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On local quasi-convexity as a three-space property in topological abelian groups

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To the memory of N.Ya. Vilenkin on the 100th anniversary of his birth

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ABSTRACT

Let X be a topological abelian group and H a subgroup of X. We find conditions under which local quasi-convexity of both H and X/H results in the same property for X. This is true for instance if H is precompact, or if X is metrizable and H is a dually embedded subgroup which is also either discrete or bounded torsion. We also give some general principles and point out some errors we have found in the existing literature on this problem.

1. Introduction

The question whether local convexity is a three-space property was first posed in the context of (metrizable and complete) topological vector spaces, and was settled independently by J. W. Roberts, N. J. Kalton and M. Ribe in the late seventies. These authors gave an example of a space X with a non-complemented, one-dimensional subspace Y such that X/Y is topologically isomorphic to the Banach space ℓ_1. Such a space X cannot be locally convex, as every finite-dimensional subspace of a locally convex space is complemented. The details can be found in Chapter 5 of the monograph [31].

This three-space problem can be reformulated in the wider context of topological abelian groups, by replacing local convexity with local quasi-convexity:

Problem 1.1. Let X be a topological abelian group and H a subgroup of X. Suppose that both H and X/H are locally quasi-convex. Is X a locally quasi-convex group?

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Below we recall the relevant definitions; let us just mention for now that the concept of a locally quasi-convex group generalizes that of a locally convex space (Proposition 1.3) in such a way that Roberts-Kalton-Ribe's example can be invoked to give a negative answer to Problem 1.1. (Another counterexample which features a discrete subgroup H is given in Example 5.3 below.)

Some particular cases where additional conditions allow for a positive answer to Problem 1.1 are known (see Proposition 1.4 and Proposition 1.5). This paper is devoted mainly to generalizing these results, giving a wider perspective on these and related problems, and also fixing some wrong statements that can be found in the existing literature on this subject. In the following summary of some of our main results, H stands for a locally quasi-convex subgroup of a topological abelian group X such that the quotient group X/H is locally quasi-convex.

- If H is equicontinuously dually embedded in X, then X is locally quasi-convex (Theorem 3.4).
- If X is locally quasi-convex, H does not need to be dually embedded in X (Example 3.10).
- If H is precompact, then X is locally quasi-convex if and only if H is dually embedded in X (Theorem 4.3).
- If H is discrete and dually embedded in X and X is first countable, then X is locally quasi-convex (Theorem 5.2). The same is true with "bounded torsion" instead of "discrete" (Corollary 6.2).

Notation and preliminaries

We denote by $\mathbb{T}$ the set $\{z \in \mathbb{C} : |z| = 1\}$ endowed with the group structure given by multiplication of complex numbers, and by $\mathbb{T}_+$ the set $\{t \in \mathbb{T} : \mathrm{Re}(t) \geq 0\}$.

All topological groups considered in this paper are abelian. We do not assume them to be Hausdorff by default. We denote by $\mathcal{N}_0(X)$ the set of all neighborhoods of zero of the topological abelian group X. We represent the (Raĭkov) completion of a Hausdorff topological group X by ϱX.

A *k-space* is a Hausdorff topological space X such that every subset of X which meets every compact subset of X in a closed subset is itself closed. Locally compact Hausdorff spaces and metrizable spaces are k-spaces.

A *character* of an abelian group X is a homomorphism $\chi : X \to \mathbb{T}$. We denote by $\mathrm{Hom}(X, \mathbb{T})$ the group of all characters of X with pointwise multiplication. If X is a topological abelian group, we represent by $X^\wedge$ the subgroup of $\mathrm{Hom}(X, \mathbb{T})$ whose elements are the continuous characters of X. (We consider on $\mathbb{T}$ the group topology induced by the usual one on $\mathbb{C}$.) Given topological abelian groups X_1 and X_2 and a continuous homomorphism $f : X_1 \to X_2$, we denote by $f^\wedge$ the adjoint mapping of f, which is a homomorphism of $X_2^\wedge$ to $X_1^\wedge$ defined by $f^\wedge(\chi) = \chi \circ f$.

For simplicity we will consider only topological vector spaces over $\mathbb{R}$. We denote by E^* the dual space of the topological vector space E, i.e. E^* is the set of all continuous linear functionals $x^* : E \to \mathbb{R}$ endowed with the vector space structure inherited from $\mathbb{R}^E$. A topological vector space E will be viewed also as an additive topological abelian group with the dual group $E^\wedge$; we will need the following key result:

Proposition 1.2. *[34, Lemma 1] If E is a topological vector space, then the natural mapping $\Phi_E : E^* \to E^\wedge$ given by $\Phi_E(x^*) = \exp(2\pi i x^*)$ is a group isomorphism.*

We will use the language of group dualities first introduced in [35]. A *group duality* is a triple $(X, Y, \langle\cdot,\cdot\rangle)$ where X and Y are abelian groups and $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{T}$ is a bicharacter (that is, $\langle x, \cdot\rangle$ is a character of Y for every $x \in X$ and $\langle\cdot, y\rangle$ is a character of X for every $y \in Y$). We will abbreviate the notation $(X, Y, \langle\cdot,\cdot\rangle)$ to $\langle X, Y\rangle$ in what follows. Also, whenever a group duality $\langle X, Y\rangle$ is defined, we will (slightly abusing notation) denote by $\langle Y, X\rangle$ the natural group duality given by $\langle y, x\rangle = \langle x, y\rangle$ for every $x \in X$ and $y \in Y$. The duality $\langle X, Y\rangle$ is *separated* if $[x \in X \mapsto \langle x, \cdot\rangle \in \mathrm{Hom}(Y, \mathbb{T})]$ and $[y \in Y \mapsto \langle\cdot, y\rangle \in \mathrm{Hom}(X, \mathbb{T})]$ are monomorphisms for every $x \in X$ and $y \in Y$. If $\langle X, Y\rangle$ is a group duality and A is a subset of X, the *polar* of A with respect to $\langle X, Y\rangle$ is the set $A^{\triangleright(X,Y)} := \{y \in Y : \langle A, y\rangle \subseteq \mathbb{T}_+\}$. It is clear that $A \subseteq (A^{\triangleright(X,Y)})^{\triangleright(Y,X)}$ for every $A \subseteq X$. We say that the subset A is $\langle X, Y\rangle$-*quasi-convex* if $A = (A^{\triangleright(X,Y)})^{\triangleright(Y,X)}$. For an arbitrary $A \subseteq X$, the set $(A^{\triangleright(X,Y)})^{\triangleright(Y,X)}$ is the smallest (with respect to the set-theoretic inclusion) $\langle X, Y\rangle$-quasi-convex superset of A.

We will mainly deal with dualities of the form $\langle X, X^\wedge\rangle$ where X is a topological abelian group and $\langle x, \chi\rangle = \chi(x)$ for every $\chi \in X^\wedge$ and $x \in X$. If the topological abelian group X is clear from the context, we will drop the references to the dualities $\langle X, X^\wedge\rangle$ and $\langle X^\wedge, X\rangle$. In particular, for every $A \subseteq X$ and $B \subseteq X^\wedge$, we will use the symbols $A^\triangleright$ and $B^\triangleleft$ for the corresponding polar sets. It is a standard fact that a character χ of X is continuous if and only if $\chi \in U^\triangleright$ for some $U \in \mathcal{N}_0(X)$. Also, for any subset $A \subseteq X$, we will denote the set $(A^{\triangleright})^{\triangleleft}$ by $\mathrm{qc}_X A$.

If the duality $\langle X, X^\wedge\rangle$ is separated, we will say that X is a *dually separated* group. Note that X is dually separated if and only if for every $x \in X \setminus \{0\}$ there exists $\chi \in X^\wedge$ with $\chi(x) \neq 1$. Every locally compact abelian Hausdorff group is dually separated.

For any group duality $\langle X, Y\rangle$, we denote by $\sigma(X, Y)$ the initial topology on X with respect to the set of characters of the form $\langle\cdot, y\rangle$ where $y \in Y$. Remarkably, these turn out to be the only continuous characters of the group $(X, \sigma(X, Y))$. The topology $\sigma(X, Y)$ is a precompact group topology which admits the family of $\langle Y, X\rangle$-polars of all finite subsets of Y as a basis of neighborhoods of zero. Conversely, any precompact group (X, τ) carries the topology $\sigma(X, (X, \tau)^\wedge)$. These facts were proved by Comfort and Ross in [17] and will be used in what follows without further comment. It is also clear that for any group duality $\langle X, Y\rangle$, all $\langle X, Y\rangle$-polars are $\sigma(Y, X)$-closed subsets of Y.

Quasi-convex subsets of a topological abelian group were defined in [36] as follows: a subset A of a topological abelian group X is *quasi-convex* if it is quasi-convex with respect to the natural duality $\langle X, X^\wedge\rangle$.

We say (again following [36]) that a topological abelian group X is *locally quasi-convex* if X has a basis of neighborhood of zero formed by quasi-convex subsets. Every subgroup of a locally quasi-convex group is locally quasi-convex.

Proposition 1.3. *[9, Proposition 2.4] A topological vector space is locally convex if and only if its underlying additive topological group is locally quasi-convex.*

As we have already mentioned, the fact that local quasi-convexity is not a three-space property is a consequence of Proposition 1.3 and the corresponding examples from the theory of metrizable linear spaces. The following results give sufficient conditions under which local quasi-convexity of the group X does follow from local quasi-convexity of both its subgroup H and the corresponding quotient group X/H.

Proposition 1.4. *[14, Proposition 4.3.5] Let H be an open subgroup of the topological abelian group X. Then X is locally quasi-convex if and only if H is locally quasi-convex.*

Proposition 1.5. *[13, Theorem 10] Let X be a dually separated topological abelian group. Let H be a compact subgroup of X. If X/H is locally quasi-convex then so is X.*

2. Subgroups, quotients and locally quasi-convex modifications

Consider a subgroup H of a topological group X such that both H and X/H are locally quasi-convex. In order for X to be locally quasi-convex it turns out to be sufficient that H be a topological subgroup of the locally quasi-convex modification of X. This consequence of Merzon's lemma was exploited by Bruguera in [13]. Here we give a general formulation of this argument which can be invoked under different sets of hypothesis. First we recall the definition of the locally quasi-convex modification of a group topology.

Proposition 2.1. *([13, Lemma 7], [2, Proposition 6.18]) Let (X, τ) be a topological abelian group. The quasi-convex neighborhoods of zero in (X, τ) form a basis of neighborhoods of zero for a group topology $\mathcal{Q}\tau$ on X. This is the finest topology among the locally quasi-convex group topologies coarser than τ. The groups (X, τ) and $(X, \mathcal{Q}\tau)$ have the same continuous characters. Moreover, $(X, \mathcal{Q}\tau)$ is Hausdorff if and only if (X, τ) is dually separated.*

In what follows, given any topological group (X, τ) and any subgroup H of X, whenever there is no risk of confusion we denote by $\tau\restriction_H$ the group topology induced by τ on H and by τ/H the quotient topology induced by τ on X/H.

Proposition 2.2. *(Merzon's Lemma, see [32] and [20, Lemma 1]) Let λ and τ be two group topologies on a group X with $\lambda \leq \tau$. If there is a subgroup H of X for which $\lambda\restriction_H = \tau\restriction_H$ and $\lambda/H = \tau/H$ then $\lambda = \tau$.*

In order to use Proposition 2.2 in our context we need to analyze the behavior of the quasi-convex modification with respect to subgroups and quotients. The following result is essentially the same as [13, Proposition 9]:

Proposition 2.3. *Let H be a subgroup of the topological abelian group (X, τ). Then*

$$\mathcal{Q}(\tau/H) \leq \mathcal{Q}\tau/H \leq \tau/H$$

In particular if X/H is locally quasi-convex (that is, if $\mathcal{Q}(\tau/H) = \tau/H$), then these three topologies coincide.

The proof of the following Proposition is immediate.

Proposition 2.4. *Let H be a subgroup of the topological abelian group (X, τ). Then*

$$\mathcal{Q}\tau\restriction_H \leq \mathcal{Q}(\tau\restriction_H) \leq \tau\restriction_H$$

Note that, in contrast with the quotient case, the identity $\mathcal{Q}\tau\restriction_H = \tau\restriction_H$ does not follow automatically from local quasi-convexity of $\tau\restriction_H$.

Theorem 2.5. *Let (X, τ) be a topological abelian group. Let H be a subgroup of X. Suppose that X/H is locally quasi-convex. The group (X, τ) is locally quasi-convex if and only if $\mathcal{Q}\tau\restriction_H = \tau\restriction_H$.*

Proof. This follows at once from Propositions 2.2, 2.3 and 2.4. □

3. Dually embedded and equicontinuously dually embedded subgroups

A subgroup H of a topological abelian group X is said to be *dually embedded* in X if every continuous character of H can be extended to a continuous character of X, that is, if the adjoint $\imath^\wedge : X^\wedge \to H^\wedge$ of the inclusion map $\imath : H \to X$ is onto.

If X is a topological vector space, a subspace H of X such that all continuous linear functionals of H can be extended to continuous linear functionals of X is usually said to satisfy the *Hahn-Banach extension property.* We call these subspaces dually embedded; it is easy to see that, because of the natural relation between dual groups and dual spaces (Proposition 1.2), the two properties are equivalent. (The same can be said about the term "dually separated" when applied to a topological vector space.)

The property of being dually embedded appears naturally when discussing three-space problems for local quasi-convexity, partially due to the following result:

Proposition 3.1. *Let H be a subgroup of the topological abelian group (X, τ). If H is dually embedded in (X, τ) then*

$$(H, \mathcal{Q}\tau \restriction_H)^\wedge = (H, \tau \restriction_H)^\wedge = (H, \mathcal{Q}(\tau \restriction_H))^\wedge.$$

Proof. The second identity is true in general (Proposition 2.1). It is also clear that $(H, \mathcal{Q}\tau \restriction_H)^\wedge \leq (H, \tau \restriction_H)^\wedge$. Let us prove the inverse inclusion. Fix $\chi \in (H, \tau \restriction_H)^\wedge$. We need to find $U \in \mathcal{N}_0(X, \tau)$ with $\chi((\mathrm{qc}_X U) \cap H) \subseteq \mathbb{T}_+$. By hypothesis there exists $\tilde{\chi} \in X^\wedge$ with $\tilde{\chi} \restriction_H = \chi$. Find $U \in \mathcal{N}_0(X)$ with $\tilde{\chi}(U) \subseteq \mathbb{T}_+$. Clearly $\tilde{\chi}(\mathrm{qc}_X U) \subseteq \mathbb{T}_+$ and in particular $\chi((\mathrm{qc}_X U) \cap H) \subseteq \mathbb{T}_+$, as required. □

However, it is not clear at all that by adding the hypothesis "H is dually embedded in X" the answer to Problem 1.1 becomes affirmative, despite this being claimed in Theorem 2.1 of [15]. (The proof of this result is incorrect.) It becomes true under a number of different extra hypothesis, as we show below.

A closer look at the argument given in Theorem 2.1 of [15] leads to the natural reformulation presented here as Theorem 3.4. We need to introduce the following notion.

Definition 3.2. Let H be a subgroup of a topological abelian group X. We say that H is *equicontinuously dually embedded* in X if every equicontinuous subset of $H^\wedge$ is contained in the image of some equicontinuous set of $X^\wedge$ by the restriction mapping $\imath^\wedge : X^\wedge \to H^\wedge$.

Note that given any topological abelian group X and any $U \in \mathcal{N}_0(X)$, the set $U^\triangleright \subseteq X^\wedge$ is equicontinuous. Conversely, given any equicontinuous subset S of $X^\wedge$ there always exists $U \in \mathcal{N}_0(X)$ with $S \subseteq U^\triangleright$. Thus the property of being equicontinuously dually embedded can be reformulated in the following way:

Proposition 3.3. *Let H be a subgroup of a topological abelian group X. H is equicontinuously dually embedded in X if and only if for every U in $\mathcal{N}_0(X)$ there is $V \in \mathcal{N}_0(X)$ such that $(U \cap H)^{\triangleright(H, H^\wedge)} \subseteq \imath^\wedge(V^{\triangleright(X, X^\wedge)})$, where $\imath^\wedge : X^\wedge \to H^\wedge$ is the restriction mapping.*

Theorem 3.4. *Let H be an equicontinously dually embedded subgroup of a topological abelian group (X, τ).*

(a) $\mathcal{Q}\tau \restriction_H = \mathcal{Q}(\tau \restriction_H)$.
(b) *If both H and X/H are locally quasi-convex, then X is locally quasi-convex, too.*

Proof. (a) The inclusion $\mathcal{Q}\tau \restriction_H \leq \mathcal{Q}(\tau \restriction_H)$ follows from Proposition 2.4. Let us prove that $\mathcal{Q}\tau \restriction_H \geq \mathcal{Q}(\tau \restriction_H)$. Fix U in $\mathcal{N}_0(X)$. By Proposition 3.3 there is $V \in \mathcal{N}_0(X)$ such that $(U \cap H)^{\triangleright(H, H^\wedge)} \subseteq \imath^\wedge(V^{\triangleright(X, X^\wedge)})$. By applying $\langle H^\wedge, H\rangle$-polars we deduce

$$(\imath^\wedge(V^{\triangleright(X, X^\wedge)}))^{\triangleright(H^\wedge, H)} \subseteq \mathrm{qc}_H(U \cap H)$$

It is clear that $(\imath^\wedge(V^{\triangleright(X, X^\wedge)}))^{\triangleright(H^\wedge, H)} = (\mathrm{qc}_X V) \cap H$. Since $U \in \mathcal{N}_0(X)$ was arbitrary, this completes the proof.

(b) This follows from (a) and Theorem 2.5. □

It is clear that if H is equicontinuously dually embedded in X then in particular it is dually embedded in X.

It is known that every subgroup of a nuclear group is equicontinuously dually embedded in it ([9, Theorem 8.2]). A similar statement was obtained for subgroups of nuclear spaces in [8, Theorem 1.6]. (Incidentally, [8] is one of the first references after the appearance of [36] where locally quasi-convex groups are mentioned.) Note that nuclear groups are always locally quasi-convex [9, Theorem 8.5].

In the following Proposition we give conditions on the subgroup H under which H is always equicontinuously dually embedded in X:

Proposition 3.5. *Let H be a subgroup of a topological abelian group X. Assume that either*

(a) *H is precompact and dually embedded in X, or*
(b) *H is open in X.*

Then H is equicontinuously dually embedded in X.

Proof. (a) Since H is dually embedded in X, the homomorphism $\imath^\wedge : X^\wedge \to H^\wedge$ is onto. Fix an equicontinuous set $S \subset H^\wedge$. Since H is precompact, S is finite ([6, Lemma 2.6(a)]). Let S' be a finite subset of $X^\wedge$ such that $S \subseteq \imath^\wedge(S')$. It is clear that S' is equicontinuous.

(b) Fix $U \in \mathcal{N}_0(X)$. Since H is dually embedded in X (Lemma 2.2(b) in [10]), we have $(U \cap H)^{\triangleright(H,H^\wedge)} = (U \cap H)^{\triangleright(H,\imath^\wedge(X^\wedge))} = \imath^\wedge((U \cap H)^{\triangleright(X,X^\wedge)})$. Since $U \cap H \in \mathcal{N}_0(X)$, Proposition 3.3 gives the result. □

Note that, since discrete groups are locally quasi-convex, Proposition 1.4 above follows from Theorem 3.4(b) and Proposition 3.5(b).

Next we will present (Theorem 3.8) another situation where this property of being equicontinuously dually embedded is somewhat automatic.

In what follows, for any subset B of a topological vector space E we denote by B° the set of those $x^* \in E^*$ such that $x^*(B) \subseteq [-1, 1]$. Recall that a subset B of a real vector space E is said to be *balanced* if $[-1, 1]B \subseteq B$. The notation Φ_E was introduced in Proposition 1.2.

Lemma 3.6. *Let E be a topological vector space and B a nonempty, balanced subset of E. Then $B^\triangleright = \Phi_E(\frac{1}{4}B^\circ)$.*

Proof. Let us prove the inclusion $B^\triangleright \subseteq \Phi_E(\frac{1}{4}B^\circ)$; the other inclusion is immediate. Fix $\chi \in B^\triangleright$. By Proposition 1.2 there exists $x^* \in E^*$ with $\chi = \Phi_E(x^*)$. Since B is balanced, the set $x^*(B)$ is an interval in $\mathbb{R}$ containing zero. On the other hand $x^*(B) \subseteq [-1/4, 1/4] + \mathbb{Z}$, hence $x^*(B) \subseteq [-1/4, 1/4]$. This means that $x^* \in \frac{1}{4}B^\circ$. □

Lemma 3.7. *Let X be a topological vector space and let H be a subspace of X. Denote by $\imath : H \to X$ the inclusion mapping and by $\imath^* : X^* \to H^*$ the corresponding restriction mapping. The following conditions are equivalent:*

(i) *H is equicontinuously dually embedded as a subgroup of X.*
(ii) *Every equicontinuous subset of H^* is contained in the image of some equicontinuous subset of X^* by $\imath^*$.*

Proof. By Proposition 3.3, *(i)* is equivalent with:

(*) for every $U \in \mathcal{N}_0(X)$ there exists $V \in \mathcal{N}_0(X)$ with $(U \cap H)^{\triangleright(H,H^\wedge)} \subseteq \imath^\wedge(V^{\triangleright(X,X^\wedge)})$.

Any topological vector space has a basis of neighborhoods of zero formed by balanced sets. Moreover by Lemma 3.6, whenever B is a nonempty balanced subset of a topological vector space E we have $\Phi_E(\frac{1}{4}B^\circ) = B^\triangleright$. Hence *(*)* is equivalent with

(**) for every balanced $U \in \mathcal{N}_0(X)$ there exists a balanced $V \in \mathcal{N}_0(X)$ with $\Phi_H(\frac{1}{4}(U \cap H)^\circ) \subseteq \imath^\wedge(\Phi_X(\frac{1}{4}V^\circ))$

(where the polars on the left-hand side and right-hand side are meant to be computed on the dualities (H, H^*) and (X, X^*), respectively). Clearly $\frac{1}{4}(U \cap H)^\circ = (4U \cap H)^\circ$ and $\frac{1}{4}V^\circ = (4V)^\circ$. On the other hand, it is easy to check that $\imath^\wedge \circ \Phi_X = \Phi_H \circ \imath^*$. Since Φ_H is an isomorphism, we deduce that *(**)* is equivalent with

(***) for every balanced $U \in \mathcal{N}_0(X)$ there exists a balanced $V \in \mathcal{N}_0(X)$ with $(U \cap H)^\circ \subseteq \imath^*(V^\circ)$

Finally, given any topological vector space E, a subset S of E^* is equicontinuous if and only if it is a subset of the polar of some neighborhood of zero. This shows that *(***)* is equivalent with *(ii)* and the Lemma is proved. □

Theorem 3.8. *Let X be a locally convex space and let H be a subspace of X. Then H is equicontinuously dually embedded in X.*

Proof. We will apply the criterion in Lemma 3.7. Fix an equicontinuous $S \subseteq H^*$. There is a closed, convex and balanced neighborhood of zero U in X such that $S \subseteq \{h^* \in H^* : h^*(U \cap H) \subseteq [-1, 1]\}$. Define $T = \{x^* \in X^* : x^*(U) \subseteq [-1, 1]\}$. It is clear that T is equicontinuous as a subset of X^*. Let us prove that $S \subseteq \imath^*(T)$. Let $p : X \to [0, \infty)$ be a continuous semi-norm such that $U = \{x \in X : p(x) \le 1\}$. Fix an arbitrary $h^* \in S$. It is easy to see that $|h^*(x)| \le p(x)$ for every $x \in H$. According to Hahn-Banach Theorem (see for instance [33, Theorem II.3.2]), there exists $x^* \in X^*$ such that $x^* \restriction_H = h^*$ and $|x^*(x)| \le p(x)$ for every $x \in X$. In particular $x^* \in T$. This completes the proof. □

Corollary 3.9. *Let H be a locally convex subspace of a topological vector space X such that X/H is locally convex. The following properties are equivalent:*

(i) *X is locally convex.*
(ii) *H is equicontinuously dually embedded in X.*

Proof. (i)⇒(ii) follows from Theorem 3.8.
(ii)⇒(i) is a particular case of Theorem 3.4(b), taking into account Proposition 1.3. □

The same statement as Corollary 3.9 with the hypothesis of “dually embedded” in place of “equicontinuously dually embedded” is presented as Corollary 2.1 in [15]. The result is a corollary of Theorem 2.1 in the same reference, whose proof, as we already have pointed out, is incorrect. In Theorem 6.6 we will show that one can indeed drop “equicontinuously” from Corollary 3.9 under the extra hypothesis that X is first countable. The general case seems to be still open; see the comments to Question 7.4 for further details.

We end this section by discussing the “only if” part of [15, Theorem 2.1]. It claims that if H is a subgroup of a locally quasi-convex group X such that the quotient group X/H is locally quasi-convex, then H is dually embedded in X. This assertion turns out to be incorrect, as the following example shows.

Example 3.10. As usual, c_0 denotes the Banach space of all sequences of real numbers converging to 0 endowed with the supremum norm. Recall that c_0^* is naturally isomorphic with the space ℓ_1 of all summable sequences of real numbers. Consider the subgroup $c_1(\mathbb{T}) \le \mathbb{T}^{\mathbb{N}}$ formed by all sequences of elements of the unit circle $\mathbb{T}$ converging to 1. Endow $c_1(\mathbb{T})$ with the *uniform* topology [22], that is, the group topology generated by the following basis of neighborhoods of the neutral element:

$$\{U^{\mathbb{N}} \cap c_1(\mathbb{T}) : U \in \mathcal{N}_1(\mathbb{T})\}.$$

This group is locally quasi-convex (Proposition 3.1(b) of [22]), as of course is c_0 (Proposition 1.3). The group $c_1(\mathbb{T})$ is topologically isomorphic with the quotient of the Banach space c_0 by the discrete subgroup D of all sequences in c_0 with values in $\mathbb{Z}$ (Remark 3.7(c) in [22]). Let us see that D is not dually embedded in c_0. Since D is the discrete direct sum of countably many copies of $\mathbb{Z}$, the mapping $[\chi \in D^\wedge \mapsto (t_n) \in \mathbb{T}^{\mathbb{N}}]$ where $\chi((m_n)) = \prod t_n^{m_n}$ is an algebraic isomorphism. The character of D defined by the constant sequence (t_n) with $t_n = -1 \in \mathbb{T}$ cannot be extended to any continuous character of c_0. Indeed, assume the contrary. By evaluating this continuous extension on the elements of the canonical basis of c_0 we would deduce that there is a sequence $(\lambda_n) \in \ell_1$ with $\lambda_n \in 1/2 + \mathbb{Z}$ for every n, which is clearly impossible.

Question 3.11.

(a) Find an example of a dually embedded subgroup H of a topological abelian group X such that H is not equicontinuously dually embedded in X.
(b) Find an example of a dually embedded locally quasi-convex subgroup H of a topological abelian group X such that X/H is locally quasi-convex but X is not.

Note that, due to Theorem 3.4(b), any example satisfying the requirements of Question 3.11(b) would also fulfill those of (a).

4. The case of precompact H

The following Lemma is a straightforward generalization of Corollary 1 in [26]; we provide the reader with a proof anyway. In what follows, for a subgroup H of a Hausdorff topological abelian group X we identify ϱH with the closure of the subgroup H in ϱX.

Lemma 4.1. *Let H be a subgroup of a Hausdorff topological abelian group (X, τ). Let $\imath : H \to X$ be the inclusion mapping. Suppose that $(H, \tau \restriction_H)$ has a basis of neighborhoods of zero formed by $\sigma(H, \imath^\wedge(X^\wedge))$-closed subsets. Then $(\varrho X)^\wedge$ separates the points of ϱH.*

Proof. For each $\chi \in X^\wedge$, denote by $\varrho\chi$ its unique extension to ϱX. Fix $y \in \varrho H$. Assume that $\varrho\chi(y) = 0$ for every $\chi \in X^\wedge$, and let us prove that $y = 0$.

Fix a net $(h_\alpha)_{\alpha \in A}$ in H which converges to y in ϱX. In particular $\chi(h_\alpha) \to 0$ for every $\chi \in X^\wedge$. Hence (h_α) is a Cauchy net in $\tau \restriction_H$ which converges to zero in the coarser topology $\sigma(H, \imath^\wedge(X^\wedge))$. Let us see that $h_\alpha \to 0$ also in $\tau \restriction_H$; since $\tau \restriction_H$ is a Hausdorff topology, this will already imply that $y = 0$. Fix a $\tau \restriction_H$-neighborhood U of 0 in H. By hypothesis we may assume that U is $\sigma(H, \imath^\wedge(X^\wedge))$-closed. Find $\alpha_0 \in A$ with $h_\alpha - h_{\alpha'} \in U$ for every $\alpha, \alpha' \geq \alpha_0$. Since U is $\sigma(H, \imath^\wedge(X^\wedge))$-closed and (h_α) converges to 0 in $\sigma(H, \imath^\wedge(X^\wedge))$, this implies that $h_\alpha \in U$ for every $\alpha \geq \alpha_0$. Since U was arbitrary, this completes the proof. □

Proposition 4.2. *Let H be a precompact subgroup of the Hausdorff topological abelian group (X, τ). Let $\imath : H \to X$ be the inclusion mapping. The following are equivalent:*

(i) $\sigma(H, \imath^\wedge(X^\wedge)) = \tau \restriction_H$
(ii) *$(H, \tau \restriction_H)$ has a basis of neighborhoods of zero formed by $\sigma(H, \imath^\wedge(X^\wedge))$-closed subsets.*
(iii) *H is dually embedded in X.*

Proof.
- (i)⇒(ii) is trivial.
- (ii)⇒(iii): By Lemma 4.1, $(\varrho X)^\wedge$ separates the points of the compact group ϱH. This implies (see for instance Lemma 3 in [11]) that ϱH is dually embedded in ϱX. Hence H is dually embedded in X.

- (iii)⇒(i): Since H is dually embedded in (X, τ), we have $H^\wedge = \imath^\wedge(X^\wedge)$. Since H is precompact, $\tau\restriction_H = \sigma(H, H^\wedge)$. The result follows. □

The following is the main result of this section.

Theorem 4.3. *Let H be a precompact subgroup of the Hausdorff topological abelian group (X, τ). Suppose that $(X/H, \tau/H)$ is locally quasi-convex. The following are equivalent:*

(i) *(X, τ) is locally quasi-convex.*
(ii) *H is dually embedded in (X, τ).*

Proof.
- (i)⇒(ii): Since X is locally quasi-convex, in particular it has a basis of $\sigma(X, X^\wedge)$-closed neighborhoods of zero. Since $\sigma(H, \imath^\wedge(X^\wedge)) = \sigma(X, X^\wedge)\restriction_H$, we deduce that H has a basis of $\sigma(H, \imath^\wedge(X^\wedge))$-closed neighborhoods of zero. It only remains to apply (ii)⇒(iii) in Proposition 4.2.
- (ii)⇒(i): By (iii)⇒(i) in Proposition 4.2, we deduce that $\sigma(H, \imath^\wedge(X^\wedge)) = \tau\restriction_H$. Since $\sigma(H, \imath^\wedge(X^\wedge)) = \sigma(X, X^\wedge)\restriction_H$ this clearly implies $\mathcal{Q}\tau\restriction_H \geq \tau\restriction_H$. Hence the topologies $\mathcal{Q}\tau\restriction_H$ and $\tau\restriction_H$ coincide. The result follows from Theorem 2.5. Alternatively, (ii)⇒(i) can be proved by invoking Proposition 3.5(a) and Theorem 3.4(b). □

Recall that a Hausdorff topological group (X, τ) is said to be *minimal* if the topology τ is minimal on the set of all Hausdorff group topologies on X, ordered by inclusion. It is clear that every compact Hausdorff group is minimal. A celebrated result by Prodanov and Stoyanov [21, Theorem 2.7.7] establishes that every minimal abelian group is precompact. Theorem 4.3 can be sharpened for minimal groups as follows:

Theorem 4.4. *Let H be a minimal subgroup of the Hausdorff topological abelian group (X, τ). Suppose that $(X/H, \tau/H)$ is locally quasi-convex. The following are equivalent:*

(i) *(X, τ) is locally quasi-convex.*
(ii) *H is dually embedded in (X, τ).*
(iii) *$(X, \tau)^\wedge$ separates the points of H.*

Proof.
- (i)⇔(ii) follows from Theorem 4.3.
- (ii)⇔(iii): By Proposition 4.2, (ii) is equivalent to the equality $\sigma(H, \imath^\wedge(X^\wedge)) = \tau\restriction_H$. Since $(H, \tau\restriction_H)$ is minimal and $\sigma(H, \imath^\wedge(X^\wedge)) \leq \tau\restriction_H$, this happens if and only if the topology $\sigma(H, \imath^\wedge(X^\wedge))$ is Hausdorff, which is equivalent to (iii). □

Note that Theorem 4.4 is in particular true for compact H, so it generalizes Proposition 1.5 above.

5. The case of first countable X

Lemma 5.1. *If a topological abelian group is a k-space and all its characters are continuous then it is discrete.*

Proof. Assume that X is a topological group which is a k-space and $X^\wedge = \mathrm{Hom}(X, \mathbb{T})$. Let us show that X has no infinite compact subsets, which clearly implies that X is discrete. Fix a compact subset K of X. Since K is $\sigma(X, X^\wedge)$-compact, and $X^\wedge = \mathrm{Hom}(X, \mathbb{T})$, we deduce that K is a compact subset of $(X, \sigma(X, \mathrm{Hom}(X, \mathbb{T})))$. By Glicksberg theorem [27, Theorem 1.2], K must be finite. □

Theorem 5.2. *Let H be a dually embedded, discrete subgroup of a first countable topological abelian group (X, τ). Suppose that $(X/H, \tau/H)$ is locally quasi-convex. Then (X, τ) is locally quasi-convex.*

Proof. Since H is dually embedded in X, Proposition 3.1 implies that $(H, \mathcal{Q}\tau \restriction_H)^\wedge = \mathrm{Hom}(H, \mathbb{T})$. In particular $(H, \mathcal{Q}\tau \restriction_H)$ is dually separated, hence $\mathcal{Q}\tau \restriction_H$ is a Hausdorff group topology. Since (X, τ) is first countable, so is $(X, \mathcal{Q}\tau)$. We conclude that $(H, \mathcal{Q}\tau \restriction_H)$ is first countable and Hausdorff, thus metrizable. By Lemma 5.1, $(H, \mathcal{Q}\tau \restriction_H)$ is discrete. By Theorem 2.5, X is locally quasi-convex. □

The following example shows that we cannot drop the hypothesis "H dually embedded" from Theorem 5.2.

Example 5.3. We keep the notation used in Example 3.10. Let D be the discrete subgroup of c_0 formed by all sequences with integer values. Let D' be the subgroup of D formed by all sequences $(m_n)_{n\in\mathbb{N}}$ with $\sum m_n \in 2\mathbb{Z}$. The (first countable) group c_0/D' is not dually separated, as shown in the final section of [29]; since it is Hausdorff, it is immediate to see that it is not locally quasi-convex. Its quotient by the discrete subgroup D/D' is topologically isomorphic to c_0/D, which is locally quasi-convex, as mentioned in Example 3.10.

Question 5.4. Can one replace "discrete" with "locally compact" in Theorem 5.2? Can the assumption of first countability be relaxed or dropped? (Cf. Remark 14 in [13].)

Question 5.5. Let H be a dually embedded, discrete subgroup of a first countable topological abelian group X. Is then H equicontinuously dually embedded in X?

6. Mackey topologies

When dealing with the three-space property for local quasi-convexity it is important to establish conditions under which, given a topological abelian group (X, τ) and its subgroup H, the topologies induced on H by τ and by $\mathcal{Q}\tau$ coincide (Theorem 2.5). When these two topologies on H have the same dual group (which happens if H is dually embedded in X by Proposition 3.1), one arrives naturally at the question whether $(H, \mathcal{Q}\tau \restriction_H)$ is a LQC-Mackey group.

If $\langle X, Y\rangle$ is a group duality, we say that a group topology τ on X is compatible with $\langle X, Y\rangle$ when $(X, \tau)^\wedge = \{\langle\cdot, y\rangle : y \in Y\}$. In [16] the least upper bound of all locally quasi-convex group topologies τ on X which are compatible with a given group duality $\langle X, Y\rangle$ is denoted by $\tau_g(X, Y)$ and it is noted that this is a locally quasi-convex topology. Only recently ([3], [25]) it has been shown that this topology is not compatible in general.

If the topology on X of uniform convergence on $\sigma(Y, X)$-compact, $\langle Y, X\rangle$-quasi-convex subsets of Y is compatible with $\langle X, Y\rangle$, then it coincides with $\tau_g(X, Y)$ (see [16, Proposition 3.14 (a)]).

According to [22] a locally quasi-convex group X is said to be *a LQC-Mackey group* if it carries the topology $\tau_g(X, X^\wedge)$. Among the classes of groups known to have this property are the separable Baire locally quasi-convex groups, Čech-complete locally quasi-convex groups, and (by Proposition 4.4 in [28]) pseudocompact groups.

Theorem 6.1. *Let (X, τ) be a topological abelian group. Let H be a dually embedded subgroup of X. Assume that both H and X/H are locally quasi-convex. If $(H, \mathcal{Q}\tau \restriction_H)$ is a LQC-Mackey group, then X is locally quasi-convex.*

Proof. Since H is dually embedded in X, the dual groups $(H, \mathcal{Q}\tau \restriction_H)^\wedge$ and $(H, \tau \restriction_H)^\wedge$ coincide (Proposition 3.1). Since $(H, \mathcal{Q}\tau \restriction_H)$ is a LQC-Mackey group by hypothesis and $\mathcal{Q}\tau \restriction_H \leq \tau \restriction_H$, these two locally quasi-convex group topologies on H must actually be the same. The result follows from Theorem 2.5. □

Corollary 6.2. *Let (X,τ) be a first countable topological abelian group and let H be a Hausdorff, dually embedded, bounded torsion subgroup of X. Assume that both H and X/H are locally quasi-convex. Then X is locally quasi-convex.*

Proof. Since $(H,\tau\restriction_H)$ is Hausdorff and locally quasi-convex, it is dually separated. Since H is dually embedded in X, Proposition 3.1 implies that $(H,\mathcal{Q}\tau\restriction_H)$ is dually separated and in particular Hausdorff. Since (X,τ) is first countable, so is $(X,\mathcal{Q}\tau)$ and hence $(H,\mathcal{Q}\tau\restriction_H)$. We conclude that $(H,\mathcal{Q}\tau\restriction_H)$ is metrizable, bounded torsion and locally quasi-convex. By Theorem B(ii) in [7] it is a LQC-Mackey group. The result follows from Theorem 6.1. □

Since every locally quasi-convex group topology on a bounded torsion group admits a basis of neighborhoods of zero formed by subgroups [5, Proposition 2.1], the following question arises naturally. A positive answer would generalize simultaneously Corollary 6.2 and Theorem 5.2.

Question 6.3. Can the hypothesis of boundedness of torsion of the subgroup H in Corollary 6.2 be weakened to that of $\tau\restriction_H$ having a basis of neighborhoods of zero formed by subgroups?

Lemma 6.4. *Let B be a nonempty, balanced subset of a real topological vector space E. Then $\mathrm{qc}_E(B)=(B^{\triangleright})^{\triangleleft}$ is balanced and convex.*

Proof. Let us denote by B°, as in Lemma 3.6, the set $\{x^*\in E^*: x^*(B)\subseteq[-1,1]\}$. We will show that actually $(B^{\triangleright})^{\triangleleft}=\{x\in E: B^\circ(x)\subseteq[-1,1]\}$, which clearly implies the result.

From Lemma 3.6 it follows at once that if $B^\circ(x)\subseteq[-1,1]$, then $x\in(B^{\triangleright})^{\triangleleft}$. Conversely, assume that there exists $x^*\in E^*$ with $x^*\in B^\circ$ and $|x^*(x)|>1$. Put $y^*=\alpha x^*$ where $\alpha\in[-1,1]$ is such that $1<\alpha x^*(x)<3$. Since y^* is still in B° we deduce, again from Lemma 3.6, that $\chi=\Phi_E(\frac{1}{4}y^*)=\exp(\frac{1}{2}\pi i y^*)$ is in $B^{\triangleright}$. Clearly $\chi(x)\notin\mathbb{T}_+$. Hence $x\notin(B^{\triangleright})^{\triangleleft}$. □

Recall that a subset B of a real vector space E is said to be *absorbing* if for every $x\in E$ there exists $\alpha>0$ with $x\in\lambda B$ whenever $|\lambda|>\alpha$.

Lemma 6.5. *Let (E,τ) be a topological vector space. The topology $\mathcal{Q}\tau$ is a locally convex vector space topology.*

Proof. It is known that any topological vector space has a basis of neighborhoods of zero formed by balanced subsets. By Lemma 6.4, the group topology $\mathcal{Q}\tau$ on E has a basis of neighborhoods of zero formed by absorbing, convex and balanced subsets. This is enough to ensure that $\mathcal{Q}\tau$ is a locally convex vector space topology [12, Ch. I, §1.5, Proposition 4]. □

The following result was essentially proved by N. J. Kalton [30, Theorem 4.10]. We include a proof for the sake of completeness.

Theorem 6.6. *Let X be a first countable topological vector space and let H be a dually embedded, locally convex subspace of X. Assume that X/H is locally convex. Then X is locally convex.*

Proof. Let τ be the original topology of X. The topology $\mathcal{Q}\tau$ is first countable and (by Lemma 6.5) locally convex; hence so is $\mathcal{Q}\tau\restriction_H$. Due to the fact that H is dually embedded in X, the spaces $(H,\mathcal{Q}\tau\restriction_H)$ and $(H,\tau\restriction_H)$ have the same dual space (Proposition 3.1). As a first countable locally convex vector space topology, $\mathcal{Q}\tau\restriction_H$ is a Mackey (vector-space) topology and hence it must be finer than the locally convex topology $\tau\restriction_H$. This means that the two topologies coincide. Taking into account the equivalence between local quasi-convexity and local convexity (Proposition 1.3), the result follows from Theorem 2.5. □

The argument in the proof of Theorem 6.6 cannot be applied *verbatim* for groups. Indeed, there are several known examples of metrizable locally quasi-convex groups which are not LQC-Mackey groups. We mention here the group $\mathbb{Q}$ with the induced topology from $\mathbb{R}$ [18,19], the precompact group from [22, Proposition 5.2(c)], the group $\mathbb{Z}$ endowed with any nondiscrete topology with a basis of neighborhoods of zero formed by subgroups [4], and the direct sum of countably many copies of $\mathbb{R}$ with the topology induced by the product topology [24]. The latter serves also as an example of a Mackey space (in the vector space sense) which is not a LQC-Mackey group.

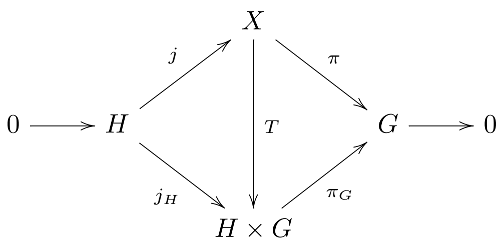
7. Splitting properties and local quasi-convexity

Our next considerations will benefit from the use of the language of extensions. An *extension* of topological abelian groups (shortly "extension") is a short exact sequence $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, where H, X, G are topological abelian groups and all maps in the sequence are assumed to be continuous and open homomorphisms when considered as maps onto their images. (Further details can be found in [11].)

If H is a subgroup of the topological abelian group X, there is a canonical extension $0 \to H \xrightarrow{\imath} X \xrightarrow{q} X/H \to 0$ where $\imath$ is the inclusion and q is the corresponding quotient map. Conversely, given any extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, there is a unique topological isomorphism $T : X/j(H) \to G$ such that $T \circ q = \pi$ where $q : X \to X/j(H)$ is the quotient map. These considerations make the following problem essentially equivalent to Problem 1.1.

Problem 7.1. Fix two locally quasi-convex groups G and H. Consider an extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ of topological abelian groups. Is X a locally quasi-convex group?

An extension of topological abelian groups $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ is said to *split* if there exists a continuous homomorphism $T : X \to H \times G$ making the following diagram commutative (j_H is the canonical inclusion of H into the product, and π_G is the canonical projection onto G). It is known that if such a T exists, it must actually be a topological isomorphism.



The extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ splits if and only if the subgroup $j(H)$ splits from X as a topological subgroup.

Given an extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ of topological abelian groups, consider the assertions

(1) X is locally quasi-convex.
(2) The extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ splits.

It is clear that (2)⇒(1). It is easy to see that in general (1) does not imply (2). It is sufficient to consider the canonical extension $0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{\pi} \mathbb{T} \to 0$. Note, however, that this extension does not split even algebraically (i.e. as an extension of abelian groups). It is not difficult to find examples of algebraically splitting extensions $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ where all the groups are locally quasi-convex and the extension does not split topologically (see for instance the extension of locally compact abelian groups in [1, Example 6.17]).

It is natural to look for further conditions connecting local quasi-convexity of the group X with splitting properties of extensions of the form $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$. Note that the natural analogue to (1)⇒(2) for topological vector spaces does hold if H is finite-dimensional, as we pointed out in the Introduction.

The following result was implicitly proved in [15]:

Proposition 7.2. *Let G be a locally quasi-convex, Hausdorff group. Let $0 \to \mathbb{T} \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. The following assertions are equivalent:*

(i) *X is locally quasi-convex.*
(ii) *X is dually separated.*
(iii) *The extension $0 \to \mathbb{T} \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ splits.*

Proof.
- (iii)⇒(i) is trivial.
- (i)⇒(ii): Since $\mathbb{T}$ and G are Hausdorff groups, X is a Hausdorff group. Since X is locally quasi-convex and Hausdorff, it is in particular dually separated.
- (ii)⇒(iii): If X is dually separated, all its compact subgroups are dually embedded [11, Lemma 3]. By extending the continuous character $j(\mathbb{T}) \to \mathbb{T}$ given by $j(t) \mapsto t$ to the group X we obtain a left inverse for j. This implies that the extension splits. □

Proposition 7.2 is not true if we replace $\mathbb{T}$ with an arbitrary compact group. Indeed, there exists a non-splitting extension of locally compact abelian groups $0 \to K \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ where K is the Pontryagin dual group of $\mathbb{Q}_d$, the group of rational numbers endowed with the discrete topology. [1, Example 6.19].

Next we give a corrected version of Theorem 4.1 in [15]. For any topological abelian groups G and H, the expression $\mathrm{Ext}(G, H) = 0$ will represent the property that every extension of topological abelian groups of the form $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ splits.

Theorem 7.3. *Let G be a locally quasi-convex, Hausdorff group. The following assertions are equivalent:*

(i) $\mathrm{Ext}(G, \mathbb{T}) = 0$
(ii) *For every extension $0 \to \mathbb{T} \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, the group X is locally quasi-convex.*
(iii) *For every precompact Hausdorff group H and every extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, the group X is locally quasi-convex.*
(iv) *For every Hausdorff topological abelian group H and every extension $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, the subgroup $j(H)$ is dually embedded in X.*

Proof.
- (iii)⇒(ii) is clear.
- (ii)⇒(i) is a corollary of Proposition 7.2.
- (i)⇒(iv) is known (see for instance the proof of (i)⇒(iv) in [15, Theorem 4.1], or [11, Theorem 21(1)]).
- (iv)⇒(iii): Suppose that (iv) is true and fix an extension of topological abelian groups $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$, where H is precompact and Hausdorff. We need to show that X is locally quasi-convex. By Theorem 4.3 it is enough to show that $j(H)$ is dually embedded in X, which is true by hypothesis. □

Question 7.4. Can one replace precompactness with local quasi-convexity of H in Theorem 7.3(iii)?

According to [15, Theorem 4.1], the answer to Question 7.4 is affirmative. This claim is based on Theorem 2.1 in the same reference, whose proof is incorrect. The authors are not aware of any counterexample. To justify the importance of this problem, let us mention that

- if just the particularization of this assertion to the topological vector space case were true, it would answer positively the long-standing conjecture of whether every locally convex $\mathcal{K}$-space is a TSC space in the sense of Domanski [23].
- on the other direction, assume that there exist a topological abelian group G satisfying the equivalent conditions in Theorem 7.3 and an extension of topological abelian groups $0 \to H \xrightarrow{j} X \xrightarrow{\pi} G \to 0$ where both G and H are locally quasi-convex but X is not. Because of condition (iv), such an example would also satisfy the requirements in Question 3.11(a) and (b).

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