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Abstract

We analyze the transitional dynamics of an endogenous growth model with physical capital, human capital, and R&D. We provide conditions for the existence of a feasible steady state equilibrium with positive long-run growth. For appropriate parameter values, the transitional dynamics of the model is represented by a two-dimensional stable manifold. This provides much richer dynamics than that of the standard two-sector endogenous growth model which is characterized by a one-dimensional stable manifold. We also show how the adjustment paths can be correctly computed by noting that the continuity of the shadow prices involves the continuity of transitional paths.

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1. Introduction

Physical capital accumulation, knowledge formation, and R&D-based technological progress are considered the three main sources of growth. Although they have been usually considered as alternative rather than complementary explanations in the theoretical literature, recently, Funke and Strulik (2000) have combined them into an endogenous growth model with physical capital, human capital and R&D. They show that along the adjustment path for a developing economy, different stages of development can be distinguished. At the first stage (the standard neoclassical model), physical capital is the only factor being accumulated; at the second stage (a developing economy in the Uzawa-Lucas framework), human capital is also being accumulated, and at the third stage (the fully industrialized or innovative economy), research is actively being conducted as well, which results in an increasing variety of goods. Transition to a higher stage of development is explained endogenously.

The purpose of this paper is to analyze the equilibrium dynamics of this model. In doing so, we generalize and correct the analyses in the papers of Funke and Strulik (2000) and Arnold (2000). We provide a necessary and sufficient condition for the existence of a feasible steady-state equilibrium with positive long-run growth that generalizes the condition stated in Funke and Strulik (2000, Proposition 1) and puts right a slight incorrectness in Arnold (2000, Theorem 1). We then analyze the local dynamics generated by the model. Under the assumptions stated by Arnold (2000, Theorem 2) and Funke and Strulik (2000, Proposition 1), the system that describes the dynamics of the model has “too many” unstable roots, i.e., the number of unstable roots exceeds the number of jump variables. Thus, it is not possible to make the system stable for arbitrary initial values of the predetermined variables. We then re-elaborate the conditions for the stability of the model, and show that its transitional dynamics is represented by a two-dimensional stable manifold. This provides a much richer dynamics for the transition paths, where variables can exhibit non-monotonic behaviour throughout the transition to the balanced growth path, relative to that of the standard two-sector endogenous growth model which is characterized by a one-dimensional stable manifold (see, e.g., Bond et al., 1996). However, it cannot be guaranteed that the stable manifold is transversal to the directions in which the system can jump for arbitrary initial values of the predetermined variables. Therefore, saddlepoint stability of the steady state cannot be ensured even if the conditions that guarantee that there exists the right number of stable roots are fulfilled.

Numerical computation of transitional paths is a difficult task when the economy evolves through different stages of development, so structural changes take place. Actually, adjustment paths are incorrectly computed in Funke and

Strulik (2000). They calculate the stable saddle-paths of the developing economy and the innovative economy towards their respective steady-state positions. Then, they use the fact that state variables cannot jump to calculate the point of transition, and cut the relevant time paths of the developing economy up to the value of the connecting state variable and link them with the already obtained paths for the innovating economy. Proceeding in this way, discontinuities at the point of transition from the developing economy to the fully industrialized economy arise that violate the continuity of the costate variables of the households' utility maximization problem. We then show how the transitional paths may be correctly computed.

The remainder of this paper is organized as follows. Section 2 recapitulates the model. Section 3 analyzes the balanced growth equilibrium, and Section 4 the stability of the model. Section 5 considers the computation of adjustment paths. Section 6 concludes.

2. The model

This section recapitulates the Funke and Strulik's (2000) model of endogenous growth. For further details, see that reference and Arnold (2000).

2.1. Setup of the model

Consider a closed economy inhabited by a constant population, normalized to one, of identical infinitely-lived households that maximize the intertemporal utility function

$$\int_0^\infty \frac{C^{1-\theta}-1}{1-\theta} \mathrm{e}^{-\rho t}\, dt, \ \ \rho > 0, \ \ \theta > 0,$$

where C denotes consumption, subject to the budget constraint and the knowledge accumulation technology. Human capital, H, can be devoted to production, education and R&D, respectively:

$$H = H_Y + H_H + H_n,$$

and is accumulated according to

$$\dot{H} = \xi H_H, \ \ \xi > 0.$$

The budget constraint faced by the household is

$$\dot{A} = rA + w(H - H_H) - C,$$

where r is the return per unit of aggregate wealth, A, and w the wage rate per unit of employed human capital. Let $g_z = \dot{z}/z$ denote the growth rate of any variable z. In this paper, we will concentrate on the regimes where human capital is being accumulated and, hence, $H_H > 0$. The first-order conditions of the household's utility maximization problem give then

$$g_C = (r-\rho)/\theta, \tag{2.1}$$

$$g_w = r-\xi. \tag{2.2}$$

A single homogenous final good Y is produced with Cobb-Douglas technology

$$Y = A_1 K^\beta D^\eta H_Y^{1-\beta-\eta},\ A_1 > 0,\ \beta > 0,\ \eta > 0,\ \beta+\eta < 1.$$

Here K is physical capital and D is an index of intermediate goods,

$$D = \left[\int_0^n x(i)^\alpha \, di\right]^{1/\alpha},\ \ 0<\alpha<1,$$

where $x(i)$ is the amount used for each one of the n intermediate goods. The market for final goods is perfectly competitive and the price for final goods is normalized to one. Profit maximization delivers the factor demands

$$r = \beta Y/K, \tag{2.3a}$$

$$w = (1-\beta-\eta)Y/H_Y, \tag{2.3b}$$

$$p(i) = \eta Y x(i)^{\alpha-1}/D^\alpha, \tag{2.3c}$$

where $p(i)$ represents the price of intermediate i.

Invention of new intermediates is determined solely by the aggregate knowledge devoted to R&D according to

$$\dot{n} = \delta H_n,\ \ \delta > 0.$$

Firms are granted infinitely-lived patents, so that there is monopolistic competition in the intermediate-goods sector. Suppose that an intermediate good costs one unit of Y to produce. Facing the price elasticity of demand for the intermediates $1/(1-\alpha)$, firms maximize operating profits, $\pi(i) = (p(i) - 1)x(i)$, by charging a constant markup price $p(i) = 1/\alpha$. Since both technology and demand are the same for all intermediates, the equilibrium is symmetric: $x(i) = x$, $p(i) = p$. Using (2.3c), this yields the quantity of intermediates employed as $xn = \alpha\eta Y$, firms profits as

$$\pi = (1-\alpha)\eta Y/n, \tag{2.4}$$

and $D = xn^{1/\alpha} = n^{(1-\alpha)/\alpha}\alpha\eta Y$. Substituting this expression into the production function yields

$$Y^{1-\eta} = A_1(\alpha\eta)^\eta K^\beta n^{(1-\alpha)\eta/\alpha}(u_1 H)^{1-\beta-\eta}, \tag{2.5}$$

where $u_1 = H_Y/H$ is the proportion of human capital employed in production.

The value of an innovation υ is the present value of the stream of monopoly profits,

$$\upsilon(t) = \int_t^\infty e^{-\bar{r}(\tau,t)}\pi(\tau)\,d\tau,$$

with $\bar{r}(\tau,t)=\int_t^\tau r(s)\,ds$. Log-differentiating this expression with respect to time gives the no-arbitrage equation

$$g_\upsilon = r-\pi/\upsilon. \tag{2.6}$$

Finally, free-entry into R&D requires

$$w=\delta\upsilon \quad \text{and} \quad H_n>0, \tag{2.7}$$

in an equilibrium with innovation, or

$$w>\delta\upsilon \quad \text{and} \quad H_n=0.$$

Some equations will be needed for solving the model. Log-differentiating the expressions for r in (2.3a), w in (2.3b), and Y in (2.5) provides, respectively,

$$g_r = g_Y - g_K, \tag{2.8}$$

$$g_w = g_Y - g_{u_1} - g_H, \tag{2.9}$$

and

$$(1-\eta)g_Y = \beta g_K + \frac{(1-\alpha)\eta}{\alpha} g_n + (1-\beta-\eta)(g_{u_1}+g_H). \tag{2.10}$$

Let $\chi \equiv C/K$ denote the consumption to physical capital ratio, $\omega \equiv K/H$, the physical to human capital ratio, and $\psi \equiv H/n$, the knowledge-ideas ratio. The economy's resource constraint,

$$\dot{K} = Y - C - nx = (1-\alpha\eta)Y - C,$$

can be expressed as

$$g_K = \frac{(1-\alpha\eta)}{\beta} r - \chi. \tag{2.11}$$

2.2. The developing economy

The developing economy is characterized by the presence of human capital accumulation ($\dot{H}>0$) but R&D is not profitable ($\dot{n}=0$). As long as $\xi>\rho$, the economy necessarily arrives at a point from which on households will invest permanently in human capital formation (see Funke and Strulik, 2000). The following system of differential equations describes the dynamics of the developing economy:

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}, \tag{2.12a}$$

$$g_{u_1} = \frac{(1-\alpha)\eta}{\beta} r - \chi + \xi u_1 + \frac{(1-\beta-\eta)\xi}{\beta}, \tag{2.12b}$$

$$g_r = -\frac{(1-\beta-\eta)}{\beta}(r-\xi). \tag{2.12c}$$

From (2.1) and (2.11), we obtain (2.12a). From (2.2), (2.8), (2.9), (2.10), (2.11), together with $g_H = \xi(1 - u_1)$ and $g_n = 0$, we obtain (2.12b) and (2.12c).

From (2.12c), the interest rate r converges to $\xi > 0$ independently of the remaining system dynamics as $\partial g_r / \partial r < 0$, whereas the growth rate of wages converges to zero from (2.2). Hence, there must be a point in time at which the value of an innovation, υ, equals its cost, w/δ. Up from this point, the economy enters the innovative stage.

2.3. The innovative economy

The fully industrialized economy is characterized by the presence of both human capital accumulation ($\dot{H} > 0$) and R&D ($\dot{n} > 0$). The following system of differential equations describes the dynamics of the fully industrialized economy:

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta} \right) r + \chi - \frac{\rho}{\theta}, \tag{2.13a}$$

$$g_r = \left[\frac{\alpha\beta - (1-\alpha)\eta}{(1-\alpha)\eta} \right]^{-1} \left\{ \frac{1-\alpha\eta}{\beta} r - \left[1 + \frac{\alpha(1-\beta-\eta)}{(1-\alpha)\eta} \right] (r - \xi) - \chi \right\}, \tag{2.13b}$$

$$g_\psi = \xi \left\{ 1 - \left[\frac{(1-\beta-\eta)\xi}{(1-\alpha)\eta} + g_n \right] \frac{1}{\delta\psi} \right\} - g_n, \tag{2.13c}$$

where the growth rate of n can be expressed as a function of r and χ as

$$g_n = \frac{\alpha((1-\alpha)\eta r + (1-\eta)\xi - \beta\chi)}{\alpha\beta - \eta(1-\alpha)}. \tag{2.14}$$

The system (2.13) characterizes the dynamics of the economy in terms of r, χ and ψ, after substituting g_n from (2.14) into (2.13c).

Log-differentiating the free-entry condition (2.7) with respect to time, and substituting g_υ from (2.6), π from (2.4), and w from (2.3b), we get

$$r - g_w = \frac{(1-\alpha)\eta\delta}{1-\beta-\eta} \frac{H_Y}{n},$$

which using (2.2) gives

$$u_1 = \frac{(1-\beta-\eta)\xi}{(1-\alpha)\eta\delta\psi}. \tag{2.15}$$

From (2.1) and (2.11), we derive (2.13a). Log-differentiating (2.15) with respect to time delivers $g_{u_1} = g_n - g_H$, which together with (2.2), (2.8), (2.9), (2.10) and (2.11) give (2.13b) and (2.14). Equation (2.13c) is obtained from $g_\psi = g_H - g_n$, $g_H = \xi(1 - u_1 - u_2)$, $g_n = \delta u_2 \psi$ and (2.15), where $u_2 = H_n / H$ is the proportion of human capital employed in R&D.

3. Balanced growth equilibrium

In this section, we shall focus on a balanced growth equilibrium in which all variables grow at constant, but possibly different, rates, and the shares of human capital in its different uses are constant. The next theorem states a necessary and sufficient condition for the existence of a feasible steady-state equilibrium with positive long-run growth, which generalizes the conditions stated in Funke and Strulik (2000, Proposition 1), and puts right a slight incorrectness in Arnold (2000, Theorem 1).

Theorem 1. Let $\xi > \rho$. For the innovating economy, there exists a unique positive steady-state equilibrium with positive long-run growth rate:

$$r^* = \frac{\theta(1+(\alpha/(1-\alpha))((1-\beta-\eta)/\eta))\xi - \rho}{\theta(1+(\alpha/(1-\alpha))((1-\beta-\eta)/\eta))-1}, \tag{3.1a}$$

$$\chi^* = \left(\frac{1-\alpha\eta}{\beta} - \frac{1}{\theta}\right) r^* + \frac{\rho}{\theta}, \tag{3.1b}$$

$$\psi^* = \frac{\xi(g_n^*(1-\alpha)\eta + (1-\beta-\eta)\xi)}{(1-\alpha)\delta\eta(\xi - g_n^*)}, \tag{3.1c}$$

where the steady-state growth rate of knowledge and human capital equals

$$g_n^* = g_H^* = \frac{\xi - \rho}{\theta + (\theta-1)((1-\alpha)/\alpha)(\eta/(1-\beta-\eta))}, \tag{3.2}$$

and the steady-state growth rate of income, consumption, and physical capital equals

$$g_K^* = g_Y^* = g_C^* = (1+((1-\alpha)/\alpha)(\eta/(1-\beta-\eta)))\, g_n^*, \tag{3.3}$$

if and only if

$$\theta > \theta_{\min} = \frac{\eta(1-\alpha) + \alpha(1-\beta-\eta)(1-\rho/\xi)}{\eta(1-\alpha) + \alpha(1-\beta-\eta)}. \tag{3.4}$$

Proof. According to (2.15), constancy of u_1 implies $g_\psi^* = 0$, and hence $g_H^* = g_n^*$. Constancy of g_C^* implies, by the Ramsey rule (2.1), the constancy of r, i.e., $g_r^* = 0$. Therefore, $g_Y^* = g_K^*$, from (2.3a), and χ is also constant in the steady state, i.e., $g_\chi^* = 0$, from (2.13a). Hence, $g_Y^* = g_K^* = g_C^*$.

Solving for the steady state of the system (2.13a)-(2.13c) yields (3.1a)-(3.1c). Substitution of (3.1a) and (3.1b) into (2.14) gives (3.2). Now, $\xi > \rho$ entails $r^* > \rho > 0$. If $g_n^* > 0$, the condition $\psi^* > 0$ is satisfied if and only if $g_n^* < \xi$, which is equivalent to (3.4). Condition (3.4) also ensures $g_n^* > 0$. The ratio of consumption to physical capital can be expressed as

$$\chi^* = \frac{(1-\alpha\eta)((1+\Phi)\theta\xi-\rho)-\beta(1+\Phi)(\xi-\rho)}{\beta(\theta(1+\Phi)-1)},$$

where $\Phi = (\alpha(1-\beta-\eta))/((1-\alpha)\eta)$. As (3.4) can be equivalently expressed as $(1+\Phi)\theta\xi > \xi + \Phi(\xi-\rho)$, the denominator of χ^* is positive. Its numerator is also positive, and hence $\chi^* > 0$, since

$$(1-\alpha\eta)((1+\Phi)\theta\xi-\rho)-\beta(1+\Phi)(\xi-\rho) >$$
$$> (1-\alpha\eta)((1+\Phi)(\xi-\rho))-\beta(1+\Phi)(\xi-\rho) = (1-\alpha\eta-\beta)(1+\Phi)(\xi-\rho) > 0.$$

As $g_H^* = g_n^*$, $g_Y^* = g_K^*$ and $g_{u_1}^* = 0$, from (2.10) we obtain (3.3). Finally, $g_n^* > 0$ and $\psi^* > 0$ entail that $u_1^* > 0$ using (2.15), $u_2^* = (H_n/H)^* = g_n^*/(\delta\psi^*) > 0$, and $u_3^* = (H_H/H)^* = g_n^*/\xi > 0$. Thus, the steady state is feasible. The transversality conditions of the household's optimization problem can be readily shown to be equivalent to $\xi > g_n^*$. This completes the proof. ■

A sufficient, but not necessary, condition for (3.4) to hold is $\theta > 1$, which is assumed by Funke and Strulik (2000). Hence, condition (i), $(1-\alpha\eta)/\beta > 1$, in Funke and Strulik (2000, Proposition 1) is not required to guarantee the existence of a positive steady-state equilibrium with positive long-run growth. Arnold (2000, Theorem 1) assumes instead the fulfilment of the condition $1/\theta < 1 + \alpha(1-\beta-\eta)/((1-\alpha)\eta)$, which is equivalent to

$$\theta > \bar{\theta} = \frac{\eta(1-\alpha)}{\eta(1-\alpha)+\alpha(1-\beta-\eta)}. \tag{3.5}$$

Condition (3.4) implies, but is not necessary for, the validity of (3.5). Although (3.5) ensures that $g_n^* > 0$, it does not guarantee that $g_n^* < \xi$, and so, it does not guarantee the positivity of ψ^* (and u_1^*). Hence, (3.5) is not a sufficient condition for the steady state to be feasible.

4. Stability analysis

We shall now analyze the dynamics of the model in the neighbourhood of the steady state. As usual, we assume that the stocks of physical and human capital, and the number of intermediates move sluggishly, so that $K(0)$, $H(0)$ and $n(0)$ are given by their historical values. The knowledge-ideas ratio, $\psi = H/n$, is therefore a predetermined variable. Although one might expect the interest rate, r, to be a jump variable, it is a predetermined variable in the innovative economy as well since, substituting u_1 from (2.15) into (2.5), and using (2.3a), it can be expressed as a function of K and n as

$$r = \beta A_1^{\frac{1}{1-\eta}} (\alpha\eta)^{\frac{\eta}{1-\eta}} K^{-\frac{1-\beta-\eta}{1-\eta}} n^{\frac{(1-\alpha)\eta+\alpha(1-\beta-\eta)}{\alpha(1-\eta)}} \left(\frac{(1-\beta-\eta)\xi}{(1-\alpha)\eta\delta} \right)^{\frac{1-\beta-\eta}{1-\eta}} .$$

Thus, the consumption to physical capital ratio, χ, is the unique jump variable. Hence, the system (2.13) has two predetermined variables, r and ψ, and only one jump variable, χ. To ensure local saddlepoint stability of the steady state (r^*,χ^*,ψ^*), we should first look for conditions that guarantee that the coefficient matrix of the linearized system around the steady state has two stable and one unstable eigenvalues, and then, check whether the stable manifold is transversal to the directions in which the system can jump for arbitrary initial values of the predetermined variables (see, e.g., Buiter, 1984). In the following discussion, the condition (3.4) in Theorem 1 will be assumed to hold.

Linearizing the system (2.13) around its steady state (r^*,χ^*,ψ^*), the dynamics may be approximated by the following third-order system:

$$\begin{pmatrix} \dot{r} \\ \dot{\chi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \left(\frac{\alpha\eta(1-\alpha)}{\alpha\beta-(1-\alpha)\eta} - \frac{1-\beta-\alpha\eta}{\beta} \right) r^* & \frac{-(1-\alpha)\eta}{\alpha\beta-(1-\alpha)\eta} r^* & 0 \\ (1/\theta - (1-\alpha\eta)/\beta)\chi^* & \chi^* & 0 \\ \bullet & \bullet & \xi - g_n^* \end{pmatrix} \begin{pmatrix} r - r^* \\ \chi - \chi^* \\ \psi - \psi^* \end{pmatrix}, \quad (4.1)$$

where dots replace those elements that are irrelevant for the analysis. Let J denote the coefficient matrix of (4.1), and let J_2 denote the upper left 2×2 submatrix of J. The structure of J entails that its eigenvalues are the two eigenvalues of J_2, and the third eigenvalue of J is its last diagonal element, which is positive since $g_n^* < \xi$ if (3.4) holds.

The real parts of the eigenvalues of J_2 are negative if and only if its determinant is positive and its trace negative. The determinant of J_2 is calculated as

$$\Delta_2 = \frac{\eta(1-\alpha)(1-\theta) - \alpha(1-\beta-\eta)\theta}{(\alpha\beta-(1-\alpha)\eta)\theta} r^* \chi^* . \quad (4.2)$$

Since the numerator of the former expression is negative if condition (3.4), and therefore (3.5), holds, Δ_2 is positive if and only if

$$\alpha\beta < (1-\alpha)\eta . \quad (4.3)$$

The trace of J_2 is calculated as $T = T_1 + T_2$, where

$$T_1 = \frac{(\eta(1-\alpha)+\alpha(1-\beta-\eta))\,(\alpha\beta-\eta(1-\alpha)+((1-\alpha)^2\eta-\alpha\beta)\theta)\xi}{(\eta(1-\alpha)-\alpha\beta)\,((\alpha(1-\beta-\eta)+\eta(1-\alpha))\theta-\eta(1-\alpha))},$$

and

$$T_2 = \frac{(\eta(1-\alpha)(1-\beta-\alpha\eta)-\alpha\beta(1-\beta-\eta))\alpha\rho}{(\eta(1-\alpha)-\alpha\beta)\,((\alpha(1-\beta-\eta)+\eta(1-\alpha))\theta-\eta(1-\alpha))}.$$

The denominator of T is positive if (4.3) and (3.4), and therefore (3.5), are satisfied. Now, two cases may arise.

i) If the coefficient of θ in the numerator of T is nonpositive, i.e., $\alpha\beta \geq (1-\alpha)^2\eta$, the fact that $\theta > \theta_{\min}$ implies that the trace is negative if (3.4) is satisfied since its numerator is less than

$$-\alpha\eta(1-\alpha)(\eta(1-\alpha)+\alpha(1-\beta-\eta))(\xi-\rho),$$

which is negative.

ii) If the coefficient of θ in the numerator of T is positive, i.e., $\alpha\beta < (1-\alpha)^2\eta$, the numerator of T is increasing in θ. Equalizing the numerator of T to zero, the value of θ such that the trace T is zero, $\theta_{T=0}$, is found to be

$$\theta_{T=0} = \frac{(\alpha\beta(1-\beta-\eta)-\eta(1-\alpha)(1-\beta-\alpha\eta))\alpha\rho}{(\eta(1-\alpha)+\alpha(1-\beta-\eta))\,((1-\alpha)^2\eta-\alpha\beta)\xi}+\frac{\eta(1-\alpha)-\alpha\beta}{(1-\alpha)^2\eta-\alpha\beta}. \qquad (4.4)$$

Hence, the trace T is negative if and only if $\theta < \theta_{T=0}$. Note that $\theta_{\min} < \theta_{T=0}$, as

$$\theta_{\min}-\theta_{T=0} = \frac{\alpha\eta(1-\alpha)(\xi-\rho)}{(\alpha\beta-(1-\alpha)^2\eta)\xi} < 0.$$

Under the former conditions, there are a single unstable and two stable roots. The *Stable Manifold Theorem* (e.g., Guckenheimer and Holmes, 1983) entails that there exists a two-dimensional differentiable stable manifold M containing (r^*,χ^*,ψ^*) such that for any point (r,χ,ψ) in M the solution through this point converges to the steady state. The following theorem summarizes the former findings.

Theorem 2. In the conditions of Theorem 1, if $\alpha\beta < (1-\alpha)\eta$ and one of the following conditions is verified:

i) $\alpha\beta \geq (1-\alpha)^2\eta$, or

ii) $\alpha\beta < (1-\alpha)^2\eta$ and $\theta < \theta_{T=0}$, with $\theta_{T=0}$ defined by Eq. (4.4),

then there exists a two-dimensional differentiable stable manifold M tangent to the stable space of (4.1) at (r^*,χ^*,ψ^*), that is invariant under the flow of system (2.13) and such that for any point in M the solution through this point converges to the steady state.

Numerical simulations show that the two stable eigenvalues may be complex conjugate or real. For instance, under the baseline $\beta = 0.4$, $\eta = 0.5$, $\alpha = 0.5$, $\xi = 0.05$, $\rho = 0.03$, $\theta = 1$, $\delta = 0.1$, and $A_1 = 1$, the eigenvalues of J are $\lambda_1 = -0.2471$, $\lambda_2 = -0.0979$, and $\lambda_3 = 0.03$. Under the baseline $\beta = 0.34$, $\eta = 0.34$, $\alpha = 0.4$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 1$, $\delta = 0.1$, and $A_1 = 1$, the eigenvalues of J are

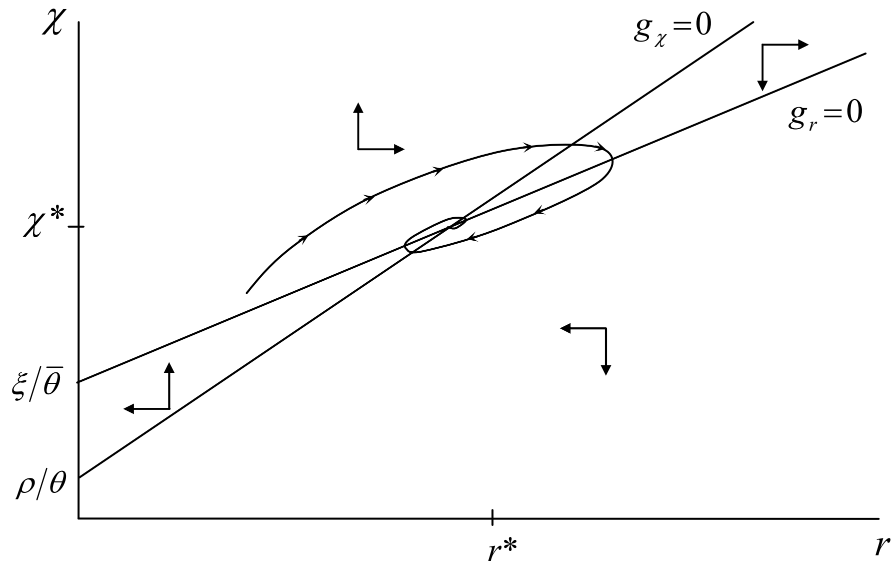

Figure 1. Convergence of r and χ in the innovative economy

$\lambda_1 = -0.044 + 0.165i$, $\lambda_2 = -0.044 - 0.165i$, and $\lambda_3 = 0.023$. Our numerical simulations suggest, however, that the stable eigenvalues are more likely to be complex conjugate for plausible parameter values.

Whereas Theorem 2 gives conditions that guarantee that there exists the right number of stable eigenvalues, this is not sufficient for saddlepoint stability. It should also be checked whether the stable manifold is transversal to the directions in which the system can jump for arbitrary initial values of the predetermined variables. It can be readily shown that the stable space of the linear system (4.1), which is the eigenspace corresponding to the two negative eigenvalues of J, is equal to the hyperplane defined by $\psi = \psi^*$. Since ψ is a predetermined variable, its initial value $\psi(0)$ will generally be different from ψ^*. In this case, $\psi(0) \neq \psi^*$, there does not exist any χ such that (r,χ,ψ) is contained in the two-dimensional stable space of the system (4.1). In other words, given the initial values $r(0)$ and $\psi(0)$, if $\psi(0) \neq \psi^*$ a jump in χ cannot move the system onto the stable eigenspace of the system (4.1). Hence, saddlepoint stability of the steady state (r^*,χ^*,ψ^*) cannot be ensured even if the conditions of Theorem 2 hold. So, we cannot be confident that for arbitrary initial values, the predetermined variables r and ψ pick off a (unique) point in the stable manifold.

From (2.13a), the $g_\chi = 0$ – locus is given by

$$\chi = \left(\frac{1-\alpha\eta}{\beta} - \frac{1}{\theta} \right) r + \frac{\rho}{\theta},$$

and from (2.13b), the $g_r = 0$ – locus is calculated as

$$\chi = \left(\frac{\eta(1-\alpha)+\alpha(1-\beta-\eta)}{(1-\alpha)\eta} \right)\xi + \left(\frac{1-\alpha\eta}{\beta} - \frac{\eta(1-\alpha)+\alpha(1-\beta-\eta)}{(1-\alpha)\eta} \right) r.$$

Note that $g_\chi > 0$ above the $g_\chi = 0$ – locus, and $g_\chi < 0$ below. The condition (4.3) assumed in Theorem 2 implies that the factor $(\alpha\beta-(1-\alpha)\eta)/((1-\alpha)\eta)$ in (2.13b) is negative, and hence, $g_r > 0$ above the $g_r = 0$ – locus, and $g_r < 0$ below. Given the condition (3.4) stated in Theorem 1, the $g_\chi = 0$ – locus is located below the $g_r = 0$ – locus at $r = 0$ and, furthermore, the slope of the $g_\chi = 0$ – locus is greater than that of the $g_r = 0$ – locus. The $g_r = 0$ – locus is increasing, and hence so the $g_\chi = 0$ – locus is, if (4.3) is satisfied since the coefficient of r is positive as

$$\frac{1-\alpha\eta}{\beta} - \frac{1}{\overline{\theta}} > \frac{1-\eta}{\beta} - \frac{1}{\overline{\theta}} = \frac{(1-\beta-\eta)((1-\alpha)\eta-\alpha\beta)}{(1-\alpha)\beta\eta} > 0.$$

Figure 1 depicts the phase diagram in the (r,χ) – plane. It should be noted that one needs full stability of the steady state (r^*,χ^*), not saddle-path stability as assumed by Arnold (2000) and Funke and Strulik (2000). The steady state (r^*,χ^*) is a stable spiral point if both stable roots are complex conjugate, and a stable node if they are real. A particular adjustment path is depicted when both stable roots are assumed to be complex.

Figure 2 depicts a phase diagram in the (g_n,ψ) – plane. From (2.13c), the $g_\psi = 0$ – locus is given by

$$\psi = \left[\frac{(1-\beta-\eta)\xi}{(1-\alpha)\eta} + g_n \right] \frac{\xi}{\delta(\xi - g_n)}.$$

The $g_\psi = 0$ – locus is upward sloping with a positive ordinate intersection, and a pole at $g_n = \xi$. It can be easily shown that $g_\psi > 0$ above the $g_\psi = 0$ – locus, and $g_\psi < 0$ below. The $\dot{g}_n = 0$ – locus is vertical at $g_n = g_n^*$ in the (g_n,ψ) – plane. Note that (2.14) entails that g_n converges since r and χ do so. However, we cannot ensure that the arrows point west (east) if $g_n > g_n^*$ ($g_n < g_n^*$), as Funke and Strulik (2000) and Arnold (2000) do, since it cannot be ensured that convergence to g_n^* is monotonic. The reason is that, as shown before, r and χ may exhibit a non-monotonic convergence to their steady-state values, and so, (2.14) entails that g_n may also converge in a non-monotonic fashion. Actually, the adjustment paths depicted in Figures 1 and 2 arise under the parametrization $\beta = 0.34$, $\eta = 0.34$, $\alpha = 0.4$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 1$, $\delta = 0.1$, and $A_1 = 1$.

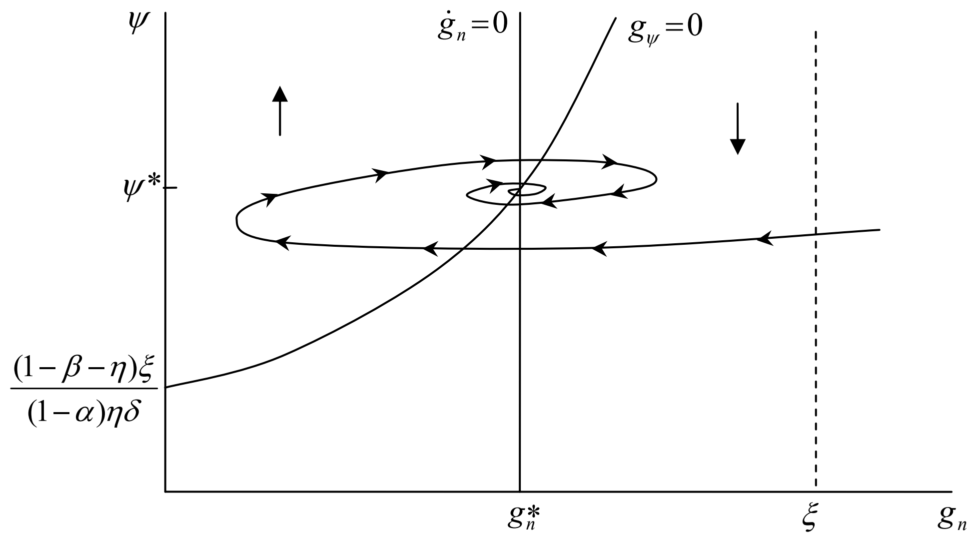

Figure 2. Convergence of ψ and g_n in the innovative economy

Figure 2 shows that, although the stable manifold of the system (2.13) is tangent to the stable eigenspace of the system (4.1) at the steady state (r^*, χ^*, ψ^*), they cannot be equal. If it were the case, the direction of the arrows shows that the system would leave the 'supposedly stable' manifold $\psi = \psi^*$, unless we were effectively at the steady state at the initial time. Thus, although the stable eigenspace of the linear system (4.1) is not transversal to the directions in which the system can jump, it might be the case that the stable manifold of the nonlinear system (2.13) be so. Actually, this is the case in all of our numerical simulations.

Figure 3 displays adjustment paths in the innovative economy, computed by backward integration (Brunner and Strulik, 2002), along the stable two-dimensional manifold in the (r, χ, ψ) – space when parameter values are $\beta = 0.36$, $\eta = 0.36$, $\alpha = 0.4$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.1$, $A_1 = 1$. Note that this parametrization differs from that considered by Funke and Strulik (2000) only in the value of the parameter α, and fulfils the conditions in Theorems 1 and 2. For a better illustration of the two-dimensional stable manifold, the constraint $H_n \geq 0$ has been ignored. The fact that the system features two stable roots provides a much richer dynamics for the transition paths relative to that of the standard two-sector endogenous growth model which is characterized by a one-dimensional stable manifold (see, e.g., Bond et al., 1996). Figures 1, 2 and 3 illustrate that the economy can exhibit non-monotonic behaviour throughout the transition, and grow through damped oscillations around the balanced growth path.

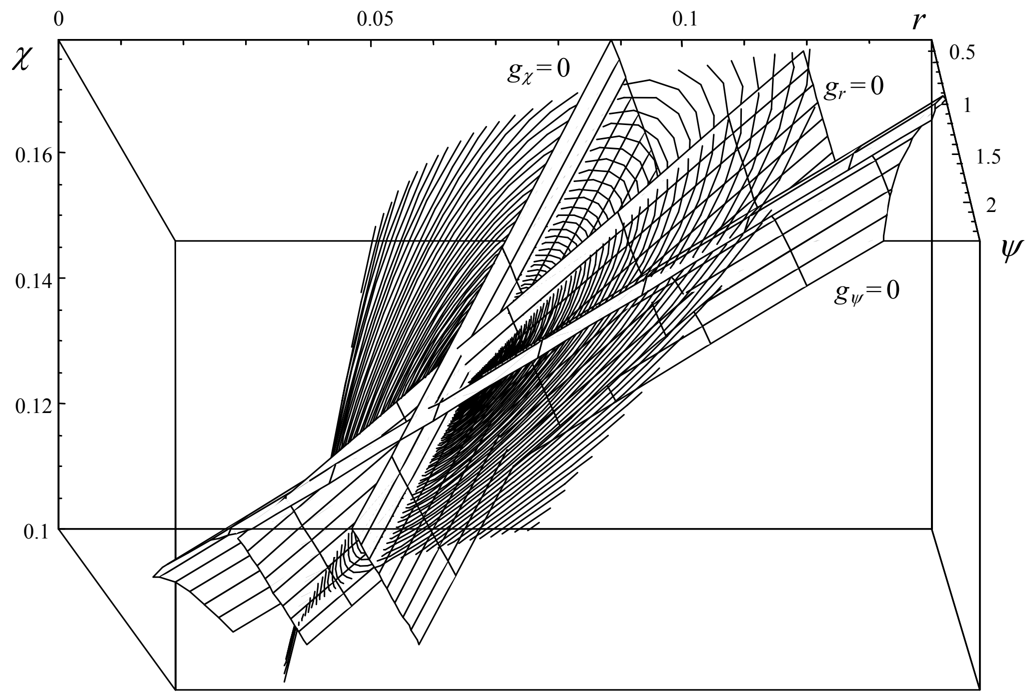

Figure 3. Adjustment dynamics along a two-dimensional stable manifold in the innovative economy.

Arnold (2000, Theorem 2) and Funke and Strulik (2000, Proposition 1) assume that the condition $\alpha\beta > (1-\alpha)\eta$ holds, which is exactly the reverse of the condition assumed in Theorem 2. With this assumption, the determinant of J_2 is negative when (3.4) is satisfied, and so, J has two unstable and one stable roots. Thus, the system that describes the dynamics of the model has "too many" unstable roots, i.e., the number of unstable roots exceeds the number of jump variables, and the dimension of the stable manifold is smaller than the number of predetermined variables. So it is not possible to make the system stable for arbitrary initial values of the predetermined variables: The system has 'too many' initial conditions, and would start in the stable manifold only by coincidence. The problem is that Arnold (2000) and Funke and Strulik (2000) fail to acknowledge that the interest rate is a state variable, not a jump variable. So, they look for parameter values that yield the wrong number of stable eigenvalues needed for stability.

5. Computation of adjustment paths

Funke and Strulik (2000) show that along the adjustment path for a evolving economy, different stages of development can be distinguished, and analyze growth in stages via simulations. In this section, we show that transition paths are incorrectly computed in their paper. By means of the backward integration method (Mulligan and Sala-i-Martín, 1993, and Brunner and Strulik, 2002), they calculate the stable saddle-paths of the developing economy and the innovative economy towards their respective steady-state positions. Then, they use the fact that state variables cannot jump to calculate the point of transition, and cut the relevant time paths of the developing economy up to the value of the connecting state variable and link them with the already obtained paths for the innovating economy. Proceeding in this way, however, discontinuities arise at the point of a regime switch that violate the continuity of the shadow prices. We shall show how the correct paths can be computed. As Funke and Strulik (2000), we shall focus on the transition from the developing to the innovative economy.

Note first that along the transition path, $\psi(t)$ is continuous as the state variables $H(t)$ and $n(t)$ are so. The current value Hamiltonian, Π, for the household's utility maximization problem is

$$\Pi = \frac{C^{1-\theta}-1}{1-\theta} + \lambda\,(rA + w(H - H_H) - C) + \mu\,(\xi H_H)\,,$$

where λ and μ are costate variables. Among the necessary conditions, we have

$$\frac{\partial \Pi}{\partial C} = C^{-\theta} - \lambda = 0\,.$$

Since the costate variable $\lambda(t)$ is continuous (e.g., Seierstad and Sydsæter, 1987), this equation implies that $C(t)$ must be continuous. Hence, the continuity of the state variable $K(t)$ entails that $\chi(t)$ must be continuous as well. If $\dot{H} > 0$ (i.e., $H_H > 0$), we also obtain the following condition:

$$\frac{\partial \Pi}{\partial H_H} = -\lambda w + \mu \xi = 0\,.$$

By the continuity of the costate variables $\lambda(t)$ and $\mu(t)$, this equation implies that the wage rate $w(t)$ is continuous. Using (2.3a), (2.3b) and (2.5), the wage rate can be expressed as a function of K, n and u_1, and therefore u_1 is continuous. Eqs. (2.3a) and (2.5) entail that r and Y are continuous as well. Hence, r, χ, and u_1 (and the remaining variables too) must be continuous at the point of transition from the developing economy to the fully industrialized economy.

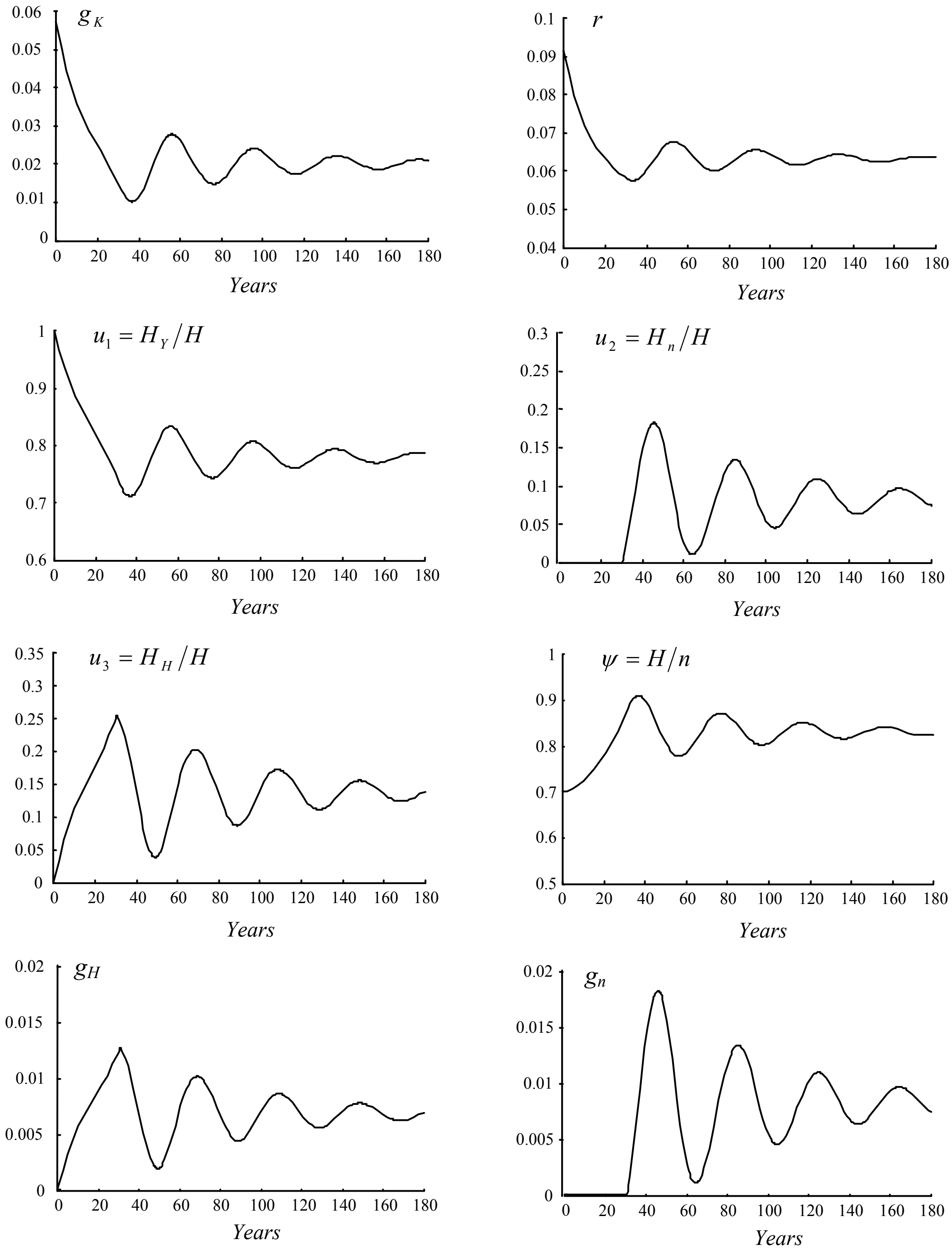

Figure 4. Transitional dynamics of several variables: the case of two stable roots

Figures 2–6 in Funke and Strulik (2000) display the transitional paths of several variables of the model under the base run scenario $\beta = 0.36$, $\eta = 0.36$, $\alpha = 0.54$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.1$, and $A_1 = 1$. These Figures clearly show that several variables are discontinuous at the point of the regime switch (e.g., their Figure 3 shows that r and u_1 are discontinuous at the transition point). Hence, adjustment paths have been incorrectly calculated. The correct transition paths can be computed as follows. First, we calculate the stable saddle-paths of the innovative economy towards their steady-state position, (r^*, χ^*, ψ^*). Backward looking from the steady state (r^*, χ^*, ψ^*), the adjustment paths come to an end at the transition point from the developing economy to the innovating economy, where $H_n = 0$. From that point backward, the system (2.12) describes the dynamics of the developing economy. We have shown that r, χ, and u_1 must be continuous at the transition point. Hence, we compute the unique trajectories determined by the system (2.12) that pass through the computed values of r, χ, and u_1 at the transition point, and link them with the already obtained for the innovative economy. A similar argument, based on the continuity of the costate variables, has been recently used by Gómez (2003) to determine the transitional dynamics in the one-sector endogenous growth model with physical and human capital when investments are irreversible.

The numerical simulations performed by Funke and Strulik (2000) are characterized by one-dimensional stable manifolds, and do not satisfy the conditions stated in Theorem 2. Thus, Figure 4 displays the transition paths of several variables under the parametrization $\beta = 0.36$, $\eta = 0.36$, $\alpha = 0.4$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.1$, $A_1 = 1$, used before to depict Figure 3. Now, the Uzawa–Lucas framework serves as a description of development dynamics for about 30 years, in which adjustment dynamics is monotonic, but oscillatory adjustment dynamics occurs when the economy enters the innovative stage.

6. Conclusions

In this paper, we examined the equilibrium dynamics of an endogenous growth model with physical capital, human capital and R&D. First, we provided a necessary and sufficient condition for the existence of a feasible steady-state equilibrium with positive long-run growth. We then analyzed the local dynamics generated by the model. The dynamics of the economy is described by a third-order system. For appropriate parameter values, this dynamical system features two stable roots, rather than only one as in the standard two-sector endogenous growth model, and two predetermined variables. Hence, the equilibrium dynamics is characterized by a two-dimensional stable manifold. This provides a much richer dynamics for the transition paths. The economy can exhibit non-monotonic behaviour throughout the transition and grow through damped oscillations around the balanced growth path. However, it cannot be guaranteed that the stable manifold is transversal to the directions in which the system can jump for arbitrary initial values of the predetermined variables. Therefore, saddlepoint stability cannot be ensured even if the conditions that guarantee that there exists the right number of stable roots are fulfilled.

Numerical computation of adjustment paths is a difficult task when the economy evolves through different stages of development, so structural changes take place. One could be tempted to calculate the stable saddle-paths of each development stage towards its respective steady state, cut the relevant saddle-paths up to the transition point and link them with the saddle-paths computed for the next stage of development. Computing in this way the transition paths in the model considered in this paper discontinuities at the point of transition from the developing economy to the fully industrialized economy arise that violate the continuity of the costate variables of the households' utility maximization problem. We then showed how the transitional paths could be correctly computed. Similar arguments may be applicable to other models of endogenous growth with temporally binding constraints.

References

Arnold, L.G. (2000): "Endogenous growth with physical capital, human capital and product variety: A comment," *European Economic Review*, 44, 1599-1605.

Bond, R.W., P. Wang, and C.K. Yip (1996): "A general two-sector model of endogenous growth with human and physical capital: Balanced growth and transitional dynamics," *Journal of Economic Theory*, 68, 149-173.

Brunner, M. and H. Strulik (2002): "Solution of perfect foresight models: a simple method and applications", *Journal of Economic Dynamics and Control*, 26, 737-753.

Buiter, W.H. (1984): "Saddlepoint problems in continuous time rational expectations models: A general method and some macroeconomic examples", *Econometrica*, 52, 665-680.

Funke, M., and H. Strulik (2000): "On endogenous growth with physical capital, human capital and product variety," *European Economic Review*, 44, 491-515.

Gómez, M.A. (2003): “Equilibrium dynamics in the one-sector endogenous growth model with physical and human capital”, *Journal of Economic Dynamics and Control*, 28, 367-375.

Guckenheimer, J., and P. Holmes (1983): *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag: New York.

Mulligan, C.B., and X. Sala-i-Martín (1993): “Transitional dynamics in two-sector models of endogenous growth”, *Quarterly Journal of Economics*, 108, 739-773.

Seierstad, A., and K. Sydsæter (1987): *Optimal Control Theory with Economic Applications*. North-Holland: Amsterdam.