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Topological groups of Lipschitz functions and Graev metrics

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A B S T R A C T

We study the properties of the free abelian topological group $A_d(X)$ on a metric space (X, d) endowed with the topology generated by the Graev extension $\hat{d}$ of a given metric d on X. We find that the group of Lipschitz functions $\mathrm{Lip}_0(X, \mathbb{T})$ is the group of continuous characters of $A_d(X)$. From this fact we derive some interesting properties of the metric groups $A_d(X)$ and $\mathrm{Lip}_0(X, \mathbb{T})$.

1. Introduction

Let (X, d) and (Y, d') be metric spaces and let $f: (X, d) \to (Y, d')$ be a mapping. We say that f is a Lipschitz mapping if there exists $\lambda \geq 0$ with $d'(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. The least such λ is called the Lipschitz constant $\|f\|_L$ of f. It is clear that $\|f\|_L = \sup_{x \neq y} \frac{d'(f(x), f(y))}{d(x,y)}$. The Lipschitz condition for functions between metric spaces allows to keep control of the distances in the target space knowing the distances in the domain. This is the key to the wide range of uses of this concept. We should mention as a fundamental prototype the Picard-Lindelöf theorem on the existence and uniqueness of solutions of ordinary differential equations [24].

If (X, d) is a metric space and 0 is a fixed element of X, $\mathrm{Lip}_0(X, \mathbb{R})$ denotes the Banach space of real-valued Lipschitz functions defined on X. It is the dual of the norm-closed subspace of $(\mathrm{Lip}_0(X, \mathbb{R}))^*$ generated by the Dirac measures, which is denoted by $\mathcal{F}(X)$ and called the Lipschitz-free Banach space over X. The Banach space $\mathcal{F}(X)$ is of interest in its own right: It is the universal Banach space that contains an

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isometric copy of X (see [25, Chapter 3]). The construction of $\mathcal{F}(X)$ can be traced back to the mid-20th century with the works of Kantorovich and Rubinstein on optimal transport problems [18]. There is an extensive literature on Lipschitz functions which has been growing during the last years (see for example [3,15,25]). Some of the recent lines of work have been the study of approximation of uniformly continuous real-valued functions by Lipschitz functions [14] and the description of the structure of Lipschitz-free spaces over different classes of metric spaces (see e.g. [1,12]). Another current problem is finding out whether the linear structure of a Banach or p-Banach space is completely determined by its Lipschitz structure [2].

Despite the importance of the concept of a Lipschitz mapping between metric spaces, Lipschitz functions with values in the unit circle $\mathbb{T}$ have not been considered yet. This paper comes to fill this gap. Given a metric space (X, d) with a fixed element $0 \in X$, the set $\mathrm{Lip}_0(X, \mathbb{T})$ of all Lipschitz functions from X into $\mathbb{T}$ has a group structure with the addition defined pointwise. If we endow it with the canonical metric given by the Lipschitz constant, we obtain a metric topological group which is related by duality with the free metric group $A_d(X)$ over the metric space X.

The group $A_d(X)$ has been considered by V. Bel'nov [8], O. Sipacheva and V. Uspenskij [22], A.V. Arhangel'skii and M. Tkachenko [6] and C. E. McPhail [20]. We establish several properties of this metric group. Making use of results of Enflo [13] and McPhail [20] we show in Theorem 2.11 that $A_d(X)$ is a metric subgroup of the Lipschitz-free Banach space $\mathcal{F}(X)$, which in particular implies that $A_d(X)$ is locally quasi-convex, and we build its completion in Theorem 2.9.

We prove in Theorem 3.1 that $\mathrm{Lip}_0(X, \mathbb{T})$ is exactly the group of continuous characters of the metric group $A_d(X)$ endowed with the topology of uniform convergence on the balls of $A_d(X)$. Thus it turns out that $\mathrm{Lip}_0(X, \mathbb{T})$ admits a predual, analogously to what happens with the Banach space $\mathrm{Lip}_0(X, \mathbb{R})$ which admits as predual the Lipschitz free Banach space $\mathcal{F}(X)$. However, the consideration of $\mathbb{T}$-valued functions forces us to work in the context of metric groups instead of the more restricted one of Banach spaces.

On the group of continuous characters of a metric group (G, d), the topology of uniform convergence on the bounded sets (which is the usual strong topology for the dual in locally convex vector spaces) may not coincide with the topology of uniform convergence on the balls defined by the metric d. However, on the dual of a Banach space E both topologies coincide. We establish in Theorems 4.2 and 4.5 that if (X, d) is a totally bounded metric space or a convex subspace of a linear normed space, then the balls of $A_d(X)$ are bounded and consequently on its dual group $\mathrm{Lip}_0(X, \mathbb{T})$, the topologies of uniform convergence on the bounded sets and on the balls coincide.

We apply the Pontryagin duality techniques to deduce several properties of the group $\mathrm{Lip}_0(X, \mathbb{T})$. In particular, we prove in Corollary 3.6 that this group is locally quasi-convex and complete.

1.1. Preliminaries

All the groups we are going to consider will be abelian.

For any metric group (G, d) the symbol d always refers to an invariant metric, and we will denote by $\|\cdot\|$ the associated group norm, i.e. $\|x\| = d(x, 0)$ for every $x \in G$. Further, for every $r > 0$ we define the closed r-ball of (G, d) with center at zero as $B_r = \{x \in G : \|x\| \leq r\}$.

We define the torus $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and denote by $\pi\colon \mathbb{R} \to \mathbb{T}$ the canonical projection. On $\mathbb{T}$ we consider the group norm $|\pi(r)| = d_{\mathbb{R}}(r + \mathbb{Z}, 0)$, where $d_{\mathbb{R}}$ is the usual metric on $\mathbb{R}$. Note that $|\pi(r)| \leq \min\{|r|, 1/2\}$ for every $r \in \mathbb{R}$, and $|\pi(r)| = |r|$ whenever $|r| \leq 1/2$. We put $\mathbb{T}_+ = \pi([-1/4, 1/4])$.

A character of an abelian group G is a homomorphism $\chi\colon G \to \mathbb{T}$. For any topological abelian group G, we denote by $\mathrm{Hom}(G, \mathbb{T})$ the group of characters of G and by $G^\wedge$ the group of continuous characters of G. We will call $G^\wedge$ the *dual group* of G. We will say that a topological abelian group G has *enough continuous characters* if for each $g \neq 0$ there is some element $\chi \in G^\wedge$ such that $\chi(g) \neq 0$.

Let G be a topological abelian group and let A be a subset of G. We say that A is *quasi-convex* if for any $x \in G \setminus A$ there exists $\chi \in G^\wedge$ with $\chi(A) \subseteq \mathbb{T}_+$ and $\chi(x) \notin \mathbb{T}_+$. We say that a topological abelian group G

is *locally quasi-convex* if G has a basis of neighborhoods of zero formed by quasi-convex sets. It is easy to see that a Hausdorff locally quasi-convex group has enough continuous characters.

Let G be an abelian group, $B \subseteq G$ a nonempty subset and $n \in \mathbb{N}$. Here and below nB will denote the set $\{nx : x \in B\}$. Obviously $nB \subseteq \underbrace{B + \cdots + B}_{n \text{ times}}$.

A nonempty family $\mathfrak{S}$ of subsets of a group G is called *well-directed* if the following conditions hold:

a) For $B_1, B_2 \in \mathfrak{S}$, there exists $B_3 \in \mathfrak{S}$ such that $B_1 \cup B_2 \subseteq B_3$.
b) For $B \in \mathfrak{S}$ and $n \in \mathbb{N}$, there exists $A \in \mathfrak{S}$ such that $nB \subseteq A$.

Examples of well-directed families are the family of all nonempty finite subsets, and the family of all compact subsets of a topological group G.

In what follows it will be convenient to fix a group duality (G, H), which consists of an abelian group G and a subgroup H of $\mathrm{Hom}(G, \mathbb{T})$.

Let (G, H) be a group duality and $\mathfrak{S}$ a well-directed family of nonempty subsets of G. Since $\mathbb{T}$ is a metric space, we can consider on $H \subseteq \mathbb{T}^G$ the topology $\tau_{\mathfrak{S}}(H, G)$ of uniform convergence on the sets $A \in \mathfrak{S}$. Clearly it is a group topology. If $\mathfrak{S}$ covers G, then $\tau_{\mathfrak{S}}(H, G)$ is Hausdorff. Moreover, the collection

$$\mathcal{A} = \{A^{\triangleright} : A \in \mathfrak{S}\}$$

is a fundamental system of neighborhoods of the neutral element e_H in the topology $\tau_{\mathfrak{S}}(H, G)$, where $A^{\triangleright} = \{\psi \in H : \psi(A) \in \mathbb{T}_+\}$. It is easy to prove that all topologies of this kind on H are locally quasi-convex.

Let G be a topological abelian group. If we consider the duality $(G, G^{\wedge})$ and $\mathfrak{S}$ is the well-directed family of compact subsets of G we obtain the classical Pontryagin duality. The Pontryagin-van Kampen Theorem says that if G is locally compact, its dual group $(G^{\wedge}, \tau_{\mathfrak{S}}(G^{\wedge}, G))$ is locally compact and the bidual group is topologically isomorphic to G through the canonical evaluation mapping $\alpha_G : G \to (G^{\wedge})^{\wedge}$ defined by $\alpha_G(g)(\chi) = \chi(g)$.

In the case of metrizable groups, the topology of uniform convergence on compact sets in the dual group is a k-space group topology that can fail to be metrizable (see [10]). As we will see later on, there are other well-directed families that we can use in the case of metric groups in order to obtain different metric topologies on the dual group.

2. The metric groups $\mathrm{Lip}_0(X, \mathbb{T})$ and $A_d(X)$

If (X, d) is a metric space with a fixed element $0 \in X$, we denote by $\mathrm{Lip}_0(X, \mathbb{T})$ the set of all Lipschitz mappings $f : (X, d) \to (\mathbb{T}, |\cdot|)$ which satisfy $f(0) = 0$. It is clearly an abelian group with the natural pointwise addition. We consider the invariant metric $(f, g) \mapsto \|f - g\|_L$ on this group. The metric group $(\mathrm{Lip}_0(X, \mathbb{T}), \|\cdot\|_L)$ is the natural group analogue of $\mathrm{Lip}_0(X, \mathbb{R})$. We are going to establish some properties of this group along the paper.

First we present some basic facts in the following propositions.

Let $\mathbb{T}_0^X$ be the group of all mappings $f : X \to \mathbb{T}$ with $f(0) = 0$, and $\|\cdot\|_\infty$ the group norm on $\mathbb{T}_0^X$ defined by $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We say that a metric space (X, d) is *uniformly discrete* if $\inf\{d(x, y) : x, y \in X, x \neq y\} > 0$.

Proposition 2.1. *If X is a uniformly discrete metric space, then* $\mathrm{Lip}_0(X, \mathbb{T})$ *and* $\mathbb{T}_0^X$ *coincide as sets, and the identity mapping* $(\mathbb{T}_0^X, \|\cdot\|_\infty) \to \mathrm{Lip}_0(X, \mathbb{T})$ *is Lipschitz.*

Proof. Put $\delta = \min\{d(x,y) : x, y \in X,\ x \neq y\}$. For every mapping $f\colon X \to \mathbb{T}$ we have

$$\sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)} \leq \frac{2\|f\|_\infty}{\delta}$$

This inequality implies that the sets $\mathbb{T}_0^X$ and $\mathrm{Lip}_0(X,\mathbb{T})$ coincide algebraically, and the identity mapping $(\mathbb{T}_0^X, \|\cdot\|_\infty) \to \mathrm{Lip}_0(X,\mathbb{T})$ is Lipschitz. □

Proposition 2.2. *Let* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *The mapping* $\Phi : \mathrm{Lip}_0(\mathbb{N}_0,\mathbb{T}) \to (\mathbb{T}^{\mathbb{N}}, \|\cdot\|_\infty)$ *defined by* $\Phi(f)(n) = f(n) - f(n-1)$ *is an isometric isomorphism.*

Proof. Note that by Proposition 2.1, $\mathrm{Lip}_0(\mathbb{N}_0,\mathbb{T})$ and $\mathbb{T}_0^{\mathbb{N}}$ coincide as sets. From this it is easy to deduce that Φ is bijective. It is clearly a homomorphism.

Given any $f \in \mathrm{Lip}_0(\mathbb{N}_0,\mathbb{T})$

$$\|f\|_L = \sup_{n,m\in\mathbb{N}_0,\ n\neq m} \frac{|f(n)-f(m)|}{|n-m|} \geq \sup_{n\in\mathbb{N}} |f(n)-f(n-1)| = \|\Phi(f)\|_\infty.$$

Conversely, if $f \in \mathrm{Lip}_0(\mathbb{N}_0,\mathbb{T})$ and $n, m \in \mathbb{N}_0$ with $n > m$,

$$\begin{aligned} \frac{|f(n)-f(m)|}{n-m} &\leq \frac{1}{n-m} \sum_{k=m+1}^{n} |f(k)-f(k-1)| \\ &\leq \frac{1}{n-m}(n-m) \sup_{k\in\mathbb{N}} |f(k)-f(k-1)| = \|\Phi(f)\|_\infty \end{aligned}$$

This implies that $\|f\|_L \leq \|\Phi(f)\|_\infty$. Hence Φ is an isometry. □

Proposition 2.3. *If* X *is a metric space with bounded metric, then the inclusion mapping* $\mathrm{Lip}_0(X,\mathbb{T}) \to (\mathbb{T}_0^X, \|\cdot\|_\infty)$ *is Lipschitz.*

Proof. Put $M = \max_{y\in X} d(y,0)$. For any $f \in \mathrm{Lip}_0(X,\mathbb{T})$ and every $x \in X \setminus \{0\}$ we have

$$|f(x)| \leq M \frac{|f(x)-f(0)|}{d(x,0)} \leq M\|f\|_L.$$

As $|f(x)| \leq M\|f\|_L$ is trivial for $x = 0$ we obtain $\|f\|_\infty \leq M\|f\|_L$. Hence the identity mapping $\mathrm{Lip}_0(X,\mathbb{T}) \to (\mathbb{T}_0^X, \|\cdot\|_\infty)$ is Lipschitz. □

Corollary 2.4. *Let* F *be a finite metric space with* $0 \in F$ *and* $|F| \geq 2$. *Then* $\mathrm{Lip}_0(F,\mathbb{T})$ *is Lipschitz isomorphic with* $(\mathbb{T}^n, \|\cdot\|_\infty)$, *where* $n = |F| - 1$.

Proof. This follows from Propositions 2.1 and 2.3. □

Observe that both the examples given in Proposition 2.2 and Corollary 2.4 have analogues for real Lipschitz functions. It is known that $\mathrm{Lip}_0(F,\mathbb{R})$ is isometrically isomorphic to $\mathbb{R}^{|F|-1}$ and $\mathrm{Lip}_0(\mathbb{N}_0,\mathbb{R})$ is isometrically isomorphic to the Banach space l_∞ (see for instance [4, Examples 2.3.6 and 2.3.7]).

Proposition 2.5. *Let* (X,d) *be a metric space with a distinguished point* $0 \in X$. *Let* D *be a dense subspace of* (X,d) *such that* $0 \in D$. *Then the restriction mapping* $f \mapsto f{\restriction_D}$, *with* $f \in \mathrm{Lip}_0(X,\mathbb{T})$, *is an isometric isomorphism of the metric group* $\mathrm{Lip}_0(X,\mathbb{T})$ *onto* $\mathrm{Lip}_0(D,\mathbb{T})$.

Proof. It suffices to show that for every $g \in \mathrm{Lip}_0(D, \mathbb{T})$, the unique uniformly continuous function $\tilde{g}\colon X \to \mathbb{T}$ extending g is a Lipschitz function with the same Lipschitz constant as g. Recall that for every $x \in X$, $\tilde{g}(x) = \lim_{n\to\infty} g(x_n)$, where $\{x_n : n \in \mathbb{N}\}$ is any sequence in D converging to x. Fix $x, y \in X$. Find sequences $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ in D with $x_n \to x$, $y_n \to y$. Letting $n \to \infty$ in the inequalities $|g(x_n) - g(y_n)| \le \|g\|_L\, d(x_n, y_n)$ we obtain $|\tilde{g}(x) - \tilde{g}(y)| \le \|g\|_L\, d(x, y)$. Since $x, y \in X$ were arbitrary, we get that $\tilde{g}$ is a Lipschitz function and $\|\tilde{g}\|_L \le \|g\|_L$. But $\tilde{g}$ extends g, so we have $\|\tilde{g}\|_L = \|g\|_L$. □

Every real-valued Lipschitz function defined on a nonempty subspace of a metric space X can be extended to a Lipschitz function defined on X (with the same Lipschitz constant). The well-known fact that the unit circle is not a retract of the unit disk shows that the analogous property does not hold for $\mathbb{T}$-valued Lipschitz functions. However, the following is true:

Proposition 2.6. *Let X be a metric space, X_0 be a nonempty subset of X and f_0 a Lipschitz function from X_0 to $\mathbb{T}$. Assume that there is a Lipschitz function $g_0\colon X_0 \to \mathbb{R}$ with $f_0 = \pi \circ g_0$. Then there exists a Lipschitz function $f\colon X \to \mathbb{T}$ which extends f_0 and satisfies $\|f\|_L \le \|g_0\|_L$.*

Proof. By [25, Theorem 1.33] there exists an extension $g : X \to \mathbb{R}$ of g_0 which has the same Lipschitz number as g_0. Define $f = \pi \circ g$. It is clear that f extends f_0. Further, given any $x, y \in X$ we have $|f(x) - f(y)| = |\pi(g(x) - g(y))| \le |g(x) - g(y)| \le \|g\|_L d(x, y) = \|g_0\|_L d(x, y)$. This completes the proof. □

Note that given a metric space (X, d) and a Lipschitz function $f : X \to \mathbb{T}$, in general f may fail to be factorizable through $\mathbb{R}$ in the sense that there exist a Lipschitz (or even continuous) function $g : X \to \mathbb{R}$ with $f = \pi \circ g$. An elementary example of this situation is the identity mapping of $\mathbb{T}$, which cannot be extended in this way by the classical Lifting Theorem for covering projections ([9, Theorem III.4.1]).

We now introduce the metric group $A_d(X)$.

Let X be a set with a fixed point $0 \in X$ and $A(X)$ be the Graev free abelian group over X with neutral element 0. Every nonzero element in $A(X)$ can be expressed as $g = \sum_{k=1}^{p} m_k z_k$, where $p \ge 1$, $m_1, \dots, m_p \in \mathbb{Z} \setminus \{0\}$ and $z_1, \dots, z_p$ are pairwise distinct elements of X. We will call this expression the *normal form* of g and $l(g) = \sum_{k=1}^{p} |m_k|$ the *length* of g. It is clear that $A(X)$ is algebraically isomorphic to the usual free abelian group $\mathbb{Z}^{(X\setminus\{0\})}$. Of course we have to adopt the convention that $0x = 0$ for each $x \in X$ and $m0 = 0$ for every $m \in \mathbb{Z}$. In particular $-0 = 0$.

For any given element $g \in A(X) \setminus \{0\}$ with normal form $g = \sum_{k=1}^{p} m_k z_k$ we denote by $\mathrm{supp}(g)$ the set $\{z_1, \dots, z_p\}$. We also put $\mathrm{supp}(0) = \emptyset$.

Let d be a pseudometric on X. Let us outline Graev's construction of an invariant pseudometric $\hat{d}$ on the group $A(X)$ which extends d. We refer the reader to [16] or [6] for the details.

For every $g \in A(X)$ define $N(g) = \inf \sum_{i=1}^{n} d(x_i, y_i)$ where the infimum is taken over all finite sets of pairs $\{(x_1, y_1), \dots, (x_n, y_n)\} \subseteq X^2$, where $n \in \mathbb{N}$ and $g = \sum_{i=1}^{n} (x_i - y_i)$.

It is clear that N is a group pseudonorm, that is, it satisfies

$$N(g) \ge 0,\ N(-g) = N(g),\ N(g + h) \le N(g) + N(h)$$

for all $g, h \in A(X)$.

Fix $g \in A(X) \setminus \{0\}$ with normal form $g = \sum_{k=1}^{p} m_k z_k$. Let $I_+ = \{k \in \{1, \dots, p\} : m_k > 0\}$ and $I_- = \{k \in \{1, \dots, p\} : m_k < 0\}$. It can be shown by a successive elimination process that the infimum defining $N(g)$ is attained on a representation $g = \sum_{i=1}^{n} (x_i - y_i)$ such that

(a) each element x_i is either 0 or z_k for some $k \in I_+$. Moreover each one of these z_k appears exactly m_k times in the representation.

(b) each y_i is either 0 or z_k for some $k \in I_-$. Moreover each one of these z_k appears exactly $|m_k|$ times in the representation.

(c) $n = \max\{\sum_{k\in I_+} m_k,\ \sum_{k\in I_-} |m_k|\} \le l(g)$ and in particular either $x_i \neq 0$ for every $i \in \{1,\dots,n\}$ or $y_i \neq 0$ for every $i \in \{1,\dots,n\}$.

For instance, if $h = x + y - z$, where $x, y, z \in X \setminus \{0\}$, then the minimum defining $N(h)$ is attained at one of the following two representations of h:

$$h = (x - z) + (y - 0), \quad h = (y - z) + (x - 0)$$

and consequently, $N(h) = \min\{d(x, z) + d(y, 0),\ d(y, z) + d(x, 0)\}$.

Define $\hat{d}(g, h) = N(g - h)$ for every $g, h \in A(X)$. It is clear from the preceding considerations that $\hat{d}$ is a maximal invariant pseudometric on $A(X)$ which extends d, and that if d is a metric, so is $\hat{d}$. The group $A(X)$ with metric $\hat{d}$ is a Hausdorff topological group with a base of neighborhoods of 0 formed by the sets

$$B_\varepsilon = \{g \in A(X) : \hat{d}(g, 0) \le \varepsilon\}, \tag{1}$$

where ε is an arbitrary positive real number. For brevity, the metric group $(A(X), \hat{d})$ will be denoted by $A_d(X)$ and will be called the *Lipschitz-free* metric group over X. We denote by $\imath\colon X \to A_d(X)$ the corresponding (isometric) inclusion mapping.

Proposition 2.7. *Let (X, d) be a metric space with a distinguished point $0 \in X$. If $f\colon X \to H$ is a Lipschitz mapping to a metric group H with $f(0) = 0$ and $\bar{f}\colon A_d(X) \to H$ is the group homomorphism extending f, then $\bar{f}$ is a Lipschitz mapping with the same Lipschitz constant as f.*

Proof. Fix $g \in A_d(X)$ and let $g = (x_1 - y_1) + \cdots + (x_n - y_n)$ be a representation of g such that $\hat{d}(g, 0) = \sum_{i=1}^n d(x_i, y_i)$. Then

$$\|\bar{f}(g)\| = \|\sum_{i=1}^n (f(x_i) - f(y_i))\| \le \sum_{i=1}^n \|f(x_i) - f(y_i)\| \le \sum_{i=1}^n \|f\|_L d(x_i, y_i) = \|f\|_L \hat{d}(g, 0).$$

This implies that $\bar{f}$ is Lipschitz and $\|\bar{f}\|_L \le \|f\|_L$. Since $\bar{f}$ extends f it is clear that $\|\bar{f}\|_L = \|f\|_L$. □

The above result justifies the name "Lipschitz-free metric group" that we will use for $A_d(X)$.

Corollary 2.8. *Let (X, d) and (Y, d') be metric spaces with distinguished points $0 \in X$ and $0' \in Y$. Let $A_d(X)$ and $A_{d'}(Y)$ be the respective Lipschitz-free metric groups. Any Lipschitz function $h\colon X \to Y$ with $h(0) = 0$ extends uniquely to a Lipschitz homomorphism $\tilde{h}\colon A_d(X) \to A_{d'}(Y)$. Moreover h and $\tilde{h}$ have the same Lipschitz constant.*

Proof. This follows easily from Proposition 2.7 by taking $\tilde{h} = \overline{\imath_Y \circ h}$. □

The extension property described in Proposition 2.7 and Corollary 2.8 allows us to carry the structure of an arbitrary Lipschitz mapping over the domain and range of a simpler mapping, namely a homomorphism between the corresponding Lipschitz-free metric groups. This construction can be invoked when trying to rule out the existence of bi-Lipschitz homeomorphisms between two metric spaces, by proving that their Lipschitz-free metric groups are not isomorphic to each other.

Next we will describe the (Raĭkov) completion of the metric group $A_d(X)$, which will be denoted by G_X. First we consider the set S_X which consists of formal infinite sums

$$(s_1^1 - s_1^2) + \cdots + (s_n^1 - s_n^2) + \cdots ,$$

where $s_n^1, s_n^2 \in X$ for each $n \geq 1$, and the sum

$$\sum_{n=1}^{\infty} d(s_n^1, s_n^2)$$

converges.

For every integer $n \geq 1$, we define the truncating mapping $\varphi_n \colon S_X \to A_d(X)$ by $\varphi_n(s) = \sum_{i=1}^{n}(s_i^1 - s_i^2)$, where $s = \sum_{i=1}^{\infty}(s_i^1 - s_i^2) \in S_X$. Consider the mapping

$$(s,t) \in S_X \times S_X \mapsto \lim_{n\to\infty} N(\varphi_n(s) - \varphi_n(t)) \in [0, \infty),$$

where N denotes the above defined group norm on $A_d(X)$ corresponding to the metric d on X. Note that this limit is well defined since $(N(\varphi_n(s) - \varphi_n(t)))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $s, t \in S_X$. Indeed, given any $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned}
&|N(\varphi_m(s) - \varphi_m(t)) - N(\varphi_n(s) - \varphi_n(t))| \leq N(\varphi_m(s) - \varphi_m(t) - \varphi_n(s) + \varphi_n(t)) \\
&\leq N(\varphi_m(s) - \varphi_n(s)) + N(\varphi_m(t) - \varphi_n(t)) = N(\sum_{i=n+1}^{m}(s_i^1 - s_i^2)) + N(\sum_{i=n+1}^{m}(t_i^1 - t_i^2)) \\
&\leq \sum_{i=n+1}^{m} d(s_i^1, s_i^2) + \sum_{i=n+1}^{m} d(t_i^1, t_i^2).
\end{aligned}$$

It is clear that the above defined mapping is a pseudometric. In particular, it induces an equivalence relation on S_X given by

$$s \approx t \Leftrightarrow \lim_{n\to\infty} N(\varphi_n(s) - \varphi_n(t)) = 0 \Leftrightarrow \lim_{n\to\infty} \hat{d}\Big(\sum_{i=1}^{n}(s_i^1 - s_i^2), \sum_{i=1}^{n}(t_i^1 - t_i^2)\Big) = 0.$$

We denote by d^* the corresponding quotient metric on the quotient set $G_X := S_X/\approx$.

Next we will define a sum on G_X. Given $s, t \in S_X$ we put

$$s + t := (s_1^1 - s_1^2) + (t_1^1 - t_1^2) + \cdots + (s_n^1 - s_n^2) + (t_n^1 - t_n^2) + \cdots$$

It is not difficult to show that the binary operation on G_X given by $[s] + [t] = [s + t]$ is well defined, and gives G_X an abelian group structure. Here $[s]$ denotes the equivalence class of an element $s \in S_X$ w.r.t. $\approx$. The neutral element of G_X is $[(0-0) + (0-0) + \cdots]$. Given any $s = (s_1^1 - s_1^2) + \cdots + (s_n^1 - s_n^2) + \cdots$ in S_X, the inverse of $[s]$ is $[-s]$, where $-s = (s_1^2 - s_1^1) + \cdots + (s_n^2 - s_n^1) + \cdots$. Clearly the metric d^* is invariant with respect to the sum operation thus defined.

Consider the mapping $j_X : A_d(X) \to G_X$ defined by

$$j_X(g) = [(x_1 - y_1) + \cdots + (x_n - y_n) + (0-0) + (0-0) + \cdots]$$

where $g = (x_1 - y_1) + \cdots + (x_n - y_n)$ with $x_i,\, y_i \in X$. This mapping is well defined (i.e. does not depend on the chosen representation of g) and it is an isometric monomorphism. In what follows we identify $A_d(X)$ with its image $j_X(A_d(X))$ in G_X.

It is easy to see that if $[s] \in G_X$, where $s = (s_1^1 - s_1^2) + (s_2^1 - s_2^2) + \cdots$, then the sequence $[\varphi_n(s)]$ converges to $[s]$. This shows that $A_d(X)$ is a dense subgroup of G_X.

Theorem 2.9. *For every metric space (X, d), the group (G_X, d^*) is complete.*

Proof. Take an arbitrary Cauchy sequence $\{[s_n] : n \in \mathbb{N}\}$ in (G_X, d^*). Choosing a subsequence of this sequence, if necessary, we can assume that $d^*([s_n], [s_m]) < 1/2^n$ if $n < m$. For every $n \in \mathbb{N}$, let $s_n = \sum_{i=1}^{\infty}(s^1_{n,i} - s^2_{n,i})$, where $s^1_{n,i}, s^2_{n,i} \in X$. Given $n \in \mathbb{N}$ there exists an integer $k_n \geq 1$ such that $\sum_{i=k_n}^{\infty} d(s^1_{n,i}, s^2_{n,i}) < 1/2^n$. Note that for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\hat{d}(\varphi_{k_{n+1}}(s_{n+1}), \varphi_{k_n}(s_n)) &= d^*([\varphi_{k_{n+1}}(s_{n+1})], [\varphi_{k_n}(s_n)]) \\ &\leq d^*([\varphi_{k_{n+1}}(s_{n+1})], [s_{n+1}]) + d^*([s_{n+1}], [s_n]) + d^*([s_n], [\varphi_{k_n}(s_n)]) \\ &< 3/2^n.\end{aligned}$$

By the definition of the metric $\hat{d}$ on $A_d(X)$, this means that for every $n \in \mathbb{N}$ we can represent the difference $\varphi_{k_{n+1}}(s_{n+1}) - \varphi_{k_n}(s_n)$ in the form

$$\varphi_{k_{n+1}}(s_{n+1}) - \varphi_{k_n}(s_n) = (a_{n,1} - b_{n,1}) + \cdots + (a_{n,l_n} - b_{n,l_n}), \tag{2}$$

where $l_n \in \mathbb{N}$, $a_{n,i}, b_{n,i} \in X$ and

$$\sum_{i=1}^{l_n} d(a_{n,i}, b_{n_i}) < 3/2^n. \tag{3}$$

For every integer $n \geq 1$, let

$$t_n = \sum_{j=1}^{n}\sum_{i=1}^{l_j}(a_{j,i} - b_{j,i}) \text{ and } t = \sum_{j=1}^{\infty}\sum_{i=1}^{l_j}(a_{j,i} - b_{j,i}).$$

It follows from (3) that $\sum_{j=1}^{\infty}\sum_{i=1}^{l_j} d(a_{j,i} - b_{j,i}) \leq 3$, so $[t] \in G_X$ and by our proof of the density claim, $[t_n] \to [t]$ in (G_X, d^*). Also, (2) and the definition of t_n together imply that $t_n = \sum_{j=1}^{n}(\varphi_{k_{j+1}}(s_{j+1}) - \varphi_{k_j}(s_j)) = \varphi_{k_{n+1}}(s_{n+1}) - \varphi_{k_1}(s_1)$. This implies that the sequence $\big([\varphi_{k_n}(s_n)]\big)_{n\in\mathbb{N}}$ converges in (G_X, d^*). Finally, it follows from the inequality $d^*([\varphi_{k_n}(s_n)], [s_n]) < 1/2^n$ that $\big([s_n]\big)_{n\in\mathbb{N}}$ is convergent, too. Hence the group (G_X, d^*) is complete. □

We devote the remaining of this section to proving that $A_d(X)$ is isometrically embedded in the Lipschitz-free Banach space $\mathcal{F}(X)$. (We only consider real vector spaces in this paper.) The following fact was established in [20] by applying a criterion due to Enflo [13] to the metric group $A_d(X)$:

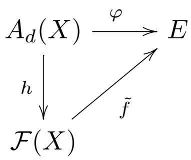
Proposition 2.10. *[20] The metric group $A_d(X)$ is isometrically isomorphic to a subgroup of a Banach space.*

We follow the definition of the Lipschitz-free Banach space $\mathcal{F}(X)$ given in [25, Chapter 3] (see also [5,21]). Recall that X is isometrically embedded in $\mathcal{F}(X)$ in such a way that the point $0 \in X$ is in correspondence with the zero element of $\mathcal{F}(X)$, and every Lipschitz mapping $f\colon X \to E$ to a Banach space E that sends 0 to the zero element of E extends to a unique linear operator $\tilde{f}\colon \mathcal{F}(X) \to E$ with norm $\|f\|_L$.

Theorem 2.11. *The group $A_d(X)$ is isometrically isomorphic to a subgroup of the Lipschitz-free Banach space $\mathcal{F}(X)$.*

Proof. By Proposition 2.10, there exists an isometric isomorphism $\varphi\colon A_d(X) \to E$ of $A_d(X)$ to a Banach space E. We identify (X, d) with the corresponding subspaces of $A_d(X)$ and $\mathcal{F}(X)$. The mapping $f = \varphi\restriction_X$

of X to E is an isometry, so it extends to a linear operator $\tilde{f}\colon \mathcal{F}(X) \to E$ with $\|\tilde{f}\| = 1$. By Proposition 2.7, the canonical isometric embedding of X to $\mathcal{F}(X)$ extends to a homomorphism $h\colon A_d(X) \to \mathcal{F}(X)$ with $\|h\|_L = 1$. Then $\varphi = \tilde{f} \circ h$, because both homomorphisms coincide on the subset X of $A_d(X)$.



Let $||\cdot||$ and $||\cdot||_E$ be the respective norms of $\mathcal{F}(X)$ and E. Take arbitrary elements $u, v \in A_d(X)$. Since $\|h\|_L = 1$ we obtain $||h(u) - h(v)|| \leq \hat{d}(u, v)$. On the other hand, $\varphi(u) = \tilde{f}(h(u))$ and $\varphi(v) = \tilde{f}(h(v))$, so

$$\hat{d}(u, v) = \hat{d}(u - v, 0) = ||\varphi(u) - \varphi(v)||_E \leq ||h(u) - h(v)||$$

We conclude that $\hat{d}(u, v) = ||h(u) - h(v)||$ and, hence, h is an isometric isomorphic embedding. □

In order to establish further properties of the groups $\mathrm{Lip}_0(X, \mathbb{T})$ and $A_d(X)$ we will use some duality techniques that we present in the next section.

3. Duality of metric groups

Let us introduce the *b-duality* in the class of metric groups.

If (G, d) is a metric group, the family of all balls B_r $(r > 0)$ of G is well-directed. The symbol $G_b^\wedge$ denotes the group $G^\wedge$ endowed with the topology $\tau_{\mathfrak{S}}(G^\wedge, G)$ of uniform convergence on the balls of G. In the next result we introduce a natural metric on $G^\wedge$ which is compatible with this topology.

Theorem 3.1. *Let (G, d) be a metric group with invariant metric d. Then every continuous character of (G, d) is Lipschitz. The topology τ_b of the dual group $G_b^\wedge$ is the one induced by the norm $\|\chi\|_L = \sup_{x \neq 0} |\chi(x)|/\|x\|$. In other words, $G^\wedge \subseteq \mathrm{Lip}_0(G, \mathbb{T})$ and $G_b^\wedge$ carries the topology induced by the restriction of the natural metric on $\mathrm{Lip}_0(G, \mathbb{T})$.*

Proof. Given a character $\chi \in G_b^\wedge$, we choose $r > 0$ such that $\chi(B_r) \subseteq \mathbb{T}_+$. Let us show that $|\chi(x)| \leq \frac{1}{2r}\|x\|$ for every $x \in G$. Indeed, the case $x = 0$ is trivial. If $0 < \|x\| \leq r$ we fix $n \in \mathbb{N}$ with $\frac{r}{n+1} < \|x\| \leq \frac{r}{n}$. Then for every $k \in \{1, 2, \dots, n\}$, we have $\|kx\| \leq r$, hence $\chi(kx) = k\chi(x) \in \mathbb{T}_+$. This implies $|\chi(x)| \leq 1/4n < (1/2r)\|x\|$. Finally if $\|x\| > r$ the inequality $|\chi(x)| \leq \frac{1}{2r}\|x\|$ is a simple consequence of the fact that $|\cdot|$ is bounded by $1/2$. This shows that $G^\wedge \subseteq \mathrm{Lip}_0(G, \mathbb{T})$, and actually the polar $(B_r)^\triangleright$ is contained in the $\|\cdot\|_L$-ball $B_{1/2r}$ for any $r > 0$. Conversely, it is immediate that the $\|\cdot\|_L$-ball B_r is contained in $(B_{1/4r})^\triangleright$, for every $r > 0$. Since, by the above observations, the family of sets $(B_r)^\triangleright$ $(r > 0)$ is a basis of neighborhoods of zero for the topology τ_b on the dual group $G_b^\wedge$, the proof is complete. □

In what follows we will always consider on $G_b^\wedge$ the metric structure described in Theorem 3.1. Some properties of the metric dual group $G_b^\wedge$ are collected in the following result:

Proposition 3.2. *Let (G, d) be a metric group with invariant metric d. Then,*

(a) *If d is bounded then $G_b^\wedge$ is discrete. If (G, d) is discrete then the metric on $G_b^\wedge$ is bounded.*
(b) *The metric group $\left(\mathbb{Z}, |\cdot|\right)_b^\wedge$ is naturally isometric with $(\mathbb{T}, |\cdot|)$ and the metric group $\left(\mathbb{T}, |\cdot|\right)_b^\wedge$ is naturally isometric with $(\mathbb{Z}, |\cdot|)$.*

(c) *The group $G_b^\wedge$ is locally quasi-convex.*
(d) *The group $G_b^\wedge$ is complete.*

Proof. (a) Assume that the metric d is bounded. Let $M > 0$ be such that $G = B_M$. Then $B_M^\triangleright = \{0\}$ is a neighborhood of zero in $G_b^\wedge$, so $G_b^\wedge$ is discrete.

Assume that (G, d) is discrete. Since the metric d is invariant (G, d) is a uniformly discrete metric space, and from Proposition 2.1 it easily follows that the metric on $\mathrm{Lip}_0(G, \mathbb{T})$ is bounded. Hence its metric subgroup $G_b^\wedge$ has a bounded metric, too.

(b) Let $\varphi\colon (\mathbb{Z}, |\cdot|)^\wedge \to \mathbb{T}$ be defined by $\chi(n) = n\varphi(\chi)$ for every $\chi \in (\mathbb{Z}, \|\cdot\|)^\wedge$ and every $n \in \mathbb{Z}$. It is known that φ is a well-defined algebraic isomorphism. We have $\|\chi\|_L = \sup_{n\neq 0} |\chi(n)|/|n| = \sup_{n\neq 0} |n\varphi(\chi)|/|n| = |\varphi(\chi)|$. We have used the elementary fact that $|t| = \sup_{n\neq 0} |nt|/|n|$ for every $t \in \mathbb{T}$.

Let $\psi\colon (\mathbb{T}, |\cdot|)^\wedge \to \mathbb{Z}$ be defined by $\chi(t) = \psi(\chi)\cdot t$ for every $\chi \in (\mathbb{T}, |\cdot|)^\wedge$ and $t \in \mathbb{T}$. It is known that ψ is a well-defined algebraic isomorphism. We have $\|\chi\|_L = \sup_{t\neq 0} |\chi(t)|/|t| = \sup_{t\neq 0} |\psi(\chi)\cdot t|/|t| = |\psi(\chi)|$. We have used the elementary fact that $|n| = \sup_{t\neq 0} |nt|/|t|$ for every $n \in \mathbb{Z}$.

(c) This is a consequence of Proposition 3.4(b) in [11].

(d) Fix a Cauchy sequence (χ_n) in $G_b^\wedge$. Let $\chi \in \mathrm{Hom}(G, \mathbb{T})$ be the pointwise limit of (χ_n). Fix any $r > 0$. The sequence $(\chi_n\restriction_{B_r})$ is a Cauchy sequence in the group $\mathcal{C}(B_r, \mathbb{T})$ with the uniform convergence topology. This group is complete (see for instance [19, Theorem 7.8]). It is clear that the limit of the sequence $(\chi_n\restriction_{B_r})$ is the restriction of χ to B_r. Since this limit is continuous on the neighborhood of zero B_r, we conclude that $\chi \in G^\wedge$. Letting r run over $(0, \infty)$ we deduce that (χ_n) converges to χ in $G_b^\wedge$. □

We next show that the metric group $A_d(X)$ is a b-predual for the Lipschitz group $\mathrm{Lip}_0(X, \mathbb{T})$.

Theorem 3.3. *The restriction mapping $\Psi\colon A_d(X)_b^\wedge \to \mathrm{Lip}_0(X, \mathbb{T})$ defined by $\Psi(\chi)(x) = \chi(x)$ for each $\chi \in A_d(X)^\wedge$, is a well defined isometry with respect to the corresponding Lipschitz metrics. In particular, the topological groups $A_d(X)_b^\wedge$ and $\mathrm{Lip}_0(X, \mathbb{T})$ are topologically isomorphic.*

Proof. By Theorem 3.1, $A_d(X)^\wedge \subseteq \mathrm{Lip}_0(A_d(X), \mathbb{T})$. Since X is isometrically embedded in $A_d(X)$, it is clear that $\chi\restriction_X$ is a Lipschitz mapping for every $\chi \in A_d(X)^\wedge$ and thus Ψ is well defined. Conversely, for any $f \in \mathrm{Lip}_0(X, \mathbb{T})$ its homomorphic extension to $A_d(X)$ is unique and it is a continuous character with the same Lipschitz constant, by Proposition 2.7. □

Proposition 3.4. *For every metric space (X, d), the metric groups $A_d(X)$ and G_X are locally quasi-convex.*

Proof. Since every Banach space is a locally quasi–convex topological group [7, Proposition 2.4] and every subgroup of a locally quasi-convex group is locally quasi-convex, we obtain from Theorem 2.11 that $A_d(X)$ is locally quasi-convex and so is its completion G_X. □

Remark 3.5. Note that the fact that $A_d(X)$ has enough continuous characters, which is a corollary of Proposition 3.4, admits a simpler proof: Fix $g \in A_d(X)$ with $g \neq 0$. Let $g = \sum_{i=1}^n m_i x_i$ be the normal form of g. From Proposition 2.6 it is clear that there exists $f \in \mathrm{Lip}_0(X, \mathbb{T})$ with $m_1 f(x_1) \neq 0$ and $f(x_i) = 0$ for every $i \in \{2, \ldots, n\}$. With the notation of Theorem 3.3, we have $\Psi^{-1}(f)(g) = \sum_{i=1}^n m_i f(x_i) = m_1 f(x_1) \neq 0$.

Corollary 3.6. *Let X be a metric space. Then*

(a) $\mathrm{Lip}_0(X, \mathbb{T})$ *is locally quasi-convex.*
(b) $\mathrm{Lip}_0(X, \mathbb{T})$ *is complete.*

Proof. By Theorem 3.3, the topological group $\mathrm{Lip}_0(X, \mathbb{T})$ is topologically isomorphic to the dual group of $A_d(X)$ endowed with the topology of uniform convergence on the balls of $A_d(X)$. By Proposition 3.2, this group is locally quasi-convex and complete. □

Proposition 3.7. *Let (X, d) be a metric space. The mapping $\Phi\colon (X, d) \to \mathrm{Lip}_0(X, \mathbb{T})^\wedge_b$ defined by $\Phi(x)(f) = f(x)$ is an isometric embedding.*

Proof. The mapping Φ is well defined — it is clear that $\Phi(x)$ is a character of $\mathrm{Lip}_0(X, \mathbb{T})$, for every $x \in X$. Let us check that $\Phi(x)$ is continuous. For every $x \in X$ and $f \in \mathrm{Lip}_0(X, \mathbb{T})$ with $f \neq 0$, we have

$$\frac{|\Phi(x)(f)|}{\|f\|_L} = \frac{|f(x)|}{\|f\|_L} \leq d(x, 0).$$

Thus $\Phi(x)$ is a Lipschitz mapping and in particular it is continuous.

Fix $x, y \in X$ and let us show that $\|\Phi(x) - \Phi(y)\| = d(x, y)$. We have

$$\|\Phi(x) - \Phi(y)\| = \sup_{f \neq 0} \frac{|f(x) - f(y)|}{\|f\|_L} \leq d(x, y).$$

Let us prove the opposite inequality. Fix $M \in \mathbb{R}$ such that $M > 2d(x, y)$. Define

$$f(z) = \pi\Big(\frac{1}{M}\big(d(x, z) - d(x, 0)\big)\Big)$$

The mapping f is clearly an element of $\mathrm{Lip}_0(X, \mathbb{T})$. Given any $z_1, z_2 \in X$ we have

$$f(z_1) - f(z_2) = \pi(\frac{1}{M}(d(x, z_1) - d(x, z_2)))$$

Hence

$$|f(z_1) - f(z_2)| \leq \frac{1}{M}|d(x, z_1) - d(x, z_2)| \leq \frac{d(z_1, z_2)}{M}$$

and we deduce that $\|f\|_L \leq 1/M$. Note that

$$|f(x) - f(y)| = |\pi(\frac{d(x, y)}{M})| = \frac{d(x, y)}{M}$$

(the last equality follows from $d(x, y)/M < 1/2$). We finally obtain

$$\frac{|f(x) - f(y)|}{\|f\|_L} \geq d(x, y). \quad \square$$

4. Metric duality and bounded subsets

Duality of locally convex spaces constitutes a well-established topic in Functional Analysis. M. F. Smith [23] was the first to relate the Pontryagin dual of a locally convex space X to the traditional notion of duality in the Functional Analysis sense. By the dual of X it is commonly understood the linear space X^* of continuous linear functionals on X endowed with the topology of uniform convergence on the family of all bounded subsets of X. If X^*_β denotes the dual so topologized then X is said to be reflexive if the canonical mapping from X to $(X^*_\beta)^*_\beta$ is a topological isomorphism. Smith points out that X^* and $X^\wedge$ are algebraically isomorphic as groups. Once the algebraic isomorphism is proved, it is easy to see that X^*_β and $X^\wedge_\beta$ are also isomorphic as topological groups. In [23,7] it is also proved that all reflexive locally convex spaces and all Banach spaces are Pontryagin reflexive as topological groups. The result is valid for completely metrizable locally convex spaces as well [7, Proposition 15.2]. Thus Pontryagin reflexivity is strictly weaker than reflexivity in the sense of Functional Analysis.

In the framework of topological groups there exists a different notion of boundedness that we recall here.

Let G be a topological group. We say that a subset B of G is bounded (notion introduced by Hejcman [17] for uniform spaces) if for every neighborhood U of zero there exist a finite set $F \subseteq G$ and some $n \in \mathbb{N}$ such that $B \subseteq F + \underbrace{U + \cdots + U}_{n \text{ times}}$.

If G is metrizable, a set $B \subseteq G$ is bounded according to this definition if and only if it is d-bounded for any compatible invariant metric d on G [17, Theorem 2.6]. If X is a locally convex space, a set $B \subseteq X$ is bounded according to this definition if and only if it is bounded in the usual sense.

The family of bounded subsets of G is well-directed and we can endow the group $G^\wedge$ with the topology of uniform convergence on the bounded subsets of G. We denote that group by $G^\wedge_\beta$. Observe that for a metric group (G, d), the identity mapping $G^\wedge_b \to G^\wedge_\beta$ is always continuous. If every ball in (G, d) is bounded, then the identity mapping $G^\wedge_b \to G^\wedge_\beta$ is a topological isomorphism. That happens of course if G is a normed vector space. In the latter case the following result is valid.

Proposition 4.1. *If $(Y, \|\cdot\|)$ is a normed space, then*

(a) *The groups $(Y, \|\cdot\|)^\wedge_\beta$ and $(Y, \|\cdot\|)^\wedge_b$ are topologically isomorphic.*
(b) *The mapping $\Phi\colon Y^* \to (Y, \|\cdot\|)^\wedge$ defined by $\Phi(f)(x) = \pi(f(x))$ for every $x \in Y$ and $f \in Y^*$ is an isometry between $(Y^*, \|\cdot\|^*)$ and $(Y, \|\cdot\|)^\wedge_b$. In particular, the metric groups $(\mathbb{R}, \|\cdot\|)$, $(\mathbb{R}, \|\cdot\|)^\wedge_b$ and $(\mathbb{R}, \|\cdot\|)^\wedge_\beta$ are isometric.*

Proof. (a) Observe that a normed space is a metric group in which every ball is bounded.

(b) It is known that Φ is an algebraic isomorphism [23]. Let us show that $\|\Phi(f)\|_L = \|f\|^*$ for every $f \in Y^*$. Indeed,

$$\|\Phi(f)\|_L = \sup_{x \neq 0} |\Phi(f)(x)|/\|x\| = \sup_{x \neq 0} |\pi(f(x))|/\|x\|.$$

We claim that the latter supremum is $\|f\|^*$. The case $f = 0$ is trivial. Assume $\|f\|^* > 0$. On the one hand, $\sup_{x \neq 0} |\pi(f(x))|/\|x\| \leq \sup_{x \neq 0} |f(x)|/\|x\| = \|f\|^*$. On the other hand, given any $\varepsilon > 0$ with $\varepsilon < \|f\|^*$ we can find $x \in Y$ with $0 < \|x\| \leq \frac{1}{2\|f\|^*}$ such that $|f(x)|/\|x\| \geq \|f\|^* - \varepsilon$; since $|f(x)| \leq \|f\|^* \|x\| \leq 1/2$ we have $|\pi(f(x))| = |f(x)|$ and hence $|\pi(f(x))|/\|x\| = |f(x)|/\|x\| \geq \|f\|^* - \varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small, we have proved that $\sup_{x \neq 0} |\pi(f(x))|/\|x\| \geq \|f\|^*$. □

We next consider the same problem (equivalence of metric boundedness and boundedness in Hejcman sense) on a different class of metric groups: those of the form $A_d(X)$ where (X, d) is a metric space.

Let us say that a subset B of a topological group G is *strongly bounded* if for every neighborhood U of zero in G, there exists an integer $n \geq 1$ such that $B \subseteq \underbrace{U + \cdots + U}_{n \text{ times}}$.

Theorem 4.2. *Let X be a convex subspace of a linear normed space L and d be a metric on X induced by the norm of L. Then for every number $t > 0$, the ball $B_t \subseteq A_d(X)$ is strongly bounded in $A_d(X)$. In particular, the group $A_d(X)$ has a base at zero of strongly bounded open sets.*

Proof. Let $||\cdot||$ be the norm of L. Then the metric d on X is defined by $d(x, y) = ||x - y||$, for all $x, y \in X$. For a given number $t > 0$, we will show that the ball

$$B_t = \{g \in A_d(X) : \hat{d}(g, 0) \leq t\}$$

is strongly bounded in $A_d(X)$, where 0 is the neutral element of $A_d(X)$ and $\hat{d}$ is Graev's extension of d over $A_d(X)$. We can assume without loss of generality that X contains the zero element of L which is identified with the neutral element of the group $A_d(X)$.

Let U be a neighborhood of 0 in $A_d(X)$. By our definition of the topology of $A_d(X)$, there exists $\varepsilon > 0$ such that $B_\varepsilon = \{g \in A_d(X) : \hat{d}(g, 0) \leq \varepsilon\} \subseteq U$. Choose an integer $n \geq 1$ such that $t \leq n\varepsilon$. We claim that $B_t \subseteq \underbrace{U + \cdots + U}_{n \text{ times}}$. Indeed, take an arbitrary element $g \in B_t$. Then one can find an integer $p > 0$ and elements $x_i, y_i \in X$ for $i = 1, \ldots, p$ such that

$$g = (x_1 - y_1) + \cdots + (x_p - y_p) \tag{4}$$

and $\sum_{i=1}^{p} d(x_i, y_i) \leq t$. Since X is a convex subspace of L, the segment $[x_i, y_i]$ in L connecting the elements x_i and y_i is a subset of X, for each $i \leq p$. Let $\{a_{i,0}, a_{i,1}, \ldots, a_{i,n}\}$, with $a_{i,0} = x_i$ and $a_{i,n} = y_i$, be the uniform partition of $[x_i, y_i]$, so $d(a_{i,j}, a_{i,j+1}) = d(x_i, y_i)/n$ for each $j = 0, \ldots, n-1$. Clearly we have the equality

$$x_i - y_i = (a_{i,0} - a_{i,1}) + \cdots + (a_{i,n-1} - a_{i,n}). \tag{5}$$

For each $j \in \{0, 1, \ldots, n-1\}$, let

$$h_j = (a_{0,j} - a_{0,j+1}) + \cdots + (a_{p,j} - a_{p,j+1}). \tag{6}$$

Then $\sum_{i=0}^{p} d(a_{i,j}, a_{i,j+1}) = \frac{1}{n} \sum_{i=0}^{p} d(x_i, y_i) \leq \frac{t}{n} \leq \varepsilon$, whence it follows that $h_j \in B_\varepsilon$ for each $j \leq n-1$. Combining (4), (5) and (6), we see that $g = h_0 + \cdots + h_{n-1} \in \underbrace{B_\varepsilon + \cdots + B_\varepsilon}_{n \text{ times}}$. This completes the proof since $B_\varepsilon \subseteq U$. □

In Theorem 4.5 below, we present a weaker version of Theorem 4.2 for totally bounded metric spaces. First we prove an auxiliary combinatorial lemma.

Lemma 4.3. *Let $k, n \geq 1$ be integers, $t > 0$ and $r_1, \ldots, r_k$ be real numbers satisfying $0 \leq r_i \leq t/n$ for each $i \leq k$. If $\sum_{i=1}^{k} r_i \leq t$, then one can find a positive integer $m \leq 2n-1$ and a partition $\{A_1, \ldots, A_m\}$ of the set $\{1, \ldots, k\}$ such that $\sum_{i \in A_j} r_i \leq t/n$, for each $j = 1, \ldots, m$.*

Proof. The conclusion of the lemma is trivially valid for $k = 1$. Assuming the validity of the lemma for a given integer $k \geq 1$, we will show that it is also valid for $k+1$. So let $r_1, \ldots, r_k, r_{k+1}$ be non-negative real numbers such that $0 \leq r_i \leq t/n$ for each $i \leq k+1$ and $\sum_{i=1}^{k+1} r_i \leq t$. We can assume that $r_{k+1} \leq r_i$ for every $i \in \{1, \ldots, k\}$. Since $\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k+1} r_i \leq t$, our induction hypothesis implies that there exists a partition $\{B_1, \ldots, B_m\}$ of $\{1, \ldots, k\}$ with $m \leq 2n-1$ and $\sum_{i \in B_j} r_i \leq t/n$ for every $j \in \{1, \ldots, m\}$. Let us consider the following two cases:

Case 1. $m \leq 2n-2$. We define a partition $\{A_1, \ldots, A_m, A_{m+1}\}$ of $\{1, \ldots, k, k+1\}$ by letting $A_j = B_j$ for each $j \leq m$ and $A_{m+1} = \{k+1\}$. Clearly this partition satisfies the requirements of the lemma.

Case 2. $m = 2n-1$. Let us show that $r_{k+1} \leq \frac{t}{2n}$. Indeed, $r_{k+1} > \frac{t}{2n}$ implies $r_i > \frac{t}{2n}$ for every $i \in \{1, \ldots, k+1\}$ and

$$t \geq \sum_{i=1}^{k+1} r_i > (k+1)\frac{t}{2n} \Rightarrow k+1 < 2n \Rightarrow m+1 < 2n,$$

which is a contradiction.

Put $S_j = \sum_{i \in B_j} r_i$. Let $j^* \in \{1, \dots, m\}$ be such $S_{j^*} \leq S_j$ for every $j \in \{1, \dots, m\}$. We next show that $r_{k+1} + S_{j^*} \leq t/n$. Once we have this, it is easy to see that the partition $\{A_1, \dots, A_m\}$ of $\{1, \dots, k, k+1\}$ defined by letting $A_j = B_j$ if $j \neq j^*$ and $A_{j^*} = B_{j^*} \cup \{k+1\}$ is as required.

So, assume by way of a contradiction that

$$r_{k+1} + S_{j^*} > t/n. \tag{7}$$

It follows from our assumptions that

$$r_{k+1} + mS_{j^*} \leq r_{k+1} + \sum_{j=1}^{m} S_j = \sum_{i=1}^{k+1} r_i \leq t. \tag{8}$$

From (7) and (8) we deduce

$$(m-1)S_{j^*} < t(n-1)/n.$$

By replacing $m = 2n - 1$ we obtain $S_{j^*} < \frac{t}{2n}$. Since $r_{k+1} \leq \frac{t}{2n}$ we arrive at $r_{k+1} + S_{j^*} < t/n$, which contradicts (7). □

Remark 4.4. The upper bound $2n - 1$ in the above proof is sharp. Indeed, consider the case $r_i = \frac{t}{2n-1}$ for every $i \leq k = 2n - 1$. Since $\frac{t}{2n-1} + \frac{t}{2n-1} > \frac{t}{n}$, the only partition satisfying the conditions of Lemma 4.3 is the one formed by one-element sets, and its size is exactly $2n - 1$.

Theorem 4.5. *Let (X, d) be a totally bounded metric space. Then for every $t > 0$, the ball $B_t \subseteq A_d(X)$ is bounded in $A_d(X)$. In particular, the group $A_d(X)$ has a local base at zero consisting of bounded open sets.*

Proof. Fix any $t > 0$ and $n \in \mathbb{N}$. Since (X, d) is totally bounded, there exists a finite $\frac{t}{2n}$-net C in (X, d). Let $F_0 = C \cup \{0\}$, where 0 is a distinguished point of X, and $F = \{x - y : x, y \in F_0\}$. Clearly F is a finite subset of $A_d(X)$. To complete the proof it suffices to show that

$$B_t \subseteq \underbrace{F + \cdots + F}_{n-1 \text{ times}} + \underbrace{B_{t/n} + \cdots + B_{t/n}}_{4n-2 \text{ times}}. \tag{9}$$

Take an arbitrary element

$$g = (x_1 - y_1) + \cdots + (x_p - y_p) \in B_t, \tag{10}$$

where $p \geq 1$ is an integer, $x_i, y_i \in X$, $x_i \neq y_i$ for each $i \leq p$ and $\sum_{i=1}^{p} d(x_i, y_i) \leq t$. We can assume without loss of generality that there exists an integer $k \leq p$ such that $d(x_i, y_i) \leq t/n$ for $i = 1, \dots, k$ and $d(x_i, y_i) > t/n$ if $k < i \leq p$. It is clear that $p - k < n$ otherwise we would have

$$t = n \cdot \frac{t}{n} < \sum_{i=k+1}^{p} d(x_i, y_i) \leq \sum_{i=1}^{p} d(x_i, y_i),$$

which contradicts our choice of the element $g \in B_t$.

According to Lemma 4.3 there exists a partition $\{A_1, \dots, A_m\}$ of the set $\{1, \dots, k\}$, where $m \leq 2n - 1$, such that $\sum_{i \in A_j} d(x_i, y_i) \leq t/n$ for each $j \leq m$. Therefore the element $a_j = \sum_{i \in A_j} (x_i - y_i)$ is in $B_{t/n}$ and

$$(x_1 - y_1) + \cdots + (x_k - y_k) = \sum_{j=1}^{m} a_j. \tag{11}$$

For every integer i with $k < i \le p$, choose elements $z_i, t_i \in F_0$ such that $d(x_i, z_i) \le \frac{t}{2n}$ and $d(y_i, t_i) \le \frac{t}{2n}$. Let $h = (z_{k+1} - t_{k+1}) + \cdots + (z_p - t_p)$. It is clear that $h \in \underbrace{F + \cdots + F}_{n-1 \text{ times}}$.

Replacing in (10) the summand $x_i - y_i$ with $(x_i - z_i) + (z_i - t_i) + (t_i - y_i)$, for each i satisfying $k < i \le p$, we obtain the representation of the element g in the form

$$g = h + (x_1 - y_1) + \cdots + (x_k - y_k) + \sum_{i=k+1}^{p} \big[(x_i - z_i) + (t_i - y_i)\big]. \tag{12}$$

Note that $\sum_{i=k+1}^{p} \big[d(x_i, z_i) + d(t_i, y_i)\big] \le (p - k)(\frac{t}{2n} + \frac{t}{2n}) < t$. Let $r_i = d(x_{i+k}, z_{i+k}) + d(t_{i+k}, y_{i+k})$, for each i with $1 \le i \le p - k$. It is clear that $r_i \le \frac{t}{n}$ for each $i \le p - k$ and $\sum_{i=1}^{p-k} r_i \le t$. Hence we can apply Lemma 4.3 to the numbers $r_1, \dots, r_{p-k}$ and find a positive integer $l \le 2n - 1$ and a partition $\{B_1, \dots, B_l\}$ of the set $\{1, \dots, p - k\}$ such that $\sum_{i \in B_j} r_i \le t/n$ for each $j \le l$.

Let $b_j = \sum_{i \in B_j} \big[(x_i - z_i) + (t_i - y_i)\big]$, where $1 \le j \le l$. Then $b_j \in B_{t/n}$, so (11) and (12) together imply that

$$g = h + \sum_{j=1}^{m} a_j + \sum_{j=1}^{l} b_j \in \underbrace{F + \cdots + F}_{n-1 \text{ times}} + \underbrace{B_{t/n} + \cdots + B_{t/n}}_{m+l \text{ times}}.$$

Since g is an arbitrary element of B_t, we conclude that

$$B_t \subseteq \underbrace{F + \cdots + F}_{n-1 \text{ times}} + \underbrace{B_{t/n} + \cdots + B_{t/n}}_{m+l \text{ times}}.$$

Finally, it follows from $m \le 2n - 1$ and $l \le 2n - 1$ that $m + l \le 4n - 2$. This completes the proof of the theorem. □

Corollary 4.6.

(a) *Let X be a convex subspace of a normed space L with $0 \in X$ and let d be the metric on X induced by the norm of L. Then $A(X)^{\wedge}_{\beta} \cong A(X)^{\wedge}_{b} \cong \mathrm{Lip}_0(X, \mathbb{T})$.*

(b) *Let (X, d) be a totally bounded metric space with a distinguished point 0. Then $A(X)^{\wedge}_{\beta} \cong A(X)^{\wedge}_{b} \cong \mathrm{Lip}_0(X, \mathbb{T})$.*

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