\sigma_0) = \frac{(\sigma_0 - 1)}{k_0}\,\alpha\,(\sigma_0). \tag{15}$$

Note that $B(\sigma_0) > 0$ if $\sigma_0 > 1$ and $B(\sigma_0) < 0$ if $\sigma_0 < 1$, and

$$\frac{dB}{d\sigma_0}\,(\sigma_0) = \frac{\pi_0}{k_0\left[\pi_0 + \sigma_0\left(1 - \pi_0\right)\right]^2} > 0,$$

so B is increasing in the initial elasticity of substitution.

We can state the following proposition.

Proposition 1: *Consider two economies with VES technology that initially differ only with respect to their initial elasticity of substitution, and share initially a common per capita income (y_0), per capita capital (k_0), population growth rate (n) and depreciation rate (δ). If the initial per capita capital is below its steady-state value, the economy with the higher initial elasticity of substitution will have a higher steady-state per capita output, capital and capital income share.*

Proof. See Appendix A.

3.2 The model with Sobelow technology

Let us now consider the Sobelow technology proposed by Jones and Manuelli (1990),

$$Y = F\,(K, L) = AK + BK^{\alpha}L^{1-\alpha}, \quad A > 0, B > 0, 0 < \alpha < 1,$$

which can be written in intensive form as

$$y = f\,(k) = F\,(k, 1) = Ak + Bk^{\alpha}. \tag{16}$$

The (variable) elasticity of substitution is always above unity,

$$\sigma(k) = \frac{Ak + \alpha Bk^{\alpha}}{\alpha Ak + \alpha Bk^{\alpha}} > 1,$$

and

$$\sigma'(k) = \frac{(1-\alpha)^2 ABk^{\alpha}}{\alpha\,(Ak + Bk^{\alpha})^2} > 0,$$

so the elasticity of substitution is increasing as k increases. Given that $\lim_{k\to\infty} f'(k) = A$, Assumption 2 amounts to $A < n + \delta + \rho$. Given that $f'' < 0$ there exists a unique and saddle-path stable steady state.

The Sobelow production function has three parameters, A, B and α. Proceeding as in the case of the VES technology, given the initial value of the elasticity of substitution, σ_0, and for given baseline values of capital per capita, k_0, income per capita, $y_0 = f(k_0, \sigma_0)$, and the capital income share $\pi_0 = k_0 \frac{\partial f}{\partial k}(k_0, \sigma_0)/f(k_0, \sigma_0)$, the normalized Sobelow production function in intensive form is

$$y = f\,(k, \sigma_0) = A\,(\sigma_0)k + B\,(\sigma_0)k^{\alpha(\sigma_0)}, \tag{17}$$

where the productivity and distribution parameters can be computed as

$$A(\sigma_0) = \frac{\pi_0\,(\sigma_0 - 1)y_0}{k_0\,(\sigma_0 - \pi_0)}, \tag{18}$$

$$\alpha(\sigma_0) = \frac{\pi_0}{\sigma_0}, \tag{19}$$

$$B(\sigma_0) = \frac{(1-\pi_0)\sigma_0 y_0}{(\sigma_0 - \pi_0)k_0^{\pi_0/\sigma_0}}. \tag{20}$$

Note that $A(\sigma_0) > 0$ if and only if $\sigma_0 > 1$ and, therefore, $B(\sigma_0) > 0$.

We can state the following result.

Proposition 2: *Consider two economies with Sobelow technology that initially differ only with respect to their initial elasticity of substitution, and share initially a common per capita income (y_O), per capita capital (k_O), population growth rate (n) and depreciation rate (δ). If the initial per capita capital is below its steady-state value, the economy with the higher initial elasticity of substitution will have a higher steady-state per capita output, capital and capital income share.*

Proof. See Appendix B.

3.3 The model with Sigmoidal technology

Let us finally consider the Sigmoidal technology proposed by Capasso, Engbers, and Torre (2010):[3]

$$Y = F(K,L) = \frac{AK^\alpha L}{BK^\alpha + L^\alpha}, \quad A > 0, B > 0, \alpha > 0,$$

which can be written in intensive form as

$$y = f(k) = F\,(k,1) = \frac{Ak^\alpha}{Bk^\alpha + 1}, \tag{21}$$

where $y = Y/L$ and $k = K/L$.[4] The (variable) elasticity of substitution is always below unity,

$$\sigma(k) = \frac{1-\alpha + Bk^\alpha}{1-\alpha + (1+\alpha)Bk^\alpha} < 1,$$

and

$$\sigma'(k) = \frac{(\alpha-1)\alpha^2 Bk^{\alpha-1}}{[1-\alpha + (1+\alpha)Bk^\alpha]^2},$$

so, as k increases the elasticity of substitution is increasing if $\alpha > 1$, constant if $\alpha = 1$, and decreasing if $\alpha < 1$.

The function f is strictly increasing because

$$f'(k) = \frac{\alpha Ak^{\alpha-1}}{(1+Bk^\alpha)^2} > 0,$$

and we have that

$$f''(k) = -\frac{\alpha Ak^{\alpha-2}[1-\alpha + (1+\alpha)Bk^\alpha]}{(1+Bk^\alpha)^3}.$$

Hence, f is strictly concave if $\alpha \leq 1$. We have that $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$ if $\alpha < 1$, and therefore, there exists a unique steady state. We have that $\lim_{k\to 0} f'(k) = A$ and $\lim_{k\to\infty} f'(k) = 0$ if $\alpha = 1$ and, therefore, a necessary and sufficient condition for the existence of a (unique) solution is that . If $\alpha > 1$ there is an inflection point at

3 This technology has also been used by Brianzoni, Mammana, and Michetti (2015), Michetti (2015), and Grassetti, Mammana, and Michetti (2018).

4 As $B \to 0$, the Sigmoidal technology approaches the Cobb-Doublas technology, $y = Ak^\alpha$.

$$\tilde{k} = \left[\frac{\alpha - 1}{B(1+\alpha)}\right]^{1/\alpha}. \tag{22}$$

Given that $f''(\tilde{k}) < 0$, the function f shows an S-shaped behavior for $\alpha > 1$, being first convex and then concave. We have that $\lim_{k\to 0} f'(k) = \lim_{k\to\infty} f'(k) = 0$ if $\alpha > 1$. As the function $f'(k)$ reaches a maximum at the inflection point $\tilde{k}$, a necessary and sufficient condition for the existence of a (not necessarily unique) steady-state solution is that

$$f'\left(\tilde{k}\right) = \frac{AB^{\frac{1-\alpha}{\alpha}}(\alpha-1)^{\frac{\alpha-1}{\alpha}}(\alpha+1)^{\frac{1+\alpha}{\alpha}}}{4\alpha} > n + \delta. \tag{23}$$

The condition $f'(k) = n + \delta$ can be rewritten as

$$p\,(k) = -B^2\,(n+\delta)k^{2\alpha} - 2B\,(n+\delta)k^{\alpha} + \alpha A k^{\alpha-1} - (n+\delta), \tag{24}$$

which is a generalized polynomial ordered according to decreasing powers of k if $\alpha > 1$. Let us denote the sequence of terms as

$$a_1(k) = -B^2\,(n+\delta)k^{2\alpha} < 0,\;\; a_2(k) = -2B(n+\delta)k^{\alpha} < 0,\;\; a_3(k) = \alpha A k^{\alpha-1} > 0,\;\; a_4(k) = -(n+\delta) < 0.$$

The sequence $\{a_1(1), a_2(1), a_3(1), a_4(1)\}$ has two sign changes and, therefore, using the Descartes rule we can conclude that $p(k)$ has none or two zeros in the interval $(0, \infty)$ (e.g., Theorem 3.1 Jameson 2006). Hence, if Eq. (23) is satisfied there exist two steady states $\overline{k}_1 < \tilde{k} < \overline{k}_2$, the lower one being unstable and the higher one locally saddle-path stable. Otherwise, if Eq. (23) is not satisfied, there is no steady-state solution. We will assume that Eq. (23) is met and, therefore, there exist two steady states, and we will focus on the saddle-path stable one, $\overline{k} = \overline{k}_2$.

Given the initial value of the elasticity of substitution, σ_0, and for given baseline values of capital per capita, k_0, income per capita, $y_0 = f(k_0, \sigma_0)$, and the capital income share

$$\pi_0 = \frac{k_0}{f\,(k_0, \sigma_0)}\,\frac{\partial f}{\partial k}\,(k_0, \sigma_0),$$

the normalized sigmoidal production function in intensive form is

$$y = f\,(k, \sigma_0) = \frac{A\,(\sigma_0)k^{\alpha(\sigma_0)}}{1 + B\,(\sigma_0)k^{\alpha(\sigma_0)}}, \tag{25}$$

where the productivity and distribution parameters are

$$A\,(\sigma_0) = \frac{[(1-\pi_0)(1-\sigma_0) + \pi_0\sigma_0]k^{1-2\pi_0-\frac{1-\pi_0}{\sigma_0}}y_0}{\pi_0\sigma_0}, \tag{26}$$

$$\alpha\,(\sigma_0) = -1 + 2\pi_0 + \frac{1-\pi_0}{\sigma_0} = \frac{1-\sigma_0-\pi_0+2\sigma_0\pi_0}{\sigma_0}, \tag{27}$$

$$B\,(\sigma_0) = \frac{(1-\pi_0)(1-\sigma_0)k^{1-2\pi_0-\frac{1-\pi_0}{\sigma_0}}}{\pi_0\sigma_0}. \tag{28}$$

The condition $\sigma_0 < 1$ is necessary and sufficient for $A(\sigma_0) > 0$, $B(\sigma_0) > 0$ and $\alpha(\sigma_0) > 0$. The condition for the sigmoidal function to exhibit a convex-concave shape; i.e., $\alpha > 1$, requires that $\sigma_0 < 1/2$.

We can state the following proposition.

Proposition 3: *Consider two economies with Sigmoidal technology that initially differ only with respect to their initial elasticity of substitution, and share initially a common per capita income (y_0), per capita capital (k_0), population growth rate (n) and depreciation rate (δ). If the initial per capita capital is below its steady-state value,* *the economy with the higher initial elasticity of substitution will have a higher steady-state per capita output, capital and capital income share.*

Proof. See Appendix C.

3.4 Summary

Propositions 1, 2 and 3 extend the results obtained by Klump and de La Grandville (2000) to an economy with VES, Sobelow and Sigmoidal technologies: if the initial per capita capital is below its steady-state value, the steady-state per capita output, capital and capital's income share is increasing in the (initial) elasticity of substitution. Intuitively, if the initial per capita capital is below its stationary value, a higher easiness to substitute labor for capital fosters capital accumulation which generates a higher steady-state per capita capital and output, as well as a higher steady-state capital's income share. Some insight on this result can be attained by noting that differentiating $\overline{\pi}(\sigma_0) = \pi(\overline{k}(\sigma_0), \sigma_0)$ with respect to σ_0 we have that

$$\frac{d\overline{\pi}}{d\sigma_0}(\sigma_0) = \frac{\partial\pi}{\partial\sigma_0}\left(\overline{k}(\sigma_0), \sigma_0\right) + \frac{\partial\pi}{\partial k}\left(\overline{k}(\sigma_0), \sigma_0\right)\frac{d\overline{k}}{d\sigma_0}(\sigma_0). \tag{29}$$

Therefore, the effect of the elasticity of substitution on the steady-state physical capital income share can be decomposed as the sum of the (direct) distribution effect, $\partial\pi/\partial\sigma_0$ (Irmen and Klump 2009), and the effect of a change in k as a consequence of the change in the elasticity of substitution, $(\partial\pi/\partial k)(dk/d\sigma_0)$. The distribution effect depends on the relative position of the baseline per capita capital with respect to its stationary value, as shown by Eq. (A.8) for the VES case, Eq. (B. 5) for the Sobelow case, and Eq. (C.4) for the Sigmoidal case. The sign of the indirect effect depends on i) the effect of σ_0 on the steady-state per capita capital, $d\overline{k}/d\sigma_0$, which is positive if the baseline per capita capital is below its steady-state value, as shown by Eq. (A.4) for the VES case, Eq. (B.6) for the Sobelow case and Eq. (C.5) for the Sigmoidal case and ii) the direct effect of capital on the capital income share, $\partial\pi/\partial k$, which is positive in the VES and Sobelow cases, as shown by Eqs. (A.7) and (B.4), respectively, and negative in the Sigmoidal case, as Eq. (C.3) shows. In any case, the direct distribution effect dominates the indirect effect, so the overall effect depends only on the relative position of the baseline ratio of per capita capital with respect to its stationary value, as shown by Eq. (A.9) for the VES case, Eq. (B.8) for the Sobelow case, and Eq. (C.6) for the Sigmoidal case.

4 Numerical results

Finally, we perform some numerical exercises to study the effect of the elasticity of substitution on economic performance. To this end we have to calibrate the model. Our strategy will be as follows. We consider the model with CES production as the baseline model. For given baseline values of capital per capita, k_0, income per capita, $y_0 = f(k_0, \sigma)$, and the capital income share $\pi_0 = k_0 \frac{\partial f}{\partial k}(k_0, \sigma)/f(k_0, \sigma)$, the normalized CES production function in intensive form is

$$y = f(k, \sigma) = A(\sigma)\left[\alpha(\sigma)k^{\frac{\sigma-1}{\sigma}} + (1 - \alpha(\sigma))\right]^{\frac{\sigma}{\sigma-1}}. \tag{30}$$

Here, the productivity and distribution parameters, A and α, can be derived by solving the system

$$y_0 = f(k_0, \sigma) = A\left[\alpha k_0^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\right]^{\frac{\sigma}{\sigma-1}},$$

$$\pi_0 = \frac{k_0 \frac{\partial f}{\partial k}(k_0, \sigma)}{f(k_0, \sigma)} = \frac{\alpha k_0^{\frac{\sigma-1}{\sigma}}}{\alpha k_0^{\frac{\sigma-1}{\sigma}} + (1-\alpha)},$$

as

$$A(\sigma) = y_0\left[\pi_0 k_0^{\frac{1-\sigma}{\sigma}} + (1-\pi_0)\right]^{\frac{\sigma}{\sigma-1}}, \tag{31}$$

$$\alpha(\sigma) = \frac{\pi_0 k_0^{\frac{1-\sigma}{\sigma}}}{\pi_0 k_0^{\frac{1-\sigma}{\sigma}} + (1-\pi_0)}. \tag{32}$$

Then, we calibrate the model to U.S. data following Gomme and Lkhagvassure (2015) (see also Gomme and Rupert 2007). Thus, we consider that the steady-state values of the capital's share of output, $\overline{\pi}$, is 0.2852; the depreciation rate, δ, is 0.0718; the relative risk aversion, θ, is 2; the (pre-tax) return to capital, $\overline{r}$, is 0.0941, and we assume that there is no population growth, $n = 0$. Data to match by the model and predetermined parameter values are displayed in Table 1. With these values we solve the system:

$$\frac{\overline{k}f'\left(\overline{k}\right)}{f\left(\overline{k}\right)} = \overline{\pi},$$
$$f'\left(\overline{k}\right) = \overline{r},$$
$$f\left(\overline{k}\right) = \overline{c} + (n+\delta)\overline{k},$$
$$f'\left(\overline{k}\right) = n + \delta + \rho.$$

Instead of normalizing the productivity parameter to unity, $A = 1$, we normalize the steady-state value of per capita capital to unity, $\overline{k} = 1$. In this way we get that the parameter a is equal to the capital's share of output, $\alpha = \overline{\pi} = 0.2852$, the productivity parameter is $A = 0.3299$ and the rate of time preference is $\rho = 0.0223$. Thus, the steady state is $\overline{k} = 1$, $\overline{c} = 0.2581$ and per capita income is $\overline{y} = f\left(\overline{k}\right) = 0.3299$. Now, we assume that the baseline ratio of per capita capital is 75% of its steady state value; i.e., $k_0 = 0.75$. With the previously computed parameter values, we get the baseline values of per capita income, $y_0 = 0.3045$, and the capital's share of output, $\pi_0 = 0.2736$.

There is no consensus in the literature about the value of the elasticity of substitution between capital and labor. In his survey, Chirinko (2008) concludes that prior evidence suggests a value between 0.40 and 0.60. An estimate below one is also obtained in more recent works by, e.g., León-Ledesma, McAdam, and Willman (2010) and León-Ledesma, McAdam, and Willman (2015). Combining a low-pass filter with panel data of US industries, Chirinko and Mallick (2017) obtain a preferred estimate of 0.40. Using a panel of 82 countries, Duffy and Papageorgiou (2000) find that the elasticity of substitution is greater than one in the subsample of well developed countries and less than one in the subsample of less developed countries. Piketty (2014), Piketty and Saez (2014), and Piketty and Zucman (2014) argue that a higher-than-one elasticity of substitution would be the case for actual rich hightech economies. Using cross-country data, Karabarbounis and Neiman (2014) report a preferred estimate of the elasticity of substitution of around 1.25. Given these controversial results, we consider two scenarios: a low-elasticity case, in which the (initial) elasticity of substitution is $\sigma_0 = 0.4$ (Chirinko and Mallick 2017), and a high-elasticity case, in which the (initial) elasticity of substitution is $\sigma_0 = 1.25$ (Karabarbounis and Neiman 2014). With the baseline values and the initial value of the elasticity of factor substitution corresponding to each scenario, we can compute the parameter values corresponding to each of the

Table 1: Data to match by the model and predetermined parameter values.

$\overline{r}$	$\overline{\pi}$	$\overline{k}$	$\boldsymbol{\delta}$	$\boldsymbol{\theta}$	σ_0	$\boldsymbol{n}$
0.0941	0.2852	1	0.0718	2	1.25, 0.4	0

Table 2: Parameter and steady-state values.

High-elasticity case, $\sigma_0 = 1.25$								
CES:	A		α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	0.3299		0.2852	0.0223	1	0.2581	0.3299	1.25
VES:	A	B	α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	0.3117	0.0772	0.2316	0.0223	1.0097	0.2584	0.3309	1.3366
Sobelow:	A	B	α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	0.0284	0.3015	0.2189	0.0223	1.0067	0.2583	0.3306	1.3090
Low-elasticity case, $\sigma_0 = 0.4$								
CES:	A		α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	0.3299		0.2852	0.0223	1	0.2581	0.3299	0.4
VES:	A	B	α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	0.4266	−0.4845	0.6056	0.0223	0.9426	0.2559	0.3236	0.2459
Sigmoidal:	A	B	α	ρ	$\bar{k}$	$\bar{c}$	$\bar{y}$	$\bar{\sigma}$
	1.5046	3.5592	1.3097	0.0223	1.0043	0.2583	0.3304	0.4109

production functions—VES and Sobelow in the high-elasticity case, and VES and Sigmoidal in the low-elasticity case—reported in Table 2.

In the following, to simplify the exposition we will interpret our simulations as comparing an economy that suffers a shock that increases the elasticity of substitution by 0.25 with the case in which there is no shock.

Table 3: Effect of a shock that increases the elasticity of substitution by 0.25 in the high-elasticity case, $\sigma_0 = 1.25$.

A. Quantities (%Δ)

	Impact		After 10 years		Intertemporal	
	Capital	Consum.	Capital	Consum.	Capital	Consum.
CES:	0	−0.6489	1.9033	−0.4316	6.2280	0.7136
VES:	0	−0.7271	2.1591	−0.5116	7.8567	0.8900
Sobelow:	0	−0.7030	2.0802	−0.4866	7.3187	0.8321

B. Growth rates and ratios (percentage point change)

	Impact			After 10 years			Intertemporal		
	g_C	g_K	C/Y	g_C	g_K	C/Y	g_C	g_K	C/Y
CES:	0	0.2013	−0.6489	0.0330	0.1692	−0.9982	0	0	−1.1775
VES:	0	0.2254	−0.7271	0.0340	0.1958	−1.1521	0	0	−1.4902
Sobelow:	0	0.2180	−0.7030	0.0337	0.1876	−1.1044	0	0	−1.3870

C. Elasticity of substitution (σ)

	Impact	After 10 years	Intertemporal
CES:	1.50	1.50	1.50
VES:	1.50	1.5997	1.7260
Sobelow:	1.50	1.5694	1.6225

D. Welfare evaluation (%Δ)

	Impact	After 10 years	Intertemporal
CES:	−0.0065	−0.5749	0.1143
VES:	−0.0073	−0.6560	0.1301
Sobelow:	−0.0071	−0.6309	0.1252

Table 4: Effect of a shock that increases the initial elasticity of substitution by 0.25 in the low elasticity case, $\sigma_0 = 0.4$.

A. Quantities (%Δ)

	Impact		After 10 years		Intertemporal	
	Capital	Consum.	Capital	Consum.	Capital	Consum.
CES:	0	−3.4800	9.4997	−1.0718	21.3448	3.7848
VES:	0	−3.6504	10.2587	−1.0331	20.2543	3.5197
Sigmoidal:	0	−3.5342	9.0606	−0.9788	19.9987	3.5952

B. Growth rates and ratios (percentagepoint change)

	Impact			After 10 years			Intertemporal		
	g_C	g_K	C/Y	g_C	g_K	C/Y	g_C	g_K	C/Y
CES:	0	0.9938	−3.4799	0.3008	0.7369	−4.6412	0	0	−3.5512
VES:	0	1.0516	−3.6504	0.3445	0.7803	−4.7716	0	0	−3.2707
Sigmoidal:	0	0.9573	−3.3542	0.2906	0.6969	−4.4074	0	0	−3.3401

C. Elasticity of substitution (σ)

	Impact	After 10 years	Intertemporal
CES:	0.65	0.65	0.65
VES:	0.65	0.5331	0.4710
Sigmoidal:	0.65	0.6386	0.6332

D. Welfare evaluation (%Δ)

	Impact	After 10 years	Intertemporal
CES:	−0.0361	−2.6344	1.4361
VES:	−0.0379	−2.7679	1.4123
Sigmoidal:	−0.0347	−2.5065	1.3830

However, these results can be equivalently interpreted as comparing two otherwise identical economies except in their initial elasticities of substitution, one higher by 0.25 than the other.

Tables 3 and 4 illustrate the effect of a shock that increases the initial elasticity of substitution by 0.25, so it changes from 1.25 to 1.50 in the high-elasticity case, and from 0.4 to 0.65 in the low-elasticity case. The effects are calculated by comparing the value of the variable when there is no shock with the corresponding value if the shock happens. For example, if $x^{\text{new}}(t)$ is the value of the variable x at time t in the economy that suffers the shock that increases the initial elasticity of substitution by 0.25, and $x^{\text{old}}(t)$ is the value of the variable x at time t if there is no shock, the percent variation at time t would be $(x^{\text{new}}(t) - x^{\text{old}}(t))/x^{\text{old}}(t)$.

In the high-elasticity case, Table 3 shows that consumption falls on impact relative to the non-shock case for all the models. The higher initial drop happens in the VES economy (0.73%) and the lower one in the CES economy (0.65%). The fall in consumption relative to the non-shock case becomes lower after 10 years and, eventually, long-run consumption is higher than that in the non-shock case for all the specifications of the production function. The higher increase in long-run consumption happens in the VES economy (0.89%) and the lower one in the CES economy (0.71%). Per capita capital increases significantly relative to the non-shock case both after 10 years (between 1.90 and 2.16%) and in the long run (between 6.23 and 7.86%) for all the models. The increase in noticeably higher in the models with variable (increasing) elasticity of substitution than in the CES model and, among them, the highest increase happens in the VES economy. This behavior can also be observed in the percentage point increases in the growth rate of consumption after 10 years, and in the growth rate of capital both on impact and after 10 years. These results reflect the fact that as the economy evolves, the elasticity of substitution increases from 1.5 to 1.73 in the VES economy and to 1.65 in the Sobelow economy. Intuitively, a shock that increases the easiness to substitute labor for capital encourages investment in capital and, consequently, a drop in consumption at early stages of development. Eventually, the increase in capital and output allows for a higher consumption relative to the non-shock case. The initial drop in consumption and the increase in capital, and so, in income, results in a lower consumption-output ratio along the transition, and eventually also in the long run, relative to the non-shock case.

The fall in consumption at earlier stages after the shock causes welfare to fall both on impact and after 10 years relative to the non-shock case. The highest drop happens in the VES economy and the lowest one in the CES economy. Eventually, the increase in consumption relative to the non-shock case leads to a gain in intertemporal welfare for all the specifications of the production function. Actually, in the CES economy it takes 80.05 years for intertemporal welfare becoming higher than the one in the non-shock model. The corresponding figures in the VES and Sobelow economies are 83.88 and 82.67 years, respectively. In the long-run, overall welfare increases in all the economies and the highest welfare gain happens in the VES economy (0.130%), followed by the Sobelow economy (0.125%) and lastly the CES economy (0.114%).

In the low-elasticity case, Table 4 shows that on impact consumption falls relative to the non-shock case in all the models. The highest initial drop happens in the VES economy (3.65%), and the lowest one in the Sigmoidal economy (3.35%). The fall in consumption relative to the non-shock case becomes lower after 10 years and, eventually, in the long-run consumption exceeds the corresponding one in the non-shock case for all the specifications. The highest increase in long-run consumption occurs in the CES economy (3.78%) and the lowest one in the VES economy (3.52%). In parallel, capital increases noticeably relative to the non-shock case both after 10 years (between 9.06 and 10.26%) and in the long run (between 20 and 21.34%) for all the models. This is also reflected in the percentage point increase in the growth rate of capital both on impact and after 10 years. The effect of the shock on the consumption-output capital ratio reflects the formerly described behavior of consumption and capital and, therefore, output. It is interesting to note that the increase in per capita capital—and consumption—as a consequence of the shock in the low-elasticity case is significantly higher than the one found in the high-elasticity case. Intuitively, the lower the initial elasticity of substitution (0.4 vs. 1.25) the higher the effect of easing the substitutability between inputs.

As it happened with consumption, the increase in long-run capital is higher in the CES model than the corresponding one in the models with variable elasticity of substitution; in particular, the VES model. So, the ranking is reversed relative to the high-elasticity case. As the economy develops with an increasing per capita capital, in the high-elasticity case—where the initial elasticity of substitution is above unity—the long-run elasticity of substitution increases in the VES and Sobelow economies relative to its initial value—to 1.73 and 1.65, respectively—whereas it remains constant in the CES economy. The higher easiness to substitute for inputs entails a higher steady-state per capita capital, income and consumption. However, in the low-elasticity case—where the initial elasticity of substitution is below unity—as the economy evolves with an increasing per capita capital, the long-run elasticity of substitution in the VES and Sigmoidal models falls relative to its initial value. Thus, the elasticity of substitution falls from 0.65 to 0.47 in the VES economy and to 0.63 in the Sigmoidal economy.[5] So, substitutability between inputs becomes harder and, as a consequence, the increase in per capita capital, income and consumption relative to the non-shock model is lower than that attainable in the CES economy with a constant elasticity of substitution.

As a consequence of the initial drop in consumption, welfare falls both on impact and after 10 years relative to the non-shock case. The highest drop happens in the VES economy, and the lowest one in the Sigmoidal economy. Eventually, the increase in consumption relative to the non-shock case leads to a gain in intertemporal welfare. Actually, in the CES economy it takes 39.53 years for intertemporal welfare becoming higher than the one in the non-shock model. The corresponding figures in the VES and Sigmoidal economies are 37.96 and 38.98 years, respectively. In the long-run, overall welfare increases in all the models and the highest welfare gain happens in the CES economy (1.44%), followed by the VES economy (1.41%) and lastly the Sigmoidal economy (1.38%). Interestingly, even though long-run consumption is higher in the Sigmoidal

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5 In contrast, it should be noted that in the non-shock case with $\sigma_0 = 0.4$, the elasticity of substitution increases as k increases in the Sigmoidal economy (to 0.41 in the long run) because $\alpha(0.4) = 1.3097 > 1$. However, after a shock that increases the initial σ_0 to 0.65 then $\alpha(0.65) = 0.7141 < 1$, and so, the elasticity of substitution becomes decreasing as k increases.

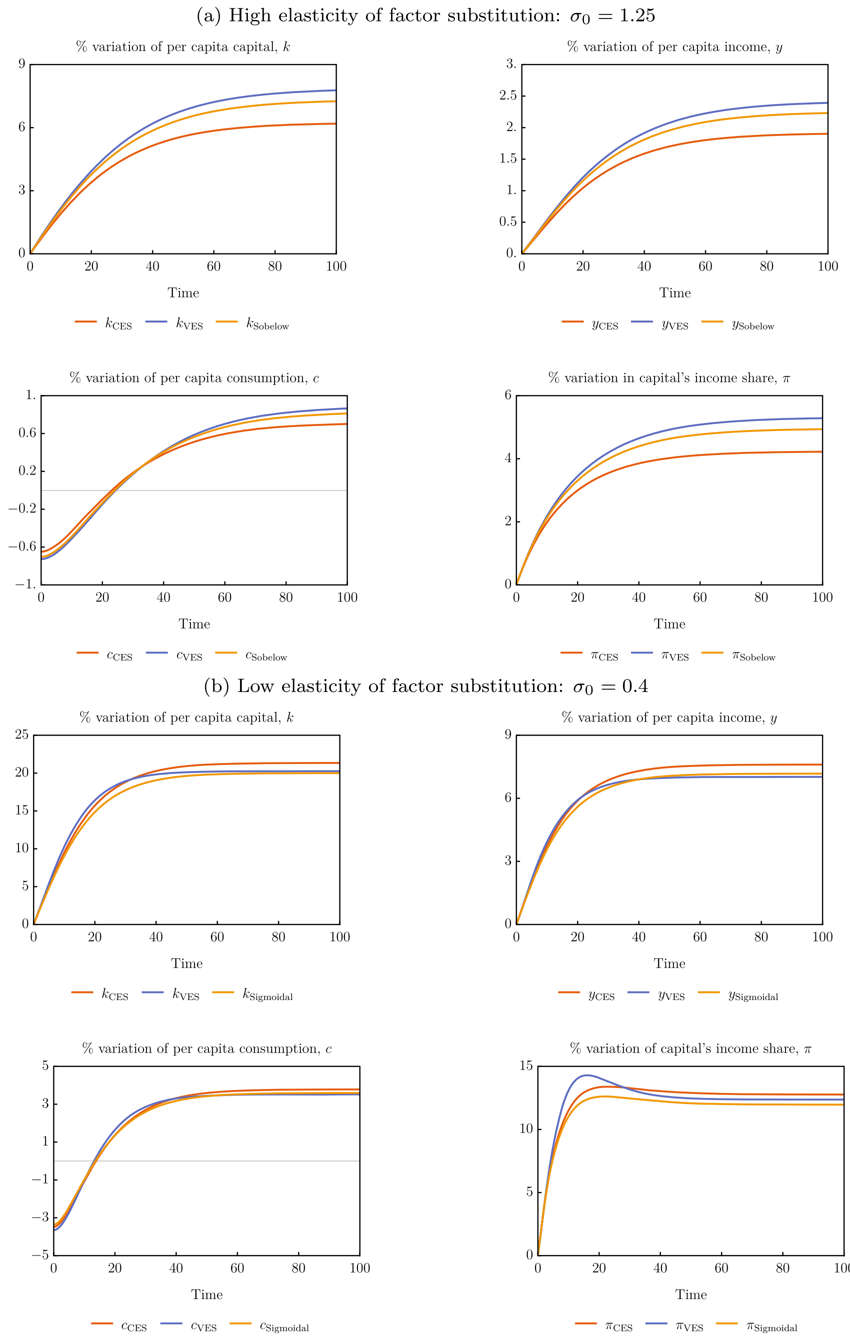

Figure 1: Dynamic effect of a shock that increases the initial elasticity of substitution by 0.25.

economy than in the VES economy, the welfare gain is greater in the VES economy due to its higher consumption along the transition to the steady state.

Figure 1 depicts the percentage variation of per capita capital, per capita income, per capita consumption and the capital's income share after an increase of 0.25 in the elasticity of factor substitution relative to the non-shock case. Thus, if -0.0379 is the value of the variable x at time t with the old elasticity of substitution and -2.7679 is the value of the variable x at time t with the new elasticity of substitution, the percent variation at time t would be -0.0347. Figure 1a illustrates their evolution in the CES, VES and Sobelow models in the high-elasticity case,[6] and Figure 1b illustrates their evolution in the CES, VES and Sigmoidal models in the low-elasticity case.[7]

In the high-elasticity case, Figure 1a confirms the results obtained in Table 3. The highest increment of per capita capital, per capita income and the capital's income share relative to the non-shock model happens in the VES economy not only in the long run but also along the transition, whereas the lowest one happens in the CES model. The highest relative fall in consumption on impact occurs in the VES economy, and the lowest one in the CES economy. Eventually, the higher capital and output allows consumption to overtake its level in the non-shock case.

Figure 1b complements the results contained in Table 4. The highest percent increase in capital and output per capita happens in the VES economy and the lowest one in the CES economy at early stages after the shock that increases the initial elasticity of substitution. Eventually, the CES economy overtakes the VES economy, which even ends with the lowest increase in long-run output relative to the non-shock case. Whereas the highest relative fall in consumption on impact occurs in the VES economy, consumption eventually recovers and the relative gain becomes the highest one to, finally, end being the lowest one in the long run. The evolution of the relative increase in the capital's income share shows a similar behavior. It can be noted that, according with our previous analytical results, the steady-state values of per capita capital, income and capital's income share are higher after an increase in the elasticity of substitution, both in the low- and the high-elasticity cases.

5 Conclusions

This paper has analyzed the link between the elasticity of substitution, economic growth and factor income distribution. The previous literature has studied this issue in growth models with CES technology, so that the elasticity of substitution is constant. In contrast, we consider the neoclassical model with three different technologies with time-varying elasticity of substitution: VES, Sobelow and Sigmoidal. We have shown that for two economies differing only in factor substitutability, if the initial per capita capital is below its steady-state value, the economy with the higher initial elasticity of substitution will have a higher steady-state per capita output, capital and capital's income share. This last result should be reversed if the initial per capita capital is above its steady-state value. Thus, the findings of Klump and de La Grandville (2000) for an economy with CES technology can be extended to an economy with VES, Sobelow or Sigmoidal technology. This shows the robustness of this result to the choice of the technology.

Our numerical simulations confirm the former theoretical results. Furthermore, if we compare two economies with elasticities of substitution above unity, the highest increase in long-run per capita capital, output, consumption and intertemporal welfare happens in the VES economy, and the lowest one happens in the CES economy. The reason is that the elasticity of substitution increases as the economy develops with an increasing per capita capital in the VES and Sobelow economies when the initial elasticity is above unity. A higher easiness to substitute labor for capital fosters investment in capital which eventually leads to more output and consumption. In contrast, the increase in long-run per capita capital, output, consumption, and intertemporal welfare are higher in the CES economy than the corresponding ones in the VES economy if the initial elasticity is below unity. This reflects that in this case the elasticity of substitution decreases as the economy develops in the VES economy.

Our results for the VES, Sobelow and Sigmoidal technologies allow us to conjecture that for an economy starting with a per capita capital below its stationary value, the steady-state income and capital per capita are increasing in the (initial) elasticity of substitution for any general neoclassical function. This will be the subject of future research.

6 In the Sigmoidal model the elasticity of substitution is below unity.

7 In the Sobelow model the elasticity of substitution is above unity.

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Appendix A: Proof of Proposition 1

Differentiating Eq. (12) with respect to −2.5065 we get that y is a an increasing function of the initial elasticity of substitution $x^{\text{old}}(t)$,

$$\frac{\partial y}{\partial \sigma_0}=\frac{\partial f}{\partial \sigma_0}\,(k,\sigma_0)=-\frac{(1-\pi_0)\pi_0}{[\pi_0+\sigma_0(1-\pi_0)]^2}f(k,\sigma_0)\left[1-\frac{(1-\pi)k}{(1-\pi_0)k_0}+\ln\frac{(1-\pi)k}{(1-\pi_0)k_0}\right]\geq 0, \tag{A.1}$$

with equality if and only if $k=k_0$. To derive the former inequality we have used that the logarithmic function is strictly concave and, therefore, $\ln x\leq x-1$, with equality if and only if $x=1$. Differentiating Eq. (12) with respect to k we have that

$$\frac{\partial f}{\partial k}\,(k,\sigma_0)=A(\sigma_0)k^{\alpha(\sigma_0)-1}[B(\sigma_0)k+1]^{-\alpha(\sigma_0)}[B(\sigma_0)k+\alpha(\sigma_0)]>0. \tag{A.2}$$

Differentiating Eq. (A.2) with respect to $k=k_0$, after simplification, we get that

$$\frac{\partial^2 f}{\partial k\,\partial\sigma_0}\,(k,\sigma_0)=\frac{(1-\pi_0)f(k,\sigma_0)}{\pi_0\alpha(\sigma_0)^2[1+B(\sigma_0)k]k}\times\left\{[1+B(\sigma_0)k+\alpha(\sigma_0)]\left[\frac{(1-\pi)k}{(1-\pi_0)k_0}-1\right]-[B(\sigma_0)k+\alpha(\sigma_0)]\ln\frac{(1-\pi)k}{(1-\pi_0)k_0}\right\}>0, \tag{A.3}$$

if x = 1. To derive the last inequality in Eq. (A.3), let us define

$$h(k)=\frac{(1-\pi)k}{(1-\pi_0)k_0}=\frac{[1-\alpha(\sigma_0)]k}{(1-\pi_0)k_0[B(\sigma_0)k+1]}.$$

The function h is strictly increasing because its derivative is strictly positive,

$$h'(k)=\frac{[1-\alpha(\sigma_0)]}{(1-\pi_0)k_0[B(\sigma_0)k+1]^2}>0,$$

and satisfies that $h(k_0)=1$. Hence, we have that $h(k)>1$ if and only if $k>k_0$. Taking into account this feature and using that the logarithmic function is strictly concave and, therefore, $\ln x\leq x-1$, we have that

$$[1+B(\sigma_0)k+\alpha(\sigma_0)]\left[\frac{(1-\pi)k}{(1-\pi_0)k_0}-1\right]-[B(\sigma_0)k+\alpha(\sigma_0)]\ln\frac{(1-\pi)k}{(1-\pi_0)k_0}\geq\frac{(1-\pi)k}{(1-\pi_0)k_0}-1=\frac{k-k_0}{[1+B(\sigma_0)k]k_0}>0,$$

if $k>k_0$, and the inequality in Eq. (A.3) follows. Differentiating Eq. (A.2) with respect to k, after simplification, we get that

$$\frac{\partial^2 f}{\partial k^2}\,(k,\sigma_0)=-[1-\alpha(\sigma_0)]\alpha(\sigma_0)A(\sigma_0)k^{\alpha(\sigma_0)-2}[B(\sigma_0)k+1]^{-\alpha(\sigma_0)-1}<0.$$

We are now ready to examine the effect of the elasticity of substitution on per capita capital and income. Taking into account the former results, differentiating Eq. (8) with respect to σ_0 we have that

$$\frac{d\bar{k}}{d\sigma_0}(\sigma_0) = -\frac{\frac{\partial^2 f}{\partial k \partial \sigma_0}\left(\bar{k}(\sigma_0), \sigma_0\right)}{\frac{\partial^2 f}{\partial k^2}\left(\bar{k}(\sigma_0), \sigma_0\right)} > 0, \tag{A.4}$$

if $\bar{k}(\sigma_0) > k_0$[8] The effect of the initial elasticity of substitution on per capita income is given by

$$\frac{d\bar{y}}{d\sigma_0}(\sigma_0) = \frac{\partial f}{\partial k}\left(\bar{k}(\sigma_0), \sigma_0\right)\frac{d\bar{k}}{d\sigma_0}(\sigma_0) + \frac{\partial f}{\partial \sigma_0}\left(\bar{k}(\sigma_0), \sigma_0\right) > 0, \tag{A.5}$$

if $\bar{k}(\sigma_0) > k_0$.

Let us now consider the effect of the elasticity of substitution on the capital income share

$$\pi = \pi(k, \sigma_0) = \frac{k\,\frac{\partial f}{\partial k}(k, \sigma_0)}{f(k, \sigma_0)} = \frac{B(\sigma_0)k + \alpha(\sigma_0)}{B(\sigma_0)k + 1} = \frac{\pi_0[k_0 + k(\sigma_0 - 1)]}{\pi_0(\sigma_0 - 1)k + [\pi_0 + \sigma_0(1 - \pi_0)]k_0}. \tag{A.6}$$

Differentiating this expression, with respect to k and σ_0, after simplification we get

$$\frac{\partial \pi}{\partial k}(k, \sigma_0) = \frac{[1 - \alpha(\sigma_0)]B(\sigma_0)}{[B(\sigma_0)k + 1]^2} > 0, \tag{A.7}$$

$$\frac{\partial \pi}{\partial \sigma_0}(k, \sigma_0) = \frac{[1 - \alpha(\sigma_0)]\alpha(\sigma_0)^2}{[B(\sigma_0)k + 1]^2[B(\sigma_0)k_0 + \alpha(\sigma_0)]k_0}(k - k_0)\begin{cases} > 0, & \text{if } k > k_0, \\ = 0, & \text{if } k = k_0, \\ < 0, & \text{if } k < k_0. \end{cases} \tag{A.8}$$

Differentiating Eq. (A.6) we get that

$$\begin{aligned}\frac{d\bar{\pi}}{d\sigma_0}(\sigma_0) &= \frac{\partial \bar{\pi}}{\partial \sigma_0}\left(\bar{k}(\sigma_0), \sigma_0\right) + \frac{\partial \bar{\pi}}{\partial k}\left(\bar{k}(\sigma_0), \sigma_0\right)\frac{d\bar{k}}{d\sigma_0}(\sigma_0) \\ &= \frac{[1 - \alpha(\sigma_0)]\alpha(\sigma_0)\left[B(\sigma_0)\bar{k}(\sigma_0) + \alpha(\sigma_0)\right]}{\left[B(\sigma_0)\bar{k}(\sigma_0) + 1\right][B(\sigma_0)k_0 + \alpha(\sigma_0)]}\left\{\frac{\bar{k}(\sigma_0)}{k_0}\right. \\ &\quad\left. -1 - B(\sigma_0)\bar{k}(\sigma_0)\ln\left[\frac{(B(\sigma_0)k_0 + 1)\bar{k}(\sigma_0)}{\left(B(\sigma_0)\bar{k}(\sigma_0) + 1\right)k_0}\right]\right\}.\end{aligned}$$

On the one hand, using that the logarithmic function is strictly concave and, therefore, $\ln x \le x - 1$, we have that

$$\begin{aligned}\frac{d\bar{\pi}}{d\sigma_0}(\sigma_0) &\ge \frac{[1 - \alpha(\sigma_0)]\alpha(\sigma_0)\left[B(\sigma_0)\bar{k}(\sigma_0) + \alpha(\sigma_0)\right]}{\left[B(\sigma_0)\bar{k}(\sigma_0) + 1\right][B(\sigma_0)k_0 + \alpha(\sigma_0)]} \\ &\quad\times\left\{\frac{\bar{k}(\sigma_0)}{k_0} - 1 - B(\sigma_0)\bar{k}(\sigma_0)\left[\frac{(B(\sigma_0)k_0 + 1)\bar{k}(\sigma_0)}{\left(B(\sigma_0)\bar{k}(\sigma_0) + 1\right)k_0} - 1\right]\right\} \\ &= \frac{[1 - \alpha(\sigma_0)]\alpha(\sigma_0)\left[B(\sigma_0)\bar{k}(\sigma_0) + \alpha(\sigma_0)\right]}{\left[B(\sigma_0)\bar{k}(\sigma_0) + 1\right]^2 k_0[B(\sigma_0)k_0 + \alpha(\sigma_0)]}\left[\bar{k}(\sigma_0) - k_0\right] > 0,\end{aligned}$$

if $\bar{k} = \bar{k}(\sigma_0) > k_0$. On the other hand, using that the function $x \ln x$ is strictly convex and, therefore, $x \ln x \ge x - 1$, we have that

$$\frac{d\bar{\pi}}{d\sigma_0}(\sigma_0) \le \frac{[1 - \alpha(\sigma_0)]\alpha(\sigma_0)\left[B(\sigma_0)\bar{k}(\sigma_0) + \alpha(\sigma_0)\right]}{\left[B(\sigma_0)\bar{k}(\sigma_0) + 1\right][B(\sigma_0)k_0 + \alpha(\sigma_0)]}$$

8 Numerical results show that the derivative can be positive or negative if $\bar{k}(\sigma_0) < k_0$, depending on the values of the parameters.

$$\times\left\{\frac{\bar{k}\,(\sigma_0)}{k_0}-1-\frac{B\,(\sigma_0)k_0\left(B\bar{k}+1\right)}{[B\,(\sigma_0)k_0+1]}\left[\frac{(B\,(\sigma_0)k_0+1)\bar{k}\,(\sigma_0)}{\left[B\,(\sigma_0)\bar{k}\,(\sigma_0)+1\right]k_0}-1\right]\right\}$$
$$=\frac{[1-\alpha\,(\sigma_0)]\alpha\,(\sigma_0)\left[B\,(\sigma_0)\bar{k}\,(\sigma_0)+\alpha\,(\sigma_0)\right]}{\left[B\,(\sigma_0)\bar{k}\,(\sigma_0)+1\right][B\,(\sigma_0)k_0+\alpha\,(\sigma_0)]\,[B\,(\sigma_0)k_0+1]k_0}\left[\bar{k}\,(\sigma_0)-k_0\right]<0,$$

if $\bar{k}=\bar{k}(\sigma_0)<k_0$. In summary, we get that

$$\frac{d\bar{\pi}}{d\sigma_0}\,(\sigma_0)\begin{cases}>0, & \text{if } \bar{k}\,(\sigma_0)>k_0,\\ =0, & \text{if } \bar{k}\,(\sigma_0)=k_0,\\ <0, & \text{if } \bar{k}\,(\sigma_0)<k_0.\end{cases} \tag{A.9}$$

Appendix B: Proof of Proposition 2

Differentiating Eq. (17) with respect to σ_0 we get that y is a an increasing function of the initial elasticity of substitution σ_0,

$$\frac{\partial y}{\partial\sigma_0}=\frac{\partial f}{\partial\sigma_0}\,(k,\sigma_0)=-\frac{(1-\pi_0)\pi_0 y_0}{(\sigma_0-\pi_0)^2}\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\left[1-\left(\frac{k}{k_0}\right)^{1-\alpha(\sigma_0)}+\ln\left(\frac{k}{k_0}\right)^{1-\alpha(\sigma_0)}\right]\geq 0, \tag{B.1}$$

with equality if and only if $k=k_0$, where we have used that the logarithmic function is strictly concave and, therefore, $\ln x\leq x-1$, with equality if and only if $x=1$. Differentiating Eq. (17) with respect to k we have that

$$\frac{\partial f}{\partial k}\,(k,\sigma_0)=A\,(\sigma_0)+\alpha\,(\sigma_0)B\,(\sigma_0)k^{\alpha(\sigma_0)-1}>0. \tag{B.2}$$

Differentiating Eq. (B.1) with respect to k we get that

$$\frac{\partial^2 f}{\partial k\,\partial\,\sigma_0}\,(k,\sigma_0)=\frac{(1-\pi_0)\pi_0 y_0}{k_0\,(\sigma_0-\pi_0)^2}\left[\frac{k}{k_0}\right]^{\frac{\pi_0-\sigma_0}{\sigma_0}}\left\{-1+\left[\frac{k_0}{k}\right]^{\frac{\pi_0-\sigma_0}{\sigma_0}}-\frac{\pi_0}{\sigma_0}\ln\left[\frac{k_0}{k}\right]^{\frac{\pi_0-\sigma_0}{\sigma_0}}\right\}>0,$$

if $k>k_0$. Here we have used that $\ln x\leq x-1$ entails that

$$\frac{\partial^2 f}{\partial k\,\partial\,\sigma_0}\,(k,\sigma_0)\geq\frac{(1-\pi_0)\pi_0 y_0}{k_0\,(\sigma_0-\pi_0)^2}\left[\frac{k_0}{k}\right]^{\frac{\pi_0-\sigma_0}{\sigma_0}}\frac{(\sigma_0-\pi_0)}{\sigma_0}\left\{\left[\frac{k_0}{k}\right]^{\frac{\pi_0-\sigma_0}{\sigma_0}}-1\right\}>0,$$

if $k>k_0$ (note that $\sigma_0>1>\pi_0$). Differentiating Eq. (B.2) with respect to k we get that

$$\frac{\partial^2 f}{\partial k^2}\,(k,\sigma_0)=-[1-\alpha\,(\sigma_0)]\alpha\,(\sigma_0)B\,(\sigma_0)k^{\alpha-2}<0.$$

The capital income share is

$$\pi=\pi\,(k,\sigma_0)=\frac{k\,\frac{\partial f}{\partial k}\,(k,\sigma_0)}{f\,(k,\sigma_0)}=\frac{\left(\frac{k}{k_0}\right)^{1-\pi_0/\sigma_0}(\sigma_0-1)\pi_0+(1-\pi_0)\pi_0}{\left(\frac{k}{k_0}\right)^{1-\pi_0/\sigma_0}(\sigma_0-1)\pi_0+(1-\pi_0)\sigma_0}<1. \tag{B.3}$$

Differentiating this expression with respect to k and σ_0, after simplification we get

$$\frac{\partial\pi}{\partial k}\,(k,\sigma_0)=\frac{[1-\alpha\,(\sigma_0)]^2A\,(\sigma_0)B\,(\sigma_0)k^{\alpha\,(\sigma_0)}}{[A\,(\sigma_0)k+B\,(\sigma_0)k^{\alpha\,(\sigma_0)}]^2}>0, \tag{B.4}$$

and

$$\frac{\partial \pi}{\partial \sigma_0}(k,\sigma_0)=\frac{\alpha(\sigma_0)^2 A(\sigma_0)B(\sigma_0)k^{\alpha(\sigma_0)+1}}{\pi_0 f(k,\sigma_0)^2}\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)-1}$$

$$\times\left\{\frac{\sigma_0(1-\pi_0)}{\pi_0(\sigma_0-1)}\left[\left(\frac{k}{k_0}\right)^{1-\alpha(\sigma_0)}-1\right]+\left(\frac{k}{k_0}\right)^{1-\alpha(\sigma_0)}\ln\left[\left(\frac{k}{k_0}\right)^{1-\alpha(\sigma_0)}\right]\right\}\begin{cases}>0, & \text{if } k>k_0,\\ =0, & \text{if } k=k_0,\\ <0, & \text{if } k<k_0.\end{cases} \tag{B.5}$$

Differentiating Eq. (8) with respect to σ_0 we have that

$$\frac{d\bar{k}}{d\sigma_0}(\sigma_0)=-\frac{\frac{\partial^2 f}{\partial k\partial\sigma_0}\left(\bar{k}(\sigma_0),\sigma_0\right)}{\frac{\partial^2 f}{\partial k^2}\left(\bar{k}(\sigma_0),\sigma_0\right)}>0, \tag{B.6}$$

if $\bar{k}(\sigma_0)>k_0$. The effect of the initial elasticity of substitution on per capita income is, therefore,

$$\frac{d\bar{y}}{d\sigma_0}(\sigma_0)=\frac{\partial f}{\partial k}\left(\bar{k}(\sigma_0),\sigma_0\right)\frac{d\bar{k}}{d\sigma_0}(\sigma_0)+\frac{\partial f}{\partial\sigma_0}\left(\bar{k}(\sigma_0),\sigma_0\right)>0, \tag{B.7}$$

if $\bar{k}(\sigma_0)>k_0$.

Differentiating Eq. (B.3) we get that

$$\begin{aligned}\frac{d\bar{\pi}}{d\sigma_0}(\sigma_0)&=\frac{\partial\pi}{\partial\sigma_0}\left(\bar{k}(\sigma_0),\sigma_0\right)+\frac{\partial\pi}{\partial k}\left(\bar{k}(\sigma_0),\sigma_0\right)\frac{d\bar{k}}{d\sigma_0}(\sigma_0)\\&=\frac{\alpha(\sigma_0)B(\sigma_0)\bar{\pi}(\sigma_0)\bar{k}(\sigma_0)^{\alpha(\sigma_0)}}{\pi_0 f\left(\bar{k}(\sigma_0),\sigma_0\right)}\left[\left(\frac{\bar{k}(\sigma_0)}{k_0}\right)^{1-\alpha(\sigma_0)}-1\right]\begin{cases}>0, & \text{if } \bar{k}(\sigma_0)>k_0,\\ =0, & \text{if } \bar{k}(\sigma_0)=k_0,\\ <0, & \text{if } \bar{k}(\sigma_0)<k_0.\end{cases}\end{aligned} \tag{B.8}$$

Appendix C: Proof of Proposition 3

We have that

$$\frac{\partial y}{\partial\sigma_0}=\frac{\partial f}{\partial\sigma_0}(k,\sigma_0)=\frac{(1-\pi_0)y_0\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}}{\pi_0\sigma_0^2[1+B(\sigma_0)k^{\alpha(\sigma_0)}]^2}\left\{-1+\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}-\ln\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\right]\right\}\geq 0,$$

with equality if and only if $k=k_0$.

After simplification, we have that

$$\frac{\partial^2 f}{\partial k^2}(k,\sigma_0)=-\frac{\pi(k,\sigma_0)^2 f(k,\sigma_0)[1-\alpha(\sigma_0)+(1+\alpha(\sigma_0))B(\sigma_0)k^{\alpha(\sigma_0)}]}{\alpha(\sigma_0)k^2},$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial k\,\partial\sigma_0}(k,\sigma_0)=&\frac{(1-\pi_0)y_0\pi(k,\sigma_0)^3}{\alpha(\sigma_0)^2\sigma_0^2\pi_0 k_0}\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)-1}\times\left\{\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\left(\frac{\alpha(\sigma_0)-\pi_0}{\pi_0}\right)-1\right]\ln\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\right]\right.\\&\left.+2\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}-1\right]\right\}.\end{aligned}$$

Using that the logarithmic function is strictly concave and, therefore, $\ln x\leq x-1$, we have that

$$\frac{\partial^2 f}{\partial k\,\partial\sigma_0}(k,\sigma_0)\geq\frac{(1-\pi_0)y_0\pi(k,\sigma_0)^3}{\alpha(\sigma_0)^2\sigma_0^2\pi_0 k_0}\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)-1}\times\left\{\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\left(\frac{\alpha(\sigma_0)-\pi_0}{\pi_0}\right)+1\right]\ln\left[\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)}\right]\right\}\geq 0, \tag{C.1}$$

if $k\geq k_0$, with equality if and only if $k=k_0$. Here, we have used that $\alpha(\sigma_0)-\pi_0=(1-\pi_0)(1-\sigma_0)/\sigma_0>0$.

Let π denote the capital income share,

$$\pi = \pi\,(k, \sigma_0) = \frac{k\,\frac{\partial f}{\partial k}\,(k, \sigma_0)}{f\,(k, \sigma_0)} = \frac{\alpha\,(\sigma_0)}{1 + B\,(\sigma_0) k^{\alpha\,(\sigma_0)}}. \tag{C.2}$$

We have that

$$\frac{\partial \pi}{\partial k}\,(k, \sigma_0) = -\frac{\alpha\,(\sigma_0)^2 B\,(\sigma_0) k^{\alpha\,(\sigma_0)-1}}{[1 + B\,(\sigma_0) k^{\alpha\,(\sigma_0)}]^2} < 0. \tag{C.3}$$

and, after simplification,

$$\frac{\partial \pi}{\partial \sigma_0}\,(k, \sigma_0) = \frac{(1-\pi_0)}{\sigma_0^2\,[1 + B\,(\sigma_0) k^{\alpha(\sigma_0)}]^2} \times \left[\frac{\alpha\,(\sigma_0) - \pi_0}{\pi_0} \left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)} \ln\left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)} + \left(\frac{k}{k_0}\right)^{\alpha(\sigma_0)} - 1\right] \begin{cases} >0, & \text{if } \bar{k}\,(\sigma_0) > k_0, \\ = 0, & \text{if } \bar{k}\,(\sigma_0) = k_0, \\ <0, & \text{if } \bar{k}\,(\sigma_0) < k_0. \end{cases} \tag{C.4}$$

Assuming that the steady state is saddle-path stable, so that $\frac{\partial^2 f}{\partial k^2}\,(\bar{k}(\sigma_0), \sigma_0) < 0$, differentiating Eq. (8) with respect to σ_0 we have that

$$\frac{d\bar{k}}{d\sigma_0}\,(\sigma_0) = -\frac{\frac{\partial^2 f}{\partial k \partial \sigma_0}\left(\bar{k}\,(\sigma_0), \sigma_0\right)}{\frac{\partial^2 f}{\partial k^2}\left(\bar{k}\,(\sigma_0), \sigma_0\right)} > 0 \tag{C.5}$$

if $\bar{k}(\sigma_0) > k_0$.

The effect of the elasticity of substitution on the capital income share can be derived as

$$\frac{d\bar{\pi}}{d\sigma_0}\,(\sigma_0) = \frac{\partial \pi}{\partial \sigma_0}\left(\bar{k}\,(\sigma_0), \sigma_0\right) + \frac{\partial \pi}{\partial k}\left(\bar{k}\,(\sigma_0), \sigma_0\right) \frac{d\bar{k}}{d\sigma_0}\,(\sigma_0)$$
$$= \frac{\pi_0^2\,\bar{\pi}(\sigma_0)\left[1 - \alpha\,(\sigma_0) + (1 + \alpha\,(\sigma_0)) B\,(\sigma_0) k_0^{\alpha}\right]^2}{\alpha\,(\sigma_0)^3 (1 - \pi_0)\left[1 - \alpha\,(\sigma_0) + (1 + \alpha\,(\sigma_0)) B\,(\sigma_0) \bar{k}\,(\sigma_0)^{\alpha(\sigma_0)}\right]}$$
$$\times \left\{ [1 - \alpha\,(\sigma_0)]\left[\left(\frac{\bar{k}\,(\sigma_0)}{k_0}\right)^{\alpha(\sigma_0)} - 1\right] + B\,(\sigma_0) k_0^{\alpha(\sigma_0)} \left[\frac{\bar{k}(\sigma_0)}{k_0}\right]^{\alpha(\sigma_0)} \ln\left[\frac{\bar{k}(\sigma_0)}{k_0}\right]^{\alpha(\sigma_0)} \right\} \begin{cases} >0, & \text{if } \bar{k}\,(\sigma_0) > k_0, \\ = 0, & \text{if } \bar{k}\,(\sigma_0) = k_0, \\ <0, & \textit{if } \bar{k}\,(\sigma_0) < k_0. \end{cases} \tag{C.6}$$

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