Applied Mathematics and Computation 413 (2022) 126647

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Total value adjustment for European options in a multi-currency setting

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ARTICLE INFO

Article history:
Received 8 January 2021
Revised 26 July 2021
Accepted 31 August 2021
Available online 22 September 2021

Keywords:
European options
Multi-currency
Counterparty risk
Total value adjustment
Monte Carlo method

ABSTRACT

In this article we mainly extend to a multi-currency setting some previous works in the literature concerning total value adjustments in a single currency framework. The motivation comes from the fact that financial institutions operate in global markets, so that the financial derivatives in their portfolios involve different currencies. More precisely, in this multi-currency setting we pose the PDE formulations for pricing the total adjustment and the financial derivative with counterparty risk. Moreover, we also formulate the problem in terms of expectations, which allows the use of suitable Monte Carlo techniques that overcome the curse of dimensionality associated to the numerical solution of PDE formulation, when a high number of stochastic factors are involved. Finally, we present some examples to illustrate the performance of the formulations and the proposed numerical techniques.

1. Introduction

After the financial crisis started in 2007, it became clear that any pricing framework should take into account the possibility of default of any counterparty involved in the trade [11,16], as well as aspects related to collateral posting, liquidity risk or funding costs [15,25]. Therefore, different valuation adjustments due to these factors must be considered when pricing a derivative. The set of these adjustments is globally referred to as Total Value Adjustment or XVA, where 'X' stands for the different letters that appear in the value adjustments associated to credit (CVA), debit (DVA), funding (FVA), capital (KVA) or collateral (CollVA), among others (see [7], for example).

Most of the literature has addressed the modelling and computation of the different adjustments or the total value adjustment for a single currency setting. In this framework, three main methodologies have been developed. A first approach, following the seminal papers by Piterbarg [23] and Burgard and Kjaer [8] that obtain PDE formulations by hedging arguments on suitable portfolios and the application of Itô lemma for jump-diffusion processes. This approach in terms of PDEs formulation has been followed in [12], where the problem is also equivalently written in terms of expectations. Moreover, it has been also addressed in [2] and [1], where PDE models with one and two stochastic factors have been mathematically analyzed and numerically solved for pricing European options with one and two stochastic factors. A second approach follows the initial ideas in [6] to obtain the CVA by means of formulations based on expectations, next extended to the

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https://doi.org/10.1016/j.amc.2021.126647

collateralized, close-out and funding costs in [20]. Also this approach has been addressed in [3] for American options and in [5] for Levy dynamics. A third approach is based on backward stochastic differential equations and it has been introduced in [9] and [10].

Recently, attention has been given to the extension of valuation adjustments from the single currency to the multi-currency setting [13]. Indeed, nowadays financial institutions may operate in different currencies, for example making investments on derivatives with underlying assets denominated in domestic currencies, and funding or posting collateral in foreign currencies. The three previously indicated methodologies that have been developed can be extended to the multi-currency setting. In this work we will focus on the formulations based on PDEs and expectations.

More precisely, in the present paper we address the European options pricing problem in a multi-currency setting when taking into account the valuation adjustments associated to counterparty risk. For this purpose, stochastic intensities of default are assumed and underlying assets denominated in different currencies are involved. Our approach is based on the same framework and assumptions as in [12], although extended to a multi-currency environment and with the additional hypothesis of a zero default intensity for the hedger.

Therefore, we are in a multidimensional setting, where the involved stochastic factors are the underlying assets denominated in different currencies and the credit spread of the investor. As in [12] for the case of a single currency, we assume that:

- All costs associated to hedging must be incorporated in the price of the financial derivative.
- The price of a derivative must only reflect the hedging costs transmitted by the hedger, in a setting where most of uncollateralized transactions involve the presence of an investor (risk taker) and a hedger (risk hedger).
- Only the variations in the derivative price while it is alive will be hedged by the hedger.
- For fully collateralized derivatives, CVA or FVA do not apply.

We also make the following market assumptions:

- A liquid credit default swap (CDS) curve for the investor exists.
- A liquid curve of bonds issued by the hedger exists.
- Continuous hedging is feasible and unlimited liquidity is available. Bid-offer spreads and trading costs are not considered.
- The risk of recovery is not taken into account because recovery rates are either assumed to be deterministic or there are recovery locks available.
- The risk associated to foreign exchange (FX) is not considered because the exchange rates are assumed to be deterministic.

Finally, we take into consideration the following assumptions on the model for the multi-currency setting:

- The investor can default, but the hedger is default-free.
- Prices of the involved underlying assets are modelled by correlated diffusion processes.
- The events of investor default do not affect the evolution of the prices of the involved underlying assets.
- The stochastic credit spread of the investor is modelled as a diffusion process, which is correlated with the processes followed by the prices of the underlying assets.

By taking into account all previous assumptions and following [13], we first infer partial differential equations (PDEs) formulations of the XVA pricing problem. For this purpose, we employ hedging and no arbitrage arguments jointly with a choice for the mark-to-market value of the derivative at default. This choice leads either to a nonlinear problem, if the mark-to-market value is equal to the price of the derivative when counterparty risk is not taken into account (risk-free derivative), or to a linear problem when the counterparty risk is considered (risky derivative).

In a second step, we deduce the corresponding formulations of the pricing problem in terms of expectations with the goal of applying a Monte Carlo method for computing the price of total value adjustment. The choice of this numerical method for the approximation of XVA is due to the fact that the Monte Carlo method is not affected by the so-called *curse of dimensionality*, that arises when using other numerical approaches to solve multidimensional PDE problems.

The article is structured as follows. In Section 2 we obtain the mathematical model for XVA based either on linear or nonlinear PDEs. In Section 3 we write the problem in terms of expectations with the purpose of applying the Monte Carlo method. In Section 4 we present and analyse the numerical results related to some examples for different choices of the derivative payoff. More precisely, we consider a sum of call options, a spread option and an exchange option. In Section 5 we point out several main conclusions.

2. Statement of partial differential equations models

In this section, following [12,13] the value of a derivative is modelled by taking into account the valuation adjustments that have to be considered in case of a possible default of the counterparties involved in the deal.

We consider a trade between a non-defaultable hedger and a defaultable investor in a multi-currency framework, where a domestic currency D and foreign currencies $C_0, \dots, C_N$ are involved. For $j = 0, \dots, N$, let X_t^{D,C_j} be the FX rate between currencies D and C_j at time t, namely the domestic price at time t of one unit of the foreign currency C_j.

We denote by Q^D the risk neutral probability measure of the domestic market. As stated in [7], the dynamics of X_t^{D,C_j} under Q^D is described by

$$dX_t^{D,C_j} = (r^D - r^{C_j})X_t^{D,C_j}dt + \sigma^{Xj}X_t^{D,C_j}dW^{Xj}, \quad j = 0,\dots,N,$$

where r^D and r^{C_j} are the risk-free rates in the domestic market and in the $j-$foreign market, respectively. Moreover σ^{Xj} is the volatility of X_t^{D,C_j} and W^{Xj} is a Q^D-Brownian motion. Nevertheless, throughout this article we consider $\sigma^{Xj} = 0$ in order to have deterministic FX rates. Furthermore, in the numerical examples we consider constant values for X_t^{D,C_j}, so that subindex t will be removed in that part of the article. The consideration of stochastic rates will be addressed in a future work.

We denote by $S_t = (S_t^1,\dots,S_t^N)$ the vector of the prices of the underlying assets S^i at time t, each one of them being denominated in its corresponding currency C_i, and by h_t the investor's credit spread at time t. We assume that under the real world measure P the evolution of the prices of the underlying assets in each currency and of the investor's credit spread are governed by the following SDEs:

$$dS_t^i = \mu^{S^i}S_t^i\,dt + \sigma^{S^i}S_t^i\,dW_t^{S^i,P}, \qquad \text{for } i = 1,\dots,N, \tag{1}$$

$$dh_t = \mu^h\,dt + \sigma^h\,dW_t^{h,P}, \tag{2}$$

where μ^{S^i} and μ^h are the real world drifts of the processes S_t^i and h_t, respectively. Moreover, σ^{S^i} and σ^h are their respective volatility functions, while $W^{S^i,P}$ and $W^{h,P}$ are Brownian motions under the real world measure P. Moreover, we assume that the assets prices and spread processes in (1) and (2) are correlated. Thus, we consider the $(N+1)\times(N+1)$ correlation matrix given by

$$\mathrm{corr}(S^1,\dots,S^N,h) = \begin{pmatrix} 1 & \rho^{S^1S^2} & \cdots & \rho^{S^1S^N} & \rho^{S^1h} \\ \rho^{S^1S^2} & 1 & \cdots & \rho^{S^2S^N} & \rho^{S^2h} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{S^1S^N} & \rho^{S^2S^N} & \cdots & 1 & \rho^{S^Nh} \\ \rho^{S^1h} & \rho^{S^2h} & \cdots & \rho^{S^Nh} & 1 \end{pmatrix}, \tag{3}$$

where the $N\times N$ submatrix contains the correlations between assets ($\rho^{S^iS^j}$) and the last row (column) contains the correlations between each asset and the spread (ρ^{S^ih}).

Although in the numerical examples we consider constant values for drifts, volatilities and correlations, time dependent functions can be assumed in all the developments.

Next, we denote by J_t^P the investor's default state at time t, i.e., $J_t^P = 1$ in case of default before or at time t, otherwise $J_t^P = 0$. By changing the probability measure from P to Q^D in (1) and (2), then the corresponding drifts of S^i and h are respectively given by $(r^i - q^i)S^i$ and $\mu^h - M^h\sigma^h$, where r^i is the risk-free rate in currency C_i, q^i is the dividend paid by S^i and M^h is the investor's market price of credit risk. Under the measure Q^D, the investor's intensity of default λ and the investor's credit spread h satisfy the relation

$$\lambda = \frac{h}{1-R},$$

where R is the constant investor's recovery rate. By considering the relationship between the drift of the spread $\mu^h - M^h\sigma^h$ and the investor's intensity of default λ, namely $\mu^h - M^h\sigma^h = -\kappa\lambda$, we get:

$$\mu^h - M^h\sigma^h = -\frac{\kappa}{1-R}h\,. \tag{4}$$

From the investor's point of view, the derivative value in domestic currency D at time t is given by $V_t^D = V^D(t,S_t,h_t,J_t^P)$. The price of the corresponding risk-free derivative in currency D, i.e. the same derivative in currency D, which is traded between two non-defaultable counterparties, is denoted by $V_t^{RF,D} = V^{RF,D}(t,S_t)$.

In case that the investor defaults, the expression of the risky derivative price is:

$$V^D(t,S_t,h_t,1) = RM^+(t,S_t,h_t) + M^-(t,S_t,h_t)\,, \tag{5}$$

with $M(t,S_t,h_t)$ representing the mark-to-market derivative price. Moreover, we have used the notation $x^+ = \max(x,0)$ and $x^- = \min(x,0)$. Next, by using (5), we define the variation of V^D at default as:

$$\Delta V^D = RM^+ + M^- - V^D\,. \tag{6}$$

In order to price the derivative, we consider a self-financing portfolio Π that hedges all the risk factors, which are:

- the market risk due to changes in $S^1, S^2,\dots,S^N$;
- the investor's spread risk due to changes in h;

- the investor's default risk.

More precisely, in order to hedge the exposure to the first risk factor, the hedger will have to trade in fully collateralized derivatives on the same underlying assets. We denote by H^i (for $i = 1, \dots, N$) the net present value of the derivative in currency C_i from the hedger's point of view, and we define $H^{i,D} = H^i X^{D/C_i}$ its net present value in currency D.

In order to hedge the spread risk and the investor's default risk, the hedger will have to trade on two credit default swaps with different maturities written on the investor: a short term credit default swap, $CDS^D(t, t+dt)$, and a long term credit default swap, $CDS^D(t, T)$. More precisely, $CDS^D(t, t+dt)$ is an overnight credit default swap with unit notional under which the protection buyer pays a premium at time $t+dt$ equal to $h_t dt$. If the investor defaults between t and $t+dt$, the protection buyer receives $(1-R)$ at time $t+dt$. We assume that $h_t dt$ is such that $CDS^D(t, t+dt) = 0$. The second credit default swap, $CDS^D(t, T)$, is a cash collateralized credit default swap maturing on T and is usually not null.

Next, we assume the existence of a collateral account C^{C_0}, which is denominated in currency C_0 and composed of a portfolio of bonds R^{C_0} and cash M^{C_0}, that is

$$C^{C_0} = R^{C_0} + M^{C_0} . \tag{7}$$

According to the self-financing condition of a replicating strategy, the hedger matches the spread duration of the uncollateralized part of the derivative by trading on short term bonds maturing on $t+dt$, $B^D(t, t+dt)$, so that the net buyback at time t is equal to $V_t^D - C_t^{C_0} X_t^{D,C_0}$, namely

$$\Omega_t B^D(t, t+dt) = V_t^D - C_t^{C_0} X_t^{D,C_0} , \tag{8}$$

where Ω_t is the number of units of $B^D(t, t+dt)$ at time t.

Hence, we consider a replicating portfolio Π that is an extension to the multi-currency framework of the portfolio in [1] and such that:

- α_t^i is the weight of the fully collateralized derivative H_t^i, for $i = 1, \dots, N$, in the portfolio composition at time t;
- γ_t and ϵ_t are the weights of the long term CDS and short term CDS, respectively, in the portfolio composition at time t;
- Ω_t represents the weight of the short term bond in the portfolio composition at time t;
- β_t^D denotes the cash in the bank account of the portfolio at time t.

Thus, the portfolio at time t is given by:

$$\Pi_t = \sum_{i=1}^{N} \alpha_t^i H_t^{i,D} + \gamma_t CDS^D(t, T) + \epsilon_t CDS^D(t, t+dt) + \Omega_t B^D(t, t+dt) + \beta_t^D . \tag{9}$$

The composition of the bank account β^D is given by

$$\beta_t^D = -\sum_{i=1}^{N} \alpha_t^i H_t^{i,D} - \gamma_t CDS^D(t, T) + C_t^{C_0} X_t^{D,C_0} ,$$

so that its variation in the time interval $[t, t+dt]$ is

$$d\beta_t^D = -\left[\sum_{i=1}^{N} \alpha_t^i \left(c^D + b^{D,C_i}\right) H_t^{i,D} + \gamma_t c^D CDS^D(t, T) \right] dt + \left[\left(r^R + b^{D,C_0}\right) R_t^{C_0} + \left(c^D + b^{D,C_0}\right) M_t^{C_0} \right] X_t^{D,C_0} dt ,$$

where r^R is the instantaneous repo rate associated to the bond R^{C_0}, b^{D,C_0} is the cross-currency basis, and c^D is the OIS rate in the domestic currency D.

From no arbitrage arguments we have $V^D(t, S_t, h_t, J_t^P) = \Pi_t$, that jointly with the self-financing condition leads to

$$dV_t^D = d\Pi_t = \sum_{i=1}^{N} \alpha_t^i dH_t^{i,D} + \gamma_t dCDS^D(t, T) + \epsilon_t dCDS^D(t, t+dt) + \Omega_t dB^D(t, t+dt) + d\beta_t^D \tag{10}$$

As $V_t^D = V^D(t, S_t, h_t, J_t)$ depends on diffusion and jump processes, we apply Itô's formula for jump-diffusion processes [22] to obtain that the variation of V^D in the time interval $[t, t+dt]$ is given by:

$$dV_t^D = \frac{\partial V^D}{\partial t} dt + \sum_{i=1}^{N} \frac{\partial V^D}{\partial S^i} dS_t^i + \frac{\partial V^D}{\partial h} dh_t + \Delta V_t^D dJ_t^P + \left[\frac{1}{2} \sum_{i,k=1}^{N} \rho^{S^i S^k} \sigma^{S^i} \sigma^{S^k} S_t^i S_t^k \frac{\partial^2 V^D}{\partial S^i \partial S^k} + \frac{1}{2} \left(\sigma^h\right)^2 \frac{\partial^2 V^D}{\partial h} + \sum_{i=1}^{N} \rho^{S^i h} \sigma^{S^i} \sigma^h S_t^i \frac{\partial^2 V^D}{\partial S^i \partial h} \right] dt,$$

where $\Delta V_t^D = V^D(t, S_t, 1) - V^D(t, S_t, 0)$ represents the jump of V_t^D in case of default at time t, which is given by (6).

The dynamics of the short term credit default swap, $CDS^D(t, t+dt)$, and of the overnight bond, $B^D(t, t+dt)$, are respectively given by:

$$dCDS^D(t, t+dt) = h_t \, dt - (1-R) \, dJ_t^P ,$$
$$dB^D(t, t+dt) = f^{H,D} B^D(t, t+dt) \, dt ,$$

where $f^{H,D}$ is the hedger's domestic funding rate.

From the self-financing condition on our strategy, stated in (8), we obtain

$$\Omega_t = \frac{V_t^D - C_t^D}{B^D(t, t+dt)}.$$

Thus, the change in Π_t from t to $t+dt$ is given by:

$$\begin{aligned} d\Pi_t &= \sum_{i=1}^N \alpha_t^i dH_t^{i,D} + \gamma_t dCDS^D(t,T) + \epsilon_t dCDS^D(t,t+dt) + \tfrac{V_t^D - C_t^{C_0} X_t^{D,C_0}}{B^D(t,t+dt)} dB^D(t,t+dt) + d\beta_t^D \\ &= \sum_{i=1}^N \alpha_t^i X_t^{D,C_i}\left(\tfrac{\partial H^i}{\partial t}dt + \tfrac{\partial H^i}{\partial S^i}dS_t^i + \tfrac{1}{2}\left(\sigma^{S^i}S_t^i\right)^2 \tfrac{\partial^2 H^i}{\partial (S^i)^2}dt\right) + \gamma_t\left[\tfrac{\partial CDS^D(t,T)}{\partial t}dt + \tfrac{\partial CDS^D(t,T)}{\partial h}dh_t + \tfrac{1}{2}\left(\sigma^h\right)^2 \tfrac{\partial^2 CDS^D(t,T)}{\partial h^2}dt\right] \\ &\quad + \gamma_t \Delta CDS^D(t,T)dJ_t^P + \epsilon_t\left[h_t dt - (1-R)dJ_t^P\right] + \left(V^D - C_t^{C_0}X_t^{D,C_0}\right)f^{H,D}dt - \sum_{i=1}^N \alpha_t^i\left(c^D + b^{D,C_i}\right)H_t^i \\ &\quad - c^D CDS^D(t,T)dt + \left[\left(r^R + b^{D,C_0}\right)R^{C_0} + \left(c^D + b^{D,C_0}\right)M^{C_0}\right]X_t^{D,C_0}dt. \end{aligned}$$

In order to hedge the risk of the portfolio Π, we assume:

$$\begin{aligned} \alpha_t^i &= \tfrac{\partial V^D/\partial S^i}{\partial H^{i,D}/\partial S^i}, \quad \text{for } i = 1,\cdots,N, \\ \gamma_t &= \tfrac{\partial V^D/\partial h}{\partial CDS^D(t,T)/\partial h}, \\ \epsilon_t &= \tfrac{1}{1-R}\left(\gamma_t \Delta CDS^D(t,T) - \Delta V^D\right). \end{aligned}$$

Next, we take into account the Black-Scholes equations that model H^i and $CDS^D(t,T)$, namely

$$\begin{aligned} &\tfrac{\partial H^i}{\partial t} + \tfrac{1}{2}\left(\sigma^{S^i}S^i\right)^2 \tfrac{\partial^2 H^i}{\partial (S^i)^2} + \left(r^i - q^i\right)S^i \tfrac{\partial H^i}{\partial S^i} = \left(c^D + b^{D,C_i}\right)H^i, \\ &\tfrac{\partial CDS^D(t,T)}{\partial t} + \tfrac{1}{2}\left(\sigma^h\right)^2 \tfrac{\partial CDS^D(t,T)}{\partial h^2} + \left(\mu^h - M^h\sigma^h\right)\tfrac{\partial CDS^D(t,T)}{\partial h} + \tfrac{h}{1-R}\Delta CDS^D(t,T) = c^D CDS^D(t,T). \end{aligned}$$

Thus, (10) turns into:

$$\begin{aligned} &\tfrac{\partial V^D}{\partial t} + \sum_{i=1}^N \tfrac{\partial V^D}{\partial S^i}\left(r^i - q^i\right)S^i + \tfrac{\partial V^D}{\partial h}\left(\mu^h - M^h\sigma^h\right) + \tfrac{1}{2}\sum_{i,k=1}^N \rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k \tfrac{\partial^2 V^D}{\partial S^i \partial S^k} + \tfrac{1}{2}\left(\sigma^h\right)^2 \tfrac{\partial^2 V^D}{\partial h^2} + \sum_{i=1}^N \rho^{S^ih}\sigma^{S^i}\sigma^h S^i \tfrac{\partial^2 V^D}{\partial S^i \partial h} \\ &\quad = -\tfrac{h}{1-R}\Delta V^D + f^{H,D}V^D + \left[\left(r^R + b^{D,C_0} - f^{H,D}\right)R^{C_0} + \left(c^D + b^{D,C_0} - f^{H,D}\right)M^{C_0}\right]X^{D,C_0}. \end{aligned}$$

Therefore, we obtain the following pricing PDE:

$$\frac{\partial V^D}{\partial t} + \mathcal{L}_{Sh}V^D - f^{H,D}V^D + \frac{h}{1-R}\Delta V^D = \left[\left(r^R + b^{D,C_0} - f^{H,D}\right)R^{C_0} + \left(c^D + b^{D,C_0} - f^{H,D}\right)M^{C_0}\right]X^{D,C_0}, \tag{11}$$

where the second order differential operator $\mathcal{L}_{Sh}$ is given by

$$\mathcal{L}_{Sh} = \frac{1}{2}\sum_{i,k=1}^N \rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k \frac{\partial^2}{\partial S^i \partial S^k} + \frac{1}{2}\left(\sigma^h\right)^2 \frac{\partial^2}{\partial h^2} + \sum_{i=1}^N \rho^{S^ih}\sigma^{S^i}\sigma^h S^i \frac{\partial^2}{\partial S^i \partial h} + \sum_{i=1}^N \left(r^i - q^i\right)S^i \frac{\partial}{\partial S^i} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial}{\partial h}. \tag{12}$$

Finally, we use (4) to write the differential operator (12) as follows:

$$\mathcal{L}_{Sh} = \frac{1}{2}\sum_{i,k=1}^N \rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k \frac{\partial^2}{\partial S^i \partial S^k} + \frac{1}{2}\left(\sigma^h\right)^2 \frac{\partial^2}{\partial h^2} + \sum_{i=1}^N \rho^{S^ih}\sigma^{S^i}\sigma^h S^i \frac{\partial^2}{\partial S^i \partial h} + \sum_{i=1}^N \left(r^i - q^i\right)S^i \frac{\partial}{\partial S^i} - \frac{\kappa h}{1-R}\frac{\partial}{\partial h}. \tag{13}$$

In the pricing Eq. (11) the variation of V^D upon default is involved and is given by (see (6)):

$$\Delta V_t^D = RM^+ + M^- - V^D.$$

Following the seminal article [8], in the literature two possible values for the mark-to-market at default, M, can be chosen: either equal to the risky value or to the risk-free value of the derivative. Thus, we derive the following PDEs for both cases:

- if $M = V^D$, (11) turns into

$$\frac{\partial V^D}{\partial t} + \mathcal{L}_{Sh}V^D - fV^D = (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0} + h(V^D)^+, \tag{14}$$

- if $M = V^{RF,D}$, (11) becomes

$$\frac{\partial V^D}{\partial t} + \mathcal{L}_{Sh}V^D - \left(\frac{h}{1-R} + f\right)V^D = \left(\bar{r}R^{C_0} + \overline{m}M^{C_0}\right)X^{D,C_0} + h\left(V^{RF,D}\right)^+ - \frac{h}{1-R}V^{RF,D}, \tag{15}$$

where $\bar{r} = r^R + b^{D,C_0} - f^{H,D}$, $\bar{m} = c^D + b^{D,C_0} - f^{H,D}$ and $f = f^{H,D}$. Note that $\bar{r}$ represents the difference between the specific rates, $r^R + b^{D,C_0}$, fixed in the Credit Support Annex (CSA) to remunerate the portfolio of bonds in the collateral account, and the hedger's domestic funding rate $f^{H,D}$ associated to the overnight bond B^D, while $\bar{m}$ represents the sum of rates, $c^D + b^{D,C_0}$, also fixed in the CSA to remunerate the cash part of the collateral account, minus the hedger's domestic funding rate $f^{H,D}$. Thus, both quantities represent remuneration minus funding rates associated to the collateral account.

Next, in order to pose the PDEs formulation for the XVA price, the risky derivative value can be split up into $V^D = V^{RF,D} + U$, where $V^{RF,D}$ and U represent the risk-free derivative price and the XVA price, respectively.

Note that the risk-free derivative price $V^{RF,D}$ satisfies the classical multidimensional Black-Scholes equation:

$$\frac{\partial V^{RF,D}}{\partial t} + \mathcal{L}_S V^{RF,D} - f V^{RF,D} = 0 \,,$$

where

$$\mathcal{L}_S = \frac{1}{2}\sum_{i,k=1}^{N} \rho^{S^i S^k} \sigma^{S^i} \sigma^{S^k} S^i S^k \frac{\partial^2}{\partial S^i \partial S^k} + \sum_{i=1}^{N} (r^i - q^i) S^i \frac{\partial}{\partial S^i} \,.$$

Moreover, since the final conditions for V^D and for $V^{RF,D}$ are given by

$$V^{RF,D}(T,S) = V^D(T,S,h) = G(S) \,,$$

the final condition for U is given by $U(T,S,h) = 0$.

Therefore, depending on the choice of the mark-to-market value at default we obtain two possible PDE problems.

- Nonlinear final value problem (case $M = V^D$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{Sh} U - fU = h\left(V^{RF,D} + U\right)^+ + \left(\bar{r} R^{C_0} + \bar{m} M^{C_0}\right) X^{D,C_0} \,, \\ U(T,S,h) = 0 \,. \end{cases} \quad (16)$$

- Linear final value problem (case $M = V^{RF,D}$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{Sh} U - \left(\frac{h}{1-R} + f\right) U = h\left(V^{RF,D}\right)^+ + \left(\bar{r} R^{C_0} + \bar{m} M^{C_0}\right) X^{D,C_0} \,, \\ U(T,S,h) = 0 \,. \end{cases} \quad (17)$$

In both cases, $(t,S,h) \in [0,T) \times (0,+\infty)^N \times (0,+\infty)$.

Note that the spatial dimension of problems (16) and (17) depends on the number of currencies, so that the PDE easily becomes high dimensional in space and the numerical solution requires specific discretization techniques to overcome the curse of dimensionality (see [19] or [18], as examples using sparse grids with recombination technique for solving high-dimension PDEs for derivatives pricing). Therefore, alternative formulations in terms of expectations are obtained in next section, so that appropriate numerical Monte Carlo techniques could be efficiently applied.

3. Formulation in terms of expectations and Monte Carlo method

In the previous section, two multidimensional problems for pricing the total valuation adjustment have been posed. Numerical approaches commonly used to solve PDE problems are affected by the so-called *curse of dimensionality* when increasing the dimension of the problem. Therefore, a first approach to the computation of the total value adjustment could be made by using a Monte Carlo method, which is suitable to approximate expectations in a multidimensional framework, thus allowing to manage problems that involve more than two stochastic factors [14].

First, in order to compute the values of U by using the Monte Carlo method in the nonlinear model (16), we apply the nonlinear Feynman-Kac theorem, that relates the solution of nonlinear PDEs with the solution of BSDEs. The statement of the nonlinear Feynman-Kac theorem dates back from the seminal paper [21]. As the nonlinear term in (16) appears in the unknown U and not in the first order derivatives, Theorem 1.1 in the recent work by Beck et al. [4] can be applied to formulate the nonlinear problem (16) in terms of a nonlinear integral equation. Note that in [4] a large number of previous references on the nonlinear Feynman-Kac theorem are indicated, probably the here treated nonlinear PDE could be framed in many of them. Secondly, the linear Feynman-Kac theorem (see [22], for example) can be applied to the linear problem (17).

- If $M = V^D$, the total value adjustment at time t satisfies the equation

$$U(t,S,h) = E_t^{Q^D}\left[-\int_t^T e^{-f(u-t)} \left(h_u \left(V^{RF,D}(u,S_u) + U(u,S_u,h_u) \right)^+ + \left(\bar{r} R_u^{C_0} + \bar{m} M_u^{C_0} \right) X_u^{D,C_0} \right) du \,\middle|\, S_t = S, h_t = h \right]. \quad (18)$$

Note that (18) is an integral equation as the unknown U appears also at the right hand side in the integral. We are interested in the XVA at the current time $t = 0$, when the derivative is priced, that is to say

$$U(0,S,h) = E_0^{Q^D}\left[-\int_0^T e^{-fu} \left(h_u \left(V^{RF,D}(u,S_u) + U(u,S_u,h_u) \right)^+ + \left(\bar{r} R_u^{C_0} + \bar{m} M_u^{C_0} \right) X_u^{D,C_0} \right) du \,\middle|\, S_0 = S, h_0 = h \right]. \quad (19)$$

- If $M = V^{RF,D}$, the total value adjustment at time t is given by

$$U(t,S,h) = E_t^{Q^D}\left[-\int_t^T e^{-\int_t^u(\frac{h_r}{1-R}+f)dr}\left(h_u(V^{RF,D}(u,S_u))^+ + \left(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\right)X_u^{D,C_0}\right)du \,\middle|\, S_t = S, h_t = h\right]. \tag{20}$$

Note that (20) gives an explicit formula for XVA. In particular, at time $t = 0$ we have

$$U(0,S,h) = E_0^{Q^D}\left[-\int_0^T e^{-\int_0^u(\frac{h_r}{1-R}+f)dr}\left(h_u(V^{RF,D}(u,S_u))^+ + \left(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\right)X_u^{D,C_0}\right)du \,\middle|\, S_0 = S, h_0 = h\right]. \tag{21}$$

We assume constant FX rates. We need a time discretization in order to discretize the dynamics of S^i ($i = 1,\ldots,N$), h, R^{C_0} and M^{C_0} by using Euler-Maruyama scheme [17]. Thus, we choose a uniform mesh with nodes $0 = t_0 < t_1 < \ldots < t_{N_T} = T$, and we denote by $\Delta t = t_m - t_{m-1}$ the distance between two consecutive nodes. Hence, denoting $S_m^i = S^i(t_m)$, $h_m = h(t_m)$, $R_m^{C_0} = R^{C_0}(t_m)$ and $M_m^{C_0} = M^{C_0}(t_m)$, we have:

$$\begin{aligned}
\Delta S_m^i &= (r^i - q^i)S_m^i\,\Delta t + \sigma^{S^i} S_m^i\,\Delta W_m^{S^i},\\
\Delta h_m &= -\tfrac{\kappa}{1-R} h_m\,\Delta t + \sigma^h\,\Delta W_m^h,\\
\Delta R_m^{C_0} &= \left(r^R + b^{D,C_0}\right)R_m^{C_0}\,\Delta t,\\
\Delta M_m^{C_0} &= \left(c^D + b^{D,C_0}\right)M_m^{C_0}\,\Delta t,
\end{aligned}$$

where $\Delta W_m^{S^i} = W_m^{S^i} - W_{m-1}^{S^i}$, for $i = 1,\ldots,N$, and $\Delta W_m^h = W_m^h - W_{m-1}^h$ are correlated Brownian increments, according to correlation matrix (3). Thus, these correlated Brownian motions can be built by Cholesky factorization.

When $M = V^{RF,D}$, (21) gives an explicit expression for the XVA price that is computed with the help of numerical formulae for the approximation of the integral that use the time discretization stated above.

When $M = V^D$, a fixed point iteration is implemented to compute the XVA price, given by the integral Eq. (19). More precisely, we start from $U^0 = 0$ and recursively compute:

$$\begin{aligned}
U^{\ell+1}(0,S,h) = E_0^{Q^D}\Bigg[-\int_0^T e^{-fu}\Big(h_u\Big(V^{RF,D}(u,S_u) + U^\ell(u,S_u,h_u)\Big)^+\\
+\left(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\right)X_u^{D,C_0}\Big)du \,\Big|\, S_0 = S, h_0 = h\Bigg]
\end{aligned} \tag{22}$$

for $\ell = 0,1,2,\ldots$ until convergence is attained.

As in the expression of the linear model (21), at each iteration (22) of the fixed point algorithm of the nonlinear model the computation of an integral term is required. In both cases we consider a simple trapezoidal quadrature formula. Note that in expression (22) the first discounting factor in the integral is deterministic, while in expression (21) the discounting factor is stochastic due to the presence of the stochastic spread. Therefore, the evaluation of the involved integral is more expensive in the linear case, while in the nonlinear one several iterations are required. As will be illustrated by the forthcoming numerical examples, the solution of both models requires the same order of computational time.

4. Numerical results

In this section we report some tests that illustrate the behaviour of the previous Monte Carlo method when it is used for the evaluation of different multiasset options [24] in the presence of XVA. Our aim is to analyse how the choice of the mark-to-market, the initial values of the underlying assets and the investor's credit spread affect the total valuation adjustment and, therefore, the price of the financial derivative.

In all examples we consider constant FX rates, so that we have dropped subindex t to use the notation X^{D,C_j} instead of X_t^{D,C_j} throughout this section.

We have used $N_P = 10000$ paths and $N_T = 1000$ time steps. The elapsed computational time depends on the number of the underlying assets and on the value assigned to the mark-to-market value M, as well as on the choice of N_P and N_T. In all numerical examples, we have considered a simple trapezoidal rule to approximate the integrals appearing in the expressions (21) and (22) related to the linear and nonlinear models, respectively. All tests have been performed by using Matlab on an Intel(R) Core(TM) i7-8550U, 1.99 GHz, 16 GB (RAM), x64-based processor.

4.1. Sum of call options

We assume the hedger H buys from a counterparty C a portfolio of European call options in different currencies, so that the portfolio payoff function is the sum of the payoff functions of the involved call options, i.e.

$$G(t,S^1,\ldots,S^N) = \sum_{i=1}^N (X^{D,C_i}S^i - K^i)^+, \tag{23}$$

Table 1
Financial data.

$r^1 = 0.30$	$q^1 = 0.24$	$\sigma^{S^1} = 0.30$	$K^1 = 12$
$r^2 = 0.24$	$q^2 = 0.18$	$\sigma^{S^2} = 0.20$	$K^2 = 15$
$\rho^{S^1S^2} = 0.15$	$\rho^{S^1h} = 0.40$	$\rho^{S^2h} = -0.2$	$\sigma^h = 0.20$
$h_0 = 0.20$	$R_C = 0.30$	$R_0^D = 15$	$M_0^D = 15$
$\kappa = 0.01$	$t_0 = 0$	$T = 0.5$	

Table 2
Interest rates.

$f = 0.06$	$r^R = 0.05$	$b^{D,C_0} = 0.02$	$c^D = 0.06$

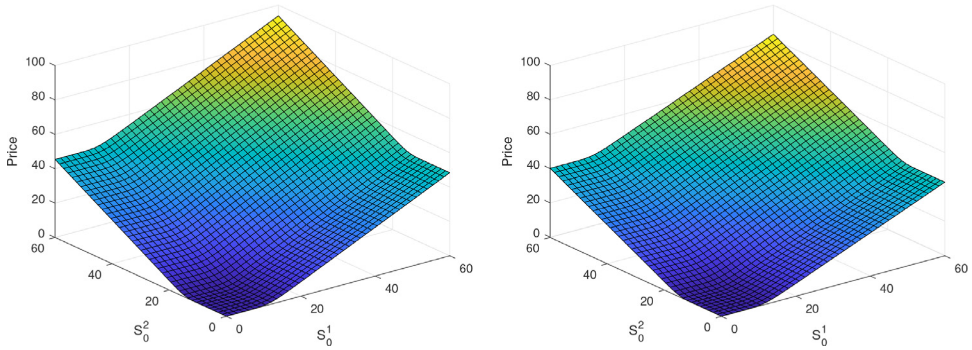

Fig. 1. Sum of call options. Risk-free price (left) and risky price (right) in the nonlinear case.

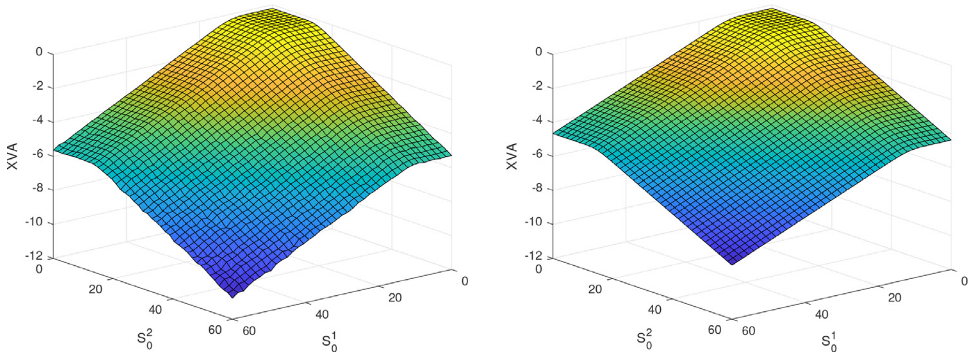

Fig. 2. Sum of call options. Total value adjustment in the nonlinear case (left) and in the linear case (right).

where $S^1, \dots, S^N$ are the N assets respectively written in currencies $C_1, \dots, C_N$, and $K^1, \dots, K^N$ are the strike values given in the domestic currency D.

We assume that the investor C is defaultable, while the hedger H is default-free. Hence, only H will charge C an adjustment on the trade, thus reducing the value of the derivative with respect to the risk-free setting.

First, in order to plot the total value adjustment as function of the initial prices of the underlying assets, we restrict our analysis to the case of two underlying assets. We set the parameters as specified in Table 1 (where we have used the notation $R_0^D = X^{D,C_0} R_0^{C_0}$ and $M_0^D = X^{D,C_0} M_0^{C_0}$) and Table 2.

Figure 1 shows the risk-free price and the risky price in the nonlinear case (similar results are obtained in the linear case), while in Fig. 2 we plot the total valuation adjustment in the nonlinear and in the linear cases when $N = 2$. For each pair of (S_0^1, S_0^2) we simulate $N_P = 10000$ paths using $N_T = 1000$ time steps. After computing the risk-free price and the XVA price for each path, we obtain the mean value for both. The risky price is the sum of the mean value of the risk free price and the mean value of the XVA price. These values are plotted in Fig. 2 and we follow the same procedure for the rest of the figures in the article. As in all forthcoming figures, prices are reported in domestic currency.

As expected, the total value adjustment is negative because, when buying the derivative, H will ask the counterparty C for a reduction in the price due to the possibility of C's default. In general, the XVA becomes more negative when the asset prices increase, namely when the option is in the money —because H would be worst affected by C's default— and the XVA approaches to zero when the asset prices decrease, namely when the option is out of the money. In the nonlinear case

Table 3
Data for the sum of call options on 32 assets in their corresponding currencies. For the case of 2, 4, 8 and 16 assets, the respective first rows of data are considered.

i	S_0^i	r^i	q^i	σ^{S^i}	K^i	i	S_0^i	r^i	q^i	σ^{S^i}	K^i
1	10	0.30	0.24	0.34	13	17	10	0.30	0.24	0.27	10
2	14	0.24	0.15	0.35	16	18	14	0.24	0.15	0.35	14
3	15	0.25	0.20	0.34	18	19	12	0.25	0.20	0.28	11
4	12	0.31	0.26	0.30	12	20	11	0.31	0.26	0.26	15
5	10	0.28	0.22	0.26	13	21	11	0.28	0.22	0.24	11
6	10	0.29	0.23	0.24	12	22	10	0.29	0.23	0.35	11
7	13	0.32	0.30	0.29	17	23	13	0.33	0.30	0.25	14
8	14	0.28	0.22	0.28	15	24	14	0.27	0.22	0.27	10
9	15	0.34	0.29	0.30	10	25	15	0.34	0.29	0.24	16
10	12	0.33	0.26	0.29	15	26	12	0.34	0.26	0.26	18
11	11	0.25	0.18	0.30	17	27	11	0.24	0.20	0.28	18
12	15	0.23	0.18	0.30	10	28	13	0.23	0.18	0.34	13
13	16	0.22	0.18	0.33	14	29	18	0.22	0.18	0.28	15
14	12	0.26	0.19	0.32	17	30	12	0.26	0.22	0.35	15
15	17	0.32	0.29	0.31	18	31	15	0.32	0.29	0.24	14
16	12	0.26	0.21	0.29	15	32	12	0.26	0.18	0.28	14

Table 4
Sum of call options. Confidence intervals for the prices and elapsed time in thenonlinear case ($M = V^D$) for different numbers of underlying assets.

Number of assets	Risk-free value	Risky value	XVA	Time (s)
2	[1.0945,1.2077]	[0.7527,0.8659]	[−0.3428, −0.3407]	0.9125
4	[2.7742,3.0062]	[2.2363,2.4681]	[−0.5407, −0.5354]	2.1732
8	[4.0586,4.3002]	[3.3775,3.6192]	[−0.6848, −0.6773]	2.8364
16	[19.0263,19.4744]	[16.6646,17.1123]	[−2.3796, −2.3442]	6.7180
32	[37.9273,38.4797]	[33.4274,33.9931]	[−4.5276, −4.4588]	13.8403

Table 5
Sum of call options. Confidence intervals for the prices and elapsed time in the linear case ($M = V^{RF,D}$) for different numbers of underlying assets.

Number of assets	Risk-free value	Risky value	XVA	Time (s)
2	[1.0945,1.2077]	[0.7780,0.8912]	[−0.3172, −0.3158]	0.8571
4	[2.7742,3.0062]	[2.2987,2.5305]	[−0.4776, −0.4736]	1.1240
8	[4.0586,4.3002]	[3.4646,3.7063]	[−0.5969, −0.5911]	3.3523
16	[19.0263,19.4744]	[17.0534,17.5011]	[−1.9873, −1.9589]	6.7890
32	[37.9273,38.4797]	[34.2119,34.7745]	[−3.7380, −3.6825]	13.4987

($M = V^D$) the total value adjustment is more negative than in the linear case ($M = V^{RF,D}$). However, the difference seems to be not so relevant, as it is shown in Fig. 2.

Next, we consider the sum of call options on different numbers of assets in their corresponding currencies. Table 3 shows data for the case of 32 assets. Note that when considering a number of assets lower than 32, we use the data of the first rows appearing in Table 3 (i.e., for the case of 2 assets we consider the first 2 rows, and so on for 4, 8 and 16 assets).

Tables 4 and 5 show the Monte Carlo 99% confidence intervals of the risk-free price, the risky price and the total value adjustment with different numbers of underlying assets, both in the (nonlinear) case where $M = V^D$ and the (linear) case where $M = V^{RF,D}$, respectively. The initial values of the underlying assets are converted into the domestic currency D. Moreover, we choose $h_0 = 0.20$, $\kappa = 0.1$, $R_C = 0.30$ and $\sigma^h = 0.2$, while $R_0^{C_0} X^{D,C_0} = M_0^{C_0} X^{D,C_0} = 15$. The chosen interest rates are again those of Table 2.

In the nonlinear case the tolerance for the fixed point iteration has been taken equal to 10^{-12} for the relative maximum error, thus requiring 11 fixed point iterations for all the numbers of assets presented in Table 4.

Finally, we show in Fig. 3 the elapsed computational time for different numbers of underlying assets in the nonlinear and linear cases. From the last column of Tables 4 and 5, as well as from Fig. 3, we can see that the computational times for the nonlinear and the linear cases are very close to each other. As indicated, integrals are computed with simple trapezoidal rules and convergence was checked by comparing the results with a higher number of intervals in a composite trapezoidal rule. Note that in the linear case expression (21) the discount factor is stochastic, so that we simulate the integrand to compute the integral in each of the 10000 paths and then compute the expectation. On the other hand, at each iteration (22) of the nonlinear case, we take advantage that the discount factor $\exp(-fu)$ and the term $(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0})X_u^{D,C_0}$ are constant and

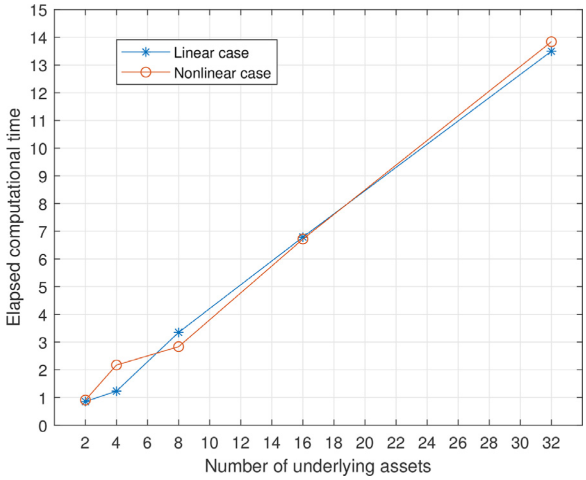

Fig. 3. Sum of call options. Elapsed time in the linear ($M = V^{RF,D}$) and nonlinear ($M = V^D$) cases.

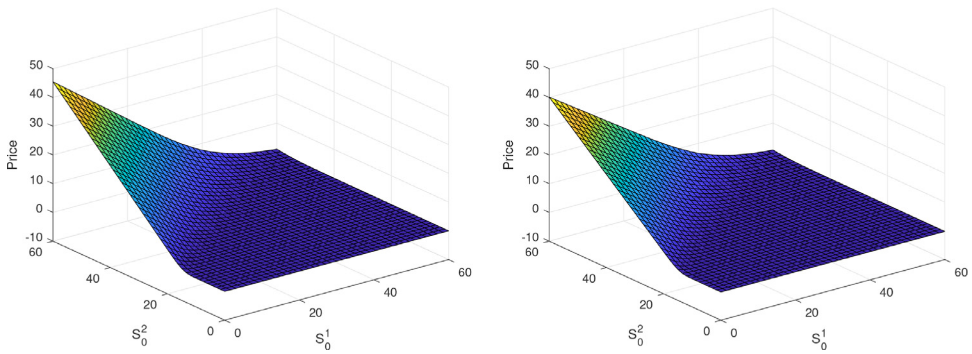

Fig. 4. Spread option. Risk-free price (left) and risky price (rigth) in the nonlinear case.

their product can be easily computed outside the expectation, while the other term in the sum requires simulation. These details explain why the computational time of the 11 iterations in the nonlinear case is very close to the integral evaluation in the linear case.

4.2. Spread option

We now assume that the hedger H buys from a counterparty C a spread option, written on two underlying assets, each of them being denominated in a different currency. The payoff function is given by

$$G(t, S^1, S^2) = (X^{D,C_2} S^2 - X^{D,C_1} S^1 - K)^+, \tag{24}$$

where K is the strike value given in the domestic currency D.

We suppose again that C is defaultable, while H is default-free. Fig. 4 shows the risk-free price and the risky price in the nonlinear case, while Fig. 5 shows the total valuation adjustment that H charges to C either in the nonlinear or in the linear case. The fixed strike value is $K = 15$, while the other parameters are shown in Tables 1 and 2.

Here we can take out similar conclusions to those drawn in the case of the sum of call options: the total value adjustment becomes more negative when the option is in the money than when the option is out of the money. Also, it becomes more negative when the mark-to-market value, M, is set to be equal to the risky value of the derivative than when M is considered equal to the risk-free value. However, the dependence of the XVA on M is not outstanding.

Tables 6 and 7 illustrate these arguments. Indeed, they exhibit the confidence intervals for the risk-free price, the risky price and the total value adjustment for fixed values of the underlying assets for the nonlinear case in Table 6 and for the linear case in Table 7. We use the notation $S^{i,D} = X^{D,C_i} S^i$, for $i = 1, 2$, so that we can display all the prices in the same currency D. For each fixed value of S^1 we analyse three different possibilities: out of the money option, at the money option and in the money option, respectively.

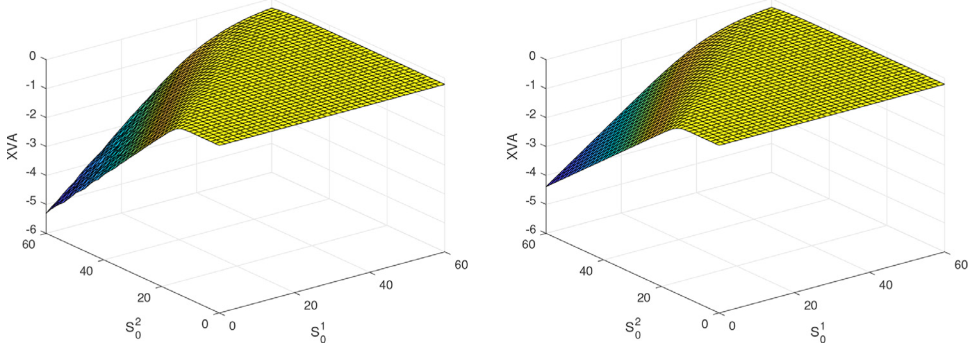

Fig. 5. Spread option. Total value adjustment in the nonlinear (left) and linear (right) cases.

Table 6
Spread option. Confidence intervals in the nonlinear case ($M = V^D$) for fixed initial prices of S^1 and S^2.

$S^{1,D}$	$S^{2,D}$	$V^{RF,D}$	V^D	XVA
	21	[0.4146, 0.4771]	[0.1504, 0.2132]	[−0.2644, −0.2636]
9	24	[1.5770, 1.6984]	[1.1800, 1.3023]	[−0.3980, −0.3951]
	27	[3.7666, 3.9483]	[3.1188, 3.3031]	[−0.6499, −0.6430]
	24	[0.5925, 0.6712]	[0.3075, 0.3866]	[−0.2854, −0.2841]
12	27	[1.8421, 1.9828]	[1.4150, 1.5569]	[−0.4282, −0.4248]
	30	[3.9308, 4.1296]	[3.2643, 3.4659]	[−0.6688, −0.6615]
	27	[0.7784, 0.8761]	[0.4705, 0.5687]	[−0.3084, −0.3069]
15	30	[2.0906, 2.2497]	[1.6340, 1.7945]	[−0.4577, −0.4538]
	33	[4.1200, 4.3376]	[3.4311, 3.6518]	[−0.6912, −0.6835]
	30	[0.9934, 1.1095]	[0.6614, 0.7781]	[−0.3327, −0.3307]
18	33	[2.3421, 2.5199]	[1.8550, 2.0345]	[−0.4884, −0.4840]
	36	[4.3293, 4.5673]	[3.6157, 3.8571]	[−0.7159, −0.7078]
	33	[1.2288, 1.3643]	[0.8683, 1.0046]	[−0.3613, −0.3589]
21	36	[2.6351, 2.8355]	[2.1143, 2.3165]	[−0.5224, −0.5174]
	39	[4.5873, 4.8488]	[3.8443, 4.1092]	[−0.7456, −0.7370]

Table 7
Spread option. Confidence intervals in the linear case ($M = V^{RF,D}$) for fixed initial prices of S^1 and S^2.

$S^{1,D}$	$S^{2,D}$	$V^{RF,D}$	V^D	XVA
	21	[0.4146, 0.4772]	[0.1626, 0.2237]	[−0.2520, −0.2516]
9	24	[1.5770, 1.6984]	[1.2664, 1.3912]	[−0.3674, −0.3652]
	27	[3.7666, 3.9483]	[3.1879, 3.3706]	[−0.5657, −0.5603]
	24	[0.5925, 0.6712]	[0.3316, 0.4106]	[−0.2701, −0.2694]
12	27	[1.8421, 1.9828]	[1.4892, 1.6324]	[-0.3910, −0.3884]
	30	[3.9308, 4.1296]	[3.3475, 3.5500]	[−0.5829, −0.5773]
	27	[0.7784, 0.8761]	[0.5206, 0.6183]	[−0.2902, −0.2892]
15	30	[2.0906, 2.2497]	[1.7201, 1.8823]	[−0.4154, −0.4125]
	33	[4.1200, 4.3376]	[3.5286, 3.7510]	[−0.6023, −0.5964]
	30	[0.9935, 1.1095]	[0.7244, 0.8414]	[−0.3119, −0.3106]
18	33	[2.3421, 2.5199]	[1.9562, 2.1378]	[−0.4404, −0.4370]
	36	[4.3293, 4.5673]	[3.7257, 3.9682]	[−0.6234, −0.6171]
	33	[1.2288, 1.3643]	[0.9377, 1.0743]	[−0.3346, −0.3329]
21	36	[2.6351, 2.8355]	[2.1957, 2.3968]	[−0.4657, −0.4620]
	39	[4.5873, 4.8488]	[3.9335, 4.1960]	[−0.6455, −0.6389]

4.3. Exchange option

Finally, we suppose that the default-free hedger H buys from the defaultable counterparty C an exchange option, written on an underlying asset S^1, denominated in the domestic currency, and an underlying asset S^2, denominated in a foreign currency C_2. Hence, the payoff function of the option is given by

$$G(t, S^1, S^2) = (S^1 - X^{D,C_2} S^2)^+. \tag{25}$$

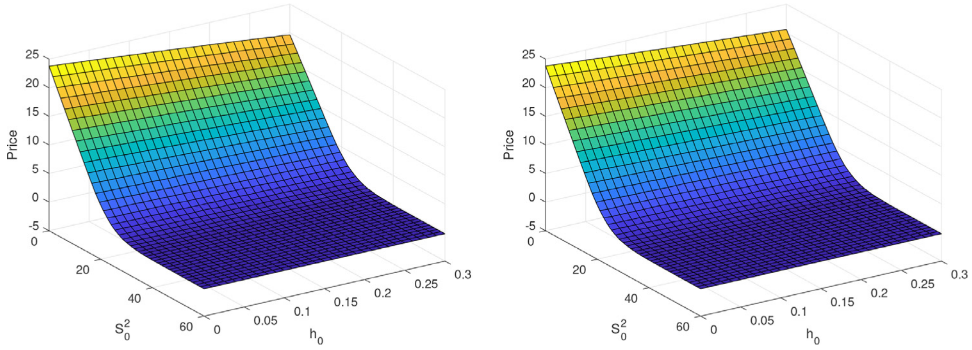

Fig. 6. Exchange option. Risky price in the nonlinear (left) and linear (right) cases.

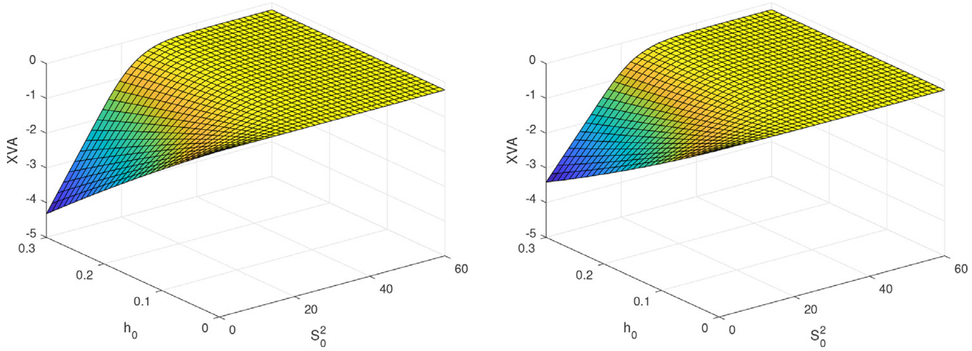

Fig. 7. Exchange option. Total value adjustment in the nonlinear (left) and linear (right) cases.

Table 8
Exchange option. Confidence intervals in the nonlinear case ($M = V^D$) for fixed initial values of $S^{2,D}$ and h.

$S^{2,D}$	h	$V^{RF,D}$	V^D	XVA
21	0.05	[3.8620, 4.0916]	[3.5341, 3.7606]	[−0.3330, −0.3259]
	0.10	[3.8620, 4.0916]	[3.4279, 3.6544]	[−0.4393, −0.4321]
	0.15	[3.8620, 4.0916]	[3.3189, 3.5454]	[−0.5483, −0.5411]
	0.20	[3.8620, 4.0916]	[3.2070, 3.4335]	[−0.6601, −0.6529]
24	0.05	[2.2162, 2.4013]	[1.9322, 2.1157]	[−0.2870, −0.2827]
	0.10	[2.2162, 2.4013]	[1.8718, 2.0552]	[−0.3474, −0.3432]
	0.15	[2.2162, 2.4013]	[1.8098, 1.9932]	[−0.4094, −0.4052]
	0.20	[2.2162, 2.4013]	[1.7461, 1.9296]	[−0.4730, −0.4688]
27	0.05	[1.1537, 1.2920]	[0.8983, 1.0359]	[−0.2569, −0.2547]
	0.10	[1.1537, 1.2920]	[0.8670, 1.0052]	[−0.2875, −0.2853]
	0.15	[1.1537, 1.2920]	[0.8362, 0.9738]	[−0.3190, −0.3167]
	0.20	[1.1537, 1.2920]	[0.8040, 0.9416]	[−0.3512, −0.3490]

Figure 6 displays the sensitivity of the option price with respect to both the probability of C's default, represented by the credit spread h, and the choice of the foreign asset. In Fig. 7 the total value adjustment is exhibited. Parameters are taken from Tables 1 and 2. We fix the initial value of the domestic underlying asset at $S_0^1 = 24$.

In order to better infer how the choice of the mark-to-market value affects the option value, we report in Tables 8 and 9 show the confidence intervals for the risk-free price, the risky price and the total value adjustment for fixed initial values of both the foreign underlying asset and the counterparty's credit spread. Table 8 corresponds to the nonlinear model and Table 9 refers to the linear model. Again, in order to display all the prices in the domestic currency D, we use the notation $S^{2,D} = X^{D,C_2}S^2$. Note that the risk-free price does not depend on h. Thus, for fixed $S^{2,D}$ the presence of different values of $V^{RF,D}$ is only due to the variability of the Monte Carlo simulations.

As expected, the total value adjustment is affected by the increasing of the probability of C's default: it becomes more negative when it is more likely that C defaults. However, it is worth mentioning that when the option is out of the money the total value adjustment remains small, even increasing the probability of C's default, although when the option is in the money the total value adjustment decays quickly when increasing C's credit spread h. Finally, once more, it is evident that

Table 9
Exchange option. Confidence intervals in the linear case ($M = V^{RF,D}$) for fixed initial values of $S^{2,D}$ and h.

$S^{2,D}$	h	$V^{RF,D}$	V^D	XVA
21	0.05	[3.8620, 4.0916]	[3.5506,3.7774]	[−0.3161, −0.3096]
	0.10	[3.8620, 4.0916]	[3.4600,3.6870]	[−0.4064, −0.4002]
	0.15	[3.8620, 4.0916]	[3.3725, 3.5996]	[−0.4937, −0.4878]
	0.2	[3.8620, 4.0916]	[3.2879, 3.5151]	[−0.5781, −0.5726]
24	0.05	[2.2162, 2.4013]	[1.9422, 2.1258]	[−0.2766, −0.2730]
	0.10	[2.2162, 2.4013]	[1.8912, 2.0750]	[−0.3274, −0.3240]
	0.15	[2.2162, 2.4013]	[1.8420, 2.0258]	[−0.3766, −0.3733]
	0.20	[2.2162, 2.4013]	[1.7943, 1.9782]	[−0.4241, −0.4210]
27	0.05	[1.1537, 1.2920]	[0.9040, 1.0417]	[−0.2509, −0.2491]
	0.10	[1.1537, 1.2920]	[0.8789, 1.0166]	[−0.2760, −0.2743]
	0.15	[1.1537, 1.2920]	[0.8545, 0.9923]	[−0.3003, −0.2987]
	0.20	[1.1537, 1.2920]	[0.8310, 0.9688]	[−0.3238, −0.3223]

the total value adjustment is greater in absolute value in the nonlinear case than in the linear one, but the difference is not so significant.

5. Conclusions

Modelling and computation of the XVA in a multicurrency setting becomes very relevant for financial institutions. In the literature, a lot of work has been recently developed in the framework of a single currency. In the present article we aim to extend some of this work to the multi-currency setting. For this purpose, we have considered financial derivatives contracts that involve assets that are denominated in different currencies as well as a stochastic spread for the investor. Moreover, we assume the existence of a collateral account. In this setting, an appropriate extension of the replicating portfolio to the multicurrency setting can be obtained. Then, following analogous methodologies to the single currency case, different linear and nonlinear formulations based either on PDEs or expectations for the pricing of different European options when including total valuation adjustments related to counterparty risk can be posed.

In order to compute the price of the valuation adjustments (XVA), due to the high dimension of the problem when the number of currencies increases, we have proposed appropriate Monte Carlo techniques for solving the formulations based on expectations. Numerical examples help us in the analysis of the XVA behavior and its dependence on the underlying assets and the investor's credit spread. As examples, we consider a sum of call options on a different number of assets, a spread option and an exchange option. In all cases, the computed XVA value is more negative when using the nonlinear model than the linear one. Also the expected qualitative behaviour is obtained and explained for each example. When using a simple trapezoidal formula for the involved integrals, the required computational times to solve the linear and nonlinear models in each example are very close to each other. Although around eleven iterations of a fixed point algorithm are necessary in the nonlinear case, the presence of a stochastic discount factor inside the integral of the solution of the linear case balances the computational cost of both cases. Conclusions about the scaling in the number of factors follows from the example of the sum of call options, where up to 32 assets are considered and a sub-linear increase in the computational cost is observed with respect to the number of assets.

As future work, we aim to develop the mathematical analysis and numerical solution of the here proposed PDE models. It seems that the existence and uniqueness of solution for an arbitrary number of currencies can be obtained with the extension of the methodology developed for European options in a single currency. For a large number of currencies, the direct numerical solution of the PDE formulation becomes more challenging, due to the curse of dimensionality. In this article, we have assumed deterministic FX rates. In a future work, we plan to develop the extension of the model to stochastic FX rates. Finally, we note that American options in a multi-currency setting can also be treated by extending some previous works developed for the single currency case.

Acknowledgements

This work has been partially funded by EU H2020-MSCA-ITN-2018 (ABC-EU-XVA Grant Agreement 813261), Spanish Ministry of Science and Innovation (Grant PID2019-108584RB-I00) and by Galician Government (Grant ED431C2018/033), both including FEDER financial support. Authors also acknowledge the support received from the Centro de Investigacin de Galicia (CITIC), funded by Xunta de Galicia and the European Union (European Regional Development Fund- Galicia 2014-2020 Program), by grant ED431G 2019/01.

Authors also thank two anonymous reviewers whose comments and suggestions helped to improve the final version of this article.

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