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On the closed-form solution of an endogenous growth model with anticipated consumption

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ABSTRACT

This paper derives a closed-form solution of the AK endogenous growth model with logarithmic preferences and anticipated future consumption which enters additively into effective consumption. We get an explicit representation of the time paths of the economic variables in level by resorting to Gaussian Hypergeometric functions. We compare the model with anticipated future consumption to the model with habit formation. The maximum utility attainable in the model with anticipation is shown to be higher than the one attainable in the model with habits. Using the derived explicit expressions, we perform some comparative-dynamics and -statics analyses with respect to relevant parameters. Numerical simulations complement the theoretical results. Thus, this work provides further support to the usefulness of especial functions in the study of economic dynamics.
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1. Introduction

Anticipation of the future may have an impact on current well-being. Individuals derive happiness from waiting for a pleasant experience such as going to the theater, on vacation or to dinner to a fancy restaurant. Starting with the seminal work of Loewenstein (1987), a large subsequent literature has followed the idea of introducing the anticipatory feelings of a future event into current utility (e.g., Elster and Loewenstein, 1992, Caplin and Leahy, 2001, Brunnermeier and Parker, 2005, Köszegi and Rabin, 2007, Baucells and Bellezza, 2017, and Guo, 2020). However, only recently the literature has started to examine the consequences of introducing anticipated future consumption in economic growth models; e.g., on portfolio choice (Kuznitz et al., 2008), on the effectiveness of monetary policy (Faria and McAdam, 2013), on the equilibrium dynamics of the neoclassical growth model (Monteiro and Turnovsky, 2016) and the AK model (Gómez and Monteiro, 2020), or on the Green Golden Rule (Faria and McAdam, 2018).

The closely related literature on habit-forming preferences assumes that current utility depends on how current consumption compares to a reference consumption level — the habits stock — which is determined by past consumption levels. This literature introduces habits into utility in two main ways: multiplicative (e.g., Carroll et al., 1997, 2000, Abel, 1990, Alonso-Carrera et al., 2005, Bambi and Gozzi, 2020) and additive (e.g., Constantinides, 1990, Boldrin et al., 2001, Alonso-Carrera et al., 2004, Augeraud-Veron and Bambi, 2015). The multiplicative specification has the drawback that the agent's utility function might fail to be concave (e.g., Alonso-Carrera et al., 2005, Hiraguchi, 2008, Bambi and Gozzi, 2020), whereas additive habits guarantee concavity but the utility function could not be defined for positive consumption levels (e.g., Carroll, 2000, Wendner, 2003).[1] This limitation would not arise in a model with additive anticipation.

However, the previous works by Faria and McAdam (2013, 2018), Monteiro and Turnovsky (2016) and Gómez and Monteiro (2020) introduced anticipated future consumption into utility in a multiplicative manner. Thus, the study of the model with additive anticipated consumption is lacking in the literature. The purpose of this paper is to fill this gap. To this end, we study the equilibrium dynamics of the AK endogenous growth model with anticipated future consumption that enters additively into utility. The assumption that technology is AK is appealing in this context because it allows to isolate the effect of anticipation. As it is well known, the AK model with standard time-separable utility does not exhibit transition dynamics (Rebelo, 1991) and, therefore, the dynamics of the economy would be driven exclusively by preferences; i.e., by the presence of anticipated future consumption. This approach has also been widely used to study the dynamics in models with habit-forming preferences (e.g., Carroll et al., 1997, 2000, Alonso-Carrera et al., 2006, Gómez, 2010, Schäfer and Valente, 2011, and Augeraud-Veron and Bambi, 2015).

[1] There are few papers that contrast the different implications of additive and multiplicative habits (e.g., Carroll, 2000, Wendner, 2003, Bossi and Gomis-Porqueras, 2009, Dennis, 2009) but there does not seem to be compelling evidence in favor of one specification or the other. Actually, some works find that the qualitative properties of models with additive and multiplicative habits are largely similar (e.g., Bunzel, 2006, and Kraft et al., 2017).

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The endogenous growth literature generally relies on the dimension reduction method to analyze the dynamic properties of a model, which is combined with qualitative phase-diagram analysis, local approximations and/or numerical simulations (e.g., Mulligan and Sala-i-Martin, 1993; Bond et al., 1996; Arnold, 2006). With the dimension reduction method, the dynamics of the economy is described in terms of new variables which are formed as ratios between the variables of the original system—states, costates and controls. The objective is to transform the original dynamic system into another system with a lower dimension and a steady-state equilibrium rather than a balanced growth path. Actually, this is what Gómez and Monteiro (2020) did to analyze the AK model with CRRA utility and multiplicative anticipated consumption. This approach is very useful — and could be the only option — in many applications. However, it has some weaknesses, as pointed out by Boucekkine and Ruiz-Tamarit (2008), most notably that the dynamics of the variables in level remains largely unknown, so they have to be explored by means of numerical simulations.

In this paper we assume that preferences are logarithmic—a particular case of the standard CRRA specification. This assumption allows us to derive closed-form expressions for the equilibrium paths of the economic variables in level as well as for the maximum attainable utility.[2] The closed-form solution obtained is based on Gaussian hypergeometric functions. Thus, this paper also adds to a still relatively scant literature that illustrates the usefulness of special functions to analyze the dynamics of economic growth models. Hypergeometric functions have been used, e.g., to find a closed-form solution of the Solow model with logistic population (Guerrini, 2006), the two-sector Lucas endogenous growth model (Boucekkine and Ruiz-Tamarit, 2008; Hiraguchi, 2009; Boucekkine et al., 2013), the AK model with logistic population (Guerrini, 2010), the AK model with external habits (Gómez, 2014), and the Ramsey–Cass–Koopmans model with leisure (Hiraguchi, 2014).

The closed-form solution of the model is used to study analytically the comparative-dynamics effect of several shocks on the time paths of the economic variables in level as well as on intertemporal utility. Furthermore, it allows to compare analytically these effects with the corresponding ones in the habit-formation model. None of these analytical studies could be made in Gómez and Monteiro (2020), which had to rely on numerical simulations. First, we show that the time path of effective consumption in the anticipation model lies above the corresponding one in the habits model. Hence, the maximum utility in the model with anticipation is higher than the one attainable in the model with habit formation. We prove analytically that, as expected, a higher productivity leads to a higher intertemporal utility in both the anticipation and habits models whereas a higher weight of anticipated future consumption in utility entails a higher overall utility in the anticipation model but a lower one in the habits model. A faster speed of adjustment of the reference consumption stock leads to a lower intertemporal utility in the anticipation model, but the effect in the habits model depends on how initial output compares to the initial habits stock. The theoretical results are complemented with numerical simulations that illustrate the effect of a shock that increases productivity.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 derives an explicit solution of the model based on Gaussian hypergeometric functions. Section 4 compares the model with anticipated future consumption to the model with habit formation. Finally, Section 5 concludes.

[2] The combination of a logarithmic utility function with additive habits is also relatively habitual in the literature (e.g., Christiano et al., 2005, Iacoviello and Neri, 2010, Altig et al., 2011, Gertler and Karadi, 2011, and Christiano et al., 2014).

2. The model

We consider a closed economy inhabited by a constant population of identical infinitely-lived agents. The intertemporal utility derived by the representative agent depends on her current consumption, $C(t)$, and a reference level of anticipated future consumption, $A(t)$, according to

$$U = \int_0^\infty \ln[C(t) + \gamma A(t)]\, e^{-\beta t} dt, \quad \beta > 0,\ \gamma > 0. \tag{1}$$

Here β is the rate of time preference and γ reflects the weight of anticipated future consumption in utility. The anticipated consumption stock at time t is formed as an exponentially declining average of future consumption according to

$$A(t) = \rho \int_t^\infty C(s) e^{-\rho(s-t)} ds, \quad \rho > 0. \tag{2}$$

Differentiating (2) with respect to time, the rate of adjustment of the anticipated consumption stock is

$$\dot{A}(t) = \rho\,[A(t) - C(t)], \qquad \lim_{t\to\infty} A(t) e^{-\rho t} = 0. \tag{3}$$

Gross output per capita Y is determined by the AK technology

$$Y(t) = BK(t), \quad B > 0,$$

where K is the per capita capital stock. In absence of depreciation, the agent's budget constraint is

$$\dot{K}(t) = BK(t) - C(t), \quad K(0) = K_0. \tag{4}$$

From (4), the intertemporal resource's constraint at time t is

$$\int_t^\infty C(s)\, \mathrm{e}^{-B(s-t)} ds = K(t). \tag{5}$$

As noted by Monteiro and Turnovsky (2016), anticipated future consumption, A, must be consistent with this constraint for all t which, using (2), entails that

$$\frac{A(t)}{\rho} = \int_t^\infty C(s)\, \mathrm{e}^{-\rho(s-t)} ds \qquad \le \int_t^\infty C(s)\, \mathrm{e}^{-B(s-t)} ds = K(t).$$

For these consumption expectations to be consistently viable we must have for all s and t that

$$\mathrm{e}^{-\rho(s-t)} \le \mathrm{e}^{-B(s-t)}$$

which reduces to $\rho \ge B$. If $\rho < B$ the present value of the agent's anticipated consumption eventually exceeds her resources and is therefore unrealizable. If $\rho = B$, the intertemporal budget constraint reduces to $A(0) = \rho K_0$, so the agent's anticipations are constrained by his current wealth, which imposes an initial condition on $A(t)$. We shall impose henceforth the following condition:

$$\rho > B > \beta > 0. \tag{6}$$

The first inequality ensures that the anticipated consumption stock is unconstrained by the agent's intertemporal resources, whereas the second inequality will ensure a positive long-run growth rate (see Proposition 6).

The representative agent chooses C, K and A to maximize the lifetime utility (1) subject to the budget constraint (4) and the constraint on the accumulation of the anticipated consumption stock (3), taking as given the initial condition on capital, $K(0) = K_0 > 0$.[3] Let $\mathcal{H}$ be the current value Hamiltonian of the agent's maximization problem,

$$\mathcal{H} = \ln(C + \gamma A) + \lambda(BK - C) + \mu\rho(A - C),$$

[3] Henceforth, the time argument will be suppressed when there is no risk of confusion.

where λ and μ are the shadow values of capital and the anticipated consumption stock, respectively.

To simplify the calculations, let $Z = C + \gamma A$ be the 'effective' consumption. The budget constraint can be equivalently expressed as

$$\dot{K} = BK + \gamma A - Z, \tag{7}$$

and the evolution of the anticipated consumption stock (3) can be rewritten as

$$\dot{A} = \rho[(1+\gamma)A - Z]. \tag{8}$$

Thus, the Hamiltonian can be expressed as

$$\mathcal{H} = \ln Z + \lambda(BK + \gamma A - Z) + \mu\rho[(1+\gamma)A - Z].$$

The first-order conditions for an interior optimum are[4]

$$\frac{\partial \mathcal{H}}{\partial Z} = 0 \quad \Longrightarrow \quad 1/Z = \lambda + \rho\mu, \tag{9}$$

$$\frac{\partial \mathcal{H}}{\partial K} = \beta\lambda - \dot{\lambda} \quad \Longrightarrow \quad B = \beta - \dot{\lambda}/\lambda, \tag{10}$$

$$\frac{\partial \mathcal{H}}{\partial A} = \beta\mu - \dot{\mu} \quad \Longrightarrow \quad \gamma\lambda/\mu + (1+\gamma)\rho = \beta - \dot{\mu}/\mu, \tag{11}$$

together with the initial conditions $K(0) = K_0$ and $\mu(0) = 0$, and the transversality condition

$$\lim_{t\to\infty} e^{-\beta t}\lambda K = \lim_{t\to\infty} e^{-\beta t}\mu A = 0. \tag{12}$$

Eq. (9) equates the marginal utility of 'effective' consumption to its cost, comprised of the (usual) shadow value of the current capital forgone plus the shadow value of future anticipated consumption. Eq. (10) equates the rate of return on capital to the rate of return on effective consumption, whereas Eq. (11) is an arbitrage condition that links the return on effective consumption expressed in terms of units of anticipation, on the right-hand side, to the return on anticipations, given by the left-hand side. Furthermore, as $A(0)$ is free the shadow value of the stock of anticipated future consumption must be zero at the initial time, $\mu(0) = 0$ (e.g., Kamien and Schwartz, 1971; Leonard and Long, 1992; Hestenes, 1996).

3. Closed-form solution of the equilibrium dynamics

In this section we derive a closed-form solution of the model. To this end, we will prove a series of lemmas which will lead us to the main proposition that describes the dynamics of the economy.

Let us define $z = Z/A$ as the ratio of effective consumption to the anticipated consumption stock, $a = A/K$ as the ratio of the anticipated consumption stock to capital, and $q = \mu/\lambda$ as the relative shadow value of the anticipated consumption stock. From (9) we get that

$$\lambda = \frac{1}{Z(1+\rho q)}, \tag{13}$$

and

$$\mu = \frac{q}{Z(1+\rho q)}. \tag{14}$$

Using (10) and (11) the dynamics of the relative shadow value of anticipated consumption is driven by

$$\dot{q} = -q[(1+\gamma)\rho - B] - \gamma, \tag{15}$$

with $q(0) = \mu(0)/\lambda(0) = 0$. Solving this ordinary differential equation, we can immediately state the following result.

Lemma 1. *The equilibrium time path of the relative shadow value, q, is given by*

$$q(t) = -\frac{\gamma}{\rho - B + \gamma\rho}\left[1 - e^{-(\rho - B + \gamma\rho)t}\right], \tag{16}$$

which converges in the long-run to

$$\bar{q} = -\frac{\gamma}{\rho - B + \gamma\rho}. \tag{17}$$

Our next step will be to compute the time path of the ratio of effective consumption to the anticipation stock, $z(t) = Z(t)/A(t)$. Differentiating (9), using (10) and (11), taking into account (13) and (14), we find that effective consumption evolves according to

$$\dot{Z} = \frac{Z[B - \beta + \gamma\rho + (\rho - \beta + \gamma\rho)\rho q]}{1 + \rho q}. \tag{18}$$

Using (18) and (8) we get after simplification that

$$\frac{\dot{z}}{z} = \frac{\dot{Z}}{Z} - \frac{\dot{A}}{A} = \frac{B - \rho}{1 + \rho q} - \beta + \rho z. \tag{19}$$

Let $f(t)$ be the Gaussian hypergeometric function (e.g., formula 15.6.1 in Olver et al., 2010),[5]

$$f(t) = F\left(1, \frac{\beta}{\rho - B + \gamma\rho} + 1; \frac{\beta}{\rho - B + \gamma\rho} + 2; -\frac{\gamma\rho}{\rho - B}e^{-(\rho - B + \gamma\rho)t}\right), \tag{20}$$

with

$$f(0) = F\left(1, \frac{\beta}{\rho - B + \gamma\rho} + 1; \frac{\beta}{\rho - B + \gamma\rho} + 2; -\frac{\gamma\rho}{\rho - B}\right).$$

The following result determines the time path of the ratio of effective consumption to the anticipation stock, $z(t) = Z(t)/A(t)$.

Lemma 2. *The equilibrium time path of the ratio of 'effective' consumption to the stock of anticipated consumption, $z(t) = Z(t)/A(t)$, is given by*

$$z(t) = \frac{(\rho - B)(\rho - B + \gamma\rho + \beta)}{\rho\left[\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}\right]f(t)}, \tag{21}$$

which converges in the long-run to

$$\bar{z} = \frac{\rho - B + \beta + \gamma\rho}{\rho}. \tag{22}$$

Proof. See Appendix A.2. □

Using this result, we can state the equilibrium paths of the anticipated consumption stock, effective consumption and current consumption.

Lemma 3. *The equilibrium time paths of anticipated consumption, $A(t)$, consumption, $C(t)$, and effective consumption, $Z(t) = C(t) + \gamma A(t)$, are given by*

$$Z(t) = \frac{(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}} Z_0 e^{(B-\beta)t}, \tag{23}$$

$$A(t) = \frac{\rho(\rho - B + \gamma\rho)f(t)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} Z_0 e^{(B-\beta)t}, \tag{24}$$

$$C(t) = Z(t) - \gamma A(t). \tag{25}$$

[4] The hamiltonian is concave in the states and the controls, and so, the first-order conditions, along with the initial and transversality conditions, characterize the interior optimal solution of the agent's problem.

[5] The hypergeometric function is also habitually notated as $F(a, b; c; z) = {}_2F_1(a, b; c; z)$; see formula 15.5.1 in Olver et al. (2010). Besides this reference, a quick overview of the Gaussian hypergeometric function is provided in Boucekkine and Ruiz-Tamarit (2008).

Proof. Eq. (23) follows from solving Eq. (18), after substituting q for (16), with (yet unknown) initial condition $Z(0) = Z_0$. Now, Eq. (24) follows from $A(t) = Z(t)/z(t)$, using (21) and (23). □

The next result provides the time path of the capital stock.

Lemma 4. *The equilibrium time path of the capital stock is given by*

$$K(t) = \left\{ \frac{1}{\beta} + \frac{\gamma\rho f(t)\left[1 - e^{-(\rho-B+\gamma\rho)t}\right]}{(\rho - B)(\rho - B + \gamma\rho + \beta)} \right\} Z_0 e^{(B-\beta)t} + \left[K_0 - \frac{Z_0}{\beta} \right] e^{Bt}, \tag{26}$$

where $f(t)$ is the Gaussian hypergeometric function defined in (20).

Proof. See Appendix A.3. □

The next result determines the still unknown initial value of effective consumption.

Lemma 5. *The initial value of effective consumption, $Z(0) = Z_0$, is*

$$Z_0 = \beta K_0. \tag{27}$$

Proof. See Appendix A.4. □

Finally, combining the previous lemmas, we can state our main result which fully characterizes the dynamics of the economy.

Proposition 6. *Let us assume that $\rho > B > \beta > 0$. The economy has a unique feasible equilibrium with positive long-run growth which is described by*

$$K(t) = \left\{ 1 + \frac{\beta\gamma\rho}{(\rho - B)(\rho - B + \gamma\rho + \beta)} \times \times \left[1 - e^{-(\rho-B+\gamma\rho)t}\right] f(t) \right\} K_0 e^{(B-\beta)t}, \tag{28}$$

$$Z(t) = \frac{\beta(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho-B+\gamma\rho)t}} K_0 e^{(B-\beta)t}, \tag{29}$$

$$A(t) = \frac{\beta\rho(\rho - B + \gamma\rho)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} f(t) K_0 e^{(B-\beta)t}, \tag{30}$$

$$C(t) = Z(t) - \gamma A(t), \tag{31}$$

where $f(t)$ is the Gaussian hypergeometric function defined in (20). These equilibrium paths approach asymptotically to the unique positive balanced growth paths given by, respectively,

$$K(t) = \left[1 + \frac{\beta\gamma\rho}{(\rho - B)(\rho - B + \gamma\rho + \beta)} \right] K_0 e^{(B-\beta)t}, \tag{32}$$

$$Z(t) = \frac{\beta(\rho - B + \gamma\rho)}{\rho - B} K_0 e^{(B-\beta)t}, \tag{33}$$

$$A(t) = \frac{\beta\rho(\rho - B + \gamma\rho)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} K_0 e^{(B-\beta)t}, \tag{34}$$

$$C(t) = \frac{\beta(\rho - B + \beta)(\rho - B + \gamma\rho)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} K_0 e^{(B-\beta)t}, \tag{35}$$

along which K, Z, A and C grow permanently at a positive constant rate

$$\bar{g} = B - \beta. \tag{36}$$

Proof. Eqs. (28), (29), (30) and (31) follow from substituting (27) into (26), (23), (24) and (25), respectively. □

Using this result it can be immediately observed that the ratio of the anticipated consumption stock to the capital stock converges to its stationary value

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} \frac{A(t)}{K(t)} = \frac{\rho\beta}{\rho - B + \beta} = \bar{a},$$

the ratio of effective consumption to the anticipated consumption stock converges to its stationary value (22),

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{Z(t)}{A(t)} = \frac{\rho - B + \beta + \gamma\rho}{\rho} = \bar{z},$$

and the ratio of current consumption to capital converges to its stationary value

$$\lim_{t\to\infty} \frac{C(t)}{K(t)} = \beta.$$

It should be noted that the initial ratio of effective consumption to capital is equal to the long-run value of current consumption to capital, $Z_0/K_0 = \beta = \lim_{t\to\infty} C(t)/K(t)$.

On the convergence to the balanced growth path, we can state the following result.

Proposition 7. *The growth rates of effective consumption, the anticipated consumption stock, and the capital stock are above its long-run value, $B - \beta$, along the transition to their respective balanced growth paths.*

Proof. See Appendix A.5. □

Actually, effective consumption grows at a positive but monotonically decreasing rate given that

$$\frac{d}{dt} g_Z(t) = \frac{d}{dt}\left[\frac{\dot{Z}(t)}{Z(t)}\right] = -\frac{\gamma\rho(\rho - B)(\rho - B + \gamma\rho)^2 e^{-(\rho-B+\gamma\rho)t}}{\left[\rho - B + \gamma\rho e^{-(\rho-B+\gamma\rho)t}\right]^2} < 0.$$

Finally, using the closed-form equilibrium solution found in Proposition 6, we can also determine an explicit expression for the maximum intertemporal utility.

Proposition 8. *The maximum attainable intertemporal utility is*

$$\bar{U} = \frac{\ln Z_0}{\beta} + \frac{B - \beta}{\beta^2} + \frac{(\rho - B + \gamma\rho)\gamma\rho}{\beta(\rho - B)(\rho - B + \gamma\rho + \beta)} f(0). \tag{37}$$

Proof. See Appendix A.6. □

4. Anticipation versus habits

In this section we shall briefly compare the models with anticipated future consumption and habit formation.

In the model with habits, the agent derives utility from her current consumption, C, and a reference consumption level or habits stock, H, according to

$$U = \int_0^\infty \ln[C(t) - \gamma H(t)] e^{-\beta t} dt, \quad \beta > 0, \quad 0 < \gamma < 1. \tag{38}$$

The habits stock is formed as an exponentially declining average of own past consumption,

$$H(t) = \rho \int_{-\infty}^{t} e^{\rho(s-t)} C(s) ds, \quad \rho > 0,$$

so its rate of adjustment is

$$\dot{H} = \rho(C - H), \qquad H(0) = H_0. \tag{39}$$

The agent chooses C, K and H to maximize the lifetime utility (38) subject to the budget constraint (4) and the constraint on the

Table 1
Benchmark parameters and steady-state values.

Production	Preferences	Initial values	Steady state
$B = 0.07$	$\beta = 0.05$, $\rho = 0.2$, $\gamma = 0.5$	$K_0 = 1$ ($H_0 = 0.05$)	$\bar{g} = 0.02$, $\overline{C/K} = 0.05$

dynamics of the habits stock (39), taking as given the initial conditions on capital, $K(0) = K_0 > 0$, and the stock of habits, $H(0) = H_0$. It can be readily noted that the setup of the habits model is similar to that of the anticipation model by replacing A, ρ and γ with H, $-\rho$ and $-\gamma$, respectively. A crucial difference is that now the habits stock, H, is a variable historically predetermined, so that $H(0) = H_0$ is given. In contrast, the anticipated consumption stock, A, is not a predetermined but a jumpable variable, so its initial value $A(0) = A_0$ is free and it is its associated co-state variable, say μ, which is zero at the initial time; i.e., $\mu(0) = 0$.

In what follows, the superscripts 'H' and 'A' will refer to the habit-formation and the anticipated consumption models, respectively. Let $Z^H(t) = C(t) - \gamma H(t)$ denote the *effective* consumption in the habit-formation model. The closed-form solution of the former (internal) habits model has been derived by Gómez (2010, Appendix C) and is given by

$$\begin{aligned}
K^H(t) &= \frac{(\rho + B - \beta)Z_0^H}{\beta(\rho + B - \beta - \rho\gamma)} e^{(B-\beta)t} \\
&\quad + \left[H(0) - \frac{\rho Z_0^H}{\rho + B - \beta - \gamma\rho} \right] \frac{\gamma}{\rho + B - \gamma\rho} e^{-\rho(1-\gamma)t}, \\
H(t) &= \frac{\rho Z_0^H}{\rho + B - \beta - \rho\gamma} e^{(B-\beta)t} \\
&\quad + \left[H(0) - \frac{\rho Z_0^H}{\rho + B - \beta - \rho\gamma} \right] e^{-\rho(1-\gamma)t}, \\
C^H(t) &= \frac{(\rho + B - \beta)Z_0^H}{\rho + B - \beta - \rho\gamma} e^{(B-\beta)t} \\
&\quad + \left[H(0) - \frac{\rho Z_0^H}{\rho + B - \beta - \rho\gamma} \right] \gamma e^{-\rho(1-\gamma)t}, \\
Z^H(t) &= Z_0^H e^{(B-\beta)t},
\end{aligned}$$

where the initial value of effective consumption is

$$Z_0^H = \beta K_0 - \frac{\beta\gamma(\rho K_0 + H_0)}{B + \rho}. \tag{40}$$

Hence, the intertemporal utility can be readily computed to be

$$\bar{U}^H = \int_0^\infty \ln Z^H(t) e^{-\beta t} dt = \frac{\ln Z_0^H}{\beta} + \frac{B - \beta}{\beta^2}. \tag{41}$$

To compare the intertemporal utility in the anticipation and habits models, note first that the initial value of 'effective' consumption in the anticipation model given by (27) is higher than the corresponding one in the habits model given by (40) as

$$Z_0^A - Z_0^H = \frac{\beta\gamma[\rho K_0 + H_0]}{B + \rho} > 0.$$

Actually, the entire path of effective consumption in the anticipation model lies above the corresponding one in the habits model as we have that

$$\begin{aligned}
\frac{Z^A(t)}{Z^H(t)} &= \frac{\frac{(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}} Z_0^A e^{(B-\beta)t}}{Z_0^H e^{(B-\beta)t}} \\
&= \frac{(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}} \frac{Z_0^A}{Z_0^H} > 1.
\end{aligned}$$

This immediately entails that the maximum utility attainable in the anticipation model, given by (37), is higher than the one attainable in the habits model, given by (41), as

$$\bar{U}^A - \bar{U}^H = \frac{1}{\beta} \ln(Z_0^A / Z_0^H) + \frac{(\rho - B + \gamma\rho)\gamma\rho}{\beta(\rho - B)(\rho - B + \gamma\rho + \beta)} f(0) > 0.$$

Now, we shall run comparative-statics and -dynamics exercises with respect to some key parameters, and compare the results in both models. We can state the following result.

Proposition 9.

(i) A shock that increases the productivity parameter, B, shifts upwards the time path of effective consumption and increases the intertemporal utility both in the models with anticipated consumption and habit formation.

(ii) A shock that increases the weight of the consumption stock (anticipated consumption stock or habits stock, respectively) in effective consumption, γ, shifts upwards the time path of effective consumption and increases the intertemporal utility in the model with anticipated consumption, and shifts downwards the time path of effective consumption and decreases the intertemporal utility in the model with habit formation.

(iii) A shock that increases the speed of adjustment of the reference consumption stock, ρ, decreases the intertemporal utility in the anticipated consumption model, whereas the effect in the habit-formation model is negative (resp., positive) if initial income is higher (resp., lower) than the initial habits stock.

Proof. See Appendix A.7. □

Intuitively, in the anticipation model a higher productivity has a positive effect on the expectations of future consumption and, therefore, on effective consumption which leads to a higher intertemporal utility. The same happens in the habit-formation model. An increase in the weight of anticipated future consumption, γ, carries on a higher effective consumption while does not affect the resources' constraint and, therefore, also leads to a higher intertemporal utility. In contrast, in the habit-formation model an increase in the weight of habits reduces effective consumption and, therefore, overall intertemporal utility. Finally, a faster speed of adjustment of the anticipated consumption stock means a higher weight of consumption in the near future which reduces the willingness to substitute present for future consumption. Hence, differently to what happens in the former cases, effective consumption does not have to increase unambiguously at each point in time in the anticipation model. Nevertheless, intertemporal utility decreases after the shock. In the habit-formation model, a higher speed of adjustment entails that the habits stock builds faster which would result in a lower effective consumption. Substituting current for future consumption can slow down the increase in the habits stock, but the effect on intertemporal utility depends on how initial income compares to the initial stock of habits.

To complement the former theoretical results, we present some numerical simulations to compare the effect of a shock on the dynamics of the models with anticipation and habits. Table 1 summarizes the baseline parameterization with the corresponding steady-state values. We follow Carroll et al. (1997) to choose the rate of time preference, $\beta = 0.05$, and the speed of adjustment of the consumption stock, $\rho = 0.2$, whereas the weight of the consumption stock in utility is chosen as $\gamma = 0.5$. The productivity parameter B is then set so that the long-run

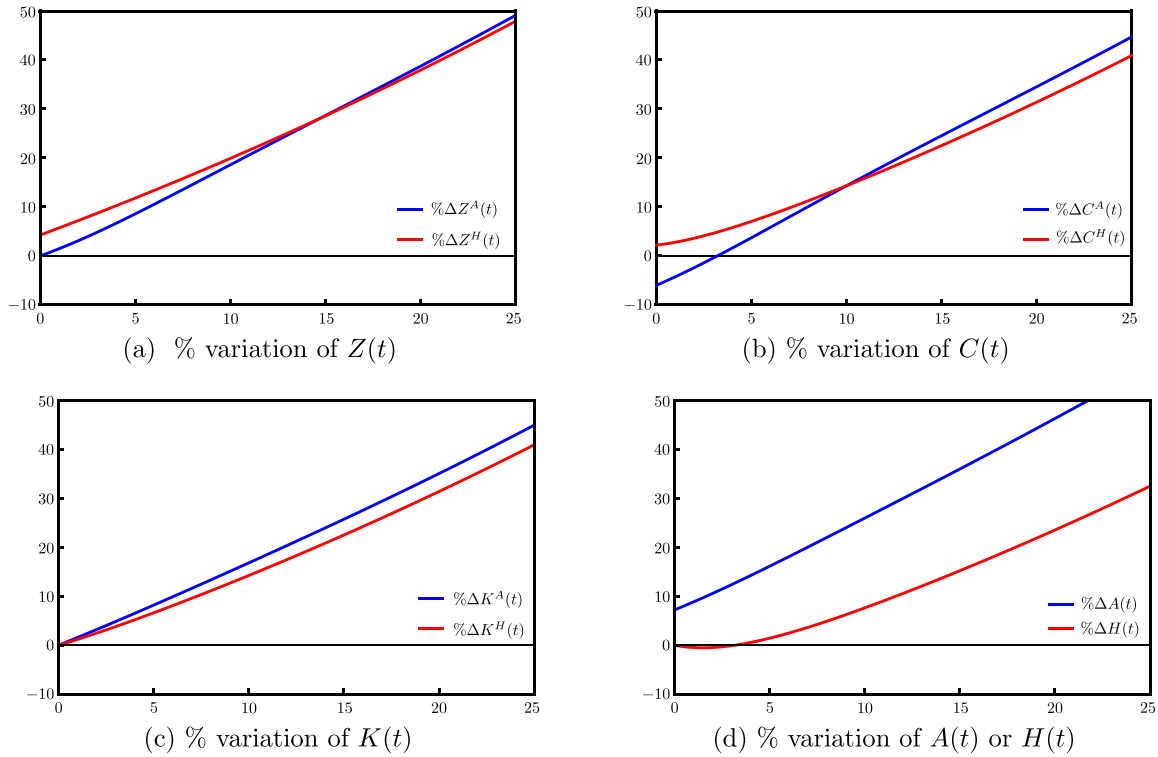


Fig. 1. Percentage variation paths after a shock that increases productivity by 20%.

growth rate is 2 percent, which yields a value of $B = 0.07$. The resulting steady-state ratio of consumption to capital is $\overline{C/K} = \beta = 0.05$ in both models.

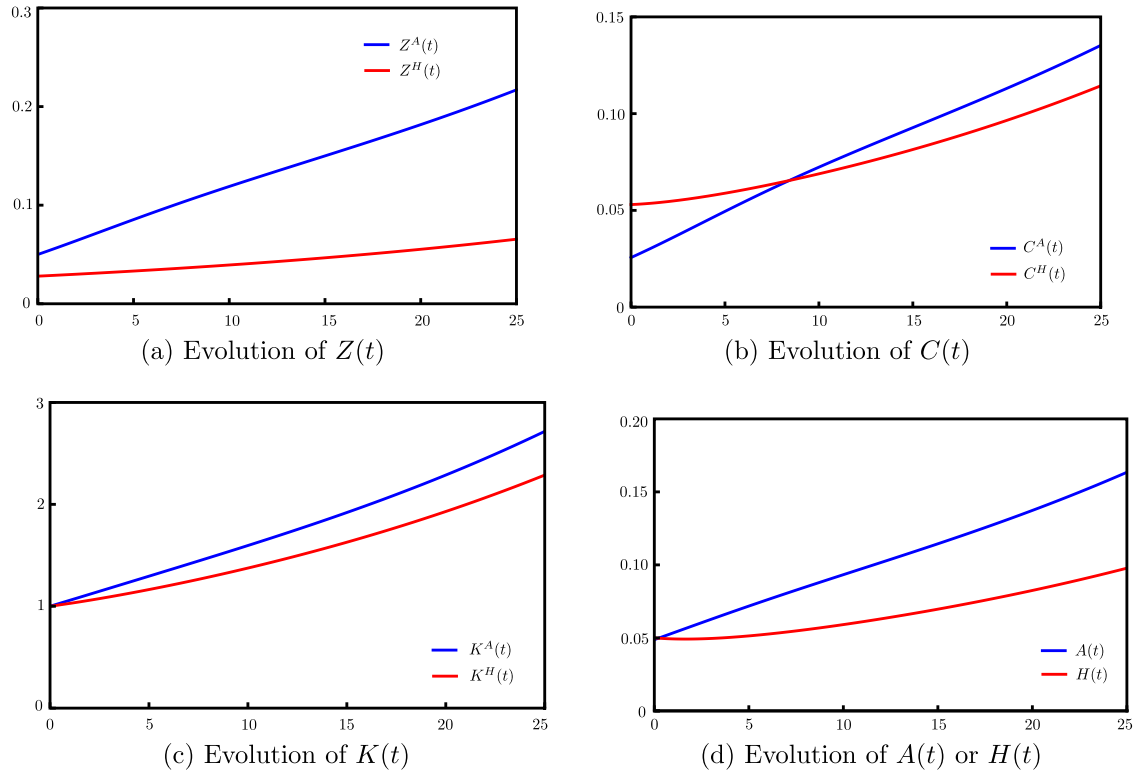
We consider the dynamic effects of a shock that increases productivity by a 20%, so that B increases from 0.07 to 0.084. This is non stationary in the sense that after this shock the long-run growth rate of the economy increases to 3.4%. Figs. 1 and 2 depict the adjustment of some variables in the models with anticipated consumption and habit formation. To start the simulations, we normalize the initial value of the capital stock at $K_0 = 1$ and, in the habit-formation model, we set the initial habits stock to $H_0 = 0.05$.

Fig. 1 depicts the percentage variation of effective consumption, current consumption, the capital stock and the reference consumption stock (anticipated consumption stock or habits stock, respectively) after an increase by a 20% in the productivity parameter. Thus, if $x^{\text{old}}(t)$ is the value of the variable x at time t with the pre-shock (old) productivity parameter and $x^{\text{new}}(t)$ is the value of the variable x at time t with the post-shock (new) productivity parameter, the percent variation at time t would be $(x^{\text{new}}(t) - x^{\text{old}}(t))/x^{\text{old}}(t)$. On impact Fig. 1(b) shows that consumption responds in the opposite way in both models, decreasing in the anticipated consumption model and increasing in the habit-formation model. In the anticipation model agents incorporate the impact that a higher productivity will have on their expected future consumption. Thus, they reduce consumption at the outset, allowing to save more and accumulate more capital in the future, as shown by Fig. 1(c), which results in a higher anticipated consumption stock. In contrast, in the habits model agents dislike large changes in consumption, and so, increase consumption both on impact and along time to reflect the rise in productivity. This results in a smaller rise in capital than in the anticipation model, as shown in Fig. 1(c). According to the theoretical results in Proposition 9, effective consumption is unambiguously above its pre-shock value in both models.

Fig. 2 depicts the time paths of effective consumption, current consumption, capital stock and reference consumption stock (anticipated consumption stock or habits stock, respectively). Although consumption in the anticipation model jumps down on impact, Fig. 2(b) shows that after approximately 10 years it surpasses consumption in the habits model. The evolution of the reference consumption stock, shown in Fig. 2(d), reveals that a higher productivity brings good expectations and thus the agents rise their anticipated consumption stock, while habits must adjust over time and never catch up to the anticipation stock. As a result, Fig. 2(a) shows that effective consumption is unambiguously higher in the anticipation model than in the habits model. The capital stock increases in both models, but it increases at a higher pace in the anticipation model, as shown by Fig. 2(c).

5. Conclusions

This paper has determined the equilibrium dynamics of the AK endogenous growth model when utility depends on anticipated future consumption. In order to get an explicit solution, we have made the simplifying assumption that utility is logarithmic and the reference consumption stock enters additively into effective consumption. We have determined the equilibrium time paths of the variables in level by means of Gaussian hypergeometric functions. Thus, this paper adds more evidence to recent research that has shown that special functions — in particular, Gaussian hypergeometric functions — may be helpful to study the dynamics of growth models. We compare the equilibrium dynamics of the model with anticipated consumption and the model with habit formation. We show that the time path of effective consumption in the anticipation model is unambiguously above the corresponding one in the habits model. The maximum utility attainable in the model with anticipation is therefore higher than that in the model with habit formation. The derived explicit



(a) Evolution of $Z(t)$ (b) Evolution of $C(t)$

(c) Evolution of $K(t)$ (d) Evolution of $A(t)$ or $H(t)$

Fig. 2. Time paths after a shock that increases productivity by 20%.

solution also allows to determine the comparative-dynamics and -static effects of some shocks on parameters in an analytical manner. A higher productivity leads to a higher intertemporal utility in both models. A higher weight of the reference consumption stock in utility entails a higher intertemporal utility in the anticipation model and a lower one in the habit-formation model. A faster speed of adjustment of the reference consumption stock leads to a lower intertemporal utility in the anticipation model whereas the effect in the habits model depends on how initial output compares with the initial stock of habits. We complement the theoretical results with numerical simulations to illustrate the effect of a productivity shock.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Proofs

A.1. Preliminary results

We shall first state some results that will be needed in the following proofs.

Lemma 10. *It holds that*

$$I = \int_0^t e^{-a_1 s}(b_1 + b_2 e^{-a_2 s})^{-1} ds = -\frac{1}{a_1 b_1}\left[e^{-a_1 t} f(t) - f(0)\right], \quad \text{(A.1)}$$

where f is the Gaussian hypergeometric function

$$f(t) = F(1, a_1/a_2, 1 + a_1/a_2, -(b_2/b_1)e^{-a_2 t}). \quad \text{(A.2)}$$

Proof. By means of the change of variable $u = e^{-a_2 s}$, so that $du = -a_2 u ds$, we get

$$\begin{aligned} I &= \frac{1}{b_1}\int_0^t e^{-a_1 s}\left[1 + \frac{b_2}{b_1}e^{-a_2 s}\right]^{-1} ds \\ &= -\frac{1}{a_2 b_1}\left\{\int_0^{e^{-a_2 t}} u^{-1+\frac{a_1}{a_2}}\left[1 + \frac{b_2}{b_1}u\right]^{-1} du \right. \\ &\quad \left. - \int_0^1 u^{-1+\frac{a_1}{a_2}}\left[1 + \frac{b_2}{b_1}u\right]^{-1} du\right\}. \end{aligned}$$

Now, by means of the change of variable $u = xe^{-a_2 t}$, so that $du = -e^{-a_2 t}dx$, in the first integral, and using the integral representation of the hypergeometric function (see, e.g., formula 15.6.1, together with formula 15.1.2, in Olver et al., 2010), we arrive at

$$\begin{aligned} I &= -\frac{1}{a_2 b_1}\left\{e^{-a_1 t}\int_0^1 x^{-1+\frac{a_1}{a_2}}\left[1 + \frac{b_2}{b_1}xe^{-a_2 t}\right]^{-1} dx \right. \\ &\quad \left. - \int_0^1 u^{-1+\frac{a_1}{a_2}}\left[1 + \frac{b_2}{b_1}u\right]^{-1} du\right\} \\ &= -\frac{1}{a_2 b_1}\frac{\Gamma(a_1/a_2)\,\Gamma(1)}{\Gamma(1 + a_1/a_2)}\left[e^{-a_1 t} f(t) - f(0)\right] \\ &= -\frac{1}{a_1 b_1}\left[e^{-a_1 t} f(t) - f(0)\right], \end{aligned}$$

where $f(t)$ is defined by (A.2), and so

$$f(0) = F(1, a_1/a_2, 1 + a_1/a_2, -b_2/b_1), \tag{A.3}$$

and Γ denotes the gamma function. □

Lemma 11. *It holds that*

$$J = \int_t^\infty e^{-\beta(s-t)} F(1, a+1; a+2; be^{-\beta s/a}) ds$$
$$= \frac{1}{\beta} F(1, a; a+2; be^{-\beta t/a}). \tag{A.4}$$

Proof. Using the integral representation of the hypergeometric function (see, e.g., formula 15.6.1, together with formula 15.1.2, in Olver et al., 2010), we have that

$$J = \frac{\Gamma(a+2)}{\Gamma(a+1)\Gamma(1)} \int_t^\infty e^{-\beta(s-t)} \int_0^1 x^a (1 - bxe^{-\beta s/a})^{-1} dx\, ds$$
$$= (1+a) \int_0^1 x^a \int_t^\infty e^{-\beta(s-t)} (1 - bxe^{-\beta s/a})^{-1} ds\, dx,$$

after changing the order of integration in the last line. Now, using the change of variable $u = e^{-\beta(s-t)/a}$, so that $du = -(\beta/a) e^{-\beta(s-t)/a} ds = -(\beta u/a) ds$, and noting that the change in the limits of integration entails a sign change, we obtain that

$$J = \frac{(a+1)a}{\beta} \int_0^1 x^a \int_0^1 u^a (1 - bxe^{-\beta t/a} u)^{-1} du\, dx$$
$$= \frac{a+1}{\beta} \int_0^1 x^a F(1, a; a+1; bxe^{-\beta t/a}) dx,$$

where we have used again the integral representation of the hypergeometric function. Now, using the extension of Euler's integral to a larger domain (see, e.g., Andrews et al., 1999, Theorem 2.2.4, p. 68) in the last expression, we get that

$$J = \frac{(a+1)\,\Gamma(a+1)}{\beta\ \Gamma(a+2)\Gamma(1)} F(1, a; a+2; be^{-\beta t/a})$$
$$= \frac{1}{\beta} F(1, a; a+2; be^{-\beta t/a}),$$

where Γ denotes the gamma function. Thus, we get the result. □

A.2. Proof of Lemma 2

Using Eq. (16) to substitute for q into (19), and after simplification, we arrive at the Riccati differential equation

$$\frac{\dot{z}}{z^2} + \left[\beta - \frac{(B-\rho)(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}}\right] \frac{1}{z} = \rho,$$

with (yet unknown) initial condition $z(0) = z_0$. Defining $w = 1/z$, so that $\dot{w} = -\dot{z}/z^2$, the former differential equation can be rewritten as

$$\dot{w} - [\beta - h(t)]\, w = -\rho, \tag{A.5}$$

where

$$h(t) = \frac{(B-\rho)(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}}.$$

The solution of this linear ODE is

$$w(t) = e^{\int_0^t (\beta - h(s)) ds} \left[w_0 - \int_0^t e^{-\int_0^s (\beta - h(\tau)) d\tau} \rho ds \right], \tag{A.6}$$

where $w_0 = 1/z_0$. Note first that

$$\int_0^s (\beta - h(\tau)) d\tau = (\rho - B + \gamma\rho + \beta)s + \ln\left[\frac{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)s}}{\rho - B + \gamma\rho}\right],$$

and, therefore,

$$e^{-\int_0^s (\beta - h(\tau)) d\tau} = \frac{(\rho - B + \gamma\rho) e^{-(\rho - B + \gamma\rho + \beta)s}}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)s}}.$$

Hence, using Eq. (A.1) in Lemma 10 with $a_1 = \rho - B + \gamma\rho + \beta$, $a_2 = \rho - B + \gamma\rho$, $b_1 = 1$, and $b_2 = \gamma\rho/(\rho - B)$, we get that

$$\int_0^t e^{-\int_0^s (\beta - h(\tau)) d\tau} \rho ds$$
$$= \frac{(\rho - B + \gamma\rho)\rho}{\rho - B} \int_0^t e^{-(\rho - B + \gamma\rho + \beta)s}$$
$$\left[1 + \frac{\gamma\rho}{\rho - B} e^{-(\rho - B + \gamma\rho)s}\right]^{-1} ds$$
$$= \frac{\rho(\rho - B + \gamma\rho)}{(B-\rho)(\rho - B + \gamma\rho + \beta)} \left[e^{-(\rho - B + \gamma\rho + \beta)t} f(t) - f(0)\right],$$

where f is the Gaussian hypergeometric function

$$f(t) = F\left(1, \frac{\beta}{\rho - B + \gamma\rho} + 1; \frac{\beta}{\rho - B + \gamma\rho} + 2; -\frac{\gamma\rho}{\rho - B} e^{-(\rho - B + \gamma\rho)t}\right),$$

defined in (20). Substituting the former results into (A.6), after simplification we get that

$$w(t) = \frac{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)t}}{\rho - B + \gamma\rho}$$
$$\times \left\{ \left[w_0 - \frac{\rho(\rho - B + \gamma\rho) f(0)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} \right] e^{(\rho - B + \gamma\rho + \beta)t} + \frac{\rho(\rho - B + \gamma\rho) f(t)}{(\rho - B)(\rho - B + \gamma\rho + \beta)} \right\}.$$

For $w(t)$ to converge to a steady-state value, $\lim_{t\to\infty} w(t) = \bar{w} = 1/\bar{z}$, (rather than diverge to infinity), noting that $\lim_{t\to\infty} f(t) = 1$, it must be that the term that multiplies $e^{(\rho - B + \gamma\rho + \beta)t}$ is zero; i.e.,

$$w_0 = \frac{1}{z_0} = \frac{\rho(\rho - B + \gamma\rho) f(0)}{(\rho - B)(\rho - B + \gamma\rho + \beta)}.$$

Then, using that $z(t) = 1/w(t)$, we finally get Eq. (21) after simplification. The long-run value (22) is then obtained from Eq. (21) by computing $\bar{z} = \lim_{t\to\infty} z(t)$.

A.3. Proof of Lemma 4

Substituting $Z(t)$ from (23) and $A(t)$ from (24) into Eq. (7), and solving the resulting linear ODE with initial condition $K(0) = K_0$, we immediately get that

$$K(t) = K_0 e^{Bt} - Z_0(\rho - B + \gamma\rho) e^{Bt} \int_0^t \frac{e^{-\beta s}}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)s}} ds$$
$$+ \frac{Z_0(\rho - B + \gamma\rho)\gamma\rho}{(\rho - B)(\rho - B + \gamma\rho + \beta)} e^{Bt} \int_0^t e^{-\beta s} f(s) ds, \tag{A.7}$$

where f is the Gaussian hypergeometric function defined in Eq. (20):

$$f(t) = F\left(1, \frac{\beta}{\rho - B + \gamma\rho} + 1; \frac{\beta}{\rho - B + \gamma\rho} + 2; -\frac{\gamma\rho}{\rho - B} e^{-(\rho - B + \gamma\rho)t}\right).$$

Step 1: Let us first solve the integral in the second term of the right-hand side of Eq. (A.7),

$$\int_0^t \frac{e^{-\beta s}}{\rho - B + \gamma\rho e^{-(\rho - B + \gamma\rho)s}} ds.$$

Using Eq. (A.1) in Lemma 10 with $a_1 = \beta$, $a_2 = \rho - B + \gamma\rho$, $b_1 = 1$, and $b_2 = \gamma\rho/(\rho - B)$, we have that

$$\begin{aligned}\int_0^t \frac{e^{-\beta s}}{\rho - B + \gamma\rho e^{-(\rho-B+\gamma\rho)s}} ds &= \frac{1}{\rho - B}\int_0^t e^{-\beta s}\left[1 + \frac{\gamma\rho}{\rho - B}e^{-(\rho-B+\gamma\rho)s}\right]^{-1} ds \\ &= \frac{1}{\beta(\rho - B)}\left[g(0) - e^{-\beta t}g(t)\right], \end{aligned} \tag{A.8}$$

where g is the Gaussian hypergeometric function

$$g(t) = F\left(1, \frac{\beta}{\rho - B + \gamma\rho}; \frac{\beta}{\rho - B + \gamma\rho} + 1; -\frac{\gamma\rho}{\rho - B}e^{-(\rho-B+\gamma\rho)t}\right). \tag{A.9}$$

Step 2: Let us now solve the integral in the third term of the right-hand side of Eq. (A.7),

$$\int_0^t e^{-\beta s}f(s)ds = \int_0^t e^{-\beta s}F\left(1, a+1; a+2; b\, e^{-\beta s/a}\right) ds,$$

where we have denoted $a = \beta/(\rho - B + \gamma\rho)$ and $b = -\gamma\rho/(\rho - B)$. Let us rewrite the former integral as

$$\int_0^t e^{-\beta s}f(s)ds = \int_0^\infty e^{-\beta s}f(s)ds - e^{-\beta t}\int_t^\infty e^{-\beta(s-t)}f(s)ds. \tag{A.10}$$

Using Eq. (A.4) in Lemma 11 with $t = 0$ we have that

$$\begin{aligned}\int_0^\infty e^{-\beta s}f(s)ds &= \int_0^\infty e^{-\beta s}F(1, a+1; a+2; b\, e^{-\beta s/a})ds \\ &= \frac{1}{\beta}F(1, a; a+2; b),\end{aligned}$$

and, therefore, application of Lemma 11 to (A.10) entails that

$$\int_0^t e^{-\beta s}f(s)ds = \frac{1}{\beta}\left[F(1, a; a+2; b) - e^{-\beta t}F(1, a; a+2; be^{-\beta t/a})\right]. \tag{A.11}$$

Using the formula 15.5.16 in Olver et al. (2010) on the relationship between contiguous hypergeometric functions we have that

$$\begin{aligned}&(a+1)(1-z)F(1, a; a+1; z) - (a+1)F(0, a; a+1; z) \\ &\quad + z\, F(1, a; a+2; z) \\ &\quad = (a+1)(1-z)F(1, a; a+1; z) - (a+1) \\ &\quad + z\, F(1, a; a+2; z) = 0,\end{aligned}$$

and, therefore, we obtain that

$$F(1, a; a+2; z) = \frac{a+1}{z}\left[1 - (1-z)F(1, a; a+1; z)\right]. \tag{A.12}$$

Substituting Eq. (A.12) with $z = b$ and $z = be^{-\beta t/a}$, respectively, into Eq. (A.11), we get that

$$\begin{aligned}\int_0^t e^{-\beta s}f(s)ds &= \frac{a+1}{\beta b} \\ &\times \{1 - (1-b)F(1, a; a+1; b) \\ &- \left[1 - (1 - be^{-\beta t/a})F(1, a; a+1; be^{-\beta t/a})\right]e^{\beta(1-a)t/a}\}.\end{aligned}$$

Substituting $a = \beta/(\rho - B + \gamma\rho)$ and $b = -\gamma\rho/(\rho - B)$, using that $g(t) = F(1, a; a+1; be^{-\beta t/a})$ from Eq. (A.9), we obtain after some algebra that

$$\begin{aligned}\int_0^t e^{-\beta s}f(s)ds &= \frac{\rho - B + \gamma\rho + \beta}{\beta\gamma\rho} \\ &\times \left\{g(0) - \frac{\rho - B}{\rho - B + \gamma\rho} + \frac{[1 - g(t)](\rho - B)e^{(\rho-B+\gamma\rho)t} - \gamma\rho g(t)}{\rho - B + \gamma\rho}e^{-\beta t}\right\}.\end{aligned} \tag{A.13}$$

Step 3: Now, substituting Eqs. (A.8) and (A.13) into Eq. (A.7), after simplification we get that

$$K(t) = \frac{1}{\beta}\left[(1 - g(t))Z_0e^{(\rho-\beta+\gamma\rho)t} + g(t)Z_0e^{(B-\beta)t} + e^{Bt}(\beta K_0 - Z_0)\right]. \tag{A.14}$$

Using the formula 15.5.16 in Olver et al. (2010) we have that

$$(a+1)(1-z)F(1, a; a+1; z) - (a+1) + zF(1, a; a+2; z) = 0. \tag{A.15}$$

Using the formula 15.5.12 in Olver et al. (2010) we have that

$$(a-1)F(1, a; a+2; z) + F(2, a; a+2; z) - aF(1, a+1; a+2; z) = 0. \tag{A.16}$$

Using the formula 15.5.11 in Olver et al. (2010) we have that

$$((a-1)z - a)F(1, a; a+2; z) + (a+1) + (z-1)F(2, a; a+2; z) = 0. \tag{A.17}$$

Solving the system (A.15)–(A.17), in particular we get that

$$F(1, a; a+1; z) = 1 + \frac{az}{a+1}F(1, a+1; a+2; z), \tag{A.18}$$

which, using $a = \beta/(\rho - B + \gamma\rho)$ and $z = -\gamma\rho e^{-(\rho-B+\gamma\rho)t}/(\rho - B)$, entails that

$$g(t) = 1 - \frac{\beta\gamma\rho}{(\rho - B + \gamma\rho + \beta)(\rho - B)}e^{-(\rho-B+\gamma\rho)t}f(t).$$

Substituting this expression into (A.14) yields Eq. (26) after simplification.

A.4. Proof of Lemma 5

Using (10), the transversality condition (12) becomes

$$\begin{aligned}\lim_{t\to\infty} e^{-\beta t}\lambda(t)K(t) &= \lim_{t\to\infty}\lambda(0)e^{(\beta-B)t}e^{-\beta t}K(t) \\ &= \lambda(0)\lim_{t\to\infty}e^{-Bt}K(t) = 0,\end{aligned}$$

which using (26) is equivalent to

$$\begin{aligned}&\lim_{t\to\infty}\left\{\frac{\gamma\beta\left[1 - e^{-(\rho-B+\gamma\rho)t}\right]}{(1+\gamma)(\rho - B)}Z_0f(t)e^{-\beta t} + Z_0e^{-\beta t} + \beta K_0 - Z_0\right\} \\ &= \beta K_0 - Z_0 = 0.\end{aligned} \tag{A.19}$$

Hence, the condition (27) follows from (A.19).

A.5. Proof of Proposition 7

(i) The fact that the growth rate of effective consumption is positive and above its long-run value, $B - \beta$, along the transition to the balanced growth path follows from

$$\begin{aligned}\rho - \beta + \gamma\rho \geq g_Z(t) = \frac{\dot{Z}(t)}{Z(t)} &= \rho - \beta + \gamma\rho \\ &- \frac{(\rho - B)(\rho - B + \gamma\rho)}{\rho - B + \gamma\rho e^{-(\rho-B+\gamma\rho)t}} \geq B - \beta > 0.\end{aligned}$$

(ii) Let us show that the growth rate of the anticipated consumption stock is positive and above its long-run value, $B - \beta$, along the transition to the balanced growth path. Let us first note that if $z < 0$ then

$$\begin{aligned}F(1, a; a+1; z) &= a\int_0^1 t^{a-1}(1 - tz)^{-1}dt \\ &\geq a(1-z)^{-1}\int_0^1 t^{a-1}dt \\ &= (1-z)^{-1} > 0.\end{aligned} \tag{A.20}$$

Using the formula 15.2.3 in Abramowitz and Stegun (1964) on the differentiation of the hypergeometric function, we have that

$$\frac{d}{dz}F(1,a,a+1,z)=\frac{1}{z}\left[F(2,a;a+1;z)-F(1,a;a+1;z)\right]$$
$$=\frac{a}{z}\left[\frac{1}{1-z}-F(1,a;a+1;z)\right]>0, \tag{A.21}$$

if $z<0$, where we have used the inequality in (A.20) and that formula 15.5.11 in Olver et al. (2010) on the relation between contiguous functions entails that

$$F(2,a;a+1;z)=\frac{a}{1-z}-(a-1)F(1,a;a+1;z).$$

Now, log-differentiating (30), we have that

$$g_A(t)=\frac{\dot{A}}{A}=\frac{d}{dt}\ln A(t)=B-\beta+\frac{d}{dt}\ln f(t)$$
$$=B-\beta+\frac{1}{f(t)}\frac{d}{dt}f(t)\geq B-\beta>0,$$

where the inequality follows from (A.21) and the fact that

$$\frac{d}{dt}f(t)=\frac{d}{dz}F(1,a;a+1,z)\frac{dz}{dt}\geq 0, \tag{A.22}$$

with $a=\beta/(\rho-B+\gamma\rho)+1$ and $z=-\gamma\rho e^{-(\rho-B+\gamma\rho)t}/(\rho-B)$.

(iii) Let us now show that the growth rate of the capital stock is above its long-run value, $B-\beta$, along the transition to the balanced growth path. Let us first define

$$x(t)=c\left[1-e^{-(\rho-B+\gamma\rho)t}\right]f(t),$$
$$c=\frac{\beta\gamma\rho}{(\rho-B)(\rho-B+\gamma\rho+\beta)},$$

so that (28) is $K(t)=[1+x(t)]K_0e^{(B-\beta)t}$. Log-differentiating this expression we get

$$g_K(t)=\frac{\dot{K}}{K}=\frac{d}{dt}\ln K(t)$$
$$=B-\beta+\frac{1}{1+x(t)}\frac{d}{dt}x(t)\geq B-\beta>0,$$

where we have used the inequality in (A.22) and that

$$\frac{d}{dt}x(t)=c(\rho-B+\gamma\rho)te^{-(\rho-B+\gamma\rho)t}f(t)$$
$$+\,c\left[1-e^{-(\rho-B+\gamma\rho)t}\right]\frac{d}{dt}f(t)\geq 0.$$

A.6. Proof of Proposition 8

Let us define $\bar{U}=\int_0^\infty \ln Z(t)e^{-\beta t}dt$, so that

$$\bar{U}=\int_0^\infty\{\ln Z_0+\ln(\rho-B+\gamma\rho)+(B-\beta)t$$
$$-\ln[\rho-B+\gamma\rho e^{-(\rho-B+\gamma\rho)t}]\}\,e^{-\beta t}dt.$$

First, we immediately have that

$$\int_0^\infty[\ln Z_0+\ln(\rho-B+\gamma\rho)]\,e^{-\beta t}dt=\frac{\ln Z_0+\ln(\rho-B+\gamma\rho)}{\beta}, \tag{A.23}$$

and

$$\int_0^\infty(B-\beta)te^{-\beta t}dt=\frac{B-\beta}{\beta^2}. \tag{A.24}$$

Now, making the change of variable $u=e^{-(\rho-B+\gamma\rho)t}$ so that $du=-(\rho-B+\gamma\rho)udt$, and integrating by parts we get

$$-\int_0^\infty\ln[\rho-B+\gamma\rho e^{-(\rho-B+\gamma\rho)t}]e^{-\beta t}dt$$
$$=-\frac{1}{\rho-B+\gamma\rho}\int_0^1 u^{\frac{\beta}{\rho-B+\gamma\rho}-1}\ln(\rho-B+\gamma\rho u)du$$
$$=-\frac{1}{\beta}\left\{u^{\frac{\beta}{\rho-B+\gamma\rho}}\ln(\rho-B+\gamma\rho u)\right]_0^1$$
$$-\int_0^1 u^{\frac{\beta}{\rho-B+\gamma\rho}}\frac{\gamma\rho}{\rho-B+\gamma\rho u}du\Bigg\}$$
$$=-\frac{1}{\beta}\left\{\ln(\rho-B+\gamma\rho)-\frac{(\rho-B+\gamma\rho)\gamma\rho}{(\rho-B)(\rho-B+\gamma\rho+\beta)}f(0)\right\}, \tag{A.25}$$

where we have used that the integral representation of the hypergeometric function (see, e.g., formula 15.6.1, together with formula 15.1.2, in Olver et al., 2010) entails that

$$\int_0^1 u^{\frac{\beta}{\rho-B+\gamma\rho}}\frac{\gamma\rho}{\rho-B+\gamma\rho u}du$$
$$=\frac{\gamma\rho}{\rho-B}\int_0^1 u^{\frac{\beta}{\rho-B+\gamma\rho}}\left[1-\frac{\gamma\rho u}{B-\rho}\right]^{-1}du$$
$$=\frac{(\rho-B+\gamma\rho)\gamma\rho}{(\rho-B)(\rho-B+\gamma\rho+\beta)}f(0).$$

Adding Eqs. (A.23), (A.24) and (A.25) we get the result (37).

A.7. Proof of Proposition 9

(i) In the anticipation model, differentiating effective consumption, $Z^A(t)$, with respect to B we have that

$$\frac{\partial}{\partial B}Z^A(t)$$
$$=\frac{\gamma\rho\left(1-e^{-(\rho-B+\gamma\rho)t}\right)+\left[\gamma\rho(\rho-B)+(\rho-B)^2\right]t}{\left[\rho-B+\gamma\rho e^{-(\rho-B+\gamma\rho)t}\right]^2}$$
$$\times\ \beta K_0e^{(B-\beta)t}>0,$$

for all $t>0$, and

$$\frac{\partial}{\partial B}Z^A(0)=0.$$

Hence, an increase in the productivity parameter B shifts the time path of effective consumption upwards at any point in time (except at time zero), so instantaneous utility increases at each point in time after the positive productivity shock and, therefore, overall utility increases too.

In the habit-formation model, we get that

$$\frac{\partial}{\partial B}Z^H(t)=\left[\frac{\beta\gamma(\rho K_0+H_0)}{(B+\rho)^2}+Z_0^Ht\right]e^{(B-\beta)t}>0,$$

so an increase in productivity also shifts the path of effective consumption upwards at any point in time. Thus, instantaneous utility increases at any point time and, therefore, intertemporal utility increases as well.

(ii) In the anticipation model, differentiating effective consumption with respect to γ we get that

$$\frac{\partial}{\partial\gamma}Z^A(t)$$
$$=\frac{(\rho-B)\left[1-e^{-(\rho-B+\gamma\rho)t}\right]+(\rho-B+\gamma\rho)\gamma\rho te^{-(\rho-B+\gamma\rho)t}}{\left[\rho-B+\gamma\rho e^{-(\rho-B+\gamma\rho)t}\right]^2}$$
$$\times\ \rho\beta K_0e^{(B-\beta)t}>0,$$

for all $t>0$, and

$$\frac{\partial}{\partial\gamma}Z^A(0)=0.$$

Hence, an increase in the weight of anticipated future consumption in effective consumption, γ, shifts the entire time path of effective consumption upwards at any point in time (except at time zero). As a result, instantaneous utility increases at each point in time and, therefore, intertemporal utility increases too.

In the habit-formation model, we get that

$$\frac{\partial}{\partial\gamma}Z^H(t) = -\frac{\beta(H_0+\rho K_0)}{B+\rho}e^{(B-\beta)t} < 0,$$

so an increase in the weight of habits in effective consumption shifts the path of effective consumption downwards at any point in time. Thus, instantaneous utility falls at any point time and, therefore, intertemporal utility decreases.

(iii) Finally, let us consider the effect of the speed of adjustment of anticipated future consumption, ρ, on the intertemporal utility. In the anticipation model, we have that

$$\frac{\partial}{\partial\rho}Z^A(t) = \frac{(\rho-B+\gamma\rho)(1+\gamma)\rho t e^{-(\rho-B+\gamma\rho)t} - B\left(1-e^{-(\rho-B+\gamma\rho)t}\right)}{\left[\rho-B+\gamma\rho e^{-(\rho-B+\gamma\rho)t}\right]^2} \times \gamma\beta K_0 e^{(B-\beta)t},$$

with

$$\frac{\partial}{\partial\rho}Z^A(0) = 0.$$

Unlike the previous cases, the former expression cannot be signed unambiguously and depends on time t. So, we have to determine the effect of a change in ρ directly on the intertemporal utility by differentiating (37). To sign the resulting expression we will need some preliminary results.

It can be easily observed from the integral representation of the Gaussian Hypergeometric function (e.g., Formula 15.6.1 in Olver et al., 2010) that $F(b,a;a+1;z) > 0$ if $z < 0$.[6] We will use the Formula 15.5.1 in Olver et al. (2010) on the differentiation of the Gaussian Hypergeometric function which entails that

$$\frac{\partial}{\partial z}F(1,a;a+1;z) = \frac{a}{a+1}F(2,a+1;a+2;z) > 0,$$

if $z < 0$. We will also use the Formula 1 in Section 1.29.2 in Brychkov (2008, p. 85) on the derivatives of $F \equiv {}_2F_1$ with respect to parameters, which entails that

$$\frac{\partial}{\partial a}F(1,a;a+1;z) = \frac{z}{(a+1)^2}G(a+1,a+1,2;a+2,a+2;z),$$

where G is the Generalized Hypergeometric function

$$\begin{aligned} G(a+1,a+1,2;a+2,a+2;z) &= {}_3F_2(a+1,a+1,2;a+2,a+2;z) \\ &= \frac{\Gamma(a+2)}{\Gamma(2)\Gamma(a)}\int_0^1 t(1-t)^{a-1}{}_2F_1(a+1,a+1;a+2;tz)dt \\ &= a(a+1)\int_0^1 t(1-t)^{a-1}F(a+1,a+1;a+2;tz)dt > 0. \end{aligned}$$

Here, we have used the integral representation of the Generalized Hypergeometric function in Formula 2.2.2 in Andrews et al. (1999, p. 67), and the sign follows from $F(a+1,a+1;a+2;tz) > 0$ as $tz < 0$. Thus, we get that

$$\frac{\partial}{\partial a}F(1,a;a+1;z) < 0,$$

if $z < 0$. Using the previous results, we have that

$$\frac{\partial\bar{U}^A}{\partial\rho} = \underbrace{-\frac{\gamma\left[(B-\beta)(\rho-B+\gamma\rho)^2+\beta\gamma(1+\gamma)\rho^2\right]}{\beta(\rho-B)^2(\rho-B+\gamma\rho+\beta)^2}f(0)}_{<0} \overbrace{-\left[\underbrace{\frac{\partial}{\partial a}F(1,a;a+1;z)}_{<0}\underbrace{\frac{\partial a}{\partial\rho}}_{<0} + \underbrace{\frac{\partial}{\partial z}F(1,a;a+1;z)}_{>0}\underbrace{\frac{\partial z}{\partial\rho}}_{>0}\right]}^{>0} \times \underbrace{\frac{(\rho-B+\gamma\rho)\gamma\rho}{\beta(\rho-B)(\rho-B+\gamma\rho+\beta)}}_{>0} < 0,$$

where $a = 1+\beta/(\rho-B+\gamma\rho)$ and $z = -\gamma\rho/(\rho-B)$. Hence, an increase in the speed of adjustment of anticipated consumption stock causes a decrease in intertemporal utility.

In the habit-formation model, we get that

$$\frac{\partial}{\partial\rho}Z^H(t) = -\frac{BK_0-H_0}{(B+\rho)^2}\beta\gamma e^{(B-\beta)t}\begin{cases} > 0, & \text{if } BK_0 < H_0, \\ = 0, & \text{if } BK_0 = H_0, \\ < 0, & \text{if } BK_0 > H_0. \end{cases}$$

Thus, the effect of an increase in the speed of adjustment of habits to current consumption cannot be unambiguously signed and depends on how initial output compares to the initial habits stock. The same happens with intertemporal utility as

$$\frac{\partial}{\partial\rho}\bar{U}^H = -\frac{\gamma(BK_0-H_0)}{(B+\rho)^2Z_0^H}\begin{cases} > 0, & \text{if } BK_0 < H_0, \\ = 0, & \text{if } BK_0 = H_0, \\ < 0, & \text{if } BK_0 > H_0. \end{cases}$$

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[6] In this paper, a and b are real numbers.

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