Exact bootstrap methods for nonparametric curve estimation

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The undersigned, Ricardo Cao Abad, certifies that he is the advisor of the Doctoral Thesis entitled ‘Exact bootstrap methods for nonparametric curve estimation’, developed by Inés Barbeito Cal at the University of A Coruña (Department of Mathematics), as part of the interuniversity PhD program (UDC, USC and UVigo) of Statistics and Operational Research, and hereby gives his consent to the author to proceed with the thesis presentation and the subsequent defense.

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Aos meus pais

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Abstract

This thesis deals with bandwidth selection for nonparametric curve estimation. In particular, closed expressions for some error criteria of kernel estimators have been proposed. Additionally, bootstrap algorithms have been reviewed (or proposed) in order to derive exact formulas for the bootstrap version of the aforementioned error criteria. This is very useful because Monte Carlo approximation is no longer needed. Moreover, bandwidth selectors for the nonparametric curve estimators studied in this thesis have been defined by means of minimizing these bootstrap exact formulas.

Specifically, we have dealt with bandwidth selection for nonparametric density estimation under dependence and hazard rate estimation. Moreover, we have dealt with bandwidth selection for statistical matching and prediction. In the last two contexts, the concept of proxy estimator is introduced in order to derive closed-form expressions for the bootstrap version of some error criteria.

The good empirical behaviour of every method proposed in this thesis is empirically checked via simulations. Furthermore, all methods are illustrated with an application to real data sets. Asymptotic results in the context of bandwidth selection for prediction considering a Nadaraya-Watson proxy estimator is also included.

Resumo

Esta tese trata sobre a selección do parámetro ventá na estimación non paramétrica de curvas. En concreto, propuxéronse expresións pechadas para algún criterio de erro de estimadores tipo núcleo. Ademais, revisáronse (ou propuxéronse) algoritmos bootstrap para establecer fórmulas exactas para a versión bootstrap do devandito criterio de erro. Isto é moi útil xa que fai que non se precise da aproximación de Monte Carlo. Así, establécense selectores de ventá para os estimadores non paramétricos das curvas estudadas nesta tese, definidos como os valores que minimizan as fórmulas bootstrap exactas.

Concretamente, considérase o caso da selección da ventá para a densidade baixo dependencia, para a razón de fallo e para o *matching* estatístico e a predición. Nos últimos dous contextos, introdúcese o concepto de estimador aproximado, para poder desenvolver expresións pechadas para a versión bootstrap do criterio de erro a considerar.

O bo comportamento empírico de todos os métodos propostos nesta tese é analizado mediante uns estudos de simulación. Ademais, a metodoloxía desenvolta é ilustrada coa aplicación a datos reais. Tamén se inclúen os resultados asintóticos no contexto de selección da ventá para a predición considerando a versión aproximada do estimador Nadaraya-Watson.

Resumen

Esta tesis trata sobre la selección del parámetro ventana en la estimación no paramétrica de curvas. En concreto, se propusieron expresiones cerradas para algún criterio de error de estimadores tipo núcleo. Además, se revisaron (o se propusieron) algoritmos bootstrap para establecer fórmulas exactas para la versión bootstrap del susodicho criterio de error. Esto es muy útil ya que no se necesita de la aproximación de Monte Carlo. Así, se establecen selectores de ventana para los estimadores no paramétricos de las curvas estudiadas en esta tesis, definidos como los valores que minimizan las fórmulas bootstrap exactas.

Concretamente, se considera el caso de la selección de ventana para la densidad bajo dependencia, para la razón de fallo y para el *matching* estadístico y la predicción. En los últimos dos contextos, se introduce el concepto de estimador aproximado, para poder desarrollar expresiones cerradas para la versión bootstrap del criterio de error a considerar.

El buen comportamiento empírico de todos los métodos propuestos en esta tesis se analiza mediante estudios de simulación. Además, la metodología desarrollada se ilustra con aplicaciones a datos reales. También se incluyen los resultados asintóticos en el contexto de selección de la ventana para la predicción considerando la versión aproximada del estimador de Nadaraya-Watson.

Preface

This work intends to summarize all the study developed along the PhD trajectory. Mainly, it is focused on using bootstrap algorithms in order to compute closed expressions for some error criteria of a curve estimator. This is very useful because Monte Carlo approximation is no longer needed. In this sense, bootstrap bandwidth selectors are defined by means of minimizing the aforementioned exact formulas.

Chapter 1 is devoted to introduce to the reader the context in which the thesis is developed: bootstrap methods for nonparametric curve estimation. An extensive review of the current state of the art regarding this the topic is included. Specifically, the kernel density estimator is presented, as well as the main dependence conditions and bootstrap algorithms in different contexts. Moreover, a review about bandwidth selection in nonparametric curve estimation is also included, focusing on bootstrap smoothing parameter selectors.

In Chapter 2, two new bootstrap procedures are presented in a dependent data context. Furthermore, closed expressions for the bootstrap version of the mean integrated squared error (MISE) are derived and two new bootstrap bandwidth selectors are proposed. A simulation study is carried out in order to check the good empirical behaviour of the new bandwidth selectors proposed, as well as compare them with the already existing ones. The simulations results show that the new bootstrap bandwidth selectors beat their main competitors. Moreover, the performance of the new bandwidth selectors is illustrated by applying them to two real data sets.

The nonparametric estimation of the hazard rate function is studied in Chapter

3. In particular, the concept of proxy estimator is introduced so as to work out a closed expression for the bootstrap MISE. Two new bootstrap bandwidth selectors are also defined. The good empirical behaviour of the new bootstrap bandwidth selectors has been analyzed via a simulation study. Furthermore, they were also compared in practice with the already existing bandwidth selectors for hazard rate estimation. According to the simulation results, it is clear that the new bootstrap bandwidth selectors display the overall best performance. Additionally, the methodology proposed has been illustrated by applying it to a diabetes data set.

In Chapter 4, it is presented a new bandwidth selector for prediction and statistical matching, in which two populations are considered: the source and the target population. In this framework, we are provided with the explanatory variable in the source and target populations, but the response variable is only observable in the source population or to predic the values of the response variable in the target population. Assuming we have a common regression function in both populations, the aim is to predict the expectation of the response variable of the target population. In addition, closed expressions for the bootstrap versions of the mean average squared error and mean squared error are derived for the proxy Nadaraya-Watson and local linear estimators. A simulation study is carried out showing the good empirical behaviour of the new bootstrap bandwidth selectors for prediction. Furthermore, the performance of the Nadaraya-Watson bandwidth selector is illustrated by applying it to a real data set. Specifically, the aim is to predict the expected salary of Spanish women in 2014, assuming women are equally paid as men, as a function of level of studies, activity sector and experience. Conclusions derived from comparing the predictive salaries with the actual ones are really interesting. Indeed, as a result of the real data application, women are lower paid than they should be, if they were equally paid as men.

Some comments about future work are given in Chapter 5, such as the extension of these ideas to the estimation of other curves (distribution, multivariate density, conditional density, regression, . . .).

Specifically, Appendix A collects the proofs of the results stated in Chapter 2, in which nonparametric density estimation under dependence is studied. Moreover, proofs of the results stated in Chapter 3, which deals with nonparametric hazard rate estimation, are collected in Appendix B. Finally, C collects the proofs of the results stated in Chapter 4, which is about nonparametric prediction and statistical matching.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Nonparametric curve estimation

Nonparametric curve estimation (often called nonparametric smoothing) is a very active area in Statistics. Indeed, one of the main objectives of Statistical Inference is precisely to obtain some conclusions for a population given a random sample of it. In order to carry out point estimation, testing hypotheses or constructing confidence intervals, it is necessary to know some of the curves that characterize the distribution of the population (such as the density function, for instance). Nevertheless, these curves sometimes remain unknown in practice. On the one hand, this can be easily done when assuming some parametric model for the distribution, and just estimating the parameters. However, it is not often straightforward to find a plausible parametric model for the data generating process. On the other hand, the problem can be approached from a nonparametric point of view, so that no parametric assumption is made and one needs to estimate the curve of interest in a totally flexible way or just assuming regularity conditions (such as continuity or differentiability) for underlying the curve.

When the curve of interest is the underlying probability density function of a continuous random variable, the seminal papers by Parzen (1962) and Rosenblatt (1956) introduced the well known kernel density estimator (KDE). Books of reference are available where kernel density estimation is studied under the L_2 (see Silverman,

1986) or L_1 view (see Devroye, 1987). The KDE is probably the simplest setting concerned with nonparametric curve estimation. However there are plenty of other population curves one may be interested in. Among them we mention the regression function, the volatility function (conditional variance), the hazard rate, the cumulative hazard function and the cumulative distribution function. Certainly, not all these curves need to be estimated in a smooth way. For instance, the cumulative distribution function (cdf) can be estimated using the empirical cdf, which does not require any smoothing.

In kernel density estimation, the definition of the estimator depends on two elements that the analyst has to choose from: the kernel function and the smoothing parameter (often called bandwidth). It is well known that the kernel function does not have a big impact on the efficiency of the estimator, although its degree of smoothness is inherited by the KDE. For instance continuous kernels give kernel density estimators that are continuous too. Something similar occurs with differentiability properties. On the contrary, the choice of the smoothing parameter has a great influence on the efficiency of the KDE. The smoothing parameter is a positive constant. As we will see below, if it is too small the bias of the KDE is small but its variance is large. In an opposite way, when the value of the bandwidth is large, the variance of the estimator is small but its bias is large. Under the L_2 view, it is very common to consider the mean integrated squared error as a standard error measure for the estimator. As it will be made clear below, such an error measure can be estimated using the bootstrap method.

Let us consider a continuous population with cumulative distribution function, F, and density function, f. Let us assume that a simple random sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ has been observed. The kernel density estimator, KDE, (see Parzen, 1962; and Rosenblatt, 1956) is defined as

$$\hat{f}_h(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i), \tag{1.1}$$

where $h > 0$ is the smoothing parameter and $K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right)$ is the rescaled version of the kernel function, K, which is typically assumed to be a non-negative function that integrates out to 1:

$$K(u) \geq 0, \forall u, \int_{-\infty}^{+\infty} K(u)\, du = 1.$$

These two properties mean that the kernel is a density function (picked by the data analyst, of course). In such a case the KDE given in (1.1) is also a density function itself, provided that the bandwidth h is constant for all the values of x. The choice of K does not have a big impact on the efficiency of the estimator, but the value of the smoothing parameter, h, is crucial.

In order to acquire a deeper understanding of the properties of the kernel density estimator given in (1.1), the expectation and the variance will be computed in the following, for a fixed value x.

$$\begin{aligned} \mathbb{E}(\hat{f}_h(x)) &= \mathbb{E}[K_h(x - X_1)] = \int K_h(x-y) f(y) dy \\ &= (K_h * f)(x) = \int K(u) f(x - hu) du, \end{aligned}$$

where $*$ stands for the convolution operator, which is defined as follows:

$$(f * g)(x) = \int f(x-y)\, g(y)\, dy.$$

This means that the expected curve obtained with the nonparametric kernel density estimator is not the true density f, but some smoothed version of it, given by $(K_h * f)$.

We will now carry on with more detailed calculations for the expectation and variance of $\hat{f}_h(x)$, given a fixed x, so as to describe the effect of h over them. Assuming some regularity conditions on the real density function f, and considering a Taylor expansion to approximate it, leads to

$$f(x - hu) = f(x) - huf'(x) + \frac{1}{2}(hu)^2 f''(x) + o(h^2).$$

Assuming that the kernel is a symmetric density and $\mu_2(K) = \int u^2 K(u)du < \infty$, then:

$$\mathbb{E}(\hat{f}_h(x)) = f(x) + \frac{1}{2}h^2\mu_2(K)f''(x) + o(h^2).$$

Focusing on the variance, it turns out:

$$\begin{aligned}
Var(\hat{f}_h(x)) &= \frac{1}{n}Var(K_h(x - X_1)) \\
&= \frac{1}{n}\left[\mathbb{E}(K_h^2(x - X_1)) - \mathbb{E}^2(K_h(x - X_1))\right] \\
&= \frac{1}{n}\left[\int K_h^2(x-y)f(y)dy - \left(\int K_h(x-y)f(y)dy\right)^2\right] \\
&= \frac{1}{nh}\int K^2(u)f(x-hu)du - \frac{1}{n}\left(\int K(u)f(x-hu)du\right)^2 \\
&= \frac{1}{nh}\int K^2(u)(f(x)+o(1))du - \frac{1}{n}(f(x)+o(1))^2 \\
&= \frac{1}{nh}R(K)f(x) + o((nh)^{-1}),
\end{aligned}$$

where $R(K) = \int K^2(u)du$.

Therefore, in order to analyze the properties of the Parzen-Rosenblatt kernel density estimator from an asymptotic point of view, we will study what happens when $n \to \infty$, $h \to 0$ and $nh \to \infty$. If $h \to 0$, the expectation turns out to be:

$$\mathbb{E}(\hat{f}_h(x)) - f(x) = \frac{1}{2}h^2\mu_2(K)f''(x) + o(h^2). \tag{1.2}$$

On the other hand, if $nh \to \infty$, the variance happens to be:

$$Var(\hat{f}_h(x)) = \frac{1}{nh}R(K)f(x) + o((nh)^{-1}). \tag{1.3}$$

As can be seen in expressions (1.2) and (1.3), the bias decreases as $h \to 0$, which means that small values of h will provide centered estimators with large variance, providing undersmoothed estimators. On the other hand, for large values of h, the variance decreases while the bias increases, leading to oversmoothed estimators.

In general terms, some error criteria must be defined so as to evaluate the behaviour of a given estimator, such as the Parzen-Rosenblatt one in the context of density estimation. In particular, both local (that is, for a given point x) or global (that is, for the whole estimated curve) error criteria may be considered. The aim is to minimize this error criteria so as to obtain the optimal value for the bandwidth, h.

Under the L_2 view, the mean squared error is a popular way to measure the estimation error for a particular point x: $MSE(h) = E[(\hat{f}_h(x) - f(x))^2]$. Thus, for estimating the density at value x, $f(x)$, one could consider using the value of the bandwidth, h_{MSE}, that minimizes $MSE(h)$ when $h > 0$. This is the so-called MSE bandwidth and it is not observable in practice. It is oriented to estimate the density at a particular point x. However, such a theoretical choice for h does not lead to a KDE that is a density function. The reason is that the integral $\int \hat{f}_{h(x)}(x)dx$ may not be 1, when the bandwidth $h(x)$ is not a constant function. In particular, considering expressions (1.2) and (1.3) leads to:

$$\begin{aligned} MSE_x(h) &= \mathbb{E}(\hat{f}_h(x) - f(x))^2 + \text{Var}(\hat{f}_h(x)) = \text{Bias}^2(\hat{f}_h(x)) + \text{Var}(\hat{f}_h(x)) \\ &= \frac{1}{nh}R(K)f(x) + \frac{1}{4}h^4\mu_2(K)^2 f''(x)^2 + o((nh)^{-1} + h^4). \end{aligned} \tag{1.4}$$

Once obtained expression (1.4), it is straightforward to think of which value for h would provide the minimum $MSE_x(h)$. Nonetheless, it it worth noticing that the asymptotic term $o((nh)^{-1} + h^4)$ does not allow to minimize expression (1.4) in h. As a consequence, the asymptotic version of the MSE (namely $AMSE$) may be considered:

$$AMSE(h) = \frac{1}{nh}R(K)f(x) + \frac{1}{4}h^4\mu_2(K)^2 f''(x)^2.$$

Whenever $f''(x) \neq 0$, the value h for which $AMSE(\hat{f}_h(x))$ attains its minimum turns out to be:

$$h_{AMSE} = \left(\frac{R(K)f(x)}{n\mu_2(K)^2 f''(x)^2}\right)^{1/5}.$$

This is a local bandwidth, since it depends on the fixed point x, at which the density is estimated. However, h_{AMSE} is not useful in practice as it depends on $f''(x)$ and $f(x)$, which remains unknown.

When considering the additional restriction that the smoothing parameter is constant along all the values of x, a global error criterion makes much more sense for bandwidth selection purposes. The mean integrated squared error, MISE, (an integrated version of MSE) is defined as follows:

$$\begin{aligned} MISE(h) &= \mathbb{E}\left[\int (\hat{f}_h(x) - f(x))^2\right] dx = \int \mathbb{E}[(\hat{f}_h(x) - f(x))^2]dx \\ &= \int [\mathbb{E}(\hat{f}_h(x) - f(x))]^2 dx + \int Var(\hat{f}_h(x))dx, \end{aligned} \tag{1.5}$$

which can be written as the sum of the integrated squared bias and the integrated variance. The bandwidth that gives the minimum of $MISE(h)$ is denoted by h_{MISE} and we refer to it as the MISE bandwidth. It is worth mentioning that the MISE can also be seen as the expectation of the integrated squared error (namely, ISE), defined as:

$$ISE(X_1, \ldots, X_n; h) = \int \left(\hat{f}_h(x) - f(x)\right)^2 dx,$$

which is a random quantity depending on the sample and the bandwidth selector.

Under some regularity conditions and then using a Taylor expansion, the following expression for MISE can be derived:

$$MISE(h) = \frac{R(K)}{nh} + \frac{h^4}{4}\mu_2(K)^2 R(f'') + o\left((nh)^{-1} + h^4\right),$$

where its asymptotic version, the AMISE, is given by:

$$AMISE(h) = \frac{R(K)}{nh} + \frac{h^4}{4}\mu_2(K)^2 R(f''). \tag{1.6}$$

An optimal global bandwidth h may be obtained by means of minimizing expression

(1.6).

$$h_{AMISE} = \left(\frac{R(K)}{\mu_2(K)^2 R(f'')n}\right)^{1/5}. \tag{1.7}$$

To find a more explicit expression for equation (1.5) it is convenient to work out closed expressions for the bias and the variance of the KDE:

$$Bias\left(\hat{f}_h(x)\right) = E\left(\hat{f}_h(x)\right) - f(x) = (K_h * f)(x) - f(x),$$

$$Var\left(\hat{f}_h(x)\right) = \frac{1}{nh^2} Var\left(K\left(\frac{x - X_1}{h}\right)\right) = \frac{1}{nh}\left[\left(K^2\right)_h * f\right](x) - \frac{1}{n}\left[(K_h * f)(x)\right]^2.$$

As a consequence an explicit formula can be derived for the MSE:

$$MSE_x(h) = \left[(K_h * f)(x) - f(x)\right]^2 + \frac{1}{nh}\left[\left(K^2\right)_h * f\right](x) - \frac{1}{n}(K_h * f)(x)^2,$$

and also for the MISE:

$$MISE(h) = \int \left[(K_h * f)(x) - f(x)\right]^2 dx + \frac{R(K)}{nh} - \frac{1}{n}\int (K_h * f)(x)^2 dx. \tag{1.8}$$

Under regularity conditions, Taylor expansions give $MISE(h) = AMISE(h) + O\left(h^6\right) + O\left(\frac{h}{n}\right)$ when $h \to 0$ and $nh \to \infty$ as $n \to \infty$, where $AMISE(h)$ is given in expression (1.6). While h_{AMISE} can be computed explicitly (see expression (1.7)), h_{MISE} has to be approximated by numerical minimization of (1.8). Conclusively, both h_{MISE} and h_{AMISE} depend on the underlying density f, so they need to be estimated in practice.

1.2 Dependent data

A common hypothesis generally assumed in several contexts is the independence of the observations collected in the random sample. That is, the random variables in the sample, $X_1, X_2, \ldots, X_n$, are independent.

Imagine, for instance, a random variable X which accounts for the number of car accidents in A Coruña (Spain) each year. Under the hypothesis of independence, the probability of observing a particular value does not depend on the value observed the previous year. Nevertheless, there is a higher similarity in the number of accidents if we consider consecutive years. Several economic, social or political factors exist, for instance, which may well change as time passes by. This implies that those observations more closely related in time may share some resemblance, while those observations separated by a long period of time do not. Hence, in these examples the data actually change over time, leading to time series or dependent data.

Formally speaking, a time series is a collection of observations of a random variable, X, sequentially assembled over time. As previously mentioned, these observations will no longer be considered as independent data, as they will depend, precisely, one on another as time passes by.

Let us consider a stochastic process discrete in time and with continuous state space (commonly, the set of real numbers), that is, a set of random variables $\{X_t\}_{t\in\mathbb{Z}}$ all of which are defined in the same probability space, where $\mathbb{Z}$ is the set of integer numbers. We will assume, additionally, that we are able to observe part of the trajectory, i.e., given a variable X which was observed in the instants $1, 2, \dots, n$, the time series observed is going to be represented as $X_1, \dots, X_n$. Furthermore, $X_1, \dots, X_n$ will be a realization of the stochastic process. In other words, $X_1, \dots, X_n$ will be a sample of dependent data.

The most common dependence conditions will be reviewed below in this section, as well as the main parametric models of dependence, already collected by Doukhan et al. (2010) and J. Fan (2003). Let us previously introduce the concept of stationary and strictly stationary process.

For some stochastic process X_t, if the joint distribution of $(X_{t_1}, X_{t_2}, \dots, X_{t_k})$ is the same as $(X_{t_1+t}, X_{t_2+t}, \dots, X_{t_k+t})$ for every t and every set of time points $(t_1, t_2, \dots, t_k)$, the process is said to be strictly stationary. This requires that the first and second moments of the distribution, if they exist, are invariant with time.

In the linear situation the second-order moments like variance and covariance play an important role. If these and the mean are invariant in time, the process is said to be weakly stationary or second-order stationary.

1.2.1 Parametric models of dependence

There are numerous parametric models to deal with dependent data. Firstly, we will consider a stationary process $\{X_t\}_{t\in\mathbb{Z}}$ as the generator of the time series. If the process admits the following representation:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t, \tag{1.9}$$

where $c, \phi_1, \phi_2, \ldots, \phi_p$ are constants and a_t are independent of $X_{t-1}, X_{t-2}, \ldots$, it will be denoted an autoregresive process of order p ($AR(p)$).

Secondly, let $\{X_t\}_{t\in\mathbb{Z}}$ be a process which admits the following representation:

$$X_t = c + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}, \tag{1.10}$$

where $c, \theta_1, \theta_2, \ldots, \theta_q$ are constants. Then it is called a moving average process of order q ($MA(q)$).

In third place, a stationary process $\{X_t\}_{t\in\mathbb{Z}}$ which admits the representation:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}, \tag{1.11}$$

where $c, \phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$ are constants, is known as an $ARMA(p, q)$ process.

A process $\{X_t\}_{t\in\mathbb{Z}}$ which admits the representation:

$$\phi(B)(1 - B)^d X_t = c + \theta(B) a_t, \tag{1.12}$$

where B stands for the lag operator, defined as $BX_t = X_{t-1}$, and

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p,$$

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q,$$

is denoted as an autoregressive integrated moving average process (or $ARIMA(p, d, q)$), with autoregresive order p, moving average order q and d regular differences.

Finally, a process $\{X_t\}_{t\in\mathbb{Z}}$ that admits the representation:

$$\phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D X_t = c + \theta(B)\Theta(B^s)a_t, \tag{1.13}$$

where B^s is the seasonal lag operator defined as $B^s X_t = X_{t-s}$, where

$$\Phi(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps},$$

$$\Theta(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \cdots + \Theta_Q B^{Qs},$$

is known as a $ARIMA(p, d, q) \times (P, D, Q)_s$ process, with autoregresive order p, moving average order q, d regular differences, seasonal autoregresive order P, seasonal moving average order Q and D seasonal differences of period s.

In expressions (1.9), (1.10), (1.11), (1.12) and (1.13), the set $\{a_t\}_{t\in\mathbb{Z}}$ denotes a collection of uncorrelated random variables, with mean 0 and finite variance σ_a^2. It is named as white noise. If it happens to be Gaussian, then the random variables which define it are iid.

1.2.2 Situations of general dependence

In other situations, no parametric structure is assumed on the stochastic process. Consequently, some type of general dependence condition is established. Let us consider a stochastic process $\{X_t\}_{t\in\mathbb{Z}}$. Usually, it will be assumed to be strong stationary, which means that the finite dimensional distribution of the random vector which consists of the stochastic process in the points of time $t_1, t_2, \ldots, t_k$ is the same as in $t_1 + h, \ldots, t_k + h$, $\forall k, h \in \mathbb{N}$. In the following, some general dependence conditions (which will be referred to in Chapter 2) are described.

Strongly mixing (or α-mixing)

Consider $\{X_t\}_{t\in\mathbb{Z}}$ a stationary process. The α-mixing condition, collected by Doukhan et al. (2010), establishes that the dependence among the random variables which form the observations of the sample weakens as the temporal instants move away from one another. In other words,

$$\sup_{A\in\mathcal{F}_1^n, B\in\mathcal{F}_{n+k}^\infty} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \alpha_k,$$

where $\alpha_k \to 0$ and $\mathcal{F}_s^t$ is the σ-algebra generated by the random variables $X_s, \ldots, X_t$.

Uniformly mixing (or ϕ-mixing)

Once again, consider $\{X_t\}_{t\in\mathbb{Z}}$ a stationary process. Then $\{X_t\}_{t\in\mathbb{Z}}$ is uniformly mixing, according to Doukhan et al. (2010), if:

$$|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \phi_k \mathbb{P}(A), \forall A \in \mathcal{F}_1^n, \forall B \in \mathcal{F}_{n+k}^\infty, \text{ with } \phi_k \to 0.$$

Thus, if a process $\{X_t\}_{t\in\mathbb{Z}}$ is uniformly mixing, this means that the dependence among the random variables which form the sample observations weakens as the temporal instants move away from each other.

m-dependence

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary process. According to Billingsley (1968), $\{X_t\}_{t\in\mathbb{Z}}$ is m-dependent if for any integer numbers k and l greater than zero, $(X_1, \ldots, X_k)$ and $(X_{k+n}, \ldots, X_{k+n+l})$ are independent whenever $n > m$.

It is worth singling out that using this terminology, an independent sequence is 0-dependent. In addition, if a stochastic process is m-dependent, it is also α-mixing with $\alpha_n = 0, \forall n > m$, and ϕ-mixing with $\phi_n = 0, \forall n > m$.

Ψ-dependence

This notion of weak dependence was introduced by Hwang and Shin (2012). In order to describe it, we need some previous definitions of class functions.

Consider $\mathbb{L}^\infty = \bigcup_{n=1}^{\infty} \mathbb{L}^\infty(\mathbb{R}^n)$ the set of limited functions taking values over $\mathbb{R}^n$, for any $n = 1, 2, \ldots$. Let us pick a function such that $G : \mathbb{R}^n \to \mathbb{R}$, considering the L_1 norm for $\mathbb{R}^n$. Thus, the Lipschitz modulus of G is defined as follows:

$$\text{Lip}(G) = \sup_{x \neq y} \frac{|G(x) - G(y)|}{||x - y||_1}.$$

Moreover, consider $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$, where $\mathcal{L}_n = \{G \in \mathbb{L}^\infty(\mathbb{R}^n); \text{Lip}(G) < \infty, ||G||_\infty \leq 1\}$, and the following two functions:

$$\Psi_1(G, H, n, m) = \min(n, m)\text{Lip}(G)\text{Lip}(H)$$

$$\Psi_2(G, H, n, m) = 4(n + m)\text{Lip}(G)\text{Lip}(H),$$

where G and H are functions defined in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Assume $\{X_t\}_{t\in\mathbb{Z}}$ to be a stationary stochastic process. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is $(\theta, \mathcal{L}, \Psi)$-dependent (or simply, Ψ-dependent), if there exist some decreasing sequence $\theta = (\theta_r)_{r\in\mathbb{Z}}$ veryfing that $\theta = (\theta_r)_{r\in\mathbb{Z}} \to 0$ as $r \to \infty$; and a function Ψ depending on $(G, H, n, m) \in \mathcal{L}_n \times \mathcal{L}_m \times \mathbb{N}^2$ such that, given a n-tuple $(i_1, \ldots, i_n)$ and a m-tuple $(j_1, \ldots, j_m)$, with $i_1 \leq \cdots \leq i_n < i_n + r \leq j_1 \leq \cdots \leq j_m$, it fulfills the following condition:

$$|\text{Cov}(G(X_{i_1}, \ldots, X_{i_n}), H(X_{j_1}, \ldots, X_{j_m}))| \leq \Psi(G, H, n, m)\theta_r.$$

1.3 Bootstrap methodology

Forty years ago Efron (1979) introduced the bootstrap method as a useful resampling technique to approximate the sampling distribution of a statistic, $\boldsymbol{R} = R(\boldsymbol{X}, F)$

that depends on the population distribution, F, and the observed sample, $\boldsymbol{X} = (X_1, \ldots, X_n)$. A remarkable example of such a type of statistic is the error ($R(\boldsymbol{X}, F) = \hat{\theta} - \theta$) of an estimator $\hat{\theta} = t(\boldsymbol{X})$ of a parameter of interest, $\theta = \theta(F)$. Other examples are test statistics for hypothesis testing or pivotal statistics used to construct confidence intervals (see Barbeito et al., 2019, as a review of several bootstrap confidence intervals for conditional density function in Markov processes, for instance).

The idea behind the bootstrap method is to use an estimator of the underlying distribution, say $\hat{F}$, and then replace the role of F in $\boldsymbol{R}$ by $\hat{F}$. This means, using the resampling distribution of $\boldsymbol{R}^* = R(\boldsymbol{X^*}, \hat{F})$ to approximate the sampling distribution of $\boldsymbol{R}$, where $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$ is a random sample obtained from $\hat{F}$, often called resample. Since $\hat{F}$ can be computed based on the observed data, resamples can be obtained by simulation (as many as desired) with the only price of computing time. This makes the resampling distribution of $\boldsymbol{R}^*$ easy to approximate by Monte Carlo just repeating B times the previous scheme.

The books by Efron and Tibishirani (1993) and Davison and Hinkley (1997) give a comprehensive overview about the bootstrap and its use in practically every area of statistical inference. Going back to the estimation error setup and considering $\boldsymbol{R} = \hat{\theta} - \theta$, using the empirical cdf as an estimator of the underlying distribution ($\hat{F}(x) = \hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(X_i \leq x)}$), the bootstrap version of the estimation error becomes $\boldsymbol{R}^* = R(\boldsymbol{X}^*, \hat{F}_n) = \hat{\theta}^* - \theta(\hat{F}_n)$, where $\hat{\theta}^* = t(\boldsymbol{X}^*)$ and $\theta(\hat{F}_n)$ uses the same expression as $\theta(F)$ but replacing F by $\hat{F}_n$. When the estimator $\hat{\theta}$ is a functional statistic $\hat{\theta} = \theta(\hat{F}_n)$, then the bootstrap version of $\boldsymbol{R}$ becomes $\boldsymbol{R}^* = \hat{\theta}^* - \hat{\theta}$. As a consequence, the bootstrap analogue $E^*(\boldsymbol{R}^{*2}) = E^*[(\hat{\theta}^* - \hat{\theta})^2]$ can be used as an estimator of the mean squared error of the estimator, $E(\boldsymbol{R}^2) = E[(\hat{\theta} - \theta)^2]$.

Of course, using the bootstrap to estimate the mean squared error in nonparametric curve estimation is a useful tool. The mean squared error (MSE) at a given point, x, where the curve is aimed to be estimated, can be considered. Other global criteria, as the mean integrated squared error (MISE), can be also used to achieve a measure of the global estimation error as a function of the bandwidth used for nonparametric estimation.

Some bootstrap resampling schemes are described afterwards, depending on the estimator considered for the theoretical distribution. In the independent data setup, we will present the uniform bootstrap, the smoothed bootstrap and the subsampling for independent data.

Uniform bootstrap

1. For each $i = 1, \ldots, n$, draw X_i^* from the empirical distribution function, given by $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}$, i.e., $\mathbb{P}^*(X_i^* = X_j) = \frac{1}{n}$, $j = 1, \ldots, n$.
2. Obtain $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$.
3. Compute the statistic $\boldsymbol{R}^* = \left(\boldsymbol{X}^*, \hat{F}_n\right)$.

Smoothed bootstrap

If F happens to be a continuous distribution, this information must be introduced in the bootstrap algorithm, which proceeds as follows:

1. Considering the sample $(X_1, \ldots, X_n)$ and using a smoothing parameter $h > 0$, compute the Parzen-Rosenblatt density estimator given in (1.1), $\hat{f}_h$.
2. Obtain the bootstrap resamples $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$ from the estimated density $\hat{f}_h$.
3. Compute the bootstrap statistic $\boldsymbol{R}^* = \left(\boldsymbol{X}^*, \hat{F}_h\right)$, where $\hat{F}_h(x) = \int_{-\infty}^{x} \hat{f}_h(t)\, dt = \frac{1}{n}\sum_{i=1}^{n} \mathbb{K}\left(\frac{x - X_i}{h}\right)$ and $\mathbb{K}(u) = \int_{-\infty}^{u} K(v)\, dv$.

Subsampling method for independent data

This method was proposed by Politis and Romano (1994a). As we will mention below, there is also another version of this method for dependent data.

Consider the observations $X_1, \ldots, X_n$ coming from the iid random variables with distribution F. Let $\theta = \theta(F)$ be a parameter, $T_n = T_n(X_1, \ldots, X_n)$ its estimator, and $J_n(\cdot, F)$ the sampling distribution function of $\tau_n(T_n - \theta)$, i.e., $J_n(u, F) = \mathbb{P}\left(\tau_n(T_n - \theta) \leq u\right)$. Fix an integer $b < n$, and define:

$$S_{n,i} = T_b(Y_i), i = 1, 2, \ldots, N,$$

where $Y_1, Y_2, \ldots, Y_N$ are the possible $N = \binom{n}{b}$ subsamples with size b without replacement of the original sample.

The empirical distribution function

$$L_n(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau_b(S_{n,i} - T_n) \leq x\}},$$

will be used as an approximation of the sampling distribution of $\tau_n(T_n - \theta)$.

Focusing now on the dependent data setup in a nonparametric context, we refer to the moving blocks bootstrap, the stationary bootstrap and the subsampling for dependent data as the main bootstrap algorithms in this context.

Moving blocks bootstrap

In absence of an explicit expression to model the time dependence in a time series, the first proposal for general dependence situations comes up: the moving blocks bootstrap (MBB), proposed by Künsch (1989) and Liu and Singh (1992). The MBB algorithm proceeds as follows:

1. Fix a positive integer b (which is the block size), and pick k as the smallest integer greater than or equal to $\frac{n}{b}$.

2. Define the blocks as follows:

$$B_{i,b} = \left(X_i, X_{i+1}, \ldots, X_{i+b-1}\right).$$

In other words, define the blocks B_i, of b consecutive values coming from the sample, beginning with the i-th observation, $\forall i = 1, 2, \ldots, q$ and $q = n - b + 1$.

3. Generate k observations (i.e., generate k blocks), $\xi_1, \xi_2, \ldots, \xi_k$, with equiprobable distribution over the set of possible blocks $\{B_1, B_2, \ldots, B_q\}$. Notice that each ξ_i is a b-dimensional vector $(\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,b})$.

4. Finally, define $\boldsymbol{X}^*$ as the vector which consists of the first n components of

$$(\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,b}, \xi_{2,1}, \xi_{2,2} \ldots, \xi_{2,b}, \ldots, \xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,b}).$$

The ordinary bootstrap is easily obtained from the MBB, just by choosing $b = 1$, and therefore, $k = n$.

Stationary bootstrap

The stationary bootstrap (or SB) was proposed by Politis and Romano (1994b) as a consequence of the lack of stationarity of the MBB. The resampling scheme of this method may be presented in two different versions (SB1 and SB2, repectively) which turn out to be equivalent. Furthermore, the choice of a parameter $p \in [0, 1]$ is necessary.

SB1:

1. Draw $X_1^{*(SB)}$ from $\hat{F}_n$, the empirical distribution function of the sample.

2. Assume we have already drawn $X_1^*, \ldots, X_i^*$ and consider the index j, for which $X_i^* = X_j$. We define a binary auxiliary random variable I_{i+1}^*, such that $P^*\left(I_{i+1}^* = 1\right) = 1 - p$ and $P^*\left(I_{i+1}^* = 0\right) = p$. We assign $X_{i+1}^* = X_{(j \mod n)+1}$ whenever $I_{i+1}^* = 1$ and we use the empirical distribution function for $X_{i+1}^*|_{I_{i+1}^*=0}$.

SB2:

1. Considering the observed sample $(X_1, \ldots, X_n)$, determine the circular blocks as follows:

$$B_{i,b} = (X_i, X_{i+1}, \ldots, X_{i+b-1}), b \in \mathbb{N}, i = 1, 2, \ldots, n,$$

where $X_t = X_{((t-1)\mathrm{mod}n)+1}$, if $t > n$.

2. Draw iid realizations, $L_1, L_2, \ldots$, with geometric distribution of parameter p, i.e.
$$\mathbb{P}(L_1 = m) = p(1-p)^m, m = 1, 2, \ldots$$

3. Obtain random integers, $I_1, I_2, \ldots$, with equiprobable distribution on the set $\{1, 2, \ldots, n\}$.

4. Define $X_1^*, X_2^*, \ldots, X_n^*$ as the first n values obtained when joining the blocks $B_{I_1,L_1}, B_{I_2,L_2}, \ldots$.

Some relevant aspects of the SB algorithm are pointed out in the following. Notice, in the first place, that for *SB2*, the minimum number of necessary blocks, k, coincides with the smallest integer k which implies that $\sum_{i=1}^{k} L_i \geq n$, so that the set of blocks $B_{I_1,L_1}, B_{I_2,L_2}, \ldots, B_{I_k,L_k}$ counts at least with n observations. Additionally, if $p = 1$, the classic bootstrap is obtained.

Secondly, the bootstrap process $\{X_i^*\}$ obtained conditionally on the observed sample happens to be stationary. Moreover, if there are no tied observations, this is a markovian process. Generally, it is a Markov process of order $r + 1$, where $r = \max\{b \in \mathbb{N} / \exists i, j, i \neq j \text{ with } B_{i,b} = B_{j,b}\}$.

In third place, *SB2* can be generalized to other contexts where the distribution of L_i is no longer geometric and the distribution of I_i is not necessarily equiprobable. Hence, MBB could be thought of as a particular case of the generalized SB method, where:

$$\mathbb{P}(L_1 = m) = \begin{cases} 1 & \text{if } m = b \\ 0 & \text{if } m \neq b \end{cases}$$
$$\mathbb{P}(I_1 = j) = \begin{cases} 1/q & \text{if } j = 1, 2, \ldots, q \\ 0 & \text{if } j = q+1, q+2, \ldots, n \end{cases}, \text{ with } q = n - b + 1.$$

However, it is worth pointing out that the choice of the distributions is a delicate issue. Only some particular choices of these distributions make the bootstrap process stationary.

Subsampling method for dependent data

This method has been proposed by Politis and Romano (1994a), and it proceeds in a similar way as compared to the independent data setup.

Consider the observations $X_1, \ldots, X_n$ coming from an α-mixing stochastic process. Let $\theta = \theta(F)$ be a parameter, $T_n = T_n(X_1, \ldots, X_n)$ its estimator, and $J_n(\cdot, F)$ the sampling distribution function of $\tau_n(T_n - \theta)$, i.e., $J_n(u, F) = \mathbb{P}\left(\tau_n(T_n - \theta) \leq u\right)$. Fix an integer $b < n$, and define:

$$S_{n,i} = T_b(B_{i,b}), i = 1, 2, \ldots, N,$$

where $B_{i,b}, i = 1, 2, \ldots, N$ are all the possible blocks of size b and $N = n - b + 1$.

The empirical distribution function to be used as an approximation of the sampling distribution of $\tau_n(T_n - \theta)$ is the following one:

$$L_n(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau_b(S_{n,i} - T_n) \leq x\}}.$$

Sometimes it is possible to work out exact expressions for the bootstrap distribution. This means that the use of Monte Carlo approximation will no longer be necessary, leading to no approximation error and less computer time required. For instance, an exact expression for the bootstrap version of the distribution of the median can be easily obtained.

1.4 Bandwidth selection in nonparametric curve estimation

The current state of the art concerning bandwidth selection for nonparametric density estimation assuming independence has been extensively studied. A great amount of cross-validation methods proposed for iid data (see Rudemo, 1982; Chow et al., 1983; Bowman, 1984; Stone, 1984; Marron, 1985, 1987; Hall, 1983; Hall and Marron, 1987a,b; Scott and Terrell, 1987; Stute, 1992; Feluch and Koronacki, 1992) triggered the development of new techniques for the purpose of bandwidth selection. Apart from the plug-in procedures which have been proposed (see Park and Marron, 1990; Hall and Marron, 1991; Sheather and Jones, 1991 or Jones et al., 1991), there was also room for the development of bootstrap methods (see some remarkable approaches such as Taylor, 1989; Hall, 1990; Faraway and Jhun, 1990; Léger and Romano, 1990; Marron, 1992; and Cao, 1993). Some critical and extensive simulation studies were carried out in this context, we only refer to Park and Marron (1990), Cao et al. (1994) and Jones et al. (1996b) for the sake of brevity.

Nonetheless, when the data are generated by a stochastic process observed in time, they will no longer be iid and the dependence structure plays an important role. Under stationarity, $(X_1, \ldots, X_n)$ is now assumed to be part of a random trajectory of this stochastic process and we focus on the problem of estimating the common marginal density, f, assumed to exist. Of course the estimator in (1.1) can still be used for dependent data. However its asymptotic properties suffer important changes under dependence. The choice of the smoothing parameter is also a very important issue for dependent data, but very few papers have dealt with data-driven bandwidth selectors under stationarity dependence for kernel density estimation. Only a few approaches appeared concerning this issue. Firstly, the classical cross-validation method was modified by Hart and Vieu (1990), in order to produce a more stable procedure under dependence. This method was also theoretically studied by Cox and Kim (1997), who investigated its convergence rate. Hall et al. (1995) also proposed an adaptation of the plug-in method when dependence is assumed, and a bandwidth parameter chosen by minimizing an asymptotic expression for the mean integrated squared error obtained by themselves was established. In addition, Hall et al. (1995) plug-in bandwidth selector has been proposed for any data generating process that is an unknown function of a Gaussian process. Some of these methods have been extended for kernel estimation of other related curves under dependence. This is the case of the plug-in bandwidth selector proposed by Quintela del Río (2007) for nonparametric hazard rate estimation. In addition, a deep simulation study was carried out by Cao et al. (1993) so as to compare well-known bandwidth selectors (most of them proposed for the iid case) in a context of serial dependence.

The bootstrap method has been used once to propose a bandwidth selector for nonparametric kernel density estimation with dependent data. This is the paper by Saavedra and Cao (2001), where a smoothed bootstrap bandwidth selector has been proposed when the data come from a moving average process and the kernel estimator is of convolution-type, adapted to the moving average structure. In this case, the bootstrap mechanism makes an extensive use of the parametric dependence structure as well. The core of that bootstrap resampling plan is just a classical iid smoothed bootstrap for the residuals of the moving average model.

The bootstrap method has been extensively used in the dependent data setup. The interested reader may consult the review papers by Cao (1999) and Kreiss and Paparoditis (2011) for a general background on the subject. When the dependence structure is general (for instance strong mixing, uniformly mixing, or even general stationary processes) bootstrap methods to deal with relevant inference problems exist. Just to mention the most popular approaches we consider the Moving Block Bootstrap (MBB), the Stationary Bootstrap (SB) and the Subsampling method. The stationary bootstrap (see Hwang and Shin, 2012) has been recently used in the context of nonparametric density estimation, but in its original version, i.e. in an unsmooth way. To the best of our knowledge, none of the existing bootstrap methods for a general stationary dependence setting has been extended to a smooth bootstrap method. As a consequence none of them have been used or modified to produce a bootstrap bandwidth selector for nonparametric density estimation under general dependence. This is precisely the gap intended to be partially filled in Chapter 2.

In the following, Section 1.4.1 presents an up-to-date review of the main bandwidth selection methods and justify our choice of the bandwidths to be compared by simulation in Section 2.4 of Chapter 2.

1.4.1 A critical review of smoothing methods for density estimation with dependent data

Leave-$(2l+1)$-out cross-validation method

This method (see Hart and Vieu, 1990) is the adaptation to dependence of the classic leave-one-out cross-validation procedure for iid data proposed by Bowman (1984). Its aim is to minimize the cross-validation function (namely, CV_l) in order to obtain the optimal bandwidth parameter, where CV_l is given by:

$$CV_l(h) = \int \hat{f}^2(x)dx - \frac{2}{n}\sum_{j=1}^{n} \hat{f}_l^j(X_j),$$

being

$$\hat{f}_l^j(x) = \frac{1}{n_l}\sum_{i:|j-i|>l}^{n} \frac{1}{h}K\left(\frac{x-X_i}{h}\right),$$

and l is a sequence of positive integers known as the 'leave-out' sequence. It is also worth mentioning that n_l is chosen as follows:

$$n_l = \frac{\#\{(i,j) : |i-j| > l\}}{n}.$$

Finally, the leave-$(2l+1)$-out cross-validation bandwidth is defined as:

$$h_{CV_l} = \arg\min_{h>0} CV_l(h).$$

The asymptotic optimality of the method for a certain class of l, assuming some regularity conditions on the stationary process, is also stated by Hart and Vieu (1990). The convergence rates of h_{CV_l} are studied by Cox and Kim (1997), where regularity conditions are also assumed. It is worth mentioning that the regularity conditions imposed by Hart and Vieu (1990) demand short-range dependence of the underlying process. This issue is taken up by Hall et al. (1995) in their plug-in procedure (see Section 1.4.1) since they assume conditions of long-range dependence.

Plug-in method

Hall et al. (1995) proposed the plug-in when assuming dependence (see Sheather and Jones, 1991, for the iid case). The key of this method is to minimize, in h, the $AMISE(h)$ expression obtained for dependent data, assuming that f is six times differentiable, and considering $R(f) = \int f(x)^2 dx$, $R(f'') = \int f''(x)^2 dx$, $R(f''') = \int f'''(x)^2 dx$ and $\mu_k = \int z^k K(z) dz$. The AMISE expression is given by:

$$\begin{aligned} AMISE(h) &= \frac{1}{nh}R(K) + \frac{1}{4}h^4\mu_2^2 R(f'') - h^6\frac{1}{24}\mu_2\mu_4 R(f''') \\ &+\frac{1}{n}\left(2\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\int g_i(x,x)dx - R(f)\right), \end{aligned} \tag{1.14}$$

where $g_i(x_1, x_2) = f_i(x_1, x_2) - f(x_1)f(x_2)$, and f_i is the density of (X_j, X_{i+j}).

Minimizing expression (1.14) in h leads to the plug-in bandwidth selector,

$$\hat{h} = \left(\frac{\hat{J}_1}{n}\right)^{1/5} + \hat{J}_2\left(\frac{\hat{J}_1}{n}\right)^{3/5},$$

where $\hat{J}_1$ is an estimator of $J_1 = \dfrac{R(K)}{\mu_2^2 R(f'')}$, and $\hat{J}_2$ is an estimator of $J_2 = \dfrac{\mu_4 R(f''')}{20\mu_2 R(f'')}$.

Thereupon, Hall et al. (1995) propose to replace directly $R(f'')$ and $R(f''')$ by their respective estimators, that is, $\hat{I}_2$ and $\hat{I}_3$, that can be obtained as follows:

$$\hat{I}_k = 2\hat{\theta}_{1k} - \hat{\theta}_{2k}, k = 2, 3$$

where $\hat{\theta}_{1k}$ and $\hat{\theta}_{2k}$ are the respective estimators of $\theta_{1k} = \int \left(\mathbb{E}(\hat{f}_1)\right) f^{(k)}$, $\theta_{2k} = \int \left(\mathbb{E}(\hat{f}_1^{(k)})\right)^2$, $k = 1, 2, 3$, and, $\hat{f}_1$ is the nonparametric Parzen-Rosenblatt density estimator obtained with a kernel K_1 and a bandwidth h_1. The upcoming expressions are obtained for $\hat{\theta}_{1k}$ and $\hat{\theta}_{2k}$:

$$\hat{\theta}_{1k} = 2\left(n(n-1)h_1^{2k+1}\right)^{-1} \sum\sum_{1\le i<j\le n} K_1^{(2k)}\left(\frac{X_i - X_j}{h_1}\right),$$

$$\hat{\theta}_{2k} = 2\left(n(n-1)h_1^{2(k+1)}\right)^{-1} \sum\sum_{1\le i<j\le n} \int K_1^{(k)}\left(\frac{x - X_i}{h_1}\right) K_1^{(k)}\left(\frac{x - X_j}{h_1}\right) dx.$$

Under some moment and regularity conditions on the stationary process, assuming differentiability of the kernel K_1, and choosing h_1 satisfying $n^{-1/(4k+1)} \le h_1 \le 1$; Hall et al. (1995) proved that this plug-in method under dependence is consistent.

1.5 Bootstrap bandwidth selection in nonparametric curve estimation

Over the last three decades, the bootstrap method has been considered so as to define data-driven bandwidth selectors for nonparametric curve estimation. An extensive and updated review of bootstrap methods to select the smoothing parameter for the nonparametric estimation of several curves has been recently carried out by Barbeito and Cao (2019a). We start by reviewing bootstrap bandwidth selection methods for different data generating processes, such as the classical iid setup as well as dependent, censored, length-biased, grouped, missing or directional data, among others. Several curves have also been considered, such as the density, regression, hazard rate, intensity, latency, incidence or distribution functions, among many others.

It is worth mentioning that situations exist where Monte Carlo methods are not needed when using the bootstrap for bandwidth selection purposes. This is precisely the main target in this thesis (see Chapter 2). This idea has also been exploited to find closed expressions for the bootstrap version of criterion functions for some proxy of the real curve estimator (see Chapters 3; and 4).

1.5.1 Data generating processes

Classical independent setting

Let us consider $(X_1, \ldots, X_n)$ a simple random sample coming from a random variable X. The aim of this section is to make an extensive review of the main approaches concerning nonparametric curve estimation within this framework.

Density estimation

The current state of the art considering nonparametric density estimation assuming independence has been extensively studied. As previously mentioned, the first bootstrap bandwidth selection approaches were proposed by Taylor (1989), Hall (1990), Faraway and Jhun (1990), Marron (1992) and Cao (1993). The crucial differences among these versions are how they choose the pilot bandwidth, g, and how they generate the bootstrap resamples. On the one hand, Taylor (1989) established the pilot bandwidth needed to carry out the bootstrap method as $g = h$. In this context, $MISE^*(h)$ tends to zero as h tends to infinity, which leads to an inconsistent global minimum of $MISE^*(h)$. Secondly, Hall (1990) proposed to resample from the empirical distribution function, and consequently, the bias is not taken into account. On the other hand, Faraway and Jhun (1990) proposed a least-square cross-validation selector for g, which turns out to be too small. Falk (1992) investigated the accuracy of bootstrap optimal bandwidth selection in kernel density estimates with respect to the L_2-error. This paper somehow supplements a simulation study by Taylor (1989).

In Marron (1992), the idea of using the smoothed bootstrap algorithm is introduced. Furthermore, the idea of obtaining explicit formulas for the functionals of the distribution is also suggested, so that no simulated resampling is required. This idea is used, afterwards, by Marron and Wand (1992), who proposed an exact expression for the $MISE(h)$ for the kernel estimator of a general normal mixture density. Moreover, this is also the idea followed by Cao (1993) in his bootstrap approach. In this paper, a bootstrap bandwidth selector is defined as the minimizer of an explicit expression obtained for the smoothed bootstrap version of the $MISE(h)$, so that Monte Carlo approximation is no longer needed. Finally, Grund and Polzehl (1997) came up with the idea of proposing a bias corrected bootstrap bandwidth selector by means of minimizing bias corrected smoothed bootstrap estimates of the $MISE(h)$. On the other hand, Chacón and Duong (2018) focused on the study of multivariate density estimation and bandwidth selection.

As previously mentioned, sometimes it is possible to work out exact expressions for the bootstrap version of the aimed function, distribution or parameter. This means that the use of Monte Carlo approximation will no longer be necessary, leading to no approximation error and less computer time required. For instance, an exact expression for the bootstrap version of the distribution of the median can be easily obtained. Furthermore, the bootstrap method can also be used to obtain optimal bandwidth selectors. Focusing on the expression previously introduced for h_{AMISE} (see (1.7)), while the so-called plug-in bandwidth selectors attempt to estimate the unknown constant $\int f''(x)^2 dx$, the bootstrap method focuses on estimating the non-asymptotic exact error $MISE(h)$ in (1.8) and then minimize it in $h > 0$. To do this, we need to use some estimator, $\hat{F}$, for the underlying distribution function, F. Since we know that F has a density f it is very natural to use the cdf corresponding to the KDE defined in (1.1). This brings us to the well known smoothed bootstrap, presented by Silverman and Young (1987), that essentially consists in drawing bootstrap resamples from $\hat{f}_g(x)$ the KDE using a suitable initial (pilot) bandwidth g, as previously mentioned.

Following Cao (1993) we present a bootstrap algorithm to compute the bootstrap MISE bandwidth, h^*_{MISE}.

1. Using the sample $(X_1, X_2, \ldots, X_n)$ and the pilot bandwidth g, compute the KDE $\hat{f}_g$.

2. Draw bootstrap resamples $\boldsymbol{X}^* = (X_1^*, X_2^*, \ldots, X_n^*)$ from the density $\hat{f}_g$.

3. For every $h > 0$, obtain the bootstrap analogue of the KDE:

$$\hat{f}_h^*(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i^*).$$

4. Construct the bootstrap version of $MISE$:

$$MISE^*(h) = \int E^*[(\hat{f}_h^*(x) - \hat{f}_g(x))^2]dx.$$

5. Minimize $MISE^*(h)$ in $h > 0$ and obtain $h_{MISE}^* = \arg\min_{h>0} MISE^*(h)$.

Other bootstrap methods have been proposed for bandwidth selection in KDE. They differ from the one proposed by Cao (1993) in Step 2 above (see Hall, 1990), or in the way the pilot bandwidth is chosen (see Taylor, 1989; Faraway and Jhun, 1990; and Marron, 1992). In particular Taylor (1989) proposed to use $g = h$ in Step 2, but this gives an inconsistent bootstrap bandwidth selector since $MISE^*(h) \to 0$ when $h \to \infty$ for fixed n, which gives $h_{MISE}^* = \infty$.

The choice of the pilot bandwidth proposed by Cao (1993) is related to the optimal one to estimate the curvature of the underlying density, i.e. the one minimizing in g

$$\mathbb{E}\left[\left(\int \hat{f}_g''(x)^2\,dx - \int f''(x)^2\,dx\right)^2\right],$$

which turns out to be asymptotically $g_0 = \left(\left(\int K''(t)^2\,dt\right)/\left(nd_K \int f^{(3)}(x)^2\,dx\right)\right)^{1/7}$.

A Monte Carlo approximation could be used in Step 4 of the algorithm to compute h_{MISE}^* by just drawing a large number, B, of bootstrap resamples, $\boldsymbol{X}^{*1}, \ldots, \boldsymbol{X}^{*B}$ and then replace the bootstrap expectation E^* by the average over the bootstrap resamples. However this is not needed since a closed formula can be found for the

$MISE^*(h)$ in Step 4:

$$\begin{aligned} MISE^*(h) &= \frac{R(K)}{nh} - \frac{1}{n^3}\sum_{i,j=1}^{n}\left[(K_h * K_g) * (K_h * K_g)\right](X_i - X_j) \\ &+ \frac{1}{n^2}\sum_{i,j=1}^{n}\left[(K_h * K_g - K_g) * (K_h * K_g - K_g)\right](X_i - X_j). \end{aligned} \tag{1.15}$$

For the particular choice of a Gaussian kernel (K is the density of a $N(0,1)$) the convolutions in expression (1.15) can be easily worked out resulting in

$$\begin{aligned} MISE^*(h) &= \frac{R(K)}{nh} + \frac{n-1}{n^3}\sum_{i,j=1}^{n}\phi\left(X_i - X_j; 2h^2 + 2g^2\right) \\ &+ \frac{1}{n^2}\sum_{i,j=1}^{n}\left[\phi\left(X_i - X_j; 2g^2\right) - 2\phi\left(X_i - X_j; h^2 + 2g^2\right)\right], \end{aligned}$$

where $\phi(u;\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-u^2}{2\sigma^2}\right)$ is the density of a $N(0,\sigma^2)$ distribution evaluated at u.

A first bloom of bootstrap techniques to select the smoothing parameter during the 1990s gave room to the development of new proposals for the last two decades. In Chacón et al. (2008), a bootstrap procedure quite similar to that of Cao (1993) is proposed. In addition, the problem of selecting the pilot bandwidth, g, is addressed by means of considering it h-dependent. On the other hand, Miecznikowski et al. (2010) proposed a new method for estimating the $MISE(h)$ for kernel density estimators. In certain situations, this method of estimating an optimal bandwidth yields a smaller $MISE(h)$. Finally, Bose and Dutta (2013) addressed the problem which arises when the shape of the reference density differs noticeably from the density itself. In this context, they proposed a bootstrap bandwidth selector where no reference distribution is used.

Some critical and extensive simulation studies were carried out in this context (see Park and Marron, 1990, Cao et al., 1994, Jones et al., 1996b, Jones et al., 1996a; and Heidenreich et al., 2013). According to these papers, the bootstrap approach proposed by Cao (1993) offers similar results to other non-bootstrap methods for selecting the bandwidth parameter. In particular, according to Cao et al. (1994), the best bandwidth selection methods happen to be the plug-in (see Sheather and Jones, 1991), the smooth cross-validation (see Hall et al., 1992) and the bootstrap approach proposed by Cao (1993). However, as can be seen in Heidenreich et al. (2013), recent cross-validation procedures seem to outperform other existing methods. We only refer to Mammen et al. (2011), for instance, for the sake of brevity.

Regression estimation

The use of bootstrap methods to select the smoothing parameter in a regression estimation context is still a current research area in Statistics. Recent papers have dealt with this issue, such as González-Manteiga et al. (2004), in which the wild bootstrap resampling method is applied to the estimated residuals. This bootstrap scheme is used to estimate the mean squared error and select an asymptotically optimal bandwidth parameter for the multidimensional regression local linear estimator. Later on, Martínez-Miranda et al. (2008) proposed a bootstrap local bandwidth selector for estimating nonparametric additive models. The smoothing parameter is based on a bootstrap approximation of the conditional mean squared error, based on applying the wild bootstrap procedure to the estimated residuals. Finally, in Feng and Heiler (2009) a new data-driven bandwidth selector with a double smoothing bias term and data-driven variance estimator is proposed based on bootstrap ideas. As for the exponential family models, the approach by Z̈ychaluk (2014) involved the proposal of a consistent bootstrap bandwidth selector.

An up-to-date review of bandwidth selection methods for kernel regression has been recently published by Köhler et al. (2014). In this paper, existing cross-validation procedures and bootstrap techniques, among others, are profoundly reviewed and then compared through a simulation study. The main conclusions derived from the review state that cross-validation methods seem to outperform their competitors.

Hazard rate estimation

In the context of hazard rate estimation in a classical independent setup, just one approach has dealt with the use of bootstrap techniques to select the bandwidth parameter. In particular, so as to get rid of the randomness of the denominator of the classical hazard rate estimator, in Chapter 3 two proxy estimators of the hazard rate function are proposed (see Barbeito and Cao, 2019b). They are not real estimators, but theoretical approximations. Accordingly, the smoothing parameter is defined as the minimizer of an exact expression for the smoothed bootstrap version of the $MISE(h)$ of some approximations of the kernel hazard rate estimator.

Parameter estimation

Two main proposals have been published in this context. On the one hand, a new bandwidth selector (as well as the kernel order) is defined in Nishiyama (2003) by means of minimizing the bootstrap version of the mean squared error for a plug-in estimator of density weighted averages, which is a nonparametric quantity expressed by expectation of a function of random variables with density weight. On the other hand, the approach presented in Chapter 4 (see Barbeito et al., 2020) is focused on defining a bootstrap bandwidth selector for estimating the mean. In particular, they consider a source population, (X^0, Y^0), as well as a target population, X^1 (Y^1 remains unknown) and they assume that both populations share a common regression function, m. In this context, one task that may well arise is to estimate the expectation of Y^1, using the source population. Indeed, the authors suggest addressing this issue from a nonparametric point of view, so that a proper bandwidth selector is necessary in order to estimate the regression function. To the best of our knowledge, no other proposal concerning this issue has been published yet.

Dependent data

Let us move on now to a dependent data setup. When the data are generated by a stochastic process observed in time, they will no longer be iid and the dependence structure plays an important role. For instance, under stationarity, $(X_1, \ldots, X_n)$ is assumed to be part of a random trajectory of a stochastic process. Our aim is to review the state of the art concerning bootstrap bandwidth selection for nonparametric curve estimation under dependence.

Density estimation

In a density estimation context, two cases may be distinguished. Firstly, when stationarity is assumed. To the best of our knowledge, only the two approaches presented in Chapter 2 have been proposed in this setting. Smoothed versions of stationary bootstrap (SSB) and moving blocks bootstrap (SMBB) are established and used to obtain closed expressions for the $MISE^*(h)$ (see Barbeito and Cao, 2016, 2017; respectively). Afterwards, these explicit formulas are minimized so as to define the SSB and SMBB smoothing parameters. Both closed expressions for the $MISE^*$ happen to be really useful, since Monte Carlo approximation is no longer required.

Secondly, when the dependence is regulated by a moving average process. In Saavedra and Cao (2001), a suitable smoothed bootstrap bandwidth selector is established for some convolution-type kernel estimator of the marginal density function of an MA process.

Regression estimation

Focusing now on regression estimation, the choice of the bandwidth in the local log periodogram regression has been studied by Arteche and Orbe (2009). They proposed a data-driven bandwidth selector based on bootstrap techniques. In particular, they defined the bootstrap bandwidth selector as the minimizer of the bootstrap approximation of the mean squared error (MSE).

Censored and truncated data

In many situations, as for example in medical or in engineering life studies, one may not be able to observe the variable of interest, X. Consider, on the one hand, a censored data setup, i.e., (X, C) random variables, where C stands for the censoring, assumed to be non-negative. Hence, the sample is of the form $(Z_1, \delta_1), \ldots, (Z_n, \delta_n)$, with $Z_i = \min\{X_i, C_i\}$ and $\delta_i = \mathbb{1}_{\{X_i \leq C_i\}}$ (right censoring) or $Z_i = \max\{X_i, C_i\}$ and $\delta_i = \mathbb{1}_{\{X_i \geq C_i\}}$ (left censoring). Furthermore, the variable δ indicates if the lifetime is censored ($\delta = 0$) or not ($\delta = 1$). Let us now move on to a truncated data setup. Let (X, T) be random variables, where T is a non-negative variable which represents the random truncation. Thus, (X, T) is only observed if $X \geq T$ (left truncation) (alternatively, if $X \leq T$ (right truncation)). In the following, an up-to-date review of the current literature concerning bootstrap bandwidth selection with censored and truncated data is carried out.

Density estimation

In a density estimation context, very few papers have dealt with bandwidth selection based on bootstrap techniques. The first one was a proposal by de Uña-Álvarez et al. (1997), in which the Koziol-Green model of proportional censorship is considered. They proposed a bootstrap asymptotic representation of the mean weighted integrated squared error for the kernel density estimator, resulting in the definition of a new bootstrap bandwidth selector in this context. Later on, Sánchez-Sellero et al. (1999) introduced a bootstrap method which enabled them to estimate the MISE for the kernel density estimator with truncated and censored data. By means of minimizing the aforementioned bootstrap version of the MISE, a new smoothing parameter is defined.

More recently, as for the context of survival analysis, Jácome and Cao (2008) came up with the idea of obtaining an asymptotic expression for the bootstrap MISE of the presmoothed density estimator from right-censored data, leading to the introduction of a bootstrap bandwidth selector. They also presented a bandwidth selector based on plug-in ideas. Finally, a new proposal by Subramanian and Bean (2008) appeared, in which they investigated smoothed bootstrap-based bandwidth selection for the sub-density function kernel estimator with right censored data and missing censoring indicators. Indeed, this estimator plays an important role for estimating a survival function. Specifically, the bootstrap smoothing parameter is introduced as the minimizer of certain estimates of the bootstrap version of the MISE.

Hazard rate estimation

As for the kernel hazard rate estimation, an asymptotic representation of the mean weighted integrated squared error in presence of right-censored samples is obtained for different bootstrap resampling methods by González-Manteiga et al. (1996). Therefore, a new bandwidth selector is introduced. These authors also carried out an extensive simulation study, obtaining very satisfactory empirical results in comparison to the cross-validation selector. They developed a binned version for the asymptotic weighted MISE so as to speed up the computation of the bootstrap smoothing parameter.

Latency and incidence estimation in cure models

Bootstrap-based ideas have recently been used in order to introduce new smoothing parameters for both the nonparametric latency and incidence estimators in mixture cure models in survival analysis. In this context, López-Cheda et al. (2017b) presented a bootstrap bandwidth selection method for the kernel nonparametric incidence estimator based on the Beran estimator of the conditional survival function. Afterwards, López-Cheda et al. (2017a) introduced a bootstrap bandwidth selection method for latency estimation. The resulting bootstrap bandwidth selector is obtained by minimizing the bootstrap version of the asymptotic MISE of the latency estimator.

Grouped data

A challenging context for kernel curve estimation is that of grouped data, which appears in systems where the observation is not in continuous time, but monitored in periodical time instants. In this framework, time between observations can be variable, so the scheme is as follows. Let $(X_1, \ldots, X_n)$ be a random sample of X. Consider a set of intervals $[y_{j-1}, y_j)\,; j = 1, 2, \ldots, k$. The j-th interval length is

$l_j = y_j - y_{j-1}$ and its midpoint is $t_j = \dfrac{y_{j-1} + y_j}{2}$. In this context we are not able to observe $(X_1, \ldots, X_n)$ but just the number of observations in the X-sample that lies in every interval. The interval lengths, their endpoints, and the number of intervals k typically depend on the sample size n. The main objective of this section is to make an extensive review of the state of the art of bootstrap oriented bandwidth selection methods for nonparametric curve estimation with grouped data.

Density estimation

In the nonparametric density estimation context, Reyes et al. (2017) proposed a modified version of the classical kernel density estimator, so as to make it computable and therefore useful in practice. However, this modified estimator still depends on a smoothing parameter, which is selected in two ways. Firstly, a plug-in approach is presented and, secondly, a bootstrap-based smoothing parameter is proposed by these authors as well. In addition, the bootstrap algorithm designed by Reyes et al. (2017) to select the bandwidth is based on the computation of an exact expression for the bootstrap MISE, so there is no need of Monte Carlo approximation.

Distribution estimation

Reyes et al. (2019) dealt with nonparametric distribution estimation. In particular, a modification of the classical nonparametric kernel distribution estimator is introduced in this paper. Moreover, Reyes et al. (2019) derived the asymptotic bias, variance and MISE of the new proposal. Since this modified estimator strongly depends on a smoothing parameter, both plug-in and bootstrap ideas are used. Specifically, a bootstrap selector is designed by means of minimizing the exact bootstrap expression of the MISE of the modified distribution estimator. As a consequence, Monte Carlo approximation is not required in order to compute it.

Parameter estimation

From an applied point of view, Cao et al. (2013) addressed the estimation of an environmental index used to predict weed emergence (namely, hydrothermal time or HTT). In this sense, Cao et al. (2013) focused on the problem of predicting weed emergence given some HTT observations from a distribution point of view, proposing a modified version of the classical kernel distribution estimator. The behaviour of this estimator is strongly linked to the correct choice of a smoothing parameter, which is selected via bootstrapping the MISE of the modified distribution estimator.

Spatial processes

We now move on to the spatial processes framework. Let X be a spatial process defined in $\mathbb{R}^2$ and $(X_1, \ldots, X_N)$ a realization of X observed on a bounded region $W \subset \mathbb{R}^2$. Our aim is to mention the different approaches for bootstrap bandwidth selection in nonparametric curve estimation given this data generating process.

Intensity estimation

Considering kernel intensity estimation, only Fuentes-Santos et al. (2016) addressed this issue from a bootstrap point of view. Before this work was published, most authors have considered kernel intensity estimators with scalar bandwidths, which may be very restrictive. Fuentes-Santos et al. (2016) focused on a consistent kernel intensity estimator with unconstrained bandwidth matrix. In addition, a smooth bootstrap algorithm for inhomogeneous spatial point processes was also introduced. It was used to obtain a bootstrap version of the MISE, the asymptotic expression of which is minimized to obtain a bandwidth selector. This may be viewed as a plug-in bandwidth selector.

Parameter estimation

On the other hand, Loh and Jang (2010) focused on selecting the optimal bandwidth for nonparametric estimation of the two point correlation function of a point pattern. In particular, these authors obtain the optimal bandwidths by using bootstrap ideas so as to select the bandwidth which minimizes the integrated squared error (ISE). A nonparametric spatial bootstrap is used to estimate the variance term, while the bias term is estimated with a plug-in approach using a pilot estimator of the two-point correlation function based on a parametric model, the choice of which is very flexible.

Miscellany

Data contaminated by random noise

The deconvolution problem is usually referred to the nonparametric estimation of the density function from a sample of size n which has been contaminated by random noise. Let $Y_1, \ldots, Y_n$ be an iid sample from a random variable Y satisfying $Y = X + Z$, where X is a random variable with density f_X, and Z is a random variable representing the measurement error, with density f_Z. A usual assumption in this context is that the distribution of the error Z is fully known. Considering this framework, Delaigle and Gijbels (2004a) proposed a bootstrap scheme to estimate the optimal bandwidth by means of minimizing the bootstrap version of MISE. Another approach dealing with this issue has also been presented by Delaigle and Gijbels (2004b), in which the authors derived a plug-in smoothing parameter. Afterwards, a simulation study is carried out in order to compare both proposals between them as well as with a cross-validation bandwidth selector.

Length-biased data

In practice, it is common that the sample (although it is supposed to have the same basic characteristics as the population it represents) may not be utterly representative of the population itself. Specifically, bias might be introduced in the sampling scheme. These types of samples are produced when the probability of choosing an observation depends on its value and/or other covariates of interest. A particular case is the length-biased data, where the probability of an observation to be sampled is directly proportional to its value in a simple linear way. Although the problem of density estimation has extensively been studied, only one approach has dealt with bootstrap ideas to select the smoothing parameter. Indeed, Borrajo et al. (2017) addressed this issue by means of proposing two consistent bootstrap methods which are used to select the bandwidth. Furthermore, a closed expression for the bootstrap MISE is obtained, so that Monte Carlo approximation is not required.

Missing data

It is common that some observations in samples are incomplete in practice. For independent observations, classical nonparametric regression estimation methods usually consider complete samples and drop the incomplete observations. Other more complex methods consist in imputing the missing values with an estimate. Raya-Miranda and Martínez-Miranda (2011) provided a simulation study evaluating the effect of some imputation methods with different causes of missingness. However, the proposal by Raya-Miranda and Martínez-Miranda (2011) also deals with the fact that nonparametric additive models with missing data have not been paid special attention to yet. Considering the missing data is in the response variable, these authors have addressed the problem of estimating nonparametrically additive models. Hence, three estimators are presented and a data-driven local bandwidth selector based on the Wild bootstrap approximation of the MSE of the estimators have been proposed.

Directional data

Let Z denote a linear random variable with support $\mathrm{Supp}(Z) \subseteq \mathbb{R}$ and consider $(Z_1, \ldots, Z_n)$ a random sample of Z. Moreover, let $\boldsymbol{X}$ denote a directional random variable with density f. The support of such a variable is the q-dimensional sphere, denoted by $\Omega_q = \left\{\boldsymbol{x} \in \mathbb{R}^{q+1} : x_1^2 + \cdots + x_{q+1}^2 = 1\right\}$ with Lebesgue measure ω_q. Considering this framework, García-Portugués et al. (2013) have dealt with the problem of nonparametric kernel density estimation. Therefore, a new nonparametric kernel denstiy estimator for directional-linear data has been presented by these authors. This approach is based on a product kernel which accounts for the different nature of directional and linear components of the random vector. Furthermore, for some particular distributions, an explicit formula for the MISE is computed both for directional and directional-linear kernel density estimators. In this context, a closed expression for the bootstrap MISE is also found, using it to define a new bootstrap bandwidth selector.

1.5.2 Resampling plans

In every context, the resampling plan used has to replicate somehow the data generating process. As previously mentioned, the smoothed bootstrap by Silverman and Young (1987) was meant to mimic the classical iid framework.

More recent bootstrap resampling plans constructed for other contexts, such as those described in Chapter 2 (see Barbeito and Cao, 2016, 2017) in the dependent data setup, need to take into account the stationarity of the process.

In the context of length-biased data, Borrajo et al. (2017) came up with the idea of drawing the resamples from a modified version of the kernel density estimator, which is indeed constructed so as to take into account the peculiarities of this data generating process. In particular, these authors proposed to draw the bootstrap resamples $(Y_1^*, \ldots, Y_n^*)$ by sampling randomly n times from the estimated density $\hat{f}_{Y,g}(y) = \dfrac{y\hat{f}_g(y)}{\hat{\mu}}$, where $\hat{f}_g(y) = \dfrac{1}{n}\hat{\mu}\sum_{i=1}^{n}\dfrac{1}{Y_i}K_g\left(y - Y_i\right)$ and $\hat{\mu} = \left((1/n)\sum_{i=1}^{n}(1/Y_i)\right)^{-1}$. Another possibility given by Borrajo et al. (2017) is to draw the bootstrap resamples $(Y_1^*, \ldots, Y_n^*)$ by sampling randomly with replacement n times from the classical kernel density estimator with pilot bandwidth $g > 0$.

Considering the censored and truncation data setup, Sánchez-Sellero et al. (1999) proposed to select three different pilot bandwidths g_1, g_2, g_3, one for each variable X, T and C. Then, independent random values of X^*, T^* and C^* are drawn following the distributions $\hat{F}_n * K_{g_1}, \hat{G}_n * K_{g_2}$ and $\hat{M}_n * K_{g_1}$, respectively ($\hat{F}_n, \hat{G}_n$ and $\hat{M}_n$ are the product-limit estimates of X, T and C, respectively). The bootstrap resample $(Y_i^*, T_i^*, \delta_i^*), i = 1, \ldots, n$ will be supplied by those values (X^*, T^*, C^*) verifying $T^* \leq Y = X^* \wedge C^*$, and as many values of (X^*, T^*, C^*) will be drawn as needed to obtain exactly n of them verifying the inequality.

Finally, the approach by Reyes et al. (2017) deals with the grouped data framework. In this context, a modified version of the classical kernel density estimator with pilot bandwidth $g > 0$ from which the bootstrap resamples are drawn is considered. This modification of the classical kernel density estimator takes into account the sample proportions $(w_1, \ldots, w_k)$, where k is the number of intervals.

1.5.3 Computing/approximating the bootstrap criterion function

Applying Monte Carlo approximation is the most straighforward way in order to obtain the bootstrap resamples. As a matter of fact, this approximation method can always be used to draw the bootstrap resamples, regardless of the difficulties brought about by the data generating process. However, Monte Carlo approximation may well be computationally expensive. This fact is strongly noticeable if the function intended to be minimized turns out to be difficult and therefore slow to compute. It is worth singling out the importance of using the same trials and resamples everytime the error criteria chosen to be minimized is evaluated in $h > 0$. Otherwise, this could lead to a direct increase of the variability along different values of h, as well as the computer time required.

As previously mentioned, it is sometimes possible to work out a closed expression for the bootstrap version of the error criterion function intended to be minimized. This is very useful because Monte Carlo approximation would no longer be needed. See, for instance, Cao (1993) approach for density estimation in the classical iid setup; García-Portugués et al. (2013) proposal for density estimation with directional data; bootstrap resampling plans presented in Chapter 2, which are focused on density estimation with dependent data; Borrajo et al. (2017) approach for density estimation with length-biased data; Reyes et al. (2017) for density estimation with grouped data; or Martínez-Miranda et al. (2008) for regression estimation with additive models.

Nevertheless, sometimes in order to obtain a closed expression for the bootstrap version for the error criterion happens to be very difficult. Indeed, it is usual that the nonparametric estimator of a curve consists of a sum of independent random variables or dependent random variables but with easily computable covariances, which are imitated by the bootstrap. However, in some cases such as the classical hazard rate estimator, this does not happen and getting rid of the randomness of the denominator leads to obtaining a simpler stochastic expression. In this sense, the proposals in Chapters 3 and 4 are based on the idea of considering a proxy estimator. It is not a real estimation, but some theoretical approximation. Moreover, as futher details will be given afterwards, this proxy estimator is similar to the real one in the bandwidth region in which both of them attain their minimum.

Bootstrap bandwidth selection for density estimation with dependent data is profoundly studied in Chapter 2, where two new resampling schemes are proposed as well as two new bootstrap bandwidth selectors based on closed expressions of the $MISE^*$. Chapter 3 is focused on the proposal of two smoothing parameter selectors based on closed expressions for the bootstrap MISE of two proxy estimators for the hazard rate function. An approach on how to obtain a bandwidth selector for the estimation of a parameter which depends on the regression function estimation is presented in Chapter 4. The main conclusions are brought together in Chapter 5, as well as some ideas deemed to be a feasible continuation of the approaches given in this thesis. Finally, proof of theorems in Chapters 2, 3 and 4 are included in Appendices A, B and C, respectively.

Chapter 2

Bandwidth selection for density estimation with dependent data

2.1 Introduction

This chapter deals with the widely known problem of data-driven choice of smoothing parameters in nonparametric density estimation, which is indeed an important research area in Statistics. It is about the estimation of the underlying probability density function of a continuous population when only smoothness conditions are assumed. In the iid case there are classical books such as Silverman (1986) and Devroye (1987), among others, that deal with this fundamental statistical problem.

We focus on the problem of estimating the density function in a nonparametric way. Let us consider a random sample, $(X_1, \ldots, X_n)$, coming from a population with density f. Throughout this chapter the kernel density estimator defined in (1.1) is studied. As previously mentioned, while the kernel is responsible of the regularity of the resulting estimate (continuity, differentiability), the bandwidth is very important to control the degree of smoothing applied to the data. In fact, given that this degree of smoothing is a crucial aspect for the quality of the kernel density estimator, the choice of the bandwidth, h, has become a cutting-edge research topic for the past three decades. The objective has been precisely to propose automatic methods for selecting the smoothing parameter. But the vast majority of them are confined to the iid case.

In the following, the adaptation of two already existing methods is established: the modified cross-validation by Stute (1992), when dependence is considered, and the penalized cross-validation proposed by Estévez-Pérez et al. (2002) for hazard rate estimation, using it for density estimation. Section 2.3.1 presents a new resampling plan in this context: the Smoothed Stationary Bootstrap (SSB), proposed in Barbeito and Cao (2016). An explicit expression for the mean integrated squared error of the kernel density under stationary dependence is included in Section 2.3.1. A closed expression for the bootstrap version of MISE is presented in that section and a bootstrap bandwidth selector is proposed. Similarly to the smoothed stationary bootstrap, a smoothed version of the moving blocks bootstrap (see Künsch, 1989; and Liu and Singh, 1992; for the unsmoothed case) is also established in Section 2.3.2 (see Barbeito and Cao, 2017). In addition, a closed expression for the smoothed moving blocks bootstrap version of the mean integrated squared error is presented in that section and a bootstrap bandwidth selector is proposed. The performance of those bandwidth parameters is analyzed via an extensive simulation study in Section 2.4, including some concluding remarks. Finally, Appendix A contains the proof of the results stated in Sections 2.3.1 and 2.3.2.

2.2 Cross-validation procedures

In the following, two cross-validation procedures are proposed. They are all related by the use of a leave-$(2l + 1)$-out device when computing the cross-validation function, which is intended to be minimized. Firstly, the well known leave-$(2l + 1)$-out cross-validation proposed by Hart and Vieu (1990) is studied. Thereupon, two adaptations to our setting of existing methods for hazard rate estimation with dependence and density estimation with iid data are established: the penalized cross-validation and the modified cross-validation.

2.2.1 Penalized cross-validation

The penalized cross-validation (PCV) method was proposed by Estévez-Pérez et al. (2002) for hazard rate estimation under dependence in order to avoid undersmoothed estimations. As a consequence, they stated a penalization for the cross-validation bandwidth, h_{CV_l}. In this chapter, we propose an adaptation to density estimation under dependence. It consists in adding to the value h_{CV_l}, obtained by means of Hart and Vieu (1990) cross-validation procedure, a parameter empirically chosen and somehow related with the estimated autocorrelation.

Specifically, the PCV bandwidth selector is

$$h_{PCV} = h_{CV_l} + \bar{\lambda},$$

where $\bar{\lambda}$ turns out to be

$$\bar{\lambda} = \left(0.8e^{7.9\hat{\rho}-1}\right) n^{-3/10} \frac{h_{CV_l}}{100},$$

and $\hat{\rho}$ is the estimated autocorrelation of order 1. In fact, as $\hat{\rho}$ increases, so does the bandwidth parameter, h_{PCV}. It is worth pointing out that h_{PCV} is obtained in such a way that it is still consistent, according to the consistency of both $\hat{\rho}$ and h_{CV_l}.

2.2.2 Modified cross-validation

An extension to dependent data on the modified cross-validation for iid data (see Stute, 1992) is described now. In the independent case, the aim of this approach consists in avoiding undersmoothed estimations of the density function, considering a way which definitely differs from the usual. In this sense, this approach is based on a finite sample rather than an asymptotic argument, focusing on a statistic whose Hajek projection contains the unknown $\int \hat{f}_h(x) f(x) dx$, which takes part in the $ISE(h)$ expression. This is precisely the function studied by cross-validation procedures.

The main difference between the dependent case (SMCV) and the iid case (MCV) is the idea of leaving out $2l+1$ points (as in Hart and Vieu, 1990) when computing the function intended to be minimized, $SMCV(h)$:

$$
\begin{aligned}
SMCV(h) &= \frac{1}{nh}\int K^2(t)dt \\
&+\frac{1}{n(n-1)h}\sum_{i\neq j}\left[\frac{1}{h}\int K\left(\frac{x-X_i}{h}\right)K\left(\frac{x-X_j}{h}\right)dx\right] \\
&-\frac{1}{nn_l h}\sum_{j=1}^{n}\sum_{i:|j-i|>l}^{n}\left[K\left(\frac{X_i-X_j}{h}\right)-dK''\left(\frac{X_i-X_j}{h}\right)\right],
\end{aligned}
$$

where $d=\frac{1}{2}\int t^2K(t)dt$. Then the SMCV bandwidth selector is

$$h_{SMCV}=\arg\min_{h>0} SMCV(h).$$

As for the consistency of the method, similar results as those for the iid case can be obtained, assuming some regularity and moment conditions on the stochastic process.

2.3 Bootstrap-based procedures

Since the introduction of the bootstrap method by Efron (1979), this technique has been widely used to approximate the sampling distribution of a statistic of interest (see Efron and Tibishirani, 1993, for a deeper insight of the bootstrap method and its applications).

The essential idea to compute a bootstrap bandwidth selector is to obtain the bootstrap version of the mean integrated squared error (namely, MISE) and to find the smoothing parameter that minimizes this bootstrap version, given by

$$
\begin{aligned}
MISE^*(h) &= \mathbb{E}^*\left[\int\left(\hat{f}_h^*(x)-\hat{f}_g(x)\right)^2dx\right] \\
&= \int\left[\mathbb{E}^*\left(\hat{f}_h^*(x)\right)-\hat{f}_g(x)\right]^2dx+\int \mathrm{Var}^*\left(\hat{f}_h^*(x)\right)dx \\
&= B^*(h)+V^*(h),
\end{aligned}
$$

with

$$B^*(h) = \int \left[\mathbb{E}^* \left(\hat{f}_h^*(x)\right) - \hat{f}_g(x)\right]^2 dx, \text{ and}$$

$$V^*(h) = \int \mathrm{Var}^* \left(\hat{f}_h^*(x)\right) dx,$$

where $\mathbb{E}^*$ denotes the expectation (Var^*, the variance) with respect to the bootstrap resample $X_1^*, \ldots, X_n^*$, g is some pilot bandwidth, $\hat{f}_g$ is a kernel density estimation based on the sample $X_1, \ldots, X_n$, and $\hat{f}_h^*$ is the bootstrap version of the kernel density estimator with bandwidth h, based on the resample $X_1^*, \ldots, X_n^*$.

In the iid case, the bootstrap method has been used to produce bandwidth selectors (see, for instance, Cao, 1993). The idea is basically to use the smoothed bootstrap proposed by Silverman and Young (1987) to approximate the MISE of the kernel density estimator.

The main disadvantage of this procedure, however, is the necessity of Monte Carlo approximation whenever the bootstrap distribution of the bootstrap version of the statistic of interest cannot be explicitly computed. Nevertheless, as shown below, when dependence is considered, both smoothed stationary bootstrap (see Barbeito and Cao, 2016) and smoothed moving blocks bootstrap (see Barbeito and Cao, 2017) techniques, need no Monte Carlo in order to implement the bootstrap bandwidths.

2.3.1 Smoothed Stationary Bootstrap

This bootstrap resampling plan was proposed by Barbeito and Cao (2016). In a density estimation context it makes much sense to build a smoothed version of the stationary bootstrap by Politis and Romano (1994b). The key idea is to preserve stationarity of the resampling plan, but producing absolutely continuous resamples, as the regular smoothed bootstrap plan does in the independent data case (see Silverman and Young, 1987).

Let us consider the observed sample, $(X_1, X_2, \ldots, X_n)$, and fix some pilot bandwidth, g. The smoothed stationary bootstrap (SSB) can be presented in two equivalent forms (see Cao, 1999 for the unsmoothed case, SB), proceeding as follows:

SSB1:

1. Draw $X_1^{*(SB)}$ from $\hat{F}_n$, the empirical distribution function of the sample.
2. Define $X_1^* = X_1^{*(SB)} + gU_1^*$, where U_1^* has been drawn with density K and independently from $X_1^{*(SB)}$.
3. Assume we have already drawn $X_1^*, \ldots, X_i^*$ (and, consequently, $X_1^{*(SB)}, \ldots, X_i^{*(SB)}$) and consider the index j, for which $X_i^{*(SB)} = X_j$. We define a binary auxiliary random variable I_{i+1}^*, such that $P^*\left(I_{i+1}^* = 1\right) = 1 - p$ and $P^*\left(I_{i+1}^* = 0\right) = p$. We assign $X_{i+1}^{*(SB)} = X_{(j \mod n)+1}$ whenever $I_{i+1}^* = 1$ and we use the empirical distribution function for $X_{i+1}^{*(SB)}|_{I_{i+1}^*=0}$.
4. Once drawn $X_{i+1}^{*(SB)}$, we define $X_{i+1}^* = X_{i+1}^{*(SB)} + gU_{i+1}^*$, where, again, U_{i+1}^* has been drawn from the density K and independently from $X_{i+1}^{*(SB)}$.

SSB2:

1. Considering the observed sample $(X_1, \ldots, X_n)$, determine the circular blocks as follows:

$$B_{i,b} = (X_i, X_{i+1}, \ldots, X_{i+b-1}), b \in \mathbb{N}, i = 1, 2, \ldots, n,$$

   where $X_t = X_{((t-1)\mathrm{mod} n)+1}$, if $t > n$.

2. Draw iid realizations, $L_1, L_2, \ldots$, with geometric distribution of parameter p, i.e.

$$\mathbb{P}(L_1 = m) = p(1-p)^m, m = 1, 2, \ldots$$

3. Obtain random integers, $I_1, I_2, \ldots$, with equiprobable distribution on the set $\{1, 2, \ldots, n\}$.

4. Define $X_1^{*(SB)}, X_2^{*(SB)}, \ldots, X_n^{*(SB)}$ as the first n values obtained when joining the blocks $B_{I_1,L_1}, B_{I_2,L_2}, \ldots$.

5. Define $X_i^* = X_i^{*(SB)} + gU_i^*$, where U_i^* has been drawn with density K and independently from $X_i^{*(SB)}$, for all $i = 1, 2, \ldots, n$.

This resampling plan is essentially a smoothed version of the so-called stationary bootstrap (SB) and also depends on a parameter $p \in [0, 1]$ to be chosen by the user. The pilot bandwidth g needs also to be chosen.

Remark 1 *Note that in Step 3 of the SSB1 algorithm, if $j = n$, the observation $X_{j \mod n+1}$ would be the first one. If not, $X_{j+1} = X_{n+1}$, which has no sense.*

Remark 2 *The bootstrap random variables $X_1^{*(SB)}, X_2^{*(SB)}, \ldots, X_n^{*(SB)}$ are bootstrap realizations of the regular stationary bootstrap proposed by Politis and Romano (1994b). As a consequence all these bootstrap random variables have distribution $\hat{F}_n$ (see Proposition 1 in Politis and Romano, 1994b).*

Remark 3 *The bootstrap density of each X_i^* is $\hat{f}_g$, the Parzen-Rosenblatt kernel estimator based on the sample $(X_1, X_2, \ldots, X_n)$ and using the bandwidth g. This is an immediate consequence of the fact that the $X_i^{*(SB)}$ are bootstrap discrete random variables with common distribution function $\hat{F}_n$:*

$$\begin{aligned} P^*\left(X_{i+1}^* \leq x\right) &= P^*\left(X_{i+1}^{*(SB)} + gU_{i+1}^* \leq x\right) \\ &= \sum_{j=1}^n P^*\left(X_{i+1}^{*(SB)} + gU_{i+1}^* \leq x|_{X_{i+1}^{*(SB)}=X_j}\right) P^*\left(X_{i+1}^{*(SB)} = X_j\right) \\ &= \frac{1}{n}\sum_{j=1}^n P^*\left(U_{i+1}^* \leq \frac{x - X_j}{g}|_{X_{i+1}^{*(SB)}=X_j}\right) = \frac{1}{n}\sum_{j=1}^n \mathbb{K}\left(\frac{x - X_j}{g}\right), \end{aligned}$$

where $\mathbb{K}$ is the distribution function corresponding to the kernel K (a density function). Differentiating the previous expression with respect to x, we obtain that $\hat{f}_g(x)$ is the bootstrap density of X_i^*:

$$\frac{1}{n}\sum_{j=1}^n \frac{1}{g}\mathbb{K}'\left(\frac{x - X_j}{g}\right) = \frac{1}{ng}\sum_{j=1}^n K\left(\frac{x - X_j}{g}\right) = \hat{f}_g(x).$$

Remark 4 *As in the unsmoothed case, the parameter p gives some freedom to the bootstrap method. The resulting bootstrap plan ranges from the classical smoothed bootstrap mechanism ($p = 1$) (see Silverman and Young, 1987) to a smoothed circular permutation of the sample ($p = 0$).*

Smoothed Stationary Bootstrap bandwidth selection

The SSB resampling plan, previously shown, can be used to produce a bootstrap bandwidth selector for the kernel density estimator under dependence. The proposal is parallel to that of Cao (1993) for independent data, where an exact expression for the bootstrap version of MISE was obtained. A key fact in the independence setup is to find a closed expression for MISE. For this reason we first explore a somewhat generalized explicit expression for the MISE of the kernel density estimation under stationary dependence.

Exact expression for $MISE(h)$

A standard measure of performance of the kernel density estimator is its mean integrated squared error, given by

$$MISE(h) = E\left[\int \left(\hat{f}_h(x) - f(x)\right)^2 dx\right] = B(h) + V(h),$$

where

$$\begin{aligned} B(h) &= \int \left[\mathbb{E}\left(\hat{f}_h(x)\right) - f(x)\right]^2 dx \text{ and} \\ V(h) &= \int Var\left(\hat{f}_h(x)\right) dx. \end{aligned}$$

are the integrated squared bias and integrated variance of the kernel density estimator.

In the iid case it is easy to obtain (see Cao, 1993) exact expressions for $B(h)$ and

$V(h)$:

$$\begin{aligned}
B(h) &= B_0(h),\\
V(h) &= V_0(h),\\
B_0(h) &= \int \left[\mathbb{E}\left(K_h(x-X_1)\right) - f(x)\right]^2 dx = \int \left[\int K_h(x-y) f(y)\, dy - f(x)\right]^2 dx\\
&= \int \left(K_h * f(x) - f(x)\right)^2 dx \text{ and} \qquad (2.1)\\
V_0(h) &= n^{-1}\int Var\left(K_h(x-X_1)\right) dx\\
&= n^{-1}\int \left\{\mathbb{E}\left(K_h(x-X_1)^2\right) - \left[\mathbb{E}\left(K_h(x-X_1)\right)\right]^2\right\} dx\\
&= n^{-1}h^{-2}\int\int K\left(\frac{x-y}{h}\right)^2 f(y)\, dy\, dx - n^{-1}\int\left[\int K_h(x-y) f(y)\, dy\right]^2 dx\\
&= n^{-1}h^{-1}\int\int K(u)^2 f(x-hu)\, du\, dx - n^{-1}\int \left(K_h * f(x)\right)^2 dx\\
&= n^{-1}h^{-1}R(K) - n^{-1}\int \left(K_h * f(x)\right)^2 dx, \qquad (2.2)
\end{aligned}$$

where, as defined in Section 1.1, $*$ stands for convolution and the functional R is defined by $R(\psi) = \int \psi(x)^2 dx$. Expressions (2.1) and (2.2), obtained for the iid case, will be used below when considering dependent data.

The following result shows an explicit expression for the MISE of the kernel density estimator when data are generated from a stationary process. To state this result we need to consider $f_\ell(z|y)$, the conditional density of $X_{\ell+1}$ given that $X_1 = y$, evaluated at the point z. As expected, the MISE expression found in Theorem 1 depends not only on the kernel, the smoothing parameter and the marginal density (f) of the process but also on the conditional densities (f_ℓ) that characterize the type of dependence.

Theorem 1 *If the sample $(X_1, X_2, \ldots, X_n)$ comes from a stationary stochastic process the MISE of $\hat{f}_h$ can be expressed as follows:*

$$MISE(h) = \int \left(K_h * f(x) - f(x)\right)^2 dx + n^{-1}h^{-1}R(K) - \int \left(K_h * f(x)\right)^2 dx$$

$$+2n^{-2}\sum_{\ell=1}^{n-1}(n-\ell)\int\int K_h(x-y)\,f(y)\,K_h * f_\ell(\bullet|y)(x)\;dx\,dy.$$

The proof of Theorem 1 can be found in Appendix A.

Smoothed Stationary Bootstrap version of MISE

As in the iid case, a natural way to compute a MISE-oriented bandwidth selector is to obtain a bootstrap version of MISE and to find the smoothing parameter that minimizes this bootstrap version. In the iid case, with the standard smooth bootstrap plan (that corresponds to SSB with $p = 1$), it is very easy to obtain an exact expression for

$$MISE^*(h) = E^*\left[\int\left(\hat{f}_h^*(x) - \hat{f}_g(x)\right)^2 dx\right], \tag{2.3}$$

the bootstrap version of $MISE(h)$. The resulting expression is (see Cao, 1993):

$$\begin{aligned} MISE^*(h) &= n^{-2}\sum_{i,j=1}^{n}\left[(K_h * K_g - K_g) * (K_h * K_g - K_g)\right](X_i - X_j) \\ &+ n^{-1}h^{-1}R(K) - n^{-3}\sum_{i,j=1}^{n}\left[(K_h * K_g) * (K_h * K_g)\right](X_i - X_j). \end{aligned}$$

This closed expression is very useful since Monte Carlo can be avoided to implement the bootstrap.

Let us consider $(X_1^*, X_2^*, \ldots, X_n^*)$, a smoothed stationary bootstrap resample based on a pilot bandwidth g and the bootstrap version, $\hat{f}_h^*(x)$, of the Parzen-Rosenblatt kernel estimator:

$$\hat{f}_h^*(x) = \frac{1}{n}\sum_{i=1}^{n}K_h(x - X_i^*).$$

The following result presents an exact expression for $MISE^*(h)$ in the general stationary case.

Theorem 2 *If the kernel K is a symmetric density function, then the smoothed* *stationary bootstrap version of MISE admits the following closed expression:*

$$
\begin{aligned}
MISE^{*}(h) &= n^{-2}\sum_{i,j=1}^{n}(K_g * K_g)(X_i - X_j) \\
&- 2n^{-2}\sum_{i,j=1}^{n}(K_h * K_g * K_g)(X_i - X_j) \\
&+ \left[\frac{n-1}{n^3} - 2\frac{1-p-(1-p)^n}{pn^3}\right. \\
&+ \left. 2\frac{(n-1)(1-p)^{n+1} - n(1-p)^n + 1 - p}{p^2 n^4}\right] \\
&\cdot \sum_{i,j=1}^{n}\left[(K_h * K_g) * (K_h * K_g)\right](X_i - X_j) \\
&+ 2n^{-3}\sum_{\ell=1}^{n-1}(n-\ell)(1-p)^{\ell} \\
&\cdot \sum_{k=1}^{n}\left[(K_h * K_g) * (K_h * K_g)\right]\left(X_k - X_{\lceil (k+\ell-1) \mod n\rceil + 1}\right) \\
&+ n^{-1}h^{-1}R(K).
\end{aligned}
\tag{2.4}
$$

Theorem 2 is proven in Appendix A.

A bootstrap bandwidth selector,

$$h_{BOOT} = h^{*}_{MISE} = \arg\min_{h>0} MISE^{*}(h),$$

is defined as the minimizer, in h, of $MISE^{*}(h)$ given by expression (2.4). It is worth mentioning that the exact expression of $MISE^{*}(h)$ is really useful since Monte Carlo approximation is not necessary to compute the bootstrap bandwidth selector.

Remark 5 *It is very easy to check that choosing $p = 1$ (classical smoothed bootstrap) equation (2.4) reduces to (2.3). Hence Theorem 2 extends the result obtained by Cao (1993) to dependent stationary data, using the smoothed stationary bootstrap.*

Remark 6 *As pointed out by Politis and Romano (1994b), the choice of p is of secondary importance for the SB whenever $p \to 0$ and $np \to \infty$. These conditions* *imply that a small value for p is convenient for the SB. In fact when considering bootstrap estimation of the variance of the sample mean for dependent data, Politis and Romano (1994b) have proven that the optimal value for p is* $p = Cn^{-3}$, *for some constant C depending on the spectral density. Similar comments for the choice of p are valid for the SSB. Different values for p were tried in the simulations of Section 2.4 below. The performance of the smoothed stationary bootstrap bandwidth selector was very insensitive to the choice of p, provided that p is small. In practice we chose* $p = \dfrac{1}{2\sqrt{n}}$.

Remark 7 *As investigated by Cao (1993) for the iid case, the choice of g is closely related to optimally estimate the curvature of the underlying density. Thus, a natural criterion to choose g is to minimize*

$$\mathbb{E}\left[\left(\int \hat{f}_g''(x)^2 dx - \int f''(x)^2 dx\right)^2\right].$$

The asymptotically optimal value of g minimizing the previous expression is

$$g_0 = \left(\frac{\int K''(t)^2 dt}{n\left(\int t^2 K(t) dt\right)\left(\int f^{(3)}(x)^2 dx\right)}\right)^{1/7},$$

and the term $\int f^{(3)}(x)^2 dx$ *needs to be estimated to find a practical choice of g. For simplicity we propose to choose g exactly in the same way proposed by Cao (1993) for the iid case.*

Similarly to the SSB, in a density estimation context it makes more sense to build a smoothed version of the moving blocks bootstrap by Künsch (1989) and Liu and Singh (1992). The method is presented in the next section, where a closed formula for its bootstrap version of MISE is also stated.

2.3.2 Smoothed Moving Blocks Bootstrap

The smoothed moving blocks bootstrap, SMBB (see Cao, 1999, for the unsmoothed case, MBB) has been proposed by Barbeito and Cao (2017). It proceeds as follows:

1. Fix the block length, $b \in \mathbb{N}$, and define $k = \min_{\ell \in \mathbb{N}} \ell \geq \frac{n}{b}$.

2. Define:
$$B_{i,b} = (X_i, X_{i+1}, \ldots, X_{i+b-1}).$$

3. Draw $\xi_1, \xi_2, \ldots, \xi_k$ with uniform discrete distribution on $\{B_1, B_2, \ldots, B_q\}$, with $q = n - b + 1$, where $\xi_j = (\xi_{j,1}, \ldots, \xi_{j,b})$ and $j \in \{1, \ldots, k\}$.

4. Define $X_1^{*(MBB)}, \ldots, X_n^{*(MBB)}$ as the first n components of
$$(\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,b}, \xi_{2,1}, \xi_{2,2} \ldots, \xi_{2,b}, \ldots, \xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,b}).$$

5. Define $X_i^* = X_i^{*(MBB)} + gU_i^*$, where U_i^* has been drawn with density K and independently from $X_i^{*(MBB)}$, for all $i = 1, 2, \ldots, n$.

This resampling plan depends on a parameter b, which is the block length, to be chosen by the user. The pilot bandwidth, g, also needs to be chosen. The following result presents an exact expression for the smoothed moving blocks bootstrap version of the $MISE(h)$.

Smoothed Moving Blocks Bootstrap bandwidth selection

The SMBB resampling plan can be used to compute a bootstrap bandwidth selector for the kernel density estimator under dependence, in a similar way as in Section 2.3.1.

Theorem 3 *If the kernel K is a symmetric density function, then the smoothed moving blocks bootstrap version of MISE admits the following closed expression, considering n an integer multiple of b:*

1. If $b < n$,

$$
\begin{aligned}
MISE^*_{SMBB}(h) \;=\; & \frac{R(K)}{nh} + \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \psi(X_i - X_j) \\
& -\frac{2}{n} \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \left[(K_h * K_g) * K_g \right] (X_i - X_j) \\
& +\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[K_g * K_g \right] (X_i - X_j) \\
& -\frac{b-1}{n(n-b+1)^2} \sum_{i=b-1}^{n-b+1} \sum_{j=b}^{n-b+2} \psi(X_i - X_j) \\
& -\frac{1}{nb(n-b+1)^2} \left[\sum_{i=1}^{b-1} \sum_{j=1}^{b-1} \min\{i,j\} \psi(X_i - X_j) \right. \\
& + \sum_{i=1}^{b-1} i \sum_{j=b}^{n-b+1} \psi(X_i - X_j) \\
& + \sum_{i=1}^{b-1} \sum_{j=n-b+2}^{n} \min\{(n-b+i-j+1), i\} \psi(X_i - X_j) \\
& + \sum_{i=b}^{n-b+1} \sum_{j=1}^{b-1} j \psi(X_i - X_j) \\
& + \sum_{i=n-b+2}^{n} \min\{(n-i+1), b\} \sum_{j=b}^{n-b+1} \psi(X_i - X_j) \\
& + \sum_{i=b}^{n-b+1} \sum_{j=n-b+2}^{n} \min\{(n-j+1), b\} \psi(X_i - X_j) \\
& + \sum_{i=n-b+2}^{n} \sum_{j=1}^{b-1} \min\{(n-b+j-i+1), j\} \psi(X_i - X_j) \\
& + b \sum_{i=b}^{n-b+1} \sum_{j=b}^{n-b+1} \psi(X_i - X_j) \\
& \left. + \sum_{i=n-b+2}^{n} \sum_{j=n-b+2}^{n} \left(n+1-\max\{i,j\} \right) \psi(X_i - X_j) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{nb(n-b+1)} \\
&\cdot \sum_{s=1}^{b-1}\sum_{j=1}^{n-s}\left(\min\{j,b-s\}-\max\{1,j+b-n\}+1\right)\psi(X_{j+s}-X_j) \\
&-\frac{2}{nb(n-b+1)^2}\left[\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\left[\sum_{i=k}^{b-2}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)+\sum_{i=n-b+2}^{n-b+k}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)\right.\right. \\
&\left.+\sum_{i=k}^{b-2}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)+\sum_{i=n-b+2}^{n-b+k}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)\right] \\
&+\sum_{k=1}^{b-1}(b-k)\sum_{i=k}^{b-2}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=\ell}^{b-1}\psi(X_i-X_j) \\
&+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j) \\
&\left.+\sum_{k=1}^{b-1}(b-k)\sum_{i=n-b+2}^{n-b+k}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)\right],
\end{aligned}
$$

where $\psi(u)=\left[(K_h*K_g)*(K_h*K_g)\right](u)$ *and:*

$$
a_j=\frac{\min\{j,n-j+1,b\}}{b(n-b+1)}, j=1,2,\ldots,n.
$$

2. *If* $b=n$,

$$
\begin{aligned}
MISE^*_{SMBB}(h) &= \frac{R(K)}{nh}+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(X_i-X_j) \\
&-\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(K_h*K_g)*K_g\right](X_i-X_j) \\
&+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[K_g*K_g\right](X_i-X_j)+\frac{\psi(0)}{n}.
\end{aligned}
$$

Theorem 3 is proven in Appendix A.

A bootstrap bandwidth selector, h^*_{SMBB}, can be defined as the minimizer, in h, of $MISE^*(h)$ given in Theorem 3.

$$h^*_{SMBB} = h^{*SMBB}_{MISE} = \arg\min_{h>0} MISE^*_{SMBB}(h).$$

It is worth mentioning that the exact expression for the $MISE^*_{SMBB}(h)$ is really useful since Monte Carlo approximation is not necessary to compute the bootstrap bandwidth selector.

Remark 8 *The closed expression for $MISE^*_{SMBB}$ in Theorem 3 has been worked out in order to alleviate its computational cost. Alternative closed expressions for $MISE^*_{SMBB}$ could be easier and shorter to write but they would imply a larger computational cost when implemented.*

Remark 9 *If n is not an integer multiple of b, a solution in practice is to consider $\bar{n}$, which is the size of a subsample contained in the original sample of size n. The aim is to reduce the sample size so that $\bar{n}$ is a integer multiple of b, although n is not. The size of the subsample, $\bar{n}$, is defined as follows:*

$$\bar{n} = \left\lfloor \frac{n}{b} \right\rfloor \cdot b.$$

The key point is to consider all the subsamples resulting from subtracting $n - \bar{n}$ consecutive observations from the original sample (that is, $n - \bar{n} + 1$ possibilities). Then, obtain $MISE^*_{SMBB}$ for each subsample of size $\bar{n}$ and compute its average, which happens to be the new criterion function. Finally, h^*_{SMBB} is given by:

$$h^*_{SMBB} = \arg\min_{h>0} \left\{ \frac{1}{n - \bar{n} + 1} \sum_{i=1}^{n-\bar{n}+1} MISE^*_{SMBB,i}(h) \right\}.$$

Remark 10 *It is worth pointing out that it is not advisable to take p too large (or b too small), since we would not be taking into account the dependence of the data.*

Remark 11 *An extension of these ideas to the multivariate data context would be quite immediate for the density function. However, more computational cost is expected, since a bandwidth matrix, H, would be considered, as well as a pilot bandwidth matrix, G.*

Remark 12 *It is worth mentioning that in order to proceed with the bootstrap methods SSB and SMBB, the only condition needed is stationarity. However, it remains to be checked if more dependence conditions are required when proving the good behaviour of both bandwidth selectors theoretically.*

Remark 13 *Although this chapter is focused on bandwidth selection for nonparametric density estimation with dependent data, the bootstrap algorithms presented could be of interest for confidence band construction or for establishing goodness of fit testing procedures. It is important to notice that bootstrap versions of error criteria for bandwidth selection can be written in a closed form because they are bootstrap expectations of rather simple expressions. However, there is no hope to find closed expressions for the bootstrap resampling distribution itself, which implies no computational advantage for constructing confidence intervals or bands using the bootstrap.*

2.4 Simulations

2.4.1 General description of the study

In order to analyze the performance of the smoothed stationary bootstrap and smoothed moving blocks bootstrap bandwidth selectors, a simulation study is carried out, using the free software `R` and package `kedd` (see Guidoum, 2015) for that purpose. In particular, package `kedd` has been used to compute the derivatives of the density function required for implementing the plug-in bandwidth selector. The practical behaviour of the smoothing parameters h^*_{SSB} and h^*_{SMBB} are empirically compared with the aforementioned plug-in and cross validation methods, namely h_{CV_l}, h_{PCV}, h_{SMCV} and h_{PI}. Subsequently, we will consider seven different populations (six of them already used by Cao et al., 1993) so as to show the empirical results of every bandwidth in different situations:

Model 1: The sample is drawn from an $AR(2)$ model given by $X_t = -0.9X_{t-1} - 0.2X_{t-2} + a_t$, where a_t are iid random variables with common distribution $N(0,1)$. The marginal distribution is $X_t \overset{d}{=} N(0, 0.42)$.

Model 2: A $MA(2)$ model given by $X_t = a_t - 0.9a_{t-1} + 0.2a_{t-2}$ is considered, where the innovations a_t are iid standard normal random variables. The marginal distribution is $X_t \stackrel{d}{=} N(0, 1.85)$.

Model 3: An $AR(1)$ model: $X_t = \phi X_{t-1} + (1-\phi^2)^{1/2} a_t$. Here a_t are iid random variables with common standard normal distribution. The autocorrelation was set to the values $\phi = \pm 0.3, \pm 0.6, \pm 0.9$. The marginal distribution is a standard normal.

Model 4: The time series is generated from an $AR(1)$ model given by $X_t = \phi X_{t-1} + a_t$. In this case, the distribution of a_t has exponential structure:

$$\mathbb{P}(I_t = 1) = \phi, \mathbb{P}(I_t = 2) = 1 - \phi, \text{ with}$$
$$a_t|_{I_t=1} \stackrel{d}{=} 0 \text{ (constant)}, a_t|_{I_t=2} \stackrel{d}{=} \exp(1).$$

The values of ϕ chosen were $\phi = 0, 0.3, 0.6, 0.9$. The marginal distribution is $X_t \stackrel{d}{=} \exp(1)$.

Model 5: An $AR(1)$ model with a double-exponential (abbreviated as Dexp) structure: $X_t = \phi X_{t-1} + a_t$, which means that the innovations have to be drawn from the following distribution:

$$\mathbb{P}(I_t = 1) = \phi^2, \mathbb{P}(I_t = 2) = 1 - \phi^2, \text{ with}$$
$$a_t|_{I_t=1} \stackrel{d}{=} 0 \text{ (constant)}, a_t|_{I_t=2} \stackrel{d}{=} \text{Dexp}(1),$$

The values of ϕ used were $\phi = \pm 0.3, \pm 0.6, \pm 0.9$. The marginal distribution is $X_t \stackrel{d}{=} \text{Dexp}(1)$.

Model 6: A mixture of two normal densities, with probability 1/2 each, associated to the model:

$$X_t = \begin{cases} X_t^{(1)} & \text{with probability } 1/2 \\ X_t^{(2)} & \text{with probability } 1/2 \end{cases},$$

where $X_t^{(j)} = (-1)^{j+1} + 0.5X_{t-1}^{(j)} + a_t^{(j)}$ with $j = 1, 2$, $\forall t \in \mathbb{Z}$, and $a_t^{(j)} \stackrel{d}{=} N(0, 0.6)$. The marginal distribution is a normal mixture $X_t \stackrel{d}{=} \frac{1}{2}N(2, 0.8) + \frac{1}{2}N(-2, 0.8)$.

Model 7: A mixture of three normal densities (Model 9 of Marron and Wand, 1992) inducing dependence using a Markovian regime change:

$$X_t = \begin{cases} X_t^{(1)} & \text{with probability } 9/20 \\ X_t^{(2)} & \text{with probability } 9/20 \\ X_t^{(3)} & \text{with probability } 1/10 \end{cases},$$

where $X_t^{(j)} = 0.9X_{t-1}^{(j)} + a_t^{(j)}$ with $j = 1,2,3$, $\forall t \in \mathbb{Z}$, and $a_t^{(1)} \stackrel{d}{=} N(-0.12, 0.0684)$, $a_t^{(2)} \stackrel{d}{=} N(0.12, 0.0684)$, $a_t^{(3)} \stackrel{d}{=} N(0, 0.011875)$. The marginal distribution is a normal mixture $X_t \stackrel{d}{=} \frac{9}{20}N\left(-\frac{6}{5}, \frac{9}{25}\right) + \frac{9}{20}N\left(\frac{6}{5}, \frac{9}{25}\right) + \frac{1}{10}N\left(0, \frac{1}{16}\right)$.

The transition matrix used is given by:

$$T = \begin{pmatrix} 0.90790619 & 0.08152211 & 0.0105717 \\ 0.08152211 & 0.90790619 & 0.0105717 \\ 0.04757265 & 0.04757265 & 0.9048547 \end{pmatrix}.$$

This produces a first order autocorrelation of $\phi = 0.6144$.

For every model, 1000 random samples of size $n = 100$ were drawn. The Gaussian kernel is used to compute the Parzen-Rosenblatt estimator. For the three cross-validation bandwidths the value of l was $l = 5$. The parameter p used in the SSB was $p = \frac{1}{2\sqrt{n}}$ and the parameter b used in the SMBB was $b = 2\sqrt{n}$, while the pilot bandwidth, g, has been chosen as in Cao (1993) in both cases. The pilot bandwidth used in the plug-in method was $h_1 = C\tilde{h}_1 n^{4/45}$, so that h_1 has order $n^{-1/9}$ (as demonstrated by Cao, 1993, to be the optimal rate for kernel density estimator), where $C = 1.12$ and $\tilde{h}_1$ is the bandwidth chosen as in Sheather and Jones (1991) for iid data. Certainly, the selection of p, b and C does not have a great impact on the empirical behaviour of the bandwidth selector, h. An additional simulation study to discuss the choice of p (SSB) and b (SMBB) has been carried out as well. This is shown for Model 3, $\phi = -0.9$ by considering the following sets for $p \in \mathcal{P} = \{0.5, 0.4, 0.3, 0.25, 0.2, 0.1, 0.075, 0.05, 0.04, 0.02, 0.01, 0.001\}$ and $b = \{2, 5, 10, 25, 50, 100\}$. The bandwidth selectors h_{SSB}^*, h_{SMBB}^*, h_{CV_l} and h_{SMCV} are the minimizers, in h, of four empirical functions. Since these minimizers do not have explicit expressions, a numerical method is used to approximate them. The algorithm proceeds as follows:

Step 1: Let us consider a set of 5 equally spaced values of h in the interval $[0.01, 10]$.

Step 2: For each method, a bandwidth h is chosen among the five given in the preceeding step, by minimizing the objective function ($MISE^*_{SSB}$, $MISE^*_{SMBB}$, CV_l or $SMCV$). We denote it by h_{OPT_1}.

Step 3: Among the set of 5 bandwidth parameters defined in Step 1, we consider the previous and the next one to h_{OPT_1}. If h_{OPT_1} is the smallest (largest) bandwidth in the grid, then h_{OPT_1} is used instead of the previous (next) value of h_{OPT_1} in the grid.

Step 4: A set of 5 equally spaced values of h is constructed within the interval whose endpoints are the two values selected in Step 3.

Step 5: Finally, Steps 2-4 are repeated 10 times, retaining the optimal bandwidth selector in the last stage.

It is worth mentioning that, in order to avoid oversmoothing of the SMCV procedure, h_{SMCV} is considered as the smallest h for which $SMCV(h)$ attains a local minimum, not its global one.

The six bandwidth selectors are compared in terms of how close they are to the optimal MISE bandwidth and also in terms of the error committed when using each one of them. Thus, using the 1000 samples, the following expressions were approximated by simulation:

$$\log\left(\frac{\hat{h}}{h_{MISE}}\right) \text{ and } \tag{2.5}$$

$$\log\left(\frac{MISE(\hat{h})}{MISE(h_{MISE})}\right), \tag{2.6}$$

where $\hat{h} = h_{CV_l}, h_{PCV}, h_{SMCV}, h_{PI}, h^*_{SSB}, h^*_{SMBB}$, and h_{MISE} is the smoothing parameter which minimizes the error criterion, $MISE(h)$.

An additional simulation study has been carried out in order to select automatically the parameter p in the SSB algorithm. Consider a set of parameters p, namely $\mathcal{P}$ and compute the SSB bandwidth selector, $h^*_{SSB}(p)$. This procedure to select automatically p assumes that the data comes from the following autorregresive of order 1 process, given by:

$$X_t = \phi X_{t-1} + a_t, \text{ where } a_t \stackrel{d}{=} N\left(\mu, \sigma^2\right). \tag{2.7}$$

Then, obtain $\widehat{MISE}_{AR(1)}\left(h^*_{SSB}(p)\right)$, where $\widehat{MISE}_{AR(1)}$ stands for the MISE function of an estimated model given in (2.7). Finally, define p as

$$p^* := \arg\min_{p \in \mathcal{P}} \widehat{MISE}_{AR(1)}\left(h^*_{SSB}(p)\right).$$

However, as can be seen in Table 2.1, different values for p do not have a great impact on the behaviour on the bandwidth selector.

2.4.2 Discussion and results

The practical behaviour of the SSB and SMBB bandwidth selectors may well vary with a different choice of p and b, respectively. However, as it can be seen in Figures 2.9 and 2.10, as well as Table 2.1, the empirical performance of both methods is not strongly affected by the choice of these parameters. Nevertheless, it is evident that the best performance is obtained when choosing p close to 0.05 and $b = 10$.

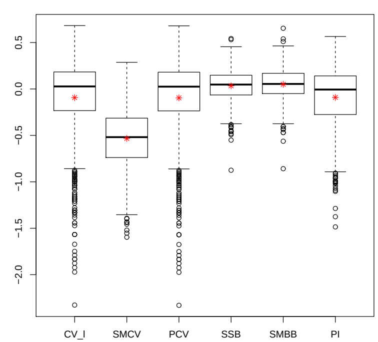
Figures 2.1-2.8 show boxplots with the results obtained for expressions (2.5) and (2.6) approximated by simulation. The simulation results remarkably show that smoothing parameters h^*_{SSB} and h^*_{SMBB} display a similar performance, actually the best ones, even for heavy dependence. For both h^*_{SSB} and h^*_{SMBB}, expression (2.5), shown in Figures 2.1-2.8 (left side) exhibit that the median of h^*_{SSB} and h^*_{SMBB} is approximately h_{MISE}. Moreover, h^*_{SSB} and h^*_{SMBB} present less variance than the three cross-validation smoothing parameters.

The three cross-validation bandwidths exhibit a worse behaviour than the bootstrap selectors. The bandwidth h_{CV_l} tends to severely undersmooth for some trials in almost every model. This is not satisfactorily corrected by h_{PCV}, which, paradoxically, sometimes shows a general tendency to oversmoothing (see Figures 2.4e and 2.8e). Although h_{SMCV} typically corrects the extreme undersmoothing cases of h_{CV_l} and h_{PCV}, on the average it tends to give smaller values than the target h_{MISE}. The undersmoothing feature is also present in h_{PI}, which, in turn, for some models, presents a remarkable proportion of trials with severe oversmoothing. The PCV bandwidth is undoubtedly the worst. On the one hand, expression (2.5) brings out that the use of h_{PCV} results on an oversmoothed density function estimation in almost every case. We can also conclude from Figures 2.1-2.8 (right side) that $MISE(h_{PCV})$ is clearly larger than the rest in all cases.

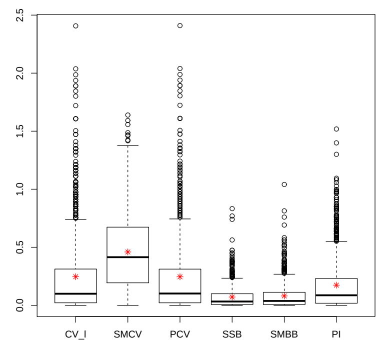
Although the behaviour of h_{CV_l} and h_{SMCV} is not spectacular they are usually placed in an intermediate position of the ranking. They sometimes produce undersmoothed density function estimations, as can be seen in Figures 2.1 - 2.8. It is also noticeable that the error criterion $MISE(h_{CV_l})$ and $MISE(h_{SMCV})$ is wider than $MISE(h^*_{SSB}), MISE(h^*_{SMBB})$ and $MISE(h_{PI})$ in all cases.

All in all, it is clear in Figures 2.1-2.8 (right side) that the two bootstrap-based bandwidth selectors present the best results in terms of MISE. It is also worth pointing out that h^*_{SMBB} actually performs slightly better than its main competitor, h^*_{SSB}, when there exists heavy and positive correlation (specifically, $\phi = 0.9$), as can be noticed in Figures 2.4f and 2.8f. Additionally, even for moderate autocorrelation, it can be easily checked by looking at Figure 2.2d that, for Model 7, the empirical behaviour presented by h^*_{SMBB} is by far the best (in terms of MISE). However, Model 7 is in itself difficult to analyze in a nonparametric way, due to the fact that its underlying theoretical density is trimodal (see Figure 2.11a). Moreover, it is also worth mentioning that Model 4, $\phi = 0$, produces samples of iid observations from an exp(1) density, which presents a discontinuity at $x = 0$. This may well be the reason that causes that the performance of the new bootstrap selectors is not very good in this case (see Figures 2.5a and 2.5b).

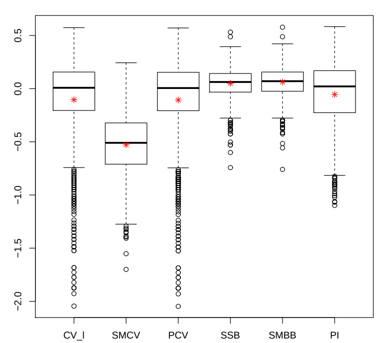
Finally, it is worth mentioning that Model 6 is a mixture of two normal densities with two dissimilar modes (see Figure 2.11a). In this situation, a large h would be needed to estimate accurately the area between the two modes. It certainly leads to a growth of the empirical bias (see Figure 2.2a) when trying to estimate the modes with the same bandwidth selector as the one used for the rest of the curve.



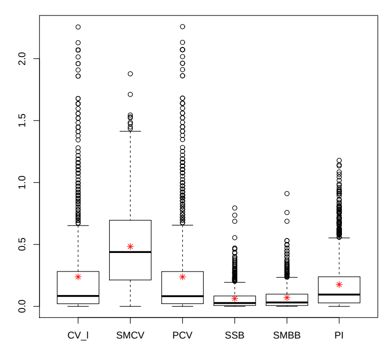
(a) Model 1



(b) Model 1

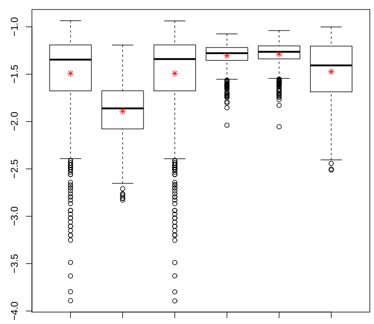


(c) Model 2

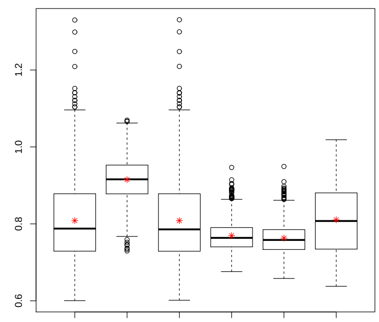


(d) Model 2

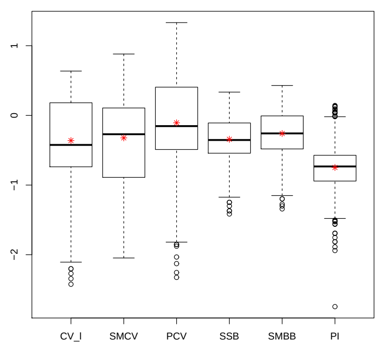
Figure 2.1: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Models 1 and 2, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



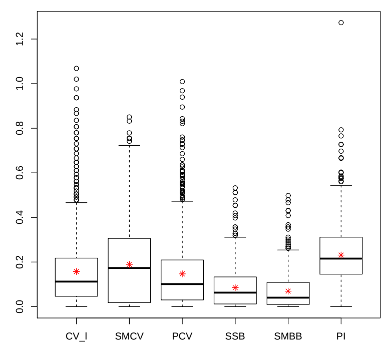
(a) Model 6



(b) Model 6

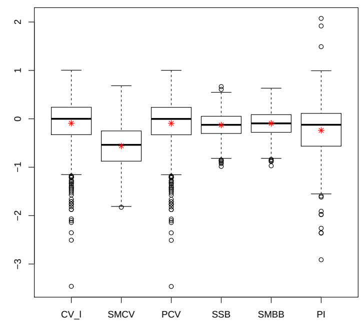


(c) Model 7

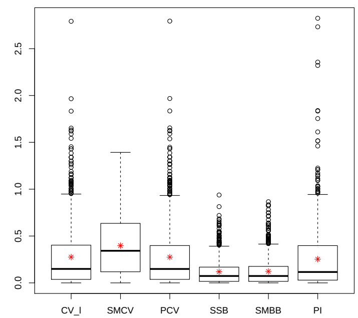


(d) Model 7

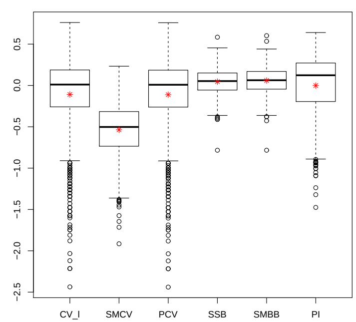
Figure 2.2: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Models 6 and 7, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



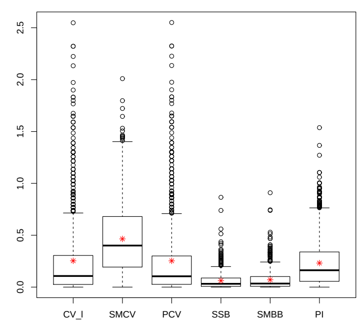
(a) Model 3, $\phi = -0.9$



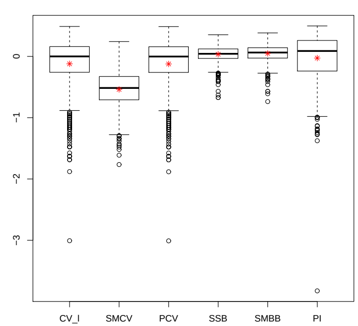
(b) Model 3, $\phi = -0.9$



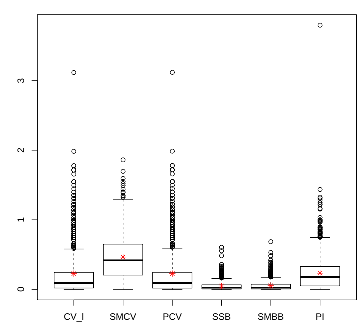
(c) Model 3, $\phi = -0.6$



(d) Model 3, $\phi = -0.6$

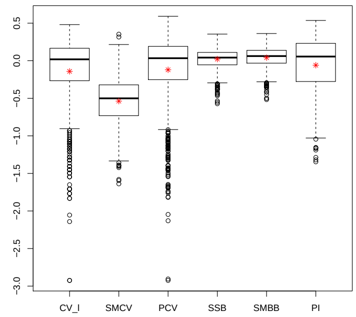


(e) Model 3, $\phi = -0.3$

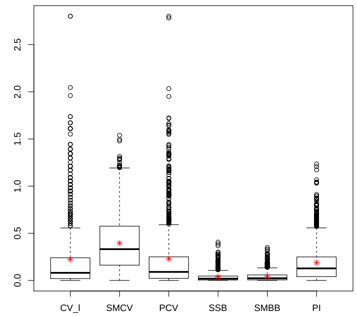


(f) Model 3, $\phi = -0.3$

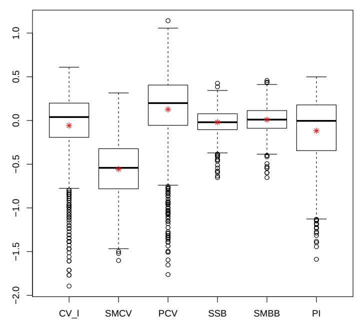
Figure 2.3: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 3 with autocorrelation $\phi = -0.9, -0.6, -0.3$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



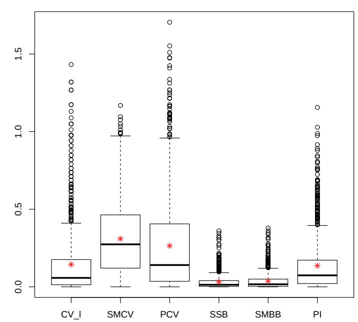
(a) Model 3, $\phi = 0.3$



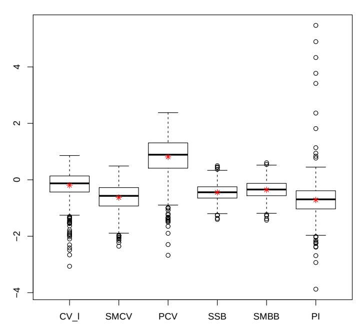
(b) Model 3, $\phi = 0.3$



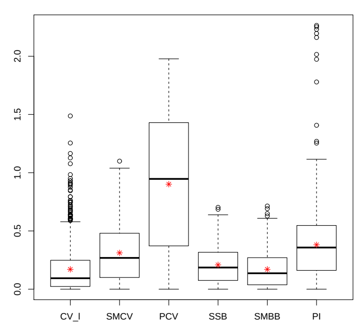
(c) Model 3, $\phi = 0.6$



(d) Model 3, $\phi = 0.6$

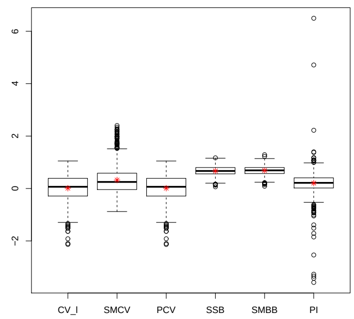


(e) Model 3, $\phi = 0.9$

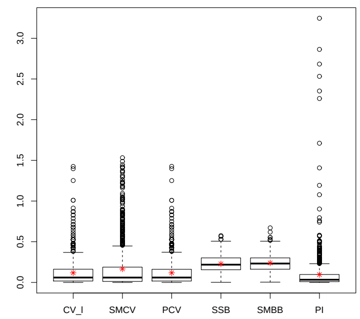


(f) Model 3, $\phi = 0.9$

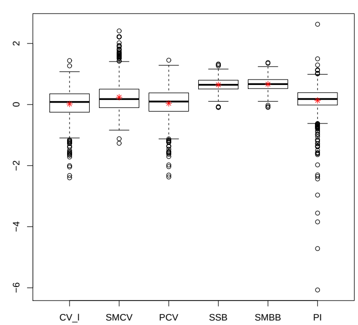
Figure 2.4: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 3 with autocorrelation $\phi = 0.3, 0.6, 0.9$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



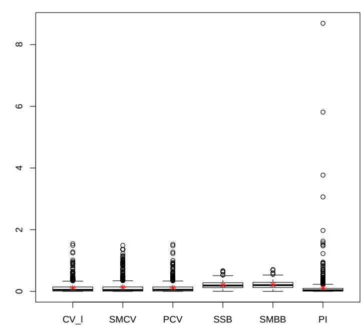
(a) Model 4, $\phi = 0$



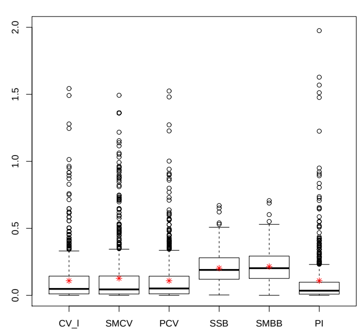
(b) Model 4, $\phi = 0$



(c) Model 4, $\phi = 0.3$

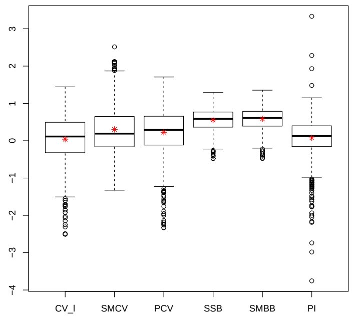


(d) Model 4, $\phi = 0.3$

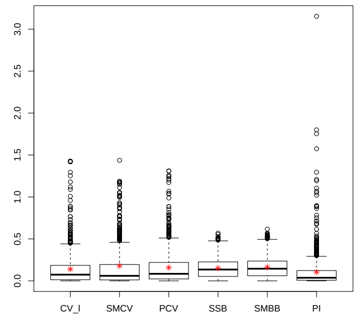


(e) Model 4, $\phi = 0.3$

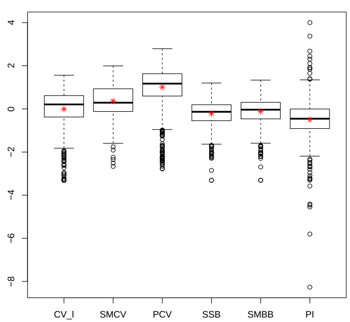
Figure 2.5: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 4 with autocorrelation $\phi = 0, 0.3$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). Plot 2.5e is just a zoom of plot 2.5d in the vertical axis. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



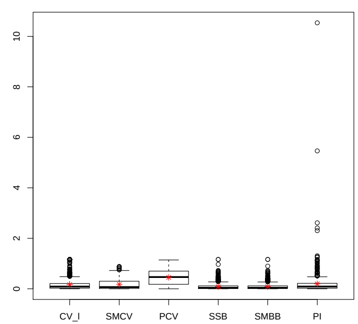
(a) Model 4, $\phi = 0.6$



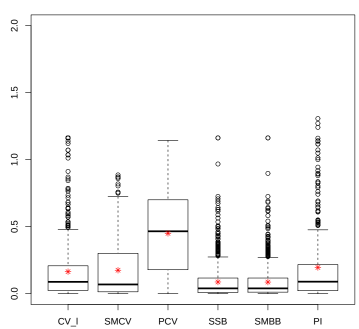
(b) Model 4, $\phi = 0.6$



(c) Model 4, $\phi = 0.9$

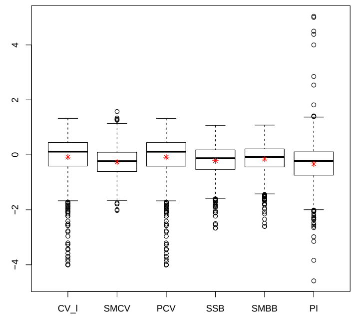


(d) Model 4, $\phi = 0.9$

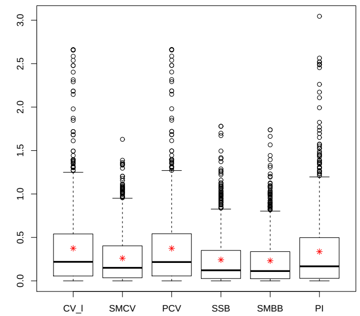


(e) Model 4, $\phi = 0.9$

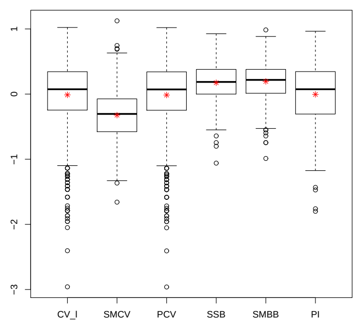
Figure 2.6: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 4 with autocorrelation $\phi = 0.6, 0.9$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). Plot 2.6e is just a zoom of plot 2.6d in the vertical axis. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



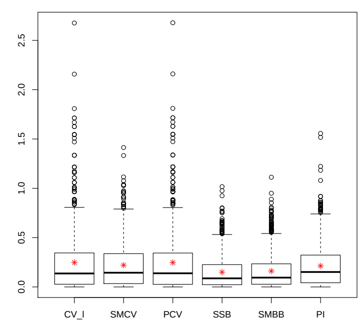
(a) Model 5, $\phi = -0.9$



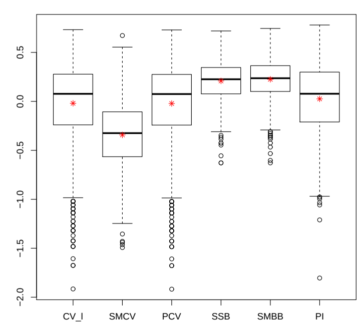
(b) Model 5, $\phi = -0.9$



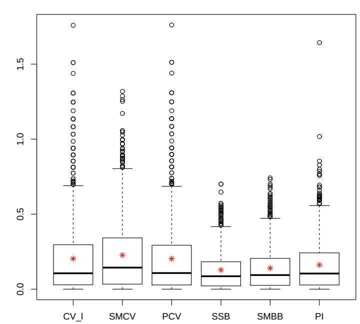
(c) Model 5, $\phi = -0.6$



(d) Model 5, $\phi = -0.6$

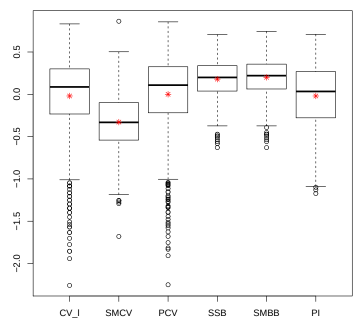


(e) Model 5, $\phi = -0.3$

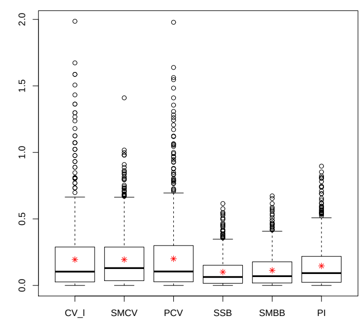


(f) Model 5, $\phi = -0.3$

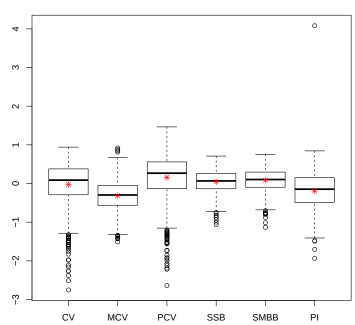
Figure 2.7: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 5 with autocorrelation $\phi = -0.9, -0.6, -0.3$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



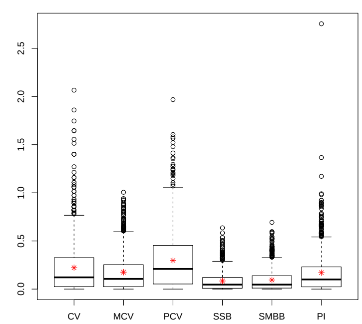
(a) Model 5, $\phi = 0.3$



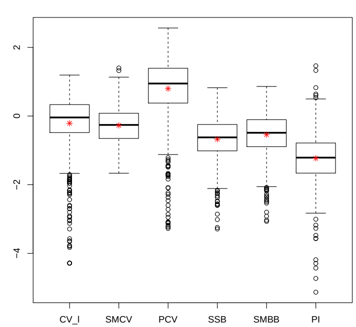
(b) Model 5, $\phi = 0.3$



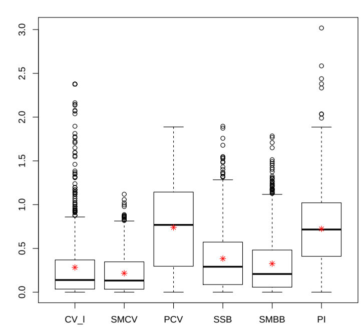
(c) Model 5, $\phi = 0.6$



(d) Model 5, $\phi = 0.6$

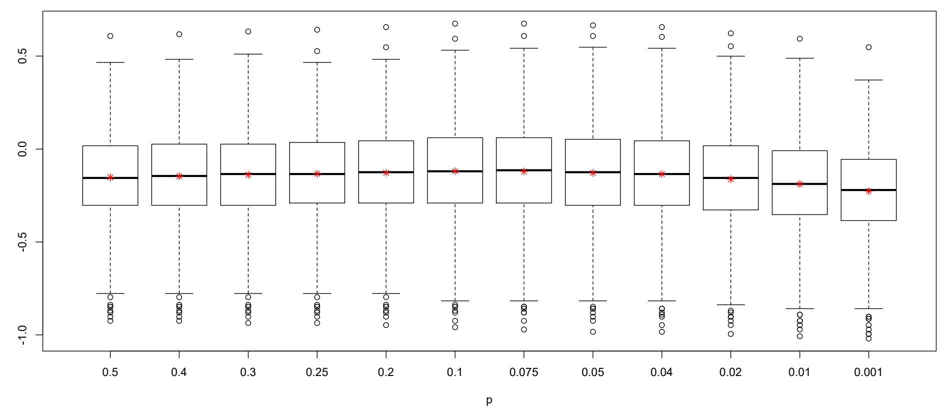


(e) Model 5, $\phi = 0.9$



(f) Model 5, $\phi = 0.9$

Figure 2.8: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (right side) for Model 5 with autocorrelation $\phi = 0.3, 0.6, 0.9$, where $\hat{h} = h_{CV_l}$ (first box), h_{SMCV} (second box), h_{PCV} (third box), h^*_{SSB} (fourth box), h^*_{SMBB} (fifth box) and h_{PI} (sixth box). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.



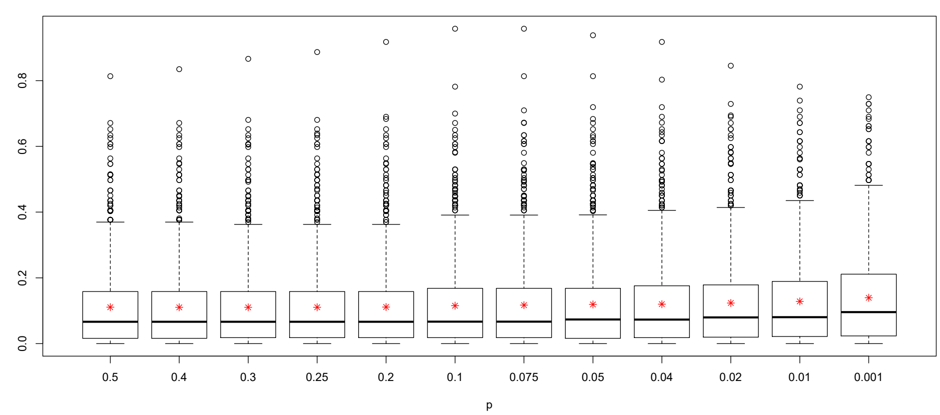


Figure 2.9: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (top row) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (bottom row) for Model 3 with $\phi = -0.9$ where $\hat{h} = h^*_{SSB}$, considering $p \in \mathcal{P} = \{0.5, 0.4, 0.3, 0.25, 0.2, 0.1, 0.075, 0.05, 0.04, 0.02, 0.01, 0.001\}$. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.

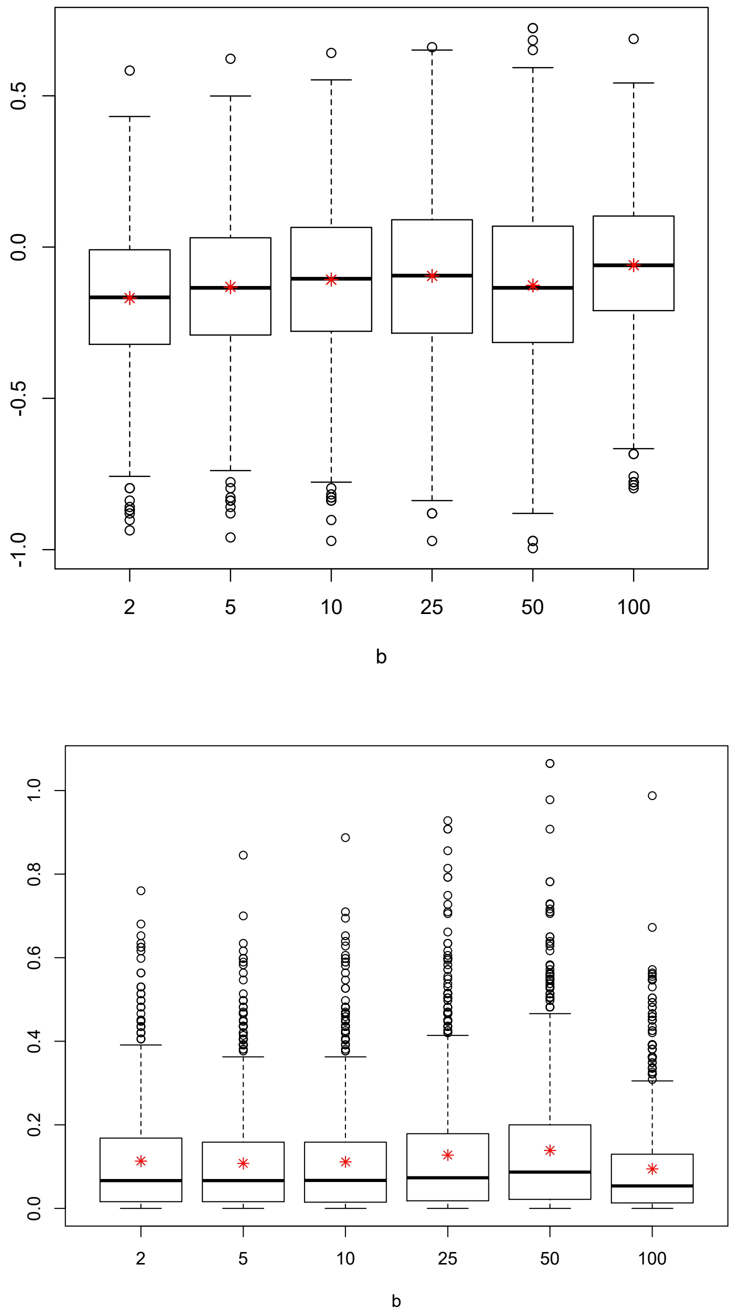


Figure 2.10: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (top row) and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$ (bottom row) for Model 3 with $\phi = -0.9$ where $\hat{h} = h^*_{SMBB}$, considering $b \in \{2, 4, 5, 10, 20, 25, 50, 100\}$. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $\log\left(MISE(\hat{h})/MISE(h_{MISE})\right)$.

	Model 3, $\phi = -0.9$		Model 3, $\phi = 0.9$	
p	h^*_{SSB}	$MISE(h^*_{SSB})$	h^*_{SSB}	$MISE(h^*_{SSB})$
0.5	0.4782	0.0111	0.4362	0.0377
0.4	0.4812	0.0111	0.4464	0.0372
0.3	0.4855	0.0111	0.4596	0.0367
0.25	0.4882	0.0111	0.4677	0.0365
0.2	0.4811	0.0111	0.4771	0.0362
0.1	0.4962	0.0112	0.4964	0.0358
0.075	0.4958	0.0112	0.4981	0.0358
0.05	0.4921	0.0112	0.4928	0.0359
0.04	0.4888	0.0113	0.4861	0.0362
0.02	0.4754	0.0113	0.4554	0.0373
0.01	0.4625	0.0114	0.4242	0.0386
0.001	0.4445	0.0115	0.3829	0.0406
p^*	0.4893	0.0112	0.4841	0.0359

Table 2.1: $MISE(h^*_{SSB})$ for Model 3, $\phi = -0.9$ and $\phi = 0.9$ considering different values of p.

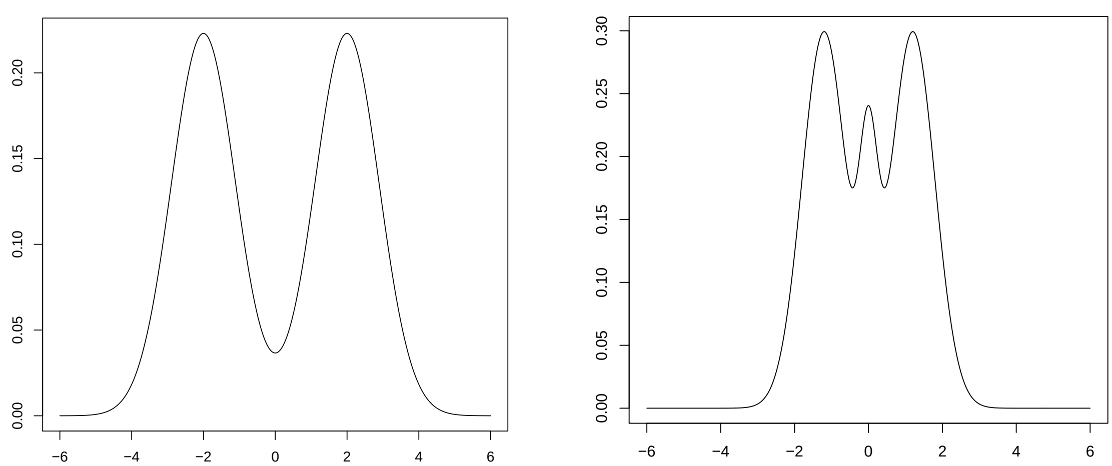


Figure 2.11: Marginal density of X_t for Models 6 (left side) and 7 (right side).

2.5 Real data analysis

In this section, an application of the smoothed stationary bootstrap and smoothed moving blocks bootstrap is illustrated by computing the non parametric density estimation for two real data sets that can be found in package TSA (see Chan and Ripley, 2012) of the free software R. A first data set consists of the annual record of the number of Canadian lynxes trapped in the Mackenzie River, district of North-West, Canada, for the period 1821-1934. It is a series of 114 observations that have been widely used along the literature. An extensive study of this series can be found in Chapter 7 of Tong (1996). We also used the data set of the yearly number of sunspots from 1700 to 1988 (see Tong, 1996, p. 471). Due to the skewed distribution of both samples a logarithmic transformation of both data sets was considered. As in Section 2.4, h^*_{SSB} and h^*_{SMBB} are empirically compared with h_{CV_l}, h_{PCV}, h_{SMCV} and h_{PI} as smoothing parameter selectors. The non parametric estimation of the density function for each data set is shown in Figures 2.12 and 2.13, using h^*_{SSB}, h^*_{SMBB}, h_{CV_l}, h_{PCV}, h_{SMCV} and h_{PI}. Specifically, values $p = \dfrac{1}{2\sqrt{n}}$, $b = 2\sqrt{n}$ and $l = 5$ were taken into account to compute h^*_{SSB}, h^*_{SMBB}, h_{CV_l} and h_{SMCV}, respectively. The values for the bandwidth selectors obtained are shown in Table 2.2. The required computing times (in seconds) for the six selectors are contained in Table 2.3. Furthermore, the `forecast` package (see Hyndman, 2015) is used in order to fit a parametric model to the selected real data sets, revealing strong dependence. Additionally, when assuming independence, smaller values for bandwidth parameters are obtained as compared to those calculated for the dependent case. However, even assuming independence, the bootstrap provides a bandwidth parameter close to the one obtained considering dependence.

Figure 2.12 shows that h^*_{SSB}, h^*_{SMBB}, h_{CV_l}, h_{SMCV} and h_{PI} produce a clear bimodal estimate for the density function, while h_{PCV} seems to overestimate the underlying density for the lynx data set. A similar overestimation performance seems to be the case of h_{PCV} for the sunspot data in Figure 2.13. Both bootstrap bandwidths, as well as h_{CV_l} and h_{PI} produce a smooth density estimation, which seems to be more accurate when looking at the data support. Furthermore, h_{SMCV} seems to underestimate the underlying density for the lynx and sunspot data sets (see

Figures 2.12 and 2.13, respectively). The bandwidth selectors h^*_{SMBB} and h_{SMCV} are considerably more tim consuming than the rest, which have a similar CPU time.

Data set	h^*_{SSB}	h^*_{SMBB}	h_{CV_l}	h_{PCV}	h_{SMCV}	h_{PI}
lynx	0.429502	0.424624	0.3173096	0.6193772	0.2585352	0.3092726
sunspot	0.3148706	0.3319434	0.3002368	0.5065466	0.1960352	0.3045369

Table 2.2: Smoothing parameters for the lynx and sunspot data set.

Data set	CPU_{SSB}	CPU_{SMBB}	CPU_{CV_l}	CPU_{PCV}	CPU_{SMCV}	CPU_{PI}
lynx	7.85	11.5	8.42	8.43	13.58	8.75
sunspot	30.32	60.89	27.20	27.24	63.84	34.33

Table 2.3: CPU times (in seconds) for the lynx and sunspot data set.

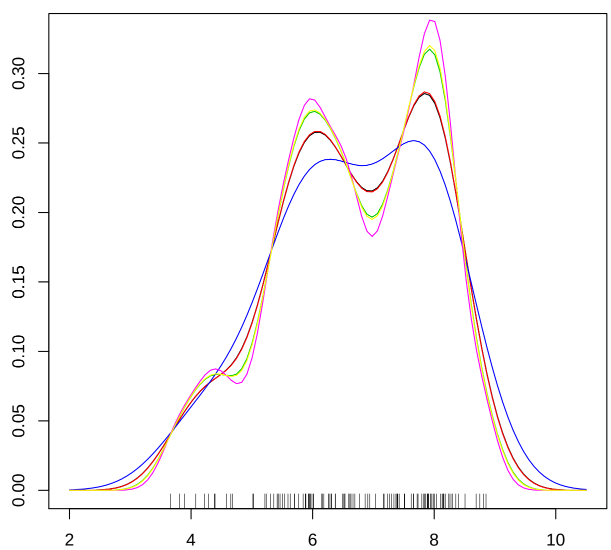


Figure 2.12: Nonparametric density estimation for the lynx data set using the smoothing parameters h^*_{SSB} (black line), h^*_{SMBB} (red line) and h_{CV_l} (green line), h_{PCV} (blue line), h_{SMCV} (pink line) and h_{PI} (yellow line).

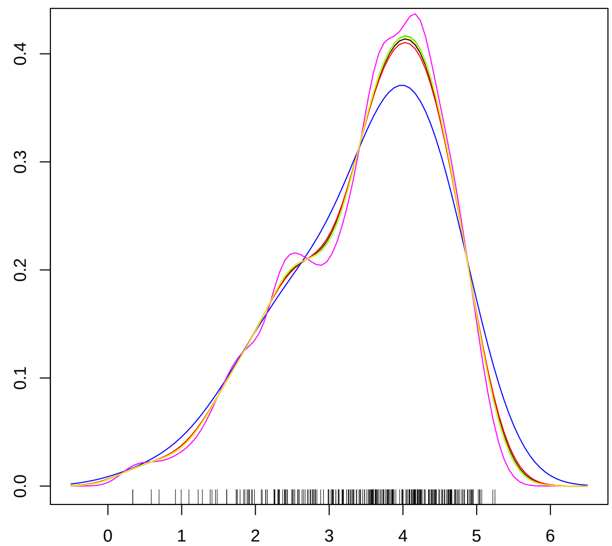


Figure 2.13: Nonparametric density estimation for the sunspot data set using the smoothing parameters h^*_{SSB} (black line), h^*_{SMBB} (red line) and h_{CV_l} (green line), h_{PCV} (blue line), h_{SMCV} (pink line) and h_{PI} (yellow line).

Chapter 3

Bandwidth selection for hazard rate estimation

3.1 Introduction

Nonparametric kernel-based hazard rate estimators have been extensively studied since its introduction in a classical iid setting by Watson and Leadbetter (1964a) and Watson and Leadbetter (1964b). Since the hazard rate is a useful tool in survival analysis, there are numerous papers dealing with this topic in the censored data case as well, such as Tanner and Wong (1983), Tanner and Wong (1984), Lo et al. (1989), Patil (1993a) or Patil (1993b). For a deeper insight in the complete data case see, for instance, Murthy (1965), Rice and Rosenblatt (1976), Singpurwalla and Wong (1980), Singpurwalla and Wong (1983), Rao (1983) or Sarda and Vieu (1991).

Let us consider $(X_1, X_2, \ldots, X_n)$, a simple random sample coming from a population with continuous density f and cumulative distribution function F. Although X is usually considered as a non-negative variable, we will deal with the general case, so as to take into account the monotonic transformations of this type of variables (which may no longer be non-negative). We focus on the problem of estimating the hazard rate function, r, which is indeed a curve that provides useful information about the variable X in survival analysis and reliability. If X is a continuous random variable, the hazard rate function can be expressed as follows:

$$r(x)=\lim_{\delta\to 0^+}\frac{\mathbb{P}\left(X\le x+\delta\,|X>x\right)}{\delta}=\frac{f(x)}{1-F(x)}.$$

When no parametric assumption is made concerning the underlying distribution, it seems natural to estimate the function r via the kernel method (see Watson and Leadbetter, 1964a,b):

$$\hat{r}_h(x)=\frac{\hat{f}_h(x)}{1-\hat{F}_h(x)} \tag{3.1}$$

or

$$\hat{r}_h^e(x)=\frac{\hat{f}_h(x)}{1-\hat{F}_n(x)}, \tag{3.2}$$

where $\hat{f}_h(x)$ stands for the kernel density estimator defined in (1.1) and

$$\hat{F}_h(x)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i}{h}\right) \tag{3.3}$$

is the kernel distribution estimator, K is a kernel function (typically a density function), $\mathbb{K}(t)=\int_{-\infty}^{t}K(u)du$ is the cdf related to K and $\hat{F}_n(x)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{X_i\le x\}}$ is the empirical cdf. Thus, the widely known problem of choosing the smoothing parameter, $h>0$, arises. The aim of this chapter is precisely the choice of the smoothing parameter for the estimator given in (3.1).

A great deal of cross-validation procedures proposed in both censored and complete data setups (see Tanner and Wong, 1984; Sarda and Vieu, 1991 or Patil, 1993a,b) triggered the development of new techniques for the purpose of bandwidth selection. More recently, a new cross-validation bandwidth selector (see Gámiz et al., 2016) has been proposed for local linear hazard rate estimation (see Nielsen and Tanggaard, 2001, for more in-depth details regarding this estimator). It is worth singling out that the main disadvantage brought about by the local linear hazard rate estimator is that it may produce negative estimations (as can be seen in Section

3.7). On the other hand, the local linear estimator is typically more efficient in the boundary of the support.

Considering the bootstrap method, González-Manteiga et al. (1996) proposed a smoothing parameter selector for censored data, based on 'smoothed bootstrap' ideas. However, this method is very time consuming and the authors proposed an asymptotic approximation for the bootstrap version of the mean integrated squared error. See Marron (1992) for an introduction and more references in the context of density estimation.

The smoothed bootstrap can also be used to produce a bootstrap bandwidth selector for the hazard rate estimator considering general independent data, which is discussed in Sections 3.3 and 3.4. The key idea is to consider approximate versions of the estimator in equation (3.1) and to approximate the mean integrated squared error (MISE) of these approximations of the kernel hazard rate estimation by using the smoothed bootstrap proposed by Silverman and Young (1987). The bandwidth which minimizes the bootstrap version of the MISE turns out to be a reasonable empirical analogue for the theoretical smoothing parameter that minimizes MISE. This proposal is parallel to that of Cao (1993) for density estimation with independent data, or those in Chapter 2 (see Barbeito and Cao, 2016, 2017) when dependence is considered. As in those three proposals, an exact expression for the bootstrap version of the MISE of two modified versions of the kernel hazard rate estimator is obtained, so Monte Carlo approximation is no longer needed. This seems not to be the case for censored data where the survival, density and hazard rate kernel estimators do not have a structure of sum of iid terms. This is the main difficulty to extend these ideas to a censored data framework. In this chapter, two new bootstrap bandwidth selectors for hazard rate estimation are proposed (see Barbeito and Cao, 2019b). They are based on some approximations of the classical kernel hazard rate estimator.

The rest of this chapter contains the novel contributions, proposed in Barbeito and Cao (2019b), and it proceeds as follows. Sections 3.3 and 3.4 present two new bootstrap smoothing parameter selectors for hazard rate estimation. Furthermore, explicit expressions for the mean integrated squared error of two modified estimators and their bootstrap versions are obtained. A closed form for these expressions in the case of Gaussian kernel is worked out in Section 3.5. In Section 3.6, the performance of these two new bootstrap bandwidth selectors is analysed via an extensive simulation study. Three other already existing bandwidth selectors for hazard rate estimation are also considered for comparison: the cross-validation selector (see Patil, 1993a), the bootstrap smoothing parameter selector proposed by González-Manteiga et al. (1996) -both for the kernel estimator given in (3.1)- and the DO-validated bandwidth selector, for the local linear estimator (see Gámiz et al., 2016). Section 3.7 presents the empirical behaviour of the five bandwidth selectors by applying them to a diabetes data set. Finally, proofs of the results established in Sections 3.3 and 3.4 are collected in Appendix B.

3.2 A critical review of smoothing methods

3.2.1 Cross-validation bandwidth selector

The aim of this method, proposed by Patil (1993a) in the context of censored data, is to minimize the cross-validation function $CV(h)$ so as to determine the optimal bandwidth parameter in terms of ISE. The criterion function $CV(h)$ proposed by this author, now considering a modified version for complete data, is given by:

$$CV(h) \;=\; \int \hat{r}_h^e(x)^2\, w(x)\, dx - \frac{2}{n}\sum_{i=1}^{n} \frac{\hat{r}_h^{e,-i}(X_i)}{1-\hat{F}_n^{-i}(X_i)}\, w(X_i),$$

where $\hat{r}_h^e$ has already been defined in (3.2), $\hat{r}_h^{e.-i}$ stands for the definition in (3.2) computed without the i-th observation, F_n^{-i} is the empirical distribution function worked out without the i-th observation and $w(\cdot)$ is a weight function.

The cross-validation bandwidth selector is defined as the minimizer of $CV(h)$, that is:

$$h_{CV} = \arg\min_{h>0} CV(h).$$

3.2.2 DO-validation bandwidth selector

Gámiz et al. (2016) proposed the double one-sided cross-validation (or simply, DO-validation) bandwidth selector of local linear hazards, when a counting process is considered. Indeed, the local linear hazard rate estimator is deemed to generalize the classical one (which makes a constant linear fitting) by means of making a local linear fitting.

The key idea of this method is to consider a combination of left- and right-sided cross-validation. In other words, the DO-validation procedure takes the average of two indirect cross-validation bandwidths. The two kernels K_L and K_R used to carry out the left- and right- sided cross-validation algorithm, correspond to local linear smoothing with one-sided kernels. That is,

$$\begin{aligned} K_L(u) &= 2K(u)\mathbb{1}_{\{u<0\}}, \\ K_R(u) &= 2K(u)\mathbb{1}_{\{u>0\}}. \end{aligned}$$

Finally, the DO-validation bandwidth selector, h_{DO}, is defined as the weighted average:

$$h_{DO} = \frac{1}{2}\left(\frac{R(K_L)\mu_2(K)^2}{R(K)\mu_2(K_L)^2}\right)^{1/5} \cdot \left(h_{CV}^{K_L} + h_{CV}^{K_R}\right),$$

where $h_{CV}^{K_L}$ and $h_{CV}^{K_R}$ are the minimizers of the left and right-sided cross-validation criteria, respectively, and $\mu_2(L) = \int u^2 L(u)\, du$.

For a deeper insight in DO-validation bandwidth selection for kernel density estimation see Mammen et al. (2014).

3.2.3 Bootstrap bandwidth selector

This method has been originally proposed in the context of censored data by González-Manteiga et al. (1996). A modified version for complete data will be considered in Sections 3.6 and 3.7 in order to carry out an empirical comparison of the behaviour of this smoothing parameter with the ones that will be proposed in Sections 3.3 and 3.4. These authors define an optimal bootstrap-based bandwidth selector as the minimizer of the bootstrap weighted asymptotic version of the MISE, that is:

$$\begin{aligned} AMISE_w^*(h) &= \int \left[K_h * \frac{\hat{f}_g}{1-\hat{F}_n}(x) - \frac{\hat{f}_g}{1-\hat{F}_n}(x) \right]^2 w(x)\,dx \\ &\quad + \frac{1}{nh} \int K^2 \int \frac{\hat{f}_g}{\left(1-\hat{F}_n\right)^2}(x)\, w(x)\,dx, \end{aligned}$$

where $g > 0$ stands for the pilot bandwidth.

Therefore, the optimal bandwidth selector, namely h^*_{GCM}, is deemed to be the minimizer of $AMISE^*_w(h)$, that is:

$$h^*_{GCM} = \arg\min_{h>0} AMISE^*_w(h).$$

3.3 First bootstrap-based bandwidth selector

As previously mentioned, our goal is to establish a closed-form expression for the bootstrap version of the mean integrated squared error (namely, MISE). Accordingly, in order to get rid of the randomness of the denominator of the classical hazard rate estimator given in (3.1), let us consider the following proxy for the kernel hazard rate estimator:

$$\tilde{r}_{h,1}(x) = \frac{\hat{f}_h(x)}{1-F(x)}, \tag{3.4}$$

where $\hat{f}_h(x)$ stands for the nonparametric kernel density estimator defined in (1.1) (see Parzen, 1962; Rosenblatt, 1956). As can be seen in (3.4), the denominator depends on the underlying theoretical distribution function F. Thus, this is not a real estimator but some theoretical approximation of (3.1). To the best of our knowledge, this is the first time the idea of constructing a 'proxy estimator' for bootstrap bandwidth selection purposes is used.

The key idea of this section is to compute a bootstrap bandwidth selector by considering the bootstrap version of a standard measure of performance of the hazard rate approximated estimator defined in (3.4), and to find the smoothing parameter which minimizes it. The MISE happens to be a commonly-used global error measure. Its expression is given by:

$$\begin{aligned} \mathrm{MISE}_{w,\tilde{r}_{h,1}}(h) &= \mathbb{E}\left[\int \left(\tilde{r}_{h,1}(x) - r(x)\right)^2 w(x)\, dx\right] \\ &= B(h) + V(h), \end{aligned}$$

where $w(x)$ is a nonnegative weight funcion, which plays an important role as it is used to eliminate endpoint effects. Furthermore, $B(h)$, $V(h)$ are the integrated squared bias and integrated variance of the hazard rate estimator, given by:

$$\begin{aligned} B(h) &= \int \left[\mathbb{E}\left(\tilde{r}_{h,1}(x)\right) - r(x)\right]^2 w(x)\, dx, \text{ and} \\ V(h) &= \int \mathrm{Var}\left(\tilde{r}_{h,1}(x)\right) w(x)\, dx. \end{aligned}$$

In order to prove our first result some conditions are needed:

(A1) The function w is non-negative, it is bounded and its support is contained in some compact inverval $[a, b]$.

(A2) K is a bounded symmetric density function.

(A3) The density f is bounded.

(A4) $F(b) < 1$.

The upcoming result shows an explicit expression for the MISE of $\tilde{r}_{h,1}$, given in (3.4).

Theorem 4 *If $X_1, \ldots, X_n$ are iid random variables and Assumptions (A1)-(A4) hold, then the MISE of $\tilde{r}_{h,1}$ in (3.4) can be expressed as follows:*

$$\begin{aligned} MISE_{w,\tilde{r}_{h,1}}(h) &= \int \left(\frac{K_h * f(x) - f(x)}{1 - F(x)} \right)^2 w(x)\, dx \\ &+ \frac{1}{n} \int \frac{(K_h)^2 * f(x)}{(1-F(x))^2} w(x)\, dx - \frac{1}{n} \int \left(\frac{K_h * f(x)}{1 - F(x)} \right)^2 w(x)\, dx, \end{aligned}$$

where $K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right)$.

The proof of Theorem 4 can be found in Appendix B.

Figure 3.1 clearly shows that $\text{MISE}_{w,\tilde{r}_{h,1}}$ (given in Theorem 4) is a good approximation of the mean integrated squared error of the estimator (3.1), which is approximated by Monte Carlo, specially focusing on the area where both functions attain their minimum. As a consequence, the approximated version given in (3.4) is a reasonable proxy for the real estimator (3.1) concerning MISE.

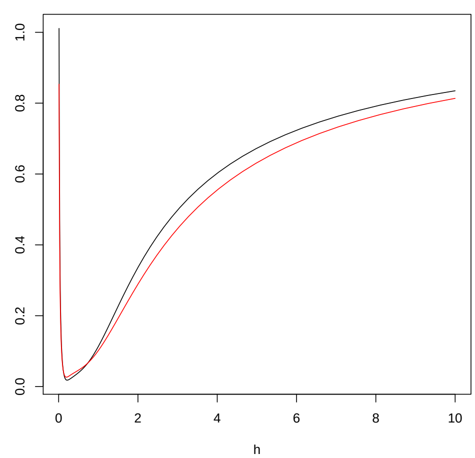


Figure 3.1: Mean integrated squared error of the estimator (3.1) (black line) and its approximated version given in (3.4) (red line) when the sample of size $n = 100$ comes from a standard normal population.

3.3.1 Bootstrap version of MISE

As previously mentioned, our goal is to compute a MISE-oriented bandwidth selector, by minimizing the bootstrap version of the MISE. As a consequence, it turns out to be of utmost importance to work out a closed expression for

$$MISE^*_{\tilde{r}_{h,1},w}(h) \;=\; \mathbb{E}^*\left[\int \left(\tilde{r}^*_{h,1}(x) - \tilde{r}_{g,1}(x)\right)^2 w(x)\,dx\right],$$

so as to avoid Monte Carlo approximation.

Let us consider $(X_1^*, \ldots, X_n^*)$ a smoothed bootstrap resample (see Silverman and Young, 1987; or Cao, 1993) based on a pilot bandwidth, g, and the bootstrap version of the aforementioned approximated estimator of the hazard rate function, $\tilde{r}_{h,1}$, given in (3.4):

$$\tilde{r}^*_{h,1}(x) = \frac{\hat{f}^*_h(x)}{1-\hat{F}_g(x)}.$$

The following result presents an exact expression for $\mathrm{MISE}^*_{\tilde{r}_{h,1},w}(h)$.

Theorem 5 *Under Assumptions (A1)-(A4), the smoothed bootstrap version of MISE for $\tilde{r}_{h,1}$ admits the following closed expression for iid data:*

$$\begin{aligned} MISE^*_{\tilde{r}_{h,1},w}(h) \;=\; & \int \left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n \left(K_h * K_g(x-X_i) - K_g(x-X_i)\right)\right]^2 w(x)dx \\ & + \frac{1}{n}\int \left[\frac{1}{n(1-\hat{F}_g(x))^2}\sum_{i=1}^n (K_h)^2 * K_g(x-X_i)\right] w(x)\,dx \\ & - \frac{1}{n}\int \left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n K_h * K_g(x-X_i)\right]^2 w(x)\,dx, \end{aligned}$$

where $(K_h)^2(u) = \frac{1}{h^2}K\left(\frac{u}{h}\right)^2$.

Theorem 5 is proven in Appendix B.

Moreover, as it can bee seen in Figure 3.2, $MISE^*_{w,\tilde{r}_{h,1}}$ (given in Theorem 5) happens to be a good approximation of the MISE of $\hat{r}_h(x)$ defined in (3.1) in the area where both functions attain their minimum.

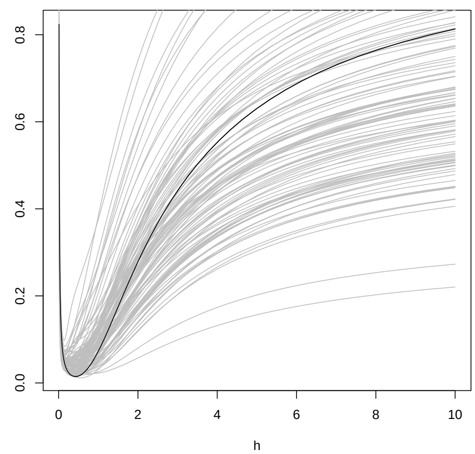


Figure 3.2: Mean integrated squared error of the estimator given in (3.1) (black line) and bootstrap version of the mean integrated squared error the approximated estimator given in (3.4) considering 100 different samples (grey lines) of size $n = 100$ simulated from a $N(0, 1)$ population.

A bootstrap bandwidth selector for the hazard rate estimator can now be defined as the minimizer, in h, of $MISE^*_{\tilde{r}_{h,1},w}(h)$, given in Theorem 5,

$$h_{BOOT1} = h_{MISE^*_{\tilde{r}_{h,1},w}} = \arg\min_{h>0} MISE^*_{\tilde{r}_{h,1},w}(h).$$

It is worth stressing that the exact expression of $\mathrm{MISE}^*_{\tilde{r}_{h,1},w}(h)$ is really useful since Monte Carlo approximation is no longer needed to compute the bootstrap bandwidth selector.

3.4 Second bootstrap-based bandwidth selector

Similarly, as in Section 3.3, our aim is to propose a new MISE-oriented smoothing parameter for hazard rate estimation. But we will do so based on a more accurate approximation, which takes into account second order terms. Let us consider $\hat{r}_h =$

$\hat{f}_h(x)/(1-\hat{F}_h(x))$ the hazard rate estimator, defined in (3.1). Our target is to get rid of the randomness of the denominator of (3.1) so as to work out a closed expression for the $MISE(h)$. The main difference with the estimator given in (3.4) is that an expanded approximated expression for $\hat{r}_h(x)-r(x)$ is now considered, using that $\hat{F}_h$ is the kernel distribution estimator of F. This approximated estimator, denoted by $\tilde{r}_{h,2}$, is given by:

$$\begin{aligned}\tilde{r}_{h,2}(x) &= (\hat{r}_h(x)-r(x))\frac{1-\hat{F}_h(x)}{1-F(x)}+r(x) \\ &= \frac{1}{1-F(x)}\hat{f}_h(x)+\frac{f(x)}{(1-F(x))^2}\hat{F}_h(x)-\frac{f(x)}{(1-F(x))^2}+r(x).\end{aligned} \quad (3.5)$$

A more detailed explanation on how to obtain this expression is described in Appendix B.

In the upcoming statement a closed-form expression for the MISE of $\tilde{r}_{h,2}$, defined in (3.5), is established.

Theorem 6 *If the sample is iid and assuming Conditions (A1)-(A4), the MISE of* $\tilde{r}_{h,2}$ *in (3.5) can be expressed as follows:*

$$\begin{aligned}MISE_{\tilde{r}_{h,2},w}(h) &= \int \frac{1}{(1-F(x))^2}\left[\frac{1}{n}(K_h)^2 * f(x)+\frac{n-1}{n}(K_h * f(x))^2\right]w(x)\,dx \\ &+2\int \frac{f(x)}{(1-F(x))^3}\left[\frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\right. \\ &\left.+\frac{n-1}{n}K_h * f(x)\int \mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\right]w(x)\,dx \\ &-2\int \frac{f(x)}{(1-F(x))^3}K_h * f(x)w(x)\,dx+\int \frac{f^2(x)}{(1-F(x))^4}w(x)\,dx \\ &+\int \frac{f^2(x)}{(1-F(x))^4}\left[\frac{1}{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 f(y)dy\right. \\ &\left.+\frac{n-1}{n}\left[\int \mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\right]^2\right]w(x)\,dx \\ &-2\int \frac{f^2(x)}{(1-F(x))^4}\int \mathbb{K}\left(\frac{x-y}{h}\right)f(y)dyw(x)\,dx.\end{aligned}$$

The proof of Theorem 6 can be found in Appendix B.

3.4.1 Bootstrap version of MISE

Let us take into account $(X_1^*, \ldots, X_n^*)$ a smoothed bootstrap resample based on a pilot bandwidth, g, and the bootstrap version of the aforementioned expression, $\tilde{r}_{h,2}$ in (3.5), which results in:

$$\tilde{r}_{h,2}^*(x) = \frac{1}{1-\hat{F}_g(x)}\hat{f}_h^*(x) + \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^2}\hat{F}_h^*(x) - \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^2} + \hat{r}_g(x).$$

Once again, it makes sense to find an explicit expression for the smoothed bootstrap version of the MISE, which is given by:

$$MISE_{\tilde{r}_{h,2},w}^*(h) = \mathbb{E}^*\left[\int \left(\tilde{r}_{h,2}^*(x) - \tilde{r}_g(x)\right)^2 w(x)\,dx\right], \tag{3.6}$$

so as to avoid Monte Carlo approximation and, therefore, establish a new bootstrap smoothing parameter. In expression (3.6), $\hat{r}_g$ is considered as the kernel hazard rate estimator of the function r, being $g > 0$ the pilot bandwidth.

Subsequently, Theorem 7 presents an exact expression for $MISE_{\tilde{r}_{h,2},w}^*(h)$ in the iid case. But let us first introduce some notation.

Definition 1 *For any integrable function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the function $\mathbb{I}_\psi$ is defined by $\mathbb{I}_\psi(t) := \int_{-\infty}^t \psi(u)du$. If the integrable function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ depends on two variables, then the function $\mathbb{I}_\psi$ is defined by $\mathbb{I}_\psi(t_1, t_2) := \int_{-\infty}^{t_2}\int_{-\infty}^{t_1} \psi(u, v)\,du\,dv$.*

Definition 2 *Given two functions $\psi, \varphi : \mathbb{R} \longrightarrow \mathbb{R}$ the tensor product function, $\psi \otimes \varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, is defined by $\psi \otimes \varphi(u, v) := \psi(u)\varphi(v)$.*

Definition 3 *Given two integrable functions $\psi, \varphi : \mathbb{R} \longrightarrow \mathbb{R}$, the convolution function $(\psi \otimes \psi) * \varphi$ is defined by $[(\psi \otimes \psi) * \varphi](u, v) := \int \psi(u - s)\psi(v - s)\varphi(s)\,ds$.*

Theorem 7 *Under Assumptions (A1)-(A4), the smoothed bootstrap version of MISE for $\tilde{r}_{h,2}$ admits the following closed expression:*

$$\begin{aligned}
MISE^*_{\tilde{r}_{h,2},w}(h) &= \frac{1}{n^2}\int \frac{1}{(1-\hat{F}_g(x))^2}\sum_{i=1}^n (K_h)^2 * K_g(x-X_i)w(x)\,dx \\
&+\frac{n-1}{n^3}\int \frac{1}{(1-\hat{F}_g(x))^2} \\
&\cdot\sum_{i,j=1}^n K_h * K_g(x-X_i)K_h * K_g(x-X_j)w(x)\,dx \\
&+\frac{2}{n^2}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^n L_h * K_g(x-X_i)w(x)\,dx \\
&-\frac{2}{n}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^n K_h * K_g(x-X_i)w(x)\,dx \\
&+\frac{n-1}{n^3}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} \\
&\cdot\sum_{i,j=1}^n \mathbb{I}_{\{K_h*K_g\}}(x-X_i)\cdot\mathbb{I}_{\{K_h*K_g\}}(x-X_j)\,w(x)\,dx+\frac{2n-2}{n^3} \\
&\cdot\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i,j=1}^n K_h * K_g(x-X_j)\mathbb{I}_{\{K_h*K_g\}}(x-X_i)\,w(x)\,dx \\
&-\frac{2}{n}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^n \mathbb{I}_{\{K_h*K_g\}}(x-X_i)\,w(x)\,dx \\
&+\frac{1}{n^2}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^n \mathbb{I}_{\{(K_h\otimes K_h)*K_g\}}(x-X_i,x-X_i)\,w(x)\,dx \\
&+\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}w(x)\,dx, \qquad (3.7)
\end{aligned}$$

where $L(z)=\mathbb{K}(z)K(z)$ *and* $L_h(z)=\frac{1}{h}L\left(\frac{z}{h}\right)$.

Theorem 7 is proven in Appendix B.

Furthermore, Figure 3.3 reveals that, considering the area where the three functions attain their minimum, both $MISE_{w,\tilde{r}_{h,2}}$ (Theorem 6) and its bootstrap version, $MISE^*_{w,\tilde{r}_{h,2}}$ (Theorem 7), are good approximations of $MISE_{F_h}$, which is the MISE of the classical hazard rate estimator given in (3.1), approximated by simulation. As a matter of fact, this approximation, given in (3.5), somewhat enhances the performance of the one given in (3.4) in a local framework, although its efficiency is worse in a global setting.

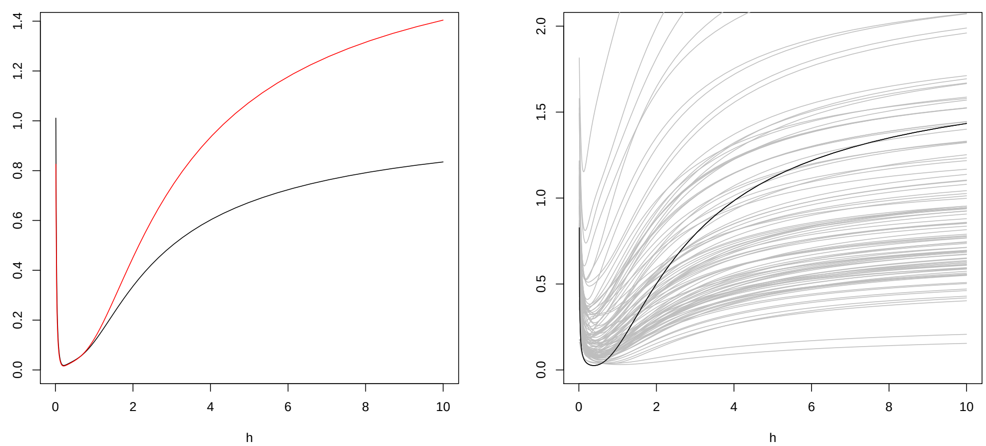


Figure 3.3: On the left, mean integrated squared error of the estimator given in (3.1) (black line) and its approximated version given in (3.5) (red line). On the right, MISE of the estimator given in (3.1) (black line) and bootstrap version of MISE for the approximated estimator given in (3.5) considering 100 different samples (grey lines) of size $n = 100$ simulated from a standard normal population.

A second bootstrap bandwidth selector for hazard rate estimation is now defined as the minimizer, in h, of $MISE^*_{\tilde{r}_{h,2},w}(h)$, given by expression (3.7) in Theorem 7,

$$h_{BOOT2} = h_{MISE^*_{\tilde{r}_{h,2},w}} = \arg\min_{h>0} MISE^*_{\tilde{r}_{h,2},w}(h).$$

It is worth singling out that the explicit expression of $MISE^*_{\tilde{r}_{h,2},w}(h)$ is truly helpful since the implementation of the bootstrap selector does not require Monte Carlo approximation.

3.5 Gaussian kernel case

In the particular case of a Gaussian kernel, K, that is, the density function of a standard normal, expressions given in Theorems 5 and 7 can be simplified, leading to Theorem 8, the proof of which can be found in Appendix B.

Theorem 8 *Let us assume Conditions (A1), (A3) and (A4). If K is the Gaussian kernel, then the smoooth bootstrap version of MISE for $\tilde{r}_{h,1}$ admits the following expression:*

$$\begin{aligned} MISE^*_{\tilde{r}_{h,1},w}(h) &= \int \left[\frac{1}{n(1-\hat{F}_g(x))} \sum_{i=1}^n \Big(K_{\sqrt{g^2+h^2}}(x-X_i) - K_g(x-X_i) \Big) \right]^2 w(x)dx \\ &+ \frac{1}{2\sqrt{\pi}nh} \int \left[\frac{1}{n(1-\hat{F}_g(x))^2} \sum_{i=1}^n K_{\sqrt{g^2+h^2/2}}(x-X_i) \right] w(x)\,dx \\ &- \frac{1}{n} \int \left[\frac{1}{n(1-\hat{F}_g(x))} \sum_{i=1}^n K_{\sqrt{g^2+h^2}}(x-X_i) \right]^2 w(x)\,dx. \end{aligned}$$

Similarly, the smooth bootstrap version of MISE for $\tilde{r}_{h,2}$ can be expressed as follows:

$$\begin{aligned} MISE^*_{\tilde{r}_{h,2},w}(h) &= \frac{1}{n^2 h} \int \frac{1}{2\sqrt{\pi}(1-\hat{F}_g(x))^2} \sum_{i=1}^n K_{\sqrt{g^2+h^2/2}}(x-X_i) w(x)\,dx \\ &+ \frac{n-1}{n^3} \int \frac{1}{(1-\hat{F}_g(x))^2} \\ &\cdot \sum_{i,j=1}^n K_{\sqrt{h^2+g^2}}(x-X_i) K_{\sqrt{h^2+g^2}}(x-X_j) w(x)\,dx \\ &+ \frac{2}{n^2} \int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3} \sum_{i=1}^n L_h * K_g(x-X_i) w(x)\,dx \\ &- \frac{2}{n} \int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3} \sum_{i=1}^n K_{\sqrt{h^2+g^2}}(x-X_i) w(x)\,dx \\ &+ \int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} w(x)\,dx + \frac{n-1}{n^3} \int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} \end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{i,j=1}^{n} \Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right) \Phi\left(\frac{x-X_j}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&+\frac{2n-2}{n^3} \int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3} \\
&\cdot \sum_{i,j=1}^{n} K_{\sqrt{h^2+g^2}}(x-X_j)\,\Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&-\frac{2}{n} \int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} \sum_{i=1}^{n} \Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&+\frac{1}{n^2} \int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} \sum_{i=1}^{n} \mathbf{\Phi}_{\mathbf{0},\tilde{H}}\left(x-X_i, x-X_i\right) w(x)\,dx,
\end{aligned}
$$

where $\Phi(z)$ *is the distribution function of a standard normal variable evaluated at* z, $\mathbf{\Phi}_{\boldsymbol{\mu},\Sigma}(z_1,z_2)$ *is the distribution function of a bivariate normal* $N_2(\boldsymbol{\mu},\Sigma)$ *evaluated at* (z_1,z_2), $\mathbf{0}=(0,0)^T$ *and*

$$
\tilde{H} = \begin{pmatrix} g^2+h^2 & g^2 \\ g^2 & g^2+h^2 \end{pmatrix}.
$$

Remark 14 *It is worth pointing out that the extension of these ideas to the censored data setup is not as simple as it seems. Indeed, the sum of iid disappears under censorship, turning out to be little prospects of finding a closed expression for MISE*. As a matter of fact, an extension of the ideas proposed in Cao (1993) to nonparametric density estimation with censored data does not exist yet, although it exists under dependence (see Barbeito and Cao, 2016, 2017).*

3.6 Simulations

3.6.1 General description of the study

A simulation study is now carried out in order to show the empirical performance of these two new bootstrap-based bandwidth selectors, h_{BOOT1} and h_{BOOT2}, described in Sections 3.3 and 3.4. Moreover, the practical behaviour of both smoothing parameters, h_{BOOT1} and h_{BOOT2}, has been empirically compared with the cross-validation bandwidth and the González-Manteiga-Cao-Marron bootstrap bandwidth originally designed for censored data, namely h_{CV} and h^*_{GCM}, proposed in Patil (1993a) and González-Manteiga et al. (1996), respectively. Furthermore, the empirical behaviour of h_{BOOT1} and h_{BOOT2} has also been compared with the DO-validation bandwidth selector proposed in Gámiz et al. (2016) for local linear kernel hazard estimation, namely h_{DO}.

It is worth pointing out that the free software `R` and the packages `kerdiest`, `Bolstad`, `mvtnorm`, `DOvalidation` and `QRM` (see Quintela del Río and Estévez-Pérez, 2012; Curran and Bolstad, 2016; Genz and Bretz, 2009; Genz et al., 2017; Gámiz et al., 2014; and Pfaff and McNeil, 2016; respectively), have been used to compute the estimators and the bandwidth selectors. Specifically, package `kerdiest` has been used to compute the nonparametric distribution estimation; `Bolstad`, to approximate numerically the integrals; `mvtnorm`, to compute the multivariate normal density; `DOvalidation`, to implement the DO-validation bandwidth selection and; finally, `QRM`, to deal with the Gumbel distribution. Subsequently, six different populations will be considered so as to show the practical results for every bandwidth selector in different situations. The hazard rate functions for all the models are presented in Figure 3.4.

Model 1: The sample is drawn from a standard normal model.

Model 2: A Gumbel model whose underlying density is $f_{\mu,\sigma}(x) = \sigma^{-1} e^{-\frac{x-\mu}{\sigma}} e^{-e^{-\frac{x-\mu}{\sigma}}}$. We chose $\mu = 0$ and $\sigma = 1$.

Model 3: The distribution of interest is taken as a Weibull with density

$$f_{\alpha,\beta} = \left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}.$$

In this case, the shape and scale parameters are $\alpha = 2, \beta = \pi$.

Model 4: The sample is generated from a χ^2_k distribution, considering $k = 2, 3$. The underlying density function is given by:

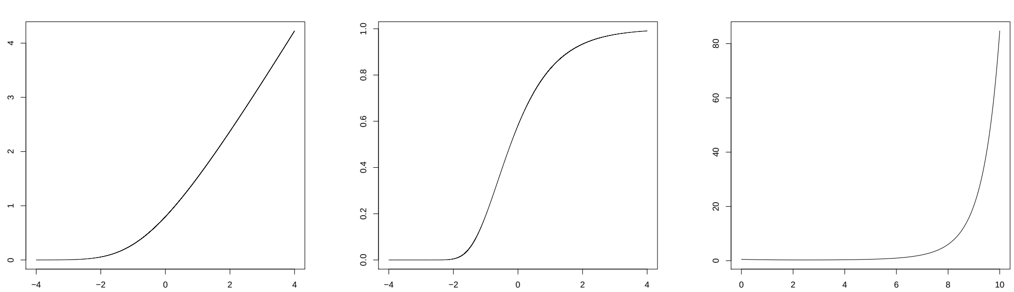
$$f_k(x) = \begin{cases} \dfrac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases},$$

where Γ stands for the Gamma function.

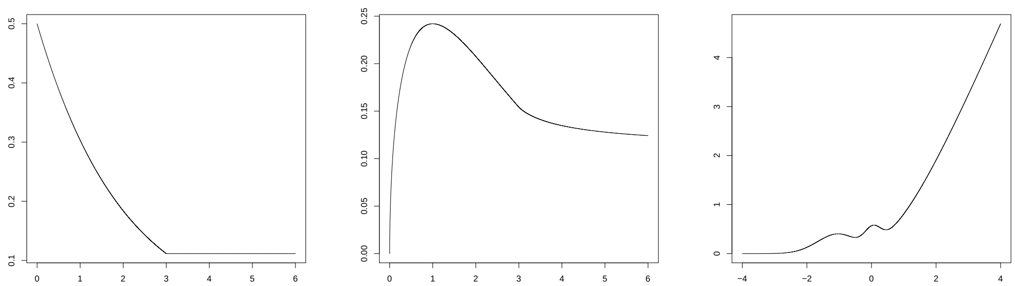
Model 5: A mixture of three normal densities (Model 9 of Marron and Wand, 1992) of the form:

$$\frac{9}{20} N\left(-\frac{6}{5}, \frac{9}{25}\right) + \frac{9}{20} N\left(\frac{6}{5}, \frac{9}{25}\right) + \frac{1}{10} N\left(0, \frac{1}{16}\right).$$

For every model, 500 random samples of size $n = 100$ were drawn. The Gaussian kernel was used to compute the Parzen-Rosenblatt estimator. The pilot bandwidth, g, used in the bootstrap methods was $g = h_{CV}\lambda n^{2/35}$, so that g has order $n^{-1/7}$ (as demonstrated by Cao, 1993, this is the optimal rate for kernel density estimation). On the other hand, λ has been picked as 0.2 (Models 1-3) and 0.1 (Models 4-5). A preliminary simulation study to discuss the choice of λ has been carried out as well. As a matter of fact, the behaviour of the three bootstrap bandwidth selectors strongly depends on λ. This is shown for Models 3 and 5 by considering twelve different values of λ, specifically, those in the set $\{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6\}$. Furthermore, the weight function w has been set as $w(x) = \mathbb{1}_{\left(-\infty, \hat{Q}_3\right]}(x)$, where $\hat{Q}_3$ is the estimator of the third quartile, namely Q_3.



(a) Model 1 (b) Model 2 (c) Model 3



(d) Model 4, $k = 2$ (e) Model 4, $k = 3$ (f) Model 5

Figure 3.4: Hazard rate functions of Models 1-5.

The four bandwidth selectors h_{CV}, h_{BOOT1}, h_{BOOT2} and h^*_{GCM} are the minimizers, in h, of four empirical functions. Since these minimizers do not have explicit expressions, numerical methods are used to approximate them. The algorithm proceeds as follows:

Step 1: Let us consider a set of 5 equally spaced values of h in the interval $[0.01, 10]$.

Step 2: For each method, a bandwidth h is chosen among the five given in the preceding step, by minimizing the objective function (CV, $\text{MISE}^*_{\tilde{r}_{h,1},w}$, $\text{MISE}^*_{\tilde{r}_{h,2},w}$ or $\widehat{\text{AMISE}}$). We denote it by h_{OPT_1}.

Step 3: Among the set of 5 bandwidth parameters defined in Step 1, we consider the previous and the next one to h_{OPT_1}. If h_{OPT_1} is the smallest (largest) bandwidth in the grid, then h_{OPT_1} is used instead of the previous (next) value of h_{OPT_1} in the grid.

Step 4: A set of 5 equally spaced values of h is constructed within the interval whose endpoints are the two values selected in Step 3.

Step 5: Finally, Steps 2-4 are repeated 10 times, retaining the optimal bandwidth selector in the last stage.

The five bandwidth selectors are compared in terms of the error committed when using each one of them. Thus, using the 500 samples, the following expressions are approximated by simulation:

$$\log\left(\frac{\hat{h}}{h_{MISE}}\right), \tag{3.8}$$

$$\text{ISE}\left(\hat{h}\right) = \int \left(\hat{r}_h(x) - r(x)\right)^2 w(x)\, dx, \tag{3.9}$$

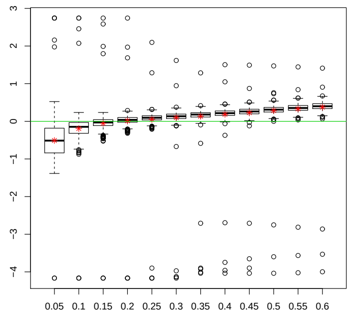
where $\hat{h} = h_{CV}, h_{DO}, h_{BOOT1}, h_{BOOT2}, h^*_{GCM}$. Moreover, $\hat{r}_h$ stands for estimator given in (3.1) when $\hat{h} = h_{CV}, h_{BOOT1}, h_{BOOT2}, h^*_{GCM}$. On the other hand, if $\hat{h} = h_{DO}$, $\tilde{r}_h$ stands for the local linear hazard rate estimator proposed by Nielsen and Tanggaard (2001) and used by Gámiz et al. (2016), now considering the context of complete data. Finally, h_{MISE} stands for the smoothing parameter selector that minimizes the theoretical MISE.

3.6.2 Discussion and results

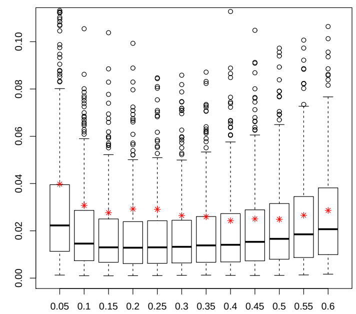
Figures 3.5-3.6 show the results for expressions (3.8) and (3.9) approximated by simulation. In particular, the empirical behaviour of h_{BOOT1}, h_{BOOT2} and h^*_{GCM} when λ takes different values is shown in Figures 3.5-3.6 for Models 3 and 5. It is evident that the best performance of the three smoothing parameters is obtained when choosing the value $\lambda = 0.2$ for Model 3, and $\lambda = 0.1$ for Model 5.

Figures 3.7-3.8 contain the results obtained for expression (3.8) approximated by simulation. It is worth mentioning that DO-validation algorithm cannot be compared with the other procedures in terms of MISE of a standard kernel estimator. However, h_{DO} can indeed be compared to the other smoothing parameter selectors in terms of the variability of expression (3.8). On the one hand, Figures 3.7 and 3.8 show that while h_{BOOT1} and h_{BOOT2} are very close in mean and median to h_{MISE}, h^*_{GCM} and h_{CV} tend to produce oversmoothed hazard rate estimations. Moreover, h_{DO} presents wider variance than its competitors. It is clear that the best smoothing parameter selector, considering Figures 3.7-3.8, is h_{BOOT2}, which is even slightly better than h_{BOOT1} in terms of median. It is also worth mentioning that h_{CV} produces a great amount of outliers which tend to produce undersmoothed versions of the hazard rate estimator.

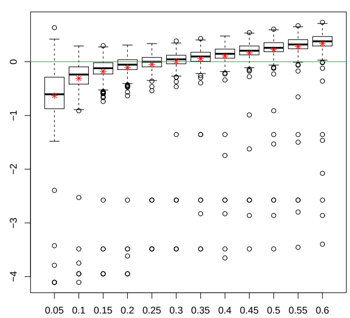
Figures 3.9-3.10 contain the results obtained approximating by simulation expression $ISE(\hat{h})$ for all the bandwidth selectors considered. According to these figures, the simulation results remarkably exhibit that the smoothing parameters h_{BOOT1} and h_{BOOT2} display the best performance in almost every case. Furthermore, it is worth singling out that both h_{BOOT1} and h_{BOOT2} distinctly beat their bootstrap competitor, h^*_{GCM}, in every case. Moreover, focusing on expression (3.9), it is noticeable that the performance of h_{BOOT1} and h_{BOOT2} is somewhat enhanced as compared to the one displayed by h_{DO} in almost every scenario considered. Consequently, it leads to place h_{DO} in the third position of the ranking. Finally, we can also conclude from Figures 3.7-3.10, that h_{CV} is undoubtedly the worst. Furthermore, h_{CV} presents the widest variance.



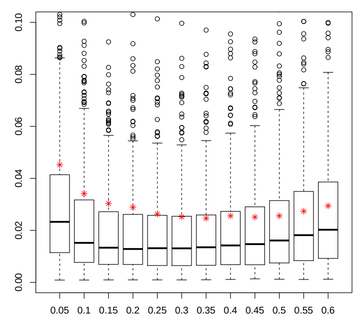
(a) $\hat{h} = h_{BOOT_1}$



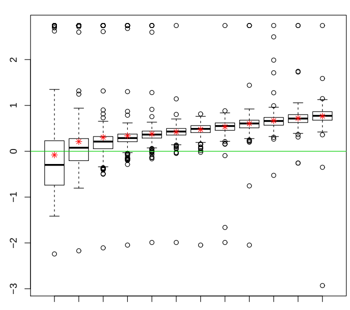
(b) $\hat{h} = h_{BOOT_1}$



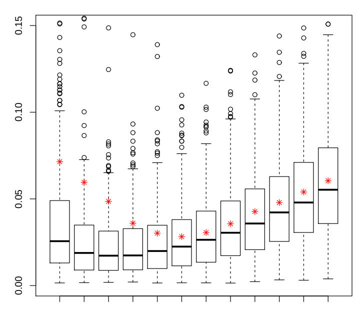
(c) $\hat{h} = h_{BOOT_2}$



(d) $\hat{h} = h_{BOOT_2}$

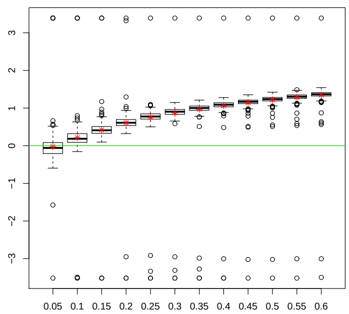


(e) $\hat{h} = h^*_{GCM}$

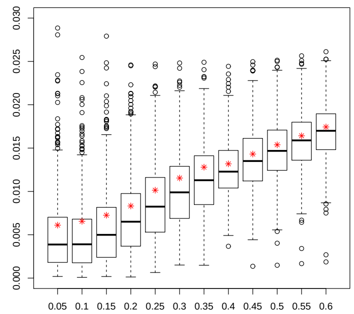


(f) $\hat{h} = h^*_{GCM}$

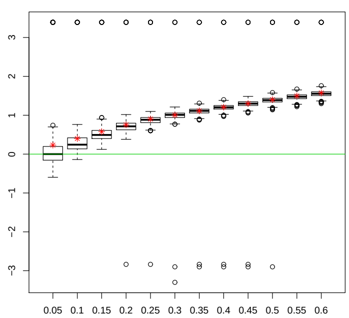
Figure 3.5: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $ISE(\hat{h})$ (right side) for Model 3, $\lambda \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6\}$ and $n = 100$, where $\hat{h} = h_{BOOT_1}$ (top row), h_{BOOT_2} (middle row) and h^*_{GCM} (bottom row). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.



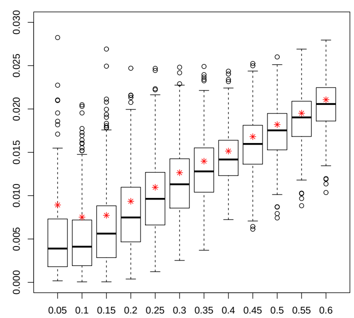
(a) $\hat{h} = h_{BOOT_1}$



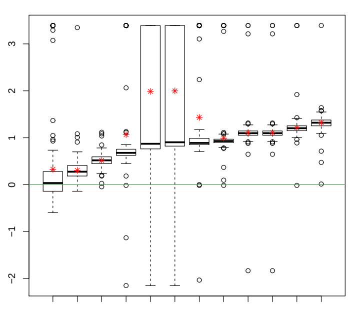
(b) $\hat{h} = h_{BOOT_1}$



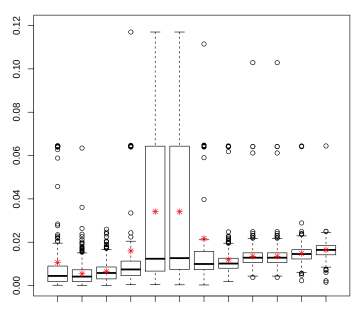
(c) $\hat{h} = h_{BOOT_2}$



(d) $\hat{h} = h_{BOOT_2}$

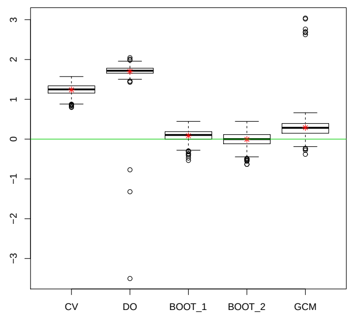


(e) $\hat{h} = h^*_{GCM}$

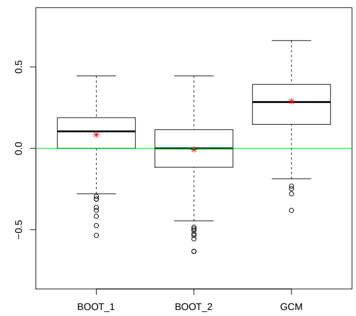


(f) $\hat{h} = h^*_{GCM}$

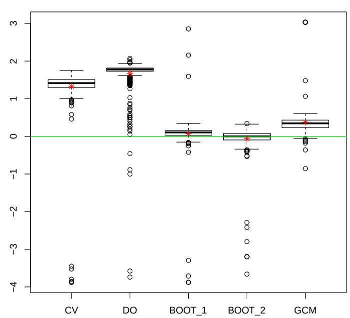
Figure 3.6: Boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ (left side) and $ISE(\hat{h})$ (right side) for Model 5, $\lambda \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6\}$ and $n = 100$, where $\hat{h} = h_{BOOT_1}$ (top row), h_{BOOT_2} (middle row) and h^*_{GCM} (bottom row). In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.



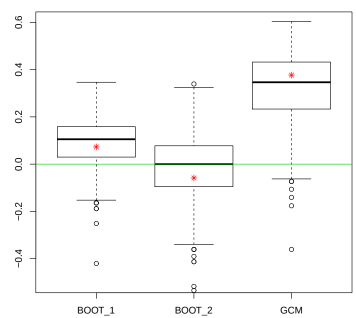
(a) Model 1



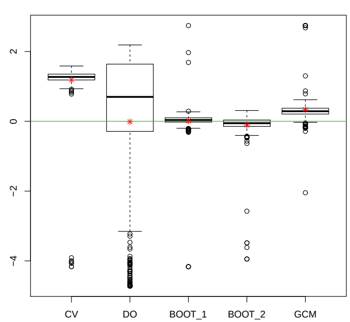
(b) Model 1



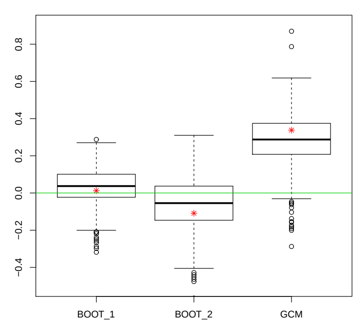
(c) Model 2



(d) Model 2

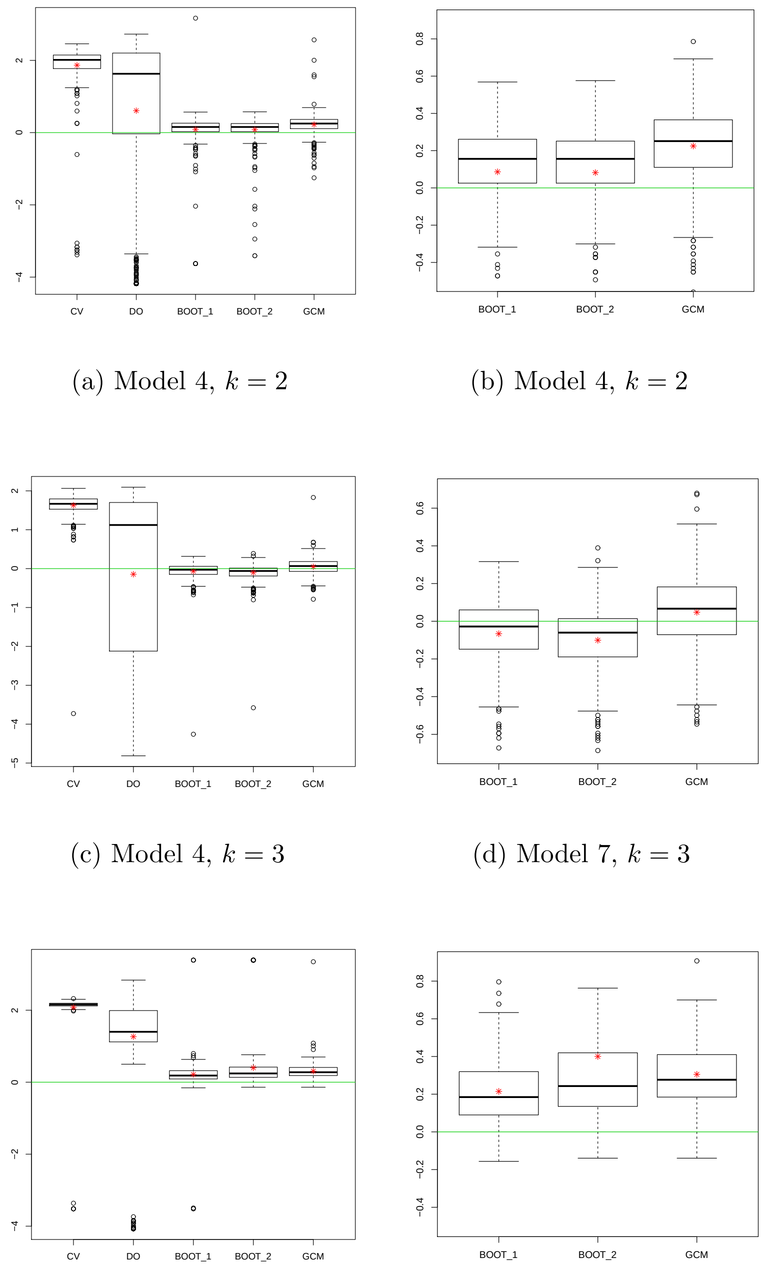


(e) Model 3



(f) Model 3

Figure 3.7: On the left, boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ for Models 1-3, $\lambda = 0.2$ and $n = 100$, where, from left to right, $\hat{h} = h_{CV}$ (first box), h_{DO} (second box), h_{BOOT_1} (third box), h_{BOOT_2} (fourth box) and h^*_{GCM} (fifth box). Boxplots on the right side are just a zoom of boxplots on the left side. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.

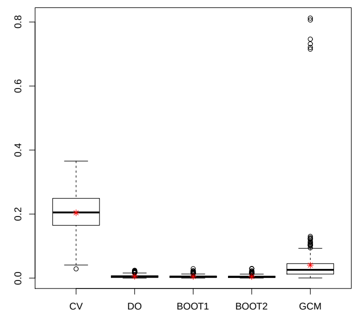


(a) Model 4, $k = 2$ (b) Model 4, $k = 2$

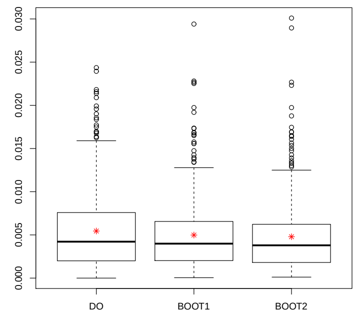
(c) Model 4, $k = 3$ (d) Model 7, $k = 3$

(e) Model 5 (f) Model 5

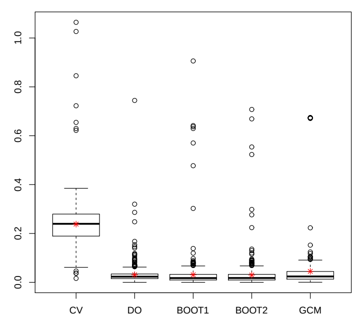
Figure 3.8: On the left, boxplots of $\log\left(\hat{h}/h_{MISE}\right)$ for Models 4-5, $\lambda = 0.1$ and $n = 100$, where, from left to right, $\hat{h} = h_{CV}$ (first box), h_{DO} (second box), h_{BOOT_1} (third box), h_{BOOT_2} (fourth box) and h^*_{GCM} (fifth box). Boxplots on the right side are just a zoom of boxplots on the left side. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.



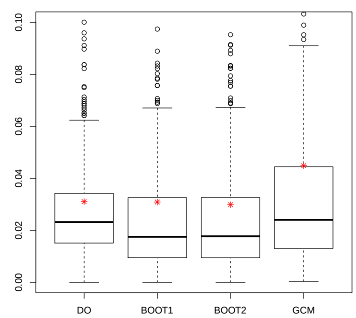
(a) Model 1



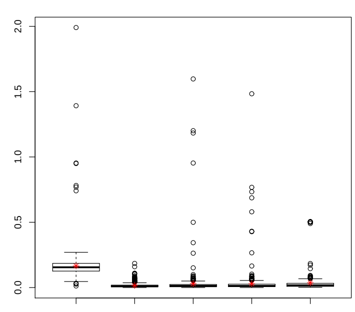
(b) Model 1



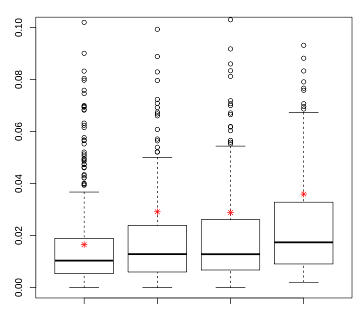
(c) Model 2



(d) Model 2

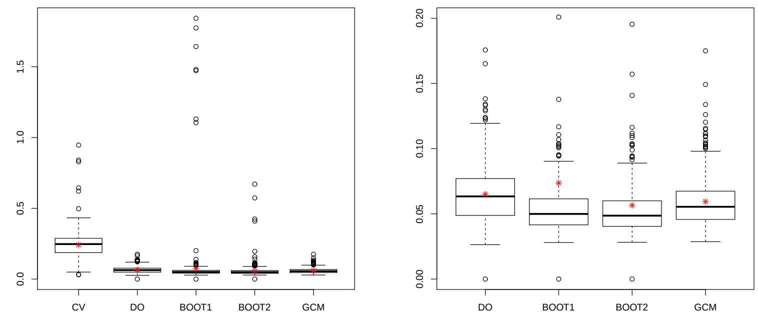


(e) Model 3

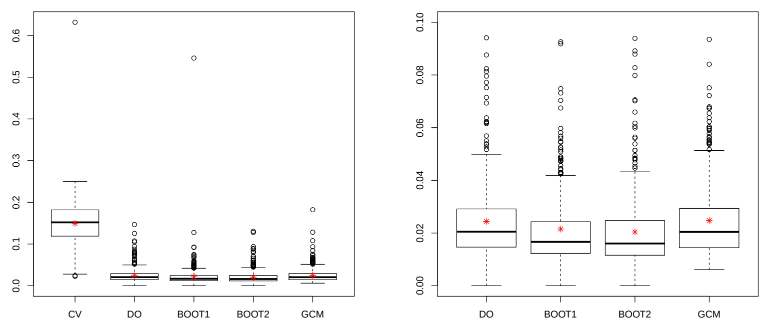


(f) Model 3

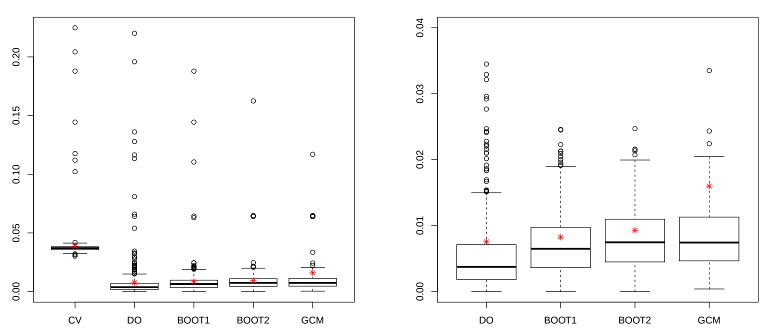
Figure 3.9: On the left, boxplots of $ISE(\hat{h})$ for Models 1-3, $\lambda = 0.2$ and $n = 100$, where, from left to right, $\hat{h} = h_{CV}$ (first box), h_{DO} (second box), h_{BOOT_1} (third box), h_{BOOT_2} (fourth box) and h^*_{GCM} (fifth box). Boxplots on the right side are just a zoom of boxplots on the left side. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.



(a) Model 4, $k = 2$ (b) Model 4, $k = 2$



(c) Model 4, $k = 3$ (d) Model 7, $k = 3$



(e) Model 5 (f) Model 5

Figure 3.10: On the left, boxplots of $ISE(\hat{h})$ for Models 4-5, $\lambda = 0.1$ and $n = 100$, where, from left to right, $\hat{h} = h_{CV}$ (first box), h_{DO} (second box), h_{BOOT_1} (third box), h_{BOOT_2} (fourth box) and h^*_{GCM} (fifth box). Boxplots on the right side are just a zoom of boxplots on the left side. In red, the mean of $\log\left(\hat{h}/h_{MISE}\right)$ and $ISE(\hat{h})$.

3.7 Real data analysis

The two new bootstrap bandwidth selectors proposed for the estimator given in (3.1) are illustrated by computing the nonparametric hazard rate estimation for a real data set that can be found in the package MASS (see Venables and Ripley, 2002) of the free software R. As in Section 3.6, h_{BOOT1} and h_{BOOT2} are empirically compared with h_{CV}, h_{DO} and h^*_{GCM} bandwidth selectors.

The chosen data set consists of the record of women with Pima Indian ancestry currently living close to Phoenix, Arizona, who were tested for diabetes. Specifically, body mass index has been analyzed via hazard rate estimation considering three different populations: those women who turned out to be diagnosed with diabetes ($n = 177$), women whose diabetes test resulted negative ($n = 355$), and finally, the whole women population tested for the disease ($n = 532$).

Figure 3.11 shows the nonparametric hazard rate estimation using: 1) h_{BOOT1}, h_{BOOT2}, h_{CV}, h^*_{GCM} considering the estimator given in (3.1), and 2) h_{DO} considering the local linear estimator (see Gámiz et al., 2016). The value $\lambda = 0.2$ was used in the pilot bandwidth to carry out the real data application and the Gaussian kernel, K, has been considered. As for the weight function, w, it has been chosen as in Section 3.6. Furthermore, the values for the bandwidth selectors obtained are shown in Table 3.1, while the required computing times (in seconds) for the five selectors are included in Table 3.2. Just for the sake of comparing both conditional and unconditional hazard rate bandwidths, Figure 3.12 presents the preceding estimations all together.

Figure 3.11 clearly shows that h_{CV} seems to overestimate excessively the underlying hazard rate. It is worth mentioning that, when considering the non diabetic women and also the whole population data sets, $\mathrm{CV}(h)$ is a decreasing function, for all $h \in [0.01, 10]$. As a consequence, h_{CV} turns out to be the largest value h in which CV is computed, that is, $h_{CV} = 10$ (see Table 3.1). On the other hand, h_{DO} seems to produce undersmoothed local linear estimations of the hazard rate function. Moreover, it is worth pointing out that the main disadvantage brought about by local linear estimators is shown in Figures 3.11 and Figure 3.12: there is an area in which the estimation happens to be negative. Finally, h_{BOOT1}, h_{BOOT2} and h^*_{GCM} produce smooth hazard rate estimations in every case, which seem the most accurate. It is worth pointing out that both h_{BOOT1} and h^*_{GCM} have similar CPU time, while h_{CV} and h_{BOOT2} are by far the worst in terms of CPU time (see Table 3.2). The bandwidth selector h_{DO} is somewhat in the middle of this range of computational efficiency.

	h_{BOOT1}	h_{BOOT2}	h_{CV}	h^*_{GCM}	h_{DO}
Diabetic women	2.127	1.9514	9.9869	2.6002	3.1099
Non diabetic women	2.4879	2.2392	10	3.0246	3.1229
Whole population	2.0294	1.8246	10	2.727	3.1179

Table 3.1: Smoothing parameters using the five methods for the Body Mass Index considering the diabetic women data set, the non diabetic women data set and the whole population.

	$CPU_{h_{BOOT1}}$	$CPU_{h_{BOOT2}}$	$CPU_{h_{CV}}$	$CPU_{h^*_{GCM}}$	$CPU_{h_{DO}}$
Diabetic women	2.04	298.58	134.78	2.24	95.98
Non diabetic women	2.49	1680.45	589.37	3.15	99.8
Whole population	5.05	6618.26	1349.5	4.73	101.2

Table 3.2: CPU times (in seconds) for the five bandwidth selectors for the Body Mass Index considering the diabetic women data set, the non diabetic women data set and the whole population.

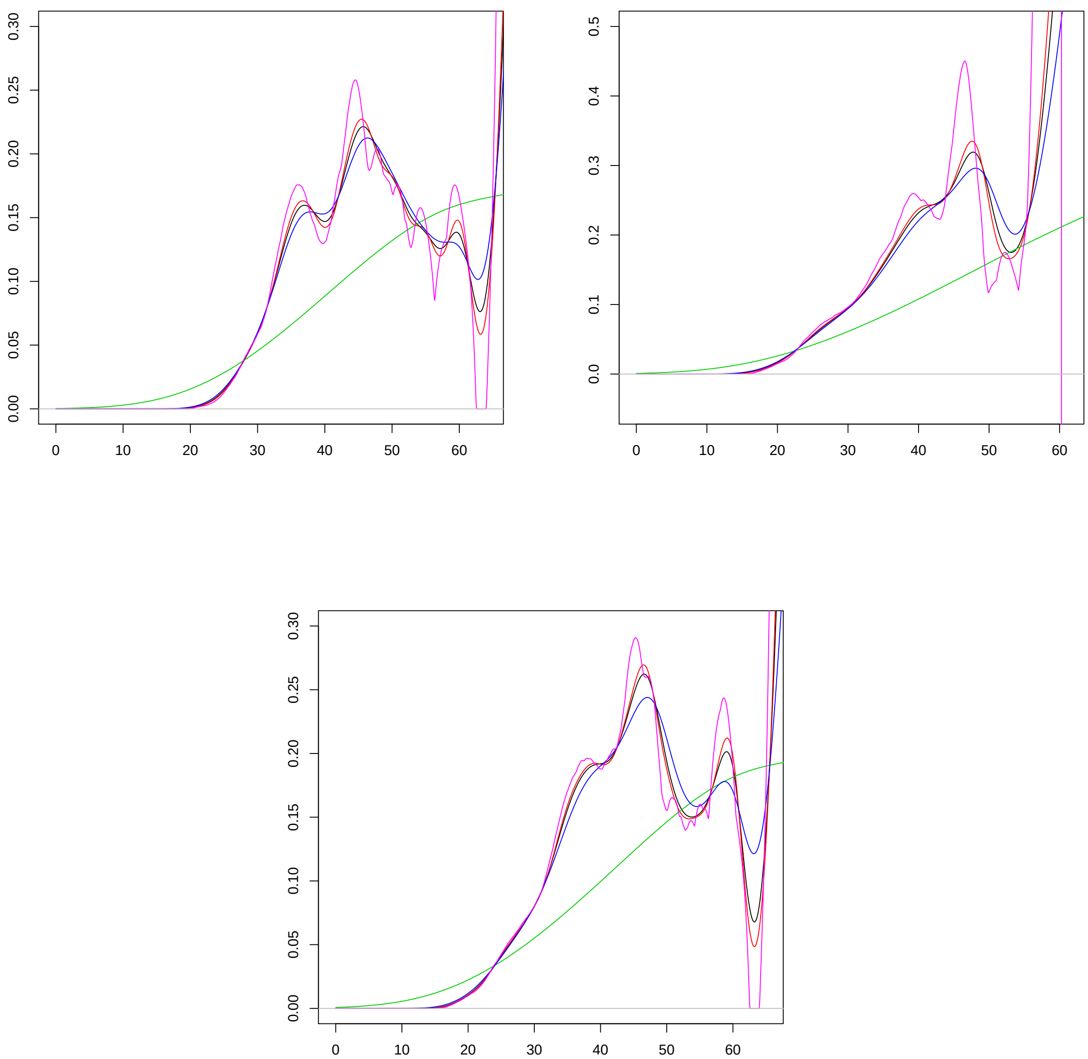
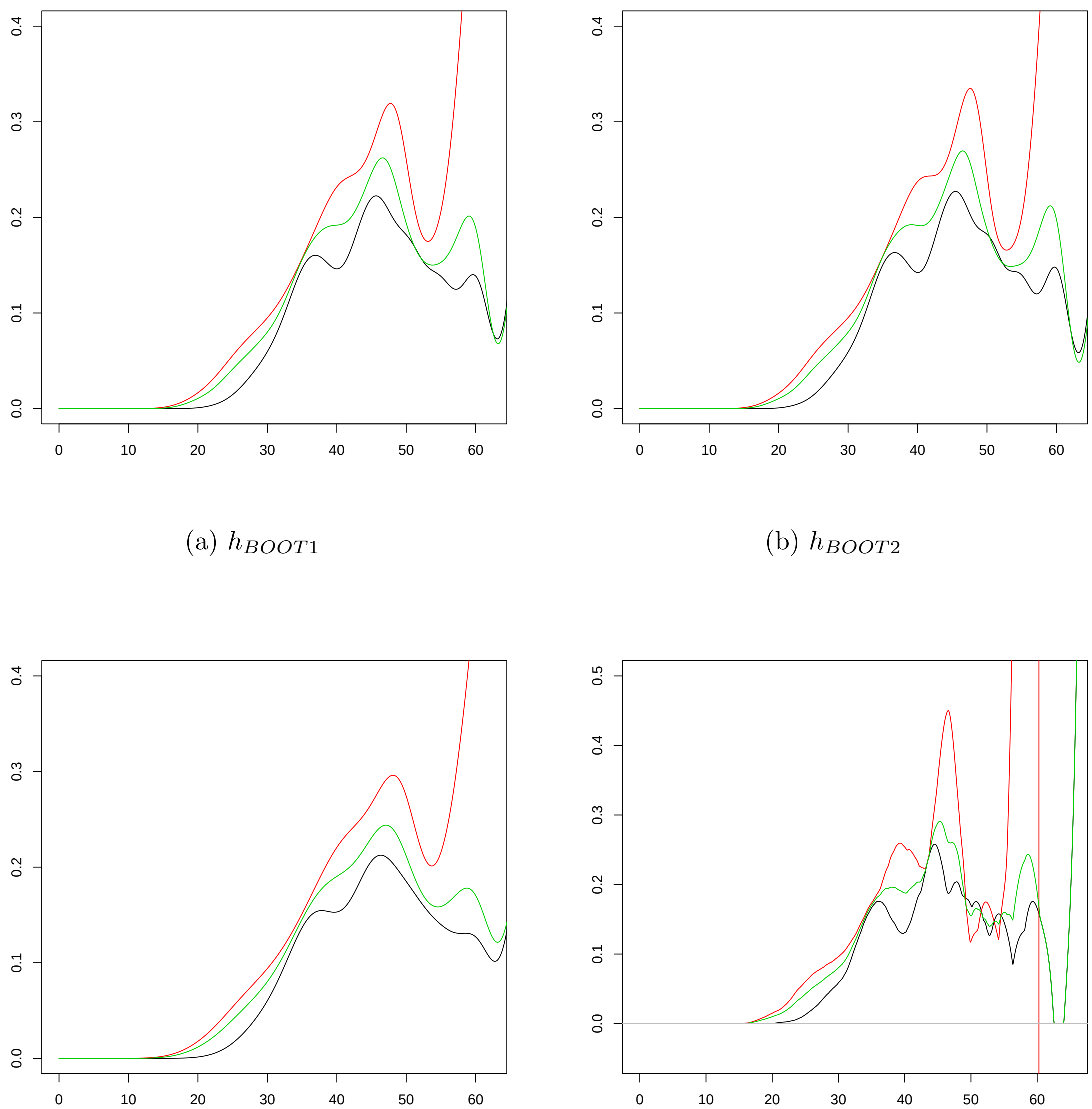


Figure 3.11: Nonparametric hazard rate estimation for the Body Mass Index considering the diabetic women data set (top left), non diabetic women data set (top right) and the whole population (bottom), using the smoothing parameters h_{BOOT1} (black line), h_{BOOT2} (red line), h_{CV} (green line), h^*_{GCM} (blue line) and h_{DO} (pink line).



(a) h_{BOOT1}

(b) h_{BOOT2}

(c) h^*_{GCM}

(d) h_{DO}

Figure 3.12: Nonparametric hazard rate estimation for the Body Mass Index data set considering the whole population (green lines), diabetic women (black lines) and non diabetic women (red lines), using smoothing parameters h_{BOOT1} (top left panel), h_{BOOT2} (top right panel), h^*_{GCM} (bottom left panel) and h_{DO} (bottom right panel).

Chapter 4

Bandwidth selection in nonparametric kernel estimation for statistical matching and prediction

4.1 Introduction

While considerable literature on bandwidth selection for kernel based nonparametric density and regression estimation exists, the problem of nonparametric prediction has largely been ignored. To the best of our knowledge, such selection methods do not exist despite the relevance and frequency of such prediction problems in practice. They include for example any situation aiming to predict counterfactuals like in impact evaluation, also known as treatment effect estimation (see the recent compendium of Frölich and Sperlich (2019)). Other examples are statistical matching or data matching (see Waal et al., 2011; Rässler, 2016; or Eurostat, 2013; and references therein), the imputation of missings (see e.g. Horton and Lipsitz, 2001; Rubin, 2004; or Su et al., 2011; and references therein), or the simulation of scenarios. Note that the problem of extrapolation far outside of the support of the observed covariates is not considered in this chapter. Indeed, this is a problem that would go beyond the here described ones (see Li and Heckman, 2003). Neither is bandwidth selection considered in stationary time series in the following. In this context, various bandwidth and other model selection methods have been developed, see e.g. the review of Antoniadis et al. (2009) or Tschernig and Yang (2000).

We will study several situations with the following three features in common: one can think of a regression model with Y being the response variable, and X the observed explanatory variables. There is a sample, denoted as 'source', in which both variables are given such that one can conduct a nonparametric regression estimation. At the same time, another sample or population, denoted as 'target' is available or simulated, for which the same (as for 'source') potential response Y is not obtained. The basic assumption is that the dependence structure between X and Y, or in this case the conditional expectation of Y given X, $m(x) := \mathbb{E}[Y|X = x]$ is the same in both populations. In data matching, and similarly when imputing missing values, the response variable Y is not sampled for the target sample. There are scenarios where the explanatory random variable, X, of the target refers to an artificial, maybe future, population, for which we just cannot observe any Y. In counterfactual exercises one typically has Y observed for the target sample, but under a different situation, called 'treatment'. Then the source sample is used to impute the potential Y of the target group for the situation 'without treatment'. The difference between the observed Y (under treatment) and the imputed (without treatment) gives the so-called 'treatment effect for the treated'. Similarly, in our application, where Y is the variable salary, we obtain from men (source population) an estimate of a wage equation with X containing 'sector', 'education', and 'experience' (measured in years and months). This estimated regression is then used to predict the corresponding wages of women (target population) to compare these predictions with the observed wages giving us an estimate of the gender wage gap. Having done all nonparametrically, this difference cannot be explained by the choice of a particular parametric wage model. It either has to be explained by unobserved differences in characteristics which are not correlated with the included ones, or it has to be recognized as a gap due to gender discrimination.

It is evident that these are relevant statistical methods for which, when executed nonparametrically, one needs to choose a bandwidth. For example, in impact evaluation this has been studied by Frölich (2005), Galdo et al. (2008) or Häggström and de Luna (2014). We concentrate here on global bandwidths, because this is the most popular approach in practice. Local bandwidths are also discussed. The presentations in this chapter are not only limited to local constant prediction methods, but local linear methods are also considered in Section 4.3. Indeed, it gets much more cumbersome for local linear methods as it is already the case for bandwidth selection in regression. It will be obvious that, though the theory for the new method is developed along the problem of bandwidth choice, the procedure applies more generally to model selection for prediction. Notice that bandwidth selection for regression and for prediction should give similar results when the source and target samples show the same distribution in X. The simulations show that the new method fulfils this.

As mentioned above, there is a vast literature on bandwidth selection for kernel density and regression estimation (see the reviews of Heidenreich et al., 2013 and Köhler et al., 2014; and references therein). The prediction problems we outline above are typically regression based; therefore one should first have a look at the existing bandwidth selection procedures for regression. As outlined in Köhler et al. (2014), they can essentially be divided into two groups: cross-validation (CV) and plug-in. As this may be misleading, we prefer to speak of those that try to minimize the integrated or averaged squared error (ISE or ASE) on the one side, and those that try to minimize the expected ISE or ASE, known as MISE and MASE, on the other side. Without pronouncing in favour of one or the other, there are good reasons in regression to be more interested in minimizing the ISE (rather than the MISE) as we may want to get the best bandwidth for their sample fit, not the population. Also the implementation seems to be simpler for CV methods which partly explains their popularity. For prediction, however, the first argument is no longer valid because here one may be more interested in minimizing the MASE (mean average squared error). Notice that, to the best of our knowledge, this is the first proposal which looks for bandwdidth for prediction. Only future research can decide on whether this will the most accepted approach.

More specifically, our proposal relies on the so-called smooth bootstrap approach, see Cao and González-Manteiga (1993). That is, its aim is to draw bootstrap resamples from a nonparametric pilot estimate of the joint distribution of (X, Y). For the original source sample, and for each bootstrap resample, $m(x)$ is estimated. This allows us to approximate the mean squared error of $\hat{m}(x)$ for any x inside the support of X. Finally, these errors are averaged over the X_i observed in the target sample. Furthermore, a closed analytical form for a proxied version of the MASE estimate is obtained. In order to compute the MASE, we construct the regression estimator with the source sample, which is evaluated in the observations of the variable X of the target sample. Then, we define the ASE (average squared error) as the average of the squared differences between the regression estimator (computed with the source sample) evaluated in the obsevations of X of the target sample and the theoretical regression function of the observations of the variable X of the target sample. Finally, the MASE is the expectation of the ASE conditional on the target sample. In other words, the ASE is the average squared error between the regression estimator computed with the source sample and the theoretical regression function, both of them evaluated in the observations of the target sample. As mentioned above, the MASE is the expectation of the ASE conditional on the target sample.

The aforementioned closed expression for the proxied MASE simplifies the procedure drastically making it quite attractive in practice. One may argue that the exactness of this MASE approximation hinges a lot upon the pilot estimate. Yet, in order to find the optimal bandwidth (or model) it suffices that our MASE approximations shown below take their minimum at the same value of the bandwidth as the true but unknown MASE. The simulation studies show that this is actually the case.

Suppose a complete sample $\{(X_i^0, Y_i^0)\}_{i=1}^{n_0}$ from the source population is provided, with $X^0 \sim f^0$ and $m(x) := \mathbb{E}\left(Y^0|X^0 = x\right)$. As for the target population, we are only provided with observations $\{X_i^1\}_{i=1}^{n_1}$ from density f^1 which is potentially different from f^0. Let's assume that $m(x) = \mathbb{E}\left(Y^1|X^1 = x\right)$. We consider two different (although related) problems: (a) predicting the unobserved $\{Y_i^1\}_{i=1}^{n_1}$ or (b) estimating $\mathbb{E}[Y^1] = \mathbb{E}[\mathbb{E}[Y^1|_{X^1}]] = \mathbb{E}[m(X^1)]$. In practice, if some 'outcomes' Y_i are observed for the target population, their conditional expectation is supposed to differ from $m(\cdot)$, but it does not have to. Recall our example of outcome under treatment vs without, or see our application in Section 4.6 were $m(x)$ is the expected wage for men given x.

For prediction, we have to estimate $m(\cdot)$ by a Nadaraya-Watson or local linear estimator $\hat{m}_h$ with bandwidth h (see Sections 4.2 and 4.3, respectively). Let us suppress for a moment the super-indices denoting from now on the source sample with Y observed. The challenge is to find a bandwidth h which is MASE optimal for our prediction problem. The pointwise MSE, and afterwards the MASE are approximated by their bootstrap versions. We follow the bootstrap method proposed by Cao and González-Manteiga (1993). Consider two pilot bandwidths, g_X, g_Y. The smoothed bootstrap method has two versions which proceed as follows:

SB 1

- Draw bootstrap resamples $(X_1^*, Y_1^*), (X_2^*, Y_2^*), \ldots, (X_n^*, Y_n^*)$ from the two dimensional distribution function

$$\hat{F}_{g_X}(x, y) \quad = \quad n^{-1} \sum_{i=1}^{n} \mathbb{1}_{\{Y_i \leq y\}} \int_{-\infty}^{x} K_{g_X}(t - X_i)\, dt.$$

SB 2

- Draw bootstrap resamples $(X_1^*, Y_1^*), (X_2^*, Y_2^*), \ldots, (X_n^*, Y_n^*)$ from the two dimensional density function

$$\hat{f}_{g_X, g_Y}(x, y) = n^{-1} \sum_{i=1}^{n} K_{g_X}(x - X_i) K_{g_Y}(y - Y_i).$$

Algorithms SB1 and SB2 fulfill the following properties:

SB 1

- X^* has bootstrap marginal density $\hat{f}_{g_X}(x)$.

- Y^* has bootstrap marginal distribution $\hat{F}_n^Y(y) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{Y_i \le y\}}$.
- $\mathbb{E}^*\left[Y^* \middle| X^* = x\right] = \hat{m}_{g_X}(x)$.
- The conditional distribution $Y^*|_{X^*=x}$ is

$$\hat{F}_{g_X}(y|_x) = \frac{\frac{1}{n}\sum_{i=1}^n K_{g_X}(x - X_i)\mathbb{1}_{\{Y_i \le y\}}}{\frac{1}{n}\sum_{i=1}^n K_{g_X}(x - X_i)}.$$

- $\sigma^{*2}(x) = \hat{\sigma}^2_{g_X}(x)$, where $\hat{\sigma}^2_{g_X}(z) = \hat{m}_{2,g_X}(z) - \hat{m}^2_{g_X}(z)$, with

$$\tilde{m}_{k,g_X} = \frac{\sum_{i=1}^n K_{g_X}\left(x - X_i\right) Y_i^k}{\sum_{i=1}^n K_{g_X}\left(x - X_i\right)}, \forall k \ge 2.$$

SB 2

- X^* has bootstrap marginal density $\hat{f}_{g_X}(x)$.
- Y^* has bootstrap marginal distribution $\hat{F}_{g_Y}^Y(y) = \frac{1}{n}\sum_{i=1}^n K_{g_Y}\left(y - Y_i\right)$.
- $\mathbb{E}^*\left[Y^* \middle| X^* = x\right] = \hat{m}_{g_X}(x)$.
- The conditional distribution $Y^*|_{X^*=x}$ is

$$\hat{F}_{g_X,g_Y}(y|_x) = \frac{\frac{1}{n}\sum_{i=1}^n K_{g_X}(x - X_i)\int_{-\infty}^y K_{g_Y}(t - Y_i)\, dt}{\frac{1}{n}\sum_{i=1}^n K_{g_X}(x - X_i)}.$$

- $\sigma^{*2}(x) = \hat{\sigma}^2_{g_X}(x) + {g_X}^2\,\mu_2(K)$.

From now on, SB2 version of the smoothed bootstrap is going to be considered. Notice that SB1 is the limit case of SB2 when $g_Y \to 0^+$, for fixed n.

This chapter proceeds as follows: Sections 4.2 and 4.3 contain the closed expressions obtained for the MSE and the MASE of the proxy Nadaraya-Watson and local linear regression estimators, respectively. Section 4.4 collects the main asymptotic results for the Nadaraya-Watson estimator. An extensive simulation study is developed in Section 4.5. Finally, a real data application about estimating the gender wage gap is carried out in Section 4.6.

4.2 Nadaraya-Watson regression estimator

4.2.1 Closed-form expressions for the $MSE_x(h)$ and $MSE_x^*(h)$

In this section, our aim is to establish a closed-form expression for the bootstrap version of the mean squared error (namely, MSE). For that purpose, consider the Nadaraya-Watson estimator of the regression function, $\hat{m}_h^{NW}(x)$, proposed by Nadaraya (1964) and Watson (1964).

Denote $\hat{m}_h^{NW}(x) = \dfrac{\hat{\Psi}_h(x)}{\hat{f}_h(x)}$, where $\hat{\Psi}_h(x) = n^{-1}\sum_{i=1}^n K_h(x - X_i)Y_i$ and $\hat{f}_h(x) = n^{-1}\sum_{i=1}^n K_h(x - X_i)$. Let us consider $f(x)$ and $\Psi(x) := f(x)m(x)$, then

$$
\begin{aligned}
\hat{m}_h^{NW}(x) - m(x) &= \frac{\hat{\Psi}_h(x)}{\hat{f}_h(x)} - \frac{\Psi(x)}{f(x)} = \left(\frac{\hat{\Psi}_h(x)}{\hat{f}_h(x)} - \frac{\Psi(x)}{f(x)}\right)\left[\frac{\hat{f}_h(x)}{f(x)} + \left(1 - \frac{\hat{f}_h(x)}{f(x)}\right)\right] \\
&= \frac{\hat{\Psi}_h(x) - m(x)\hat{f}_h(x)}{f(x)} + \frac{\left(\hat{m}_h^{NW}(x) - m(x)\right)\left(f(x) - \hat{f}_h(x)\right)}{f(x)},
\end{aligned}
\tag{4.1}
$$

since $\hat{\Psi}_h(x) - m(x)\hat{f}_h(x) = \hat{\Psi}_h(x) - \Psi(x) + m(x)\left(f(x) - \hat{f}_h(x)\right)$, the second term in the right hand side of (4.1) seems to be negligible with respect to the first one. Thus, we will consider the proxy estimator of $m(x)$, $\tilde{m}_h^{NW}(x)$, that corresponds to considering only the first term in (4.1):

$$
\tilde{m}_h^{NW}(x) := m(x) + \frac{\hat{\Psi}_h(x) - m(x)\hat{f}_h(x)}{f(x)},
$$

which gives

$$\begin{aligned}\tilde{m}_h^{NW}(x) - m(x) &= \frac{\hat{\Psi}_h(x) - m(x)\hat{f}_h(x)}{f(x)} \\ &= \frac{1}{nf(x)}\sum_{i=1}^{n} K_h(x - X_i)(Y_i - m(x)). \qquad (4.2)\end{aligned}$$

As can be seen in (4.2), the first term in the right hand side is unknown ($m(x)$) and the denominator of the second summand depends on the underlying theoretical density f. Hence, this is not a real estimator but some theoretical approximation of $\hat{m}_h^{NW}(x)$. A detailed motivation focused on how to obtain this expression is included in Appendix C.

The key idea (previously introduced in Chapter 3) of using (4.2) instead of the classical Nadaraya-Watson estimator is to get rid of the randomness of the denominator. The main reason is that, by doing so, we are able to work out an exact expression for the bootstrap version of a standard measure of performance of the proxy estimator given in (4.2). Finally, we minimize it in order to define a bandwidth selector. We will begin working out a closed expression for the bootstrap mean squared error of (4.2) (namely, MSE^*), given a fixed point x. Thus, we can define a local bandwidth selector by minimizing MSE^*.

Specifically, given that $\mathbb{E}^*\left[Y^*|_{X^*=x}\right] = \hat{m}_{g_X}(x)$, the smoothed bootstrap version of (4.2) is just:

$$\tilde{m}_h^{NW*}(x) = \hat{m}_{g_X}^{NW}(x) + \frac{1}{n\hat{f}_{g_X}(x)}\sum_{i=1}^{n} K_h(x - X_i^*)(Y_i^* - \hat{m}_{g_X}^{NW}(x)), \qquad (4.3)$$

where X^* has bootstrap marginal density $\hat{f}_{g_X}$.

Exact expression for $MSE_x(h)$

Consider the explanatory random variable X, and x an interior point of the support of X. The mean squared error of the proxy estimator in (4.2) is given by:

$$\begin{aligned} MSE_x(h) &= \mathbb{E}\left[\left(\tilde{m}_h^{NW}(x) - m(x)\right)^2\right] \\ &= \left(\mathbb{E}\left[A_1(x)\right]\right)^2 + Var[A_1(x)], \end{aligned} \tag{4.4}$$

where $A_1(x) = \tilde{m}_h^{NW}(x) - m(x)$.

The following result shows an explicit expression for the MSE of $\tilde{m}_h^{NW}$, given in (4.2). The proof of Theorem 9 is included in Appendix C.

Theorem 9 *If x is an interior point of the support of X, the sample $(X_i, Y_i), i = 1, \ldots, n$ is iid, K is a bounded symmetric density function and $f(x) \neq 0$, then the MSE_x of $\tilde{m}_h^{NW}$ (4.2) can be expressed as follows:*

$$MSE_x(h) = \left(\frac{n-1}{nf(x)^2}\right)\left[(K_h * q_x)(x)\right]^2 + \frac{1}{nf(x)^2}\left[(K_h)^2 * p_x\right](x),$$

where $p_x(z) = \left(\sigma^2(z) + (m(z) - m(x))^2\right) f(z)$, $q_x(z) = (m(z) - m(x))f(z)$ and $\sigma^2(x) := Var\left(Y|_{X=x}\right)$ stands for the volatility function.

Closed-form expression for the bootstrap version of the MSE

In order to avoid Monte Carlo approximation as well as lots of computations, a local MSE-oriented bandwidth selector is proposed by minimizing the smoothed bootstrap version of the MSE_x. The aim is to work out a closed expression for:

$$\begin{aligned} MSE_x^*(h) &= \mathbb{E}^*\left[\left(\tilde{m}_h^{NW*}(x) - \hat{m}_{g_X}^{NW}(x)\right)^2\right] \\ &= \left(\mathbb{E}^*\left[A_1^*(x)\right]\right)^2 + Var^*[A_1^*(x)], \end{aligned}$$

where $A_1^*(x) = \tilde{m}_h^{NW*}(x) - \hat{m}_{g_X}^{NW}(x)$ given in expression (4.3).

Theorem 10 *Consider the explanatory random variable X, and x an interior point of the support of X, K a symmetric bounded density function, $\hat{f}_{g_X}(x) \neq 0$, and $(X_i, Y_i), i = 1, \ldots, n$ a simple random sample. The smoothed bootstrap version of the*

MSE_x of the proxy estimator given in (4.2) results in:

$$
\begin{aligned}
MSE_x^*(h) &= \frac{1}{n\hat{f}_{g_X}^2(x)}\left[\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^{n}\left[(K_h)^2 * K_{g_X}\right](x-X_i) + \left[(K_h)^2 * \hat{p}_{x,g_X}\right](x)\right] \\
&\quad + \left(\frac{n-1}{n\hat{f}_{g_X}^2(x)}\right)\left[(K_h * \hat{q}_{x,g_X})(x)\right]^2, \qquad (4.5)
\end{aligned}
$$

where $\hat{p}_{x,g_X}(z) = \left(\hat{\sigma}_{g_X}^2(z) + (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}(z)$ and $\hat{q}_{x,g_X}(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(z)$.

Theorem 10 is proven in Appendix C. It is to be emphasised that (4.5) is exact and not just an approximation. In other words, it is not needed to draw any bootstrap resamples and (4.5) can be calculated directly.

Remark 15 *All convolutions in equation (4.5) can be expressed in terms of convolutions of K_h (or $(K_h)^2$) with K_g. For instance, carrying on with computations with the last term in (4.5) leads to:*

$$
\begin{aligned}
\hat{q}_{x,g_X}(z) &= (\hat{m}_{g_X}(z) - \hat{m}_{g_X}(x))\,\hat{f}_{g_X}(z) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(K_{g_X}(z-X_i)Y_i - \hat{m}_{g_X}(x)\sum_{i=1}^{n}K_{g_X}(z-X_i)\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}K_{g_X}(z-X_i)\cdot(Y_i - \hat{m}_{g_X}(x)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left[K_h * \hat{q}_{x,g_X}\right]^2(x) &= K_h * \hat{q}_{x,g_X}(x)^2 \\
&= \left[K_h * \frac{1}{n}\sum_{i=1}^{n}K_{g_X}(\cdot - X_i)\cdot(Y_i - \hat{m}_{g_X}(x))\right](x)^2 \\
&= \left[\frac{1}{n}\sum_{i=1}^{n}\left[K_h * K_{g_X}(\cdot - X_i)\right](x)\cdot(Y_i - \hat{m}_{g_X}(x))\right]^2 \\
&= \left[\frac{1}{n}\sum_{i=1}^{n}K_h * K_{g_X}(x - X_i)\cdot(Y_i - \hat{m}_{g_X}(x))\right]^2.
\end{aligned}
$$

Similar arguments are used with the remaining terms in equation (4.5). The resulting expression is given by:

$$
\begin{aligned}
MSE_x^*(h) &= \frac{1}{n\hat{f}_{g_X}^2(x)}\left[\frac{n-1}{n^3}\cdot\left[\sum_{i=1}^{n} K_h * K_{g_X}(x-X_i)\right.\right. \\
&\quad \cdot\left(Y_i-\hat{m}_{g_X}(x)\right)\Big]^2+\frac{1}{n^2}\sum_{i=1}^{n}\left[(K_h)^2 * K_{g_X}\right](x-X_i) \\
&\quad \left.\cdot\left[Y_i-\hat{m}_{g_X}(x)\right]^2+\frac{g_Y^2\mu_2(K)}{n^2}\sum_{i=1}^{n}\left[(K_h)^2 * K_{g_X}\right](x-X_i)\right].
\end{aligned}
$$

Given an interior point x of the support of X, a local bootstrap bandwidth selector for the regression estimator can now be defined as the minimizer, in h, of $MSE_x^*(h)$, given in Theorem 10,

$$
h_{MSE_x^*}^{NW} = h_{MSE_x^*} = \arg\min_{h>0} MSE_x^*(h). \tag{4.6}
$$

4.2.2 Closed-form expressions for $MASE(h)$ and $MASE^*(h)$

The next step is notationally cumbersome because for the MASE we need to carefully distinguish between source and target sample, and have therefore to use the super-indices again. Consider $\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\}$ a simple random sample coming from a source population, (X^0, Y^0), and $\left(X_1^1, \ldots, X_{n_1}^1\right)$ a simple random sample coming from a target population, X^1. We are interested in estimating the mean of Y^1 in the target population. Let $f^0(x)$ be the density of X^0 in the source population and $f^1(x)$ be the density in the target population. Let $m(x) = \mathbb{E}\left[Y^0 | X^0 = x\right]$ be the conditional mean function in the source population, which is assumed to be the same in the target population. Consider $\mathbb{E}_0[\cdot]$ the expectation in the source population conditional on the target sample, i.e., for any random variable Z, $\mathbb{E}_0\left[Z\right] = \mathbb{E}\left[Z \mid X_j^1, \forall j \in \{1, \ldots, n_1\}\right]$. The mean of Y^1 in the target population is:

$$
\mathbb{E}\left[Y^1\right] = \int m(x) f^1(x)\, dx,
$$

which can be estimated by a matching estimator (see Heckman et al., 1997):

$$\widehat{\mathbb{E}\left[Y^1\right]} = \frac{1}{n_1}\sum_{j=1}^{n_1} \hat{m}_h\left(X_j^1\right), \tag{4.7}$$

where $\hat{m}_h(x)$ is a nonparametric regression estimator of $m(x)$. Although both plug-in and cross-validation procedures have been used in the past to select the optimal local bandwidth (see Frölich, 2005), given a fixed point x, to the best of our knowledge, no one has already dealt with the problem of selecting the optimal global bandwidth. In the following, a bootstrap approach is introduced in order to define a global smoothing parameter for bandwidth selection in regression estimation in terms of prediction.

Exact expression for $MASE(h)$

In order to select an appropiate bandwidth for the estimator in (4.7), our aim is to minimize the error criteria given by:

$$\mathbb{E}\left[\left(\hat{\theta}_h - \theta\right)^2\right],$$

where $\theta = \mathbb{E}\left[Y^1\right]$, $\hat{\theta}_h = \frac{1}{n_1}\sum_{i=1}^{n_1} \hat{m}_h(X_i^1)$ and $\hat{m}_h(\cdot)$ is the Nadaraya-Watson regression estimator computed with the source sample. Carrying on with calculations and denoting by $\hat{\theta}^{(1)} = \frac{1}{n_1}\sum_{i=1}^{n_1} m(X_i^1)$, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\left(\hat{\theta}_h - \theta\right)^2\right] &= \mathbb{E}\left[\left(\hat{\theta}_h - \hat{\theta}^{(1)} + \hat{\theta}^{(1)} - \theta\right)^2\right] \\
&= \mathbb{E}\left[\left(\hat{\theta}_h - \theta^{(1)}\right)^2\right] + \mathbb{E}\left[\left(\theta^{(1)} - \theta\right)^2\right] + 2\mathbb{E}\left[\left(\hat{\theta}_h - \theta^{(1)}\right) \cdot \left(\theta^{(1)} - \theta\right)\right].
\end{aligned}$$

Specifically,

$$\mathbb{E}\left[\left(\hat{\theta}_h - \hat{\theta}^{(1)}\right)^2\right] = \mathbb{E}\left[\left(\frac{1}{n_1}\sum_{i=1}^{n_1} \hat{m}_h\left(X_i^1\right) - \frac{1}{n_1}\sum_{i=1}^{n_1} m\left(X_i^1\right)\right)^2\right]$$

$$= \mathbb{E}\left[\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\left(\hat{m}_h\left(X_i^1\right) - m\left(X_i^1\right)\right)\right)^2\right]$$

$$= \mathbb{E}\left[\left(\int\left(\hat{m}_h(x) - m(x)\right)\, dF_{X^1,n_1}(x)\right)^2\right], \tag{4.8}$$

where F_{X^1,n_1} is the empirical distribution function of $\left(X_1^1, \ldots, X_{n_1}^1\right)$. Expression (4.8) happens to be the empirical version of

$$\mathbb{E}\left[\left(\int\left(\hat{m}_h(x) - m(x)\right)\, dF_1(x)\right)^2\right], \tag{4.9}$$

which is quite different to we are focused on in this chapter, given by:

$$\mathbb{E}\left[\int\left(\hat{m}_h(x) - m(x)\right)^2\, dF_1(x)\right]. \tag{4.10}$$

In Proposition 1, an upper bound for expression (4.9) is given. Moreover, expression (4.10) looks for an estimated regression function close to the theoretical m. This is a way of considering this issue from a 'prediction' point of view. In the following, a definition is introduced and a proposition is stated.

Definition 4 *Denote $Y_1^1, \ldots, Y_{n_1}^1$ the unobservable values of the variable Y coming from the target population. Given that $\hat{m}\left(X_i^1\right)$ is the prediction of Y_i^1, $i = 1, \ldots, n_1$, the average prediction error is given by:*

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right].$$

Proposition 1 *If F_1 is the distribution function of the target population and $\hat{m}_h$, the estimated regression function. Then, an upper bound for expression (4.9) is given by:*

$$\mathbb{E}\left[\left(\int\left(\hat{m}_h(x) - m(x)\right)\, dF_1(x)\right)^2\right] \;\leq\; \mathbb{E}\left[\int\left(\hat{m}_h(x) - m(x)\right)^2\, dF_1(x)\right].$$

On the other hand, the average prediction error is given by:

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] = \int \sigma^2(x)\, dF_1(x) + \mathbb{E}\left[\int \left(\hat{m}_h(x) - m(x)\right)^2 dF_1(x)\right],$$

in other terms, expression (4.10) plus a constant which does not depend on h.

The proof of Propositon 1 is included in Appendix C. According to Proposition 1, expression (4.10) seems to be an appropriate error criteria to consider. On the one hand, the average prediction error is equal to (4.10) plus a constant which does not depend on h, which means that the minimum in h of expression (4.10) is the minimum in h of the average prediction error. On the other hand, expression (4.10) is an upper bound of expression (4.9), which means that a minimum in h of expression (4.10) would produce good results for expression (4.9).

We now focus on the $MSE_x(h)$ given in expression (4.4). Our objective is to somehow take into account the target population, so that a fair error criterion to minimize seems to be:

$$\int MSE_x(h)\, dF_1(x) = \int \mathbb{E}\left[\tilde{m}_h^{NW}(x) - m(x)\right]^2 dF_1(x), \tag{4.11}$$

where F_1 stands for the distribution function of the target population.

If the integral in (4.11) is approximated by a sum, a straightforward estimation of expression (4.11) is given by:

$$\frac{1}{n_1}\sum_{j=1}^{n_1}\mathbb{E}_0\left[\left(\tilde{m}_h^{NW}(X_j^1) - m(X_j^1)\right)^2\right],$$

which is the mean average squared error (namely, MASE) of the proxy estimator defined in (4.2).

The key idea is to work out a closed expression for the bootstrap version of the

MASE of the proxy estimator introduced in (4.2). This is a prediction error criteria:

$$\begin{aligned} MASE_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\mathbb{E}_0\left[\left(\tilde{m}_h^{NW}(X_j^1)-m(X_j^1)\right)^2\right] \\ &= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_1^0\left(X_j^1\right)]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\left(X_j^1\right)\right]\right)^2, \end{aligned}$$

where $A_1^0\left(X_j^1\right)=\dfrac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))$.

The next result presents an explicit expression for the MASE of $\tilde{m}_h^{NW}$, given in (4.2). The proof of Theorem 11 is detailed in Appendix C.

Theorem 11 *Let $\{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)\}$ be a simple random sample coming from the source population, (X^0,Y^0), and $\left(X_1^1,\ldots,X_{n_1}^1\right)$, a simple random sample coming from the target population, X^1. Consider K a symmetric bounded density. Then, the prediction error MASE admits the following representation:*

$$\begin{aligned} MASE_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left(\left[K_h*q_{X_j^1}^0\right](X_j^1)\right)^2\right. \\ &\quad\left.+\frac{1}{n_0}\left[(K_h)^2*p_{X_j^1}^0\right](X_j^1)\right], \end{aligned} \tag{4.12}$$

where $q_x^0(z)=(m(z)-m(x))f^0(z)$ and $p_x^0(z)=\left(\sigma_0^2(z)+(m(z)-m(x))^2\right)f^0(z)$.

Closed-form expression for the bootstrap version of the MASE

Our goal is to compute a MASE-oriented bandwidth selector for prediction in regression, by minimizing the bootstrap version of the MASE of the proxy estimator given in (4.2). Accordingly, we focus on working out an exact expression for the bootstrap MASE, given by:

$$\begin{aligned} MASE^*_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\mathbb{E}_0^*\left[\left(\tilde{m}_h^{NW*}(X_j^1)-\hat{m}_{g_X}^{NW}(X_j^1)\right)^2\right]\right]= \\ &= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*[A_1^{0*}\left(X_j^1\right)]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\left(X_j^1\right)\right]\right)^2, \end{aligned}$$

where $A_1^{0*}\left(X_j^1\right) = \dfrac{1}{n_0 \hat{f}_{g_X}^0(X_j^1)} \sum_{i=1}^{n_0} K_h(X_j^1 - X_i^{0*})(Y_i^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))$ and $\hat{f}_{g_X}^0, \hat{m}_{g_X}^{NW}$ stand for the Parzen-Rosenblatt kernel estimator of the source density and the Nadaraya-Watson estimator of the regression function, respectively, with pilot bandwidth $g_X > 0$.

Theorem 12 *If $\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\}$ is a simple random sample coming from the source population, (X^0, Y^0); $\left(X_1^1, \ldots, X_{n_1}^1\right)$ is a simple random sample coming from the target population, X^1; and K a symmetric bounded density, then the bootstrap version of the prediction error MASE admits the following representation:*

$$\begin{aligned} MASE^*_{\tilde{m}_h^{NW}, X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1} \frac{1}{\hat{f}_{g_X}^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left(\left[K_h * \hat{q}^0_{X_j^1, g_X}\right](X_j^1)\right)^2\right. \\ &\quad + \frac{1}{n_0}\left[(K_h)^2 * \hat{p}^0_{X_j^1, g_X}\right](X_j^1) \\ &\quad \left. + \frac{g_Y^2 \mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2 * K_{g_X}\right](X_j^1 - X_i^0)\right], \end{aligned} \tag{4.13}$$

where $\hat{p}^0_{x,g_X}(z) = \left(\hat{\sigma}^2_{0,g_X}(z) + (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}^0(z)$ and $\hat{q}_x^0(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}^0(z)$ and $\hat{\sigma}^2_{0,g_X}(z) = \hat{m}_{2,g_X}(z) - \hat{m}^2_{g_X}(z)$, where, $\forall k \geq 2$, $\hat{m}_{k,g_X}(z) = \dfrac{\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)(Y^0)^k}{\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)}$.

The proof is included in Appendix C.

In the particular case of considering SB1, that is, if $g_Y = 0$, expression (4.13) can be simplified as shown in the next result.

Corollary 1 *If $g_Y = 0$, then expression (4.13) happens to be:*

$$\begin{aligned} MASE^*_{\tilde{m}_h^{NW}, X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1} \frac{1}{\hat{f}_{g_X}^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left(\left[K_h * \hat{q}^0_{X_j^1, g_X}\right](X_j^1)\right)^2\right. \\ &\quad \left. + \frac{1}{n_0}\left[(K_h)^2 * \hat{p}^0_{X_j^1, g_X}\right](X_j^1)\right]. \end{aligned}$$

The proof of Corollary 1 is straightforward. On the other hand, Corollary 2 collects the closed expression for $MASE^*_{\hat{m}_h, X^1}$ if we work out the convolutions in expression (4.13) further.

Corollary 2 *If K is a Gaussian kernel, then expression (4.13) can be rewritten as follows:*

$$\begin{aligned} MASE^*_{\tilde{m}_h, X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X^1_j)^2}\left[\frac{n_0-1}{n_0^3}\cdot\left[\sum_{i=1}^{n_0}K_h * K_{g_X}\left(X^1_j - X^0_i\right)\right.\right. \\ &\quad \cdot\left(Y^0_i - \hat{m}_{g_X}\left(X^1_j\right)\right)\Big]^2 + \frac{1}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2 * K_{g_X}\right](X^1_j - X^0_i) \\ &\quad \left.\cdot\left[Y^0_i - \hat{m}_{g_X}\left(X^1_j\right)\right]^2 + \frac{g_Y^2\mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2 * K_{g_X}\right](X^1_j - X^0_i)\right]. \end{aligned} \tag{4.14}$$

The proof of Corollary 2 is collected in Appendix C. For the sake of simplicity, we will consider $g = g_X = g_Y$ in the following. A global bandwidth selector for prediction in regression based on bootstrap ideas is now defined as the minimizer, in h, of $MASE^*_{\tilde{m}_h^{NW}, X^1}(h)$:

$$h^{NW}_{BOOT} = h_{MASE^*_{\tilde{m}_h^{NW}, X^1}} = \arg\min_{h>0} MASE^*_{\tilde{m}_h^{NW}, X^1}(h). \tag{4.15}$$

Remark 16 *Note that the computation of h^{NW}_{BOOT} does no longer require the use of Monte Carlo approximation nor the computation of $\hat{f}^1_g$, which is the nonparametric estimation of the density function of the target population, f^1, considering the target sample and pilot bandwidth $g > 0$.*

In order to understand whether this procedure could work, one has first to answer two questions: Is the MASE of the approximation (4.2), in which $\tilde{m}_h^{NW}$ substitutes $\hat{m}_h^{NW}$, a useful approximation for the MASE of $\hat{m}_h^{NW}$? And is also the MASE bootstrap analogue a useful approximation for it? Figure 4.1 reveals that the optimal bandwidths for both estimators are very close. It also shows that the first approximation is much more sensitive to the bandwidth than the original one, which makes it numerically much easier to find the minimizer. Furthermore, for almost all simulations the minimizer in the bootstrap world is very close to the true one. This answers both questions with a clear 'yes'.

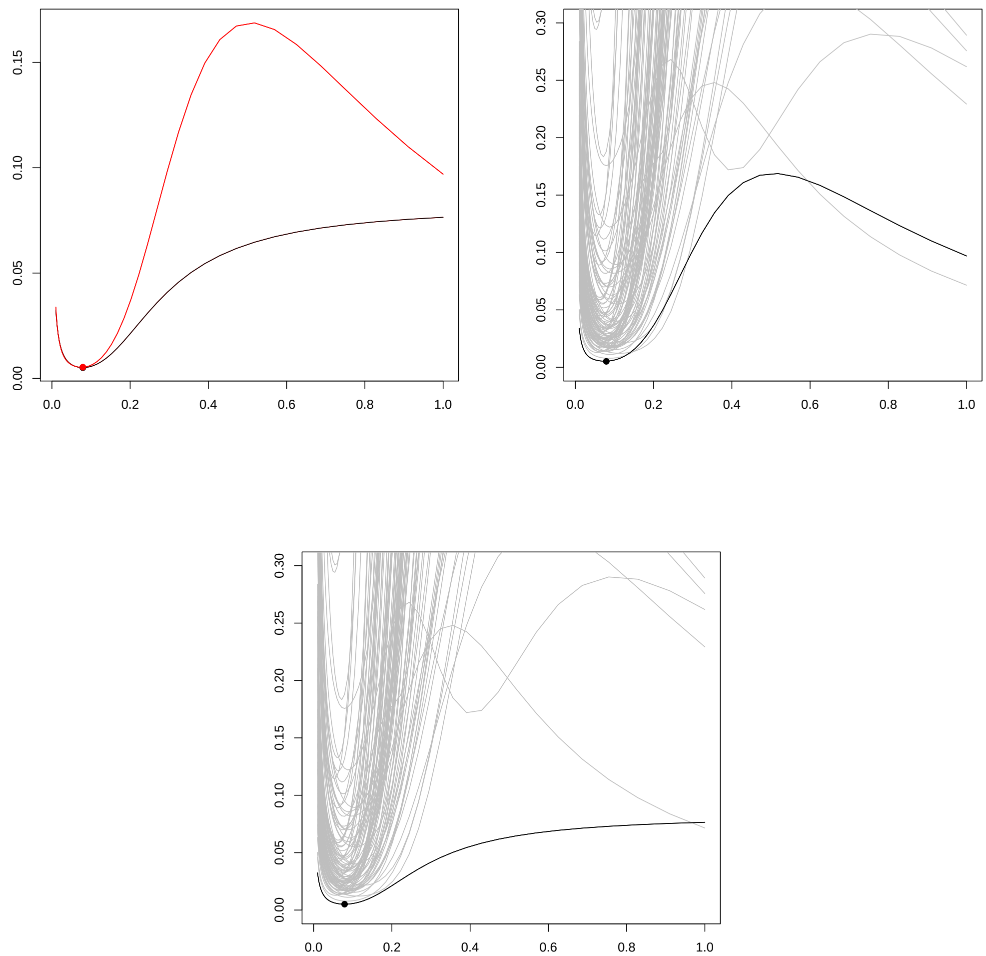


Figure 4.1: On the top left, mean average squared error of the Nadaraya-Watson regression estimator using Monte Carlo approximation (black line) and of its approximated version given in (4.2) (red line). On the top right, MASE of the approximated estimator given in (4.2) (black line) and bootstrap version of MASE for the approximated estimator given in (4.2) considering 100 different samples (grey lines) $\{X^0, Y^0\}$ of size $n_0 = 100$ and $\{X^1\}$ of size $n_1 = 100$. At the bottom, mean average squared error of the Nadaraya-Watson regression estimator approximated by simulation (black line) and bootstrap version of MASE for the approximated estimator given in (4.2) (grey lines). Specifically, X_0 has been simulated from a $\beta(2,4)$ population, and $Y_0 = m(X^0) + 0.4\epsilon$, where $m(x) = 2x^{1/2}$ and ϵ comes from a standard normal population. On the other hand, X_1 has been simulated from a $\beta(4,2)$ population.

4.3 Local linear regression estimator

4.3.1 Closed-form expressions for $MSE_x(h)$ and $MSE^*_x(h)$

Similarly as in Section 4.2, our aim is to propose a new MASE-oriented smoothing parameter for prediction in regression estimation. We will do it considering the local linear regression estimator, $\hat{m}_h^{LL}$ (see Fan and Gijbels (1992) for a deeper insight on the nonparametric estimator), given by

$$\hat{m}_h^{LL}(x) = \frac{\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h) - \hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)}{\hat{s}_2(x;h)\hat{s}_0(x;h) - h^2\hat{s}_1(x;h)^2}, \tag{4.16}$$

where

$$\begin{aligned}
\hat{\Psi}_h^0(x;h) &= n^{-1}\sum_{i=1}^n K_h(X_i - x)Y_i,\\
\hat{\Psi}_h^1(x;h) &= n^{-1}\sum_{i=1}^n (X_i - x)K_h(X_i - x)Y_i,\\
\hat{s}_0(x;h) &= n^{-1}\sum_{i=1}^n K_h(X_i - x) = \hat{f}_h(x),\\
\hat{s}_1(x;h) &= n^{-1}h^{-2}\sum_{i=1}^n (X_i - x)K_h(X_i - x), \text{ and}\\
\hat{s}_2(x;h) &= n^{-1}h^{-2}\sum_{i=1}^n (X_i - x)^2K_h(X_i - x).
\end{aligned}$$

Firstly, the target is to establish an explicit expression for the bootstrap version of the mean squared error (namely, MSE). For that purpose, consider $(X_i, Y_i), i = 1, \ldots, n$ a simple random sample. Assume that f is two times differentiable, the kernel K is symmetric and bounded with zero mean. Consider, additionally, the following approximation of the local linear kernel regression estimator, given by:

$$\tilde{m}_h^{LL}(x) = \frac{\Theta^1}{\Theta^0} + \frac{\hat{\Theta}_h^1\Theta^0 - \Theta^1\hat{\Theta}_h^0}{(\Theta^0)^2}, \tag{4.17}$$

where $\Theta^0 = f(x)^2\mu_2(K)$, $\Theta^1 = m(x)f(x)^2\mu_2(K)$, $\hat{\Theta}_h^0 = \hat{s}_2(x;h)\hat{s}_0(x;h) - h^2\hat{s}_1(x;h)$

and $\hat{\Theta}_h^1 = \hat{\Psi}_h^0(x;h)\hat{s}_2(x;h) - \hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)$.

As can be seen in (4.17), both the numerator and the denominator depend on the underlying theoretical density, f. Thus, this is not a real estimator but some theoretical approximation of $\hat{m}_h^{LL}(x)$. In Appendix C, a detailed motivation describing how to obtain this expression can be found.

Remark 17 *Other straightforward approximations have also been considered, such as:*

$$\tilde{m}_h^{LL}(x) = \frac{\gamma^1}{\gamma^0} + \frac{\hat{\gamma}_h^1\gamma^0 - \gamma^1\hat{\gamma}_h^0}{\left(\gamma^0\right)^2}, \tag{4.18}$$

where $\gamma^0 = f(x)^2\mu_2(K) - f'(x)\mu_2(K)$*,* $\gamma^1 = m(x)f(x)^2\mu_2(K)$*,* $\hat{\gamma}_h^0 = \hat{s}_2(x;h)\hat{s}_0(x;h) - h^2\hat{s}_1(x;h)$ *and* $\hat{\gamma}_h^1 = \hat{\Psi}_h^0(x;h)\hat{s}_2(x;h) - \hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)$*. However, they all led to numerical problems due to the great deal of zeros brought about when computing* γ^0 *and* $\left(\gamma^0\right)^2$*, expressions placed in the denominator of expression (4.18).*

Once again, the key idea of using (4.17) instead of the classical local linear estimator is to get rid of the randomness of the denominator in order to work out an explicit expression for the bootstrap version of a standard measure of performance of the proxy estimator given in (4.17), and then minimize it so as to define an optimal bandwidth selector. Following the same steps as in Section 4.2, we will start working out a closed-form expression for the bootstrap mean squared error of (4.17) (MSE^*), given a fixed point x. Afterwards, a local bandwidth selector can be defined by minimizing the objective function previously computed.

Consider the SB2 version of the smoothed bootstrap (see Cao and González-Manteiga, 1993) to compute the proxy smoothed bootstrap local linear estimator, $\hat{m}_h^{LL*}(x)$. For the sake of simplicity, consider a pilot bandwidth $g > 0$. The smoothed bootstrap version of (4.17) is just:

$$\tilde{m}_h^{LL*}(x) = \frac{\hat{\Theta}_g^1}{\hat{\Theta}_g^0} + \frac{\hat{\Theta}_h^{1*}\hat{\Theta}_g^0 - \hat{\Theta}_g^1\hat{\Theta}_h^{0*}}{\left(\hat{\Theta}_g^0\right)^2}, \tag{4.19}$$

where $\hat{\Theta}_g^0 = \hat{f}_g(x)^2\mu_2(K)$, $\hat{\Theta}_g^1 = \hat{m}_g(x)\hat{f}_g(x)^2\mu_2(K)$, $\hat{\Theta}_h^{0*} = \hat{s}_2^*(x;h)\hat{s}_0^*(x;h) - h^2\hat{s}_1^*(x;h)^2$, $\hat{\Theta}_h^{1*} = \hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h) - \hat{s}_1^*(x;h)\hat{\Psi}_h^{1*}(x;h)$ and

$$\begin{aligned}
\hat{\Psi}_h^{0*}(x;h) &= n^{-1}\sum_{i=1}^n K_h(X_i^* - x)Y_i^*, \\
\hat{\Psi}_h^{1*}(x;h) &= n^{-1}\sum_{i=1}^n (X_i^* - x)K_h(X_i^* - x)Y_i^*, \\
\hat{s}_0^*(x;h) &= n^{-1}\sum_{i=1}^n K_h(X_i^* - x) = \hat{f}_h^*(x), \\
\hat{s}_1^*(x;h) &= n^{-1}h^{-2}\sum_{i=1}^n (X_i^* - x)K_h(X_i^* - x), \text{ and} \\
\hat{s}_2^*(x;h) &= n^{-1}h^{-2}\sum_{i=1}^n (X_i^* - x)^2K_h(X_i^* - x).
\end{aligned}$$

Moreover, $\{(X_1^*, Y_1^*), \ldots, (X_n^*, Y_n^*)\}$ are bootstrap resamples drawn from the two dimensional density function $\hat{f}_g(x,y) = n^{-1}\sum_{i=1}^n K_g(x - X_i)K_g(y - Y_i)$.

Exact expression for $MSE_x(h)$

Consider x an interior point of the support of the random variable X^0. The mean squared error of the proxy estimator in (4.17) is given by:

$$\begin{aligned}
MSE_x(h) &= \mathbb{E}\left[\left(\tilde{m}_h^{LL}(x) - m(x)\right)^2\right] \\
&= \left(\Theta^0\right)^{-2}\mathbb{E}\left[\hat{\Psi}_h^0(x;h)^2\hat{s}_2^2(x;h)\right] + \left(\Theta^0\right)^{-2}\mathbb{E}\left[\hat{s}_1^2(x;h)\hat{\Psi}_h^1(x;h)^2\right] \\
&\quad -2\left(\Theta^0\right)^{-2}\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)\right] \\
&\quad -2\left(\Theta^0\right)^{-3}\Theta^1\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2^2(x;h)\hat{s}_0(x;h)\right] \\
&\quad +2\left(\Theta^0\right)^{-3}\Theta^1h^2\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1^2(x;h)\right] \\
&\quad +2\left(\Theta^0\right)^{-3}\Theta^1\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_0(x;h)\hat{s}_1(x;h)\hat{s}_2(x;h)\right] \\
&\quad -2\left(\Theta^0\right)^{-3}\Theta^1h^2\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_1^3(x;h)\right]
\end{aligned}$$

$$+\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_2^2(x;h)\hat{s}_0^2(x;h)\right]+h^4\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_1^4(x;h)\right]$$
$$-2h^2\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_2(x;h)\hat{s}_0(x;h)\hat{s}_1^2(x;h)\right].$$

In the following theorem, a closed expression for the MSE of the proxy local linear estimator is worked out, given a fixed point x.

Theorem 13 *Consider x an interior point of the support of the random variable X^0, K a symmetric bounded density function and $\{(X_1,Y_1),\ldots(X_n,Y_n)\}$ a simple random sample. The MSE of the proxy estimator given in (4.17) results in:*

$$\begin{aligned}
MSE_x(h) \;=\;& \left(\Theta^0\right)^{-2}h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*b_x^0\right]^2(x)\left[K_h*d_x^2\right]^2(x)\\
&+\frac{(n-1)(n-2)}{n^3}\Big(\left[(K_h)^2*a_x^0\right](x)\left[K_h*d_x^2\right]^2(x)\\
&+\left[(K_h)^2*d_x^4\right](x)\left[K_h*b_x^0\right]^2(x)\\
&+2\left[(K_h)^2*b_x^2\right](x)\left[K_h*b_x^0\right](x)\left[K_h*d_x^2\right](x)\Big)\\
&+\frac{n-1}{n^3}\left[(K_h)^2*a_x^2\right](x)\left[(K_h)^2*d_x^4\right](x)\Bigg]\\
&-2\left(\Theta^0\right)^{-2}h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*b_x^0\right](x)\\
&\left[K_h*d_x^2\right](x)\left[K_h*d_x^1\right](x)\left[K_h*b_x^1\right](x)\\
&+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2*b_x^1\right](x)\left[K_h*d_x^2\right](x)\left[K_h*b_x^1\right](x)\\
&+\left[(K_h)^2*a_x^1\right](x)\left[K_h*d_x^2\right](x)\left[K_h*d_x^1\right](x)\\
&+\left[(K_h)^2*d_x^3\right](x)\left[K_h*b_x^1\right](x)\left[K_h*b_x^0\right](x)\\
&+\left[(K_h)^2*b_x^3\right](x)\left[K_h*b_x^0\right](x)\left[K_h*d_x^1\right](x)\Big]\\
&+\frac{n-1}{n^3}\Big(\left[(K_h)^2*b_x^1\right](x)\left[(K_h)^2*b_x^3\right](x)\\
&+\left[(K_h)^2*a_x^1\right](x)\left[(K_h)^2*d_x^3\right](x)\Big)\Bigg]\\
&+\left(\Theta^0\right)^{-2}h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*d_x^1\right]^2(x)\left[K_h*b_x^1\right]^2(x)\\
&+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2*a_x^2\right](x)\left[K_h*d_x^1\right]^2(x)
\end{aligned}$$

$$+\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^1\right]^2(x)$$
$$+2\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^1\right](x)\left[K_h * b_x^1\right](x)\Big]$$
$$+\frac{n-1}{n^3}\left(2\left[(K_h)^2 * b_x^2\right]^2(x)+\left[(K_h)^2 * d_x^2\right](x)\left[(K_h)^2 * a_x^2\right](x)\right)\Bigg]$$
$$-2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^0\right](x)$$
$$\left[K_h * d_x^2\right]^2(x)\left[K_h * d_x^0\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)$$
$$+\left[K_h * b_x^0\right](x)\left[(K_h)^2 * d_x^4\right](x)\left[K_h * d_x^0\right](x)+\left[(K_h)^2 * d_x^2\right](x)$$
$$\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)+\left[(K_h)^2 * b_x^0\right](x)\left[K_h * d_x^2\right]^2(x)\Big]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * b_x^4\right](x)\left[(K_h)^2 * \hat{f}_g^0\right](x)\Bigg]$$
$$+2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)$$
$$\left[K_h * d_x^1\right]^2(x)+\frac{(n-1)(n-2)}{n^3}\Big[3\left[(K_h)^2 * b_x^1\right](x)\left[K_h * d_x^2\right](x)$$
$$\left[K_h * d_x^1\right](x)+2\left[(K_h)^2 * d_x^3\right](x)\left[K_h * b_x^0\right](x)\left[K_h * d_x^1\right](x)\Big]$$
$$+3\frac{n-1}{n^3}\left[(K_h)^2 * b_x^1\right](x)\left[(K_h)^2 * d_x^3\right](x)\Bigg]$$
$$+2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^1\right](x)\left[K_h * d_x^0\right](x)$$
$$\left[K_h * d_x^1\right](x)\left[K_h * d_x^2\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2 * d_x^3\right](x)\left[K_h * d_x^0\right](x)\left[K_h * b_x^1\right](x)$$
$$+\left[(K_h)^2 * b_x^3\right](x)\left[K_h * d_x^1\right](x)\left[K_h * d_x^0\right](x)$$
$$+\left[(K_h)^2 * d_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * b_x^1\right](x)\Big]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * b_x^3\right](x)\left[(K_h)^2 * d_x^1\right](x)\Bigg]$$
$$-2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * d_x^1\right]^3(x)\left[K_h * b_x^1\right](x)$$

$$+2\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^1\right](x)\left[K_h * d_x^1\right](x)\right.$$
$$\left.+\left[(K_h)^2 * b_x^2\right](x)\left[K_h * \hat{d}_x^1\right]^2(x)\right] + 3\frac{n-1}{n^3}\left[(K_h)^2 * b_x^2\right](x)\left[(K_h)^2 * d_x^2\right](x)\Bigg]$$
$$+\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * d_x^2\right]^2(x)\left[K_h * d_x^0\right]^2(x)\right.$$
$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * d_x^4\right](x)\left[K_h * d_x^0\right]^2(x) + \left[(K_h)^2 * d_x^0\right](x)\right.$$
$$\left.\left[K_h * d_x^2\right]^2(x) + 2\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\right]$$
$$\left.+\frac{n-1}{n^3}\left[(K_h)^2 * d_x^4\right](x)\left[(K_h)^2 * \hat{f}_g^0\right](x)\right]$$
$$+h^{-4}\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * d_x^1\right]^4(x)\right.$$
$$\left.+4\frac{(n-1)(n-2)}{n^3}\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^1\right]^2(x) + 3\frac{n-1}{n^3}\left[(K_h)^2 * d_x^2\right]^2(x)\right]$$
$$-2\,h^{-4}\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\right.$$
$$\left[K_h * d_x^1\right]^2(x) + \frac{(n-1)(n-2)}{n^3}\left[2\left[(K_h)^2 * d_x^3\right](x)\left[K_h * d_x^0\right](x)\left[K_h * d_x^1\right](x)\right.$$
$$\left.+2\left[(K_h)^2 * d_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^1\right](x)\right]$$
$$\left.+2\frac{n-1}{n^3}\left[(K_h)^2 * d_x^3\right](x)\left[(K_h)^2 * d_x^1\right](x)\right],$$

where $a_x^j(y) = (y-x)^j\left(\sigma^2(y) + m^2(y)\right)f(y)$, $b_x^j(y) = (y-x)^j m(y) f(y)$ *and* $d_x^j(y) = (y-x)^j f(y)$, *with* $j \in \mathbb{N}$.

A detailed proof of Theorem 13 can be found in Appendix C.

Closed-form expression for the bootstrap version of $MSE_x(h)$

Similarly as in Section 4.2, the target is to define a local bandwidth selector by means of minimizing an explicit expression of the MSE. Thus, Monte Carlo approximation is no longer required. Consider x an interior point of the support of the random variable X^0. The smoothed bootstrap version of the MSE of the proxy estimator given in (4.17) results in:

$$
\begin{aligned}
MSE_x^*(h) &= \mathbb{E}^*\left[\left(\tilde{m}_h^{LL*}(x) - \hat{m}_g^{LL}(x)\right)^2\right] \\
&= \left(\hat{\Theta}_g^0\right)^{-2}\mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)^2\hat{s}_2^*(x;h)^2\right] \\
&\quad + \left(\hat{\Theta}_g^1\right)^2\left(\hat{\Theta}_g^0\right)^{-4}\mathbb{E}^*\left[\hat{s}_2^*(x;h)^2\hat{s}_0^*(x;h)^2\right] \\
&\quad -2\,h^2\left(\hat{\Theta}_g^1\right)^2\left(\hat{\Theta}_g^0\right)^{-4}\mathbb{E}^*\left[\hat{s}_2^*(x;h)\hat{s}_0^*(x;h)\hat{s}_1^*(x;h)^2\right] \\
&\quad -2\left(\hat{\Theta}_g^0\right)^{-2}\mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)\hat{s}_1^*(x;h)\hat{\Psi}_h^{1*}(x;h)\right] \\
&\quad + \left(\hat{\Theta}_g^0\right)^{-2}\mathbb{E}^*\left[\hat{s}_1^*(x;h)^2\hat{\Psi}_h^{1*}(x;h)^2\right] \\
&\quad +h^4\left(\hat{\Theta}_g^1\right)^2\left(\hat{\Theta}_g^0\right)^{-4}\mathbb{E}^*\left[\hat{s}_1^*(x;h)^4\right] \\
&\quad -2\left(\hat{\Theta}_g^0\right)^{-3}\hat{\Theta}_g^1\,\mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)^2\hat{s}_0^*(x;h)\right] \\
&\quad +2\left(\hat{\Theta}_g^0\right)^{-3}\hat{\Theta}_g^1h^2\mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)\hat{s}_1^*(x;h)^2\right] \\
&\quad +2\left(\hat{\Theta}_g^0\right)^{-3}\hat{\Theta}_g^1\mathbb{E}^*\left[\hat{\Psi}_h^{1*}(x;h)\hat{s}_0^*(x;h)\hat{s}_1^*(x;h)\hat{s}_2^*(x;h)\right] \\
&\quad -2\left(\hat{\Theta}_g^0\right)^{-3}\hat{\Theta}_g^1h^2\mathbb{E}^*\left[\hat{\Psi}_h^{1*}(x;h)\hat{s}_1^*(x;h)^3\right], \qquad (4.20)
\end{aligned}
$$

where $\hat{\Theta}_g^0 = \hat{f}_g(x)^2\mu_2(K)$, $\hat{\Theta}_g^1 = \hat{m}_g^{LL}(x)\hat{f}_g(x)^2\mu_2(K)$, $\hat{\Theta}_h^{0*} = \hat{s}_2^*(x;h)\hat{s}_0^*(x;h) - h^2\hat{s}_1^*(x;h)^2$ and $\hat{\Theta}_h^{1*} = \hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h) - \hat{s}_1^*(x;h)\hat{\Psi}_h^{1*}(x;h)$.

In the following theorem, an exact expression for the bootstrap MSE is presented, given a fixed point x.

Theorem 14 *Given x an interior point of the support of X^0, K a symmetric bounded density function and $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ a simple random sample, the bootstrap version of the MSE of the proxy estimator given in (4.17) admits the following representation:*

$$
MSE_x^*(h) = \left(\hat{\Theta}_g^0\right)^{-2} h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}_{x,g}^0\right]^2(x)\left[K_h * \hat{d}_{x,g}^2\right]^2(x)\right.
$$

$$
\begin{aligned}
&+\frac{(n-1)(n-2)}{n^3}\Bigg(\left[(K_h)^2 * \hat{a}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x) \\
&+\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right]^2(x) \\
&+2\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\Bigg) \\
&+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{a}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\Bigg] \\
&-2\left(\hat{\Theta}^0_g\right)^{-2} h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x) \\
&\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x) \\
&+\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x) \\
&+\left[(K_h)^2 * \hat{a}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x) \\
&+\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x) \\
&+\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\Bigg] \\
&+\frac{n-1}{n^3}\Bigg(\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x) \\
&+\left[(K_h)^2 * \hat{a}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\Bigg)\Bigg] \\
&+\left(\hat{\Theta}^0_g\right)^{-2} h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^2(x)\left[K_h * \hat{b}^1_{x,g}\right]^2(x) \\
&+\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{a}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x) \\
&+\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right]^2(x) \\
&+2\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\Bigg] \\
&+\frac{n-1}{n^3}\left(2\left[(K_h)^2 * \hat{b}^2_{x,g}\right]^2(x)+\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{a}^2_{x,g}\right](x)\right)\Bigg] \\
&-2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g\, h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x) \\
&\left[K_h * \hat{d}^0_{x,g}\right](x)+\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) \\
&+\left[K_h * \hat{b}^0_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)+\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)
\end{aligned}
$$

$$\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) + \left[(K_h)^2 * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\Bigg]$$

$$+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}^0_g\right](x)\Bigg]$$

$$+2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)$$

$$\left[K_h * \hat{d}^1_{x,g}\right]^2(x) + \frac{(n-1)(n-2)}{n^3}\Bigg[3\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)$$

$$\left[K_h * \hat{d}^1_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\Bigg]$$

$$+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\Bigg]$$

$$+2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)$$

$$\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)$$

$$+\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)$$

$$+\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)$$

$$+\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\Bigg]$$

$$+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\Bigg]$$

$$-2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^3(x)\left[K_h * \hat{b}^1_{x,g}\right](x)$$

$$+2\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)$$

$$+\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x}\right]^2(x)\Bigg]$$

$$+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\Bigg]$$

$$+\left(\hat{\Theta}^1_g\right)^2\left(\hat{\Theta}^0_g\right)^{-4}h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\left[K_h * \hat{d}^0_{x,g}\right]^2(x)$$

$$+\frac{(n-1)(n-2)}{n^3}\Bigg[\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right]^2(x)$$

$$
\begin{aligned}
&+\left[(K_h)^2 * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x) \\
&+2\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\Big] \\
&+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}^0_g\right](x)\Bigg] \\
&+h^{-4}\left(\hat{\Theta}^1_g\right)^2\left(\hat{\Theta}^0_g\right)^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^4(x) \\
&+4\frac{(n-1)(n-2)}{n^3}\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x) \\
&+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{d}^2_{x,g}\right]^2(x)\Bigg] \\
&-2\,h^{-4}\left(\hat{\Theta}^1_g\right)^2\left(\hat{\Theta}^0_g\right)^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) \\
&\left[K_h * \hat{d}^1_{x,g}\right]^2(x)+\frac{(n-1)(n-2)}{n^3}\Big[2\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) \\
&\left[K_h * \hat{d}^1_{x,g}\right](x)+2\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\Big] \\
&+2\frac{n-1}{n^3}\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\Bigg], \qquad (4.21)
\end{aligned}
$$

where $\hat{a}^j_{x,g}(y) = (y-x)^j\left(\hat{\sigma}^2_g(y) + g^2\mu_2(K) + \hat{m}^{LL}_g(y)^2\right)\hat{f}_g(y)$, $\hat{d}^j_{x,g}(y) = (y-x)^j\hat{f}_g(y)$ *and* $\hat{b}^j_{x,g}(y) = (y-x)^j\hat{m}^{LL}_g(y)\hat{f}_g(y)$, *with* $j \in \mathbb{N}$.

Theorem 14 is proven in Appendix C.

Given an interior point x of the support of X^0, a local bootstrap bandwidth selector for the local linear regression estimator can now be defined as the minimizer, in h, of $MSE^*_x(h)$, given in Theorem 14,

$$h^{LL}_{MSE^*_x} = h_{MSE^*_x} = \arg\min_{h>0} MSE^*_x(h).$$

It is worth singling out that minimizing the closed-form expression of $MSE^*_x(h)$ provides a local bandwidth selector for regression estimation avoiding the use of Monte Carlo approximation.

4.3.2 Closed-form expressions for $MASE(h)$ and $MASE^*(h)$

In a similar way as in Section 4.2, consider $\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\}$ a simple random sample coming from a source population, (X^0, Y^0), and $\left(X_1^1, \ldots, X_{n_1}^1\right)$ a simple random sample coming from a target population, X^1. Let $f^0(x)$ be the density of X^0 in the source population and $f^1(x)$ be the density in the target population. Let $m(x) = \mathbb{E}\left[Y^0 | X^0 = x\right]$ be the conditional mean function in the source population, which is assumed to be the same in the target population. Consider $\mathbb{E}_0[\cdot]$ the expectation in the source population conditional on the target sample, i.e., $\mathbb{E}_0\left[Y\right] = \mathbb{E}\left[Y \mid X_j^1, \forall j \in \{1, \ldots, n_1\}\right]$. Our aim is to estimate the mean of Y^1 by means of the matching estimator given in (4.7), now using the local linear regression estimator. As a consequence, selecting an appropriate bandwidth selector to estimate the regression function in terms of prediction is deemed to be of utmost importance. In this sense, the mean average squared error (namely, MASE) is going to be considered and afterwards minimized so as to define the global bootstrap smoothing parameter. Once again, the key idea is to compute an exact expression for the bootstrap version of the MASE so as to avoid Monte Carlo approximation. In order to be able to do so, the approximation given in expression (4.17) is going to be considered so as to get rid of the randomness of the denominator of the classical local linear regression estimator. To the best of our knowledge, this proposal as well as the one introduced in Section 4.2 are the first global bandwidth selectors proposed for prediction in nonparametric regression.

Exact expression for $MASE(h)$

Let us denote $\Theta_j^0 = f(X_j^1)^2 \mu_2(K)$, $\Theta_j^1 = m(X_j^1) f(X_j^1)^2 \mu_2(K)$, $\hat{\Theta}_{h,j}^0 = \hat{s}_2(X_j^1; h) \cdot \hat{s}_0(X_j^1; h) - h^2 \hat{s}_1(X_j^1; h)^2$ and $\hat{\Theta}_{h,j}^1 = \hat{\Psi}_h^0(X_j^1; h) \hat{s}_2(X_j^1; h) - \hat{s}_1(X_j^1; h) \hat{\Psi}_h^1(X_j^1; h)$, $j \in \{1, \ldots, n_1\}$. The mean average squared error, which is a prediction error, of the proxy estimator in (4.17) is given by:

$$MASE_{\tilde{m}_h^{LL}, X^1}(h) \quad = \quad \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbb{E}_0 \left[\left(\tilde{m}_h^{LL}(X_j^1) - m(X_j^1) \right)^2 \right]$$

$$
\begin{aligned}
= & \frac{1}{n_1}\sum_{j=1}^{n_1}\Big[\left(\Theta_j^0\right)^{-2}\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)^2\hat{s}_2^2(X_j^1;h)\right]\\
& -2\left(\Theta_j^0\right)^{-2}\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)\hat{s}_2(X_j^1;h)\hat{s}_1(X_j^1;h)\hat{\Psi}_h^1(X_j^1;h)\right]\\
& +\left(\Theta_j^0\right)^{-2}\mathbb{E}_0\left[\hat{s}_1^2(X_j^1;h)\hat{\Psi}_h^1(X_j^1;h)^2\right]\\
& -2\left(\Theta_j^0\right)^{-3}\Theta_j^1\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)\hat{s}_2^2(X_j^1;h)\hat{s}_0(X_j^1;h)\right]\\
& +2\left(\Theta_j^0\right)^{-3}\Theta_j^1h^2\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)\hat{s}_2(X_j^1;h)\hat{s}_1^2(X_j^1;h)\right]\\
& +2\left(\Theta_j^0\right)^{-3}\Theta_j^1\mathbb{E}_0\left[\hat{\Psi}_h^1(X_j^1;h)\hat{s}_0(X_j^1;h)\hat{s}_1(X_j^1;h)\hat{s}_2(X_j^1;h)\right]\\
& -2\left(\Theta_j^0\right)^{-3}\Theta_j^1h^2\mathbb{E}_0\left[\hat{\Psi}_h^1(X_j^1;h)\hat{s}_1^3(X_j^1;h)\right]\\
& +\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}\mathbb{E}_0\left[\hat{s}_2^2(X_j^1;h)\hat{s}_0^2(X_j^1;h)\right]\\
& +h^4\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}\mathbb{E}_0\left[\hat{s}_1^4(X_j^1;h)\right]\\
& -2h^2\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}\mathbb{E}_0\left[\hat{s}_2(X_j^1;h)\hat{s}_0(X_j^1;h)\hat{s}_1^2(X_j^1;h)\right]\Big]. \qquad (4.22)
\end{aligned}
$$

Computing each expectation in (4.22) separately and in a similar manner as in expressions (C.28)-(C.38), and plugging them in (4.22) leads to state a closed expression for $MASE$, which is collected in Theorem 15.

Theorem 15 *Given K a symmetric bounded density function, a simple random sample known as source sample, $\left\{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)\right\}$, and $(X_1^1,\ldots,X_{n_1}^1)$ a simple random sample coming from the target population, then the MASE of the proxy estimator given in (4.17) is given by:*

$$
\begin{aligned}
MASE_{\tilde{m}_h^{LL},X^1}(h) = & \frac{1}{n_1}\sum_{j=1}^{n_1}\Big[\left(\Theta_j^0\right)^{-2}h^{-4}\Big[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\\
& \left[K_h*b^0_{X_j^1}\right]^2(X_j^1)\left[K_h*d^2_{X_j^1}\right]^2(X_j^1)\\
& +\frac{(n_0-1)(n_0-2)}{n_0^3}\Big(\left[(K_h)^2*a^0_{X_j^1}\right](X_j^1)\left[K_h*d^2_{X_j^1}\right]^2(X_j^1)\\
& +\left[(K_h)^2*d^4_{X_j^1}\right](X_j^1)\left[K_h*b^0_{X_j^1}\right]^2(X_j^1)\\
& +2\left[(K_h)^2*b^2_{X_j^1}\right](X_j^1)\left[K_h*b^0_{X_j^1}\right](X_j^1)\left[K_h*d^2_{X_j^1}\right](X_j^1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * a^2_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * d^4_{X^1_j}\right]\left(X^1_j\right)\bigg]\\
&-2\left(\Theta^0_j\right)^{-2} h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^0_{X^1_j}\right](X^1_j)\right.\\
&\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * b^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\right.\\
&+\left[(K_h)^2 * a^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\\
&+\left[(K_h)^2 * d^3_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\left[K_h * b^0_{X^1_j}\right](X^1_j)\\
&+\left.\left[(K_h)^2 * b^3_{X^1_j}\right](X^1_j)\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\right]\\
&+\frac{n_0-1}{n_0^3}\left(\left[(K_h)^2 * b^1_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * b^3_{X^1_j}\right]\left(X^1_j\right)\right.\\
&+\left.\left.\left[(K_h)^2 * a^1_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * d^3_{X^1_j}\right]\left(X^1_j\right)\right)\right]\\
&+\left(\Theta^0_j\right)^{-2} h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)\left[K_h * b^1_{X^1_j}\right]^2(X^1_j)\right.\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * a^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)\right.\\
&+\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right]^2(X^1_j)\\
&+2\left.\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\right]\\
&+\frac{n_0-1}{n_0^3}\left(2\left[(K_h)^2 * b^2_{X^1_j}\right]^2\left(X^1_j\right)\right.\\
&+\left.\left.\left[(K_h)^2 * d^2_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * a^2_{X^1_j}\right]\left(X^1_j\right)\right)\right]\\
&-2\left(\Theta^0_j\right)^{-3}\Theta^1_j\, h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^0_{X^1_j}\right](X^1_j)\right.\\
&\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\right.\\
&+\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[(K_h)^2 * d^4_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)+\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\\
&\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)+\left.\left[(K_h)^2 * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)\right]
\end{aligned}
$$

$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^4_{X_j^1}\right]\left(X_j^1\right)\left[(K_h)^2 * \hat{f}^0_g\right]\left(X_j^1\right)\Bigg]$$

$$+2\left(\Theta_j^0\right)^{-3}\Theta_j^1 h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^0_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)\right.$$

$$\left[K_h * d^1_{X_j^1}\right]^2(X_j^1)+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[3\left[(K_h)^2 * b^1_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)\right.$$

$$\left.\left[K_h * d^1_{X_j^1}\right](X_j^1)+2\left[(K_h)^2 * d^3_{X_j^1}\right](X_j^1)\left[K_h * b^0_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right](X_j^1)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^1_{X_j^1}\right]\left(X_j^1\right)\left[(K_h)^2 * d^3_{X_j^1}\right]\left(X_j^1\right)\right]$$

$$+2\left(\Theta_j^0\right)^{-3}\Theta_j^1 h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^1_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right](X_j^1)\right.$$

$$\left[K_h * d^1_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * d^3_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right](X_j^1)\left[K_h * b^1_{X_j^1}\right](X_j^1)\right.$$

$$+\left[(K_h)^2 * b^3_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right](X_j^1)$$

$$\left.+\left[(K_h)^2 * d^1_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)\left[K_h * b^1_{X_j^1}\right](X_j^1)\right]$$

$$\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^3_{X_j^1}\right]\left(X_j^1\right)\left[(K_h)^2 * d^1_{X_j^1}\right]\left(X_j^1\right)\right]$$

$$-2\left(\Theta_j^0\right)^{-3}\Theta_j^1 h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^1_{X_j^1}\right]^3(X_j^1)\left[K_h * b^1_{X_j^1}\right](X_j^1)\right.$$

$$+2\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * d^2_{X_j^1}\right](X_j^1)\left[K_h * b^1_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right](X_j^1)\right.$$

$$\left.+\left[(K_h)^2 * b^2_{X_j^1}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1}\right]^2(X_j^1)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^2_{X_j^1}\right](X_j^1)\left[(K_h)^2 * d^2_{X_j^1}\right](X_j^1)\right]$$

$$+\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^2_{X_j^1}\right]^2(X_j^1)\left[K_h * d^0_{X_j^1}\right]^2(X_j^1)\right.$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * d^4_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right]^2(X_j^1)\right.$$

$$+\left[(K_h)^2 * d^0_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right]^2(X_j^1)$$

$$\left.+2\left[(K_h)^2 * d^2_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right](X_j^1)\right]$$

$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^4_{X_j^1}\right](X_j^1)\left[(K_h)^2 * \hat{f}^0_g\right](X_j^1)\Big]$$
$$+h^{-4}\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^1_{X_j^1}\right]^4(X_j^1)$$
$$+4\frac{(n_0-1)(n_0-2)}{n_0^3}\left[(K_h)^2 * d^2_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right]^2(X_j^1)$$
$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^2_{X_j^1}\right]^2(X_j^1)\Bigg]$$
$$-2\,h^{-4}\left(\Theta_j^1\right)^2\left(\Theta_j^0\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^2_{X_j^1}\right](X_j^1)$$
$$\left[K_h * d^0_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right]^2(X_j^1)$$
$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[2\left[(K_h)^2 * d^3_{X_j^1}\right](X_j^1)\left[K_h * d^0_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right](X_j^1)$$
$$+2\left[(K_h)^2 * d^1_{X_j^1}\right](X_j^1)\left[K_h * d^2_{X_j^1}\right](X_j^1)\left[K_h * d^1_{X_j^1}\right](X_j^1)\Big]$$
$$+2\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^3_{X_j^1}\right](X_j^1)\left[(K_h)^2 * d^1_{X_j^1}\right](X_j^1)\Big]\Bigg], \tag{4.23}$$

where $a_x^j(y) = (y-x)^j\left(\sigma_0^2(y) + m^2(y)\right)f^0(y)$, $b_x^j(y) = (y-x)^j m(y) f^0(y)$ *and* $d_x^j(y) = (y-x)^j f^0(y)$, *with* $j \in \mathbb{N}$.

Closed-form expression for the bootstrap version of $MASE(h)$. Our aim is to define a bandwidth selector which is MASE optimal for our predicting problem. For that purpose, consider the smoothed bootstrap version of the MASE of the proxy estimator given in (4.17), which is given by:

$$\begin{aligned}
MASE^*_{\tilde{m}_h^{LL},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\mathbb{E}_0\left[\left(\tilde{m}_h^{LL}(X_j^1) - \hat{m}_g^{LL}(X_j^1)\right)^2\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\Bigg[\left(\hat{\Theta}^0_{g,j}\right)^{-2}\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)^2\hat{s}_2^2(X_j^1;h)\right] \\
&\quad -2\left(\hat{\Theta}^0_{g,j}\right)^{-2}\mathbb{E}_0\left[\hat{\Psi}_h^0(X_j^1;h)\hat{s}_2(X_j^1;h)\hat{s}_1(X_j^1;h)\hat{\Psi}_h^1(X_j^1;h)\right] \\
&\quad +\left(\hat{\Theta}^0_{g,j}\right)^{-2}\mathbb{E}_0\left[\hat{s}_1^2(X_j^1;h)\hat{\Psi}_h^1(X_j^1;h)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\Theta^1_j\mathbb{E}_0\left[\hat{\Psi}^0_h(X^1_j;h)\hat{s}^2_2(X^1_j;h)\hat{s}_0(X^1_j;h)\right]\\
&+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^2\mathbb{E}_0\left[\hat{\Psi}^0_h(X^1_j;h)\hat{s}_2(X^1_j;h)\hat{s}^2_1(X^1_j;h)\right]\\
&+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}\mathbb{E}_0\left[\hat{\Psi}^1_h(X^1_j;h)\hat{s}_0(X^1_j;h)\hat{s}_1(X^1_j;h)\hat{s}_2(X^1_j;h)\right]\\
&-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^2\mathbb{E}_0\left[\hat{\Psi}^1_h(X^1_j;h)\hat{s}^3_1(X^1_j;h)\right]\\
&+\left(\hat{\Theta}^0_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\mathbb{E}_0\left[\hat{s}^2_2(X^1_j;h)\hat{s}^2_0(X^1_j;h)\right]+h^4\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\mathbb{E}_0\left[\hat{s}^4_1(X^1_j;h)\right]\\
&-2h^2\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\mathbb{E}_0\left[\hat{s}_2(X^1_j;h)\hat{s}_0(X^1_j;h)\hat{s}^2_1(X^1_j;h)\right]\Bigg], \qquad (4.24)
\end{aligned}
$$

where $\hat{\Theta}^0_{g,j}=\hat{f}_g(X^1_j)^2\mu_2(K)$, $\hat{\Theta}^1_{g,j}=\hat{m}^{LL}_g(X^1_j)\hat{f}_g(X^1_j)^2\mu_2(K)$, $\hat{\Theta}^{0*}_{h,j}=\hat{s}^*_2(X^1_j;h)\hat{s}^*_0(X^1_j;h)-h^2\hat{s}^*_1(X^1_j;h)^2$ and $\hat{\Theta}^{1*}_{h,j}=\hat{\Psi}^{0*}_h(X^1_j;h)\hat{s}^*_2(X^1_j;h)-\hat{s}^*_1(X^1_j;h)\hat{\Psi}^{1*}_h(X^1_j;h)$. Computing each expectation in (4.24) separately and similarly as in expressions (C.39)-(C.50), and plugging them in (4.24) leads to state a closed expression for the bootstrap version of the $MASE$, which is collected in Theorem 16.

Theorem 16 *Consider K, a symmetric bounded density function, a simple random sample known as source sample, $\left\{(X^0_1,Y^0_1),\ldots,(X^0_{n_0},Y^0_{n_0})\right\}$, a simple random sample coming from the target population, $(X^1_1,\ldots,X^1_{n_1})$, and $g>0$ a pilot bandwidth, then the MASE of the proxy estimator given in (4.17) is given by:*

$$
\begin{aligned}
MASE^*_{\tilde{m}^{LL}_h,X^1}(h) \;=\; & \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\left(\hat{\Theta}^0_{g,j}\right)^{-2}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.\right.\\
&\left[K_h*\hat{b}^0_{X^1_j,g}\right]^2(X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\Bigg(\left[(K_h)^2*\hat{a}^0_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\\
&+\left[(K_h)^2*\hat{d}^4_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^0_{X^1_j,g}\right]^2(X^1_j)\\
&+2\left[(K_h)^2*\hat{b}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\Bigg)\\
&\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{a}^2_{X^1_j,g}\right](X^1_j)\left[(K_h)^2*\hat{d}^4_{X^1_j,g}\right](X^1_j)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\right.\\
&\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^1_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^1_{X^1_j,g}\right](X^1_j)\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{b}^1_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^1_{X^1_j,g}\right](X^1_j)\right.\\
&+\left[(K_h)^2*\hat{a}^1_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^1_{X^1_j,g}\right](X^1_j)\\
&+\left[(K_h)^2*\hat{d}^3_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^1_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\\
&\left.+\left[(K_h)^2*\hat{b}^3_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^1_{X^1_j,g}\right](X^1_j)\right]\\
&+\frac{n_0-1}{n_0^3}\left(\left[(K_h)^2*\hat{b}^1_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2*\hat{b}^3_{X^1_j,g}\right]\left(X^1_j\right)\right.\\
&\left.\left.+\left[(K_h)^2*\hat{a}^1_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2*\hat{d}^3_{X^1_j,g}\right]\left(X^1_j\right)\right)\right]\\
&+\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{d}^1_{X^1_j,g}\right]^2(X^1_j)\right.\\
&\left[K_h*\hat{b}^1_{X^1_j,g}\right]^2(X^1_j)+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{a}^2_{X^1_j,g}\right](X^1_j)\right.\\
&\left[K_h*\hat{d}^1_{X^1_j,g}\right]^2(X^1_j)+\left[(K_h)^2*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^1_{X^1_j,g}\right]^2(X^1_j)\\
&\left.+2\left[(K_h)^2*\hat{b}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^1_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^1_{X^1_j,g}\right](X^1_j)\right]+\frac{n_0-1}{n_0^3}\\
&\left.\cdot\left(2\left[(K_h)^2*\hat{b}^2_{X^1_j,g}\right]^2\left(X^1_j\right)+\left[(K_h)^2*\hat{d}^2_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2*\hat{a}^2_{X^1_j,g}\right]\left(X^1_j\right)\right)\right]\\
&-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}\, h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\right.\\
&\left[K_h*\hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\left[K_h*\hat{d}^0_{X^1_j,g}\right](X^1_j)\\
&+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{b}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^0_{X^1_j,g}\right](X^1_j)\right.\\
&+\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\left[(K_h)^2*\hat{d}^4_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^0_{X^1_j,g}\right](X^1_j)\\
&+\left[(K_h)^2*\hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right](X^1_j)\\
&\left.+\left[(K_h)^2*\hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h*\hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\right]\\
&\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^4_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2*\hat{f}^0_g\right]\left(X^1_j\right)\right]\\
&+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.
\end{aligned}
$$

$$\left[K_h * \hat{b}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right]^2(X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[3\left[(K_h)^2 * \hat{b}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\right.$$

$$\left.\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1) + 2\left[(K_h)^2 * \hat{d}^3_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^1_{X_j^1,g}\right]\left(X_j^1\right)\left[(K_h)^2 * \hat{d}^3_{X_j^1,g}\right]\left(X_j^1\right)\right]$$

$$+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * \hat{d}^3_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$+\left[(K_h)^2 * \hat{b}^3_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)$$

$$\left.+\left[(K_h)^2 * \hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1)\right]$$

$$\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^3_{X_j^1,g}\right]\left(X_j^1\right)\left[(K_h)^2 * \hat{d}^1_{X_j^1,g}\right]\left(X_j^1\right)\right]$$

$$-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h * \hat{d}^1_{X_j^1,g}\right]^3(X_j^1)\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1)$$

$$+2\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$\left.+\left[(K_h)^2 * \hat{b}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1}\right]^2(X_j^1)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^2_{X_j^1,g}\right](X_j^1)\left[(K_h)^2 * \hat{d}^2_{X_j^1,g}\right](X_j^1)\right]$$

$$+\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h * \hat{d}^2_{X_j^1,g}\right]^2(X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right]^2(X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * \hat{d}^4_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right]^2(X_j^1)\right.$$

$$+\left[(K_h)^2 * \hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right]^2(X_j^1)$$

$$+2\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)\Bigg]$$
$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^4_{X^1_j,g}\right](X^1_j)\left[(K_h)^2 * \hat{f}^0_g\right](X^1_j)\Bigg]$$
$$+h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{d}^1_{X^1_j,g}\right]^4(X^1_j)\right.$$
$$+4\frac{(n_0-1)(n_0-2)}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right]^2(X^1_j)$$
$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\right]$$
$$-2\,h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$
$$\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right]^2(X^1_j)$$
$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[2\left[(K_h)^2 * \hat{d}^3_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right](X^1_j)\right.$$
$$\left.+2\left[(K_h)^2 * \hat{d}^1_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right](X^1_j)\right]$$
$$\left.+2\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^3_{X^1_j,g}\right](X^1_j)\left[(K_h)^2 * \hat{d}^1_{X^1_j,g}\right](X^1_j)\right]\Bigg], \tag{4.25}$$

where $\hat{a}^j_{x,g}(y) = (y-x)^j\left(\hat{\sigma}^2_g(y) + g^2\mu_2(K) + \hat{m}^{LL}_g(y)^2\right)\hat{f}_g(y)$, $\hat{d}^j_{x,g}(y) = (y-x)^j\hat{f}_g(y)$ *and* $\hat{b}^j_{x,g}(y) = (y-x)^j\hat{m}^{LL}_g(y)\hat{f}_g(y)$, *with* $j \in \mathbb{N}$.

A bootstrap bandwidth selector for prediction in regression is now defined as the minimizer, in h, of $MASE^*_{\tilde{m}^{LL}_h}(h)$, given by expression (4.25):

$$h^{LL}_{BOOT} = h_{MASE^*_{\tilde{m}^{LL}_h}} = \arg\min_{h>0} MASE^*_{\tilde{m}^{LL}_h}(h).$$

Remark 18 *Note that the computation of* h^{LL}_{BOOT} *does no longer require the use of Monte Carlo approximation nor the computation of* $\hat{f}^1_g$, *which is the nonparametric estimation of the density function of the target population,* f^1, *considering the target sample and pilot bandwidth* $g > 0$.

Remark 19 *The approximation given in (4.17) (the bootstrap version of which is given in (4.19)), is an accurate one in terms of* $MASE$ *whenever the range of the* *sample coming from the source population contains the range of the sample coming from the target population.*

Furthermore, Figure 4.2 reveals that, considering the area where the three functions attain their minimum, both $MASE_{\tilde{m}_h^{LL}}(h)$ (expression (4.23)) and its bootstrap version, $MASE^*_{\tilde{m}_h^{LL}}(h)$ (expression (4.25)), are good approximations of $MASE_{\hat{m}_h^{LL}}$.

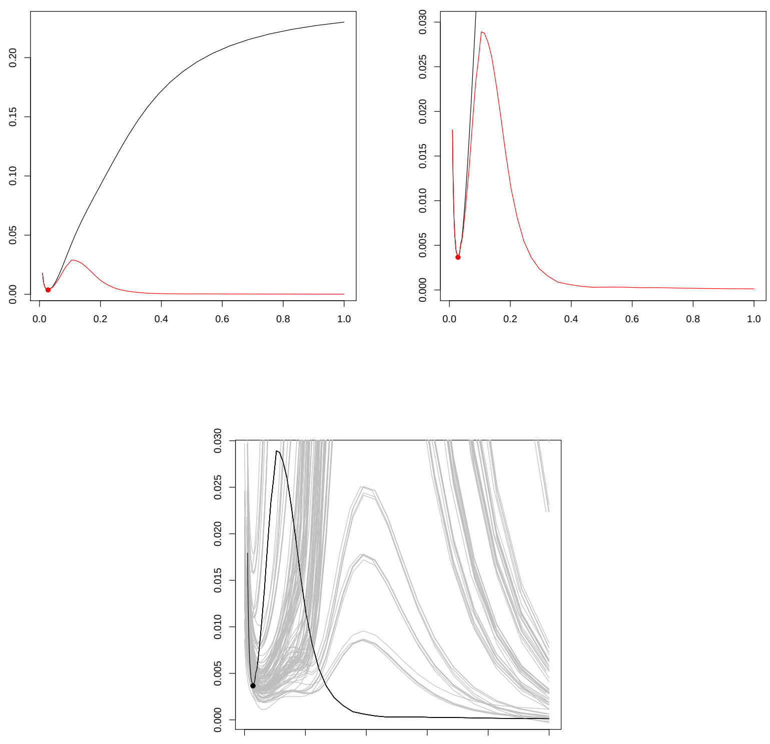


Figure 4.2: On the top left, mean average squared error of the local linear regression estimator using Monte Carlo approximation (black line) and of its approximated version given in (4.17) (red line). Plot on the top right is just a zoom of plot at the top on the left. At the bottom, MASE of the approximated estimator given in (4.17) (black line) and bootstrap version of MASE for the approximated estimator given in (4.17) considering 100 different samples (grey lines) $\{X^0, Y^0\}$ of size $n_0 = 100$ and $\{X^1\}$ of size $n_1 = 100$. Specifically, X^0 has been simulated from a $\beta(2,2)$ population, and $Y^0 = m(X^0) + 0.1\epsilon$, where $m(x) = 1 + \sin(1 + \pi x^4)$ and ϵ comes from a standard normal distribution. On the other hand, X^1 has been simulated from a $\beta(4,2)$ population.

4.4 Asymptotic theory: Nadaraya-Watson regression estimator

In this section, the aim is to establish the asymptotic rate of convergence for the bootstrap bandwidth selector which minimizes the MISE of the proxy Nadaraya-Watson estimator. In order to do so, we start by analysing the theoretical analogue to $MASE_{\tilde{m}_h^{NW},X^1}$, and afterwards its bootstrap version. The proof of theorems, lemmas, propositions and corollaries of this section are collected in Appendix C.

4.4.1 Asymptotic expression for the crierion function

We start by computing the theoretical analogue to $MASE_{\tilde{m}_h^{NW},X^1}(h)$ given in (4.12), which is going to be denoted by $MISE^a(h)$ since it is the MISE of the proxy estimator given in (4.2). That is, $MISE^a(h) := \mathbb{E}\left[\int \left(\tilde{m}_h^{NW}(x) - m(x)\right)^2 dF_1(x)\right]$. This is a proxy MISE in terms of prediction. Using a Taylor expansion and a change of variable, we obtain the result in Lemma 1. From now on, in order to state the results in Section 4.4, we need to assume some regularity conditions on the kernel, the source density function and the regression function, which are introduced below.

(K1) K is a symmetric density function with mean zero.

(D1) f^0 is four times differentiable and $f^0(x) \neq 0$, $\forall x \in \textit{support}\,(X^1)$.

(M1) m is four times differentiable.

(V1) σ^2 is two times differentiable.

Lemma 1 *Under regularity conditions (K1), (D1), (M1) and (V1), the function* $MISE^a$ *admits the following representation:*

$$
\begin{aligned}
MISE^a(h) \;=\; & \frac{R(K)}{n_0\, h} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\
& + \frac{h^4}{4}\mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right. \\
& \left. + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + \mathcal{O}(h^6) + \mathcal{O}\left(\frac{h}{n_0}\right). \qquad (4.26)
\end{aligned}
$$

Moreover, the asymptotic version of expression (4.26), namely $AMISE^a(h)$, is given by:

$$\begin{aligned} AMISE^a(h) &= \frac{R(K)}{n_0\,h}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \\ &+\frac{h^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \\ &\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx. \end{aligned} \tag{4.27}$$

The proof of Lemma 1 is included in Appendix C. By means of minimizing expression (4.27), we can obtain the $AMISE^a$ bandwidth. We just have to set to zero the derivative of expression (4.27):

$$\begin{aligned} 0 &= h^3\,\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx \\ &-R(K)n_0^{-1}h^{-2}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx, \end{aligned}$$

which leads to

$$h_{AMISE^a} = c_0\,n_0^{-1/5}, \tag{4.28}$$

where

$$c_0 := \left(\frac{R(K)\int\sigma^2(x)f^1(x)\left(f^0(x)\right)^{-1}dx}{\mu_2(K)^2\int\beta(x)\,f^1(x)\,dx}\right)^{1/5},$$

and $\beta(x) = m''(x)^2+4m'(x)m''(x)\left(f^0\right)'(x)\left(f^0(x)\right)^{-1}+4m'(x)^2\left(\left(f^0\right)'(x)\right)^2\left(f^0(x)\right)^{-2}$.

Considering expressions (4.26) and (4.28), the $MISE^a$ of h_{AMISE^a} admits the following representation:

$$\begin{aligned} MISE^a\left(h_{AMISE^a}\right) &= R(K)n_0^{-4/5}c_0^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \\ &+\frac{c_0^4\,n_0^{-4/5}}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \end{aligned}$$

$$+\frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2}\Bigg] f^1(x)\,dx + \mathcal{O}(n_0^{-6/5}),$$

which tends to zero as $n_0 \longrightarrow \infty$.

Finally, the approximation error between the minimizer of $MISE^a$ and its asymptotic counterpart is presented in the next result.

Theorem 17 *Under regularity conditions (K1), (D1), (M1), the bandwidth selector which minimizes the function $MISE^a$ happens to be:*

$$h_{MISE^a} \;=\; c_0\, n_0^{-1/5} + \mathcal{O}\left(n_0^{-2/5}\right). \tag{4.29}$$

It remains to be seen how accurate the theoretical approximation we have considered for the function $MISE$ is, which results in $MISE^a$. We will show below that the aforementioned approximation is a good one in terms of ISE.

Assume conditions on the kernel (C1) and the density function (C2) of Silverman (1978); as well as conditions on the regression function (C3) of Mack and Silverman (1982).

(C1) (a) K is uniformly continuous with modulus of continuity w_K and of bounded variation $V(K)$.

(b) K is absolutely integrable with respect to the Lebesgue measure on the line.

(c) $K(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

(d) $\int |x \log |x||^{1/2} |dK(x)| < \infty$.

(C2) (a) f^0 is uniformly continuous.

(C3) (a) $\mathbb{E}\,|Y|^s < \infty$ and $\sup_x \int |y|^s f_{XY}(x,y)\,dy < \infty$, $s \geq 2$, where f_{XY} is the joint density function.

(b) The marginal density of X^0, f^0; the joint density function, f_{XY}; and $m(x)f^0(x)$, the theoretical analogue of $\hat{\Psi}_h$, are continuous on an open interval containing J, where J is a bounded interval on which f^0 is bounded away from zero.

Proposition 2 *Suppose conditions (C1), (C2) and (C3) are fulfilled. Consider* h_{n_0} *is a sequence of bandwidths such that* $\sum_{n_0=1}^{+\infty} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$. *Assume* $(n_0\, h)^{-1/2} \log(h^{-1}) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\, h \longrightarrow \infty$. *Then,*

$$ISE(h) \;=\; ISE^a(h) + \mathcal{O}_P\left(h^6\right) + \mathcal{O}_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathcal{O}_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathcal{O}_P\left(\frac{\log\frac{1}{h}}{n_0^{3/2}h^{3/2}}\right), \tag{4.30}$$

where $ISE(h) = \int \left(\hat{m}_h^{NW}(x) - m(x)\right)^2 dF^1(x)$ *and, on the other hand,* $ISE^a(h) = \int \left(\tilde{m}_h^{NW}(x) - m(x)\right)^2 dF^1(x)$.

4.4.2 Asymptotic expression for the bootstrap criterion function

Up to now, the function $MISE^a$ has been analyzed, providing the theoretical bandwidth for which the function attains its minimum, given in (4.29). From now on, the smoothed bootstrap version of $MISE^a$ will be studied. First, we start by computing $MISE^{a*}(h) := \mathbb{E}^*\left[\int \left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)^2\right]$, which is the theoretical analogue of $MASE^*_{\tilde{m}_h^{NW},X^1}$. This is a proxy MISE in terms of prediction. Thus, the function $MISE^{a*}$ (which is the bootstrap MISE of $\tilde{m}_h^{NW}$, given in (4.2)) is collected in Lemma 2. Let us firstly denote, for the sake of brevity,

$$\begin{aligned}
\hat{A}_g &= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx,\\
\hat{B}_g &= \int \left[\hat{m}_g''(x)^2 + \frac{4\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)} + \frac{4\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^2}\right]\hat{f}_g^1(x)\,dx,
\end{aligned}$$

$$A = \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx, \text{ and}$$
$$B = \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \qquad (4.31)$$

Lemma 2 *Under the regularity conditions (K1), (D1), (M1) and (V1), the function $MISE^{a*}$ admits the following representation:*

$$\begin{aligned} MISE^{a*}(h) &= \frac{R(K)}{n_0 h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2 \hat{B}_g \\ &+ \mathcal{O}_P\left(h^6 n_1^{-1} g^{-7}\left(g^{-2} + g^{-1} + 1\right)\right) + \mathcal{O}_P(h^8 n_1^{-1} g^{-9}) \\ &+ \mathcal{O}_P\left(h^{-1} g^2 n_1^{-1}\right) + \mathcal{O}_P\left(h\, n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right). \end{aligned} \qquad (4.32)$$

Besides, the asymptotic version of expression (4.32), namely $AMISE^{a*}(h)$, is given by:

$$AMISE^{a*}(h) = \frac{R(K)}{n_0 h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2 \hat{B}_g. \qquad (4.33)$$

By means of minimizing expression (4.33), we can define an $AMISE^{a*}$ bandwidth. We just have to set to zero the first derivative of expression (4.33):

$$0 = h^3 \mu_2(K)^2 \hat{B}_g - R(K) n_0^{-1} h^{-2} \hat{A}_g,$$

which leads to

$$h^*_{AMISE^a} = \hat{c}_0 \cdot n_0^{-1/5}, \qquad (4.34)$$

where

$$\hat{c}_0 := \left(\frac{R(K)\hat{A}_g}{\mu_2(K)^2 \hat{B}_g} \right)^{1/5}.$$

Expression (4.32) reveals that the pilot bandwidth g should be selected so as to minimize the error produced by estimating expression (4.27) with expression (4.33).

In other words, denote by $\hat{\delta}_g(h) := AMISE^{a*}(h) - AMISE^{a}(h)$, then:

$$\hat{\delta}_g(h) = \frac{R(K)}{n_0\, h}\left(\hat{A}_g - A\right) + \frac{h^4}{4}\mu_2(K)^2\left(\hat{B}_g - B\right). \tag{4.35}$$

The aim is precisely to obtain the minimizer in $g > 0$ of the mean squared error of expression (4.35), that is:

$$g_{OPT} := \arg\min_{g>0} \mathbb{E}\left[\hat{\delta}_g^2(h)\right] \tag{4.36}$$

Nonetheless, as can be seen in expression (4.35), the optimal pilot bandwidth (g_{OPT}, defined in (4.36)) depends on the bandwidth h. Given that the area in which one should be focused is where the objective function attains its minimum, the optimal pilot bandwidth g is defined from now on as follows:

$$g_{OPT} := \arg\min_{g>0} \mathbb{E}\left[\hat{\delta}_g^2(h_{AMISE^a})\right],$$

where h_{AMISE^a} has been defined in (4.28), and

$$\hat{\delta}_g(h_{AMISE^a}) = \frac{R(K)}{n_0\, h_{AMISE^a}}\left(\hat{A}_g - A\right) + \frac{(h_{AMISE^a})^4}{4}\mu_2(K)^2\left(\hat{B}_g - B\right).$$

Standard calculations give

$$\begin{aligned}\mathbb{E}\left[\hat{\delta}_g^2(h_{AMISE^a})\right] &= \mu_2(K)^{4/5}R(K)^{8/5}n_0^{-8/5}A^{-2/5}B^{2/5} \\ &\cdot\mathbb{E}\left\{\left[\left(\hat{A}_g - A\right) + \frac{A}{4\,B}\cdot\left(\hat{B}_g - B\right)\right]^2\right\},\end{aligned} \tag{4.37}$$

which leads to set the optimal pilot bandwidth g as the minimizer in $g > 0$ of

$$g_{OPT} = \arg\min_{g>0} \mathbb{E}\left\{\left[\left(\hat{A}_g - A\right) + \frac{A}{4B}\left(\hat{B}_g - B\right)\right]^2\right\}$$

$$\begin{aligned} &= \arg\min_{g>0} \left\{ \mathbb{E}\left[\left(\hat{A}_g - A \right)^2 \right] \right. \\ &\quad \left. + \frac{A^2}{16B}\mathbb{E}\left[\left(\hat{B}_g - B \right)^2 \right] + \frac{A}{2B}\mathbb{E}\left[\left(\hat{A}_g - A \right) \cdot \left(\hat{B}_g - B \right) \right] \right\} \\ &= \arg\min_{g>0} \left\{ \mathbb{E}\left[\alpha^2 \right] + \frac{A^2}{16B}\mathbb{E}\left[\beta^2 \right] + \frac{A}{2B}\mathbb{E}\left[\alpha \cdot \beta \right] \right\}, \end{aligned} \tag{4.38}$$

where $\alpha = \hat{A}_g - A$ and $\beta = \hat{B}_g - B$.

Remark 20 (Approximation considered in (4.2)) *Given $\hat{\psi}_1, \hat{\psi}_2$ consistent estimators of ψ_1, ψ_2, respectively, then:*

$$\begin{aligned} \frac{\hat{\psi}_1}{\hat{\psi}_2} - \frac{\psi_1}{\psi_2} &= \frac{\hat{\psi}_1}{\hat{\psi}_2}\left(\frac{\hat{\psi}_2}{\psi_2} + 1 - \frac{\hat{\psi}_2}{\psi_2} \right) - \frac{\psi_1}{\psi_2} = \frac{\hat{\psi}_1 - \psi_1}{\psi_2} + \frac{\hat{\psi}_1}{\hat{\psi}_2} \cdot \frac{\psi_2 - \hat{\psi}_2}{\psi_2} \\ &= \frac{\hat{\psi}_1 - \psi_1}{\psi_2} + \frac{\hat{\psi}_1\left(\hat{\psi}_2 - \psi_2 \right)}{\psi_2^2} + \frac{\hat{\psi}_1}{\hat{\psi}_2}\left(\frac{\psi_2 - \hat{\psi}_2}{\psi_2} \right)^2 \\ &= \frac{\hat{\psi}_1 - \psi_1}{\psi_2} + \frac{\hat{\psi}_1\left(\hat{\psi}_2 - \psi_2 \right)}{\psi_2^2} + \frac{\left(\hat{\psi}_1 - \psi_1 \right) \cdot \left(\psi_2 - \hat{\psi}_2 \right)}{\hat{\psi}_2^2} + \frac{\hat{\psi}_1\left(\psi_2 - \hat{\psi}_2 \right)^2}{\hat{\psi}_2\psi_2^2}, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{\hat{\psi}_1}{\hat{\psi}_2} - \frac{\psi_1}{\psi_2} &= \frac{\psi_2\left(\hat{\psi}_1 - \psi_1 \right) - \psi_1\left(\hat{\psi}_2 - \psi_2 \right)}{\psi_2^2} \\ &\quad + \mathcal{O}\left(\left(\hat{\psi}_2 - \psi_2 \right)^2 + \left(\hat{\psi}_2 - \psi_2 \right) \cdot \left(\hat{\psi}_1 - \psi_1 \right) \right). \end{aligned} \tag{4.39}$$

Denote $\hat{\Psi}_{\ell,g}(x) = n_0^{-1} \sum_{i=1}^{n_0} K_g\left(x - X_i^0 \right)\left(Y_i^0 \right)^\ell$ and $\Psi_\ell(x) = m_\ell(x) f^0(x), \forall \ell \in \{0,1,2\}$, where $m_\ell(x) = \mathbb{E}\left(Y^{0\ell}\big|_{X^0=x} \right)$ if $\ell \in \{1,2\}$, and $m_\ell = 1$ if $\ell = 0$. Unfortunately, the computation of the expectations in expression (4.38) is extremely tedious and it would imply the calculation of more than three hundred U-statistics. Our approach then is to find an upper bound for expression (4.37), given by:

$$\mathbb{E}\left[\hat{\delta}_g^2\left(h_{AMISE^a} \right) \right] \leq 2\mathbb{E}\left[\left(\hat{A}_g - A \right)^2 \right] + \frac{A^2}{8B^2}\mathbb{E}\left[\left(\hat{B}_g - B \right)^2 \right].$$

This requires to work with $\hat{A}_g - A$ and $\hat{B}_g - B$ using the approximation given in Remark 20. Some technical results concerning the quantification of the error of both approximations are collected in the following. In particular, Lemma 3 collects an upper bound for $\hat{A}_g - A$ and Lemma 4, for $\hat{B}_g - B$. We need to introduce one definition first.

Definition 5 *In order to analyze the MSE of $\hat{A}_g$ and $\hat{B}_g$, consider the following expression:*

$$C_{\nu,\ell,r}^{[s]} \quad := \quad \int \nu(x) \left(\hat{\Psi}_{s,\ell}^{(r)}(x) - \Psi_{s,\ell}^{(r)}(x) \right) dx, \tag{4.40}$$

where $\nu(x)$ is a function verifying that $\nu : \mathbb{R} \to \mathbb{R}; x \to \nu(x)$, $\ell \in \mathbb{Z}^+$, $r \in \mathbb{Z}^+$, $s \in \{0,1\}$, $\hat{\Psi}_{s,\ell}^{(r)}(x) = \dfrac{1}{n_s\, g^{r+1}} \sum_{i=1}^{n_s} K^{(r)} \left(\dfrac{x - X_i^{[s]}}{g} \right) Y_i^{[s]\ell}$ and $\Psi_{s,\ell}^{(r)}(x) := \dfrac{\partial^r \left(m_\ell(x) f^{[s]}(x) \right)}{\partial x^r}$.

Lemma 3 *Consider the approximation given in (4.39) and expressions for $\hat{A}_g$ and A, given in (4.31). Then,*

$$\begin{aligned}
\hat{A}_g - A \quad = \quad & \int \frac{\sigma^2(x)}{f^0(x)} \left[\hat{f}_g^1(x) - f^1(x) \right] dx - \int \frac{\sigma^2(x) f^1(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& + \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x) \right] dx - \int \frac{f^1(x) \Psi_2(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& -2 \int \frac{f^1(x) \Psi_1(x)}{f^0(x)^3} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right] dx \\
& -2 \int \frac{f^1(x) \Psi_1^2(x)}{f^0(x)^4} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& + \mathcal{O} \left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x) \right) dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{\sigma}_g^2(x) \hat{f}_g^1(x) - \sigma^2(x) f^1(x) \right) dx \right)
\end{aligned}$$

$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right). \quad (4.41)$$

In other terms,

$$\hat{A}_g - A = \sum_{i=1}^{k_0} a_i C_{\nu_i,\ell_i,r_i}^{[s]_i} + A_1, \quad (4.42)$$

where $k_0 = 6$, $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, a_5 = -2, a_6 = -2$, $\nu_1(x) = \frac{\sigma^2(x)}{f^0(x)}$,$\nu_2(x) = \frac{\sigma^2(x) f^1(x)}{f^0(x)^2}$,$\nu_3(x) = \frac{f^1(x)}{f^0(x)^2}$,$\nu_4(x) = \frac{f^1(x)\Psi_2(x)}{f^0(x)^3}$, $\nu_5(x) = \frac{f^1(x)\Psi_1(x)}{f^0(x)^3}$, $\nu_6(x) = \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4}$,$\ell_1 = 0$,$\ell_2 = 0$,$\ell_3 = 2$,$\ell_4 = 0$,$\ell_5 = 1$,$\ell_6 = 0$, $r_1 = 0$,$r_2 = 0$,$r_3 = 0$, $r_4 = 0$,$r_5 = 0$,$r_6 = 0$, $[s]_1 = 1$, $[s]_2 = 0$, $[s]_3 = 0$, $[s]_4 = 0$, $[s]_5 = 0$, $[s]_6 = 0$ *and* $A_1 = \mathcal{O}(r_{0,n_0})$, *with*

$$\begin{aligned} r_{0,n_0} &= \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx \\ &+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) dx \\ &+ \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx \\ &+ \int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx. \end{aligned}$$

Lemma 4 *Given the approximation in (4.39) and expressions for* $\hat{B}_g$ *and* B *in (4.38), then we have:*

$$\hat{B}_g - B = \sum_{i=7}^{k_1} a_i C_{\nu_i,\ell_i,r_i}^{[s]_i} + B_1, \quad (4.43)$$

where $k_1 = 66$. The values of $\nu(x)$, r, ℓ, a and $[s]$ are collected in Tables 4.1, 4.2, 4.3 and 4.4,

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
7	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}$	0	1	1	8
8	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1'(x)}{f^0(x)^4}$	0	0	0	-8
9	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	1	0	-8
10	$\frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}$	1	0	0	4
11	$\frac{m'(x)^2 f^1(x)\left(f^0\right)'(x)}{f^0(x)^2}$	0	0	1	8
12	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^4}$	0	0	0	-16
13	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1(x)}{f^0(x)^4}$	0	0	1	-8
14	$\frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}$	0	0	0	-8
15	$\frac{f^1(x) m''(x)}{f^0(x)}$	0	1	2	2
16	$\frac{f^1(x) m''(x)\Psi_1''(x)}{f^0(x)^2}$	0	0	0	-2
17	$\frac{f^1(x) m''(x)\Psi_1(x)}{f^0(x)^2}$	0	0	2	-2
18	$\frac{f^1(x) m''(x)\left(f^0\right)''(x)}{f^0(x)^2}$	0	1	0	-2
19	$\frac{f^1(x) m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}$	0	0	0	8
20	$\frac{f^1(x) m''(x)\Psi_1'(x)}{f^0(x)^2}$	0	0	1	-4

Table 4.1: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{7, \ldots, 20\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
21	$\frac{f^1(x)m^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^2}$	0	1	1	-4
22	$\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}$	0	0	0	8
23	$\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}$	0	0	1	8
24	$\frac{f^1(x)m^{''}(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^3}$	0	1	0	4
25	$\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^4}$	0	0	0	-12
26	$m^{''}(x)^2$	1	0	0	1
27	$\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}$	1	0	0	4
28	$\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^3}$	0	0	1	4
29	$\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}$	0	1	2	4
30	$\frac{\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}$	0	1	1	4
31	$\frac{m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^2}$	0	0	0	-4
32	$\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}$	0	0	0	-8
33	$\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)}{f^0(x)^5}$	1	0	0	8
34	$\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3f^1(x)}{f^0(x)^5}$	0	1	0	8
35	$\frac{\Psi_1^{'}(x)\Psi_1(x)f^1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^5}$	0	0	1	24
36	$\frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
37	$\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	0	0	-32

Table 4.2: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{21, \ldots, 37\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
38	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	1	0	0	-4
39	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	0	1	-4
40	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
41	$\frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	2	-4
42	$\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	1	-4
43	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}$	0	0	0	-12
44	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}$	1	0	0	-8
45	$\frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-16
46	$\frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}$	0	1	1	-16
47	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}$	0	0	0	24
48	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}$	1	0	0	-4
49	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	1	2	-4
50	$\frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
51	$\frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-8
52	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}$	0	0	0	12
53	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}$	1	0	0	-8
54	$\frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}$	0	1	0	-16

Table 4.3: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{38, \ldots, 54\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
55	$\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}$	0	0	1	-32
56	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}$	0	0	0	32
57	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}$	1	0	0	4
58	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}$	0	0	2	4
59	$\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}$	0	1	0	8
60	$\frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}$	0	0	1	8
61	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$	0	0	0	-16
62	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}$	1	0	0	8
63	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
64	$\frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}$	0	1	0	8
65	$\frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}$	0	0	1	24
66	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}$	0	0	0	-32

Table 4.4: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{55, \ldots, 66\}$.

Furthermore, term B_1 is given by $B_1 = \mathcal{O}\left(r_{1,n_0}\right)$ and r_{1,n_0} is detailed in Appendix C.

In the next Corollary, an expression for $MISE^{a*}$ is introduced given that the pilot bandwidth $g > 0$ is of exact order $n_0^{-1/2}$. The proof of Corollary 3 is included in Appendix C.

Corollary 3 *Consider a pilot bandwidth $g > 0$ of exact order $n_0^{-1/2}$ and the definitions in (4.31). Assume that f^0 and its derivatives tend to zero as $x \to \infty$ and m_ℓ, ν*

and their derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Then:

$$\begin{aligned} MISE^{a*}(h) &= \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g \\ &+\mathcal{O}_P\left(h^6\, n_1^{-1}\left(n_0^{7/2}+n_0^4+n_0^{9/2}\right)\right)+\mathcal{O}_P\left(h\, n_1^{-1}\left(1+n_0^{1/2}+n_0\right)\right) \\ &+\mathcal{O}_P\left(h^{-1}\, n_0^{-1}\, n_1^{-1}\right). \end{aligned} \tag{4.44}$$

Therefore, considering expression (4.34) for $h_{AMISE^{a*}}$, it follows that:

$$\begin{aligned} MISE^{a*}(\hat{c}_0\, n_0^{-1/5}) &= R(K)\hat{c}_0^{-1}\, n_0^{-4/5}\hat{A}_g + \frac{\hat{c}_0^4\, n_0^{-4/5}}{4}\mu_2(K)^2\hat{B}_g \\ &+\mathcal{O}_P\left(n_1^{-1}\left(n_0^{23/10}+n_0^{14/5}+n_0^{33/10}\right)\right)+\mathcal{O}_P\left(n_0^{-4/5}\, n_1^{-1}\right) \\ &+\mathcal{O}_P\left(n_1^{-1}\left(n_0^{-1/5}+n_0^{3/10}+n_0^{4/5}\right)\right). \end{aligned}$$

Thanks to Tchebycheff inequality together with expression (C.128), we can conclude:

Lemma 5 *An upper bound for expressions $\hat{A}_g - A$ and $\hat{B}_g - B$ in (4.31), under regularity conditions (K1), (D1), (M1) and (V1), is given by:*

$$\begin{aligned} \hat{A}_g - A &= \mathcal{O}_P\left(n_0^{-1/2}\right), \text{ and} \\ \hat{B}_g - B &= \mathcal{O}_P\left(n_0^{-1/2}\right). \end{aligned} \tag{4.45}$$

Finally, similarly as in the non bootstrap context, it remains to be seen the accuracy of the theoretical approximation we have considered for the function $MISE^*$, which results in $MISE^{a*}$. Afterwards, we show that the aforementioned approximation is a good one in terms of ISE^*.

In order to state Proposition 3 included below, consider conditions on the kernel (C1) and the density function (C2) of Silverman (1978); as well as conditions on the regression function (C3) of Mack and Silverman (1982).

Proposition 3 *Assume conditions (C1), (C2), (C3). Suppose, additionally, that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\, h \longrightarrow \infty$. Consider h_{n_0} a sequence of bandwidths such that*

$\sum_{n_0=1}^{+\infty} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$. *Assume, moreover, that* $(n_0\, h)^{-1/2} \log(h^{-1}) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\, h \longrightarrow \infty$. *Then,*

$$\begin{aligned} ISE^*(h) &= ISE^{a*}(h) + \mathcal{O}_{P^*}\left(h^6\right) + \mathcal{O}_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathcal{O}_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) \\ &\quad + \mathcal{O}_{P^*}\left(\frac{\log\frac{1}{h}}{n_0^{3/2}h^{3/2}}\right), \end{aligned} \tag{4.46}$$

almost sure with respect to P, *where*

$$\begin{aligned} ISE^*(h) &= \int \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x), \text{ and} \\ ISE^{a*}(h) &= \int \left(\tilde{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x). \end{aligned}$$

4.4.3 Rate of convergence for the bootstrap bandwidth

Our aim now is to establish the convergence rate for $(h_{MISE^{a*}} - h_{MISE^a})/h_{MISE^a}$, using expressions (4.32), (4.34), (4.44) and (4.45). It is stated in Theorem 18, the proof of which is collected in Appendix C.

Theorem 18 *Consider* h_{MISE^a} *and its bootstrap version,* $h^*_{MISE^a}$, *which are the minimizers of expressions (4.26) and (4.32), respectively. Under regularity conditions (K1), (D1), (M1), (V1) and assuming that* g *is of order* $n_0^{-1/2}$, *it turns out*

$$h^*_{MISE^a} - h_{MISE^a} = \mathcal{O}_P\left(n_0^{-7/10}\right),$$

or rather,

$$\frac{h^*_{MISE^a} - h_{MISE^a}}{h_{MISE^a}} = \mathcal{O}_P\left(n_0^{-1/2}\right).$$

4.5 Simulations

4.5.1 Nadaraya-Watson regression estimator

General description of the study

A simulation study is now carried out in order to show the empirical performance of this new smoothing parameter selector, namely h_{BOOT}^{NW}. Eleven different scenarios have been considered:

Scenario 1: In the source population, the distribution of X^0 is a $\beta(2,4)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin(\pi x)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 2: In the source population, X^0 has distribution $\beta(2,5)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(5,2)$.

Scenario 3: In the source population, X^0 has distribution $\beta(5,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(2,5)$.

Scenario 4: In the source population, the distribution of X^0 is a $\beta(2,5)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(5,2)$.

Scenario 5: In the source population, X^0 has distribution $\beta(5,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(2,5)$.

Scenario 6: In the source population, X^0 has distribution $\beta(2,5)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin(1 + \pi x^4)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(5,2)$.

Scenario 7: In the source population, X^0 has distribution $\beta(5,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin(1 + \pi x^4)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(2,5)$.

Scenario 8: In the source population, X^0 has distribution $\beta(2,6)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(6,2)$.

Scenario 9: In the source population, X^0 has distribution $\beta(6,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(2,6)$.

Scenario 10: In the source population, X^0 has distribution $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(2,4)$.

Scenario 11: In the source population, X^0 has distribution $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(1.5,4)$.

For every scenario, 100 random samples of size $n_0 = n_1 = 500$ were drawn. The Gaussian kernel has been considered. The smoothed bootstrap bandwidth selector, h_{BOOT}^{NW} in (4.15), is the minimizer, in h, of an empirical function. Since this minimizer does not have an explicit expression, numerical methods are used to approximate it. The algorithm proceeds as follows:

Step 1: Let us consider a grid of 50 values of h in the interval $[0.01, 0.2]$ equispaced in logarithmic scale.

Step 2: A bandwidth value h_{OPT_1} is chosen among the ones given in the preceeding step, by miniminizing the objective function (that is, $MASE^*_{\tilde{m}_h}$)

Step 3: Among the grid of 50 bandwidth values defined in Step 1, we consider the previous and the next one to h_{OPT_1}.

Step 4: A set of 5 equally spaced in logarithmic scale values of h is constructed within the interval whose endpoints are the two values selected in Step 3.

Step 5: Finally, Steps 2-4 are repeated three times, retaining the optimal bandwidth value in the last stage. This bandwidth value will be the numerically approximated value of the bootstrap version of the optimal one, namely h_{BOOT}.

It is worth mentioning that, in order to avoid oversmoothing of the bootstrap procedure, h_{BOOT} is considered as the local minimizer of $MASE^*_{\tilde{m}_h}$ with the minimal value of this objective function.

Since the target is to check the good empirical behaviour of the new bootstrap bandwidth selector in terms of prediction, we have considered the aforementioned smoothing parameter selector, h_{BOOT} (h_1 in the following). Our aim now is to compare the empirical behaviour of the bandwidth selector we have defined in terms of prediction, h_1, with an unobservable bandwidth selector (namely, h_0) so as to check the importance of using the target sample $\{X_j^1\}_{j=1}^{n_1}$ in computing a prediction bandwidth selector (h_1) instead of ignoring this information. Indeed, h_0 is the result of minimizing the function $MASE^*_{\tilde{m}_h, X^{0'}}(h)$, which is the bootstrap version of expression (4.10) integrated with respect to F_0 instead of F_1, that is:

$$\mathbb{E}\left[\int \left(\hat{m}_h(x) - m(x)\right)^2 \, dF_0(x)\right].$$

Accordingly, the function $MASE^*_{\tilde{m}_h, X^{0'}}(h)$ is given by:

$$\begin{aligned} MASE^*_{\tilde{m}_h, X^{0'}}(h) &= \frac{1}{n_0} \sum_{j=1}^{n_0} \frac{1}{\hat{f}^0_{g_X}(X_j^{0'})^2} \left[\left(1 - \frac{1}{n_0}\right) \cdot \left(\left[K_h * \hat{q}^0_{X_j^{0'}, g_X}\right](X_j^{0'}) \right)^2 \right. \\ &\left. + \frac{1}{n_0} \left[(K_h)^2 * \hat{p}^0_{X_j^{0'}, g_X}\right](X_j^{0'}) + \frac{g_Y^2 \mu_2(K)}{n_0^2} \sum_{i=1}^{n_0} \left[(K_h)^2 * K_{g_X}\right](X_j^{0'} - X_i^0) \right]. \end{aligned} \quad (4.47)$$

In particular, estimators in expression (4.47) have been worked out with the original source sample, $\left\{\left(X_j^0, Y_j^0\right)\right\}_{j=1}^{n_0}$, but evaluated in a new source sample $\left\{\left(X_j^{0'}, Y_j^{0'}\right)\right\}_{j=1}^{n_0}$, which is unobservable in practice, and independent from the original source sample. The bandwidth selector h_0 is not defined in terms of prediction as it does not require the information provided from the target sample to compute it. Thus, the objective of defining expression (4.47), and therefore h_0, is to check the importance of using the target sample $\{X_j^1\}_{j=1}^{n_1}$ so as to compute a prediction bandwidth selector (h_1) instead of ignoring this information. We now define the bandwidth, h_0, as the minimizer of expression (4.47), that is:

$$h_0 = h_{MASE^*_{\tilde{m}_h, X^{0'}}} = \arg\min_{h>0} MASE^*_{\tilde{m}_h, X^{0'}}(h).$$

Both bandwidths, h_0 and h_1, are compared by means of the expressions:

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left[m(X_i^1) - \hat{m}_{h_j^{NW}}(X_i^1)\right]^2 \text{ and} \tag{4.48}$$

$$\int \left[m(x) - \hat{m}_{h_j^{NW}}(x)\right]^2 dF_1(x), \tag{4.49}$$

where $j = 0, 1$ and $\hat{m}_{h_j^{NW}}$ stands for the Nadaraya-Watson regression estimator obtained with the sample coming from the source population, (X^0, Y^0). The results of a simulation study are presented in Figures 4.3-4.5.

The pilot bandwidth, g, assuming $g = g_X = g_Y$, needed to carry out the bootstrap method was $g = h_{SJ} n_0^{4/45}$, where h_{SJ} is the plug-in bandwith selector proposed by Sheather and Jones (1991) for kernel density estimation. As desired, the previous choice of g has order $n_0^{-1/9}$. This was demonstrated by Cao and González-Manteiga (1993) to be the optimal rate for the smoothed bootstrap method. An additional simulation study has also been carried out in order to select the pilot bandwidth g. In particular, let us consider $g = h_C n_0^{1/5-\alpha}$, where $h_C = h_{SJ}, h_{CV}$, h_{CV} is the cross-validation bandwidth selector for regression estimation proposed by Härdle and Marron (1985). Moreover, $\alpha \in \left\{\frac{1}{5}, \frac{1}{9}\right\}$, in order to check the impact of a different order for g ($n_0^{-1/5}$ and $n_0^{-1/9}$, respectively) in the behaviour of h_{BOOT}. Consider the following expressions as well:

$$MSE\left[\tilde{\theta}\right] = \mathbb{E}\left[\left(\tilde{\theta} - \theta\right)^2\right], \text{ and} \tag{4.50}$$

$$MeSE\left[\tilde{\theta}\right] = Median\left[\left(\tilde{\theta} - \theta\right)^2\right], \tag{4.51}$$

where $\theta = \mathbb{E}(Y^1)$, $\tilde{\theta} = \frac{1}{n_1}\sum_{i=1}^{n_1} \hat{m}_{h_j}(X_i^1)$, $j = 0, 1$ and $\hat{m}_{h_j}$ is the Nadaraya-Watson estimator computed with the source sample, $(X_i^0, Y_i^0), i = 1, \ldots, n_0$. The simulation results are collected in Tables 4.5 and 4.6. According to them, it is clear that the selection of α and h_C does not have a great impact on the behaviour of the bandwidth h_{BOOT}. However, the values for expressions (4.50) and (4.51) in Table 4.5 suggest to choose $\alpha = 1/9$ and $h_C = h_{SJ}$ in order to minimize them.

		Expression (4.48)		Expression (4.49)		Expr. (4.50)	Expr. (4.51)
$\alpha = 1/5$	Scenario	Mean	Median	Mean	Median	-	-
CV	1	0.0168	0.0129	0.0194	0.0151	0.0065	0.0045
	2	0.0419	0.0365	0.0434	0.0375	0.0103	0.0064
	3	0.0133	0.0078	0.0144	0.0078	0.0126	0.0119
SJ	1	0.0163	0.0124	0.0189	0.0142	0.0073	0.0046
	2	0.0202	0.0113	0.0215	0.0119	0.0234	0.0229
	3	0.0149	0.0079	0.0162	0.0082	0.0127	0.0120

Table 4.5: Mean and median of expressions (4.48) (columns 3-4) and (4.49) (columns 5-6), expression (4.50) (column 7) and expression (4.51) (column 8) for $\alpha = 1/5$. Moreover, CV stands for $h_{CV} n_0^{1/5-\alpha}$ and, SJ, $h_{SJ} n_0^{1/5-\alpha}$.

		Expression (4.48)		Expression (4.49)		Expr. (4.50)	Expr. (4.51)
$\alpha = 1/9$	Scenario	Mean	Median	Mean	Median	-	-
CV	1	0.0318	0.0317	0.0361	0.0355	0.0030	0.0010
	2	0.0776	0.0903	0.0809	0.0939	0.0027	0.0016
	3	0.0176	0.0072	0.0161	0.0079	0.0116	0.0111
SJ	1	0.0157	0.0118	0.0181	0.0128	0.0075	0.0044
	2	0.0206	0.0113	0.0219	0.0119	0.0234	0.0023
	3	0.0161	0.0080	0.0174	0.0089	0.0128	0.0121

Table 4.6: Mean and median of expressions (4.48) (columns 3-4) and (4.49) (columns 5-6), expression (4.50) (column 7) and expression (4.51) (column 8) for $\alpha = 1/9$. Moreover, CV stands for $h_{CV} n_0^{1/5-\alpha}$ and, SJ, $h_{SJ} n_0^{1/5-\alpha}$.

Furthermore, simulation results concerning expressions (4.48) and (4.49) in Figure 4.3 reveal that h_1 clearly beats its competitor, h_0. In particular, this statement is somehow strengthened in situation 2 (see Figure 4.5). This means that using the information provided by the target sample to select the bandwidth in terms of prediction brings about less error.

Discussion and results

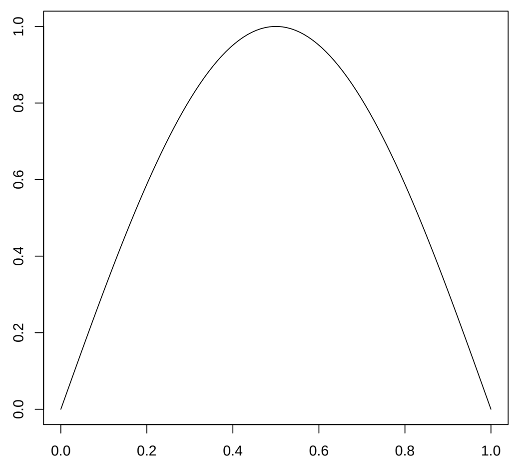
Empirical results for the simulation study described in Section 4.5.1 are assembled in Figures 4.3-4.13. As a matter of fact, we can distinguish among three different situations:

1. h_0^{NW} is much larger than h_1^{NW} (see Figures 4.4, 4.6, 4.8, 4.10, 4.12). This is logical because the regression function considered is almost flat in the support of the source population, but oscillates in the support of the target population.

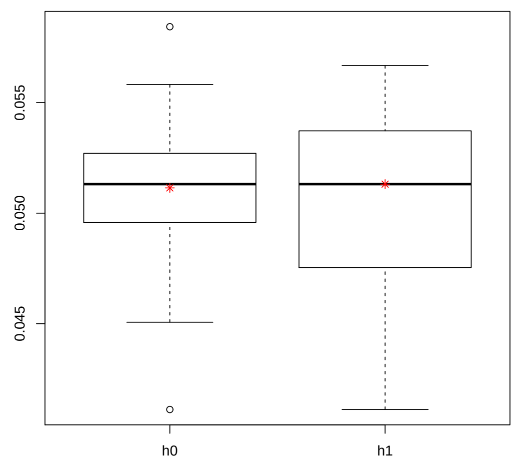
2. h_0^{NW} turns out to be smaller than h_1^{NW} (see Figures 4.5, 4.7, 4.9, 4.11, 4.13). This is the opposite situation as the mentioned above. Indeed, the empirical behaviour of h_0^{NW} and h_1^{NW} is understandable since the regression function taken into account is practically flat or with a slight fluctuation in the support of the target population, but oscillates in the support of the source population.

3. h_0^{NW} and h_1^{NW} are close (see Figure 4.3). This is quite reasonable as the regression function m is similar in both supports of the source and target populations.

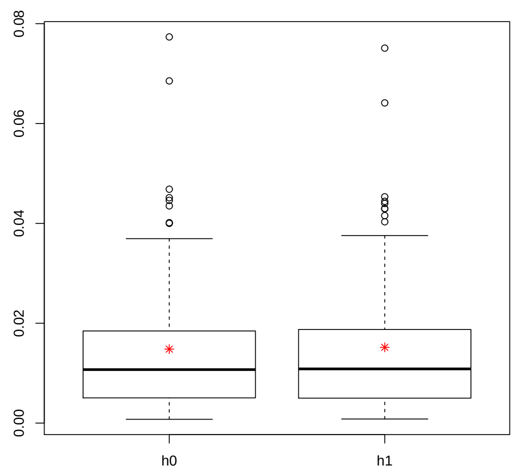
Furthermore, simulation results concerning expressions (4.48) and (4.49) in Figures 4.3-4.13 reveal that h_1 clearly beats its competitor, h_0. In particular, this statement is somehow strengthened in situation 3 (see Figures 4.5, 4.7, 4.9, 4.11, 4.13). This means that using the information provided by the target sample to select the bandwidth in terms of prediction brings about less error.



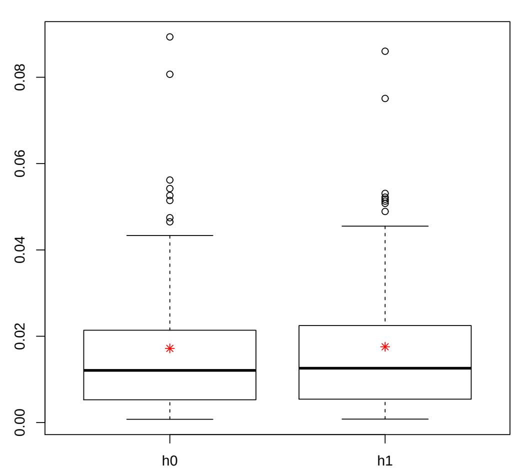
(a) $m(x) = \sin(\pi x)$



(b) Bandwidth selectors

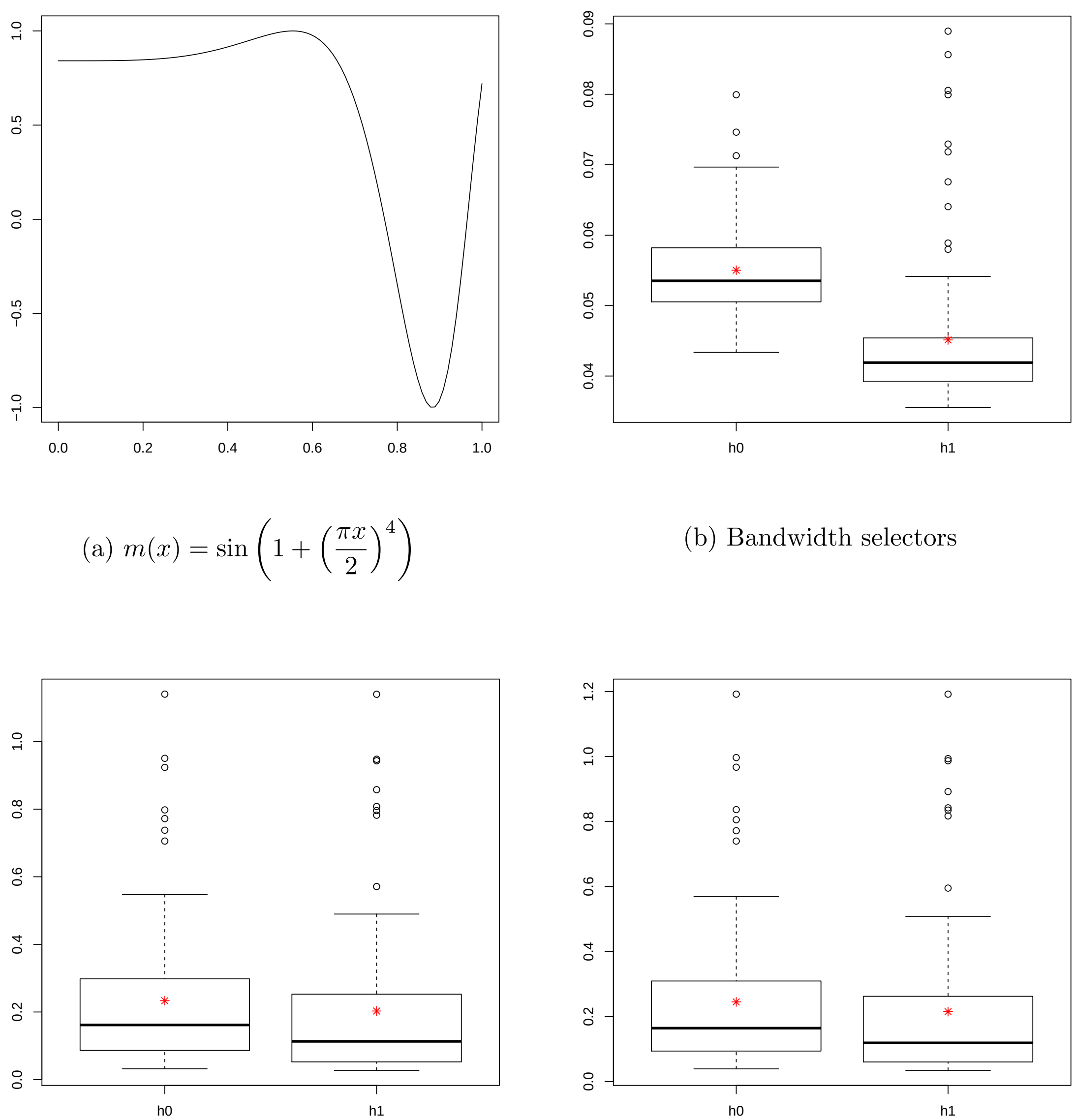


(c) Expression (4.48)



(d) Expression (4.49)

Figure 4.3: Boxplots obtained considering Scenario 1. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



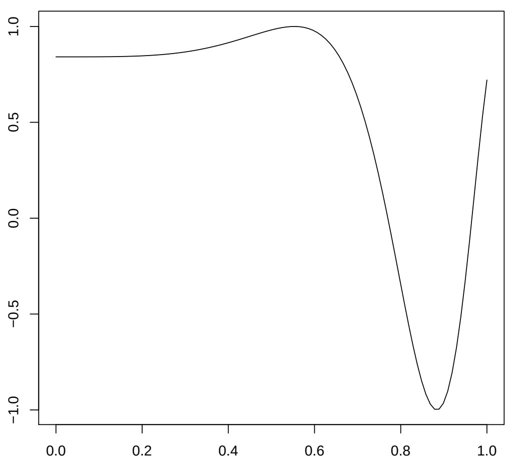
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$

(b) Bandwidth selectors

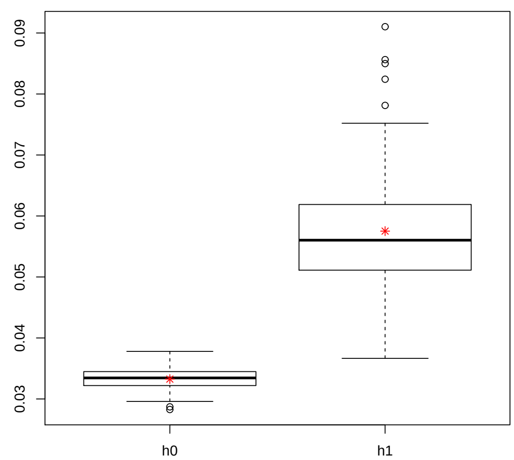
(c) Expression (4.48)

(d) Expression (4.49)

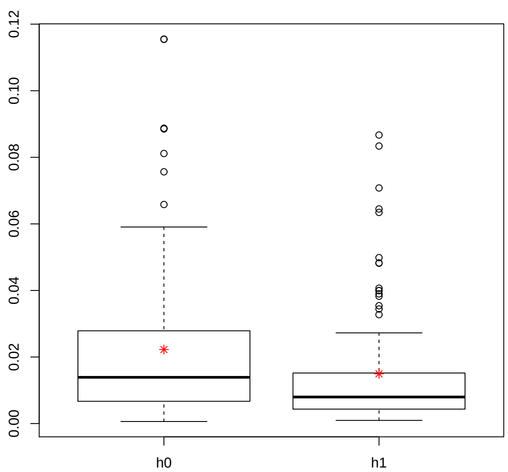
Figure 4.4: Boxplots obtained considering Scenario 2. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



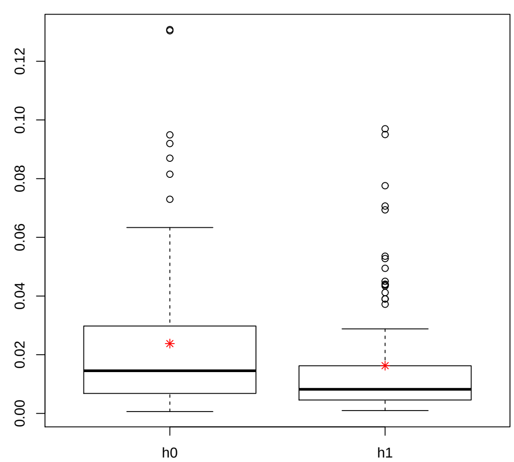
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

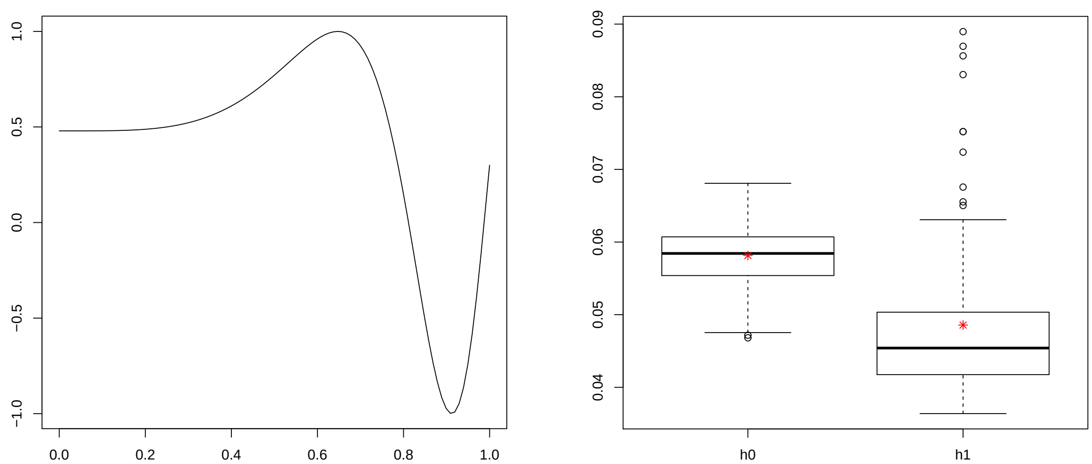


(c) Expression (4.48)



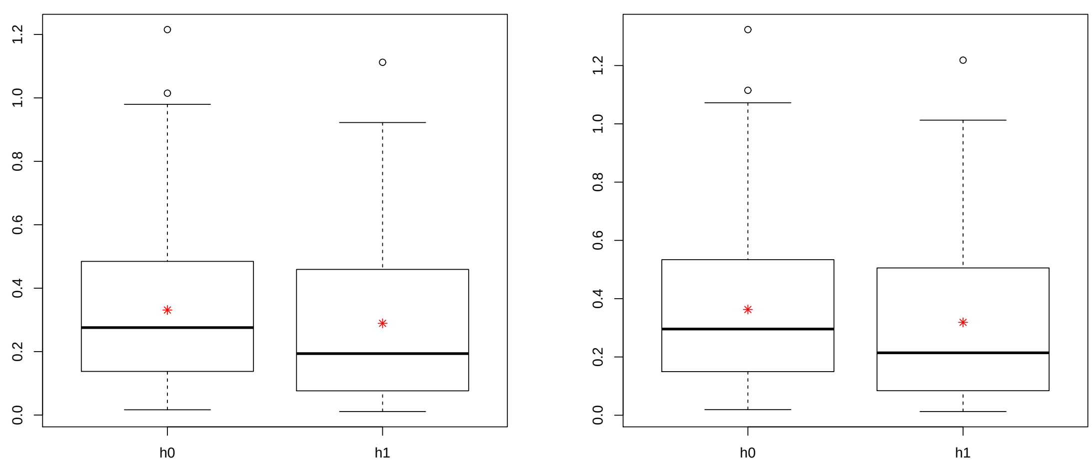
(d) Expression (4.49)

Figure 4.5: Boxplots obtained considering Scenario 3. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



(a) $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$

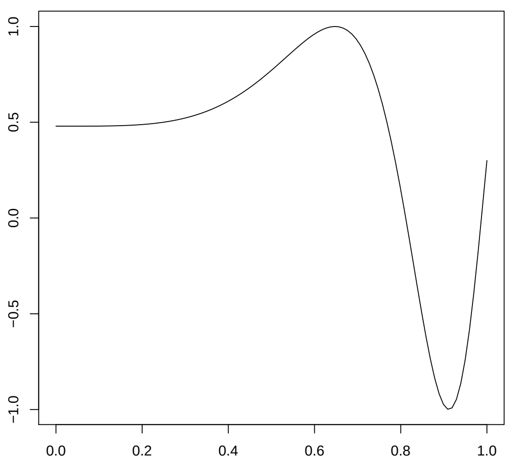
(b) Bandwidth selectors



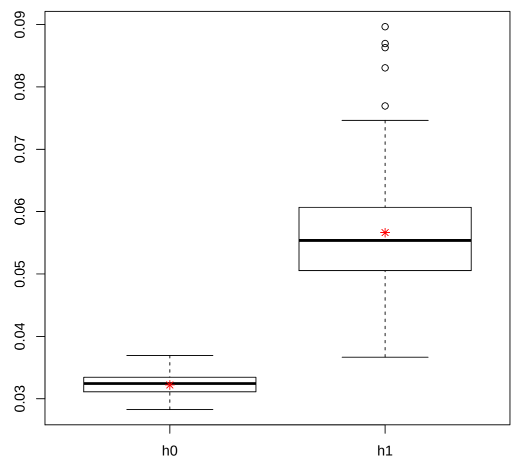
(c) Expression (4.48)

(d) Expression (4.49)

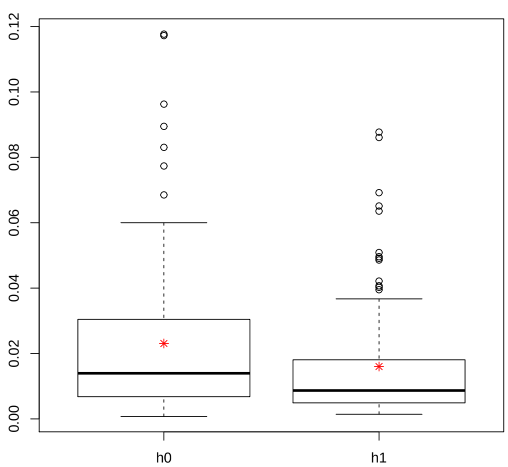
Figure 4.6: Boxplots obtained considering Scenario 4. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



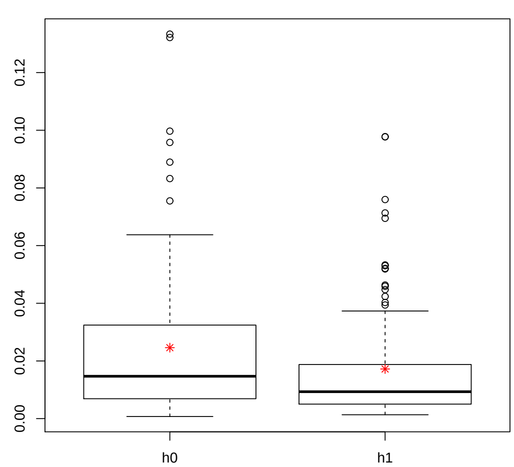
(a) $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

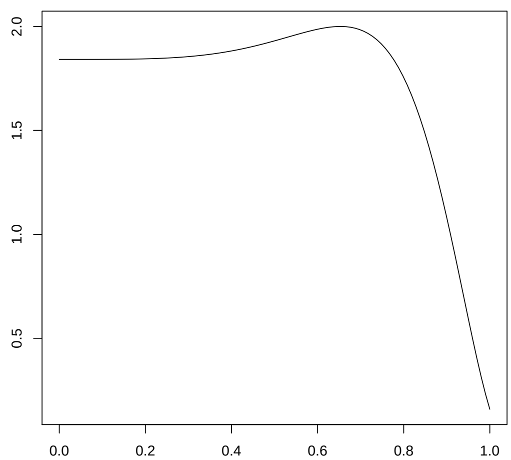


(c) Expression (4.48)

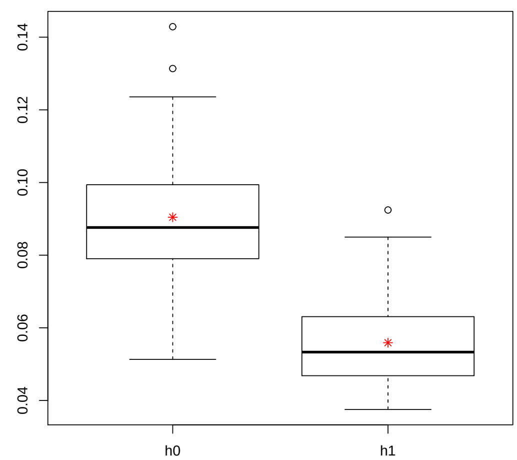


(d) Expression (4.49)

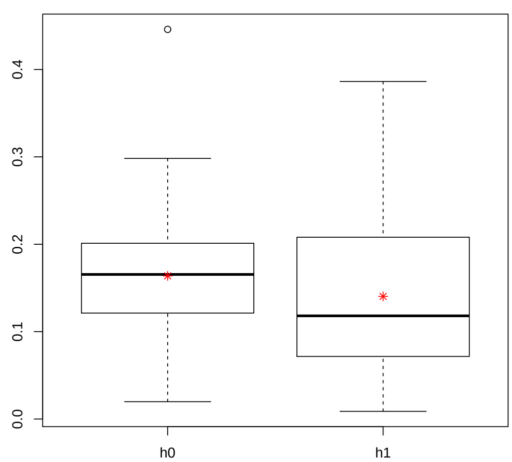
Figure 4.7: Boxplots obtained considering Scenario 5. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



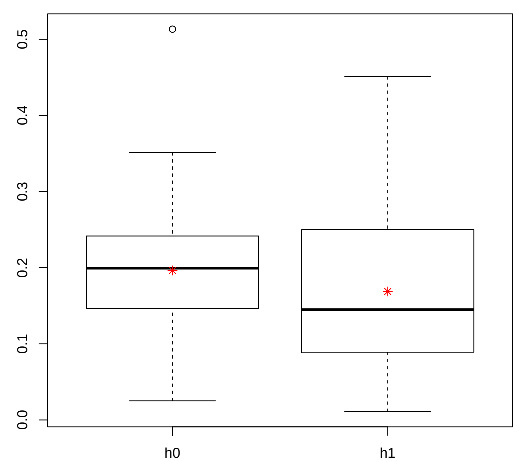
(a) $m(x) = \sin\left(1 + \sin\left(1 + \pi x^4\right)\right)$



(b) Bandwidth selectors

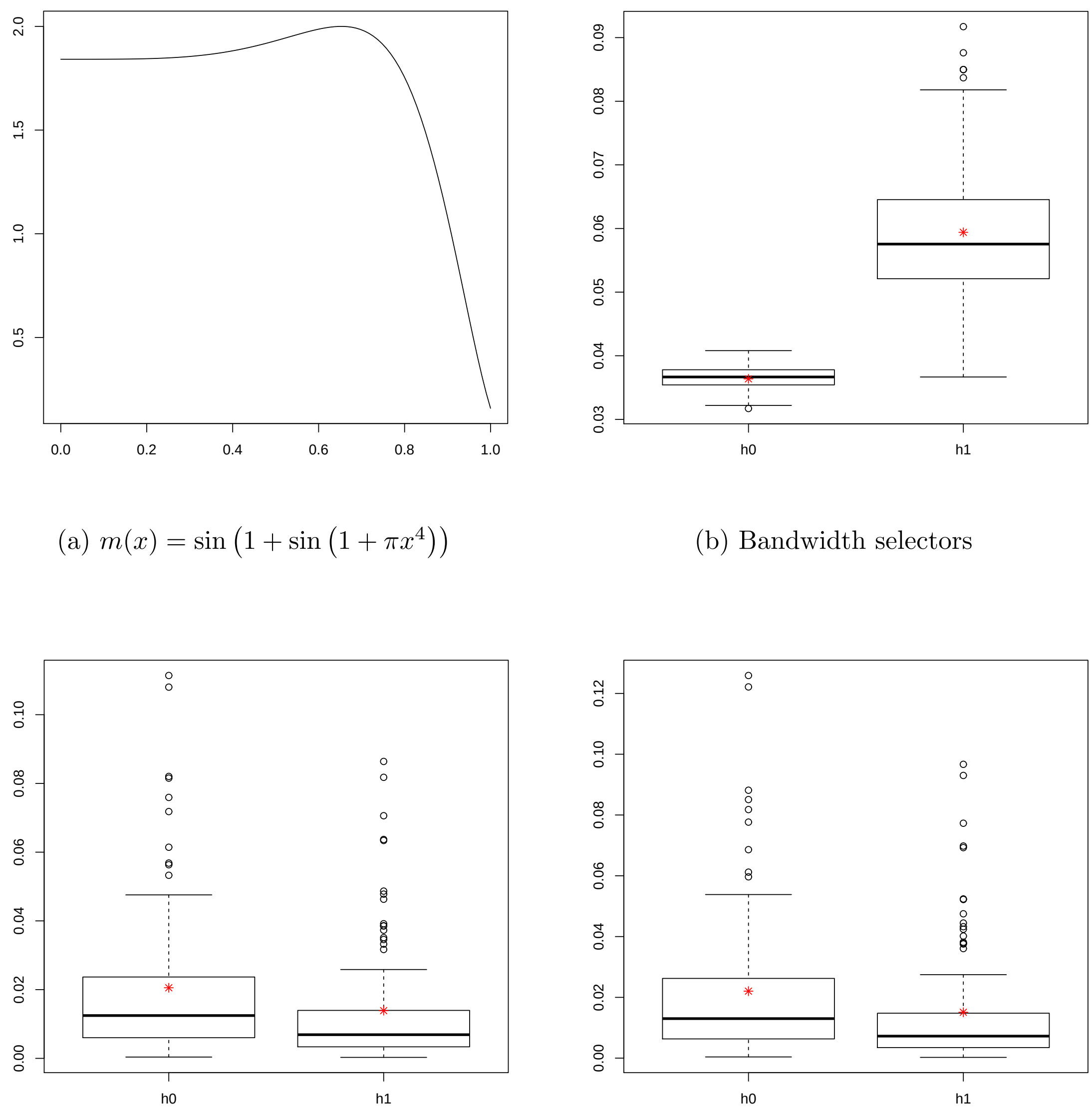


(c) Expression (4.48)



(d) Expression (4.49)

Figure 4.8: Boxplots obtained considering Scenario 6. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



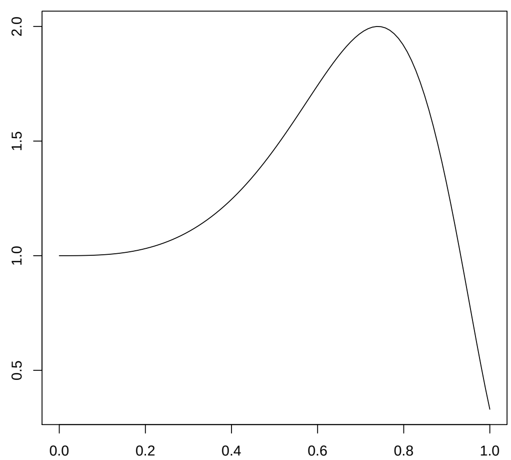
(a) $m(x) = \sin\left(1 + \sin\left(1 + \pi x^4\right)\right)$

(b) Bandwidth selectors

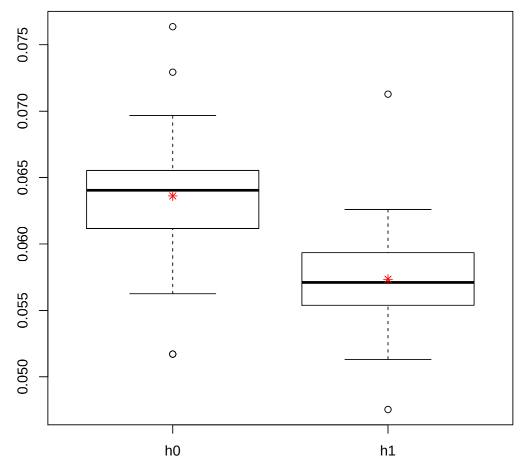
(c) Expression (4.48)

(d) Expression (4.49)

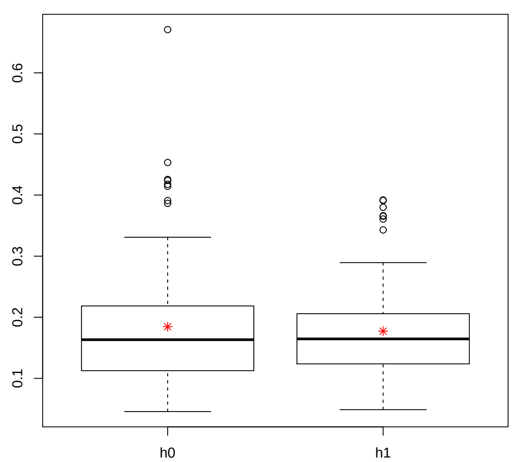
Figure 4.9: Boxplots obtained considering Scenario 7. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



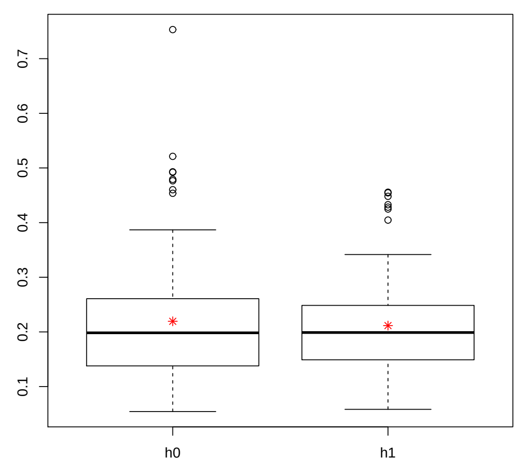
(a) $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$



(b) Bandwidth selectors

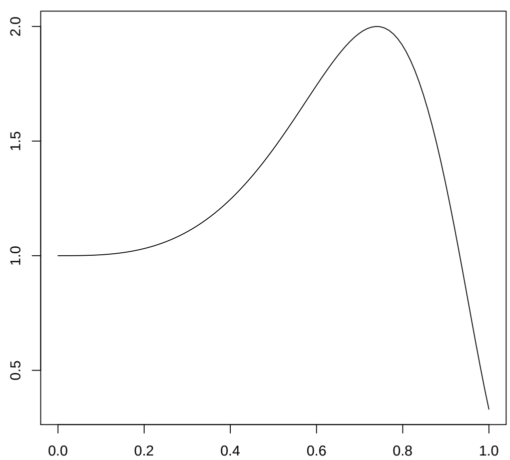


(c) Expression (4.48)

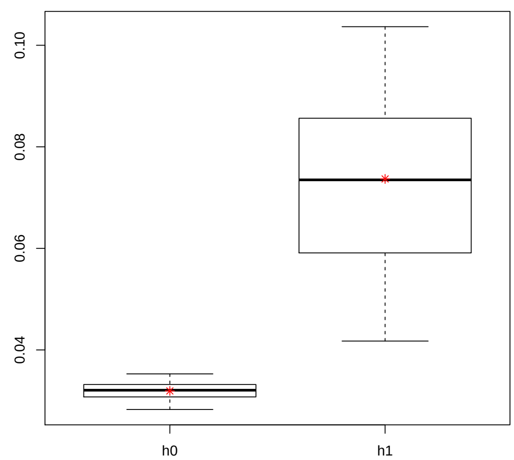


(d) Expression (4.49)

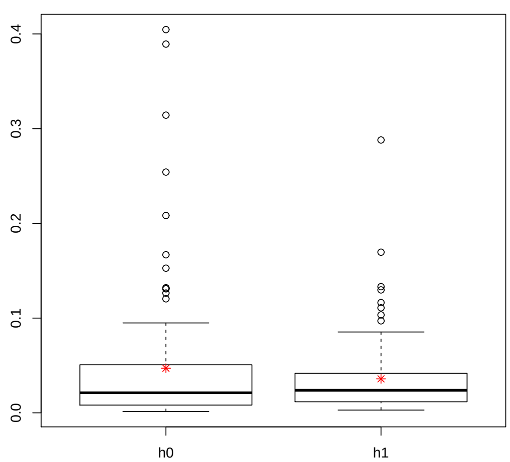
Figure 4.10: Boxplots obtained considering Scenario 8. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



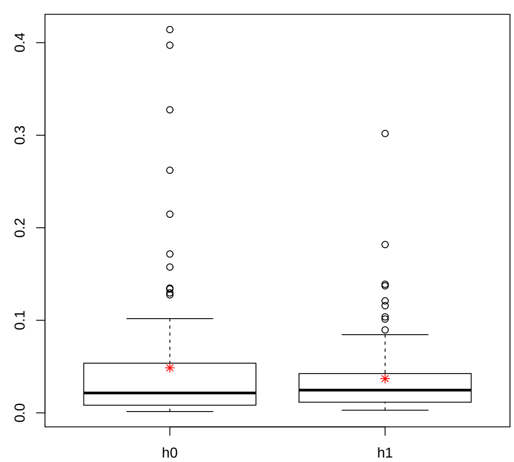
(a) $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$



(b) Bandwidth selectors

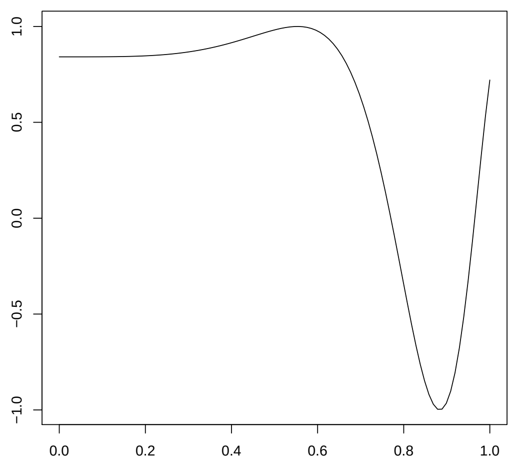


(c) Expression (4.48)

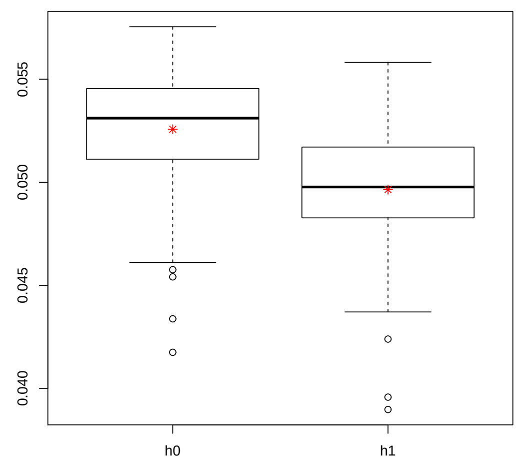


(d) Expression (4.49)

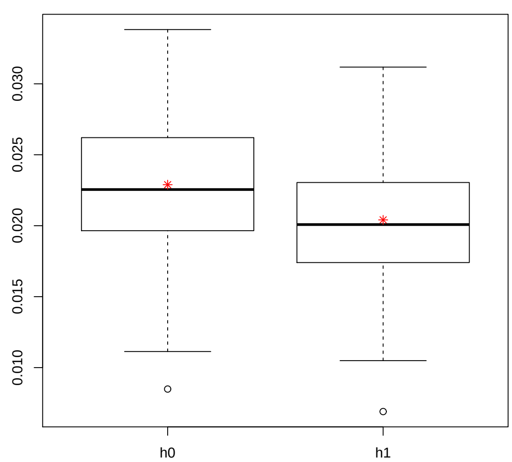
Figure 4.11: Boxplots obtained considering Scenario 9. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



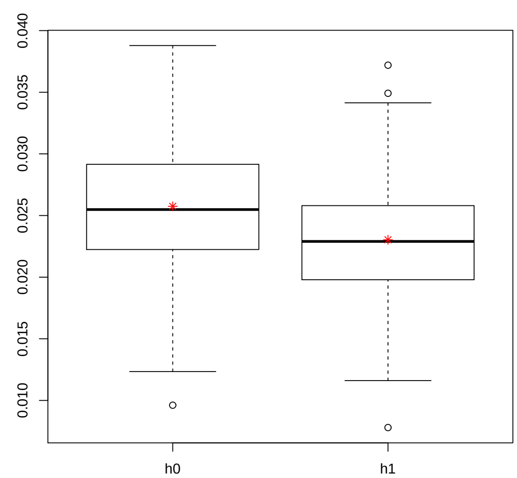
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

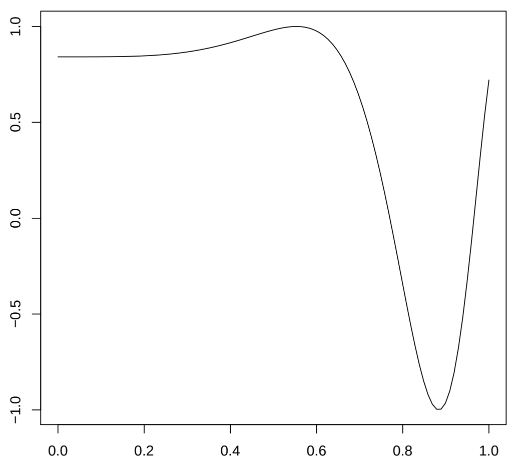


(c) Expression (4.48)

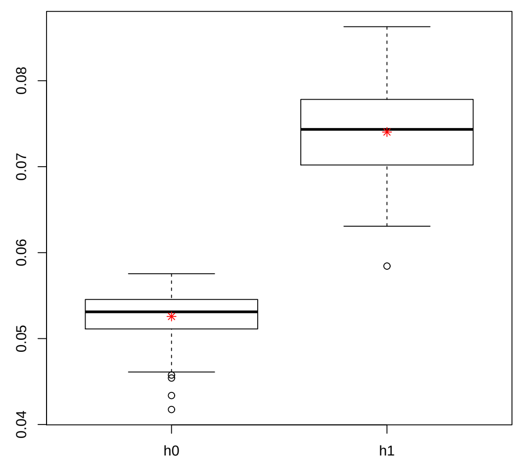


(d) Expression (4.49)

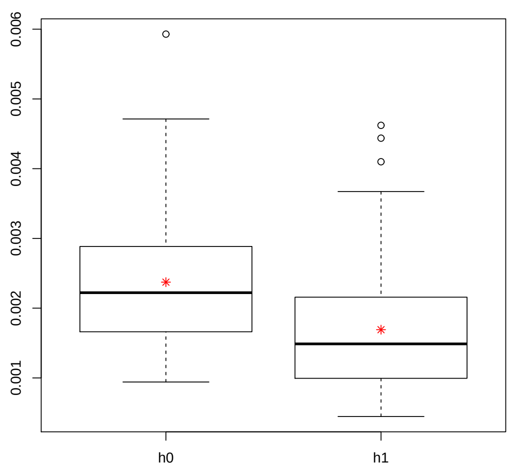
Figure 4.12: Boxplots obtained considering Scenario 10. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



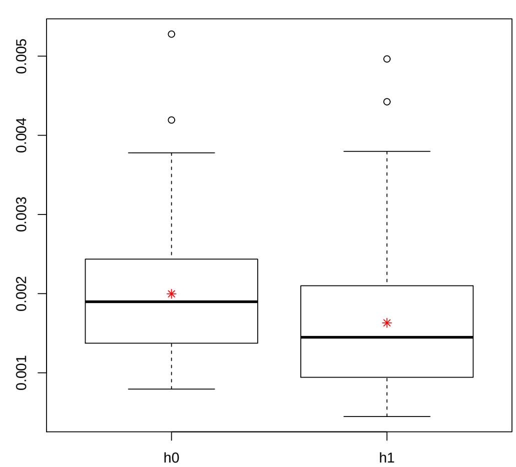
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors



(c) Expression (4.48)



(d) Expression (4.49)

Figure 4.13: Boxplots obtained considering Scenario 11. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).

4.5.2 Local linear regression estimator

General description of the study

A simulation study is now carried out in order to show the empirical performance of this new smoothing parameter selector, namely h_{BOOT}^{LL} in (4.26). Eleven different scenarios have been considered:

Scenario 1: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin(\pi x)$ and ϵ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 2: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 3: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(1.5,4)$.

Scenario 4: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 5: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(1.5,4)$.

Scenario 6: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(1 + \pi x^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 7: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(1 + \pi x^4\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 8: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin(1 + 2\pi x^4)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 9: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin(1 + 2\pi x^4)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(1.5,4)$.

Scenario 10: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(4,2)$.

Scenario 11: In the source population, the distribution of X^0 is a $\beta(2,2)$ and $Y_0 = m(X^0) + 0.3\epsilon$, where $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$ has been drawn from a standard normal distribution. The target population, X^1, has distribution $\beta(1.5,4)$.

For every scenario, 100 random samples of size $n_0 = n_1 = 500$ were drawn. The pilot bandwidth, g, used in the bootstrap method was $g = h_{SJ} n_0^{4/45}$, so that g has order $n_0^{-1/9}$.

The smoothed bootstrap bandwidth selector is the minimizer, in h, of an empirical function. Since this minimizer does not have an explicit expression, numerical methods are used to approximate it. The algorithm proceeds as the one described in Section 4.5.1 for the Nadaraya-Watson estimator.

It is worth mentioning that, in order to avoid oversmoothing of the bootstrap procedure, h_{BOOT}^{LL} is considered as the smallest h for which $MASE^*_{\tilde{m}_h^{LL}}$ attains a local minimum closest to the origin, not its global one. As explained in Section 4.5.1, our aim is to check the good empirical behaviour of the new bootstrap bandwidth selector in terms of prediction, we have considered the aforementioned smoothing parameter selector, h_{BOOT}^{LL} (h_1^{LL} in the following). Additionally, h_0^{LL} stands for the minimizer of expression $MASE^*_{\tilde{m}_h^{LL},X^{0'}}(h)$, which is given by:

$$
\begin{aligned}
MASE^*_{\tilde{m}_h^{LL},X^{0'}}(h) \;=\;& \frac{1}{n_0}\sum_{j=1}^{n_0}\Bigg[\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3} \\
& \left[K_h * \hat{b}^0_{X_j^{0'},g}\right]^2 (X_j^{0'}) \left[K_h * \hat{d}^2_{X_j^{0'},g}\right]^2 (X_j^{0'}) \\
& +\frac{(n_0-1)(n_0-2)}{n_0^3}\Bigg(\left[(K_h)^2 * \hat{a}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^2_{X_j^{0'},g}\right]^2 (X_j^{0'}) \\
& +\left[(K_h)^2 * \hat{d}^4_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^0_{X_j^{0'},g}\right]^2 (X_j^{0'}) \\
& +2\left[(K_h)^2 * \hat{b}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\Bigg) \\
& +\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{a}^2_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2 * \hat{d}^4_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\Bigg] \\
& -2\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{b}^0_{X_j^{0'},g}\right](X_j^{0'}) \\
& \left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^1_{X_j^{0'},g}\right](X_j^{0'}) \\
& +\frac{(n_0-1)(n_0-2)}{n_0^3}\Bigg[\left[(K_h)^2 * \hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'}) \\
& \left[K_h * \hat{b}^1_{X_j^{0'},g}\right](X_j^{0'}) \\
& +\left[(K_h)^2 * \hat{a}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'}) \\
& +\left[(K_h)^2 * \hat{d}^3_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^0_{X_j^{0'},g}\right](X_j^{0'}) \\
& +\frac{n_0-1}{n_0^3}\Bigg(\left[(K_h)^2 * \hat{b}^1_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2 * \hat{b}^3_{X_j^{0'},g}\right]\left(X_j^{0'}\right) \\
& +\left[(K_h)^2 * \hat{a}^1_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2 * \hat{d}^3_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\Bigg)\Bigg] \\
& +\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{d}^1_{X_j^{0'},g}\right]^2 (X_j^{0'}) \\
& \left[K_h * \hat{b}^1_{X_j^{0'},g}\right]^2 (X_j^{0'}) + \frac{(n_0-1)(n_0-2)}{n_0^3}\Bigg[\left[(K_h)^2 * \hat{a}^2_{X_j^{0'},g}\right](X_j^{0'}) \\
& \left[K_h * \hat{d}^1_{X_j^{0'},g}\right]^2 (X_j^{0'}) + \left[(K_h)^2 * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h * \hat{b}^1_{X_j^{0'},g}\right]^2 (X_j^{0'})
\end{aligned}
$$

$$+2\left[(K_h)^2*\hat{b}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\right]+\frac{n_0-1}{n_0^3}$$

$$\cdot\left(2\left[(K_h)^2*\hat{b}^2_{X_j^{0'},g}\right]^2\left(X_j^{0'}\right)+\left[(K_h)^2*\hat{d}^2_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2*\hat{a}^2_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\right)\right]$$

$$-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}\,h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\right.$$

$$\left[K_h*\hat{d}^2_{X_j^{0'},g}\right]^2(X_j^{0'})\left[K_h*\hat{d}^0_{X_j^{0'},g}\right](X_j^{0'})$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{b}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^0_{X_j^{0'},g}\right](X_j^{0'})\right.$$

$$+\left[K_h*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[(K_h)^2*\hat{d}^4_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^0_{X_j^{0'},g}\right](X_j^{0'})$$

$$+\left[(K_h)^2*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})$$

$$+\left[(K_h)^2*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right]^2(X_j^{0'})\right]$$

$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^4_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2*\hat{f}^0_g\right]\left(X_j^{0'}\right)\right]$$

$$+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^1_{X_j^{0'},g}\right]^2(X_j^{0'})$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[3\left[(K_h)^2*\hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})\right.$$

$$\left[K_h*\hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})+2\left[(K_h)^2*\hat{d}^3_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{b}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\right]$$

$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^1_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\left[(K_h)^2*\hat{d}^3_{X_j^{0'},g}\right]\left(X_j^{0'}\right)\right]$$

$$+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\right.$$

$$\left[K_h*\hat{d}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^2_{X_j^{0'},g}\right](X_j^{0'})+\frac{(n_0-1)(n_0-2)}{n_0^3}$$

$$\left[\left[(K_h)^2*\hat{d}^3_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{d}^0_{X_j^{0'},g}\right](X_j^{0'})\left[K_h*\hat{b}^1_{X_j^{0'},g}\right](X_j^{0'})\right.$$

$$+\left[(K_h)^2 * \hat{b}^3_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^1_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^0_{X_j^{0\prime},g}\right](X_j^{0\prime})$$
$$+\left[(K_h)^2 * \hat{d}^1_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{b}^1_{X_j^{0\prime},g}\right](X_j^{0\prime})\Big]$$
$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^3_{X_j^{0\prime},g}\right]\left(X_j^{0\prime}\right)\left[(K_h)^2 * \hat{d}^1_{X_j^{0\prime},g}\right]\left(X_j^{0\prime}\right)\Big]$$
$$-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}$$
$$\left[K_h * \hat{d}^1_{X_j^{0\prime},g}\right]^3(X_j^{0\prime})\left[K_h * \hat{b}^1_{X_j^{0\prime},g}\right](X_j^{0\prime}) + 2\frac{(n_0-1)(n_0-2)}{n_0^3}$$
$$\Big[\left[(K_h)^2 * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{b}^1_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^1_{X_j^{0\prime},g}\right](X_j^{0\prime})$$
$$+\left[(K_h)^2 * \hat{b}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^1_{X_j^{0\prime}}\right]^2(X_j^{0\prime})\Big]$$
$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[(K_h)^2 * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\Big]$$
$$+\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}$$
$$\left[K_h * \hat{d}^2_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})\left[K_h * \hat{d}^0_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})$$
$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[\left[(K_h)^2 * \hat{d}^4_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^0_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})$$
$$+\left[(K_h)^2 * \hat{d}^0_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^2_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})$$
$$+2\left[(K_h)^2 * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^0_{X_j^{0\prime},g}\right](X_j^{0\prime})\Big]$$
$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^4_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[(K_h)^2 * \hat{f}^0_g\right](X_j^{0\prime})\Big]$$
$$+h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{d}^1_{X_j^{0\prime},g}\right]^4(X_j^{0\prime})$$
$$+4\frac{(n_0-1)(n_0-2)}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X_j^{0\prime},g}\right](X_j^{0\prime})\left[K_h * \hat{d}^1_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})$$
$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X_j^{0\prime},g}\right]^2(X_j^{0\prime})\Big]$$
$$-2\,h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}$$

$$\left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^0_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^1_{X_j^{0'},g}\right]^2 (X_j^{0'}) + \frac{(n_0-1)(n_0-2)}{n_0^3}$$
$$\left[2\left[(K_h)^2 * \hat{d}^3_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^0_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\right.$$
$$\left. +2\left[(K_h)^2 * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^2_{X_j^{0'},g}\right](X_j^{0'}) \left[K_h * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\right]$$
$$+2\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^3_{X_j^{0'},g}\right](X_j^{0'}) \left[(K_h)^2 * \hat{d}^1_{X_j^{0'},g}\right](X_j^{0'})\Bigg]\Bigg], \tag{4.52}$$

where $\{X_j^{0'}\}_{j=1}^{n_0}$ is a sample coming from the source population but different to the sample $\{X_j^0\}_{j=1}^{n_0}$ used to compute the nonparametric estimators involved in expression (4.52). Using the same source sample to compute the estimations and then to evaluate them may well lead to numerical errors. We have defined a bandwidth selector, h_0^{LL}, as the minimizer of expression (4.52), that is:

$$h_0^{LL} = h_{MASE^*_{\tilde{m}_h^{LL},X^{0'}}} = \arg\min_{h>0} MASE^*_{\tilde{m}_h^{LL},X^{0'}}(h).$$

Both bandwidth selectors are compared by means of expressions (4.48) and (4.49), where $j = 0, 1$ and $\hat{m}_{h_j}$ stands for the local linear regression estimator obtained with the sample coming from the source population, that is $\{X^0, Y^0\}$. Both expressions were approximated by simulation in Figures 4.14-4.24.

Discussion and results

Figures 4.14-4.24 collect the empirical results for the simulation study previously described in Section 4.5.2. Indeed, we can distinguish among three different situations:

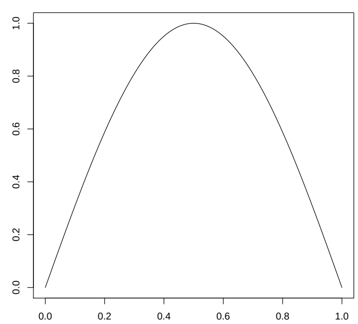
1. h_0^{LL} is much larger than h_1^{LL} (see Figures 4.15, 4.17, 4.19, 4.21, 4.23). This is logical because the regression function m oscillates in the support of the target population. On the other hand, considering the support of the source population, the regression function fluctuates within an interval but remains flat in the rest of the support.

2. h_0^{LL} turns out to be smaller than h_1^{LL} (see Figures 4.16, 4.18, 4.19, 4.20, 4.22, 4.24). This is the opposite situation as the mentioned above. Indeed, the empirical behaviour of h_0^{LL} and h_1^{LL} is understandable since m fluctuates in

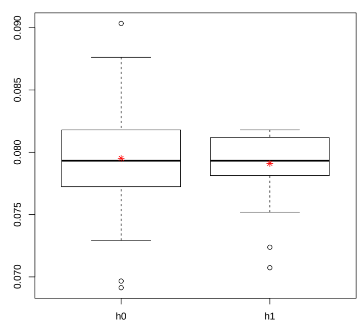
the support of the source population. Furthermore, m oscillates within an interval of the support of the target population, but is flat in the rest of it.

3. h_0^{LL} and h_1^{LL} are close (see Figure 4.14). This is quite reasonable as the regression function m is similar in both supports of the source and target populations.

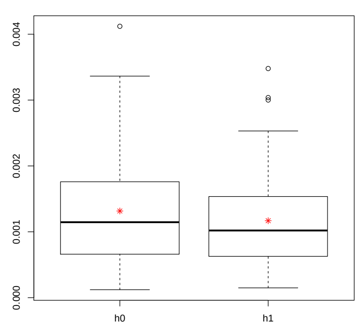
Furthermore, according to expressions (4.48) and (4.49) in Figures 4.14-4.24, it is clear that h_1 beats its competitor, h_0. This means that using the information provided by the target sample to select the bandwidth in terms of prediction brings about less error.



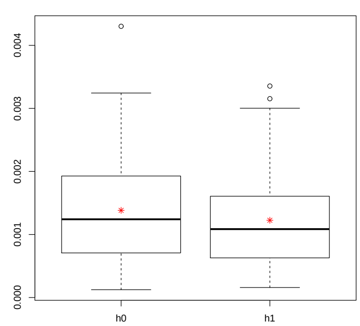
(a) $m(x) = \sin(\pi x)$



(b) Bandwidth selectors

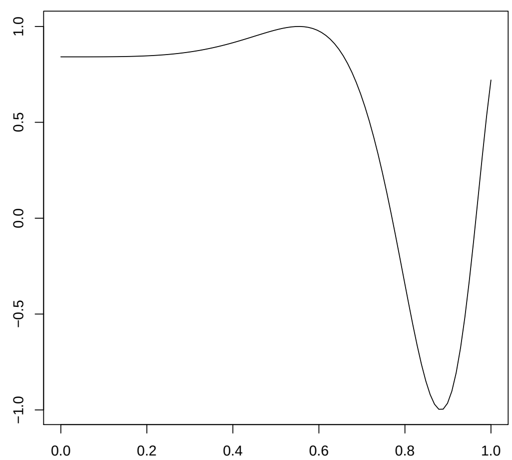


(c) Expression (4.48)

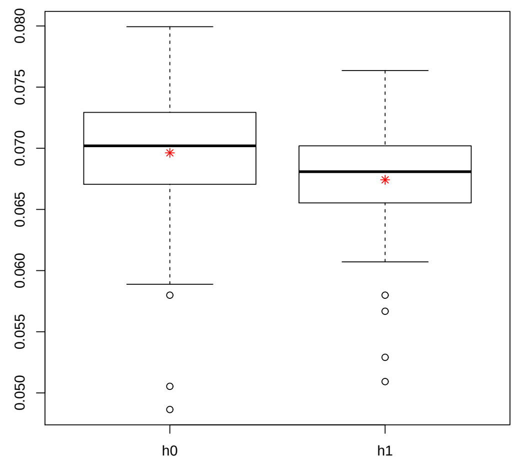


(d) Expression (4.49)

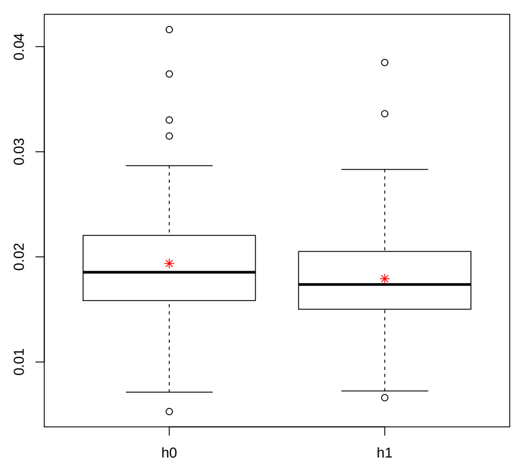
Figure 4.14: Boxplots obtained considering Scenario 1. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



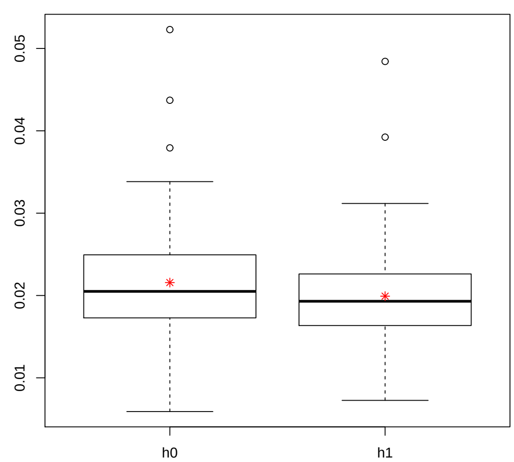
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

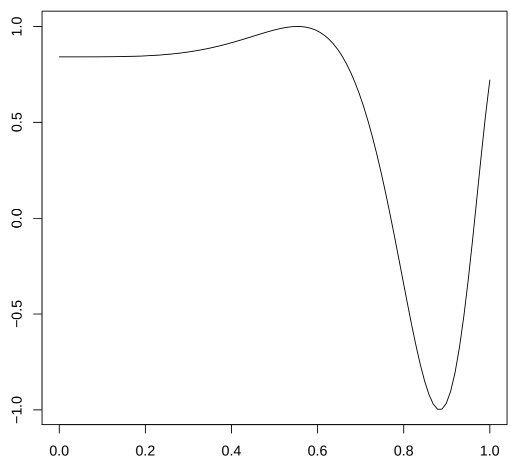


(c) Expression (4.48)

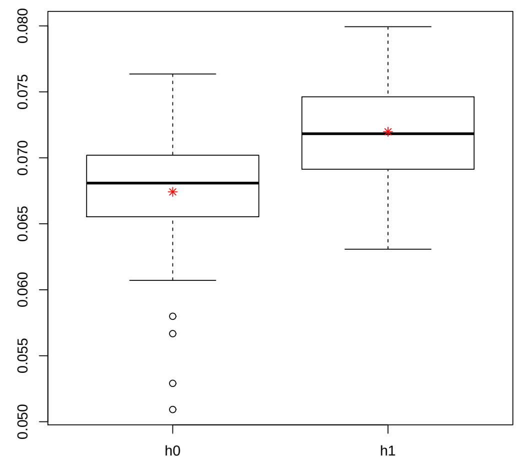


(d) Expression (4.49)

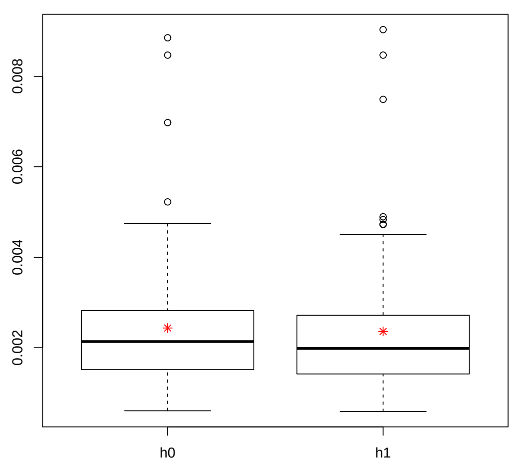
Figure 4.15: Boxplots obtained considering Scenario 2. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



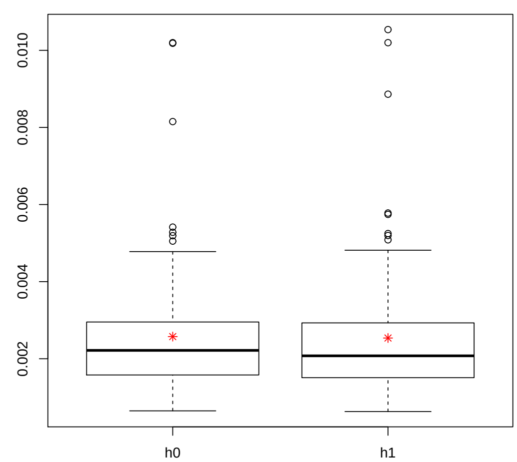
(a) $m(x) = \sin\left(1 + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

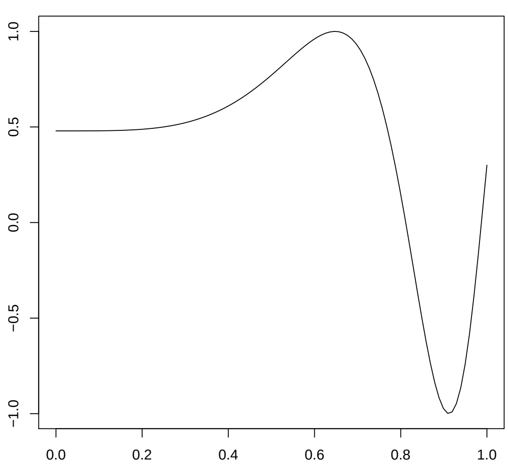


(c) Expression (4.48)

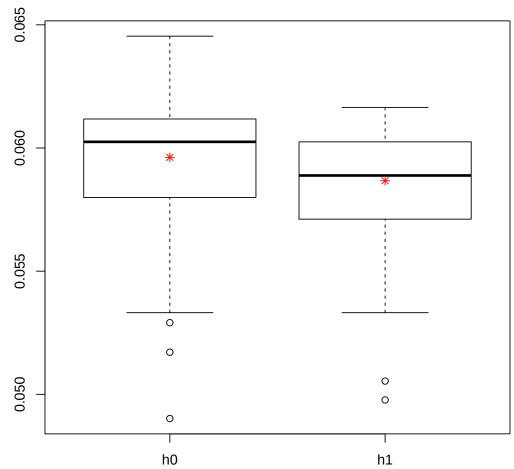


(d) Expression (4.49)

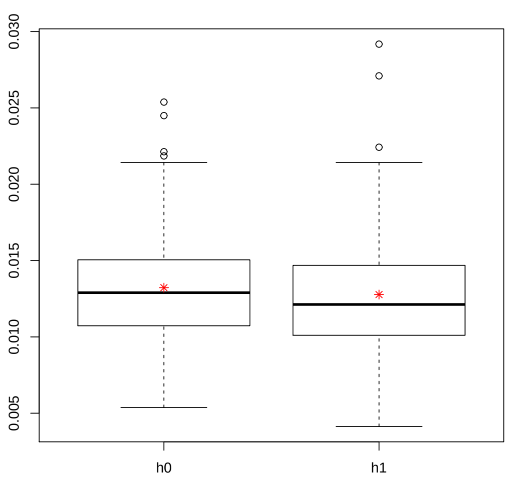
Figure 4.16: Boxplots obtained considering Scenario 3. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



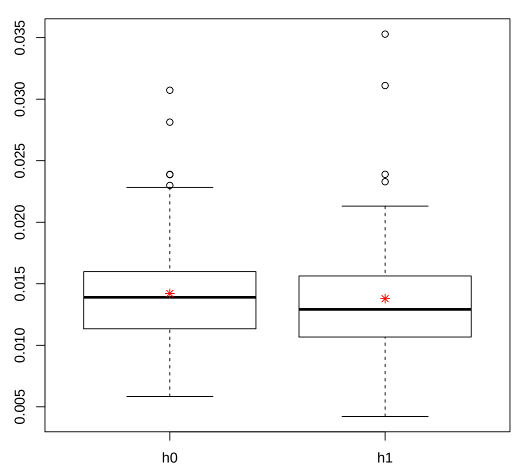
(a) $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

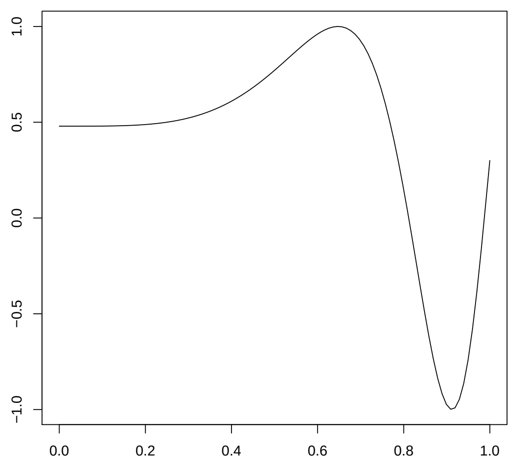


(c) Expression (4.48)

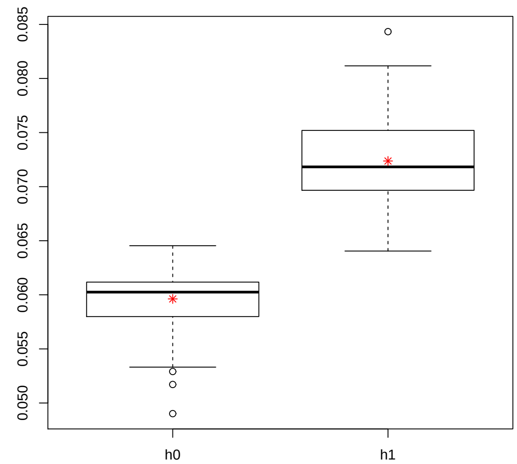


(d) Zoom expression (4.49)

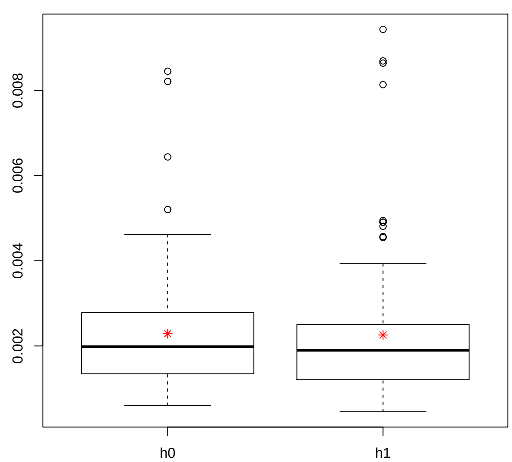
Figure 4.17: Boxplots obtained considering Scenario 4. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



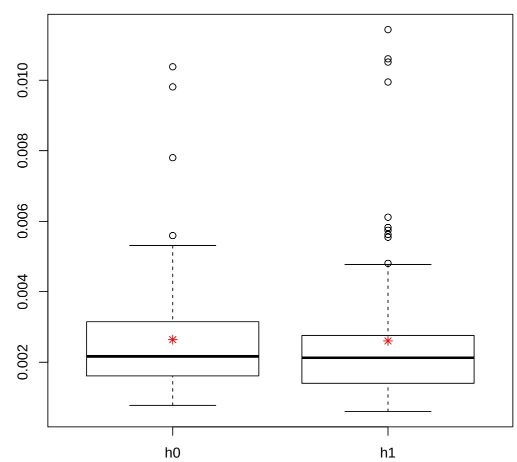
(a) $m(x) = \sin\left(\frac{1}{2} + \left(\frac{\pi x}{2}\right)^4\right)$



(b) Bandwidth selectors

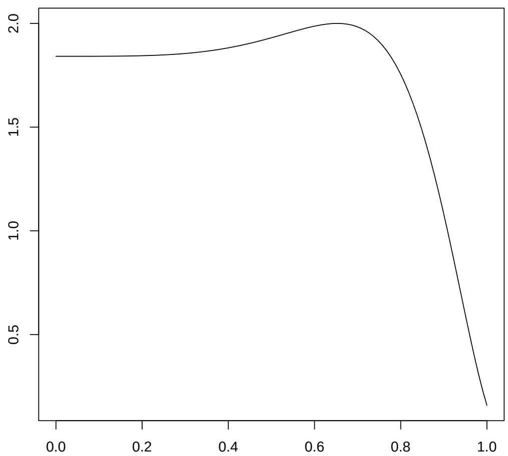


(c) Expression (4.48)

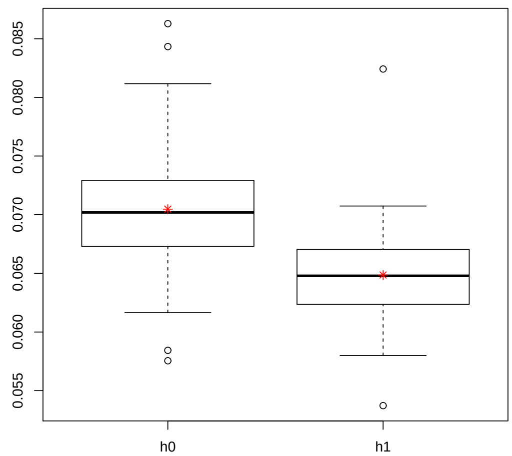


(d) Expression (4.49)

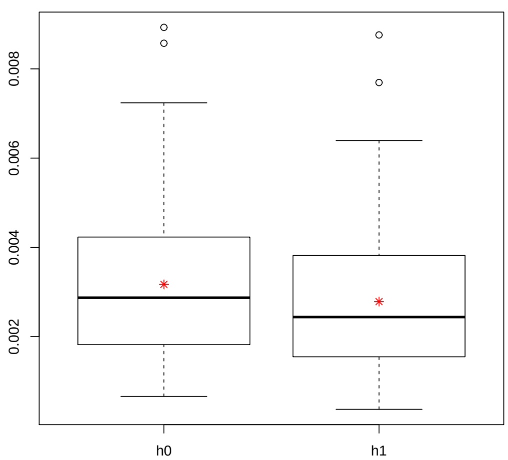
Figure 4.18: Boxplots obtained considering Scenario 5. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



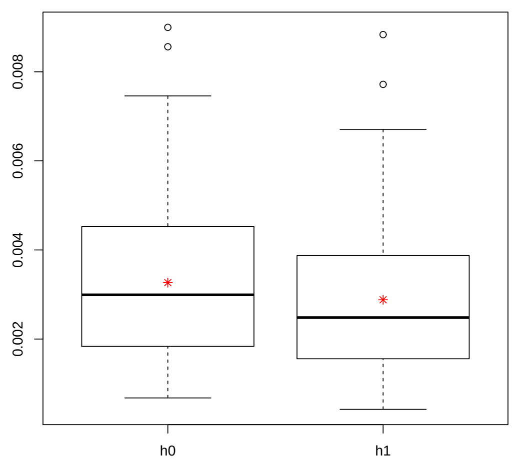
(a) $m(x) = 1 + \sin\left(1 + \pi x^4\right)$



(b) Bandwidth selectors

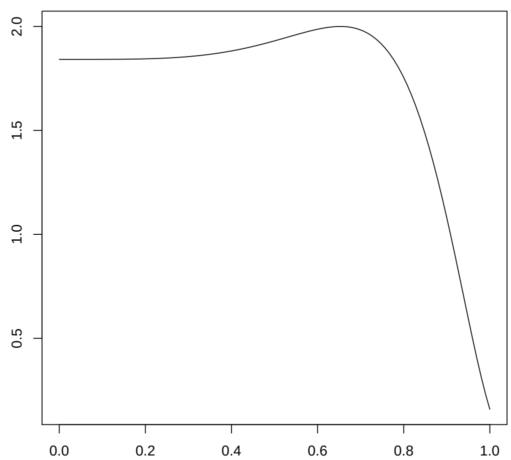


(c) Expression (4.48)

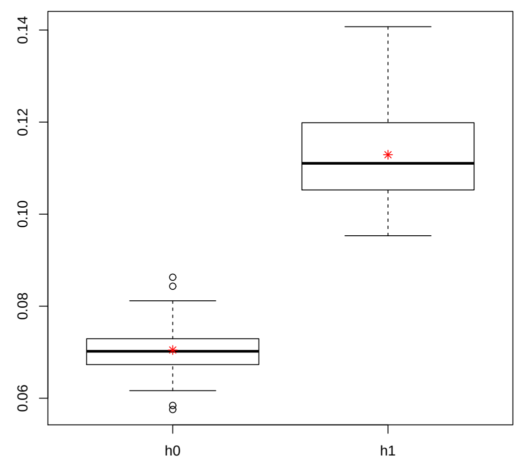


(d) Expression (4.49)

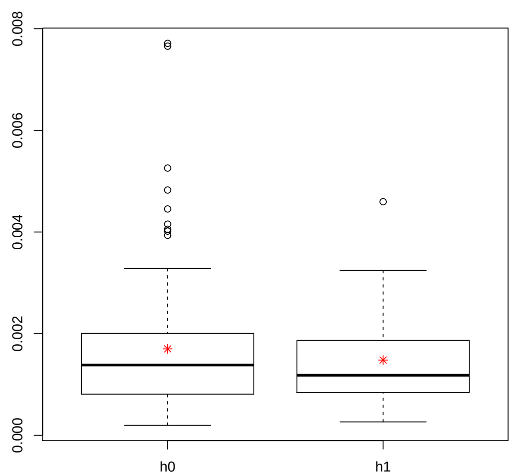
Figure 4.19: Boxplots obtained considering Scenario 6. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



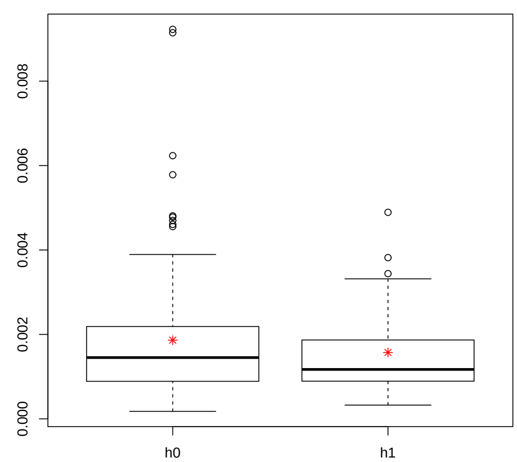
(a) $m(x) = 1 + \sin\left(1 + \pi x^4\right)$



(b) Bandwidth selectors

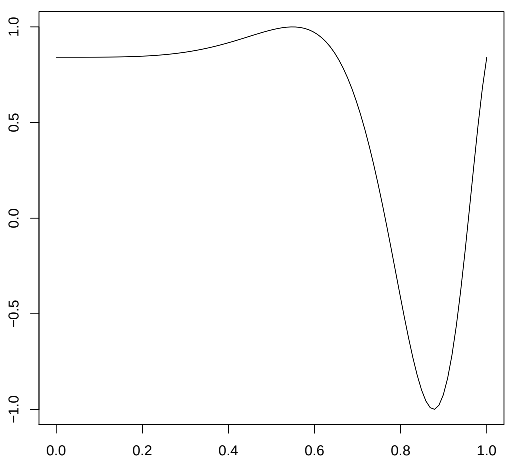


(c) Expression (4.48)

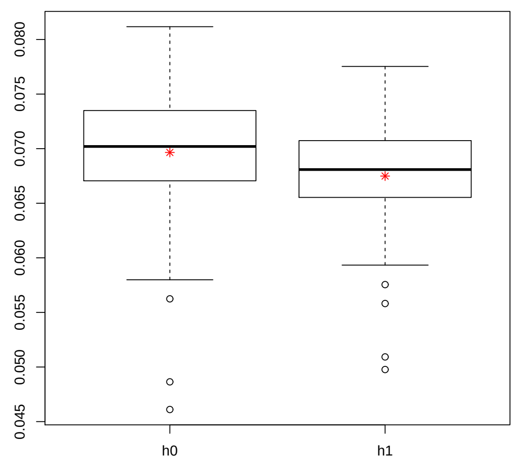


(d) Expression (4.49)

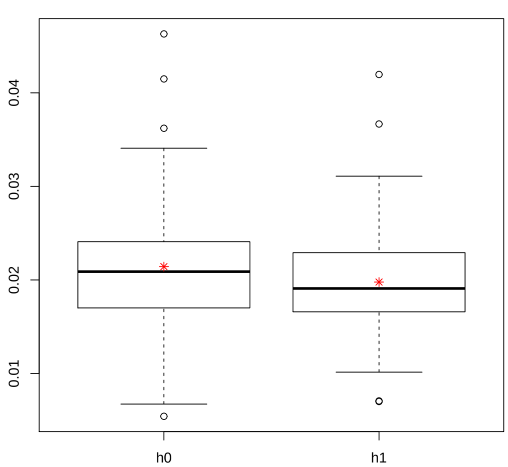
Figure 4.20: Boxplots obtained considering Scenario 7. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



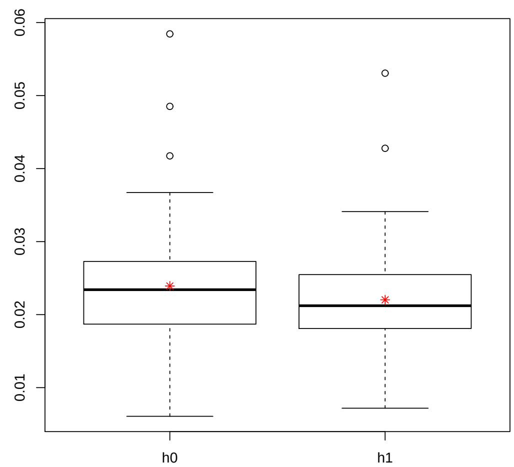
(a) $m(x) = \sin\left(1 + 2\pi x^4\right)$



(b) Bandwidth selectors

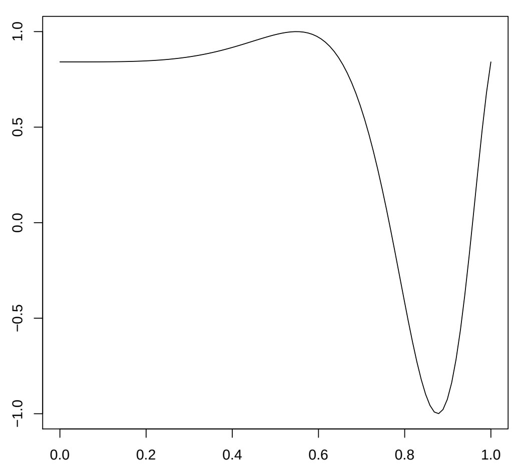


(c) Expression (4.48)

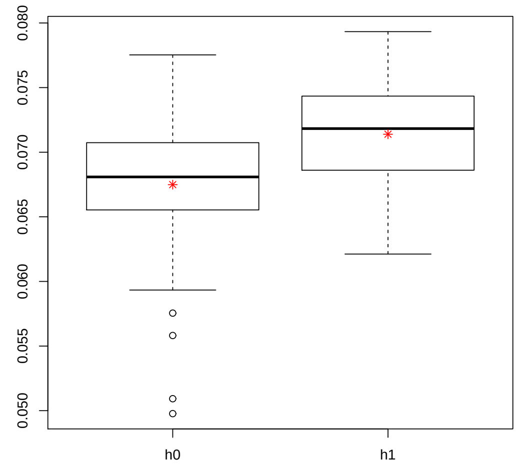


(d) Expression (4.48)

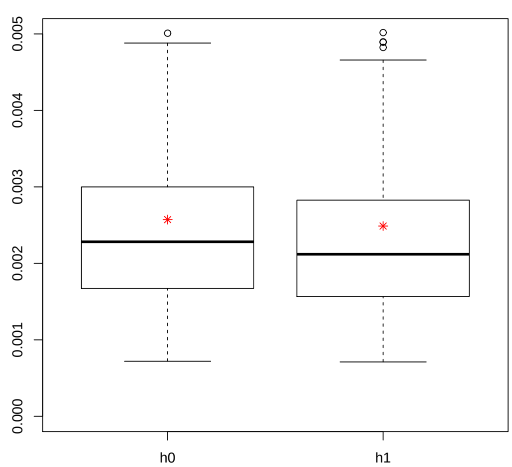
Figure 4.21: Boxplots obtained considering Scenario 8. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



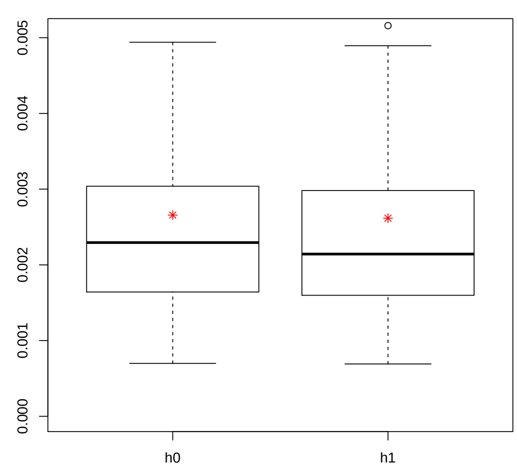
(a) $m(x) = \sin\left(1 + 2\pi x^4\right)$



(b) Bandwidth selectors

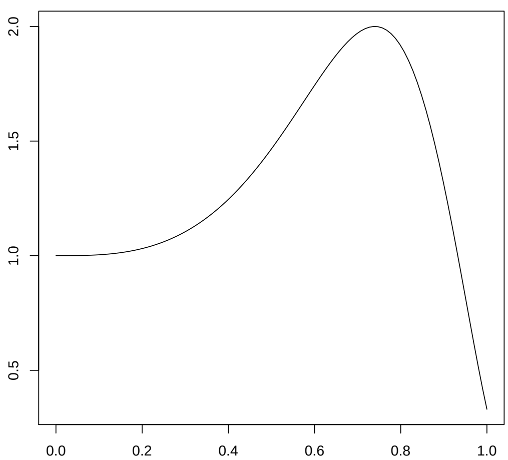


(c) Expression (4.48)

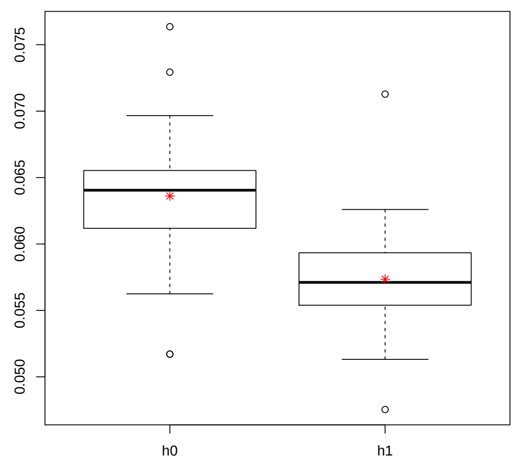


(d) Expression (4.48)

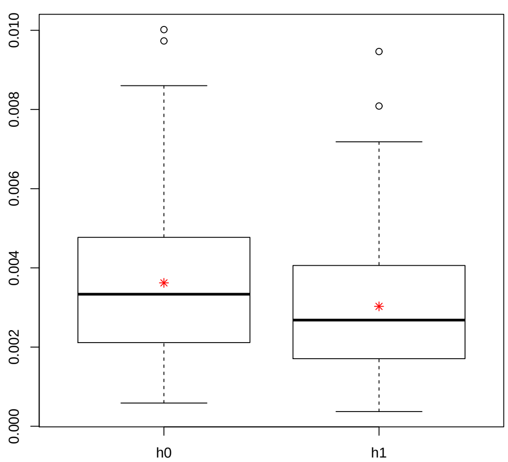
Figure 4.22: Boxplots obtained considering Scenario 9. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



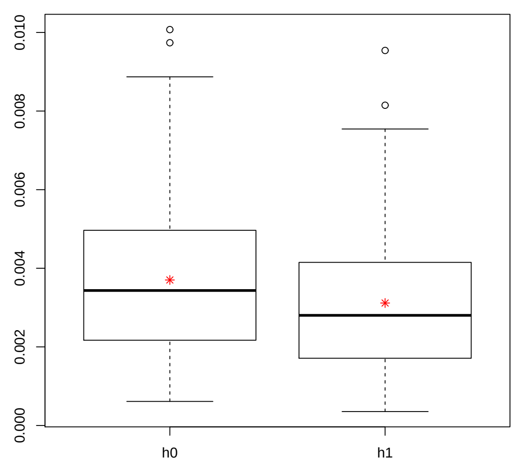
(a) $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$



(b) Bandwidth selectors

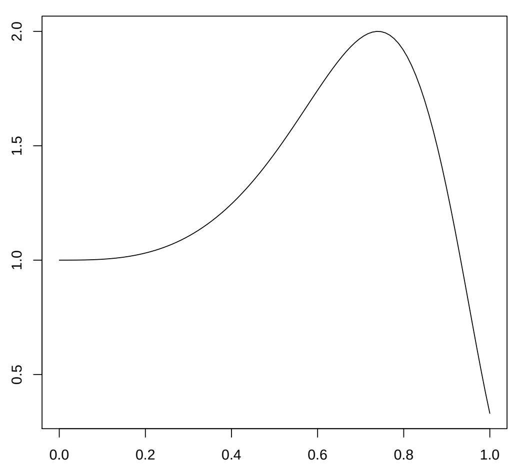


(c) Expression (4.48)

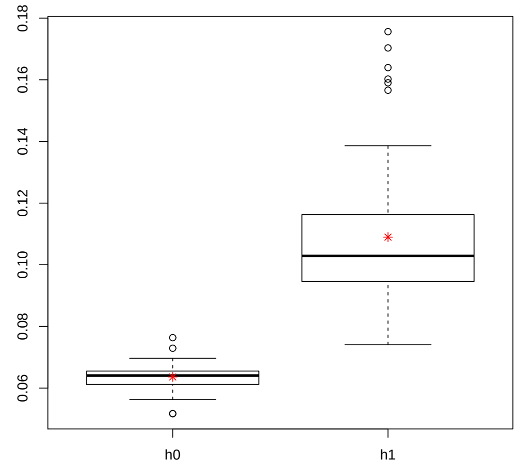


(d) Expression (4.49)

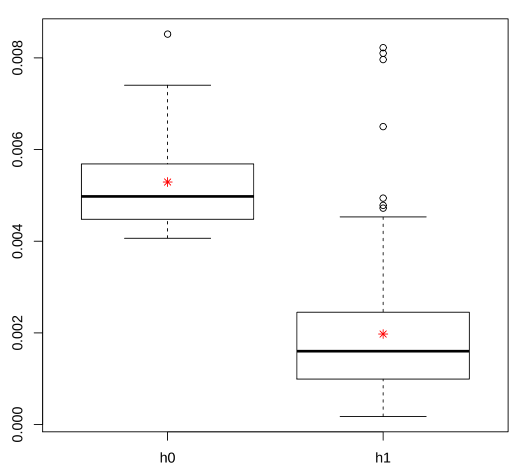
Figure 4.23: Boxplots obtained considering Scenario 10. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).



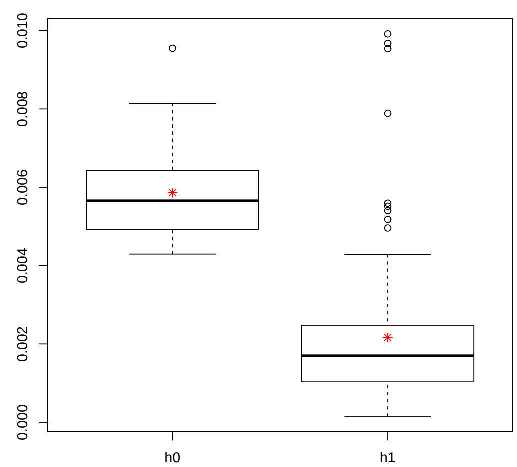
(a) $m(x) = 1 + \sin\left(\left(\frac{\pi x}{2}\right)^3\right)$



(b) Bandwidth selectors



(c) Expression (4.48)



(d) Expression (4.49)

Figure 4.24: Boxplots obtained considering Scenario 11. At the top, regression function considered (left side) and bandwidth selectors obtained (right side). At the bottom, expressions (4.48) - left side - and (4.49) - right side - approximated by simulation. The red crosses are the mean of expressions (4.48) and (4.49).

4.5.3 CPU times

Results of CPU times are collected in Tables 4.7-4.8. In particular, Table 4.7 shows the practical behaviour of the closed expression obtained for $MASE^{*}_{\tilde{m}_h^{NW},X^1}$ of the Nadaraya-Watson estimator given in expression (4.14) in terms of computer time. Moreover, it is empirically compared with the efficiency in terms of computer time of expression (4.5), which is the closed expression of MSE^{*}_{x}. Similar results are collected in Table 4.8 for the local linear regression estimator considering expressions (4.24) and (4.21). Packages `parallel` and `Bolstad` of the free software `R` have been used. The fist one, `parallel`, in order to parallelize the code. The second one, `Bolstad`, in order to approximate numerically the integrals.

Indeed, h^{NW}_{BOOT} clearly beats its main competitor, h^{LL}_{BOOT}, in terms of CPU time required (see Tables 4.7 and 4.8). This is natural considering expressions (4.13) and (4.25), since the first one is much simpler than the second one. In addition, using a global bandwidth instead of a local one turns out to be far more efficient as well.

	$n_0 = n_1 = 100$		$n_0 = n_1 = 500$	
Trials	h^{NW}_{MSE}	h^{NW}_{MASE}	h^{NW}_{MSE}	h^{NW}_{MASE}
1	0.75	0.71	18.01	17.47
100	72.19	79.36	1784.98	1758.63

Table 4.7: CPU times (in seconds) obtained using Scenario 2 of Section 4.5.1, $n_0 = n_1 = 100$ (second multicolumn), $n_0 = n_1 = 500$ (third multicolumn) and Nadaraya-Watson regression estimator, where h^{NW}_{MSE} is the bandwidth selector that minimizes (4.5) and h^{NW}_{MASE} is the bandwidth selector which minimizes (4.14) (namely h^{NW}_{BOOT}).

	$n_0 = n_1 = 100$		$n_0 = n_1 = 500$	
Trials	h^{LL}_{MSE}	h^{LL}_{MASE}	h^{LL}_{MSE}	h^{LL}_{MASE}
1	179.5	108.8	4434.22	2748.29
100	18826.64	12771.60	439513.35	290180.17

Table 4.8: CPU times (in seconds) obtained using Scenario 2 of Section 4.5.2, $n_0 = n_1 = 100$ (second multicolumn), $n_0 = n_1 = 500$ (third multicolumn) and local linear regression estimator, where h^{LL}_{MSE} is the bandwidth selector that minimizes (4.21) and h^{LL}_{MASE} is the bandwidth selector which minimizes (4.25) (namely h^{LL}_{BOOT}).

4.6 Real data analysis

The bootstrap bandwidth selector proposed for prediction in regression considering the Nadaraya-Watson estimator, defined in (4.15), is illustrated by computing the estimated mean for a real data set that can be found at INE webpage (see `http://www.ine.es`), which is the National Institute of Statistics in Spain. The aim is to analyze the gender gap for salaries in Spain.

The data set consists of 64383 data from women and 108134 data from men of the gross annual wage in 2014, which is the response variable in our model for both men and women (namely, **Salary**). Additionally, we consider some explanatory variables:

- **Years and months of service**, which is a quantitavie variable.
- **CNAE**, which stands for the National Classification of Economical Activities in Spain. It is a qualitative variable, split into eigthteen groups.
  - B: Extractive industry (anthracite, oil, coal and lignite extraction)
  - C: Manufacturing industry
  - D: Electricity and gas supply
  - E: Water supply
  - F: Construction
  - G: Wholesale and retail trade activities

  - H: Transport and storage
  - I: Hotel industry
  - J: Information and communication
  - K: Financial activities and insurances
  - L: Real-estate sector
  - M: Professional, scientific and technical activities
  - N: Administrative activities
  - O: Public, Defence and Social Security Administration
  - P: Education
  - Q: Health care system activities
  - R: Arts
  - S: Other services (computer repair ...)

- **Studies**, which is a qualitative variable. It is divided into seven groups.
  - 1: Primary Education
  - 2: First stage of Secondary Education
  - 3: Second stage of Secondary Education
  - 4: Vocational Training (FP)
  - 5: Bachelor's degree
  - 6: Master's degree and Ph.D.

Our target is to analyze the gender wage gap, which happens to be a current affair issue. Indeed, the gender wage gap has been of long-standing political concern as an indicator of discrimination against women. The fact that women are paid substantially lower wages than men may well be the result of wage discrimination in the labour market. On the other hand, part of this wage gap might also be due to differences in education, experience and other skills, the distribution of which differs between men and women.

Therefore, we consider two different populations: men (source population) and women (target population) in Spain. We also consider an independent variable, such as *Years and months of service* (X^0 in the source population, X^1 in the target population). This variable has density f^0 for men and density f^1 for women. Moreover, we know the men's salary (Y^0). Assuming that the regression function is the same for both populations, women's salary (Y^1) can be predicted in terms of the non-parametric estimation of the regression function previously computed using the men population. Our aim is to compare this prediction with the actual average salary for women, and conclude whether the gender wage gap can be explained in terms of differences in a wide range of skills, or not.

Nevertheless, as a result of using some qualitative variables as independent variables, we need to modify the definition of (4.15) so as to take them into account. Let us denote X the quantitative explanatory variable, which is 1-dimensional; Z the qualitative explanatory variable, which is r-dimensional; and Y, the quantitative response variable, which is 1-dimensional. As previously mentioned, we observe (X^0, Z^0, Y^0) in the source population (men) and (X^1, Z^1) in the target population (women), while Y^1 remains unknown. Consequently, we can estimate the mean of Y^1 as follows:

$$\hat{\theta} \;=\; \widehat{\mathbb{E}\left[Y^1\right]} = \frac{1}{n_1}\sum_{j=1}^{n_1} \tilde{m}_h\left(X_j^1, Z_j^1\right),$$

where $m(x,z) = \mathbb{E}\left[Y^0|_{X^0=x,Z^0=z}\right] = m_z(x)$. In other words, m_z happens to be the regression of $Y^0|_{Z^0=z}$ over $X^0|_{Z^0=z}$.

Thus, if we define I as the set of values which takes the variable Z^1,

$$n_{0,z} = \sum_{i=1}^{n_0} \mathbb{I}_{\left\{Z_i^0=z\right\}} = \#\left\{i \in \{1,\ldots,n_0\}\,/Z_i^0 = z\right\}, \text{ and}$$

$$n_{1,z} = \sum_{j=1}^{n_1} \mathbb{I}_{\left\{Z_j^1=z\right\}} = \#\left\{j \in \{1,\ldots,n_1\}\,/Z_j^1 = z\right\},$$

it follows that:

$$\begin{aligned}\tilde{m}_h(x,z) &= \hat{m}_{z,h}(x) = \frac{\frac{1}{n_{0,z}}\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right) Y_i^0 \mathbb{I}_{\{Z_i^0=z\}}}{\frac{1}{n_{0,z}}\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right) \mathbb{I}_{\{Z_i^0=z\}}}, \text{ and} \\ \hat{\theta}_z &= \frac{1}{n_{1,z}}\sum_{j=1}^{n_1} \tilde{m}_h(X_j^1, Z_j^1)\mathbb{I}_{\{Z_j^1=z\}} \\ &= \frac{1}{n_{1,z}}\sum_{j=1}^{n_1} \frac{\sum_{i=1}^{n_0} K_h\left(X_j^1 - X_i^0\right) Y_i^0 \mathbb{I}_{\{Z_i^0=Z_j^1\}}}{\sum_{i=1}^{n_0} K_h\left(X_j^1 - X_i^0\right) \mathbb{I}_{\{Z_i^0=Z_j^1\}}} \mathbb{I}_{\{Z_j^1=z\}}. \end{aligned} \tag{4.53}$$

Hence, using expression (4.53) it leads to define the following estimator

$$\begin{aligned}\hat{\theta} &= \frac{1}{n_1}\sum_{j=1}^{n_1} \tilde{m}_h(X_j^1, Z_j^1) = \frac{1}{n_1}\sum_{z\in I}\sum_{j=1}^{n_1} \tilde{m}_h(X_j^1, Z_j^1)\mathbb{I}_{\{Z_j^1=z\}} \\ &= \frac{1}{n_1}\sum_{z\in I} n_{1,z}\cdot\frac{1}{n_{1,z}}\sum_{j=1}^{n_1} \tilde{m}_h(X_j^1, Z_j^1)\mathbb{I}_{\{Z_j^1=z\}} = \sum_{z\in I}\frac{n_{1,z}}{n_1}\hat{\theta}_z. \end{aligned} \tag{4.54}$$

Consider the following independent variables: *Years and months of service*, *Studies*, *CNAE*. The estimation in (4.54) is computed and then used to construct the plots in Figures 4.25 and 4.26, below. The SB1 resampling plan is used to compute the bandwidth selector $h_{BOOT,Z}^{NW}$ for every $z \in I$ using the pilot bandwidth $g_Z = h_{SJ,z} n_0^{4/45}$ as in Section 4.5. Figure 4.25 collects the values for $\bar{y}_1 - \hat{\theta}$, where $\bar{y}_1$ and $\hat{\theta}$ stand for the average women's salary and the predicted women's salary in euros, respectively. Figure 4.26 shows similar information but for the relative differences: $(\bar{y}_1 - \hat{\theta})/\hat{\theta}$. Finally, Figures 4.27 and 4.28 show the differences in the distribution of the variable *Studies* among men and women, considering Sectors F (Construction) and R (Arts), respectively, of CNAE. Indeed, although the distribution of *Studies* considering Sector R seems quite similar for both genders, there are clear differences in the distribution of *Studies* among men and women if Sector F is considered.

A relevant question in this context is the following one. Does the expected salary for women with a full-time job equal men's pay with a full-time job provided that they have identical studies, time of service and CNAE? According to the results shown below, it turns out that there is indeed a gender wage gap in the labour market. For most of levels of studies and activity sectors, mean salaries of women are between 10% and 30% lower than they should be if the payment of women and men as a function of experience, studies and activity sector would be homogeneus (see Figure 4.26 and Tables 4.9-4.11). It is worth mentioning that for some particular sectors and level of studies, the sample sizes n_0, n_1 are very small (see Tables 4.9-4.11). Therefore, the estimation of $\hat{\theta}$ might somewhat lose reliability. Table 4.12 collects the values of $h_{BOOT,Z}^{NW}$ for all the combinations of Studies and CNAE. Furthermore, Sector P (Education) provides less differences between $\bar{y}_1$ and $\hat{\theta}$, for all levels of education.

An estimation of (4.54) is provided in the following. Tables 4.9-4.11 collect the estimated $\hat{\theta}$ conditional on the qualitative variables considered.

$$\bar{y}_0 = 30000.16€, \bar{y}_1 = 25092.06€, \hat{\theta} = 31241.8€.$$

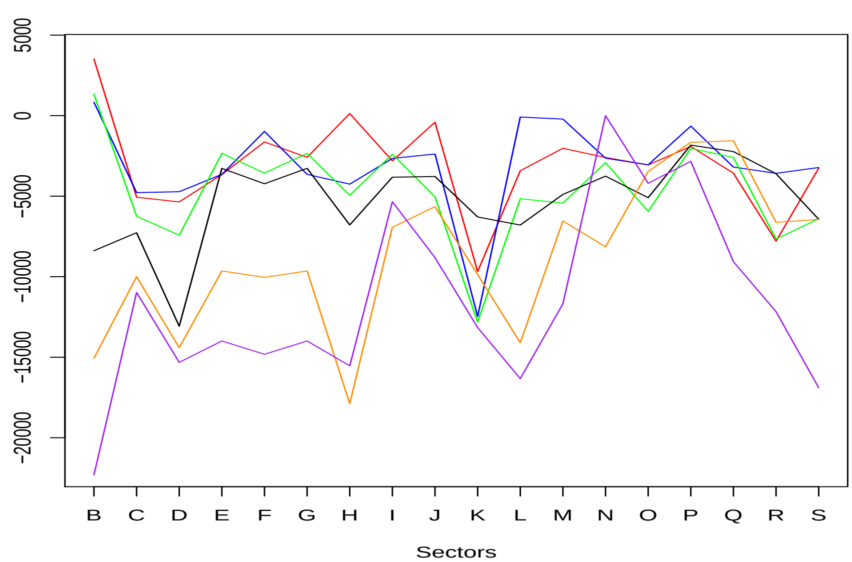


Figure 4.25: Plot of $\bar{y}_1 - \hat{\theta}$ for the 18 activity sectors and 6 levels of studies (1: red, 2: blue, 3: green, 4: black, 5: orange, 6: purple).

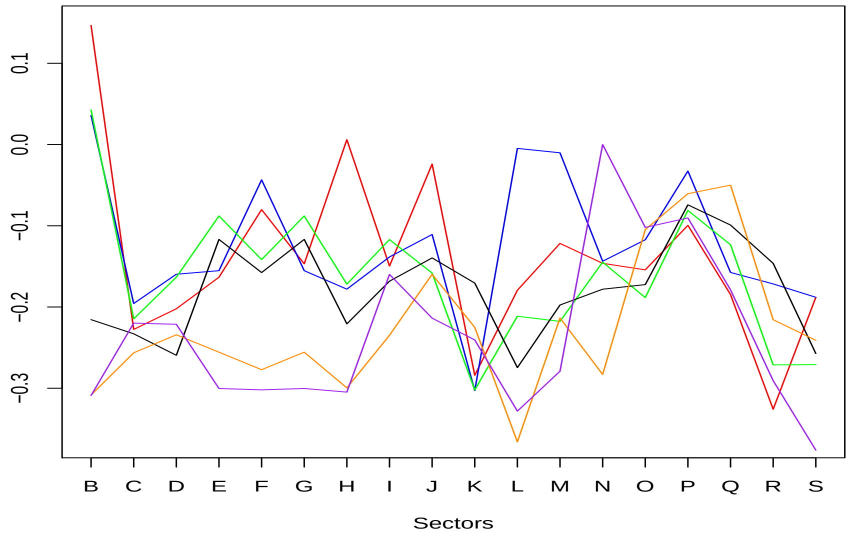


Figure 4.26: Plot of $\frac{\bar{y}_1 - \hat{\theta}}{\hat{\theta}}$ for the 18 activity sectors and 6 levels of studies (1: red, 2: blue, 3: green, 4: black, 5: orange, 6: purple).

	Studies					
CNAE	1	2	3	4	5	6
B	$n_{0,z} = 474$ $\bar{y}_0 = 24441$ $\bar{y}_1 = 27524$ $\hat{\theta} = 24004$ $n_{1,z} = 13$	$n_{0,z} = 415$ $\bar{y}_0 = 25156$ $\bar{y}_1 = 24254$ $\hat{\theta} = 23416$ $n_{1,z} = 14$	$n_{0,z} = 321$ $\bar{y}_0 = 30621$ $\bar{y}_1 = 32923$ $\hat{\theta} = 31579$ $n_{1,z} = 39$	$n_{0,z} = 92$ $\bar{y}_0 = 34414$ $\bar{y}_1 = 30525$ $\hat{\theta} = 38913$ $n_{1,z} = 23$	$n_{0,z} = 67$ $\bar{y}_0 = 48489$ $\bar{y}_1 = 33806$ $\hat{\theta} = 48876$ $n_{1,z} = 28$	$n_{0,z} = 202$ $\bar{y}_0 = 76457$ $\bar{y}_1 = 49893$ $\hat{\theta} = 72202$ $n_{1,z} = 97$
C	$n_{0,z} = 6592$ $\bar{y}_0 = 22367$ $\bar{y}_1 = 17186$ $\hat{\theta} = 22250$ $n_{1,z} = 1712$	$n_{0,z} = 9276$ $\bar{y}_0 = 23962$ $\bar{y}_1 = 19641$ $\hat{\theta} = 24420$ $n_{1,z} = 2778$	$n_{0,z} = 7479$ $\bar{y}_0 = 28976$ $\bar{y}_1 = 22864$ $\hat{\theta} = 29111$ $n_{1,z} = 2222$	$n_{0,z} = 5096$ $\bar{y}_0 = 31823$ $\bar{y}_1 = 23935$ $\hat{\theta} = 310205$ $n_{1,z} = 1227$	$n_{0,z} = 2134$ $\bar{y}_0 = 41119$ $\bar{y}_1 = 28997$ $\hat{\theta} = 38996$ $n_{1,z} = 882$	$n_{0,z} = 3504$ $\bar{y}_0 = 53101$ $\bar{y}_1 = 38967$ $\hat{\theta} = 49947$ $n_{1,z} = 2213$
D	$n_{0,z} = 43$ $\bar{y}_0 = 27904$ $\bar{y}_1 = 21145$ $\hat{\theta} = 26504$ $n_{1,z} = 5$	$n_{0,z} = 67$ $\bar{y}_0 = 30304$ $\bar{y}_1 = 24854$ $\hat{\theta} = 29576$ $n_{1,z} = 14$	$n_{0,z} = 219$ $\bar{y}_0 = 47763$ $\bar{y}_1 = 38044$ $\hat{\theta} = 45472$ $n_{1,z} = 48$	$n_{0,z} = 348$ $\bar{y}_0 = 51176$ $\bar{y}_1 = 37326$ $\hat{\theta} = 50403$ $n_{1,z} = 45$	$n_{0,z} = 360$ $\bar{y}_0 = 62343$ $\bar{y}_1 = 47109$ $\hat{\theta} = 61516$ $n_{1,z} = 73$	$n_{0,z} = 419$ $\bar{y}_0 = 72919$ $\bar{y}_1 = 53910$ $\hat{\theta} = 69229$ $n_{1,z} = 157$
E	$n_{0,z} = 1491$ $\bar{y}_0 = 23046$ $\bar{y}_1 = 18739$ $\hat{\theta} = 22398$ $n_{1,z} = 145$	$n_{0,z} = 1636$ $\bar{y}_0 = 23978$ $\bar{y}_1 = 19834$ $\hat{\theta} = 23480$ $n_{1,z} = 174$	$n_{0,z} = 609$ $\bar{y}_0 = 26245$ $\bar{y}_1 = 24309$ $\hat{\theta} = 26653$ $n_{1,z} = 165$	$n_{0,z} = 350$ $\bar{y}_0 = 28395$ $\bar{y}_1 = 24748$ $\hat{\theta} = 28021$ $n_{1,z} = 109$	$n_{0,z} = 186$ $\bar{y}_0 = 37477$ $\bar{y}_1 = 28085$ $\hat{\theta} = 37729$ $n_{1,z} = 91$	$n_{0,z} = 354$ $\bar{y}_0 = 47916$ $\bar{y}_1 = 32603$ $\hat{\theta} = 46596$ $n_{1,z} = 230$
F	$n_{0,z} = 3413$ $\bar{y}_0 = 18109$ $\bar{y}_1 = 18736$ $\hat{\theta} = 20368$ $n_{1,z} = 88$	$n_{0,z} = 3148$ $\bar{y}_0 = 20388$ $\bar{y}_1 = 21569$ $\hat{\theta} = 22549$ $n_{1,z} = 115$	$n_{0,z} = 1672$ $\bar{y}_0 = 22589$ $\bar{y}_1 = 21591$ $\hat{\theta} = 25154$ $n_{1,z} = 338$	$n_{0,z} = 815$ $\bar{y}_0 = 25678$ $\bar{y}_1 = 22610$ $\hat{\theta} = 26840$ $n_{1,z} = 189$	$n_{0,z} = 677$ $\bar{y}_0 = 37361$ $\bar{y}_1 = 26185$ $\hat{\theta} = 36226$ $n_{1,z} = 267$	$n_{0,z} = 858$ $\bar{y}_0 = 51061$ $\bar{y}_1 = 34250$ $\hat{\theta} = 49068$ $n_{1,z} = 424$
G	$n_{0,z} = 1044$ $\bar{y}_0 = 18419$ $\bar{y}_1 = 15062$ $\hat{\theta} = 17654$ $n_{1,z} = 618$	$n_{0,z} = 2402$ $\bar{y}_0 = 23978$ $\bar{y}_1 = 19834$ $\hat{\theta} = 23480$ $n_{1,z} = 1907$	$n_{0,z} = 1996$ $\bar{y}_0 = 26245$ $\bar{y}_1 = 24309$ $\hat{\theta} = 26653$ $n_{1,z} = 1623$	$n_{0,z} = 635$ $\bar{y}_0 = 28395$ $\bar{y}_1 = 24748$ $\hat{\theta} = 28021$ $n_{1,z} = 357$	$n_{0,z} = 346$ $\bar{y}_0 = 37477$ $\bar{y}_1 = 28085$ $\hat{\theta} = 37729$ $n_{1,z} = 327$	$n_{0,z} = 913$ $\bar{y}_0 = 47916$ $\bar{y}_1 = 32603$ $\hat{\theta} = 46596$ $n_{1,z} = 716$

Table 4.9: Source sample size ($n_{0,z}$), average men's salary ($\bar{y}_0$) in €, average women's salary ($\bar{y}_1$) in €, predicted women's salary ($\hat{\theta}$) in € and target sample size ($n_{1,z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group (B-G).

	Studies					
CNAE	1	2	3	4	5	6
H	$n_{0,z} = 1156$ $\bar{y}_0 = 21668$ $\bar{y}_1 = 21546$ $\hat{\theta} = 21417$ $n_{1,z} = 171$	$n_{0,z} = 2402$ $\bar{y}_0 = 23108$ $\bar{y}_1 = 19641$ $\hat{\theta} = 23895$ $n_{1,z} = 913$	$n_{0,z} = 1754$ $\bar{y}_0 = 31556$ $\bar{y}_1 = 23830$ $\hat{\theta} = 28775$ $n_{1,z} = 640$	$n_{0,z} = 629$ $\bar{y}_0 = 31844$ $\bar{y}_1 = 23956$ $\hat{\theta} = 30743$ $n_{1,z} = 196$	$n_{0,z} = 414$ $\bar{y}_0 = 59043$ $\bar{y}_1 = 41815$ $\hat{\theta} = 59699$ $n_{1,z} = 210$	$n_{0,z} = 597$ $\bar{y}_0 = 51011$ $\bar{y}_1 = 35436$ $\hat{\theta} = 50969$ $n_{1,z} = 374$
I	$n_{0,z} = 472$ $\bar{y}_0 = 19230$ $\bar{y}_1 = 15963$ $\hat{\theta} = 18771$ $n_{1,z} = 526$	$n_{0,z} = 653$ $\bar{y}_0 = 19969$ $\bar{y}_1 = 16529$ $\hat{\theta} = 19184$ $n_{1,z} = 601$	$n_{0,z} = 455$ $\bar{y}_0 = 21676$ $\bar{y}_1 = 18094$ $\hat{\theta} = 20491$ $n_{1,z} = 372$	$n_{0,z} = 76$ $\bar{y}_0 = 22717$ $\bar{y}_1 = 18894$ $\hat{\theta} = 22715$ $n_{1,z} = 63$	$n_{0,z} = 115$ $\bar{y}_0 = 29379$ $\bar{y}_1 = 22588$ $\hat{\theta} = 29520$ $n_{1,z} = 157$	$n_{0,z} = 106$ $\bar{y}_0 = 32773$ $\bar{y}_1 = 28079$ $\hat{\theta} = 33422$ $n_{1,z} = 117$
J	$n_{0,z} = 107$ $\bar{y}_0 = 17141$ $\bar{y}_1 = 16417$ $\hat{\theta} = 16821$ $n_{1,z} = 61$	$n_{0,z} = 306$ $\bar{y}_0 = 20814$ $\bar{y}_1 = 19132$ $\hat{\theta} = 21514$ $n_{1,z} = 169$	$n_{0,z} = 1385$ $\bar{y}_0 = 31729$ $\bar{y}_1 = 26799$ $\hat{\theta} = 31835$ $n_{1,z} = 850$	$n_{0,z} = 950$ $\bar{y}_0 = 25313$ $\bar{y}_1 = 23296$ $\hat{\theta} = 27075$ $n_{1,z} = 321$	$n_{0,z} = 1269$ $\bar{y}_0 = 38484$ $\bar{y}_1 = 29731$ $\hat{\theta} = 35384$ $n_{1,z} = 522$	$n_{0,z} = 2540$ $\bar{y}_0 = 41458$ $\bar{y}_1 = 32448$ $\hat{\theta} = 41267$ $n_{1,z} = 1783$
K	$n_{0,z} = 78$ $\bar{y}_0 = 35467$ $\bar{y}_1 = 24396$ $\hat{\theta} = 34077$ $n_{1,z} = 115$	$n_{0,z} = 278$ $\bar{y}_0 = 43266$ $\bar{y}_1 = 28701$ $\hat{\theta} = 41173$ $n_{1,z} = 227$	$n_{0,z} = 948$ $\bar{y}_0 = 44714$ $\bar{y}_1 = 29613$ $\hat{\theta} = 42435$ $n_{1,z} = 916$	$n_{0,z} = 227$ $\bar{y}_0 = 37669$ $\bar{y}_1 = 30588$ $\hat{\theta} = 36878$ $n_{1,z} = 352$	$n_{0,z} = 708$ $\bar{y}_0 = 44879$ $\bar{y}_1 = 33971$ $\hat{\theta} = 43836$ $n_{1,z} = 786$	$n_{0,z} = 1934$ $\bar{y}_0 = 55891$ $\bar{y}_1 = 41544$ $\hat{\theta} = 54703$ $n_{1,z} = 1815$
L	$n_{0,z} = 85$ $\bar{y}_0 = 18075$ $\bar{y}_1 = 15646$ $\hat{\theta} = 19067$ $n_{1,z} = 66$	$n_{0,z} = 85$ $\bar{y}_0 = 17302$ $\bar{y}_1 = 18217$ $\hat{\theta} = 18304$ $n_{1,z} = 73$	$n_{0,z} = 259$ $\bar{y}_0 = 22864$ $\bar{y}_1 = 19243$ $\hat{\theta} = 24400$ $n_{1,z} = 225$	$n_{0,z} = 36$ $\bar{y}_0 = 24623$ $\bar{y}_1 = 17943$ $\hat{\theta} = 24736$ $n_{1,z} = 53$	$n_{0,z} = 65$ $\bar{y}_0 = 38296$ $\bar{y}_1 = 24417$ $\hat{\theta} = 38516$ $n_{1,z} = 88$	$n_{0,z} = 181$ $\bar{y}_0 = 51158$ $\bar{y}_1 = 33430$ $\hat{\theta} = 49761$ $n_{1,z} = 199$
M	$n_{0,z} = 243$ $\bar{y}_0 = 16628$ $\bar{y}_1 = 14642$ $\hat{\theta} = 16671$ $n_{1,z} = 172$	$n_{0,z} = 403$ $\bar{y}_0 = 20671$ $\bar{y}_1 = 20845$ $\hat{\theta} = 21059$ $n_{1,z} = 411$	$n_{0,z} = 1022$ $\bar{y}_0 = 24593$ $\bar{y}_1 = 19519$ $\hat{\theta} = 24951$ $n_{1,z} = 1273$	$n_{0,z} = 791$ $\bar{y}_0 = 24673$ $\bar{y}_1 = 19866$ $\hat{\theta} = 24757$ $n_{1,z} = 606$	$n_{0,z} = 1077$ $\bar{y}_0 = 30685$ $\bar{y}_1 = 24094$ $\hat{\theta} = 30636$ $n_{1,z} = 913$	$n_{0,z} = 3123$ $\bar{y}_0 = 43773$ $\bar{y}_1 = 30183$ $\hat{\theta} = 41867$ $n_{1,z} = 2971$

Table 4.10: Source sample size ($n_{0,z}$), average men's salary ($\bar{y}_0$) in €, average women's salary ($\bar{y}_1$) in €, predicted women's salary ($\hat{\theta}$) in € and target sample size ($n_{1,z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group (H-M).

	Studies					
CNAE	1	2	3	4	5	6
	$n_{0,z} = 1278$	$n_{0,z} = 2857$	$n_{0,z} = 1627$	$n_{0,z} = 412$	$n_{0,z} = 349$	$n_{0,z} = 614$
	$\bar{y}_0 = 16314$	$\bar{y}_0 = 17866$	$\bar{y}_0 = 19669$	$\bar{y}_0 = 20114$	$\bar{y}_0 = 27671$	$\bar{y}_0 = 40845$
N	$\bar{y}_1 = 15170$	$\bar{y}_1 = 15759$	$\bar{y}_1 = 17252$	$\bar{y}_1 = 17339$	$\bar{y}_1 = 20630$	$\bar{y}_1 = 26957$
	$\hat{\theta} = 17768$	$\hat{\theta} = 18403$	$\hat{\theta} = 20172$	$\hat{\theta} = 21096$	$\hat{\theta} = 28773$	$\hat{\theta} = 26957$
	$n_{1,z} = 842$	$n_{1,z} = 1390$	$n_{1,z} = 1118$	$n_{1,z} = 333$	$n_{1,z} = 593$	$n_{1,z} = 670$
	$n_{0,z} = 384$	$n_{0,z} = 1339$	$n_{0,z} = 1223$	$n_{0,z} = 373$	$n_{0,z} = 457$	$n_{0,z} = 721$
	$\bar{y}_0 = 19897$	$\bar{y}_0 = 25292$	$\bar{y}_0 = 31343$	$\bar{y}_0 = 29584$	$\bar{y}_0 = 33636$	$\bar{y}_0 = 41898$
O	$\bar{y}_1 = 16778$	$\bar{y}_1 = 22996$	$\bar{y}_1 = 25586$	$\bar{y}_1 = 24450$	$\bar{y}_1 = 29597$	$\bar{y}_1 = 36960$
	$\hat{\theta} = 19838$	$\hat{\theta} = 26050$	$\hat{\theta} = 31526$	$\hat{\theta} = 29556$	$\hat{\theta} = 33059$	$\hat{\theta} = 41161$
	$n_{1,z} = 206$	$n_{1,z} = 882$	$n_{1,z} = 1030$	$n_{1,z} = 310$	$n_{1,z} = 812$	$n_{1,z} = 1019$
	$n_{0,z} = 41$	$n_{0,z} = 84$	$n_{0,z} = 175$	$n_{0,z} = 86$	$n_{0,z} = 322$	$n_{0,z} = 985$
	$\bar{y}_0 = 19592$	$\bar{y}_0 = 18205$	$\bar{y}_0 = 24897$	$\bar{y}_0 = 25582$	$\bar{y}_0 = 28023$	$\bar{y}_0 = 31989$
P	$\bar{y}_1 = 17319$	$\bar{y}_1 = 19290$	$\bar{y}_1 = 23074$	$\bar{y}_1 = 22799$	$\bar{y}_1 = 25723$	$\bar{y}_1 = 28612$
	$\hat{\theta} = 19232$	$\hat{\theta} = 19941$	$\hat{\theta} = 25111$	$\hat{\theta} = 24627$	$\hat{\theta} = 27380$	$\hat{\theta} = 31454$
	$n_{1,z} = 73$	$n_{1,z} = 120$	$n_{1,z} = 281$	$n_{1,z} = 183$	$n_{1,z} = 1001$	$n_{1,z} = 1239$
	$n_{0,z} = 460$	$n_{0,z} = 615$	$n_{0,z} = 709$	$n_{0,z} = 223$	$n_{0,z} = 551$	$n_{0,z} = 1173$
	$\bar{y}_0 = 19670$	$\bar{y}_0 = 20364$	$\bar{y}_0 = 20791$	$\bar{y}_0 = 22752$	$\bar{y}_0 = 30989$	$\bar{y}_0 = 53054$
Q	$\bar{y}_1 = 15765$	$\bar{y}_1 = 17077$	$\bar{y}_1 = 18438$	$\bar{y}_1 = 20297$	$\bar{y}_1 = 29582$	$\bar{y}_1 = 41606$
	$\hat{\theta} = 19338$	$\hat{\theta} = 20269$	$\hat{\theta} = 21033$	$\hat{\theta} = 22530$	$\hat{\theta} = 31140$	$\hat{\theta} = 50676$
	$n_{1,z} = 631$	$n_{1,z} = 1144$	$n_{1,z} = 2927$	$n_{1,z} = 739$	$n_{1,z} = 2752$	$n_{1,z} = 1712$
	$n_{0,z} = 344$	$n_{0,z} = 569$	$n_{0,z} = 698$	$n_{0,z} = 208$	$n_{0,z} = 163$	$n_{0,z} = 395$
	$\bar{y}_0 = 23981$	$\bar{y}_0 = 22093$	$\bar{y}_0 = 28837$	$\bar{y}_0 = 25883$	$\bar{y}_0 = 31122$	$\bar{y}_0 = 41967$
R	$\bar{y}_1 = 16128$	$\bar{y}_1 = 17296$	$\bar{y}_1 = 20587$	$\bar{y}_1 = 21062$	$\bar{y}_1 = 24103$	$\bar{y}_1 = 29706$
	$\hat{\theta} = 23923$	$\hat{\theta} = 20878$	$\hat{\theta} = 28249$	$\hat{\theta} = 24671$	$\hat{\theta} = 30729$	$\hat{\theta} = 41877$
	$n_{1,z} = 188$	$n_{1,z} = 327$	$n_{1,z} = 441$	$n_{1,z} = 92$	$n_{1,z} = 137$	$n_{1,z} = 414$
	$n_{0,z} = 300$	$n_{0,z} = 428$	$n_{0,z} = 597$	$n_{0,z} = 280$	$n_{0,z} = 173$	$n_{0,z} = 279$
	$\bar{y}_0 = 17525$	$\bar{y}_0 = 16522$	$\bar{y}_0 = 24197$	$\bar{y}_0 = 24734$	$\bar{y}_0 = 29158$	$\bar{y}_0 = 46428$
S	$\bar{y}_1 = 14075$	$\bar{y}_1 = 13907$	$\bar{y}_1 = 17230$	$\bar{y}_1 = 18572$	$\bar{y}_1 = 20296$	$\bar{y}_1 = 28020$
	$\hat{\theta} = 17337$	$\hat{\theta} = 17128$	$\hat{\theta} = 23632$	$\hat{\theta} = 25006$	$\hat{\theta} = 26741$	$\hat{\theta} = 44914$
	$n_{1,z} = 242$	$n_{1,z} = 354$	$n_{1,z} = 546$	$n_{1,z} = 164$	$n_{1,z} = 266$	$n_{1,z} = 425$

Table 4.11: Source sample size ($n_{0,z}$), average men's salary ($\bar{y}_0$) in €, average women's salary ($\bar{y}_1$) in €, predicted women's salary ($\hat{\theta}$) in € and target sample size ($n_{1,z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group (N-S).

	Studies					
CNAE	1	2	3	4	5	6
B	10.34427	0.05311245	0.06555036	0.2250906	10.32863	11.46244
C	0.7827234	0.1570725	0.2240537	0.379917	1.036544	1.006386
D	10.23072	8.540002	11.09998	10.90716	11.87939	11.15287
E	4.034317	1.54504	3.425392	6.90935	8.561086	9.706537
F	0.1979207	0.1042053	0.1779684	0.7073216	2.416007	2.031799
G	10.59052	0.7064822	2.42826	5.206829	5.918734	10.74743
H	1.049699	0.8470396	2.464503	6.063806	6.250508	4.231824
I	7.865014	0.7109818	1.295192	4.747013	3.477528	2.816318
J	6.05535	1.539357	1.665482	0.6700639	1.620418	0.6657766
K	11.36344	11.8833	12.25746	11.09947	11.41423	6.882228
L	3.848873	1.101402	2.196453	3.877995	7.955811	5.016808
M	2.164264	1.776538	1.112405	1.194964	1.119651	0.4577655
N	0.4322325	0.1414672	0.193088	0.2586702	0.814847	0.4869169
O	10.67815	2.554323	11.71623	11.92024	10.88268	11.16583
P	10.27285	3.853947	10.83386	9.703337	8.053916	1.865665
Q	10.0757	4.310891	1.978999	9.339508	7.023242	3.641688
R	11.75589	4.868713	10.85612	10.14804	8.112695	9.912113
S	3.342381	0.8564468	5.826307	2.773484	4.111988	5.707007

Table 4.12: Bandwidth selector ($h^{NW}_{BOOT,Z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group (B-S).

Finally, Figures 4.27 and 4.28 show the differences in the distribution of the variable *Studies* among men and women, considering sector F (Construction) and R (Arts), respectively, of CNAE. Indeed, although the distribution of *Studies* considering sector R seems quite similar for both genders, there are plausible differences in the distribution of *Studies* among men and women if sector F is considered.

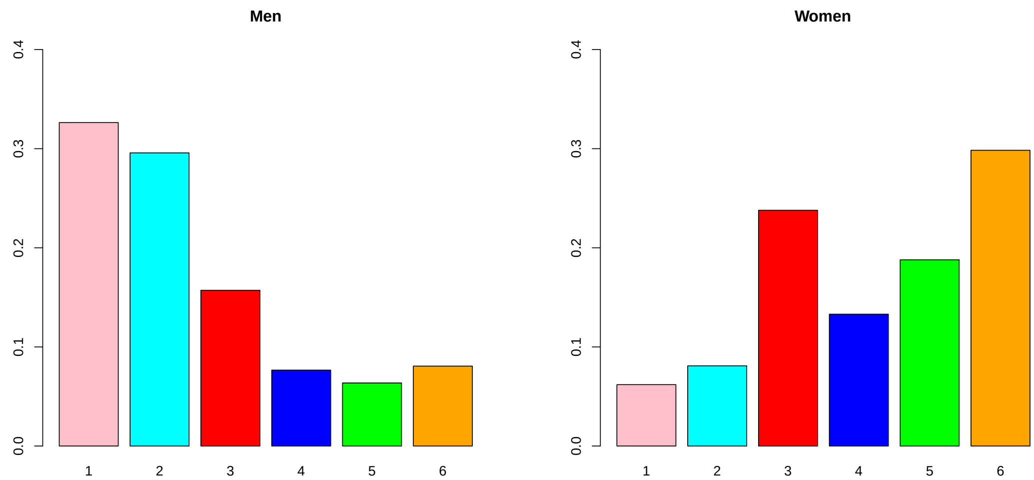


Figure 4.27: Distribution of *Studies* considering sector F for men (left side) and women (right side).

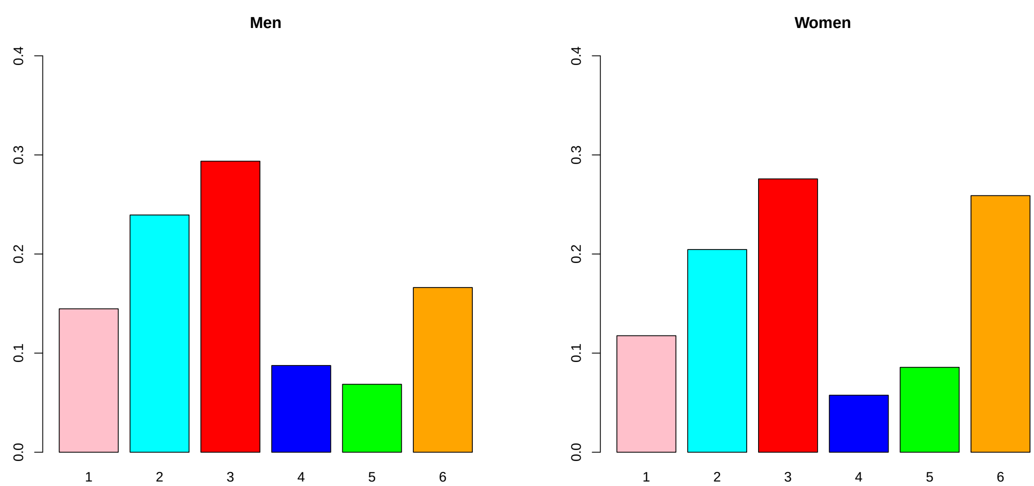


Figure 4.28: Distribution of *Studies* considering sector R for men (left side) and women (right side).

Chapter 5

Conclusions and future work

Over the last three decades, several bootstrap methods have been proposed for the purpose of bandwidth selection in nonparametric curve estimation, considering different curves and contexts. Monte Carlo approximation turns out to be a useful tool to approximate the bootstrap version of some error criteria, such as the MISE. Additionally, sometimes it may be feasible to obtain exact expressions for the MISE (or its bootstrap version, $MISE^*$). In this thesis, the definition of relatively simple proxy estimators has been proposed, which still take into account the variability of the real estimator. The aim of these proxied estimators is precisely to obtain closed bootstrap expressions of the error criteria, so that Monte Carlo approximation is no longer needed. The idea of using proxy estimators may well be a very useful tool in the future, since it can certainly lead to more computationally efficient bootstrap bandwidth selection methods.

Indeed, a research topic to carry on with in the future could be the proposal of a bootstrap bandwidth selector via minimizing a closed expression of some error criteria in the context of multivariate density function. This would be an extension to the multivariate context of the ideas in Cao (1993) and Chapter 2. Closed expressions for some error criteria have already been derived by Chacón and Duong (2018), which gives us high hopes of finding closed expressions for their bootstrap versions. Presumably, all results would be similar but less computational efficient since a bandwidth matrix, H, as well as a pilot bandwidth matrix, G, would now be considered. As a matter of fact, for bandwidth selection, we would be working in $\frac{d(d+1)}{2}$-dimensional spaces, with some restrictions, where d is the dimension of the random vector whose density is estimated.

On the other hand, a bootstrap bandwidth selector can also be studied in the future in the context of the conditional density function. Sometimes, the estimators have a simple structure, which let us compute closed expressions for the bootstrap version of an error criteria in a straightforward manner. However, most of the times, an appropiate proxy estimator needs to be defined for these purposes. In this case, a proxy estimator would need to be defined following similar ideas as those explained in this thesis. The aim is to find an exact expression for some bootstrapped error criteria of the proxy estimator. Both dependent and independent data may be considered for this issue.

Furthermore, the smoothed estimator of the distribution function in the classical independent setting has not been studied yet in terms of bandwidth selection via finding a closed expression for the bootstrap version of some error criteria. Once again, this problem is also similar to that of Cao (1993). In this context, the methods to be derived for iid data could be extended to the dependent data setup.

The context of the regression function may well also be a future line to focus on. Indeed, in Chapter 4 a global bandwidth selector in terms of prediction has been defined, using the classical and the local linear regression estimators. However, the classical problem of selecting a local bandwidth for regression has not been yet taken up. Moreover, the proxy estimators proposed in Chapter 4 are deemed to work satisfactorily also in this context, under the view presented in this thesis.

Proxy estimators may also be needed in the context of censored data. As a future cutting-edge research topic, the proposal of a proxy estimator coming from the almost sure representations of the Kaplan-Meier estimator in the context of density, distribution and hazard rate estimation under censoring can be also considered.

Finally, following the ideas proposed in Chapter 4, we can think of a setup in which the conditional distribution function is common in the source and the target population (instead of the regression function). In this context, our aim would be to estimate the distribution function, G_1, of the target response Y^1, which is unobservable.

Appendix A

Proofs of the results of Chapter 2

Theorem 1 *If the sample $(X_1, X_2, \ldots, X_n)$ comes from a stationary stochastic process the MISE of $\hat{f}_h$ can be expressed as follows:*

$$\begin{aligned} MISE(h) &= \int (K_h * f(x) - f(x))^2 \, dx + n^{-1}h^{-1}R(K) - \int (K_h * f(x))^2 \, dx \\ &+ 2n^{-2} \sum_{\ell=1}^{n-1} (n-\ell) \int \int K_h(x-y) f(y) K_h * f_\ell(\bullet|y)(x) \, dx \, dy. \end{aligned}$$

Proof of Theorem 1 In the dependent data case, under stationarity, the calculations for $B(h)$ are exactly the same as for the iid case. So $B(h) = B_0(h)$ as defined in (2.1). To compute $V(h)$ we need to take into account all possible covariances. First of all, it is useful to consider the stationarity of the process and the term for the integrated variance in the iid case (i.e., $V_0(h)$ in (2.2)):

$$\begin{aligned} V(h) &= n^{-1} \int Var(K_h(x - X_1)) \, dx \\ &+ n^{-2} \sum_{i \neq j} \int Cov(K_h(x - X_i), K_h(x - X_j)) \, dx \\ &= V_0(h) + 2n^{-2} \sum_{i<j} \int Cov(K_h(x - X_i), K_h(x - X_j)) \, dx \\ &= V_0(h) + 2n^{-2} \sum_{i<j} \int Cov(K_h(x - X_1), K_h(x - X_{j-i+1})) \, dx \end{aligned}$$

$$= \quad V_0(h) + 2n^{-2} \sum_{\ell=1}^{n-1} (n-\ell) \int Cov\left(K_h(x-X_1), K_h(x-X_{\ell+1})\right) dx. \quad \text{(A.1)}$$

Now,

$$\begin{aligned}
& \int Cov\left(K_h(x-X_1), K_h(x-X_{\ell+1})\right) dx \\
= & \int E\left(K_h(x-X_1) K_h(x-X_{\ell+1})\right) dx - \int \left[E\left(K_h(x-X_1)\right)\right]^2 dx \\
= & \int\int\int K_h(x-y) K_h(x-z) f(y) f_\ell(z|y)\, dx\, dy\, dz - \int \left(K_h * f(x)\right)^2 dx \\
= & \int\int K_h(x-y) f(y) K_h * f_\ell(\bullet|y)(x)\; dx\, dy - \int \left(K_h * f(x)\right)^2 dx. \qquad \text{(A.2)}
\end{aligned}$$

Using (A.2), (A.1), (2.1) and (2.2), the exact expression for the mean integrated squared error is obtained.

$$\begin{aligned}
MISE(h) &= B(h) + V(h), \text{ where} \\
B(h) &= \int \left(K_h * f(x) - f(x)\right)^2 dx \text{ and} \\
V(h) &= n^{-1}h^{-1}R(K) - \int \left(K_h * f(x)\right)^2 dx \\
&\quad + 2n^{-2} \sum_{\ell=1}^{n-1} (n-\ell) \int\int K_h(x-y) f(y) K_h * f_\ell(\bullet|y)(x)\; dx\, dy.
\end{aligned}$$

Theorem 2 *If the kernel K is a symmetric density function, then the smoothed stationary bootstrap version of MISE admits the following closed expression:*

$$\begin{aligned}
MISE^*(h) &= n^{-2} \sum_{i,j=1}^{n} (K_g * K_g)(X_i - X_j) \\
&- 2n^{-2} \sum_{i,j=1}^{n} (K_h * K_g * K_g)(X_i - X_j) \\
&+ \left[\frac{n-1}{n^3} - 2\frac{1-p-(1-p)^n}{pn^3}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\frac{(n-1)(1-p)^{n+1} - n(1-p)^n + 1 - p}{p^2 n^4}\Bigg] \\
&\cdot \quad \sum_{i,j=1}^{n} \left[(K_h * K_g) * (K_h * K_g)\right](X_i - X_j) \\
&+ \quad 2n^{-3}\sum_{\ell=1}^{n-1}(n-\ell)(1-p)^{\ell} \\
&\cdot \quad \sum_{k=1}^{n} \left[(K_h * K_g) * (K_h * K_g)\right]\left(X_k - X_{\lceil (k+\ell-1) \mod n\rceil + 1}\right) \\
&+ \quad n^{-1}h^{-1}R(K).
\end{aligned}
$$

Proof of Theorem 2 The bootstrap mean integrated squared error is just

$$
\mathbb{E}^*\left[\int \left(\hat{f}_h^*(x) - \hat{f}_g(x)\right)^2 dx\right] = B^*(h) + V^*(h),
$$

where

$$
\begin{aligned}
B^*(h) &= \int \left[E^*\left(\hat{f}_h^*(x)\right) - \hat{f}_g(x)\right]^2 dx \text{ and} \\
V^*(h) &= \int Var^*\left(\hat{f}_h^*(x)\right) dx.
\end{aligned}
$$

Now, straightforward calculations (including several changes of variable and using the symmetry of K) lead to

$$
\begin{aligned}
B^*(h) &= \int \left[\int K_h(x-y)\,\hat{f}_g(y)\,dy - \hat{f}_g(x)\right]^2 dx \\
&= n^{-2}\int \left[\sum_{i=1}^{n}\left(\int K_h(x-y)\,K_g(y-X_i)\,dy - K_g(x-X_i)\right)\right]^2 dx \\
&= n^{-2}\int \left[\sum_{i=1}^{n}\left(\int K_h(x-X_i-u)\,K_g(u)\,du - K_g(x-X_i)\right)\right]^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-2} \int \left[\sum_{i=1}^{n} \left(K_h * K_g \left(x - X_i \right) - K_g \left(x - X_i \right) \right) \right]^2 dx \\
&= n^{-2} \sum_{i,j=1}^{n} \int \left[\left(K_h * K_g - K_g \right) \left(x - X_i \right) \right] \left[\left(K_h * K_g - K_g \right) \left(x - X_j \right) \right] dx \\
&= n^{-2} \sum_{i,j=1}^{n} \int \left[\left(K_h * K_g - K_g \right) \left(-v \right) \right] \left[\left(K_h * K_g - K_g \right) \left(X_i - X_j - v \right) \right] dx \\
&= n^{-2} \sum_{i,j=1}^{n} \left[\left(K_h * K_g - K_g \right) * \left(K_h * K_g - K_g \right) \right] \left(X_i - X_j \right),
\end{aligned}
$$

that coincides, as expected, with the expression for the integrated bootstrap squared bias in the iid case.

The integrated bootstrap variance needs a deeper insight:

$$
\begin{aligned}
V^* \left(h \right) &= \int Var^* \left(n^{-1} \sum_{i=1}^{n} K_h \left(x - X_i^* \right) \right) dx \\
&= n^{-1} \int Var^* \left(K_h \left(x - X_1^* \right) \right) dx \\
&+ n^{-2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int Cov^* \left(K_h \left(x - X_i^* \right), K_h \left(x - X_j^* \right) \right) dx. \qquad \text{(A.3)}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& Cov^* \left(K_h \left(x - X_i^* \right), K_h (x - X_j^*) \right) \\
= \;& \mathbb{E}^* \left[Cov^* \left(K_h(x - X_i^*), K_h(x - X_j^*) \big|_{B_{ij}^*} \right) \right] \\
+ \;& Cov^* \left[\mathbb{E}^* \left(K_h(x - X_i^*) \big|_{B_{ij}^*} \right), \mathbb{E}^* \left(K_h(x - X_j^*) \big|_{B_{ij}^*} \right) \right], \qquad \text{(A.4)}
\end{aligned}
$$

where B_{ij}^* is the indicator that X_i^* and X_j^* belong to the same bootstrap block, i.e., for $i < j$,

$$
B_{ij}^* = I_{i+1}^* \cdot I_{i+2}^* \cdots I_j^*.
$$

and the I_i^*'s are the auxiliary binary bootstrap random variables defined in the algorithm SSB1 in Section 2.3.1.

Since,

$$\mathbb{E}^*\left[K_h\left(x-X_i^*\right)\big|_{B_{ij}^*}\right]=\mathbb{E}^*\left[K_h\left(x-X_j^*\right)\big|_{B_{ij}^*}\right]=\mathbb{E}^*\left[K_h\left(x-X_i^*\right)\right]=\left(K_h*\hat{f}_g\right)(x), \tag{A.5}$$

the last term in (A.4) is zero. Now using (A.4) and (A.5) in (A.3) results in

$$\begin{aligned} V^*(h) &= n^{-1}\int\left\{\mathbb{E}^*\left(K_h\left(x-X_1^*\right)^2\right)-\left[\mathbb{E}^*\left(K_h\left(x-X_1^*\right)\right)\right]^2\right\}dx \\ &+ 2n^{-2}\sum_{\substack{i,j=1\\ i<j}}^{n}\int\mathbb{E}^*\left[Cov^*\left(K_h\left(x-X_i^*\right),K_h\left(x-X_j^*\right)\big|_{B_{ij}^*}\right)\right]dx. \end{aligned} \tag{A.6}$$

The first term in (A.6) is exactly the same as that for the iid case, leading to

$$\begin{aligned} & n^{-1}\int\left\{\mathbb{E}^*\left(K_h\left(x-X_1^*\right)^2\right)-\left[\mathbb{E}^*\left(K_h\left(x-X_1^*\right)\right)\right]^2\right\}dx \\ = \; & n^{-1}h^{-1}R(K)-n^{-3}\sum_{i,j=1}^{n}\left[\left(K_h*K_g\right)*\left(K_h*K_g\right)\right]\left(X_i-X_j\right). \end{aligned} \tag{A.7}$$

So we investigate the covariance term further.

It is clear that the bootstrap indicator B_{ij}^* has a Bernoulli distribution with

$$P^*\left(B_{ij}^*=1\right)=(1-p)^{j-i}.$$

It is also evident that X_i^* and X_j^* are independent conditionally on $B_{ij}^*=0$ and that if $X_i^{*(SB)}=X_k$ and $B_{ij}^*=1$ then $X_j^{*(SB)}=X_{[(k+j-i-1) \mod n]+1}$. As a consequence

$$\begin{aligned} & \mathbb{E}^*\left[Cov^*\left(K_h\left(x-X_i^*\right),K_h\left(x-X_j^*\right)\right)\big|_{B_{ij}^*}\right] \\ = \; & (1-p)^{j-i}\cdot Cov^*\left(K_h\left(x-X_i^*\right),K_h\left(x-X_j^*\right)\big|_{B_{ij}^*=1}\right) \\ = \; & (1-p)^{j-i}\cdot\left[\mathbb{E}^*\left(K_h\left(x-X_i^*\right)\cdot K_h\left(x-X_j^*\right)\big|_{B_{ij}^*=1}\right)\right. \end{aligned}$$

$$
\begin{aligned}
- \quad & \mathbb{E}^* \Big(K_h \left(x - X_i^* \right) \big|_{B_{ij}^* = 1} \Big) \cdot \mathbb{E}^* \Big(K_h \left(x - X_j^* \right) \big|_{B_{ij}^* = 1} \Big) \Big] \\
= \quad & (1-p)^{j-i} \cdot \left[\mathbb{E}^* \left[\mathbb{E}^* \Big(K_h \left(x - X_i^* \right) \cdot K_h \left(x - X_j^* \right) \big|_{U_i^*, U_j^*, B_{ij}^* = 1} \Big) \Big|_{B_{ij}^* = 1} \right] \right. \\
- \quad & \left. \left[\mathbb{E}^* \left(K_h \left(x - X_1^* \right) \right) \right]^2 \right] \\
= \quad & (1-p)^{j-i} \cdot \left[\mathbb{E}^* \left[\frac{1}{n} \sum_{k=1}^{n} K_h \left(x - X_k - g U_i^* \right) \right. \right. \\
\cdot \quad & \left. \left. K_h \left(x - X_{\lceil (k+j-i-1) \mod n \rceil + 1} - g U_j^* \right) \right] - \left[\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}^* \left(K_h \left(x - X_k - g U_1^* \right) \right) \right]^2 \right] \\
= \quad & (1-p)^{j-i} \cdot \left[\frac{1}{n} \sum_{k=1}^{n} \int \int K_h \left(x - X_k - g u \right) \right. \\
\cdot \quad & K_h \left(x - X_{\lceil (k+j-i-1) \mod n \rceil + 1} - g v \right) \cdot K(u) \cdot K(v) \; du \, dv \\
- \quad & \left. \left(\frac{1}{n} \sum_{k=1}^{n} \int K_h \left(x - X_k - g u \right) \cdot K(u) \; du \right)^2 \right] \\
= \quad & (1-p)^{j-i} \cdot \left[\frac{1}{n} \sum_{k=1}^{n} \int \int K_h \left(x - X_k - s \right) \right. \\
\cdot \quad & K_h \left(x - X_{\lceil (k+j-i-1) \mod n \rceil + 1} - t \right) \cdot K_g(s) \cdot K_g(t) \; ds \, dt \\
- \quad & \left. \left(\frac{1}{n} \sum_{k=1}^{n} \int K_h \left(x - X_k - s \right) \cdot K_g(s) \; ds \right)^2 \right] \\
= \quad & (1-p)^{j-i} \cdot \left[\frac{1}{n} \sum_{k=1}^{n} K_h * K_g \left(x - X_k \right) \cdot K_h * K_g \left(x - X_{\lceil (k+j-i-1) \mod n \rceil + 1} \right) \right. \\
- \quad & \left. \left[\frac{1}{n} \sum_{k=1}^{n} K_h * K_g \left(x - X_k \right) \right]^2 \right].
\end{aligned}
$$

Therefore, the second term in (A.6) is

$$
2n^{-2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \int \mathbb{E}^* \left[Cov^* \Big(K_h \left(x - X_i^* \right), K_h \left(x - X_j^* \right) \big|_{B_{ij}^*} \Big) \right] dx
$$

$$
\begin{aligned}
&= 2n^{-2} \sum_{\substack{i,j=1\\ i<j}}^{n} (1-p)^{j-i} \cdot \\
&\quad \left[n^{-1} \sum_{k=1}^{n} \int K_h * K_g (x - X_k) \cdot K_h * K_g \left(x - X_{\lceil (k+j-i-1) \mod n \rceil + 1}\right) dx \right. \\
&- \left. n^{-2} \sum_{k,\ell=1}^{n} \int K_h * K_g (x - X_k) \cdot K_h * K_g (x - X_\ell)\; dx \right] \\
&= 2n^{-3} \sum_{\substack{i,j=1\\ i<j}}^{n} (1-p)^{j-i} \cdot \sum_{k=1}^{n} \left[(K_h * K_g) * (K_h * K_g) \right] \left(X_k - X_{\lceil (k+j-i-1) \mod n \rceil + 1} \right) \\
&- 2n^{-4} \sum_{\substack{i,j=1\\ i<j}}^{n} (1-p)^{j-i} \cdot \sum_{k,\ell=1}^{n} \left[(K_h * K_g) * (K_h * K_g) \right] (X_k - X_\ell) \\
&= 2n^{-3} \sum_{\ell=1}^{n-1} (n-\ell)(1-p)^{\ell} \cdot \sum_{k=1}^{n} \left[(K_h * K_g) * (K_h * K_g) \right] \left(X_k - X_{\lceil (k+\ell-1) \mod n \rceil + 1} \right) \\
&- 2n^{-4} \left(\sum_{\ell=1}^{n-1} (n-\ell)(1-p)^{\ell} \right) \cdot \sum_{k,\ell=1}^{n} \left[(K_h * K_g) * (K_h * K_g) \right] (X_k - X_\ell) \\
&= 2n^{-3} \sum_{\ell=1}^{n-1} \sum_{k=1}^{n} (n-\ell)(1-p)^{\ell} \left[(K_h * K_g) * (K_h * K_g) \right] \left(X_k - X_{\lceil (k+\ell-1) \mod n \rceil + 1} \right) \\
&- 2n^{-4} \left(n \frac{1-p-(1-p)^n}{p} - \frac{(n-1)(1-p)^{n+1} - n(1-p)^n + 1 - p}{p^2} \right) \\
&\cdot \sum_{k,\ell=1}^{n} \left[(K_h * K_g) * (K_h * K_g) \right] (X_k - X_\ell) . \qquad \text{(A.8)}
\end{aligned}
$$

As a last step, we need to calculate the following sum in order to prove the final equality.

$$
\sum_{\ell=1}^{n-1} (n-\ell)(1-p)^{\ell} = n \sum_{\ell=1}^{n-1} (1-p)^{\ell} - \sum_{\ell=1}^{n-1} \ell (1-p)^{\ell}
$$

$$= n\frac{(1-p)^{n-1}(1-p)-(1-p)}{(1-p)-1} - (1-p)\sum_{\ell=1}^{n-1}\ell(1-p)^{\ell-1}$$
$$= n\frac{1-p-(1-p)^n}{p} - (1-p)\sum_{\ell=1}^{n-1}\ell(1-p)^{\ell-1}. \tag{A.9}$$

So as to obtain the value of $\sum_{\ell=1}^{n-1}\ell(1-p)^{\ell-1}$, let us consider the subsequent function:

$$g(r) = \sum_{\ell=1}^{n-1}\ell r^{\ell-1}.$$

Obviously, $g(r)$ is the derivative function of the one given by $G(r)$:

$$G(r) = \sum_{\ell=1}^{n-1} r^{\ell} = \frac{r^{n-1}r - r}{r-1} = \frac{r^n - r}{r-1}.$$

Thus, we have:

$$\begin{aligned} g(r) = \frac{dG(r)}{dr} &= \frac{(nr^{n-1}-1)(r-1)-(r^n-r)1}{(r-1)^2} \\ &= \frac{(nr^{n-1}-1)(r-1)-(r^n-r)}{(r-1)^2} \\ &= \frac{(n-1)r^n - nr^{n-1}+1}{(r-1)^2}. \end{aligned}$$

As a consequence:

$$\begin{aligned} \sum_{\ell=1}^{n-1}\ell(1-p)^{\ell-1} = g(1-p) &= \frac{(n-1)(1-p)^n - n(1-p)^{n-1}+1}{(1-p-1)^2} \\ &= \frac{(n-1)(1-p)^n - n(1-p)^{n-1}+1}{p^2}. \end{aligned}$$

Using the previous equation in (A.9), we end up obtaining:

$$\sum_{\ell=1}^{n-1}(n-\ell)(1-p)^{\ell} = n\frac{1-p-(1-p)^n}{p} - \frac{(n-1)(1-p)^{n+1} - n(1-p)^n + 1 - p}{p^2}.$$

Collecting (A.7) and (A.8) and plugging them in (A.6) expression (2.4) follows.

Theorem 3 *If the kernel K is a symmetric density function, then the smoothed moving blocks bootstrap version of MISE admits the following closed expression, considering n an integer multiple of b:*

1. *If $b < n$,*

$$\begin{aligned}
MISE^*_{SMBB}(h) \quad = \quad & \frac{R(K)}{nh} + \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \psi(X_i - X_j) \\
& -\frac{2}{n} \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \left[(K_h * K_g) * K_g\right](X_i - X_j) \\
& +\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[K_g * K_g\right](X_i - X_j) \\
& -\frac{b-1}{n(n-b+1)^2} \sum_{i=b-1}^{n-b+1} \sum_{j=b}^{n-b+2} \psi(X_i - X_j) \\
& -\frac{1}{nb(n-b+1)^2} \left[\sum_{i=1}^{b-1} \sum_{j=1}^{b-1} \min\{i,j\} \psi(X_i - X_j) \right. \\
& + \sum_{i=1}^{b-1} i \sum_{j=b}^{n-b+1} \psi(X_i - X_j) \\
& + \sum_{i=1}^{b-1} \sum_{j=n-b+2}^{n} \min\{(n-b+i-j+1), i\} \psi(X_i - X_j) \\
& + \sum_{i=b}^{n-b+1} \sum_{j=1}^{b-1} j \psi(X_i - X_j) \\
& + \sum_{i=n-b+2}^{n} \min\{(n-i+1), b\} \sum_{j=b}^{n-b+1} \psi(X_i - X_j) \\
& + \sum_{i=b}^{n-b+1} \sum_{j=n-b+2}^{n} \min\{(n-j+1), b\} \psi(X_i - X_j)
\end{aligned}$$

$$+\sum_{i=n-b+2}^{n}\sum_{j=1}^{b-1}\min\{(n-b+j-i+1),j\}\psi(X_i-X_j)$$
$$+b\sum_{i=b}^{n-b+1}\sum_{j=b}^{n-b+1}\psi(X_i-X_j)$$
$$+\sum_{i=n-b+2}^{n}\sum_{j=n-b+2}^{n}(n+1-\max\{i,j\})\,\psi(X_i-X_j)\Bigg]$$
$$+\frac{2}{nb(n-b+1)}$$
$$\cdot\sum_{s=1}^{b-1}\sum_{j=1}^{n-s}(\min\{j,b-s\}-\max\{1,j+b-n\}+1)\,\psi(X_{j+s}-X_j)$$
$$-\frac{2}{nb(n-b+1)^2}\left[\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\left[\sum_{i=k}^{b-2}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)+\sum_{i=n-b+2}^{n-b+k}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)\right.\right.$$
$$\left.+\sum_{i=k}^{b-2}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)+\sum_{i=n-b+2}^{n-b+k}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)\right]$$
$$+\sum_{k=1}^{b-1}(b-k)\sum_{i=k}^{b-2}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)$$
$$+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)$$
$$\left.+\sum_{k=1}^{b-1}(b-k)\sum_{i=n-b+2}^{n-b+k}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)\right],$$

where $\psi(u)=\left[(K_h*K_g)*(K_h*K_g)\right](u)$ *and:*

$$a_j=\frac{\min\{j,n-j+1,b\}}{b(n-b+1)},j=1,2,\ldots,n. \tag{A.10}$$

2. *If $b = n$,*

$$
\begin{aligned}
MISE^*_{SMBB}(h) &= \frac{R(K)}{nh} + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(X_i - X_j) \\
&-\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(K_h * K_g) * K_g\right](X_i - X_j) \\
&+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[K_g * K_g\right](X_i - X_j) + \frac{\psi(0)}{n}.
\end{aligned}
$$

Proof of Theorem 3 Let us take into account $(X_1, \ldots, X_n)$, a random sample which comes from a stationary process and the smoothed moving blocks bootstrap version of the kernel density estimator, $\hat{f}_h^*(x)$. The bootstrap version of the mean integrated squared error is given by:

$$
MISE^*(h) = B^*(h) + V^*(h), \tag{A.11}
$$

where

$$
\begin{aligned}
B^*(h) &= \int \left[\mathbb{E}^*\left(\hat{f}_h^*(x)\right) - \hat{f}_g(x)\right]^2 dx, \text{ and} \\
V^*(h) &= \int \operatorname{Var}^*\left(\hat{f}_h^*(x)\right) dx.
\end{aligned}
$$

Now, straight forward calculations lead to

$$
\begin{aligned}
B^*(h) &= \int \left[\mathbb{E}^*\left(\frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i^*)\right) - \hat{f}_g(x)\right]^2 dx \\
&= \int \left[\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left(K_h(x - X_i^*)\right) - \hat{f}_g(x)\right]^2 dx \\
&= \int \left[\frac{1}{n}\sum_{i=1}^{n}\int K_h(x-y)\hat{f}_g^{(i)}(y)dy - \hat{f}_g(x)\right]^2 dx,
\end{aligned}
$$

where

$$\hat{f}_g^{(i)}(y) = \frac{1}{n-b+1} \sum_{j=t_i}^{n-b+t_i} K_g(y - X_j),$$

considering $t_i = [(i-1)\text{mod } b] + 1$.

Let us now assume that n is an integer multiple of b:

$$\begin{aligned}
& \int \left[\frac{1}{n} \sum_{i=1}^{n} \int K_h(x-y) \hat{f}_g^{(i)}(y) dy - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{n} \sum_{i=1}^{b} \frac{n}{b} \left(K_h * \hat{f}_g^{(i)} \right)(x) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b} \sum_{i=1}^{b} \left(K_h * \hat{f}_g^{(i)} \right)(x) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b} \sum_{i=1}^{b} \left(\frac{1}{n-b+1} \sum_{j=t_i}^{n-b+t_i} K_h * K_g(\cdot - X_j) \right)(x) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b} \sum_{i=1}^{b} \left(\frac{1}{n-b+1} \sum_{j=t_i}^{n-b+t_i} \int K_h(x-y) K_g(y - X_j) dy \right) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b} \sum_{i=1}^{b} \left(\frac{1}{n-b+1} \sum_{j=t_i}^{n-b+t_i} \int K_h(x-u-X_j) K_g(u) du \right) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b} \sum_{i=1}^{b} \left(\frac{1}{n-b+1} \sum_{j=t_i}^{n-b+t_i} K_h * K_g(x - X_j) \right) - \hat{f}_g(x) \right]^2 dx \\
= & \int \left[\frac{1}{b(n-b+1)} \sum_{i=1}^{b} \sum_{j=t_i}^{n-b+t_i} K_h * K_g(x - X_j) - \hat{f}_g(x) \right]^2 dx.
\end{aligned}$$

Furthermore, if $b < n$

$$
\begin{aligned}
& \frac{1}{b(n-b+1)} \sum_{i=1}^{b} \sum_{j=t_i}^{n-b+t_i} K_h * K_g(x - X_j) \\
= \; & \frac{1}{b(n-b+1)} \left(b \sum_{j=b}^{n-b+1} K_h * K_g(x - X_j) \right) + \frac{1}{b(n-b+1)} \sum_{j=1}^{b-1} j (K_h * K_g)(x - X_j) \\
& + \frac{1}{b(n-b+1)} \sum_{j=n-b+2}^{n} (n-j+1)(K_h * K_g)(x - X_j) \\
= \; & \frac{1}{n-b+1} \sum_{j=b}^{n-b+1} K_h * K_g(x - X_j) + \frac{1}{b(n-b+1)} \sum_{j=1}^{b-1} j (K_h * K_g)(x - X_j) \\
& + \frac{1}{b(n-b+1)} \sum_{j=n-b+2}^{n} (n-j+1)(K_h * K_g)(x - X_j) \\
= \; & \sum_{j=1}^{n} a_j (K_h * K_g)(x - X_j),
\end{aligned}
$$

where a_j has been defined in (A.10).

If $b = n$,

$$
\begin{aligned}
& \frac{1}{b(n-b+1)} \sum_{i=1}^{b} \sum_{j=t_i}^{n-b+t_i} K_h * K_g(x - X_j) = \frac{1}{n} \sum_{j=1}^{n} K_h * K_g(x - X_j) \\
= \; & \sum_{j=1}^{n} a_j \left(K_h * K_g \right) (x - X_j),
\end{aligned}
$$

considering $a_j = \dfrac{1}{n}$, if $b = n$.

Hence, carrying on with the calculations of the integrated bootstrap bias (including several changes of variable and using the symmetry of K) results in:

$$
\begin{aligned}
B^*(h) &= \int \left[\sum_{j=1}^{n} a_j \left(K_h * K_g \right)(x - X_j) - \hat{f}_g(x) \right]^2 dx \\
&= \int \left[\sum_{j=1}^{n} a_j \left(K_h * K_g \right)(x - X_j) - \frac{1}{n} \sum_{j=1}^{n} K_g(x - X_j) \right]^2 dx \\
&= \int \left[\sum_{j=1}^{n} a_j \left(K_h * K_g \right)(x - X_j) - \frac{1}{n} \sum_{j=1}^{n} K_g(x - X_j) \right] \\
&\quad \cdot \left[\sum_{k=1}^{n} a_k \left(K_h * K_g \right)(x - X_k) - \frac{1}{n} \sum_{k=1}^{n} K_g(x - X_k) \right] dx \\
&= \int \left[\sum_{j=1}^{n} \left(a_j \left(K_h * K_g \right)(x - X_j) - \frac{1}{n} K_g(x - X_j) \right) \right] \\
&\quad \cdot \left[\sum_{k=1}^{n} \left(a_k \left(K_h * K_g \right)(x - X_k) - \frac{1}{n} K_g(x - X_k) \right) \right] dx \\
&= \int \sum_{j=1}^{n} \sum_{k=1}^{n} \left[a_j \left(K_h * K_g \right)(x - X_j) - \frac{1}{n} K_g(x - X_j) \right] \\
&\quad \cdot \left[a_k \left(K_h * K_g \right)(x - X_k) - \frac{1}{n} K_g(x - X_k) \right] dx \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[a_j \left(K_h * K_g \right)(x - X_j) - \frac{1}{n} K_g(x - X_j) \right] \\
&\quad \times \left[a_k \left(K_h * K_g \right)(x - X_k) - \frac{1}{n} K_g(x - X_k) \right] dx \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[a_j \cdot a_k \left(K_h * K_g \right)(x - X_j) \cdot \left(K_h * K_g \right)(x - X_k) \right. \\
&\quad - \left(\frac{a_j}{n} \right) \left(K_h * K_g \right)(x - X_j) \cdot K_g(x - X_k) - \left(\frac{a_k}{n} \right) K_g(x - X_j) \\
&\quad \left. \cdot \left(K_h * K_g \right)(x - X_k) + \frac{1}{n^2} K_g(x - X_j) \cdot K_g(x - X_k) \right] dx \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[a_j \cdot a_k \left(K_h * K_g \right)(x - X_j) \cdot \left(K_h * K_g \right)(x - X_k) \right. \\
&\quad - \left(\frac{a_j}{n} \right) \left(K_h * K_g \right)(x - X_j) \cdot K_g(x - X_k) - \left(\frac{a_j}{n} \right)
\end{aligned}
$$

$$\cdot \left(K_h * K_g\right)(x - X_j) K_g(x - X_k) + \frac{1}{n^2} K_g(x - X_j) \cdot K_g(x - X_k)\Bigg] \, dx$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int \Big[a_j a_k \left(K_h * K_g\right)(x - X_j) \left(K_h * K_g\right)(x - X_k)$$

$$- \frac{2a_j}{n} \left(K_h * K_g\right)(x - X_j) K_g(x - X_k) + \frac{1}{n^2} K_g(x - X_j) K_g(x - X_k)\Bigg] \, dx$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[a_j \cdot a_k \left(K_h * K_g\right)(x - X_j) \cdot \left(K_h * K_g\right)(x - X_k)\right] dx$$

$$- \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[\frac{2a_j}{n} \left(K_h * K_g\right)(x - X_j) \cdot K_g(x - X_k)\right] dx$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[\frac{1}{n^2} K_g(x - X_j) \cdot K_g(x - X_k)\right] dx$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \int \left[\left(K_h * K_g\right)(x - X_j) \left(K_h * K_g\right)(x - X_k)\right] dx$$

$$- \frac{2}{n} \sum_{j=1}^{n} a_j \sum_{k=1}^{n} \int \left[\left(K_h * K_g\right)(x - X_j) K_g(x - X_k)\right] dx$$

$$+ \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[K_g(x - X_j) K_g(x - X_k)\right] dx$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \int \left[\left(K_h * K_g\right)(-v) \left(K_h * K_g\right)(X_j - X_k - v)\right] dv$$

$$- \frac{2}{n} \sum_{j=1}^{n} a_j \sum_{k=1}^{n} \int \left[\left(K_h * K_g\right)(-v) K_g(X_j - X_k - v)\right] dv$$

$$+ \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \int \left[K_g(-v) K_g(X_j - X_k - v)\right] dv$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \left[\left(K_h * K_g\right) * \left(K_h * K_g\right)\right](X_j - X_k)$$

$$- \frac{2}{n} \sum_{j=1}^{n} a_j \sum_{k=1}^{n} \left[\left(K_h * K_g\right) * K_g\right](X_j - X_k) + \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left[K_g * K_g\right](X_j - X_k).$$

Thus,

$$B^*(h) = \sum_{j=1}^n a_j \sum_{k=1}^n a_k \left[(K_h * K_g) * (K_h * K_g)\right](X_j - X_k) - \frac{2}{n}\sum_{j=1}^n a_j \cdot \sum_{k=1}^n \left[(K_h * K_g) * K_g\right](X_j - X_k) + \frac{1}{n^2}\sum_{j=1}^n \sum_{k=1}^n \left[K_g * K_g\right](X_j - X_k). \tag{A.12}$$

We now focus on the integrated bootstrap variance, which needs a deeper insight:

$$\begin{aligned}
V^*(h) &= \int Var^*\left(n^{-1}\sum_{i=1}^n K_h\left(x - X_i^*\right)\right)dx \\
&= n^{-2}\int \sum_{i=1}^n Var^*\left(K_h\left(x - X_i^*\right)\right)dx \\
&\quad + n^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^n \int Cov^*\left(K_h\left(x - X_i^*\right), K_h\left(x - X_j^*\right)\right)dx \\
&= n^{-2}\sum_{i=1}^n \int \left\{\mathbb{E}^*\left(K_h\left(x - X_i^*\right)^2\right) - \left[\mathbb{E}^*\left(K_h\left(x - X_i^*\right)\right)\right]^2\right\}dx \\
&\quad + n^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^n \int Cov^*\left(K_h\left(x - X_i^*\right), K_h\left(x - X_j^*\right)\right)dx \\
&= n^{-2}\sum_{i=1}^n \int \mathbb{E}^*\left(K_h\left(x - X_i^*\right)^2\right)dx - n^{-2}\sum_{i=1}^n \int \left[\mathbb{E}^*\left(K_h\left(x - X_i^*\right)\right)\right]^2 dx \\
&\quad + n^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^n \int Cov^*\left(K_h\left(x - X_i^*\right), K_h\left(x - X_j^*\right)\right)dx.
\end{aligned} \tag{A.13}$$

The first term in (A.13), after some changes of variable, is given by:

$$n^{-2}\sum_{i=1}^n \int \mathbb{E}^*\left(K_h\left(x - X_i^*\right)^2\right)dx = n^{-2}\sum_{i=1}^n \int \left[\int K_h(x-y)^2 \hat{f}_g^{(i)}(y)dy\right]dx$$

$$
\begin{aligned}
&= n^{-2}\sum_{i=1}^{n}\int\left[\int K_h(x-y)^2\left[\frac{1}{n-b+1}\sum_{j=t_i}^{n-b+t_i}K_g(y-X_j)\right]dy\right]dx \\
&= n^{-2}\sum_{i=1}^{n}\int\left[\frac{1}{n-b+1}\sum_{j=t_i}^{n-b+t_i}\int K_h(x-y)^2\cdot K_g(y-X_j)dy\right]dx \\
&= \frac{1}{n^2(n-b+1)}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)\left[\int K_h(x-y)^2dx\right]dy \\
&= \frac{1}{n^2\cdot(n-b+1)}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)\left[\int\left[\frac{1}{h}K\left(\frac{x-y}{h}\right)\right]^2dx\right]dy \\
&= \frac{1}{n^2\cdot(n-b+1)}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)\left[\int\frac{1}{h}K\left(z\right)^2dz\right]dy \\
&= \frac{1}{n^2(n-b+1)}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)\left[\frac{1}{h}\int K\left(z\right)^2dz\right]dy \\
&= \frac{1}{n^2\cdot(n-b+1)}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)\frac{1}{h}R(K)dy \\
&= \frac{R(K)}{n^2(n-b+1)h}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K_g(y-X_j)dy \\
&= \frac{R(K)}{n^2\cdot(n-b+1)\cdot h}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int\left[\frac{1}{g}K\left(\frac{y-X_j}{g}\right)\right]dy \\
&= \frac{R(K)}{n^2(n-b+1)h}\sum_{i=1}^{n}\sum_{j=t_i}^{n-b+t_i}\int K\left(u\right)du=\frac{R(K)}{nh}.
\end{aligned}
\tag{A.14}
$$

Focusing now on the second term, including several changes of variable and using the symmetry of K:

$$
n^{-2}\sum_{i=1}^{n}\int\left[\mathbb{E}^*\left(K_h\left(x-X_i^*\right)\right)\right]^2dx = n^{-2}\sum_{i=1}^{n}\int\left[\int K_h(x-y)\hat{f}_g^{(i)}(y)dy\right]^2dx
$$

$$
= n^{-1}b^{-1}\sum_{i=1}^{b}\int\left[\left(K_h*\hat{f}_g^{(i)}\right)(x)\right]^2dx
$$

$$\begin{aligned}
&= n^{-1}b^{-1}\sum_{i=1}^{b}\int\left[\sum_{j=t_i}^{n-b+t_i}\frac{1}{n-b+1}(K_h*K_g)(x-X_j)\right]^2dx \\
&= n^{-1}b^{-1}\sum_{i=1}^{b}\int\left[\sum_{j=t_i}^{n-b+t_i}\frac{1}{n-b+1}(K_h*K_g)(x-X_j)\right] \\
&\quad\times\left[\sum_{k=t_i}^{n-b+t_i}\frac{1}{n-b+1}(K_h*K_g)(x-X_k)\right]dx \\
&= n^{-1}b^{-1}\sum_{i=1}^{b}\int\sum_{j=t_i}^{n-b+t_i}\sum_{k=t_i}^{n-b+t_i}\left(\frac{1}{n-b+1}\right)^2 \\
&\quad\cdot(K_h*K_g)(x-X_j)\cdot(K_h*K_g)(x-X_k)dx \\
&= \frac{1}{nb(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=t_i}^{n-b+t_i}\sum_{k=t_i}^{n-b+t_i}\int(K_h*K_g)(x-X_j)(K_h*K_g)(x-X_k)dx \\
&= \frac{1}{nb\cdot(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=t_i}^{n-b+t_i}\sum_{k=t_i}^{n-b+t_i}\int(K_h*K_g)(-v) \\
&\quad\cdot(K_h*K_g)(X_j-X_k-v)dv \\
&= \frac{1}{nb(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=t_i}^{n-b+t_i}\sum_{k=t_i}^{n-b+t_i}\int(K_h*K_g)(v)(K_h*K_g)(X_j-X_k-v)dv \\
&= \frac{1}{nb\cdot(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=t_i}^{n-b+t_i}\sum_{k=t_i}^{n-b+t_i}\left[(K_h*K_g)*(K_h*K_g)\right](X_j-X_k) \\
&= \frac{1}{nb(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=i}^{n-b+i}\sum_{k=i}^{n-b+i}\left[(K_h*K_g)*(K_h*K_g)\right](X_j-X_k).
\end{aligned}$$

Let us consider the function $\psi(u)=\left[(K_h*K_g)*(K_h*K_g)\right](u)$ defined in Theorem 3. Whenever $b<n$, we have:

$$\begin{aligned}
&n^{-2}\sum_{i=1}^{n}\int\left[\mathbb{E}^*\left(K_h(x-X_j^*)\right)\right]^2dx \\
&= \frac{1}{nb(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=i}^{n-b+i}\sum_{k=i}^{n-b+i}\left[(K_h*K_g)*(K_h*K_g)\right](X_j-X_k)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{nb(n-b+1)^2} \sum_{i=1}^{b} \sum_{j=i}^{n-b+i} \sum_{k=i}^{n-b+i} \psi(X_j - X_k) \\
&= \frac{1}{nb(n-b+1)^2} \Bigg[\sum_{i=1}^{b} \sum_{j=i}^{b-1} \sum_{k=i}^{b-1} \psi(X_j - X_k) \\
&\quad + \sum_{i=1}^{b} \sum_{j=i}^{b-1} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) + \sum_{i=1}^{b} \sum_{j=i}^{b-1} \sum_{k=n-b+2}^{n-b+i} \psi(X_j - X_k) \\
&\quad + \sum_{i=1}^{b} \sum_{j=b}^{n-b+1} \sum_{k=i}^{b-1} \psi(X_j - X_k) + \sum_{i=1}^{b} \sum_{j=b}^{n-b+1} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) \\
&\quad + \sum_{i=1}^{b} \sum_{j=b}^{n-b+1} \sum_{k=n-b+2}^{n-b+i} \psi(X_j - X_k) + \sum_{i=1}^{b} \sum_{j=n-b+2}^{n-b+i} \sum_{k=i}^{b-1} \psi(X_j - X_k) \\
&\quad + \sum_{i=1}^{b} \sum_{j=n-b+2}^{n-b+i} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) + \sum_{i=1}^{b} \sum_{j=n-b+2}^{n-b+i} \sum_{k=n-b+2}^{n-b+i} \psi(X_j - X_k) \Bigg] \\
&= \frac{1}{nb(n-b+1)^2} \Bigg[\sum_{j=1}^{b-1} \sum_{k=1}^{b-1} \sum_{i=1}^{\min\{j,k\}} \psi(X_j - X_k) \\
&\quad + \sum_{j=1}^{b-1} \sum_{k=b}^{n-b+1} \sum_{i=1}^{j} \psi(X_j - X_k) + \sum_{j=1}^{b-1} \sum_{k=n-b+2}^{n} \sum_{i=\max\{(k+b-n),1\}}^{j} \psi(X_j - X_k) \\
&\quad + \sum_{j=b}^{n-b+1} \sum_{k=1}^{b-1} \sum_{i=1}^{k} \psi(X_j - X_k) + \sum_{j=b}^{n-b+1} \sum_{k=b}^{n-b+1} \sum_{i=1}^{b} \psi(X_j - X_k) \\
&\quad + \sum_{j=b}^{n-b+1} \sum_{k=n-b+2}^{n} \sum_{i=\max\{(k-n+b),1\}}^{b} \psi(X_j - X_k) + \sum_{j=n-b+2}^{n} \sum_{k=1}^{b-1} \\
&\quad \sum_{i=\max\{(j+b-n),1\}}^{k} \psi(X_j - X_k) + \sum_{j=n-b+2}^{n} \sum_{k=b}^{n-b+1} \sum_{i=\max\{(j-n+b),1\}}^{b} \psi(X_j - X_k) \\
&\quad + \sum_{j=n-b+2}^{n} \sum_{k=n-b+2}^{n} \sum_{i=\max\{(j-n+b),(k-n+b)\}}^{b} \psi(X_j - X_k) \Bigg] \\
&= \frac{1}{nb(n-b+1)^2} \Bigg[\sum_{j=1}^{b-1} \sum_{k=1}^{b-1} \min\{j,k\} \psi(X_j - X_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{b-1} j \sum_{k=b}^{n-b+1} \psi(X_j - X_k) + \sum_{j=1}^{b-1} \sum_{k=n-b+2}^{n} \min\{(n-b+j-k+1), j\} \\
&\psi(X_j - X_k) + \sum_{j=b}^{n-b+1} \sum_{k=1}^{b-1} k\psi(X_j - X_k) + b \sum_{j=b}^{n-b+1} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) \\
&+ \sum_{j=b}^{n-b+1} \sum_{k=n-b+2}^{n} \min\{(n-k+1), b\}\psi(X_j - X_k) \\
&+ \sum_{j=n-b+2}^{n} \sum_{k=1}^{b-1} \min\{(n-b+k-j+1), k\}\psi(X_j - X_k) \\
&+ \sum_{j=n-b+2}^{n} \min\{(n-j+1), b\} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) \\
&+ \left. \sum_{j=n-b+2}^{n} \sum_{k=n-b+2}^{n} \min\{(n-j+1), (n-k+1)\}\psi(X_j - X_k) \right] \\
= \; & \frac{1}{nb(n-b+1)^2} \left[\sum_{j=1}^{b-1} \sum_{k=1}^{b-1} \min\{j, k\}\psi(X_j - X_k) \right. \\
&+\sum_{j=1}^{b-1} j \sum_{k=b}^{n-b+1} \psi(X_j - X_k) + \sum_{j=1}^{b-1} \sum_{k=n-b+2}^{n} \min\{(n-b+j-k+1), j\}\psi(X_j - X_k) \\
&+ \sum_{j=b}^{n-b+1} \sum_{k=1}^{b-1} k\psi(X_j - X_k) + b \sum_{j=b}^{n-b+1} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) \\
&+ \sum_{j=b}^{n-b+1} \sum_{k=n-b+2}^{n} \min\{(n-k+1), b\}\psi(X_j - X_k) \\
&+ \sum_{j=n-b+2}^{n} \sum_{k=1}^{b-1} \min\{(n-b+k-j+1), k\}\psi(X_j - X_k) \\
&+ \sum_{j=n-b+2}^{n} \min\{(n-j+1), b\} \sum_{k=b}^{n-b+1} \psi(X_j - X_k) \\
&+ \left. \sum_{j=n-b+2}^{n} \sum_{k=n-b+2}^{n} (n+1-\max\{j,k\})\, \psi(X_j - X_k) \right].
\end{aligned}
\tag{A.15}
$$

On the other hand, if $b=n$:

$$\frac{1}{nb(n-b+1)^2}\sum_{i=1}^{b}\sum_{j=i}^{n-b+i}\sum_{k=i}^{n-b+i}\psi\left(X_j-X_k\right)=\frac{1}{n^2}\sum_{i=1}^{n}\psi(X_i-X_i)=\frac{\psi(0)}{n}.$$

Finally, we investigate the covariance term further. It is now necessary to take into account the following notation, naming the n/b blocks as follows:

$$J_r=\left\{(r-1)b+1,(r-1)b+2,\dots,rb\right\},r=1,2,\dots,n/b.$$

Thus, X_i^* and X_j^* turn out to be independent (in the bootstrap universe) whenever it does not exist $r\in\{1,2,\dots,n/b\}$ which satisfies $i,j\in J_r$. In that case, X_i^* and X_j^* do not belong to the same bootstrap block, implying:

$$Cov^*\left(K_h(x-X_i^*),K_h(x-X_j^*)\right)=0.$$

On the other hand, if there exists $r\in\{1,2,\dots,n/b\}$ satisfying $i,j\in J_r$, then the bootstrap distribution of the pair (X_i^*,X_j^*) is exactly identical to that of the pair $(X_{t_i}^*,X_{t_j}^*)$, where $t_i=[(i-1)\text{mod }b]+1$. Let us consider $r\in\{1,2,\dots,n/b\}$ satisfying $i,j\in J_r$, then X_i^* e X_j^* belong to the same bootstrap block. As a consequence,

$$Cov^*\left(K_h(x-X_i^*),K_h(x-X_j^*)\right)=Cov^*\left(K_h(x-X_{t_i}^*),K_h(x-X_{t_j}^*)\right).$$

Denote $D=Cov^*\left(K_h(x-X_i^*),K_h(x-X_j^*)\right)$, then we have:

$$D=\begin{cases} Cov^*\left(K_h(x-X_{t_i}^*),K_h(x-X_{t_j}^*)\right), & \text{if } \exists r/i,j\in J_r \\ \\ 0, & \text{otherwise}\end{cases}.$$

Notice that:

$$\begin{aligned}\mathbb{E}^*\left[K_h\left(x-X_i^*\right)\right] &= \left(K_h*\hat{f}_g^{(i)}\right), \text{ and}\\ \mathbb{E}^*\left[K_h\left(x-X_j^*\right)\right] &= \left(K_h*\hat{f}_g^{(j)}\right).\end{aligned}$$

Now, consider $k,\ell\in\{1,2,\dots,b\}$ satisfying $k<\ell$. Carrying on with the calculations of the covariance term and using:

$$\mathbb{P}^*\left(\left(X_k^{*(d)}, X_\ell^{*(d)}\right) = (X_j, X_{j+\ell-k})\right) = \frac{1}{n-b+1}, j = k, k+1, \ldots, n-b+k,$$

leads to:

$$\begin{aligned}
& \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} Cov^*\left(K_h\left(x - X_i^*\right), K_h\left(x - X_j^*\right)\right) \\
= \; & \frac{1}{n^2}\frac{n}{b}\sum_{\substack{k,\ell=1\\ k\neq \ell}}^{b} Cov^*\left(K_h\left(x - X_k^*\right), K_h\left(x - X_\ell^*\right)\right) \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} Cov^*\left(K_h\left(x - X_k^*\right), K_h\left(x - X_\ell^*\right)\right) \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} \left[\mathbb{E}^*\left(K_h\left(x - X_k^*\right)K_h\left(x - X_\ell^*\right)\right) - \mathbb{E}^*\left(K_h\left(x - X_k^*\right)\right)\mathbb{E}^*\left(K_h\left(x - X_\ell^*\right)\right)\right] \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} \left[\mathbb{E}^*\left[\mathbb{E}^*\left(K_h\left(x - X_k^*\right)K_h\left(x - X_\ell^*\right)|_{U_k^*,U_\ell^*}\right)\right] - \left(K_h * \hat{f}_g^{(k)}\right)\left(K_h * \hat{f}_g^{(\ell)}\right)\right] \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} \left[\mathbb{E}^*\left[\mathbb{E}^*\left(K_h\left(x - X_k^{*(d)} - gU_k^*\right)K_h\left(x - X_\ell^{*(d)} - gU_\ell^*\right)\Big|_{U_k^*,U_\ell^*}\right)\right]\right. \\
& \left. - \left(K_h * \hat{f}_g^{(k)}(x)\right)\left(K_h * \hat{f}_g^{(\ell)}(x)\right)\right] \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} \left[\mathbb{E}^*\left[\frac{1}{n-b+1}\sum_{j=k}^{n-b+k} K_h\left(x - X_j - gU_k^*\right)\cdot K_h\left(x - X_{j+\ell-k} - gU_\ell^*\right)\right]\right. \\
& \left. - \left(K_h * \hat{f}_g^{(k)}(x)\right)\cdot\left(K_h * \hat{f}_g^{(\ell)}(x)\right)\right] \\
= \; & \frac{2}{nb}\sum_{\substack{k,\ell=1\\ k<\ell}}^{b} \left[\frac{1}{n-b+1}\sum_{j=k}^{n-b+k} \mathbb{E}^*\left[K_h\left(x - X_j - gU_k^*\right)K_h\left(x - X_{j+\ell-k} - gU_\ell^*\right)\right]\right. \\
& \left. - \left(K_h * \hat{f}_g^{(k)}(x)\right)\left(K_h * \hat{f}_g^{(\ell)}(x)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} \int \int K_h(x - X_j - gu) K_h(x - X_{j+\ell-k} - gv) \right. \\
&\quad \left. K(u)K(v)dudv - \left(K_h * \hat{f}_g^{(k)}(x)\right) \left(K_h * \hat{f}_g^{(\ell)}(x)\right) \right] \\
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} \int \int K_h(x - X_j - s) K_h(x - X_{j+\ell-k} - t) \right. \\
&\quad \left. K_g(s)K_g(t)dsdt - \left(K_h * \hat{f}_g^{(k)}(x)\right) \left(K_h * \hat{f}_g^{(\ell)}(x)\right) \right] \\
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j)(K_h * K_g)(x - X_{j+\ell-k}) \right. \\
&\quad \left. - \left(K_h * \hat{f}_g^{(k)}(x)\right) \left(K_h * \hat{f}_g^{(\ell)}(x)\right) \right] \\
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j) \cdot (K_h * K_g)(x - X_{j+\ell-k}) \right. \\
&\quad - \left(\frac{1}{n-b+1} \sum_{i=k}^{n-b+k} (K_h * K_g(\cdot - X_i))(x) \right) \\
&\quad \left. \cdot \left(\frac{1}{n-b+1} \sum_{j=\ell}^{n-b+\ell} (K_h * K_g(\cdot - X_j))(x) \right) \right] \\
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j)(K_h * K_g)(x - X_{j+\ell-k}) \right. \\
&\quad - \left(\frac{1}{n-b+1} \sum_{i=k}^{n-b+k} \int K_h(x-y) K_g(y - X_i) dy \right) \\
&\quad \left. \times \left(\frac{1}{n-b+1} \sum_{j=\ell}^{n-b+\ell} \int K_h(x-y) K_g(y - X_j) dy \right) \right] \\
&= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j)(K_h * K_g)(x - X_{j+\ell-k}) \right. \\
&\quad - \left(\frac{1}{n-b+1} \sum_{i=k}^{n-b+k} \int K_h(x - X_i - u) K_g(u) du \right)
\end{aligned}
$$

$$\times \left(\frac{1}{n-b+1} \sum_{j=\ell}^{n-b+\ell} \int K_h(x - X_j - u) K_g(u) du \right) \Bigg]$$

$$= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j) \cdot (K_h * K_g)(x - X_{j+\ell-k}) \right.$$

$$- \left(\frac{1}{n-b+1} \sum_{i=k}^{n-b+k} K_h * K_g(x - X_i) \right)$$

$$\left. \cdot \left(\frac{1}{n-b+1} \sum_{j=\ell}^{n-b+\ell} K_h * K_g(x - X_j) \right) \right]$$

$$= \frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j)(K_h * K_g)(x - X_{j+\ell-k}) \right.$$

$$\left. - \frac{1}{(n-b+1)^2} \sum_{i=k}^{n-b+k} \sum_{j=\ell}^{n-b+\ell} (K_h * K_g)(x - X_i)(K_h * K_g)(x - X_j) \right].$$

The integral with respect to x is now computed (using some changes of variable and the symmetry of the kernel K):

$$\int \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} Cov^* \left(K_h\left(x - X_i^*\right), K_h\left(x - X_j^*\right) \right) dx$$

$$= \int \left[\frac{2}{nb} \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} (K_h * K_g)(x - X_j)(K_h * K_g)(x - X_{j+\ell-k}) \right. \right.$$

$$\left. \left. - \frac{1}{(n-b+1)^2} \sum_{i=k}^{n-b+k} \sum_{j=\ell}^{n-b+\ell} (K_h * K_g)(x - X_i)(K_h * K_g)(x - X_j) \right] \right] dx$$

$$= \frac{2}{nb} \left[\sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \left[\frac{1}{n-b+1} \sum_{j=k}^{n-b+k} \int (K_h * K_g)(x - X_j) \cdot (K_h * K_g)(x - X_{j+\ell-k}) dx \right. \right.$$

$$\left. \left. - \frac{1}{(n-b+1)^2} \sum_{i=k}^{n-b+k} \sum_{j=\ell}^{n-b+\ell} \int (K_h * K_g)(x - X_i) \cdot (K_h * K_g)(x - X_j) dx \right] \right]$$

$$
\begin{aligned}
&= \frac{2}{nb}\left[\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\left[\frac{1}{n-b+1}\sum_{j=k}^{n-b+k}\int (K_h*K_g)(X_{j+\ell-k}-X_j-u)\cdot(K_h*K_g)(-u)du\right.\right.\\
&\quad\left.\left.-\frac{1}{(n-b+1)^2}\sum_{i=k}^{n-b+k}\sum_{j=\ell}^{n-b+\ell}\int (K_h*K_g)(-u)\cdot(K_h*K_g)(X_i-X_j-u)du\right]\right]\\
&= \frac{2}{nb}\left[\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\left[\frac{1}{n-b+1}\sum_{j=k}^{n-b+k}\int (K_h*K_g)(X_{j+\ell-k}-X_j-u)(K_h*K_g)(u)du\right.\right.\\
&\quad\left.\left.-\frac{1}{(n-b+1)^2}\sum_{i=k}^{n-b+k}\sum_{j=\ell}^{n-b+\ell}\int (K_h*K_g)(u)(K_h*K_g)(X_i-X_j-u)du\right]\right]\\
&= \frac{2}{nb}\left[\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\left[\frac{1}{n-b+1}\sum_{j=k}^{n-b+k}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+\ell-k}-X_j)\right.\right.\\
&\quad\left.\left.-\frac{1}{(n-b+1)^2}\sum_{i=k}^{n-b+k}\sum_{j=\ell}^{n-b+\ell}\left[(K_h*K_g)(u)*(K_h*K_g)(X_i-X_j)\right]\right]\right]\\
&= \frac{2}{nb(n-b+1)}\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{j=k}^{n-b+k}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+\ell-k}-X_j)\\
&\quad-\frac{2}{nb(n-b+1)^2}\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=k}^{n-b+k}\sum_{j=\ell}^{n-b+\ell}\left[(K_h*K_g)*(K_h*K_g)\right](X_i-X_j).
\end{aligned}
$$

Notice that, whenever $b<n$:

$$
\begin{aligned}
&\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{j=k}^{n-b+k}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+\ell-k}-X_j)\\
&= \sum_{k=1}^{b-1}\sum_{j=k}^{n-b+k}\sum_{\ell=k+1}^{b}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+\ell-k}-X_j)\\
&= \sum_{k=1}^{b-1}\sum_{j=k}^{n-b+k}\sum_{s=1}^{b-k}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+s}-X_j)
\end{aligned}
$$

$$= \sum_{s=1}^{b-1} \sum_{j=1}^{n-s} \sum_{k=\max\{1,j+b-n\}}^{\min\{j,b-s\}} \left[(K_h * K_g) * (K_h * K_g)\right](X_{j+s} - X_j)$$
$$= \sum_{s=1}^{b-1} \sum_{j=1}^{n-s} (\min\{j, b-s\} - \max\{1, j+b-n\} + 1) \left[(K_h * K_g) * (K_h * K_g)\right](X_{j+s} - X_j).$$

Now, using the function ψ, and considering $b < n$, we have:

$$n^{-2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int Cov^* \left(K_h(x - X_i^*), K_h(x - X_j^*)\right) dx$$
$$= \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{n-b+k} \sum_{j=\ell}^{n-b+\ell} \psi(X_i - X_j)$$
$$= \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{b-2} \sum_{j=\ell}^{b-1} \psi(X_i - X_j) + \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{b-2} \sum_{j=b}^{n-b+2} \psi(X_i - X_j)$$
$$+ \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{b-2} \sum_{j=n-b+3}^{n-b+\ell} \psi(X_i - X_j)$$
$$+ \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=b-1}^{n-b+1} \sum_{j=n-b+3}^{n-b+\ell} \psi(X_i - X_j) + \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=n-b+2}^{n-b+k} \sum_{j=\ell}^{b-1} \psi(X_i - X_j)$$
$$+ \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=n-b+2}^{n-b+k} \sum_{j=b}^{n-b+2} \psi(X_i - X_j) + \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=n-b+2}^{n-b+k} \sum_{j=n-b+3}^{n-b+\ell} \psi(X_i - X_j)$$
$$= \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{b-2} \sum_{j=\ell}^{b-1} \psi(X_i - X_j) + \sum_{k=1}^{b-1} \sum_{\ell=k+1}^{b} \sum_{i=k}^{b-2} \sum_{j=b}^{n-b+2} \psi(X_i - X_j)$$
$$+ \sum_{\substack{k,\ell=1 \\ k<\ell}}^{b} \sum_{i=k}^{b-2} \sum_{j=n-b+3}^{n-b+\ell} \psi(X_i - X_j)$$

$$
\begin{aligned}
&+\sum_{\ell=2}^{b}\sum_{k=1}^{\ell-1}\sum_{i=b-1}^{n-b+1}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)+\sum_{i=b-1}^{n-b+1}\sum_{j=b}^{n-b+2}\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\psi(X_i-X_j)\\
&+\sum_{\ell=2}^{b}\sum_{k=1}^{\ell-1}\sum_{i=b-1}^{n-b+1}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)+\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=n-b+2}^{n-b+k}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)\\
&+\sum_{k=1}^{b-1}\sum_{\ell=k+1}^{b}\sum_{i=n-b+2}^{n-b+k}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)+\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=n-b+2}^{n-b+k}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)\\
=\;&\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=k}^{b-2}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)+\sum_{k=1}^{b-1}(b-k)\sum_{i=k}^{b-2}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)\\
&+\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=k}^{b-2}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)\\
&+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)+\frac{b(b-1)}{2}\sum_{i=b-1}^{n-b+1}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)\\
&+\sum_{\ell=2}^{b}(\ell-1)\sum_{i=b-1}^{n-b+1}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j)+\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=n-b+2}^{n-b+k}\sum_{j=\ell}^{b-1}\psi(X_i-X_j)\\
&+\sum_{k=1}^{b-1}(b-k)\sum_{i=n-b+2}^{n-b+k}\sum_{j=b}^{n-b+2}\psi(X_i-X_j)+\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=n-b+2}^{n-b+k}\sum_{j=n-b+3}^{n-b+\ell}\psi(X_i-X_j).
\end{aligned}
\tag{A.16}
$$

On the other hand, if $b = n$ and using the symmetry of the kernel K, we obtain:

$$
\begin{aligned}
&\frac{2}{nb(n-b+1)}\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{j=k}^{n-b+k}\left[(K_h*K_g)*(K_h*K_g)\right](X_{j+\ell-k}-X_j)\\
&-\frac{2}{nb(n-b+1)^2}\sum_{\substack{k,\ell=1\\k<\ell}}^{b}\sum_{i=k}^{n-b+k}\sum_{j=\ell}^{n-b+\ell}\left[(K_h*K_g)*(K_h*K_g)\right](X_i-X_j)
\end{aligned}
$$

$$= \frac{2}{n^2}\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n}\psi\left(X_\ell - X_k\right) - \frac{2}{n^2}\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n}\psi\left(X_\ell - X_k\right) = 0. \qquad \text{(A.17)}$$

Using (A.14) , (A.15) and (A.16) in (A.13), and this and (A.12) in (A.11) gives the statement of Theorem 3 for $b < n$. The case $b = n$ is even simpler using (A.17).

Appendix B

Proofs of the results of Chapter 3

Theorem 4 *If $X_1, \ldots, X_n$ are iid random variables and Assumptions (A1)-(A4) hold, then the MISE of $\tilde{r}_{h,1}$ in (3.4) can be expressed as follows:*

$$\begin{aligned} MISE_{w,\tilde{r}_{h,1}}(h) &= \int \left(\frac{K_h * f(x) - f(x)}{1 - F(x)} \right)^2 w(x)\, dx \\ &+ \frac{1}{n} \int \frac{(K_h)^2 * f(x)}{(1 - F(x))^2} w(x)\, dx - \frac{1}{n} \int \left(\frac{K_h * f(x)}{1 - F(x)} \right)^2 w(x)\, dx, \end{aligned}$$

where $K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right)$.

Proof of Threorem 4 Let us take into account $(X_1, X_2, \ldots, X_n)$, a simple random sample, and the proxy estimator $\tilde{r}_{h,1}$ given in (3.4). The mean integrated squared error is given by:

$$\begin{aligned} \mathrm{MISE}_{\tilde{r}_{h,1},w}(h) &= \mathbb{E}\left[\int (\tilde{r}_{h,1}(x) - r(x))^2 w(x)\, dx \right] \\ &= B(h) + V(h), \end{aligned}$$

where

$$\begin{aligned} B(h) &= \int \left[\mathbb{E}\left(\tilde{r}_{h,1}(x)\right) - r(x)\right]^2 w(x)\,dx, \text{ and} \\ V(h) &= \int \mathrm{Var}\left(\tilde{r}_{h,1}(x)\right) w(x)\,dx. \end{aligned}$$

Now, straightforward calculations lead to

$$\begin{aligned} B(h) &= \int \frac{1}{(1-F(x))^2}\left(\mathbb{E}\left[\hat{f}_h(x) - f(x)\right]\right)^2 w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2}\left(\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n K_h(x-X_i) - f(x)\right]\right)^2 w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2}\left(\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[K_h(x-X_i)\right] - f(x)\right)^2 w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2}\left(\mathbb{E}\left[K_h(x-X_1)\right] - f(x)\right)^2 w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2}\left(\int K_h(x-y)f(y)dy - f(x)\right)^2 w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2}\left(K_h * f(x) - f(x)\right)^2 w(x)\,dx \\ &= \int \left(\frac{K_h * f(x) - f(x)}{1-F(x)}\right)^2 w(x)\,dx. \end{aligned} \tag{B.1}$$

On the other hand, the integrated variance results in:

$$\begin{aligned} V(h) &= \int \frac{Var(\hat{f}_h(x))}{(1-F(x))^2} w(x)\,dx \\ &= \int \frac{1}{(1-F(x))^2} Var\left(\frac{1}{n}\sum_{i=1}^n K_h(x-X_i)\right) w(x)\,dx \\ &= \int \frac{1}{n^2(1-F(x))^2}\sum_{i=1}^n Var\left[K_h(x-X_i)\right] w(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{n(1-F(x))^2} Var\left[K_h(x-X_1)\right] w(x)\,dx \\
&= \frac{1}{n}\int \frac{1}{(1-F(x))^2}\left[\mathbb{E}\left[K_h(x-X_1)^2\right] - \left(\mathbb{E}\left[K_h(x-X_1)\right]\right)^2\right] w(x)\,dx \\
&= \frac{1}{nh^2}\int \frac{1}{(1-F(x))^2}\int K\left(\frac{u}{h}\right)^2 f(x-u)du\, w(x)\,dx \\
&\quad -\frac{1}{n}\int \frac{1}{(1-F(x))^2}\left[\int K_h(x-y)f(y)dy\right]^2 w(x)\,dx \\
&= \frac{1}{n}\int \frac{(K_h)^2 * f(x)}{(1-F(x))^2} w(x)\,dx - \frac{1}{n}\int \left(\frac{K_h * f(x)}{1-F(x)}\right)^2 w(x)\,dx. \qquad \text{(B.2)}
\end{aligned}
$$

Therefore, using (B.1) and (B.2), the exact expression for the MISE is obtained.

Theorem 5 *Under Assumptions (A1)-(A4), the smoothed bootstrap version of MISE for $\tilde{r}_{h,1}$ admits the following closed expression:*

$$
\begin{aligned}
MISE^*_{\tilde{r}_{h,1},w}(h) &= \int \left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n \left(K_h * K_g(x-X_i) - K_g(x-X_i)\right)\right]^2 w(x)dx \\
&\quad +\frac{1}{n}\int \left[\frac{1}{n(1-\hat{F}_g(x))^2}\sum_{i=1}^n (K_h)^2 * K_g(x-X_i)\right] w(x)\,dx \\
&\quad -\frac{1}{n}\int \left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n K_h * K_g(x-X_i)\right]^2 w(x)\,dx,
\end{aligned}
$$

where $(K_h)^2(u) = \frac{1}{h^2}K\left(\frac{u}{h}\right)^2$.

Proof of Theorem 5 The bootstrap mean integrated squared error is just:

$$
\begin{aligned}
\mathrm{MISE}^*_{\tilde{r}_{h,1},w}(h) &= \mathbb{E}^*\left[\int \left(\tilde{r}^*_{h,1}(x) - \tilde{r}_{g,1}(x)\right)^2 w(x)\,dx\right] \\
&= B^*(h) + V^*(h),
\end{aligned}
$$

where

$$
\begin{aligned}
B^*(h) &= \int \left[\mathbb{E}^*\left(\tilde{r}^*_{h,1}(x)\right) - \tilde{r}_{g,1}(x)\right]^2 w(x)\,dx, \text{ and} \\
V^*(h) &= \int \mathrm{Var}^*\left(\tilde{r}^*_{h,1}(x)\right) w(x)\,dx.
\end{aligned}
$$

Now, focusing on the bootstrap version of the integrated squared bias term:

$$
\begin{aligned}
B^*(h) &= \int \left(\frac{\mathbb{E}^*[K_h(x-X_1)]}{1-\hat{F}_g(x)} - \frac{\hat{f}_g(x)}{1-\hat{F}_g(x)}\right)^2 w(x)\,dx \\
&= \int \left(\frac{\int K_h(x-y)\hat{f}_g(y)dy - \hat{f}_g(x)}{1-\hat{F}_g(x)}\right)^2 w(x)\,dx \\
&= \int \frac{1}{(1-\hat{F}_g(x))^2}\left(\frac{1}{n}\sum_{i=1}^n \int K_h(x-y)K_g(y-X_i)dy\right. \\
&\quad \left. -\frac{1}{n}\sum_{i=1}^n K_g(x-X_i)\right)^2 w(x)\,dx = \frac{1}{n^2}\int \frac{1}{(1-\hat{F}_g(x))^2} \\
&\quad \cdot\left(\sum_{i=1}^n \int K_h(x-y)K_g(y-X_i)dy - \sum_{i=1}^n K_g(x-X_i)\right)^2 w(x)\,dx \\
&= \frac{1}{n^2}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\sum_{i=1}^n \int K_h(u)K_g(x-X_i-u)du\right. \\
&\quad \left. -\sum_{i=1}^n K_g(x-X_i)\right)^2 w(x)\,dx \\
&= \int \left[\frac{1}{n(1-\hat{F}_g(x))}\left(\sum_{i=1}^n [K_h * K_g(x-X_i)\right.\right. \\
&\quad \left.\left. -K_g(x-X_i)]\right)\right]^2 w(x)\,dx. \qquad \text{(B.3)}
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
V^*(h) &= \int Var^*\left(\frac{\hat{f}_h^*(x)}{1-\hat{F}_g(x)}\right) w(x)\,dx = \int \frac{1}{(1-\hat{F}_g(x))^2} Var^*\left[f_h^*(x)\right] = \\
&= \int \frac{1}{(1-\hat{F}_g(x))^2} Var^*\left[\frac{1}{n}\sum_{i=1}^{n} K_h(x-X_i^*)\right] \\
&= \int \frac{1}{n^2(1-\hat{F}_g(x))^2} \sum_{i=1}^{n} Var^*\left[K_h(x-X_i^*)\right] = \int \frac{Var^*\left[K_h(x-X_1^*)\right]}{n(1-\hat{F}_g(x))^2} \\
&= \int \frac{1}{n(1-\hat{F}_g(x))^2}\left[\mathbb{E}^*\left[K_h(x-X_1^*)^2\right] - \left(\mathbb{E}^*[K_h(x-X_1^*)]\right)^2\right] w(x)\,dx \\
&= \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\mathbb{E}^*\left[K_h(x-X_1^*)^2\right] w(x)\,dx \\
&\quad -\frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\mathbb{E}^*[K_h(x-X_1^*)]\right)^2 w(x)\,dx \\
&= \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\int (K_h)^2(x-y)\hat{f}_g(y)dy\right) w(x)\,dx \\
&\quad -\frac{1}{n}\int \frac{\left(\mathbb{E}^*[K_h(x-X_1^*)]\right)^2}{(1-\hat{F}_g(x))^2} w(x)\,dx.
\end{aligned}$$

The first term of the integrated bootstrap variance results in:

$$\begin{aligned}
&\frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\int (K_h)^2(x-y)\hat{f}_g(y)dyw(x)\,dx \\
&= \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\int (K_h)^2(x-y)\frac{1}{n}\sum_{i=1}^{n} K_g(y-X_i)dyw(x)\,dx \\
&= \frac{1}{n}\int \frac{1}{n(1-\hat{F}_g(x))^2}\sum_{i=1}^{n}\int (K_h)^2(u)K_g(x-X_i-u)duw(x)\,dx \\
&= \frac{1}{n}\int \left[\frac{1}{n(1-\hat{F}_g(x))^2}\sum_{i=1}^{n}(K_h)^2 * K_g(x-X_i)\right] w(x)\,dx. \qquad \text{(B.4)}
\end{aligned}$$

Now, we investigate the second term of the bootstrap integrated variance, leading to:

$$\begin{aligned}
& \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\mathbb{E}^*[K_h(x-X_1^*)]\right)^2 w(x)\,dx \\
= & \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\int K_h(x-y)\hat{f}_g(y)dy\right)^2 w(x)\,dx \\
= & \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\int K_h(x-y)\frac{1}{n}\sum_{i=1}^n K_g(y-X_i)dy\right)^2 w(x)\,dx \\
= & \frac{1}{n}\int \frac{1}{(1-\hat{F}_g(x))^2}\left(\frac{1}{n}\sum_{i=1}^n \int K_h(x-X_i-u)K_g(u)du\right)^2 w(x)\,dx \\
= & \frac{1}{n}\int \left(\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n K_h * K_g(x-X_i)\right)^2 w(x)\,dx. \qquad \text{(B.5)}
\end{aligned}$$

Collecting terms (B.3), (B.4) and (B.5), the proof concludes.

Motivation of Expression (3.5) The approximated expanded expression of (3.5), which is given by:

$$\begin{aligned}
\tilde{r}_{h,2}(x) &= (\hat{r}_h(x)-r(x))\frac{1-\hat{F}_h(x)}{1-F(x)} + r(x) \\
&= \frac{1}{1-F(x)}\hat{f}_h(x) + \frac{f(x)}{(1-F(x))^2}\hat{F}_h(x) - \frac{f(x)}{(1-F(x))^2} + r(x).
\end{aligned}$$

Consider $\hat{F}_h$ defined in expression (3.3), then:

$$\begin{aligned}
\hat{r}_h(x)-r(x) &= \left(\frac{\hat{f}_h(x)}{1-\hat{F}_h(x)} - \frac{f(x)}{1-F(x)}\right)\cdot\left(\frac{1-\hat{F}_h(x)}{1-F(x)}\right) \\
&\quad + \left(\frac{\hat{f}_h(x)}{1-\hat{F}_h(x)} - \frac{f(x)}{1-F(x)}\right)\cdot\left(\frac{\hat{F}_h(x)-F(x)}{1-F(x)}\right) \\
&\simeq \left(\frac{\hat{f}_h(x)}{1-\hat{F}_h(x)} - \frac{f(x)}{1-F(x)}\right)\cdot\left(\frac{1-\hat{F}_h(x)}{1-F(x)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hat{f}_h(x)}{1-F(x)} - \frac{f(x)}{1-F(x)} \cdot \frac{1-\hat{F}_h(x)}{1-F(x)} \\
&= \frac{\hat{f}_h(x)}{1-F(x)} - \frac{f(x)}{1-F(x)} + \frac{f(x)}{1-F(x)} - \frac{f(x)}{1-F(x)} \cdot \frac{1-\hat{F}_h(x)}{1-F(x)} \\
&= \frac{\hat{f}_h(x)-f(x)}{1-F(x)} + \frac{f(x)}{1-F(x)} \cdot \frac{1-F(x)-(1-\hat{F}_h(x))}{1-F(x)} \\
&= \frac{\hat{f}_h(x)}{1-F(x)} - \frac{f(x)}{1-F(x)} + \frac{f(x)}{1-F(x)} \left(1 - \frac{1-\hat{F}_h(x)}{1-F(x)}\right) \\
&= \frac{\hat{f}_h(x)-f(x)}{1-F(x)} + \frac{f(x)}{(1-F(x))^2} \left(\hat{F}_h(x) - F(x)\right) \\
&= \frac{\hat{f}_h(x)}{1-F(x)} + \frac{f(x)}{(1-F(x))^2}\hat{F}_h(x) - \frac{f(x)}{1-F(x)} - \frac{f(x)F(x)}{(1-F(x))^2}.
\end{aligned}
$$

Therefore, expression (3.5) holds.

Theorem 6 *If the sample is iid and assuming Conditions (A1)-(A4), the MISE of $\tilde{r}_{h,2}$ in (3.5) can be expressed as follows:*

$$
\begin{aligned}
MISE_{\tilde{r}_{h,2},w}(h) \;=\; & \int \frac{1}{(1-F(x))^2} \left[\frac{1}{n}(K_h)^2 * f(x) + \frac{n-1}{n}(K_h * f(x))^2\right] w(x)\,dx \\
& +2\int \frac{f(x)}{(1-F(x))^3} \left[\frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right) f(y)dy \right.\\
& \left. +\frac{n-1}{n} K_h * f(x) \int \mathbb{K}\left(\frac{x-y}{h}\right) f(y)dy\right] w(x)\,dx \\
& -2\int \frac{f(x)}{(1-F(x))^3} K_h * f(x) w(x)\,dx + \int \frac{f^2(x)}{(1-F(x))^4} w(x)\,dx \\
& +\int \frac{f^2(x)}{(1-F(x))^4} \left[\frac{1}{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 f(y)dy \right.\\
& \left. +\frac{n-1}{n}\left[\int \mathbb{K}\left(\frac{x-y}{h}\right) f(y)dy\right]^2\right] w(x)\,dx \\
& -2\int \frac{f^2(x)}{(1-F(x))^4} \int \mathbb{K}\left(\frac{x-y}{h}\right) f(y)dy w(x)\,dx.
\end{aligned}
$$

Proof of Theorem 6 Using expression (3.5),

$$\begin{aligned}
\tilde{r}_{h,2}(x) - r(x) &= \frac{\hat{f}_h(x)}{1-F(x)} + \frac{f(x)}{(1-F(x))^2}\hat{F}_h(x) - \frac{f(x)(1-F(x)) + f(x)F(x)}{(1-F(x))^2} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{K_h(x-X_i)}{1-F(x)} + \frac{f(x)}{(1-F(x))^2}\mathbb{K}\left(\frac{x-X_i}{h}\right)\right] \\
&\quad - \frac{f(x)(1-F(x)) + f(x)F(x)}{(1-F(x))^2} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{K_h(x-X_i)}{1-F(x)} + \frac{f(x)}{(1-F(x))^2}\mathbb{K}\left(\frac{x-X_i}{h}\right)\right] - \frac{f(x)}{(1-F(x))^2}.
\end{aligned}$$

Therefore, the mean integrated squared error of $\tilde{r}_{h,2}$ is given by:

$$\begin{aligned}
\mathrm{MISE}_{\tilde{r}_{h,2},w}(h) &= \int \mathbb{E}\left[(\hat{r}_h(x) - r(x))^2\right] w(x)\,dx \\
&= \int \mathbb{E}\left[\left(\frac{1}{1-F(x)}\hat{f}_h(x) + \frac{f(x)}{(1-F(x))^2}\hat{F}_h(x)\right.\right. \\
&\quad \left.\left. - \frac{f(x)}{(1-F(x))^2}\right)^2\right] w(x)\,dx \\
&= \int \mathbb{E}\left[\frac{1}{(1-F(x))^2}\hat{f}_h^2(x) + \frac{2f(x)}{(1-F(x))^3}\hat{F}_h(x)\hat{f}_h(x)\right. \\
&\quad - \frac{2f(x)}{(1-F(x))^3}\hat{f}_h(x) + \frac{f(x)^2}{(1-F(x))^4}\hat{F}_h^2(x) \\
&\quad \left. - \frac{2f^2(x)}{(1-F(x))^4}\hat{F}_h(x) + \frac{f^2(x)}{(1-F(x))^4}\right] w(x)\,dx \\
&= \int \frac{1}{(1-F(x))^2}\mathbb{E}\left(\hat{f}_h^2(x)\right) w(x)\,dx \\
&\quad + 2\int \frac{f(x)}{(1-F(x))^3}\mathbb{E}\left(\hat{F}_h(x)\hat{f}_h(x)\right) w(x)\,dx \\
&\quad - 2\int \frac{f(x)}{(1-F(x))^3}\mathbb{E}\left(\hat{f}_h(x)\right) w(x)\,dx \\
&\quad + \int \frac{f(x)^2}{(1-F(x))^4}\mathbb{E}\left(\hat{F}_h^2(x)\right) w(x)\,dx + \int \frac{f^2(x)}{(1-F(x))^4} w(x)\,dx \\
&\quad - 2\int \frac{f^2(x)}{(1-F(x))^4}\mathbb{E}\left(\hat{F}_h(x)\right) w(x)\,dx. \qquad \text{(B.6)}
\end{aligned}$$

Computing each expectation separately leads to:

$$\begin{aligned}
\mathbb{E}\left(\hat{f}_h(x)\right) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} K_h(x-X_i)\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(K_h(x-X_i)\right) \\
&= \mathbb{E}\left(K_h(x-X_1)\right) = \int K_h(x-y)f(y)dy = K_h * f(x). \qquad \text{(B.7)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left(\hat{F}_h(x)\right) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i}{h}\right)\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_i}{h}\right)\right) \\
&= \mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right) = \int \mathbb{K}\left(\frac{x-y}{h}\right) f(y)dy. \qquad \text{(B.8)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left(\hat{f}_h^2(x)\right) &= Var\left(\hat{f}_h(x)\right) + \left(\mathbb{E}\left(\hat{f}_h(x)\right)\right)^2 \\
&= Var\left(\frac{1}{n}\sum_{i=1}^{n} K_h(x-X_i)\right) + \left(\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} K_h(x-X_i)\right)\right)^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(K_h(x-X_i)\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(K_h(x-X_i)\right)\right)^2 \\
&= \frac{1}{n} Var\left(K_h(x-X_1)\right) + \left(\mathbb{E}\left(K_h(x-X_1)\right)\right)^2 \\
&= \frac{1}{n}\left[\mathbb{E}\left[K_h(x-X_1)^2\right] - \left[\mathbb{E}\left(K_h(x-X_1)\right)\right]^2\right] + \left[\mathbb{E}\left(K_h(x-X_1)\right)\right]^2 \\
&= \frac{1}{n}\int K_h(x-y)^2 f(y)dy + \frac{n-1}{n}\left[\int K_h(x-y)f(y)dy\right]^2 \\
&= \frac{1}{n}\int K_h(x-y)^2 f(y)dy + \frac{n-1}{n}\left[K_h * f(x)\right]^2 \\
&= \frac{1}{n}(K_h)^2 * f(x) + \frac{n-1}{n}\left[K_h * f(x)\right]^2. \qquad \text{(B.9)}
\end{aligned}$$

$$\mathbb{E}\left(\hat{F}_h^2(x)\right) = Var\left(\hat{F}_h(x)\right) + \left(\mathbb{E}\left(\hat{F}_h(x)\right)\right)^2$$

$$
\begin{aligned}
&= Var\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i}{h}\right)\right)+\left(\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i}{h}\right)\right)\right)^2\\
&= \frac{1}{n^2}\sum_{i=1}^{n}Var\left(\mathbb{K}\left(\frac{x-X_i}{h}\right)\right)+\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_i}{h}\right)\right)\right)^2\\
&= \frac{1}{n}Var\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)+\left(\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)\right)^2\\
&= \frac{1}{n}\left[\mathbb{E}\left[\mathbb{K}\left(\frac{x-X_1}{h}\right)^2\right]-\left[\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)\right]^2\right]\\
&\quad +\left[\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)\right]^2\\
&= \frac{1}{n}\mathbb{E}\left[\mathbb{K}\left(\frac{x-X_1}{h}\right)^2\right]+\frac{n-1}{n}\left[\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)\right]^2\\
&= \frac{1}{n}\int\mathbb{K}\left(\frac{x-y}{h}\right)^2 f(y)dy+\frac{n-1}{n}\left[\int\mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\right]^2.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left(\hat{f}_h(x)\hat{F}_h(x)\right) &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\right)\cdot\left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{K}\left(\frac{x-X_j}{h}\right)\right)\right]\\
&= \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[K_h(x-X_i)\cdot\mathbb{K}\left(\frac{x-X_j}{h}\right)\right]\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[K_h(x-X_i)\cdot\mathbb{K}\left(\frac{x-X_i}{h}\right)\right]\\
&\quad +\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\mathbb{E}\left[K_h(x-X_i)\cdot\mathbb{K}\left(\frac{x-X_j}{h}\right)\right]\\
&= \frac{1}{n}\mathbb{E}\left(K_h(x-X_1)\cdot\mathbb{K}\left(\frac{x-X_1}{h}\right)\right)\\
&\quad +\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\mathbb{E}\left(K_h(x-X_i)\right)\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_j}{h}\right)\right)\\
&= \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy+\frac{n-1}{n}\mathbb{E}(K_h(x-X_1))
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\mathbb{E}\left(\mathbb{K}\left(\frac{x-X_1}{h}\right)\right) = \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\\
+&\quad \frac{n-1}{n}\left(\int K_h(x-y)f(y)dy\right)\left(\int \mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\right)\\
=&\quad \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy\\
&+\frac{n-1}{n}K_h * f(x)\int \mathbb{K}\left(\frac{x-y}{h}\right)f(y)dy. \qquad \text{(B.10)}
\end{aligned}
$$

Collecting (B.7)-(B.10) and plugging them in (B.6), the proof of Theorem 6 is concluded.

Theorem 7 *Under Assumptions (A1)-(A4), the smoothed bootstrap version of MISE for $\tilde{r}_{h,2}$ admits the following closed expression:*

$$
\begin{aligned}
MISE^*_{\tilde{r}_{h,2},w}(h) \quad = \quad & \frac{1}{n^2}\int \frac{1}{(1-\hat{F}_g(x))^2}\sum_{i=1}^{n}(K_h)^2 * K_g(x-X_i)w(x)\,dx\\
&+\frac{n-1}{n^3}\int \frac{1}{(1-\hat{F}_g(x))^2}\\
&\cdot\sum_{i,j=1}^{n} K_h * K_g(x-X_i)K_h * K_g(x-X_j)w(x)\,dx\\
&+\frac{2}{n^2}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^{n} L_h * K_g(x-X_i)w(x)\,dx\\
&-\frac{2}{n}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^{n} K_h * K_g(x-X_i)w(x)\,dx\\
&+\frac{n-1}{n^3}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\cdot\sum_{i,j=1}^{n}\mathbb{I}_{\{K_h*K_g\}}(x-X_i)\\
&\cdot\mathbb{I}_{\{K_h*K_g\}}(x-X_j)\,w(x)\,dx + \frac{2n-2}{n^3}\cdot\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\\
&\sum_{i,j=1}^{n} K_h * K_g(x-X_j)\mathbb{I}_{\{K_h*K_g\}}(x-X_i)\,w(x)\,dx\\
&-\frac{2}{n}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^{n}\mathbb{I}_{\{K_h*K_g\}}(x-X_i)\,w(x)\,dx
\end{aligned}
$$

$$+\frac{1}{n^2}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^n \mathbb{I}_{\{(K_h\otimes K_h)*K_g\}}(x-X_i,x-X_i)$$
$$w(x)\,dx+\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}w(x)\,dx,$$

where $L(z)=\mathbb{K}(z)K(z)$ *and* $L_h(z)=\frac{1}{h}L\left(\frac{z}{h}\right)$.

Proof of Theorem 7 The bootstrap version of the mean integrated squared error is given by:

$$\begin{aligned}
\mathrm{MISE}^*_{\tilde{r_h},2,w}(h) &= \int \mathbb{E}^*\left[(\tilde{r}_h^*(x)-\tilde{r}_g(x))^2\right]w(x)\,dx=\int \mathbb{E}^*\Bigg[\Bigg(\frac{1}{1-\hat{F}_g(x)}\hat{f}_h^*(x)\\
&\quad+\frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^2}\hat{F}_h^*(x)-\frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^2}\Bigg)^2\Bigg]w(x)\,dx\\
&= \int \mathbb{E}^*\Bigg[\frac{1}{(1-\hat{F}_g(x))^2}\hat{f}_h^{*2}(x)+\frac{2\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\hat{F}_h^*(x)\hat{f}_h^*(x)\\
&\quad-\frac{2\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\hat{f}_h^*(x)+\frac{\hat{f}_g(x)^2}{(1-\hat{F}_g(x))^4}\hat{F}_h^{*,2}(x)-\frac{2\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\hat{F}_h^*(x)\\
&\quad+\frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\Bigg]w(x)\,dx\\
&= \int \frac{1}{(1-\hat{F}_g(x))^2}\mathbb{E}^*\left(\hat{f}_h^{*,2}(x)\right)w(x)\,dx-2\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\\
&\quad\cdot\mathbb{E}^*\left(\hat{F}_h^*(x)\right)w(x)\,dx+2\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\mathbb{E}^*\left(\hat{F}_h^*(x)\hat{f}_h^*(x)\right)w(x)\,dx\\
&\quad-2\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\mathbb{E}^*\left(\hat{f}_h^*(x)\right)w(x)\,dx+\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}w(x)\,dx\\
&\quad+\int \frac{\hat{f}_g(x)^2}{(1-\hat{F}_g(x))^4}\mathbb{E}^*\left(\hat{F}_h^{*2}(x)\right)w(x)\,dx. \qquad \text{(B.11)}
\end{aligned}$$

Similarly as in the proof of Theorem 6, each bootstrap expectation is computed separately. To work out expressions (B.13), (B.16) and (B.19), three independent random variables with density K (hence, with distribution $\mathbb{K}$), Z_1, Z_2 and Z_3, are considered.

$$\begin{aligned}
\mathbb{E}^*\left(\hat{f}_h^*(x)\right) &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left(K_h(x-X_i^*)\right) = \mathbb{E}^*\left(K_h(x-X_1^*)\right) = \int K_h(x-y)\hat{f}_g(y)dy \\
&= \int K_h(x-y)\frac{1}{n}\sum_{i=1}^{n}K_g(y-X_i)dy = \frac{1}{n}\sum_{i=1}^{n}\int K_h(x-y)K_g(y-X_i)dy \\
&= \frac{1}{n}\sum_{i=1}^{n}\int K_h(u)K_g(x-X_i-u)du = \frac{1}{n}\sum_{i=1}^{n}K_h * K_g(x-X_i). \quad \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^*\left(\hat{F}_h^*(x)\right) &= \mathbb{E}^*\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right) \\
&= \mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right) = \int \mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy \\
&= \int \mathbb{K}\left(\frac{x-y}{h}\right)\frac{1}{n}\sum_{i=1}^{n}K_g(y-X_i)dy = \frac{1}{n}\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-y}{h}\right) \\
&\quad \cdot K_g(y-X_i)dy = \frac{1}{n}\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)\frac{1}{g}K_g\left(\frac{y-X_i}{g}\right)dy \\
&= \frac{1}{n}\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-X_i-gu}{h}\right)K(u)du = \mathbb{E}^*\left[\mathbb{K}\left(\frac{x-X_i-gZ_1}{h}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left[\mathbb{P}^*\left(Z_2 \leq \frac{x-X_i-gZ_1}{h}\Big|_{Z_1}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left[\mathbb{E}^*\left(\mathbb{1}_{\left\{Z_2 \leq \frac{x-X_i-gZ_1}{h}\right\}}\Bigg|_{Z_1}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}^*\left(Z_2 \leq \frac{x-X_i-gZ_1}{h}\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}^*\left(hZ_2+gZ_1 \leq x-X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{x-X_i}m(u)du,
\end{aligned}$$

(B.13)

where m stands for the density of $hZ_2 + gZ_1$. In other words, $m(u) = K_h * K_g(u)$, and using Definition 1, we have

$$\mathbb{E}^*\left(\hat{F}_h^*(x)\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{K_h * K_g\}}(x - X_i). \tag{B.14}$$

$$\begin{aligned}
\mathbb{E}^*\left(\hat{f}_h^{2*}(x)\right) &= Var^*\left(\hat{f}_h^*(x)\right) + \left(\mathbb{E}^*\left(\hat{f}_h^*(x)\right)\right)^2 \\
&= Var^*\left(\frac{1}{n}\sum_{i=1}^n K_h(x - X_i^*)\right) + \left(\mathbb{E}^*\left(\frac{1}{n}\sum_{i=1}^n K_h(x - X_i^*)\right)\right)^2 \\
&= \frac{1}{n^2}\sum_{i=1}^n Var^*\left(K_h(x - X_i^*)\right) + \left(\frac{1}{n}\sum_{i=1}^n \mathbb{E}^*\left(K_h(x - X_i^*)\right)\right)^2 \\
&= \frac{1}{n} Var^*\left(K_h(x - X_1^*)\right) + \left(\mathbb{E}^*\left(K_h(x - X_1^*)\right)\right)^2 \\
&= \frac{1}{n}\left[\mathbb{E}^*\left[K_h(x - X_1^*)^2\right] - \left[\mathbb{E}^*\left(K_h(x - X_1^*)\right)\right]^2\right] + \left[\mathbb{E}^*\left(K_h(x - X_1^*)\right)\right]^2 \\
&= \frac{1}{n}\left[\int K_h(x - y)^2 \hat{f}_g(y)dy - \left[\int K_h(x - y)\hat{f}_g(y)dy\right]^2\right] \\
&\quad + \left[\int K_h(x - y)\hat{f}_g(y)dy\right]^2 \\
&= \frac{1}{n}\int K_h(x - y)^2 \frac{1}{n}\sum_{i=1}^n K_g(y - X_i)dy \\
&\quad + \frac{n-1}{n}\left[\int K_h(x - y)\frac{1}{n}\sum_{i=1}^n K_g(y - X_i)dy\right]^2 \\
&= \frac{1}{n}\left[\int K_h(x - y)^2 \frac{1}{n}\sum_{i=1}^n K_g(y - X_i)dy\right] \\
&\quad + \frac{n-1}{n}\left[\int K_h(x - y)\frac{1}{n}\sum_{i=1}^n K_g(y - X_i)dy\right]^2 \\
&= \frac{1}{n}\int K_h(u)^2 \frac{1}{n}\sum_{i=1}^n K_g(x - X_i - u)du
\end{aligned}$$

$$
\begin{aligned}
& +\frac{n-1}{n}\left[\int K_h(u)\frac{1}{n}\sum_{i=1}^{n} K_g(x-X_i-u)du\right]^2 \\
= \quad & \frac{1}{n^2}\sum_{i=1}^{n}(K_h)^2 * K_g(x-X_i) + \frac{n-1}{n^3}\left[\sum_{i=1}^{n} K_h * K_g(x-X_i)\right]^2 \\
= \quad & \frac{1}{n^2}\sum_{i=1}^{n}(K_h)^2 * K_g(x-X_i) \\
& +\frac{n-1}{n^3}\sum_{i,j=1}^{n} K_h * K_g(x-X_i) K_h * K_g(x-X_j). \qquad \text{(B.15)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}^*\left(\hat{F}_h^{2*}(x)\right) \quad = \quad & Var^*\left(\hat{F}_h^*(x)\right) + \left(\mathbb{E}^*\left(\hat{F}_h^*(x)\right)\right)^2 \\
= \quad & Var^*\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right) + \left(\mathbb{E}^*\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right)\right)^2 \\
= \quad & \frac{1}{n^2}\sum_{i=1}^{n} Var^*\left(\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right)\right)^2 \\
= \quad & \frac{1}{n}Var^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right) + \left(\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right)\right)^2 \\
= \quad & \frac{1}{n}\left[\mathbb{E}^*\left[\mathbb{K}\left(\frac{x-X_1^*}{h}\right)^2\right] - \left[\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right)\right]^2\right] \\
& + \left[\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right)\right]^2 \\
= \quad & \frac{1}{n}\left[\mathbb{E}^*\left[\mathbb{K}\left(\frac{x-X_1^*}{h}\right)^2\right]\right] + \frac{n-1}{n}\left[\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right)\right]^2 \\
= \quad & \frac{1}{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 \hat{f}_g(y)dy + \frac{n-1}{n}\left[\int \mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy\right]^2 \\
= \quad & \frac{1}{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 \frac{1}{n}\sum_{i=1}^{n} K_g(y-X_i)dy \\
& +\frac{n-1}{n}\left[\int \mathbb{K}\left(\frac{x-y}{h}\right)\frac{1}{n}\sum_{i=1}^{n} K_g(y-X_i)dy\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 K_g(y-X_i)dy \\
&\quad +\frac{n-1}{n^3}\left[\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-y}{h}\right) K_g(y-X_i)dy\right]^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 K_g(y-X_i)dy \\
&\quad +\frac{n-1}{n^3}\sum_{i,j=1}^{n}\left(\int \mathbb{K}\left(\frac{x-y}{h}\right) K_g(y-X_i)dy\right)\cdot \\
&\quad \left(\int \mathbb{K}\left(\frac{x-y}{h}\right) K_g(y-X_j)dy\right).
\end{aligned}
\tag{B.16}
$$

Now, we investigate the first integral in (B.16), leading to:

$$
\begin{aligned}
&\int \mathbb{K}\left(\frac{x-y}{h}\right)^2 K_g(y-X_i)dy = \int \mathbb{K}\left(\frac{x-X_i-gu}{h}\right)^2 K_g(u)\,du \\
&= \mathbb{E}^*\left[\mathbb{K}\left(\frac{x-X_i-gZ_1}{h}\right)^2\right] = \mathbb{E}^*\left[\mathbb{P}^*\left(Z_2 \le \frac{x-X_i-gZ_1}{h}\bigg|_{Z_1}\right)\right. \\
&\quad \left.\cdot\mathbb{P}^*\left(Z_3 \le \frac{x-X_i-gZ_1}{h}\bigg|_{Z_1}\right)\right] \\
&= \mathbb{E}^*\left[\mathbb{P}^*\left(Z_2 \le \frac{x-X_i-gZ_1}{h}, Z_3 \le \frac{x-X_i-gZ_1}{h}\bigg|_{Z_1}\right)\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left(\mathbb{1}_{\left\{Z_2 \le \frac{x-X_i-gZ_1}{h}, Z_3 \le \frac{x-X_i-gZ_1}{h}\right\}}\Bigg|_{Z_1}\right)\right] \\
&= \mathbb{E}^*\left[\mathbb{1}_{\left\{Z_2 \le \frac{x-X_i-gZ_1}{h}, Z_3 \le \frac{x-X_i-gZ_1}{h}\right\}}\right] \\
&= \mathbb{P}^*\left(hZ_2+gZ_1 \le x-X_i, hZ_3+gZ_1 \le x-X_i\right) \\
&= \int_{-\infty}^{x-X_i}\int_{-\infty}^{x-X_i} p(u,v)\,du\,dv,
\end{aligned}
$$

where $p(u,v)$ stands for the density function of the two-dimensional random variable (hZ_2+gZ_1, hZ_3+gZ_1), which is actually the same as the density function of $(\tilde{Z}_2^h + \tilde{Z}_1^g, \tilde{Z}_3^h + \tilde{Z}_1^g)$, being $\tilde{Z}_1^g$ a random variable with density K_g; $\tilde{Z}_2^h$ a random variable with density K_h and $\tilde{Z}_3^h$ a random variable with density K_h. Note that $\tilde{Z}_1^g, \tilde{Z}_2^h$ and $\tilde{Z}_3^h$ are independent. Thus, the distribucion function of $(\tilde{Z}_2^h + \tilde{Z}_1^g, \tilde{Z}_3^h + \tilde{Z}_1^g)$ conditional on $\tilde{Z}_1^g$ is just:

$$\begin{aligned}
\mathbb{P}^*\left(\tilde{Z}_2^h + \tilde{Z}_1^g \leq t_1, \tilde{Z}_3^h + \tilde{Z}_1^g \leq t_2 \Big|_{\tilde{Z}_1^g}\right) &= \mathbb{P}^*\left(\tilde{Z}_2^h \leq t_1 - \tilde{Z}_1^g, \tilde{Z}_3^h \leq t_2 - \tilde{Z}_1^g \Big|_{\tilde{Z}_1^g}\right) \\
&= \mathbb{P}^*\left(\tilde{Z}_2^h \leq t_1 - \tilde{Z}_1^g \Big|_{\tilde{Z}_1^g}\right) \\
&\quad \cdot \mathbb{P}^*\left(\tilde{Z}_3^h \leq t_2 - \tilde{Z}_1^g \Big|_{\tilde{Z}_1^g}\right) \\
&= \mathbb{K}\left(\frac{t_1 - \tilde{Z}_1^g}{h}\right)\mathbb{K}\left(\frac{t_2 - \tilde{Z}_1^g}{h}\right).
\end{aligned}$$

As a consequence, the joint conditional density is given by:

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left[\mathbb{K}\left(\frac{t_1 - \tilde{Z}_1^g}{h}\right)\mathbb{K}\left(\frac{t_2 - \tilde{Z}_1^g}{h}\right)\right] = K_h\left(t_1 - \tilde{Z}_1^g\right) K_h\left(t_2 - \tilde{Z}_1^g\right).$$

Then,

$$p(t_1, t_2) = \int K_h(t_1 - s) K_h(t_2 - s) K_g(s) ds. \tag{B.17}$$

Using expression (B.17), similar arguments as in expression (B.14) and Definition 1, we can conclude that

$$\begin{aligned}
\mathbb{E}^*\left(\hat{F}_h^{2*}(x)\right) &= \frac{1}{n^2}\sum_{i=1}^{n}\int_{-\infty}^{x-X_i}\int_{-\infty}^{x-X_i}\int K_h(u-s)K_h(v-s)K_g(s)dsdudv \\
&+\frac{n-1}{n^3}\sum_{i,j=1}^{n}\left(\int_{-\infty}^{x-X_i} K_h * K_g(u)du\right)\cdot\left(\int_{-\infty}^{x-X_j} K_h * K_g(u)du\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{I}_{\{(K_h\otimes K_h)*K_g\}}\left(x-X_i, x-X_i\right) \\
&+\frac{n-1}{n^3}\sum_{i,j=1}^{n}\mathbb{I}_{\{K_h*K_g\}}\left(x-X_i\right)\cdot\mathbb{I}_{\{K_h*K_g\}}\left(x-X_j\right). \qquad \text{(B.18)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^*\left(\hat{f}_h^*(x)\hat{F}_h^*(x)\right) &= \mathbb{E}^*\left[\left(\left(\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i^*)\right)\cdot\left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{K}\left(\frac{x-X_j^*}{h}\right)\right)\right)\right] \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}^*\left[K_h(x-X_i^*)\cdot\mathbb{K}\left(\frac{x-X_j^*}{h}\right)\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}^*\left[K_h(x-X_i^*)\cdot\mathbb{K}\left(\frac{x-X_i^*}{h}\right)\right] \\
&+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\mathbb{E}^*\left[K_h(x-X_i^*)\cdot\mathbb{K}\left(\frac{x-X_j^*}{h}\right)\right] \\
&= \frac{1}{n}\mathbb{E}^*\left(K_h(x-X_1^*)\cdot\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right) \\
&+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\mathbb{E}^*\left(K_h(x-X_i^*)\right)\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_j^*}{h}\right)\right) \\
&= \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy \\
&+\frac{n-1}{n}\mathbb{E}^*(K_h(x-X_1^*))\mathbb{E}^*\left(\mathbb{K}\left(\frac{x-X_1^*}{h}\right)\right) \\
&= \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy \\
&+\frac{n-1}{n}\left(\int K_h(x-y)\hat{f}_g(y)dy\right)\cdot\left(\int\mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)\frac{1}{n}\sum_{i=1}^{n}K_h(y-X_i)dy \\
&\quad +\frac{n-1}{n}\left(\int K_h(x-y)\frac{1}{n}\sum_{i=1}^{n}K_h(y-X_i)dy\right) \\
&\quad \cdot\left(\int \mathbb{K}\left(\frac{x-y}{h}\right)\frac{1}{n}\sum_{j=1}^{n}K_h(y-X_j)dy\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)K_h(y-X_i)dy \\
&\quad +\frac{n-1}{n^3}\sum_{i,j=1}^{n}\left(\int K_h(x-y)K_h(y-X_i)dy\right)\cdot\left(\int \mathbb{K}\left(\frac{x-y}{h}\right)K_h(y-X_j)dy\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)K_h(y-X_i)dy \\
&\quad +\frac{n-1}{n^3}\sum_{i,j=1}^{n}\left(\int K_h(u)K_h(x-X_i-u)du\right)\cdot\left(\int \mathbb{K}\left(\frac{x-y}{h}\right)K_h(y-X_j)dy\right) \\
&= \frac{1}{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy+\frac{n-1}{n}(K_h*\hat{f}_g)(x)\int \mathbb{K}\left(\frac{x-y}{h}\right)\hat{f}_g(y)dy \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)K_g(y-X_i)dy \\
&\quad +\frac{n-1}{n^3}\sum_{i,j=1}^{n}K_h*K_g(x-X_i)\int \mathbb{K}\left(\frac{x-y}{h}\right)K_g(y-X_j)dy. \qquad \text{(B.19)}
\end{aligned}
$$

Let us consider the function $L(z)=\mathbb{K}(z)K(z)$, defined in Theorem 7. Then, we have:

$$
L_h(z) = \frac{1}{h}L\left(\frac{z}{h}\right)=\frac{1}{h}\mathbb{K}\left(\frac{z}{h}\right)K\left(\frac{z}{h}\right)=\mathbb{K}\left(\frac{z}{h}\right)K_h\left(z\right).
$$

Consequently,

$$
\begin{aligned}
\int K_h(x-y)\mathbb{K}\left(\frac{x-y}{h}\right)K_g(y-X_i)dy &= \int L_h(x-y)K_g(y-X_i)dy \\
&= \int L_h(x-X_i-u)K_g(u)du \\
&= L_h*K_g(x-X_i). \qquad \text{(B.20)}
\end{aligned}
$$

Considering expression (B.20), the arguments used in expression (B.14) and Definition 1, lead to:

$$\begin{aligned}
\mathbb{E}^*\left(\hat{f}_h^*(x)\hat{F}_h^*(x)\right) &= \frac{1}{n^2}\sum_{i=1}^n L_h * K_g(x-X_i) \\
&+\frac{n-1}{n^3}\sum_{i,j=1}^n K_h * K_g(x-X_i)\int_{-\infty}^{x-X_i} K_h * K_g(u)du \\
&= \frac{1}{n^2}\sum_{i=1}^n L_h * K_g(x-X_i) \\
&+\frac{n-1}{n^3}\sum_{i,j=1}^n K_h * K_g(x-X_j)\mathbb{I}_{\{K_h*K_g\}}(x-X_i). \quad \text{(B.21)}
\end{aligned}$$

Plugging terms (B.12)-(B.15), (B.18) and (B.21) in (B.11), Theorem 7 is proven.

Theorem 8 *Let us assume Conditions (A1), (A3) and (A4). If K is the Gaussian kernel, then the smoooth bootstrap version of MISE for $\tilde{r}_{h,1}$ admits the following expression:*

$$\begin{aligned}
MISE^*_{\tilde{r}_{h,1},w}(h) &= \int\left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n\left(K_{\sqrt{g^2+h^2}}(x-X_i)-K_g(x-X_i)\right)\right]^2 w(x)dx \\
&+\frac{1}{2\sqrt{\pi}nh}\int\left[\frac{1}{n(1-\hat{F}_g(x))^2}\sum_{i=1}^n K_{\sqrt{g^2+h^2/2}}(x-X_i)\right] w(x)\,dx \\
&-\frac{1}{n}\int\left[\frac{1}{n(1-\hat{F}_g(x))}\sum_{i=1}^n K_{\sqrt{g^2+h^2}}(x-X_i)\right]^2 w(x)\,dx.
\end{aligned}$$

Similarly, the smooth bootstrap version of MISE for $\tilde{r}_{h,2}$ can be expressed as follows:

$$\begin{aligned}
MISE^*_{\tilde{r}_{h,2},w}(h) &= \frac{1}{n^2h}\int\frac{1}{2\sqrt{\pi}(1-\hat{F}_g(x))^2}\sum_{i=1}^n K_{\sqrt{g^2+h^2/2}}(x-X_i)w(x)\,dx \\
&+\frac{n-1}{n^3}\int\frac{1}{(1-\hat{F}_g(x))^2}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{i,j=1}^{n} K_{\sqrt{h^2+g^2}}(x-X_i) K_{\sqrt{h^2+g^2}}(x-X_j) w(x)\,dx \\
&+\frac{2}{n^2}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^{n} L_h * K_g(x-X_i)w(x)\,dx \\
&-\frac{2}{n}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3}\sum_{i=1}^{n} K_{\sqrt{h^2+g^2}}(x-X_i)w(x)\,dx \\
&+\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} w(x)\,dx + \frac{n-1}{n^3}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4} \\
&\cdot \sum_{i,j=1}^{n} \Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right)\Phi\left(\frac{x-X_j}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&+\frac{2n-2}{n^3}\int \frac{\hat{f}_g(x)}{(1-\hat{F}_g(x))^3} \\
&\cdot \sum_{i,j=1}^{n} K_{\sqrt{h^2+g^2}}(x-X_j)\,\Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&-\frac{2}{n}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^{n} \Phi\left(\frac{x-X_i}{\sqrt{h^2+g^2}}\right) w(x)\,dx \\
&+\frac{1}{n^2}\int \frac{\hat{f}_g^2(x)}{(1-\hat{F}_g(x))^4}\sum_{i=1}^{n} \boldsymbol{\Phi}_{\mathbf{0},\tilde{H}}\left(x-X_i, x-X_i\right) w(x)\,dx,
\end{aligned}
$$

where $\Phi(z)$ is the distribution function of a standard normal variable evaluated at z, $\boldsymbol{\Phi}_{\boldsymbol{\mu},\Sigma}(z_1, z_2)$ is the distribution function of a bivariate normal $N_2(\boldsymbol{\mu}, \Sigma)$ evaluated at (z_1, z_2), $\mathbf{0} = (0,0)^T$ and

$$
\tilde{H} = \begin{pmatrix} g^2+h^2 & g^2 \\ g^2 & g^2+h^2 \end{pmatrix}.
$$

Proof of Theorem 8 Let us consider expressions (B.14), (B.18), (B.21) and two independent standard normal variables, Z_1 and Z_2. Then, $Z_{12} = hZ_2 + gZ_1$ comes from a $N(0, h^2+g^2)$ distribution and,

$$
\mathbb{I}_{\{K_h * K_g\}}(x-a) \;=\; \mathbb{P}\left(Z_{12} \leq x - X_i\right) = \mathbb{P}\left(\frac{Z_{12}}{\sqrt{h^2+g^2}} \leq \frac{x-a}{\sqrt{h^2+g^2}}\right)
$$

$$= \Phi\left(\frac{x-a}{\sqrt{h^2+g^2}}\right), \tag{B.22}$$

which is used in the proof for $a = X_i, X_j$.

Now consider $\tilde{Z}_1^g, \tilde{Z}_2^h, \tilde{Z}_3^h$, the random variables used to obtain expression (B.18). When K is a Gaussian kernel, it is obvious that $\tilde{Z}_1^g \stackrel{d}{=} N(0, g^2), \tilde{Z}_2^h \stackrel{d}{=} N(0, h^2), \tilde{Z}_3^h \stackrel{d}{=} N(0, h^2)$ are independent variables and thus,

$$\begin{pmatrix} \tilde{Z}_1^g \\ \tilde{Z}_2^h \\ \tilde{Z}_3^h \end{pmatrix} \stackrel{d}{=} N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix}\right) = N_3(\mathbf{0}, H), \text{ where } H = \begin{pmatrix} g^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix}.$$

Consider $T_1 = \tilde{Z}_1^g, T_2 = \tilde{Z}_2^h + \tilde{Z}_1^g, T_3 = \tilde{Z}_3^h + \tilde{Z}_1^g$, i.e.,

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = A \cdot \begin{pmatrix} \tilde{Z}_1^g \\ \tilde{Z}_2^h \\ \tilde{Z}_3^h \end{pmatrix}, \text{ with } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Standard properties of the multidimensional normal distribution lead to

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \stackrel{d}{=} N_3\left(A \cdot \mathbf{0}, AHA^T\right) = N_3\left(\mathbf{0}, \begin{pmatrix} g^2 & g^2 & g^2 \\ g^2 & g^2+h^2 & g^2 \\ g^2 & g^2 & g^2+h^2 \end{pmatrix}\right).$$

Specifically,

$$\begin{pmatrix} T_2 \\ T_3 \end{pmatrix} \stackrel{d}{=} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g^2+h^2 & g^2 \\ g^2 & g^2+h^2 \end{pmatrix}\right) = N_2(\mathbf{0}, \tilde{H}),$$

where

$$\tilde{H} = \begin{pmatrix} g^2+h^2 & g^2 \\ g^2 & g^2+h^2 \end{pmatrix}.$$

Finally, using expression (B.18), we have

$$\begin{aligned}\mathbb{I}_{\{(K_h \otimes K_h) * K_g\}}\left(x - X_i, x - X_i\right) &= \mathbb{P}\left(T_2 \leq x - X_i, T_3 \leq x - X_i\right) \\ &= \mathbf{\Phi}_{\mathbf{0},\tilde{H}}\left(x - X_i, x - X_i\right). \end{aligned} \tag{B.23}$$

Plugging terms (B.22) and (B.23) in (3.7), and knowing that $K_h * K_g$ is the density of a $N(0, h^2 + g^2)$, the proof concludes.

Appendix C

Proofs of the results of Chapter 4

Motivation of Expression (4.2)

Consider $(X_1, \ldots, X_n)$ a simple random sample and the Nadaraya-Watson estimator of the regression function, $\hat{m}_h^{NW}(x)$ (see Nadaraya, 1964; and Watson, 1964); and the following approximation of the kernel regression estimator, given by:

$$
\begin{aligned}
\hat{m}_h^{NW}(x) - m(x) &= \left(\hat{m}_h^{NW}(x) - m(x)\right)\left[\frac{\hat{f}_h(x)}{f(x)} + 1 - \frac{\hat{f}_h(x)}{f(x)}\right] \\
&= \left(\hat{m}_h^{NW}(x) - m(x)\right)\frac{\hat{f}_h(x)}{f(x)} + \left(\hat{m}_h^{NW}(x) - m(x)\right)\frac{f(x) - \hat{f}_h(x)}{f(x)} \\
&= \left(\frac{\hat{\Psi}_h(x)}{f(x)} - \frac{m(x)}{f(x)}\hat{f}_h(x)\right) + (\hat{m}_h^{NW}(x) - m(x))\frac{f(x) - \hat{f}_h(x)}{f(x)}, \\
&= A_1(x) + A_2(x),
\end{aligned}
$$

where $\hat{\Psi}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i)Y_i$, $A_2(x) := (\hat{m}_h^{NW}(x) - m(x))\frac{f(x) - \hat{f}_h(x)}{f(x)}$ and

$$
\begin{aligned}
A_1 &:= \frac{1}{f(x)}(\hat{\Psi}_h(x) - m(x)\hat{f}_h(x)) \\
&= \frac{1}{f(x)}\left[\frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i)Y_i - m(x)\frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i)\right] \\
&= \frac{1}{nf(x)}\sum_{i=1}^{n} K_h(x - X_i)(Y_i - m(x)).
\end{aligned}
$$

Then, an approximation of $\hat{m}_h^{NW}(x)$ is given in expression (4.2), that is:

$$\tilde{m}_h^{NW}(x) = m(x) + \frac{1}{nf(x)} \sum_{i=1}^{n} K_h(x - X_i)(Y_i - m(x)),$$

where $m(x) := \mathbb{E}\left[Y|_{X=x}\right]$.

Theorem 9 *If x is an interior point of the support of X, the sample $(X_i, Y_i), i = 1 \ldots, n$ is iid, K is a bounded symmetric density function and $f(x) \neq 0$, then the MSE_x of $\tilde{m}_h^{NW}$ (4.2) can be expressed as follows:*

$$MSE_x(h) = \left(\frac{n-1}{nf(x)^2}\right)\left[(K_h * q_x)(x)\right]^2 + \frac{1}{nf(x)^2}\left[(K_h)^2 * p_x\right](x),$$

where $p_x(z) = \left(\sigma^2(z) + (m(z) - m(x))^2\right) f(z)$, $q_x(z) = (m(z) - m(x))f(z)$ and $\sigma^2(x) := Var\left(Y|_{X=x}\right)$ stands for the volatility function.

Proof of Theorem 9 Focusing firstly on the bias term:

$$\begin{aligned}
\mathbb{E}\left[A_1(x)\right] &= \frac{1}{nf(x)} \sum_{i=1}^{n} \mathbb{E}\left[K_h(x - X_i)(Y_i - m(x))\right] \\
&= \frac{1}{f(x)} \mathbb{E}\left[K_h(x - X_1)(Y_1 - m(x))\right] \\
&= \frac{1}{f(x)} \mathbb{E}\left[\mathbb{E}\left[K_h(x - X_1)(Y_1 - m(x))|_{X_1}\right]\right] \\
&= \frac{1}{f(x)} \mathbb{E}\left[K_h(x - X_1)(\mathbb{E}[Y_1 \mid X_1] - m(x))\right] \\
&= \frac{1}{f(x)} \mathbb{E}\left[K_h(x - X_1)(m(X_1) - m(x))\right] \\
&= \frac{1}{f(x)} \int K_h(x - y)(m(y) - m(x))f(y)\, dy \\
&= \frac{1}{f(x)} \int K_h(x - y)q_x(y)\, dy = \frac{1}{f(x)}\left[K_h * q_x\right](x), \qquad \text{(C.1)}
\end{aligned}$$

where $q_x(z) = (m(z) - m(x))f(z)$.

On the other hand,

$$\begin{aligned}
Var\left[A_1(x)\right] &= \frac{1}{n^2 f(x)^2}\sum_{i=1}^{n} Var\left[K_h(x-X_i)(Y_i-m(x))\right] \\
&= \frac{1}{nf(x)^2} Var\left[K_h(x-X_1)(Y_1-m(x))\right] \\
&= \frac{1}{nf(x)^2}\mathbb{E}\left[K_h(x-X_1)^2(Y_1-m(x))^2\right] \qquad \text{(C.2)} \\
&\quad -\frac{1}{nf(x)^2}\left(\mathbb{E}\left[K_h(x-X_1)(Y_1-m(x))\right]\right)^2.
\end{aligned}$$

The first term of (C.2) turns out to be:

$$\begin{aligned}
\mathbb{E}\left[K_h(x-X_1)^2(Y_1-m(x))^2\right] &= \mathbb{E}\left[\mathbb{E}\left[K_h(x-X_1)^2(Y_1-m(x))^2 \mid_{X_1}\right]\right] \\
&= \mathbb{E}\left[K_h(x-X_1)^2\mathbb{E}\left[(Y_1-m(x))^2 \mid_{X_1}\right]\right] \\
&= \mathbb{E}\Big[K_h(x-X_1)^2\left[Var(Y_1-m(x)\mid X_1)\right. \\
&\quad \left.+\left(\mathbb{E}\left[Y_1-m(x)\mid_{X_1}\right]\right)^2\right]\Big] \\
&= \mathbb{E}\Big[K_h(x-X_1)^2 \\
&\quad \cdot\left[Var(Y_1\mid X_1)+(m(X_1)-m(x))^2\right]\Big] \\
&= \int K_h(x-y)^2\left(\sigma^2(y)+(m(y)-m(x))^2\right)f(y)\,dy \\
&= \left[(K_h)^2 * p_x\right](x), \qquad \text{(C.3)}
\end{aligned}$$

where $p_x(y) = \left(\sigma^2(y) + (m(y) - m(x))^2\right) f(y)$.

Collecting terms (C.1) and (C.3) and plugging them in (C.2), and then plugging (C.1) and (C.2) in (4.4), Theorem 9 is proven.

Theorem 10 *Consider the explanatory random variable X, and x an interior point of the support of X, K a symmetric bounded density function, $\hat{f}_{gX}(x) \neq 0$, and $(X_i, Y_i), i = 1, \ldots, n$ a simple random sample. The smoothed bootstrap version of the*

MSE_x of the proxy estimator given in (4.2) results in:

$$
\begin{aligned}
MSE_x^*(h) &= \frac{1}{n\hat{f}_{g_X}^2(x)}\left[\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2 * K_{g_X}\right](x-X_i) + \left[(K_h)^2 * \hat{p}_{x,g_X}\right](x)\right] \\
&+ \left(\frac{n-1}{n\hat{f}_{g_X}^2(x)}\right)\left[(K_h * \hat{q}_{x,g_X})(x)\right]^2,
\end{aligned}
$$

where $\hat{p}_{x,g_X}(z) = \left(\hat{\sigma}_{g_X}^2(z) + (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}(z)$ and $\hat{q}_{z,g_X}(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(z)$.

Proof of Theorem 10 Focusing on the bootstrap version of the bias term,

$$
\begin{aligned}
\mathbb{E}^*\left[A_1^*(x)\right] &= \frac{1}{n\hat{f}_{g_X}(x)}\sum_{i=1}^n \mathbb{E}^*\left[K_h(x-X_i^*)(Y_i^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[\mathbb{E}^*\left[(K_h(x-X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x)) \mid_{X_1^*}\right]\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(\mathbb{E}^*[Y_1^* \mid X_1^*] - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(\hat{m}_{g_X}^{NW}(X_1^*) - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)}\int K_h(x-y)(\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(y)\,dy \\
&= \frac{1}{\hat{f}_{g_X}(x)}\int K_h(x-y)\hat{q}_{x,g_X}(y)\,dy = \frac{1}{\hat{f}_{g_X}(x)}\left[K_h * \hat{q}_{x,g_X}\right](x), \quad \text{(C.4)}
\end{aligned}
$$

where $\hat{q}_{x,g_X}(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(z)$.

On the other hand,

$$
Var^*[A_1^*(x)] = \frac{1}{n^2\hat{f}_{g_X}(x)^2}\sum_{i=1}^n Var^*\left[K_h(x-X_i^*)(Y_i^* - \hat{m}_{g_X}^{NW}(x))\right]
$$

$$
\begin{aligned}
&= \frac{1}{n\hat{f}_{g_X}(x)^2} Var^* \left[K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{n\hat{f}_{g_X}(x)^2} \mathbb{E}^* \left[K_h(x - X_1^*)^2(Y_1^* - \hat{m}_{g_X}^{NW}(x))^2\right] \\
&\quad - \frac{1}{n\hat{f}_{g_X}(x)^2} \left(\mathbb{E}^* \left[K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right]\right)^2. \qquad \text{(C.5)}
\end{aligned}
$$

The first term of (C.5) turns out to be:

$$
\begin{aligned}
&\mathbb{E}^* \left[K_h(x - X_1^*)^2(Y_1^* - \hat{m}_{g_X}^{NW}(x))^2\right] \\
&= \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(x - X_1^*)^2(Y_1^* - \hat{m}_{g_X}^{NW}(x))^2 \mid_{X_1^*}\right]\right] \\
&= \mathbb{E}^* \left[K_h(x - X_1^*)^2 \mathbb{E}^* \left[(Y_1^* - \hat{m}_{g_X}^{NW}(x))^2 \mid_{X_1^*}\right]\right] \\
&= \mathbb{E}^* \left[K_h(x - X_1^*)^2 \left[Var^*(Y_1^* - \hat{m}_{g_X}^{NW}(x) \mid X_1^*) + \left(\mathbb{E}^* \left[Y_1^* - \hat{m}_{g_X}^{NW}(x) \mid_{X_1^*}\right]\right)^2\right]\right] \\
&= \mathbb{E}^* \left[K_h(x - X_1^*)^2 \left[Var^*(Y_1^* \mid X_1^*) + \left(\hat{m}_{g_X}^{NW}(X_1^*) - \hat{m}_{g_X}^{NW}(x)\right)^2\right]\right] \\
&= \mathbb{E}^* \left[K_h(x - X_1^*)^2 \left[\sigma^{*2}(X_1^*) + \left(\hat{m}_{g_X}^{NW}(X_1^*) - \hat{m}_{g_X}^{NW}(x)\right)^2\right]\right] \\
&= \mathbb{E}^* \left[K_h(x - X_1^*)^2 \left[\hat{\sigma}_{g_X}^2(X_1^*) + g_Y^2\mu_2(K) + \left(\hat{m}_{g_X}^{NW}(X_1^*) - \hat{m}_{g_X}^{NW}(x)\right)^2\right]\right] \\
&= \int K_h(x - y)^2 \left(\hat{\sigma}_{g_X}^2(y) + g_Y^2\mu_2(K) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))^2\right) \hat{f}_{g_X}(y)\, dy \\
&= g_Y^2\mu_2(K) \int K_h(x - y)^2 \hat{f}_{g_X}(y) dy \\
&\quad + \int K_h(x - y)^2 \left(\hat{\sigma}_{g_X}^2(y) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))^2\right) \hat{f}_{g_X}(y)\, dy \\
&= \frac{g_Y^2\mu_2(K)}{n} \sum_{i=1}^n \int K_h(x - y)^2 K_{g_X}(y - X_i) dy + \int K_h(x - y)^2 \hat{p}_{x,g_X}(y)\, dy \\
&= \frac{g_Y^2\mu_2(K)}{n} \sum_{i=1}^n \left[(K_h)^2 * K_{g_X}\right](x - X_i) + \int K_h(x - y)^2 \hat{p}_{x,g_X}(y)\, dy \\
&= \frac{g_Y^2\mu_2(K)}{n} \sum_{i=1}^n \left[(K_h)^2 * K_{g_X}\right](x - X_i) + \left[(K_h)^2 * \hat{p}_{x,g_X}\right](x), \qquad \text{(C.6)}
\end{aligned}
$$

where $\hat{p}_{x,g_X}(y) = \left(\hat{\sigma}_{g_X}^2(y) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))^2\right) \hat{f}_{g_X}(y)$ and $\mu_2(K) = \int t^2 K(t)\, dt$.

Collecting terms (C.4) and (C.6) and plugging them in (C.5), and then plugging (C.4) and (C.5) in (4.5), Theorem 10 is proven.

Proposition 1 *If F_1 is the distribution function of the target population and $\hat{m}_h$, the estimated regression function. Then, an upper bound for expression (4.10) is given by:*

$$\mathbb{E}\left[\left(\int (\hat{m}_h(x) - m(x))\, dF_1(x)\right)^2\right] \le \mathbb{E}\left[\int (\hat{m}_h(x) - m(x))^2\, dF_1(x)\right]. \quad \text{(C.7)}$$

On the other hand, the average prediction error is given by:

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] = \int \sigma^2(x)\, dF_1(x) + \mathbb{E}\left[\int (\hat{m}_h(x) - m(x))^2\, dF_1(x)\right], \quad \text{(C.8)}$$

i.e., expression (4.10) plus a constant which does not depend oh h.

Proof of Proposition 1 On the one hand, consider Jensen's inequality and the convex function $\psi(z) = z^2$. Then,

$$\begin{aligned} \int (\hat{m}_h(x) - m(x))^2\, dF_1(x) &= \mathbb{E}\left[\psi\left(\hat{m}_h\left(X^1\right) - m\left(X^1\right)\right)\Big|_{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)}\right] \\ &\ge \psi\left(\mathbb{E}\left[\hat{m}_h\left(X^1\right) - m\left(X^1\right)\Big|_{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)}\right]\right) \\ &= \left(\int (\hat{m}_h(x) - m(x))\, dF_1(x)\right)^2. \end{aligned}$$

Then, expression (C.7) holds.

On the other hand, the average prediction error introduced in Definition 4 can be expressed as:

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] = \frac{1}{n_1}\sum_{i=1}^{n_1}\mathbb{E}\left[\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left(Y_1^1 - \hat{m}_h\left(X_1^1\right)\right)^2\right] = \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right) + m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\
&= A_1 + A_2 + 2A_3,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\right], \\
A_2 &:= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right], \\
A_3 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right) \cdot \left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\right].
\end{aligned}
$$

Term A_1 does not depend on h. In fact,

$$
\begin{aligned}
A_1 &= \mathbb{E}\left[\mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\Big|_{X_1^1}\right]\right] = \mathbb{E}\left[Var\left(Y_1^1|_{X_1^1}\right)\right] \\
&= \mathbb{E}\left[\sigma^2\left(X_1^1\right)\right] = \int \sigma^2(x)\, dF_1(x).
\end{aligned}
$$

Moreover, carrying on with further computations in term A_2:

$$
\begin{aligned}
A_2 &= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\Big|_{\left(X_1^0, Y_1^0\right), \ldots, \left(X_{n_0}^0, Y_{n_0}^0\right)}\right]\right] \\
&= \mathbb{E}\left[\int \left(\hat{m}_h(x) - m(x)\right)^2\, dF_1(x)\right].
\end{aligned}
$$

Finally, term A_3 results in:

$$
\begin{aligned}
A_3 &= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right) \cdot \left(Y_1^1 - m\left(X_1^1\right)\right)|_{\left(X_1^0, Y_1^0\right), \ldots, \left(X_{n_0}^0, Y_{n_0}^0\right), X_1^1}\right]\right] \\
&= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right) \cdot \left(\mathbb{E}\left[Y_1^1|_{X_1^1}\right] - m\left(X_1^1\right)\right)\right] = 0,
\end{aligned}
$$

as a consequence of $\mathbb{E}\left[Y_1^1|_{X_1^1}\right] = m\left(X_1^1\right)$. Then, expression (C.8) holds.

Theorem 11 *Let* $\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\}$ *be a simple random sample coming from the source population,* (X^0, Y^0)*, and* $\left(X_1^1, \ldots, X_{n_1}^1\right)$*, a simple random sample coming from the target population,* X^1*. Consider* K *a symmetric bounded density. Then,* *the prediction error MASE admits the following representation:*

$$
\begin{aligned}
MASE_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left(\left[K_h * q^0_{X_j^1}\right](X_j^1)\right)^2\right.\\
&\quad\left.+\frac{1}{n_0}\left[(K_h)^2 * p^0_{X_j^1}\right](X_j^1)\right],
\end{aligned}
$$

where $q_x^0(z) = (m(z)-m(x))f^0(z)$ *and* $p_x^0(z) = \left(\sigma_0^2(z) + (m(z)-m(x))^2\right) f^0(z)$.

Proof of Theorem 11 The target is to work out an explicit expression for the prediction error MASE, given by:

$$
\begin{aligned}
MASE_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\mathbb{E}_0\left[\left(\tilde{m}_h^{NW}(X_j^1)-m(X_j^1)\right)^2\right]\\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right.\\
&\quad\left.+\left(\mathbb{E}_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2\right] =\\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0[A_{1,j}^0] + \left(\mathbb{E}_0\left[A_{1,j}^0\right]\right)^2\right]\\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_{1,j}^0] + \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_{1,j}^0\right]\right)^2. \qquad \text{(C.9)}
\end{aligned}
$$

Focusing now on the second term of (C.9):

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\left(X_j^1\right)\right]\right)^2\\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0\left[K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2\\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right)^2
\end{aligned}
$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\mid_{X_1^0}\right]\right]\right)^2$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(\mathbb{E}_0\left[Y_1^0\mid_{X_1^0}\right]-m(X_j^1)\right)\right]\right)^2$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(m(X_1^0)-m(X_j^1)\right)\right]\right)^2$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)\left(m(y)-m(X_j^1)\right)f^0(y)\,dy\right)^2$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)q^0_{X_j^1}(y)\,dy\right)^2$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\left[K_h * q^0_{X_j^1}\right](X_j^1)\right)^2 \tag{C.10}$$

where $q_x^0(z) = (m(z)-m(x))f^0(z)$.

On the other hand,

$$\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_1^0\left(X_j^1\right)] = \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{n_0 f^0(X_j^1)^2}Var_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]$$

$$= \frac{1}{n_0 n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\right]-\right.$$

$$\left.\left(\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right)^2\right]. \tag{C.11}$$

The first term of (C.11) turns out to be:

$$\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\right]$$

$$= \mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\mid_{X_1^0}\right]\right]$$

$$= \mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\mathbb{E}_0\left[(Y_1^0-m(X_j^1))^2\mid_{X_1^0}\right]\right]$$

$$
\begin{aligned}
&= \mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\left[Var_0(Y_1^0-m(X_j^1)\mid_{X_1^0})+\left(\mathbb{E}_0\left[Y_1^0-m(X_j^1)\mid_{X_1^0}\right]\right)^2\right]\right]\\
&= \mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\left[Var_0(Y_1^0\mid_{X_1^0})+\left(m(X_1^0)-m(X_j^1)\right)^2\right]\right]\\
&= \int K_h(X_j^1-y)^2\left(\sigma_0^2(y)+(m(y)-m(X_j^1))^2\right)f^0(y)\,dy\\
&= \left[(K_h)^2 * p^0_{X_j^1}\right](X_j^1), \qquad \text{(C.12)}
\end{aligned}
$$

where $p_x^0(y)=(\sigma_0^2(y)+(m(y)-m(x))^2)\,f^0(y)$.

Collecting terms (C.10) and (C.12) and plugging them in (C.11), and then plugging (C.10) and (C.11) in (C.9), Theorem 11 holds.

Theorem 12 *If $\{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)\}$ is a simple random sample coming from the source population, (X^0,Y^0); $\left(X_1^1,\ldots,X_{n_1}^1\right)$ is a simple random sample coming from the target population, X^1; and K a symmetric bounded density, then the bootstrap version of the prediction error MASE admits the following representation:*

$$
\begin{aligned}
MASE^*_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left(\left[K_h*\hat{q}^0_{X_j^1,g_X}\right](X_j^1)\right)^2\right.\\
&\quad+\frac{1}{n_0}\left[(K_h)^2*\hat{p}^0_{X_j^1,g_X}\right](X_j^1)\\
&\quad\left.+\frac{g_Y^2\mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2*K_{g_X}\right](X_j^1-X_i^0)\right],
\end{aligned}
$$

where $\hat{p}^0_{x,g_X}(z)=\left(\hat{\sigma}^2_{0,g_X}(z)+(\hat{m}^{NW}_{g_X}(z)-\hat{m}^{NW}_{g_X}(x))^2\right)\hat{f}^0_{g_X}(z)$ and $\hat{q}^0_x(z)=(\hat{m}^{NW}_{g_X}(z)-\hat{m}^{NW}_{g_X}(x))\hat{f}^0_{g_X}(z)$ and $\hat{\sigma}^2_{0,g_X}(z)=\hat{m}_{2,g_X}(z)-\hat{m}^2_{g_X}(z)$, where, $\forall k\geq 2$, $\hat{m}_{k,g_X}(z)=\dfrac{\sum_{i=1}^{n_0}K_{g_X}(z-X_i^0)(Y^0)^k}{\sum_{i=1}^{n_0}K_{g_X}(z-X_i^0)}$.

Proof of Theorem 12 The aim is to compute a closed-form expression for the bootstrap version of the MASE:

$$
MASE^*_{\tilde{m}_h^{NW},X^1}(h) = \frac{1}{n_1}\sum_{j=1}^{n_1}\mathbb{E}_0^*\left[\left(\tilde{m}_h^{NW*}(X_j^1)-\hat{m}^{NW}_{g_X}(X_j^1)\right)^2\right]
$$

$$\begin{aligned}
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\Bigg[Var_0^*\Bigg[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1 - X_i^{0*}) \\
&\quad \cdot(Y_i^{0*} - \hat{m}_{gX}^{NW}(X_j^1))\Bigg] + \Bigg(\mathbb{E}_0^*\Bigg[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1 - X_i^{0*}) \\
&\quad \cdot(Y_i^{0*} - \hat{m}_{gX}^{NW}(X_j^1))\Bigg]\Bigg)^2\Bigg] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0^*[A_1^{0*}\left(X_j^1\right)] + \left(\mathbb{E}_0^*\left[A_1^{0*}\left(X_j^1\right)\right]\right)^2\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*[A_1^{0*}\left(X_j^1\right)] + \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\left(X_j^1\right)\right]\right)^2. \qquad \text{(C.13)}
\end{aligned}$$

Focusing now on the second term of (C.13):

$$\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\left(X_j^1\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0^*\left[K_h(X_j^1 - X_i^{0*})(Y_i^{0*} - \hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})(Y_1^{0*} - \hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})(Y_1^{0*} - \hat{m}_{gX}^{NW}(X_j^1)) \mid_{X_1^{0*}}\right]\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^{0*}(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})\left(\mathbb{E}_0^*\left[Y_1^{0*} \mid_{X_1^{0*}}\right] - \hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})\left(\hat{m}_{gX}^{NW}(X_1^{0*}) - \hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1 - y)\left(\hat{m}_{gX}^{NW}(y) - \hat{m}_{gX}^{NW}(X_j^1)\right)\hat{f}_{gX}^0(y)\,dy\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1 - y)\hat{q}_{X_j^1,gX}^0(y)\,dy\right)^2
\end{aligned}$$

$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}^0_{g_X}(X^1_j)}\left[K_h * \hat{q}^0_{X^1_j,g_X}\right](X^1_j)\right)^2, \tag{C.14}$$

where $\hat{q}^0_x(z) = (\hat{m}^{NW}_{g_X}(z) - \hat{m}^{NW}_{g_X}(x))\hat{f}^0_{g_X}(z)$.
On the other hand,

$$\begin{aligned}
& \frac{1}{n_1}\sum_{j=1}^{n_1} Var^*_0\left[A^{0*}_1\left(X^1_j\right)\right] \\
= & \frac{1}{n_1}\sum_{j=1}^{n_1} Var^*_0\left[\frac{1}{n_0\hat{f}^0_{g_X}(X^1_j)}\sum_{i=1}^{n_0} K_h(X^1_j - X^{0*}_i)(Y^{0*}_i - \hat{m}^{NW}_{g_X}(X^1_j))\right] \\
= & \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\frac{1}{n_0\hat{f}^0_{g_X}(X^1_j)^2} Var^*_0\left[K_h(X^1_j - X^{0*}_1)(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))\right]\right] \\
= & \frac{1}{n_0 n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X^1_j)^2}\left[\mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))^2\right] - \right. \\
& \left.\left(\mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))\right]\right)^2\right].
\end{aligned} \tag{C.15}$$

The first term of (C.15) turns out to be:

$$\begin{aligned}
& \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))^2\right] \\
= & \mathbb{E}^*_0\left[\mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))^2 \mid_{X^{0*}_1}\right]\right] \\
= & \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2\mathbb{E}^*_0\left[(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j))^2 \mid_{X^{0*}_1}\right]\right] \\
= & \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2\left[Var^*_0(Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j) \mid_{X^{0*}_1})\right.\right. \\
& \left.\left. + \left(\mathbb{E}^*_0\left[Y^{0*}_1 - \hat{m}^{NW}_{g_X}(X^1_j) \mid_{X^{0*}_1}\right]\right)^2\right]\right] \\
= & \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2\left[Var^*_0(Y^{0*}_1 \mid_{X^{0*}_1}) + \left(\hat{m}^{NW}_{g_X}(X^{0*}_1) - \hat{m}^{NW}_{g_X}(X^1_j)\right)^2\right]\right] \\
= & \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2\left[\sigma^{*2}_0(X^{0*}_1) + \left(\hat{m}^{NW}_{g_X}(X^{0*}_1) - \hat{m}^{NW}_{g_X}(X^1_j)\right)^2\right]\right] \\
= & \mathbb{E}^*_0\left[K_h(X^1_j - X^{0*}_1)^2\left[\hat{\sigma}^2_{0,g_X}(X^{0*}_1) + g_Y^2\mu_2(K) + \left(\hat{m}^{NW}_{g_X}(X^{0*}_1) - \hat{m}^{NW}_{g_X}(X^1_j)\right)^2\right]\right] \\
= & \int K_h(X^1_j - y)^2\left(\hat{\sigma}^2_{0,g_X}(y) + g_Y^2\mu_2(K) + (\hat{m}^{NW}_{g_X}(y) - \hat{m}^{NW}_{g_X}(X^1_j))^2\right)\hat{f}^0_{g_X}(y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&= g_Y^2\mu_2(K)\int K_h(X_j^1-y)^2\hat{f}^0_{g_X}(y)\,dy \\
&\quad + \int K_h(X_j^1-y)^2\left(\hat{\sigma}^2_{0,g_X}(y)+(\hat{m}^{NW}_{g_X}(y)-\hat{m}^{NW}_{g_X}(X_j^1))^2\right)\hat{f}^0_{g_X}(y)\,dy \\
&= \frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\int K_h(X_j^1-y)^2K_{g_X}(y-X_i^0)\,dy \\
&\quad + \int K_h(X_j^1-y)^2\hat{p}^0_{X_j^1,g_X}(y)\,dy \\
&= \frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\left[(Kh)^2*K_{g_X}\right](X_j^1-X_i^0)+\left[K_h*\hat{p}^0_{X_j^1,g_X}\right](X_j^1), \qquad \text{(C.16)}
\end{aligned}
$$

where $\hat{p}^0_{x,g_X}(y)=\left(\hat{\sigma}^2_{0,g_X}(y)+(\hat{m}^{NW}_{g_X}(y)-\hat{m}^{NW}_{g_X}(x))^2\right)\hat{f}^0_{g_X}(y)$.

Collecting terms (C.14) and (C.16) and plugging them in (C.15), and then plugging (C.14) and (C.15) in (C.13), Theorem 12 holds.

Corollary 2 *If K is a Gaussian kernel, then expression (4.13) can be rewritten as follows:*

$$
\begin{aligned}
MASE^*_{\tilde{m}_h,X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X_j^1)^2}\left[\frac{n_0-1}{n_0^3}\cdot\left[\sum_{i=1}^{n_0}K_h*K_{g_X}\left(X_j^1-X_i^0\right)\right.\right. \\
&\quad \left.\cdot\left(Y_i^0-\hat{m}_{g_X}\left(X_j^1\right)\right)\right]^2+\frac{1}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2*K_{g_X}\right](X_j^1-X_i^0) \\
&\quad \left.\cdot\left[Y_i^0-\hat{m}_{g_X}\left(X_j^1\right)\right]^2+\frac{g_Y^2\mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2*K_{g_X}\right](X_j^1-X_i^0)\right].
\end{aligned}
$$

Proof of Corollary 2 Carrying on with computations in expression (4.13), we have:

$$
\begin{aligned}
\hat{q}^0_{x,g_X}(z) &= \left(\hat{m}_{g_X}(z)-\hat{m}_{g_X}(x)\right)\hat{f}^0_{g_X}(z) \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\left(K_{g_X}(z-X_i^0)Y_i^0-\hat{m}_{g_X}(x)\sum_{i=1}^{n_0}K_{g_X}\left(z-X_i^0\right)\right) \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}K_{g_X}\left(z-X_i^0\right)\cdot\left(Y_i^0-\hat{m}_{g_X}(x)\right).
\end{aligned}
$$

Then,

$$\begin{aligned}
\left[K_h * \hat{q}^0_{X_j^1,g_X}\right]^2(X_j^1) &= K_h * \hat{q}^0_{X_j^1,g_X}(X_j^1)^2 \\
&= \left[K_h * \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}\left(\cdot - X_i^0\right)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right](X_j^1)^2 \\
&= \left[\frac{1}{n_0}\sum_{i=1}^{n_0}\left[K_h * K_{g_X}\left(\cdot - X_i^0\right)\right](X_j^1)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right]^2 \\
&= \left[\frac{1}{n_0}\sum_{i=1}^{n_0} K_h * K_{g_X}\left(X_j^1 - X_i^0\right)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right]^2. \quad \text{(C.17)}
\end{aligned}$$

On the other hand, using that $\hat{\sigma}^2_{0,g_X}(z) := \hat{m}_{2,g_X}(z) - \hat{m}_{g_X}(z)^2$, where $\hat{m}_{k,g_X}(z)$ is defined as $\hat{m}_{k,g_X}(z) = \dfrac{\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)(Y_i^0)^k}{\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)}$, then:

$$\begin{aligned}
\hat{p}^0_{x,g_X}(z) &= \left[\hat{\sigma}^2_{0,g_X}(z) + (\hat{m}_{g_X}(z) - \hat{m}_{g_X}(x))^2\right]\hat{f}^0_{g_X}(z) \\
&= \left[\hat{m}_{2,g_X}(z) - 2\hat{m}_{g_X}(z)\hat{m}_{g_X}(x) + \hat{m}_{g_X}(x)^2\right]\hat{f}_{g_X}(z) = \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)(Y_i^0)^2 \\
&\quad -2\hat{m}_g(x)\frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)Y_i^0 + \hat{m}_{g_X}(x)^2\frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(y - X_i^0) \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z - X_i^0)\left[\left(Y_i^0\right)^2 - 2\hat{m}_{g_X}(x)Y_i^0 + \hat{m}_{g_X}(x)^2\right].
\end{aligned}$$

Therefore, straightforward calculations lead to:

$$\begin{aligned}
(K_h)^2 * \hat{p}^0_{X_j^1,g_X}(X_j^1) &= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{g_X}(X_j^1 - X_i^0) \\
&\quad \cdot\left[\left(Y_i^0\right)^2 - 2\hat{m}_{g_X}(X_j^1)Y_i^0 + \hat{m}_{g_X}(X_j^1)^2\right] \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{g_X}(X_j^1 - X_i^0)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)^2. \quad \text{(C.18)}
\end{aligned}$$

Finally, collecting expressions (C.17) and (C.18) and plugging them in expression (4.13), Corollary 2 is concluded.

Motivation of Expression (4.17)

Consider the local linear regression estimator, $\hat{m}_h^{LL}$, and $(X_1, \ldots, X_n)$, a simple random sample and the following approximation of the local linear kernel regression estimator, given by:

$$\tilde{m}_h^{LL}(x) \;=\; \frac{\Theta^1}{\Theta^0} + \frac{\hat{\Theta}_h^1\Theta^0 - \Theta^1\hat{\Theta}_h^0}{\left(\Theta^0\right)^2},$$

where $\Theta^0 = f(x)^2\mu_2(K)$, $\Theta^1 = m(x)f(x)^2\mu_2(K)$, $\hat{\Theta}_h^0 = \hat{s}_2(x;h)\hat{s}_0(x;h) - h^2\hat{s}_1(x;h)$ and $\hat{\Theta}_h^1 = \hat{\Psi}_h^0(x;h)\hat{s}_2(x;h) - \hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)$.

In the following, the computations required to work out expression (4.17) are described in detail. On the one hand,

$$\begin{aligned}
\mathbb{E}\left[\hat{s}_1(x;h)\right] &= \mathbb{E}\left[n^{-1}h^{-2}\sum_{i=1}^{n}(X_i - x)K_h(X_i - x)\right] \\
&= \mathbb{E}\left[n^{-1}h^{-2}\sum_{i=1}^{n}\left(\frac{X_i - x}{h}\right)K\left(\frac{X_i - x}{h}\right)\right] \\
&= h^{-2}\mathbb{E}\left[\left(\frac{X_1 - x}{h}\right)K\left(\frac{X_1 - x}{h}\right)\right] \\
&= h^{-2}\int\left(\frac{y - x}{h}\right)K\left(\frac{y - x}{h}\right)f(y)\,dy \\
&= h^{-1}\int uK\left(u\right)f(x + hu)\,du. \qquad \text{(C.19)}
\end{aligned}$$

Applying Taylor approximation to f, we have

$$f(x + hu) \;=\; f(x) + huf'(x) + \frac{1}{2}(hu)^2f''(x) + o(h^2). \qquad \text{(C.20)}$$

Now plugging (C.20) in (C.19) leads to,

$$\begin{aligned}
\mathbb{E}\left[\hat{s}_1(x;h)\right] &= f'(x)\int u^2K\left(u\right)\,du + o(h) \\
&= f'(x)\mu_2(K) + o(h). \qquad \text{(C.21)}
\end{aligned}$$

Focusing now on $\hat{s}_2$,

$$\begin{aligned}
\mathbb{E}\left[\hat{s}_2(x;h)\right] &= \mathbb{E}\left[n^{-1}h^{-2}\sum_{i=1}^{n}(X_i-x)^2K_h(X_i-x)\right] \\
&= \mathbb{E}\left[n^{-1}h^{-1}\sum_{i=1}^{n}\left(\frac{X_i-x}{h}\right)^2K\left(\frac{X_i-x}{h}\right)\right] \\
&= h^{-1}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2K\left(\frac{X_1-x}{h}\right)\right] \\
&= h^{-1}\int\left(\frac{y-x}{h}\right)^2K\left(\frac{y-x}{h}\right)f(y)\,dy \\
&= \int u^2K\left(u\right)f(x+hu)\,du. \qquad \text{(C.22)}
\end{aligned}$$

Using (C.20) in (C.22), we obtain

$$\mathbb{E}\left[\hat{s}_2(x;h)\right] = f(x)\int u^2K\left(u\right)\,du + o(h^2) = f(x)\mu_2(K) + o(h^2). \qquad \text{(C.23)}$$

As for the term $\hat{\Psi}_h^0$, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\right] &= \mathbb{E}\left[n^{-1}\sum_{i=1}^{n}K_h(X_i-x)Y_i\right] = n^{-1}h^{-1}\mathbb{E}\left[\sum_{i=1}^{n}K\left(\frac{X_i-x}{h}\right)Y_i\right] \\
&= h^{-1}\mathbb{E}\left[K\left(\frac{X_1-x}{h}\right)Y_1\right] \\
&= h^{-1}\mathbb{E}\left[\mathbb{E}\left[K\left(\frac{X_1-x}{h}\right)Y_1\Big|_{X_1}\right]\right] \\
&= h^{-1}\mathbb{E}\left[K\left(\frac{X_1-x}{h}\right)\mathbb{E}\left[Y_1|_{X_1}\right]\right] \\
&= h^{-1}\mathbb{E}\left[K\left(\frac{X_1-x}{h}\right)m(X_1)\right] \\
&= h^{-1}\int K\left(\frac{y-x}{h}\right)m(y)f(y)\,dy \\
&= \int K(u)(mf)(x+hu)\,du. \qquad \text{(C.24)}
\end{aligned}$$

Using (C.20) in (C.24), we obtain

$$\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\right] \;=\; m(x)f(x) + o(h^2). \tag{C.25}$$

Considering now the remaining term, $\hat{\Psi}_h^1$,

$$\begin{aligned}
\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\right] &= \mathbb{E}\left[n^{-1}\sum_{i=1}^{n}(X_i-x)K_h(X_i-x)Y_i\right] \\
&= n^{-1}\mathbb{E}\left[\sum_{i=1}^{n}\left(\frac{X_i-x}{h}\right)K\left(\frac{X_i-x}{h}\right)Y_i\right] \\
&= \mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K\left(\frac{X_1-x}{h}\right)Y_1\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K\left(\frac{X_1-x}{h}\right)Y_1\Big|_{X_1}\right]\right] \\
&= \mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K\left(\frac{X_1-x}{h}\right)\mathbb{E}\left[Y_1\,|_{X_1}\right]\right] \\
&= \mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K\left(\frac{X_1-x}{h}\right)m(X_1)\right] \\
&= \int\left(\frac{y-x}{h}\right)K\left(\frac{y-x}{h}\right)m(y)f(y)\,dy \\
&= h\int uK(u)(mf)(x+hu)\,du.
\end{aligned}$$

Applying (C.20) to (C.26) leads to

$$\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\right] \;=\; h^2\mu_2(K)(mf)'(x) + o(h^3). \tag{C.26}$$

Expressions (C.21), (C.23), (C.25) and (C.26) give way to the approximation of the estimator given in (4.16),

$$\begin{aligned}\hat{m}_h^{LL}(x) &= \frac{\hat{\Theta}_h^1}{\hat{\Theta}_h^0}\left[\frac{\hat{\Theta}_h^0}{\Theta^0}+\left(1-\frac{\hat{\Theta}_h^0}{\Theta^0}\right)\right] = \frac{\hat{\Theta}_h^1}{\Theta^0}+\frac{\hat{\Theta}_h^1}{\hat{\Theta}_h^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)\\
&= \frac{\hat{\Theta}_h^1}{\Theta^0}+\frac{\hat{\Theta}_h^1}{\hat{\Theta}_h^0}\left[\frac{\hat{\Theta}_h^0}{\Theta^0}+\left(1-\frac{\hat{\Theta}_h^0}{\Theta^0}\right)\right]\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)\\
&= \frac{\hat{\Theta}_h^1}{\Theta^0}+\frac{\hat{\Theta}_h^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)+\frac{\hat{\Theta}_h^1}{\hat{\Theta}_h^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)^2\\
&= A_1+A_2,\end{aligned}$$

where $A_1 := \frac{\hat{\Theta}_h^1}{\Theta^0}+\frac{\hat{\Theta}_h^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)$ and $A_2 := \frac{\hat{\Theta}_h^1}{\hat{\Theta}_h^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)^2$. Now, working out A_1 term leads to

$$\begin{aligned}A_1 &= \frac{\hat{\Theta}_h^1-\Theta^1}{\Theta^0}+\frac{\Theta^1}{\Theta^0}+\frac{\hat{\Theta}_h^1-\Theta^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)+\frac{\Theta^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)\\
&= A_1'+A_2',\end{aligned}$$

where $A_1' = \frac{\hat{\Theta}_h^1-\Theta^1}{\Theta^0}+\frac{\Theta^1}{\Theta^0}+\frac{\Theta^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)$ and $A_2' = \frac{\hat{\Theta}_h^1-\Theta^1}{\Theta^0}\left(\frac{\Theta^0-\hat{\Theta}_h^0}{\Theta^0}\right)$. Finally,

$$A_1' = \frac{\Theta^1}{\Theta^0}+\frac{\hat{\Theta}_h^1\Theta^0-\Theta^1\hat{\Theta}_h^0}{\left(\Theta^0\right)^2}.$$

Then, considering that A_2 and A_2' are negligible, expression given in (4.17) is an approximation of $\hat{m}_h^{LL}(x)$.

Theorem 13 *Consider x an interior point of the support of the random variable X^0, K a symmetric bounded density function and $\{(X_1,Y_1),\ldots(X_n,Y_n)\}$ a simple* *random sample. The MSE of the proxy estimator given in (4.17) results in:*

$$
\begin{aligned}
MSE_x(h) \;=\; & \left(\Theta^0\right)^{-2} h^{-4} \left[\frac{(n-1)(n-2)(n-3)}{n^3} \left[K_h * b_x^0\right]^2 (x) \left[K_h * d_x^2\right]^2 (x) \right. \\
& + \frac{(n-1)(n-2)}{n^3} \Big(\left[(K_h)^2 * a_x^0\right](x) \left[K_h * d_x^2\right]^2 (x) \\
& + \left[(K_h)^2 * d_x^4\right](x) \left[K_h * b_x^0\right]^2 (x) \\
& +2 \left[(K_h)^2 * b_x^2\right](x) \left[K_h * b_x^0\right](x) \left[K_h * d_x^2\right](x) \Big) \\
& \left. + \frac{n-1}{n^3} \left[(K_h)^2 * a_x^2\right](x) \left[(K_h)^2 * d_x^4\right](x) \right] \\
& -2 \left(\Theta^0\right)^{-2} h^{-4} \left[\frac{(n-1)(n-2)(n-3)}{n^3} \left[K_h * b_x^0\right](x) \right. \\
& \left[K_h * d_x^2\right](x) \left[K_h * d_x^1\right](x) \left[K_h * b_x^1\right](x) \\
& + \frac{(n-1)(n-2)}{n^3} \Big[\left[(K_h)^2 * b_x^1\right](x) \left[K_h * d_x^2\right](x) \left[K_h * b_x^1\right](x) \\
& + \left[(K_h)^2 * a_x^1\right](x) \left[K_h * d_x^2\right](x) \left[K_h * d_x^1\right](x) \\
& + \left[(K_h)^2 * d_x^3\right](x) \left[K_h * b_x^1\right](x) \left[K_h * b_x^0\right](x) \\
& + \left[(K_h)^2 * b_x^3\right](x) \left[K_h * b_x^0\right](x) \left[K_h * d_x^1\right](x) \Big] \\
& + \frac{n-1}{n^3} \Big(\left[(K_h)^2 * b_x^1\right](x) \left[(K_h)^2 * b_x^3\right](x) \\
& \left. + \left[(K_h)^2 * a_x^1\right](x) \left[(K_h)^2 * d_x^3\right](x) \Big) \right] \\
& + \left(\Theta^0\right)^{-2} h^{-4} \left[\frac{(n-1)(n-2)(n-3)}{n^3} \left[K_h * d_x^1\right]^2 (x) \left[K_h * b_x^1\right]^2 (x) \right. \\
& + \frac{(n-1)(n-2)}{n^3} \Big[\left[(K_h)^2 * a_x^2\right](x) \left[K_h * d_x^1\right]^2 (x) \\
& + \left[(K_h)^2 * d_x^2\right](x) \left[K_h * b_x^1\right]^2 (x) \\
& +2 \left[(K_h)^2 * b_x^2\right](x) \left[K_h * d_x^1\right](x) \left[K_h * b_x^1\right](x) \Big] \\
& \left. + \frac{n-1}{n^3} \Big(2 \left[(K_h)^2 * b_x^2\right]^2 (x) + \left[(K_h)^2 * d_x^2\right](x) \left[(K_h)^2 * a_x^2\right](x) \Big) \right]
\end{aligned}
$$

$$-2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^0\right](x)$$
$$\left[K_h * d_x^2\right]^2(x)\left[K_h * d_x^0\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\right.$$
$$+\left[K_h * b_x^0\right](x)\left[(K_h)^2 * d_x^4\right](x)\left[K_h * d_x^0\right](x)+\left[(K_h)^2 * d_x^2\right](x)$$
$$\left.\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)+\left[(K_h)^2 * b_x^0\right](x)\left[K_h * d_x^2\right]^2(x)\right]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * b_x^4\right](x)\left[(K_h)^2 * \hat{f}_g^0\right](x)\Bigg]$$
$$+2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)$$
$$\left[K_h * d_x^1\right]^2(x)+\frac{(n-1)(n-2)}{n^3}\left[3\left[(K_h)^2 * b_x^1\right](x)\left[K_h * d_x^2\right](x)\right.$$
$$\left.\left[K_h * d_x^1\right](x)+2\left[(K_h)^2 * d_x^3\right](x)\left[K_h * b_x^0\right](x)\left[K_h * d_x^1\right](x)\right]$$
$$+3\frac{n-1}{n^3}\left[(K_h)^2 * b_x^1\right](x)\left[(K_h)^2 * d_x^3\right](x)\Bigg]$$
$$+2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^1\right](x)\left[K_h * d_x^0\right](x)$$
$$\left[K_h * d_x^1\right](x)\left[K_h * d_x^2\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * d_x^3\right](x)\left[K_h * d_x^0\right](x)\left[K_h * b_x^1\right](x)\right.$$
$$+\left[(K_h)^2 * b_x^3\right](x)\left[K_h * d_x^1\right](x)\left[K_h * d_x^0\right](x)$$
$$\left.+\left[(K_h)^2 * d_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * b_x^1\right](x)\right]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * b_x^3\right](x)\left[(K_h)^2 * d_x^1\right](x)\Bigg]$$
$$-2\left(\Theta^0\right)^{-3}\Theta^1 h^{-4}\Bigg[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * d_x^1\right]^3(x)\left[K_h * b_x^1\right](x)$$
$$+2\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^1\right](x)\left[K_h * d_x^1\right](x)\right.$$
$$\left.+\left[(K_h)^2 * b_x^2\right](x)\left[K_h * \hat{d}_x^1\right]^2(x)\right]+3\frac{n-1}{n^3}\left[(K_h)^2 * b_x^2\right](x)\left[(K_h)^2 * d_x^2\right](x)\Bigg]$$

$$+\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*d_x^2\right]^2(x)\left[K_h*d_x^0\right]^2(x)\right.$$

$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2*d_x^4\right](x)\left[K_h*d_x^0\right]^2(x)+\left[(K_h)^2*d_x^0\right](x)\right.$$

$$\left.\left[K_h*d_x^2\right]^2(x)+2\left[(K_h)^2*d_x^2\right](x)\left[K_h*d_x^2\right](x)\left[K_h*d_x^0\right](x)\right]$$

$$\left.+\frac{n-1}{n^3}\left[(K_h)^2*d_x^4\right](x)\left[(K_h)^2*\hat{f}_g^0\right](x)\right]$$

$$+h^{-4}\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*d_x^1\right]^4(x)\right.$$

$$\left.+4\frac{(n-1)(n-2)}{n^3}\left[(K_h)^2*d_x^2\right](x)\left[K_h*d_x^1\right]^2(x)+3\frac{n-1}{n^3}\left[(K_h)^2*d_x^2\right]^2(x)\right]$$

$$-2\,h^{-4}\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*d_x^2\right](x)\left[K_h*d_x^0\right](x)\right.$$

$$\left[K_h*d_x^1\right]^2(x)+\frac{(n-1)(n-2)}{n^3}\left[2\left[(K_h)^2*d_x^3\right](x)\left[K_h*d_x^0\right](x)\left[K_h*d_x^1\right](x)\right.$$

$$\left.+2\left[(K_h)^2*d_x^1\right](x)\left[K_h*d_x^2\right](x)\left[K_h*d_x^1\right](x)\right]$$

$$\left.+2\frac{n-1}{n^3}\left[(K_h)^2*d_x^3\right](x)\left[(K_h)^2*d_x^1\right](x)\right],$$

where $a_x^j(y)=(y-x)^j\left(\sigma^2(y)+m^2(y)\right)f(y)$, $b_x^j(y)=(y-x)^j m(y)f(y)$ *and* $d_x^j(y)=(y-x)^j f(y)$, *with* $j\in\mathbb{N}$.

Proof of Theorem 13 The aim is to work out a closed expression for the MSE of the proxy estimator given in (4.17), given a fixed point x.

$$\begin{aligned}
MSE_x(h) &= \mathbb{E}\left[\left(\tilde{m}_h^{LL}(x)-m(x)\right)^2\right]=\mathbb{E}\left[\left(\frac{\Theta^1}{\Theta^0}+\frac{\hat{\Theta}_h^1\Theta^0-\Theta^1\hat{\Theta}_h^0}{\left(\Theta^0\right)^2}-m(x)\right)^2\right]\\
&= \mathbb{E}\left[\left(\frac{\hat{\Theta}_h^1\Theta^0-\Theta^1\hat{\Theta}_h^0}{\left(\Theta^0\right)^2}\right)^2\right]=\mathbb{E}\left[\left(\Theta^0\right)^{-4}\left(\hat{\Theta}_h^1\Theta^0-\Theta^1\hat{\Theta}_h^0\right)^2\right]\\
&= \left(\Theta^0\right)^{-4}\mathbb{E}\left[\left(\hat{\Theta}_h^1\Theta^0-\Theta^1\hat{\Theta}_h^0\right)^2\right]\\
&= \left(\Theta^0\right)^{-4}\mathbb{E}\left[\left(\Theta^0\right)^2\left(\hat{\Psi}_h^0(x;h)^2\hat{s}_2^2(x;h)\right.\right.\\
&\quad \left.-2\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)+\hat{s}_1^2(x;h)\hat{\Psi}_h^1(x;h)^2\right)
\end{aligned}$$

$$
\begin{aligned}
& -2\Theta^0\Theta^1 \Big(\hat{\Psi}_h^0(x;h)\hat{s}_2^2(x;h)\hat{s}_0(x;h) - h^2\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1^2(x;h) \\
& -\hat{\Psi}_h^1(x;h)\hat{s}_0(x;h)\hat{s}_1(x;h)\hat{s}_2(x;h) + h^2\hat{\Psi}_h^1(x;h)\hat{s}_1^3(x;h)\Big) \\
& + \left(\Theta^1\right)^2 \left(\hat{s}_2^2(x;h)\hat{s}_0^2(x;h) + h^4\hat{s}_1^4(x;h) - 2h^2\hat{s}_2(x;h)\hat{s}_0(x;h)\hat{s}_1^2(x;h)\right)\Big] \\
= \; & \left(\Theta^0\right)^{-2} \mathbb{E}\left[\hat{\Psi}_h^0(x;h)^2\hat{s}_2^2(x;h)\right] \\
& -2\left(\Theta^0\right)^{-2} \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)\right] \\
& + \left(\Theta^0\right)^{-2} \mathbb{E}\left[\hat{s}_1^2(x;h)\hat{\Psi}_h^1(x;h)^2\right] \\
& -2\left(\Theta^0\right)^{-3}\Theta^1\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2^2(x;h)\hat{s}_0(x;h)\right] \\
& +2\left(\Theta^0\right)^{-3}\Theta^1 h^2\mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1^2(x;h)\right] \\
& +2\left(\Theta^0\right)^{-3}\Theta^1\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_0(x;h)\hat{s}_1(x;h)\hat{s}_2(x;h)\right] \\
& -2\left(\Theta^0\right)^{-3}\Theta^1 h^2\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_1^3(x;h)\right] \\
& + \left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_2^2(x;h)\hat{s}_0^2(x;h)\right] + h^4\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_1^4(x;h)\right] \\
& -2h^2\left(\Theta^1\right)^2\left(\Theta^0\right)^{-4}\mathbb{E}\left[\hat{s}_2(x;h)\hat{s}_0(x;h)\hat{s}_1^2(x;h)\right]. \qquad \text{(C.27)}
\end{aligned}
$$

We now have to work out each expectation in (C.27) separately. Splitting a four-tuple sum into different cases according to ties in the indices, computing expectations via conditional expectations and performing changes of variables in the integrals can be used to prove:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{\Psi}_h^0(x;h)^2\hat{s}_2^2(x;h)\right] \\
= \; & \mathbb{E}\left[\left(n^{-1}\sum_{i=1}^n K_h(X_i-x)Y_i\right)^2\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k-x)^2K_h(X_k-x)\right)^2\right] \\
= \; & n^{-4}h^{-4}\mathbb{E}\Bigg[\sum_{i=1}^n K_h(X_i-x)Y_i\sum_{j=1}^n K_h(X_j-x)Y_j\sum_{k=1}^n (X_k-x)^2K_h(X_k-x) \\
& \sum_{\ell=1}^n (X_\ell-x)^2K_h(X_\ell-x)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-4}h^{-4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\mathbb{E}\left[K_h(X_i-x)Y_iK_h(X_j-x)Y_j(X_k-x)^2K_h(X_k-x)\right.\\
&\quad\left.(X_\ell-x)^2K_h(X_\ell-x)\right]\\
&= h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)Y_1K_h(X_2-x)Y_2\right.\\
&\quad\left.K_h(X_3-x)(X_3-x)^2K_h(X_4-x)(X_4-x)^2\right]\\
&\quad+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1^2(X_2-x)^2K_h(X_2-x)(X_3-x)^2\right.\\
&\quad\left.K_h(X_3-x)\right]\\
&\quad+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^4K_h(X_2-x)Y_2K_h(X_3-x)Y_3\right]\\
&\quad+4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2K_h(X_2-x)Y_2\right.\\
&\quad\left.(X_3-x)^2K_h(X_3-x)\right]\\
&\quad+2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3Y_1^2(X_1-x)^2K_h(X_2-x)(X_2-x)^2\right]\\
&\quad+2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3Y_1(X_1-x)^4K_h(X_2-x)Y_2\right]\\
&\quad+h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h\left(X_1-x\right)^2Y_1^2K_h\left(X_2-x\right)^2\left(X_2-x\right)^4\right]\\
&\quad+2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2K_h(X_2-x)^2Y_2(X_2-x)^2\right]\\
&\quad+n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4Y_1^2(X_1-x)^4\right]\\
&= h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)Y_1\right]\mathbb{E}\left[K_h(X_2-x)Y_2\right]\\
&\quad\mathbb{E}\left[K_h(X_3-x)(X_3-x)^2\right]\mathbb{E}\left[K_h(X_4-x)(X_4-x)^2\right]\\
&\quad+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\\
&\quad\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\\
&\quad+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^4\right]\mathbb{E}\left[K_h(X_2-x)Y_2\right]\\
&\quad\mathbb{E}\left[K_h(X_3-x)Y_3\right]\\
&\quad+4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)Y_2\right]\\
&\quad\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]+n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4Y_1^2(X_1-x)^4\right]
\end{aligned}
$$

$$
\begin{aligned}
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3Y_1^2(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)^2\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3Y_1(X_1-x)^4\right]\mathbb{E}\left[K_h(X_2-x)Y_2\right] \\
& +h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h\left(X_1-x\right)^2Y_1^2\right]\mathbb{E}\left[K_h\left(X_2-x\right)^2\left(X_2-x\right)^4\right] \\
& +2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)^2Y_2(X_2-x)^2\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[K_h(X_3-x)(X_3-x)^2\right]\mathbb{E}\left[K_h(X_4-x)(X_4-x)^2\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^4\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[\mathbb{E}\left[K_h(X_3-x)Y_3\,|_{X_3}\right]\right] \\
& +4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\,|_{X_1}\right]\right] \\
& \mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3Y_1^2(X_1-x)^2\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)^2\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3Y_1(X_1-x)^4\,|_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^4Y_1^2(X_1-x)^4\,|_{X_1}\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[\mathbb{E}\left[\left.K_h\left(X_1-x\right)^2Y_1^2\right|_{X_1}\right]\right]\mathbb{E}\left[K_h\left(X_2-x\right)^2\left(X_2-x\right)^4\right] \\
& +2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[\mathbb{E}\left[\left.K_h(X_1-x)^2Y_1(X_1-x)^2\right|_{X_1}\right]\right] \\
& \mathbb{E}\left[\mathbb{E}\left[\left.K_h(X_2-x)^2Y_2(X_2-x)^2\right|_{X_2}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[K_h(X_3-x)(X_3-x)^2\right]\mathbb{E}\left[K_h(X_4-x)(X_4-x)^2\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^4\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[K_h(X_3-x)\mathbb{E}\left[Y_3\,|_{X_3}\right]\right] \\
& +4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right] \\
& \mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)^2\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4(X_1-x)^4\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h\left(X_1-x\right)^2\mathbb{E}\left[Y_1^2\big|_{X_1}\right]\right]\mathbb{E}\left[K_h\left(X_2-x\right)^2\left(X_2-x\right)^4\right] \\
& +2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right] \\
& \mathbb{E}\left[K_h(X_2-x)^2(X_2-x)^2\mathbb{E}\left[Y_2|_{X_2}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[K_h(X_3-x)(X_3-x)^2\right]\mathbb{E}\left[K_h(X_4-x)(X_4-x)^2\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(\sigma^2(X_1)+m^2(X_1))\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^4\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[K_h(X_3-x)m(X_3)\right] \\
& +4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2(\sigma^2(X_1)+m^2(X_1))\right] \\
& \mathbb{E}\left[K_h(X_2-x)(X_2-x)^2\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& +h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h\left(X_1-x\right)^2\left(\sigma^2(X_1)+m^2(X_1)\right)\right]\mathbb{E}\left[K_h\left(X_2-x\right)^2\left(X_2-x\right)^4\right]
\end{aligned}
$$

$$
\begin{aligned}
& +2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2m(X_1)\right] \\
& \mathbb{E}\left[K_h(X_2-x)^2(X_2-x)^2m(X_2)\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4(X_1-x)^4\left(\sigma^2(X_1)+m^2(X_1)\right)\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\left(\int K_h(y-x)m(y)f(y)\,dy\right)^2 \\
& \cdot\left(\int K_h(y-x)(y-x)^2f(y)\,dy\right)^2 \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\int K_h(y-x)^2(\sigma^2(y)+m(y)^2)f(y)\,dy \\
& \cdot\left(\int (y-x)^2K_h(y-x)f(y)\,dy\right)^2 \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\int K_h(y-x)^2(y-x)^4f(y)\,dy \\
& \cdot\left(\int K_h(y-x)m(y)f(y)\,dy\right)^2 \\
& +4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\int K_h(y-x)^2(y-x)^2m(y)f(y)\,dy \\
& \int K_h(y-x)m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy \\
& +2\frac{n-1}{n^3}h^{-4}\int K_h(y-x)^3(y-x)^2(\sigma^2(y)+m(y)^2)f(y)\,dy \\
& \int K_h(y-x)(y-x)^2f(y)\,dy \\
& +2\frac{n-1}{n^3}h^{-4}\int K_h(y-x)^3(y-x)^4m(y)f(y)\,dy\int K_h(y-x)m(y)f(y)\,dy \\
& +h^{-4}\frac{n-1}{n^3}\int K_h\left(y-x\right)^2\left(\sigma^2(y)+m^2(y)\right)f(y)\,dy \\
& \int K_h\left(y-x\right)^2\left(y-x\right)^4f(y)\,dy \\
& +2\,h^{-4}\frac{n-1}{n^3}\int K_h(y-x)^2(y-x)^2m(y)f(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
& \int K_h(y-x)^2(y-x)^2 m(y) f(y)\,dy \\
& +n^{-3}h^{-4}\int K_h(y-x)^4(y-x)^4\left(\sigma^2(y)+m(y)^2\right) f(y)\,dy \\
= \;& h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * b_x^0\right]^2(x)\left[K_h * d_x^2\right]^2(x) \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\Big(\left[(K_h)^2 * a_x^0\right](x)\left[K_h * d_x^2\right]^2(x)+\left[(K_h)^2 * d_x^4\right](x) \\
& \left[K_h * b_x^0\right]^2(x)+4\left[(K_h)^2 * b_x^2\right](x)\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)\Big) \\
& +h^{-4}\frac{n-1}{n^3}\left(2\left[(K_h)^3 * a_x^2\right](x)\left[K_h * d_x^2\right](x)+2\left[(K_h)^3 * b_x^4\right](x)\left[K_h * b_x^0\right](x)\right. \\
& \left.+\left[(K_h)^2 * a_x^0\right](x)\left[(K_h)^2 * d_x^4\right](x)+2\left[(K_h)^2 * b_x^2\right]^2(x)\right) \\
& +n^{-3}h^{-4}\left[(K_h)^4 * a_x^4\right](x), \qquad \text{(C.28)}
\end{aligned}
$$

where $a_x^j(y)=(y-x)^j\left(\sigma^2(y)+m^2(y)\right)f(y)$, $b_x^j(y)=(y-x)^j m(y)f(y)$ and $d_x^j(y)=(y-x)^j f(y)$, with $j\in\mathbb{N}$.

Carrying on with similar steps as for the previous expression, it follows:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)\right] \\
= \;& \mathbb{E}\left[\left(n^{-1}\sum_{i=1}^n K_h(X_i-x)Y_i\right)\left(n^{-1}h^{-2}\sum_{j=1}^n (X_j-x)^2 K_h(X_j-x)\right)\right. \\
& \left.\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k-x)K_h(X_k-x)\right)\left(n^{-1}\sum_{\ell=1}^n K_h(X_\ell-x)Y_\ell\right)\right] \\
= \;& n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[K_h(X_i-x)Y_i(X_j-x)^2K_h(X_j-x)\right. \\
& \left.(X_k-x)K_h(X_k-x)K_h(X_\ell-x)Y_\ell\right] \\
= \;& h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)Y_1(X_2-x)^2K_h(X_2-x)\right. \\
& \left.(X_3-x)K_h(X_3-x)(X_4-x)K_h(X_4-x)Y_4\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2(X_2-x)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& K_h(X_2-x)(X_3-x)K_h(X_3-x)Y_3] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)Y_3\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1^2(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3K_h(X_2-x)Y_2(X_3-x)K_h(X_3-x)Y_3\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3Y_1K_h(X_2-x)Y_2(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1K_h(X_2-x)Y_2(X_3-x)^2K_h(X_3-x)\right]] \\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3Y_1(X_2-x)K_h(X_2-x)Y_2\right]\right. \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2Y_1^2(X_2-x)^2K_h(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4Y_1K_h(X_2-x)Y_2\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3Y_1^2(X_1-x)^3K_h(X_2-x)(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\right]^2 \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1^2(X_2-x)^3K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)K_h(X_2-x)^2(X_2-x)^3Y_2\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4Y_1^2(X_1-x)^4\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_4-x)K_h(X_4-x)Y_4\,|_{X_4}\right]\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\,|_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)\,|_{X_3}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\mid_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\mid_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\Big] \\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3Y_1\mid_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2\mid_{X_2}\right]\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2Y_1^2\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\mid_{X_2}\right]\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^3Y_1^2(X_1-x)^3\mid_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)^2\mid_{X_1}\right]\right]^2 \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)Y_1^2\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_1-x)\Big|_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)^2(X_2-x)^3Y_2\Big|_{X_2}\right]\right]\Bigg] \\
& +n^{-3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^4Y_1^2(X_1-x)^4\mid_{X_1}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)\mathbb{E}\left[Y_4\mid_{X_4}\right]\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1\mid_{X_1}\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\right. \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right]\Big] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1^2\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\mid_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\mid_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\Big] \\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]^2 \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1^2\big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)^2(X_2-x)^3\mathbb{E}\left[Y_2|_{X_2}\right]\right]\Big] \\
& +n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4(X_1-x)^4\mathbb{E}\left[Y_1^2\,|_{X_1}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)m(X_4)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2m(X_1)\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)m(X_1)\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\right. \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right]] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)(\sigma^2(X_1)+m^2(X_1))\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^3m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\Big]
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right]\right. \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^2(\sigma^2(X_1)+m^2(X_1))\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^4m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^3(X_1-x)^3(\sigma^2(X_1)+m^2(X_1))\right]\mathbb{E}\left[K_h(X_2-x)(X_2-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]^2 \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1^2\big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)^2(X_2-x)^3\mathbb{E}\left[Y_2|_{X_2}\right]\right]\Big] \\
& +n^{-3}h^{-4}\mathbb{E}\left[K_h(X_1-x)^4(X_1-x)^4(\sigma^2(X_1)+m^2(X_1))\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\int K_h(y-x)m(y)f(y)\,dy\int(y-x)^2K_h(y-x) \\
& f(y)\,dy\int(y-x)K_h(y-x)f(y)\,dy\int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\int K_h(y-x)^2(y-x)^2m(y)f(y)\,dy\right. \\
& \int(y-x)K_h(y-x)f(y)\,dy\int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& +\int K_h(y-x)^2(y-x)m(y)f(y)\,dy\int(y-x)^2K_h(y-x)f(y)\,dy \\
& \int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& +\int K_h(y-x)^2(y-x)(\sigma^2(y)+m^2(y))f(y)\,dy\int(y-x)^2K_h(y-x)f(y)\,dy \\
& \int(y-x)K_h(y-x)f(y)\,dy \\
& +\int K_h(y-x)^2(y-x)^3f(y)\,dy\int K_h(y-x)m(y)f(y)\,dy \\
& \int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& +\int K_h(y-x)^2(y-x)^3m(y)f(y)\,dy\int K_h(y-x)m(y)f(y)\,dy \\
& \int(y-x)K_h(y-x)f(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
& + \int K_h(y-x)^2 (y-x)^2 m(y) f(y)\, dy \int K_h(y-x) m(y) f(y)\, dy \\
& \int (y-x)^2 K_h(y-x) f(y)\, dy \Big] \\
& + h^{-4} \frac{n-1}{n^3} \left[\int K_h(y-x)^3 (y-x)^3 m(y) f(y)\, dy \right. \\
& \int (y-x) K_h(y-x) m(y) f(y)\, dy \\
& + \int K_h(y-x)^3 (y-x)^2 (\sigma^2(y) + m^2(y)) f(y)\, dy \int (y-x)^2 K_h(y-x) f(y)\, dy \\
& + \int K_h(y-x)^3 (y-x)^4 m(y) f(y)\, dy \int K_h(y-x) m(y) f(y)\, dy \\
& + \int K_h(y-x)^3 (y-x)^3 (\sigma^2(y) + m^2(y)) f(y)\, dy \int K_h(y-x)(y-x) f(y)\, dy \\
& + \left(\int K_h(y-x)^2 (y-x)^2 m(y) f(y)\, dy \right)^2 \\
& + \int K_h(y-x)^2 (y-x) \left(\sigma^2(y) + m^2(y) \right) f(y)\, dy \int (y-x)^3 K_h(y-x)^2 f(y)\, dy \\
& \left. + \int K_h(y-x)^2 (y-x) m(y) f(y)\, dy \int K_h(y-x)^2 (y-x)^3 m(y) f(y)\, dy \right] \\
& + n^{-3} h^{-4} \int K_h(y-x)^4 (y-x)^4 (\sigma^2(y) + m^2(y)) f(y)\, dy \\
= \; & h^{-4} \frac{(n-1)(n-2)(n-3)}{n^3} \left[K_h * b_x^0 \right](x) \left[K_h * d_x^2 \right](x) \left[K_h * d_x^1 \right](x) \\
& \left[K_h * b_x^1 \right](x) + h^{-4} \frac{(n-1)(n-2)}{n^3} \Big[\left[(K_h)^2 * b_x^2 \right](x) \left[K_h * d_x^1 \right](x) \left[K_h * b_x^1 \right](x) \\
& + \left[(K_h)^2 * b_x^1 \right](x) \left[K_h * d_x^2 \right](x) \left[K_h * b_x^1 \right](x) \\
& + \left[(K_h)^2 * a_x^1 \right](x) \left[K_h * d_x^2 \right](x) \left[K_h * d_x^1 \right](x) \\
& + \left[(K_h)^2 * d_x^3 \right](x) \left[K_h * b_x^1 \right](x) \left[K_h * b_x^0 \right](x) \\
& + \left[(K_h)^2 * b_x^3 \right](x) \left[K_h * b_x^0 \right](x) \left[K_h * d_x^1 \right](x) \\
& + \left[(K_h)^2 * b_x^2 \right](x) \left[K_h * b_x^0 \right](x) \left[K_h * d_x^2 \right](x) \Big] \\
& + h^{-4} \frac{n-1}{n^3} \Big[\left[(K_h)^3 * b_x^3 \right](x) \left[K_h * b_x^1 \right](x) + \left[(K_h)^3 * a_x^2 \right](x) \left[K_h * d_x^2 \right](x)
\end{aligned}
$$

$$+\left[(K_h)^3 * b_x^4\right](x)\left[K_h * b_x^0\right](x) + \left[(K_h)^3 * a_x^3\right](x)\left[K_h * d_x^1\right](x)$$
$$+\left[(K_h)^2 * b_x^2\right]^2(x) + \left[(K_h)^2 * b_x^1\right](x)\left[(K_h)^2 * b_x^3\right](x)$$
$$+\left[(K_h)^2 * a_x^1\right](x)\left[(K_h)^2 * d_x^3\right](x)\Big] + n^{-3}h^{-4}\left[(K_h)^4 * a_x^4\right](x). \qquad (C.29)$$

Following similar steps as for the previous expressions leads to:

$$\begin{aligned}
&\mathbb{E}\left[\hat{\Psi}_h^1(x;h)^2\hat{s}_1^2(x;h)\right]\\
=\ &\mathbb{E}\left[\left(n^{-1}\sum_{i=1}^n (X_i-x)K_h(X_i-x)Y_i\right)^2\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k-x)K_h(X_k-x)\right)^2\right]\\
=\ &n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[(X_i-x)K_h(X_i-x)Y_i(X_j-x)K_h(X_j-x)Y_j\right.\\
&\left.(X_k-x)K_h(X_k-x)(X_\ell-x)K_h(X_\ell-x)\right]\\
=\ &h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)(X_2-x)K_h(X_2-x)\right.\\
&\left.(X_3-x)K_h(X_3-x)Y_3(X_4-x)K_h(X_4-x)Y_4\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1^2(X_2-x)K_h(X_2-x)\right.\right.\\
&\left.(X_3-x)K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)K_h(X_2-x)Y_2\right.\\
&\left.(X_3-x)K_h(X_3-x)Y_3\right]+4\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1(X_2-x)K_h(X_2-x)\right.\\
&\left.\left.(X_3-x)K_h(X_3-x)Y_3\right]\right]+\ h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1\right.\right.\\
&\left.(X_2-x)K_h(X_2-x)Y_2\right]+2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1^2(X_2-x)K_h(X_2-x)\right]\\
&+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2Y_2\right]\\
&\left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)^2Y_2^2\right]\right]\\
&+n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1^2\right]\\
=\ &h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\\
&\mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right]\mathbb{E}\left[\mathbb{E}\left[(X_4-x)K_h(X_4-x)Y_4\,|_{X_4}\right]\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1^2\,|_{X_1}\right]\right]\right.
\end{aligned}$$

$$
\begin{aligned}
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2 \mid_{X_2}\right]\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3 \mid_{X_3}\right]\right] \\
& +4\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2Y_1 \mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3 \mid_{X_3}\right]\right]\Big] \\
& + h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1 \mid_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2 \mid_{X_2}\right]\right] \\
& +2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1^2 \mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2Y_2\Big|_{X_2}\right]\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2Y_2^2\Big|_{X_2}\right]\right]\Big] \\
& +n^{-3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1^2 \mid_{X_1}\right]\right] \\
= \; & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3 \mid_{X_3}\right]\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)\mathbb{E}\left[Y_4 \mid_{X_4}\right]\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1^2 \mid_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2 \mid_{X_2}\right]\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3 \mid_{X_3}\right]\right] \\
& +4\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2\mathbb{E}\left[Y_1 \mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3 \mid_{X_3}\right]\right]\Big] \\
& + h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\mathbb{E}\left[Y_1 \mid_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2 \mid_{X_2}\right]\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\mathbb{E}\left[Y_1^2 \mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\mathbb{E}\left[Y_2|_{X_2}\right]\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\mathbb{E}\left[Y_2^2\Big|_{X_2}\right]\right]\Big] \\
& +n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\mathbb{E}\left[Y_1^2 \mid_{X_1}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)m(X_4)\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2(\sigma^2(X_1)+m^2(X_1))\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] \\
& +4\mathbb{E}\left[K_h(X_1-x)^2(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \left.\mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right]\right]+h^{-4}\frac{n-1}{n^3} \\
& \left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right]\right. \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3(\sigma^2(X_1)+m^2(X_1))\right] \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2m(X_2)\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\left(\sigma^2(X_2)+m^2(X_2)\right)\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4(\sigma^2(X_1)+m^2(X_1))\right] \\
= \quad & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left(\int(y-x)K_h(y-x)f(y)\,dy\right)^2 \\
& \cdot\left(\int(y-x)K_h(y-x)m(y)f(y)\,dy\right)^2 \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\int K_h(y-x)^2(y-x)^2(\sigma^2(y)+m^2(y))f(y)\,dy\right. \\
& \cdot\left(\int(y-x)K_h(y-x)f(y)\,dy\right)^2 \\
& +\int(y-x)^2K_h(y-x)^2f(y)\,dy\left(\int(y-x)K_h(y-x)m(y)f(y)\,dy\right)^2 \\
& +4\int(y-x)^2K_h(y-x)^2m(y)f(y)\,dy\int(y-x)K_h(y-x)f(y)\,dy \\
& \left.\int(y-x)K_h(y-x)m(y)f(y)\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-4}\frac{n-1}{n^3}\left[2\int (y-x)^3K_h(y-x)^3m(y)f(y)\,dy\right.\\
& \int (y-x)K_h(y-x)m(y)f(y)\,dy\\
& +2\int (y-x)^3K_h(y-x)^3(\sigma^2(y)+m^2(y))f(y)\,dy\int (y-x)K_h(y-x)f(y)\,dy\\
& +2\int (y-x)^2K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^2K_h(y-x)^2m(y)f(y)\,dy\\
& \left.+\int (y-x)^2K_h(y-x)^2f(y)\,dy\int (y-x)^2K_h(y-x)^2\left(\sigma^2(y)+m^2(y)\right)f(y)\,dy\right]\\
& +n^{-3}h^{-4}\int (y-x)^4K_h(y-x)^4(\sigma^2(y)+m^2(y))f(y)\,dy\\
= \;& n^{-3}h^{-4}\left[(K_h)^4 * a_x^4\right](x)\\
& +h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * d_x^1\right]^2(x)\left[K_h * b_x^1\right]^2(x)\\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * a_x^2\right](x)\left[K_h * d_x^1\right]^2(x)\right.\\
& \left.+\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^1\right]^2(x)+4\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^1\right](x)\left[K_h * b_x^1\right](x)\right]\\
& +h^{-4}\frac{n-1}{n^3}\left[2\left[(K_h)^3 * b_x^3\right](x)\left[K_h * b_x^1\right](x)+2\left[(K_h)^3 * a_x^3\right](x)\left[K_h * d_x^1\right](x)\right.\\
& \left.+2\left[(K_h)^2 * b_x^2\right]^2(x)+\left[(K_h)^2 * d_x^2\right](x)\left[(K_h)^2 * a_x^2\right](x)\right]. \qquad \text{(C.30)}
\end{aligned}
$$

Splitting a four-tuple sum into different cases according to ties in the indices, computing expectations via conditional expectations and carrying out changes of variables in the integrals, it follows that:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{s}_0^2(x;h)\hat{s}_2^2(x;h)\right]\\
= \;& \mathbb{E}\left[\left(n^{-1}\sum_{i=1}^n K_h(X_i-x)\right)^2\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k-x)^2K_h(X_k-x)\right)^2\right]\\
= \;& n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[K_h(X_i-x)K_h(X_j-x)(X_k-x)^2K_h(X_k-x)\right.
\end{aligned}
$$

$$
\begin{aligned}
& (X_\ell - x)^2 K_h(X_\ell - x)\big] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)(X_2-x)^2K_h(X_2-x)\right. \\
& \left.K_h(X_3-x)K_h(X_4-x)\right] + n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^2K_h(X_2-x)K_h(X_3-x)\right]\right. \\
& +\mathbb{E}\left[K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)(X_3-x)^2K_h(X_3-x)\right] + \\
& \left.+4\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)K_h(X_3-x)\right]\right] \\
& + h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3K_h(X_2-x)\right]\right. \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3(X_2-x)^2K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^2K_h(X_2-x)^2\right]\right] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[K_h(X_3-x)\right]\mathbb{E}\left[K_h(X_4-x)\right] + n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]\right. \\
& +\mathbb{E}\left[K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] + \\
& \left.+4\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]\right] \\
& + h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)\right]\right. \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)^2\right]\right] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left(\int (y-x)^2K_h(y-x)f(y)\,dy\right)^2 \\
& \cdot\left(\int K_h(y-x)f(y)\,dy\right)^2 + n^{-3}h^{-4}\int (y-x)^4K_h(y-x)^4f(y)\,dy \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\int (y-x)^4K_h(y-x)^2f(y)\,dy\left(\int K_h(y-x)f(y)\,dy\right)^2\right.
\end{aligned}
$$

$$
\begin{aligned}
& + \int K_h(y-x)^2 f(y)\,dy \left(\int (y-x)^2 K_h(y-x) f(y)\,dy \right)^2 \\
& +4 \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy \\
& \int K_h(y-x) f(y)\,dy \Bigg] \\
& + h^{-4} \frac{n-1}{n^3} \left[2 \int (y-x)^4 K_h(y-x)^3 f(y)\,dy \int K_h(y-x) f(y)\,dy \right. \\
& +2 \int (y-x)^2 K_h(y-x)^3 f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy \\
& +2 \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \\
& \left. + \int (y-x)^4 K_h(y-x)^2 f(y)\,dy \int K_h(y-x)^2 f(y)\,dy \right] \\
= \; & h^{-4} \frac{(n-3)(n-2)(n-1)}{n^3} \left[K_h * d_x^2 \right]^2 (x) \left[K_h * d_x^0 \right]^2 (x) \\
& + h^{-4} \frac{n-1}{n^3} \left[2 \left[(K_h)^3 * d_x^4 \right] (x) \left[K_h * d_x^0 \right] (x) + 2 \left[(K_h)^3 * d_x^2 \right] (x) \right. \\
& \left. \left[K_h * d_x^2 \right] (x) + 2 \left[(K_h)^2 * d_x^2 \right]^2 (x) + \left[(K_h)^2 * d_x^4 \right] (x) \left[(K_h)^2 * f \right] (x) \right] \\
& + n^{-3} h^{-4} \left[(K_h)^4 * d_x^4 \right] (x) + h^{-4} \frac{(n-2)(n-1)}{n^3} \left[\left[(K_h)^2 * d_x^4 \right] (x) \right. \\
& \left[K_h * d_x^0 \right]^2 (x) + \left[(K_h)^2 * d_x^0 \right] (x) \left[K_h * d_x^2 \right]^2 (x) \\
& \left. +4 \left[(K_h)^2 * d_x^2 \right] (x) \left[K_h * d_x^2 \right] (x) \left[K_h * d_x^0 \right] (x) \right]. \qquad \text{(C.31)}
\end{aligned}
$$

Similar steps as those used in the previous expressions lead to:

$$
\begin{aligned}
& \mathbb{E} \left[\hat{s}_1^4(x;h) \right] \\
= \; & \mathbb{E} \left[\left(n^{-1} h^{-2} \sum_{i=1}^{n} (X_i - x) K_h(X_i - x) \right)^4 \right] \\
= \; & n^{-4} h^{-8} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \mathbb{E} \left[(X_i - x) K_h(X_i - x) \right.
\end{aligned}
$$

$$
\begin{aligned}
& (X_j - x)K_h(X_j - x)(X_k - x)K_h(X_k - x) \\
& (X_\ell - x)K_h(X_\ell - x)] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1 - x)K_h(X_1 - x)(X_2 - x)K_h(X_2 - x)\right. \\
& \left.(X_3 - x)K_h(X_3 - x)(X_4 - x)K_h(X_4 - x)\right] \\
& + n^{-3}h^{-4}\mathbb{E}\left[(X_1 - x)^4 K_h(X_1 - x)^4\right] + 6\,h^{-8}\frac{(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[(X_1 - x)^2 K_h(X_1 - x)^2 (X_2 - x)K_h(X_2 - x)(X_3 - x)K_h(X_3 - x)\right] \\
& + h^{-8}\frac{n-1}{n^3}\left[4\mathbb{E}\left[(X_1 - x)^3 K_h(X_1 - x)^3 (X_2 - x)K_h(X_2 - x)\right]\right. \\
& \left. + 3\mathbb{E}\left[(X_1 - x)^2 K_h(X_1 - x)^2 (X_2 - x)^2 K_h(X_2 - x)^2\right]\right] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1 - x)K_h(X_1 - x)\right]\mathbb{E}\left[(X_2 - x)K_h(X_2 - x)\right] \\
& \mathbb{E}\left[(X_3 - x)K_h(X_3 - x)\right]\mathbb{E}\left[(X_4 - x)K_h(X_4 - x)\right] \\
& + n^{-3}h^{-4}\mathbb{E}\left[(X_1 - x)^4 K_h(X_1 - x)^4\right] \\
& + 6\,h^{-8}\frac{(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1 - x)^2 K_h(X_1 - x)^2\right]\mathbb{E}\left[(X_2 - x)K_h(X_2 - x)\right] \\
& \mathbb{E}\left[(X_3 - x)K_h(X_3 - x)\right] \\
& + h^{-8}\frac{n-1}{n^3}\left[4\mathbb{E}\left[(X_1 - x)^3 K_h(X_1 - x)^3\right]\mathbb{E}\left[(X_2 - x)K_h(X_2 - x)\right]\right. \\
& \left. + 3\mathbb{E}\left[(X_1 - x)^2 K_h(X_1 - x)^2\right]\mathbb{E}\left[(X_2 - x)^2 K_h(X_2 - x)^2\right]\right] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\left(\int (y - x)K_h(y - x)f(y)\,dy\right)^4 \\
& + n^{-3}h^{-4}\int (y - x)^4 K_h(y - x)^4 f(y)\,dy \\
& + 6\,h^{-8}\frac{(n-2)(n-1)}{n^3}\int (y - x)^2 K_h(y - x)^2 f(y)\,dy \\
& \cdot\left(\int (y - x)K_h(y - x)f(y)\,dy\right)^2 \\
& + h^{-8}\frac{n-1}{n^3}\left[4\int (y - x)^3 K_h(y - x)^3 f(y)\,dy\int (y - x)K_h(y - x)f(y)\,dy\right. \\
& \left. + 3\int (y - x)^2 K_h(y - x)^2 f(y)\,dy\int (y - x)^2 K_h(y - x)^2 f(y)\,dy\right]
\end{aligned}
$$

$$= h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * d_x^1\right]^4(x) + n^{-3}h^{-4}\left[(K_h)^4 * d_x^4\right](x)$$
$$+\, h^{-8}\frac{n-1}{n^3}\left[4\left[(K_h)^3 * d_x^3\right](x)\left[K_h * d_x^1\right](x) + 3\left[(K_h)^2 * d_x^2\right]^2(x)\right]$$
$$+6\, h^{-8}\frac{(n-2)(n-1)}{n^3}\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^1\right]^2(x). \tag{C.32}$$

Carrying on with similar computations worked out in previous expressions, it follows:

$$\mathbb{E}\left[\hat{s}_1^2(x;h)\hat{s}_0(x;h)\hat{s}_2(x;h)\right]$$
$$= \mathbb{E}\left[\left(n^{-1}h^{-2}\sum_{i=1}^n (X_i-x)K_h(X_i-x)\right)^2 n^{-1}\sum_{k=1}^n K_h(X_k-x)\right.$$
$$\left. n^{-1}h^{-2}\sum_{\ell=1}^n (X_\ell-x)^2K_h(X_\ell-x)\right]$$
$$= n^{-4}h^{-6}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[(X_i-x)K_h(X_i-x)(X_j-x)K_h(X_j-x)\right.$$
$$\left. K_h(X_k-x)(X_\ell-x)^2K_h(X_\ell-x)\right]$$
$$= h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)K_h(X_2-x)(X_3-x)\right.$$
$$\left. K_h(X_3-x)(X_4-x)K_h(X_4-x)\right] + n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\right]$$
$$+h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)K_h(X_2-x)(X_3-x)\right.\right.$$
$$\left. K_h(X_3-x)\right] + 2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)\right]$$
$$+2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)\right]$$
$$\left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)K_h(X_3-x)\right]\right]$$
$$+h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3(X_2-x)^2K_h(X_2-x)\right]\right.$$
$$+2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3(X_2-x)K_h(X_2-x)\right] + \mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right.$$
$$\left. K_h(X_2-x)\right] + \mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)^2\right]$$

$$
\begin{aligned}
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2(X_2-x)K_h(X_2-x)^2\right]\Big] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)\right]\mathbb{E}\left[K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)\right] \\
& +n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\right] \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\Big[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right] \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]\Big] \\
& +h^{-6}\frac{n-1}{n^3}\Big[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)^2\right]\Big] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\int (y-x)^2K_h(y-x)f(y)\,dy\int K_h(y-x)f(y)\,dy \\
& \left(\int (y-x)K_h(y-x)f(y)\,dy\right)^2 + n^{-3}h^{-6}\int (y-x)^4K_h(y-x)^4f(y)\,dy \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\int (y-x)^2K_h(y-x)^2f(y)\,dy\right. \\
& \left(\int (y-x)K_h(y-x)f(y)\,dy\right)^2 + 2\int (y-x)^3K_h(y-x)^2f(y)\,dy \\
& \int K_h(y-x)f(y)\,dy\int (y-x)K_h(y-x)f(y)\,dy \\
& +2\int (y-x)K_h(y-x)^2f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy \\
& \int (y-x)K_h(y-x)f(y)\,dy + \int (y-x)^2K_h(y-x)^2f(y)\,dy \\
& \left.\int (y-x)^2K_h(y-x)f(y)\,dy\int K_h(y-x)f(y)\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-6}\frac{n-1}{n^3}\left[\int (y-x)^2 K_h(y-x)^3 f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy\right. \\
& +2\int (y-x)^3 K_h(y-x)^3 f(y)\,dy \int (y-x) K_h(y-x) f(y)\,dy \\
& +\int (y-x)^4 K_h(y-x)^3 f(y)\,dy \int K_h(y-x) f(y)\,dy \\
& +\int (y-x)^2 K_h(y-x)^2 f(y)\,dy \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \\
& \left. +2\int (y-x)^3 K_h(y-x)^2 f(y)\,dy \int (y-x) K_h(y-x)^2 f(y)\,dy\right] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\left[K_h * d_x^1\right]^2(x) \\
& +n^{-3}h^{-6}\left[(K_h)^4 * d_x^4\right](x) \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^1\right]^2(x)\right. \\
& +2\left[(K_h)^2 * d_x^3\right](x)\left[K_h * d_x^0\right](x)\left[K_h * d_x^1\right](x) \\
& +2\left[(K_h)^2 * d_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^1\right](x) \\
& \left.+\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\left[(K_h)^3 * d_x^2\right](x)\left[K_h * d_x^2\right](x) + 2\left[(K_h)^3 * d_x^3\right](x)\left[K_h * d_x^1\right](x)\right. \\
& +\left[(K_h)^3 * d_x^4\right](x)\left[K_h * d_x^0\right](x) + \left[(K_h)^2 * d_x^2\right]^2(x) \\
& \left.+2\left[(K_h)^2 * d_x^3\right](x)\left[(K_h)^2 * d_x^1\right](x)\right]. \qquad \text{(C.33)}
\end{aligned}
$$

Carrying out analogous calculations as in previous steps, it leads to:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2^2(x;h)\hat{s}_0(x;h)\right] \\
= \; & \mathbb{E}\left[n^{-1}\sum_{i=1}^n K_h(X_i-x)Y_i\left(n^{-1}h^{-2}\sum_{j=1}^n (X_j-x)^2 K_h(X_j-x)\right)^2\right. \\
& \left. n^{-1}\sum_{\ell=1}^n K_h(X_\ell-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-4}h^{-4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\mathbb{E}\left[K_h(X_i-x)Y_i(X_j-x)^2K_h(X_j-x)\right.\\
&\quad\left.(X_k-x)^2K_h(X_k-x)K_h(X_\ell-x)\right]\\
&= h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[K_h(X_1-x)Y_1(X_2-x)^2K_h(X_2-x)\right.\\
&\quad\left.(X_3-x)^2K_h(X_3-x)K_h(X_4-x)\right]+n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\right]\\
&\quad+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)\right.\right.\\
&\quad\left.K_h(X_3-x)\right]+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2K_h(X_2-x)Y_2(X_3-x)^2\right.\\
&\quad\left.K_h(X_3-x)\right]+\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)(X_3-x)^2\right.\\
&\quad\left.\left.K_h(X_3-x)\right]+\mathbb{E}\left[K_h(X_1-x)Y_1(X_2-x)^4K_h(X_2-x)^2K_h(X_3-x)\right]\right]\\
&\quad+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3Y_1K_h(X_2-x)\right]\right.\\
&\quad+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3K_h(X_2-x)Y_2\right]\\
&\quad+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3Y_1(X_2-x)^2K_h(X_2-x)\right]\\
&\quad+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right]\\
&\quad\left.+\mathbb{E}\left[K_h(X_1-x)^2Y_1(X_2-x)^4K_h(X_2-x)^2\right]\right]\\
&= h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\\
&\quad\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\mathbb{E}\left[K_h(X_4-x)\right]\\
&\quad+n^{-3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\,|_{X_1}\right]\right]\\
&\quad+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\right.\\
&\quad\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]\\
&\quad+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\\
&\quad\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]+\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\\
&\quad\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\\
&\quad\left.+\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\mathbb{E}\left[K_h(X_3-x)\right]\right]\\
&\quad+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\right]\right.\\
&\quad+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\\
&\quad+2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)^2Y_1\big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\Big] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\mathbb{E}\left[K_h(X_4-x)\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\mathbb{E}\left[Y_1\mid_{X_1}\right]\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\Big[2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right] \\
& \mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\mid_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\mathbb{E}\left[K_h(X_3-x)\right]\Big] \\
& +h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\mid_{X_2}\right]\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\Big] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]\mathbb{E}\left[K_h(X_4-x)\right] \\
& +n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4m(X_1)\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\Big[2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\mathbb{E}\left[K_h(X_3-x)\right]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)\right]\right.\\
&+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right]\\
&+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\\
&+2\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]\\
&\left.+\mathbb{E}\left[K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^2\right]\right]\\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\int K_h(y-x)m(y)f(y)\,dy\\
&\left(\int (y-x)^2K_h(y-x)f(y)\,dy\right)^2\int K_h(y-x)f(y)\,dy\\
&+n^{-3}h^{-4}\int (y-x)^4K_h(y-x)^4m(y)f(y)\,dy\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\int (y-x)^2K_h(y-x)^2m(y)f(y)\,dy\right.\\
&\int (y-x)^2K_h(y-x)f(y)\,dy\int K_h(y-x)f(y)\,dy\\
&+2\int (y-x)^2K_h(y-x)^2f(y)\,dy\\
&\int K_h(y-x)m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy\\
&+\int K_h(y-x)^2m(y)f(y)\,dy\left(\int (y-x)^2K_h(y-x)f(y)\,dy\right)^2\\
&\left.+\int K_h(y-x)m(y)f(y)\,dy\int (y-x)^4K_h(y-x)^2f(y)\,dy\int K_h(y-x)f(y)\,dy\right]\\
&+h^{-4}\frac{n-1}{n^3}\left[\int (y-x)^4K_h(y-x)^3m(y)f(y)\,dy\int K_h(y-x)f(y)\,dy\right.\\
&+\int (y-x)^4K_h(y-x)^3f(y)\,dy\int K_h(y-x)m(y)f(y)\,dy\\
&+2\int (y-x)^2K_h(y-x)^3m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy\\
&+2\int (y-x)^2K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^2K_h(y-x)^2f(y)\,dy\\
&\left.+\int K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^4K_h(y-x)^2f(y)\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right]^2(x)\left[K_h * d_x^0\right](x) \\
& + n^{-3}h^{-4}\left[(K_h)^4 * b_x^4\right](x) \\
& + h^{-4}\frac{(n-2)(n-1)}{n^3}\Big[2\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x) \\
& + \left[K_h * b_x^0\right](x)\left[(K_h)^2 * d_x^4\right](x) + 2\left[(K_h)^2 * d_x^2\right](x) \\
& \left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x) + \left[(K_h)^2 * b_x^0\right](x)\left[K_h * d_x^2\right]^2(x)\Big] \\
& + h^{-4}\frac{n-1}{n^3}\Big[\left[(K_h)^3 * b_x^4\right](x)\left[K_h * d_x^0\right](x) + \left[(K_h)^3 * d_x^4\right](x)\left[K_h * b_x^0\right](x) \\
& + 2\left[(K_h)^3 * b_x^2\right](x)\left[K_h * d_x^2\right](x) + 2\left[(K_h)^2 * b_x^2\right](x)\left[(K_h)^2 * d_x^2\right](x) \\
& + \left[(K_h)^2 * b_x^4\right](x)\left[(K_h)^2 * f\right](x)\Big]. \qquad \text{(C.34)}
\end{aligned}
$$

Following similar steps as for the previous expression, it follows:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\hat{s}_1^2(x;h)\right] \\
= \;& \mathbb{E}\Bigg[n^{-1}\sum_{i=1}^n K_h(X_i - x)Y_i\, n^{-1}h^{-2}\sum_{j=1}^n (X_j - x)^2 K_h(X_j - x) \\
& \left(n^{-1}h^{-2}\sum_{k=1}^n (X_k - x)K_h(X_k - x)\right)^2\Bigg] \\
= \;& n^{-4}h^{-6}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[K_h(X_i - x)Y_i(X_j - x)^2K_h(X_j - x)\right. \\
& \left.(X_k - x)K_h(X_k - x)(X_\ell - x)K_h(X_\ell - x)\right] \\
= \;& n^{-3}h^{-6}\mathbb{E}\left[(X_1 - x)^4K_h(X_1 - x)^4Y_1\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[K_h(X_1 - x)Y_1(X_2 - x)^2K_h(X_2 - x)(X_3 - x)K_h(X_3 - x)\right. \\
& \left.(X_4 - x)K_h(X_4 - x)\right] \\
& + h^{-6}\frac{(n-2)(n-1)}{n^3}\Big[\mathbb{E}\left[(X_1 - x)^2K_h(X_1 - x)^2Y_1(X_2 - x)K_h(X_2 - x)\right. \\
& \left.(X_3 - x)K_h(X_3 - x)\right] \\
& + \mathbb{E}\left[(X_1 - x)^2K_h(X_1 - x)^2K_h(X_2 - x)Y_2(X_3 - x)^2K_h(X_3 - x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2K_h(X_2-x)Y_2(X_3-x)K_h(X_3-x)\right]\Big] \\
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1(X_2-x)K_h(X_2-x)\right]\right. \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)(X_2-x)^2K_h(X_2-x)^3Y_2\right] \\
& +\mathbb{E}\left[K_h(X_1-x)Y_1(X_2-x)^4K_h(X_2-x)^3\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1(X_2-x)^3K_h(X_2-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\,|_{X_1}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[(X_4-x)K_h(X_4-x)\right] \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right] \\
& \mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]+2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right] \\
& \left.\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\right. \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^3Y_2\,|_{X_2}\right]\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^3\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+2\mathbb{E}\left[\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\mathbb{E}\left[(X_4-x)K_h(X_4-x)\right] \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right] \\
& \mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]+2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right] \\
& \left.\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\right. \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^3\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
& +\mathbb{E}\left[K_h(X_1-x)\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^3\right]\Big] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right]\Big] \\
= \; & n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4m(X_1)\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[(X_4-x)K_h(X_4-x)\right]+h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \Big[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)\right. \\
& \left.K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
& \mathbb{E}\left[(X_3-x)^2K_h(X_3-x)\right]+2\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2m(X_1)\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\Big] \\
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\right. \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^3m(X_2)\right] \\
& +\mathbb{E}\left[K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[(X_2-x)^4K_h(X_2-x)^3\right]\Big]
\end{aligned}
$$

$$
\begin{aligned}
& +\mathbb{E}\left[(X_1-x)^2 K_h(X_1-x)^2 m(X_1)\right]\mathbb{E}\left[(X_2-x)^2 K_h(X_2-x)^2\right] \\
& +2\mathbb{E}\left[(X_1-x) K_h(X_1-x)^2 m(X_1)\right]\mathbb{E}\left[(X_2-x)^3 K_h(X_2-x)^2\right]\Big] \\
= \;& n^{-3}h^{-6}\int (y-x)^4 K_h(y-x)^4 m(y) f(y)\,dy + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \int K_h(y-x) m(y) f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy \\
& \cdot\left(\int (y-x) K_h(y-x) f(y)\,dy\right)^2 + h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \Bigg[\int (y-x)^2 K_h(y-x)^2 m(y) f(y)\,dy \left(\int (y-x) K_h(y-x) f(y)\,dy\right)^2 \\
& + \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \\
& \int K_h(y-x) m(y) f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy \\
& +2\int (y-x) K_h(y-x)^2 m(y) f(y)\,dy \int (y-x)^2 K_h(y-x) f(y)\,dy \\
& \int (y-x) K_h(y-x) f(y)\,dy. + 2\int (y-x)^3 K_h(y-x)^2 f(y)\,dy \\
& \int K_h(y-x) m(y) f(y)\,dy \int (y-x) K_h(y-x) f(y)\,dy\Bigg] \\
& +h^{-6}\frac{n-1}{n^3}\Bigg[2\int (y-x)^3 K_h(y-x)^3 m(y) f(y)\,dy \int (y-x) K_h(y-x) f(y)\,dy \\
& + \int (y-x)^2 K_h(y-x) f(y)\,dy \int (y-x)^2 K_h(y-x)^3 m(y) f(y)\,dy \\
& + \int K_h(y-x) m(y) f(y)\,dy \int (y-x)^4 K_h(y-x)^3 f(y)\,dy \\
& + \int (y-x)^2 K_h(y-x)^2 m(y) f(y)\,dy \int (y-x)^2 K_h(y-x)^2 f(y)\,dy \\
& +2\int (y-x) K_h(y-x)^2 m(y) f(y)\,dy \int (y-x)^3 K_h(y-x)^2 f(y)\,dy\Bigg]
\end{aligned}
$$

$$= n^{-3}h^{-6}\left[(K_h)^4 * b_x^4\right](x) + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x)$$
$$\left[K_h * d_x^1\right]^2(x) + h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^1\right]^2(x)\right.$$
$$+\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^0\right](x)\left[K_h * d_x^2\right](x) + 2\left[(K_h)^2 * b_x^1\right](x)\left[K_h * d_x^2\right](x)$$
$$\left.\left[K_h * d_x^1\right](x) + 2\left[(K_h)^2 * d_x^3\right](x)\left[K_h * b_x^0\right](x)\left[K_h * d_x^1\right](x)\right]$$
$$+h^{-6}\frac{n-1}{n^3}\left[2\left[(K_h)^3 * b_x^3\right](x)\left[K_h * d_x^1\right](x) + \left[K_h * d_x^2\right](x)\left[(K_h)^3 * b_x^2\right](x)\right.$$
$$+\left[K_h * b_x^0\right](x)\left[(K_h)^3 * d_x^4\right](x) + \left[(K_h)^2 * b_x^2\right](x)\left[(K_h)^2 * d_x^2\right](x)$$
$$\left.+2\left[(K_h)^2 * b_x^1\right](x)\left[(K_h)^2 * d_x^3\right](x)\right].$$

Working out similar calculations as in previous expressions, it follows that:

$$\mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_0(x;h)\hat{s}_1(x;h)\hat{s}_2(x;h)\right]$$
$$= \mathbb{E}\left[n^{-1}\sum_{i=1}^n (X_i-x)K_h(X_i-x)Y_i\, n^{-1}\sum_{j=1}^n K_h(X_j-x)\right.$$
$$\left.n^{-1}h^{-2}\sum_{k=1}^n (X_k-x)K_h(X_k-x)n^{-1}h^{-2}\sum_{\ell=1}^n (X_\ell-x)^2K_h(X_\ell-x)\right]$$
$$= n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}\left[(X_i-x)K_h(X_i-x)Y_iK_h(X_j-x)\right.$$
$$\left.(X_k-x)K_h(X_k-x)(X_\ell-x)^2K_h(X_\ell-x)\right]$$
$$= n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\right]$$
$$+h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)Y_1K_h(X_2-x)\right.$$
$$\left.(X_3-x)K_h(X_3-x)(X_4-x)^2K_h(X_4-x)\right]$$
$$+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2K_h(X_2-x)\right.\right.$$
$$\left.(X_3-x)K_h(X_3-x)Y_3\right]$$
$$+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)K_h(X_2-x)(X_3-x)K_h(X_3-x)Y_3\right]$$

$$
\begin{aligned}
&+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2Y_1(X_2-x)K_h(X_2-x)K_h(X_3-x)\right]\\
&+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)Y_3\right]\\
&+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)K_h(X_3-x)\right]\\
&+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)(X_3-x)K_h(X_3-x)\right]\Big]\\
&+h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3(X_2-x)K_h(X_2-x)Y_2\right]\\
&+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3Y_1K_h(X_2-x)\right]\\
&+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1(X_2-x)K_h(X_2-x)\right]\\
&+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3Y_1(X_2-x)^2K_h(X_2-x)\right]\\
&+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1(X_2-x)^3K_h(X_2-x)^2\right]\\
&+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right]\\
&+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2Y_1(X_2-x)K_h(X_2-x)^2\right]\Big]\\
=\;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)K_h(X_1-x)Y_1\,|_{X_1}\right]\right]\\
&\mathbb{E}\left[K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\mathbb{E}\left[(X_4-x)^2K_h(X_4-x)\right]\\
&+n^{-3}h^{-4}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\,|_{X_1}\right]\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\Big[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)\right]\\
&\mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\\
&\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right]\\
&+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\\
&\mathbb{E}\left[K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\\
&\mathbb{E}\left[\mathbb{E}\left[(X_3-x)K_h(X_3-x)Y_3\,|_{X_3}\right]\right]+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\\
&\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]\\
&+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\\
&\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\Big]\\
&+h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2\,|_{X_2}\right]\right]
\end{aligned}
$$

$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\right]$$
$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]$$
$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\Big]$$
$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right]$$
$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]$$
$$+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)^2\right]\Big]$$
$$= h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]$$
$$\mathbb{E}\left[K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]$$
$$\mathbb{E}\left[(X_4-x)^2K_h(X_4-x)\right]+n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]$$
$$+h^{-4}\frac{(n-2)(n-1)}{n^3}\Big[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)\right]$$
$$\mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]$$
$$\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right]$$
$$+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]$$
$$+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]$$
$$\mathbb{E}\left[(X_3-x)K_h(X_3-x)\mathbb{E}\left[Y_3\mid_{X_3}\right]\right]+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]$$
$$\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right]+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]$$
$$\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\Big]$$
$$+h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2\mid_{X_2}\right]\right]$$
$$+\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[K_h(X_2-x)\right]$$
$$+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]$$
$$+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3\mathbb{E}\left[Y_1\mid_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]$$
$$+\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right]$$
$$+\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)^2\right]$$
$$+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]\Big]$$

$$
\begin{aligned}
= \quad & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)K_h(X_1-x)m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[(X_4-x)^2K_h(X_4-x)\right] + n^{-3}h^{-4}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4m(X_1)\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2\right]\mathbb{E}\left[K_h(X_2-x)\right]\right. \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] + \mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^2m(X_1)\right] \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)m(X_3)\right] + \mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right] \\
& \mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\mathbb{E}\left[K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& \left.\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right]\right. \\
& +\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right]\right] \\
& +\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^3K_h(X_2-x)^2\right] \\
= \quad & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& \int K_h(y-x)f(y)\,dy\int(y-x)K_h(y-x)f(y)\,dy \\
& \int(y-x)^2K_h(y-x)f(y)\,dy + n^{-3}h^{-4}\int(y-x)^4K_h(y-x)^4m(y)f(y)\,dy
\end{aligned}
$$

$$+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\int (y-x)^3K_h(y-x)^2f(y)\,dy\int K_h(y-x)f(y)\,dy\right.$$
$$\int (y-x)K_h(y-x)m(y)f(y)\,dy+\int (y-x)^2K_h(y-x)^2f(y)\,dy$$
$$\int (y-x)K_h(y-x)f(y)\,dy\int (y-x)K_h(y-x)m(y)f(y)\,dy$$
$$+\int (y-x)^3K_h(y-x)^2m(y)f(y)\,dy\int (y-x)K_h(y-x)f(y)\,dy$$
$$\int K_h(y-x)f(y)\,dy+\int (y-x)K_h(y-x)^2f(y)\,dy$$
$$\int (y-x)^2K_h(y-x)f(y)\,dy\int (y-x)K_h(y-x)m(y)f(y)\,dy$$
$$+\int (y-x)^2K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy$$
$$\int K_h(y-x)f(y)\,dy+\int (y-x)K_h(y-x)^2m(y)f(y)\,dy$$
$$\left.\int (y-x)^2K_h(y-x)f(y)\,dy\int (y-x)K_h(y-x)f(y)\,dy\right]$$
$$+h^{-4}\frac{n-1}{n^3}\left[\int (y-x)^3K_h(y-x)^3f(y)\,dy\int (y-x)K_h(y-x)m(y)f(y)\,dy\right.$$
$$+\int (y-x)^4K_h(y-x)^3m(y)f(y)\,dy\int K_h(y-x)f(y)\,dy$$
$$+\int (y-x)^3K_h(y-x)^3m(y)f(y)\,dy\int (y-x)K_h(y-x)f(y)\,dy$$
$$+\int (y-x)^2K_h(y-x)^3m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy$$
$$+\int (y-x)K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^3K_h(y-x)^2f(y)\,dy$$
$$+\int (y-x)^2K_h(y-x)^3m(y)f(y)\,dy\int (y-x)^2K_h(y-x)f(y)\,dy$$
$$\left.+\int (y-x)K_h(y-x)^2m(y)f(y)\,dy\int (y-x)^3K_h(y-x)^2f(y)\,dy\right]$$

$$
\begin{aligned}
= \quad & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * b_x^1\right](x)\left[K_h * d_x^0\right](x)\left[K_h * d_x^1\right](x)\\
& \left[K_h * d_x^2\right](x) + n^{-3}h^{-4}\left[(K_h)^4 * b_x^4\right](x)\\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * d_x^3\right](x)\left[K_h * d_x^0\right](x)\left[K_h * b_x^1\right](x)\right.\\
& +\left[(K_h)^2 * d_x^2\right](x)\left[K_h * d_x^1\right](x)\left[K_h * b_x^1\right](x)\\
& +\left[(K_h)^2 * b_x^3\right](x)\left[K_h * d_x^1\right](x)\left[K_h * d_x^0\right](x)\\
& +\left[(K_h)^2 * d_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * b_x^1\right](x)\\
& +\left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x)\\
& \left.+\left[(K_h)^2 * b_x^1\right](x)\left[K_h * d_x^2\right](x)\left[K_h * d_x^1\right](x)\right] + h^{-4}\frac{n-1}{n^3}\\
& \left[\left[(K_h)^3 * d_x^3\right](x)\left[K_h * b_x^1\right](x) + \left[(K_h)^3 * b_x^4\right](x)\left[K_h * d_x^0\right](x)\right.\\
& +\left[(K_h)^3 * b_x^3\right](x)\left[K_h * d_x^1\right](x) + \left[(K_h)^3 * b_x^2\right](x)\left[K_h * d_x^2\right](x)\\
& +\mathbb{E}\left[(K_h)^2 * b_x^1\right](x)\left[(K_h)^2 * d_x^3\right](x) + \left[(K_h)^2 * b_x^2\right](x)\\
& \left.\left[(K_h)^2 * d_x^2\right](x) + \left[(K_h)^2 * b_x^3\right](x)\left[(K_h)^2 * d_x^1\right](x)\right]. \qquad \text{(C.35)}
\end{aligned}
$$

By splitting the four-tuple sum into different cases depending on the ties in the indices, working out expectations by means of conditional expectations and performing changes of variables in the integrals, we have:

$$
\begin{aligned}
& \mathbb{E}\left[\hat{\Psi}_h^1(x;h)\hat{s}_1^3(x;h)\right]\\
= \quad & \mathbb{E}\left[n^{-1}\sum_{i=1}^{n}(X_i-x)K_h(X_i-x)Y_i\left(n^{-1}h^{-2}\sum_{j=1}^{n}(X_j-x)K_h(X_j-x)\right)^3\right]\\
= \quad & n^{-4}h^{-6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\mathbb{E}\left[(X_i-x)K_h(X_i-x)Y_i(X_j-x)K_h(X_j-x)\right.\\
& \left.(X_k-x)K_h(X_k-x)(X_\ell-x)K_h(X_\ell-x)\right]\\
= \quad & n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}\left[(X_1-x)\right.\\
& \left.K_h(X_1-x)(X_2-x)K_h(X_2-x)(X_3-x)K_h(X_3-x)(X_4-x)K_h(X_4-x)Y_4\right]
\end{aligned}
$$

$$
\begin{aligned}
& +3\,h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2(X_2-x)K_h(X_2-x)Y_2(X_3-x)K_h(X_3-x)\right]\right. \\
& \left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)K_h(X_2-x)(X_3-x)K_h(X_3-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3(X_2-x)K_h(X_2-x)Y_2\right]\right. \\
& +3\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1(X_2-x)K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1(X_2-x)^2K_h(X_2-x)^2\right]\right] \\
= \;& n^{-3}h^{-6}\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4Y_1\,|_{X_1}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[(X_4-x)K_h(X_4-x)Y_4\,|_{X_4}\right]\right]+3\,h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\right. \\
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\,|_{X_1}\right]\right] \\
& \left.\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\right]+h^{-6}\frac{n-1}{n^3} \\
& \left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[\mathbb{E}\left[(X_2-x)K_h(X_2-x)Y_2\,|_{X_2}\right]\right]\right. \\
& +3\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2Y_1\Big|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]\right] \\
= \;& n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[(X_4-x)K_h(X_4-x)\mathbb{E}\left[Y_4\,|_{X_4}\right]\right]+3\,h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] + \mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1\,|_{X_1}\right]\right] \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\big] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right]\right. \\
& +3\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\mathbb{E}\left[Y_1\,|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right] \\
& \left.+\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\mathbb{E}\left[Y_1|_{X_1}\right]\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}\left[(X_1-x)^4K_h(X_1-x)^4m(X_1)\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}\left[(X_1-x)K_h(X_1-x)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& \mathbb{E}\left[(X_4-x)K_h(X_4-x)m(X_4)\right] \\
& +3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2\right]\right. \\
& \mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right]\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right] \\
& +\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \left.\mathbb{E}\left[(X_3-x)K_h(X_3-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)m(X_2)\right]\right. \\
& +3\mathbb{E}\left[(X_1-x)^3K_h(X_1-x)^3m(X_1)\right]\mathbb{E}\left[(X_2-x)K_h(X_2-x)\right] \\
& \left.+3\mathbb{E}\left[(X_1-x)^2K_h(X_1-x)^2m(X_1)\right]\mathbb{E}\left[(X_2-x)^2K_h(X_2-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\int(y-x)^4K_h(y-x)^4m(y)f(y)\,dy + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \left(\int(y-x)K_h(y-x)f(y)\,dy\right)^3\int(y-x)K_h(y-x)m(y)f(y)\,dy \\
& +3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\int(y-x)^2K_h(y-x)^2f(y)\,dy\right. \\
& \int(y-x)K_h(y-x)m(y)f(y)\,dy\int(y-x)K_h(y-x)f(y)\,dy \\
& \left.+\int(y-x)^2K_h(y-x)^2m(y)f(y)\,dy\left(\int(y-x)K_h(y-x)f(y)\,dy\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+h^{-6}\frac{n-1}{n^3}\left[\int (y-x)^3 K_h(y-x)^3 f(y)\,dy \int (y-x) K_h(y-x) m(y) f(y)\,dy\right.\\
&+3\int (y-x)^3 K_h(y-x)^3 m(y) f(y)\,dy \int (y-x) K_h(y-x) f(y)\,dy\\
&\left.+3\int (y-x)^2 K_h(y-x)^2 m(y) f(y)\,dy \int (y-x)^2 K_h(y-x)^2 f(y)\,dy\right]\\
=\;& n^{-3}h^{-6}\left[(K_h)^4 * b_x^4\right](x) + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * d_x^1\right]^3(x)\\
&\left[K_h * b_x^1\right](x) + 3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * d_x^2\right](x)\left[K_h * b_x^1\right](x)\right.\\
&\left.\left[K_h * d_x^1\right](x) + \left[(K_h)^2 * b_x^2\right](x)\left[K_h * d_x^1\right]^2(x)\right]\\
&+h^{-6}\frac{n-1}{n^3}\left[\left[(K_h)^3 * d_x^3\right](x)\left[K_h * b_x^1\right](x) + 3\left[(K_h)^3 * b_x^3\right](x)\right.\\
&\left.\left[K_h * d_x^1\right](x) + 3\left[(K_h)^2 * b_x^2\right](x)\left[(K_h)^2 * d_x^2\right](x)\right]. \qquad \text{(C.36)}
\end{aligned}
$$

Splitting a double sum into two different cases according to ties in the indices, computing expectations via conditional expectations and performing changes of variables in the integrals can be used to prove:

$$
\begin{aligned}
\mathbb{E}\left[\hat{\Theta}_h^0\right] &= \mathbb{E}\left[\hat{s}_2(x;h)\hat{s}_0(x;h) - h^2\hat{s}_1^2(x;h)\right]\\
&= \mathbb{E}\left[\hat{s}_2(x;h)\hat{s}_0(x;h)\right] - h^2\mathbb{E}\left[\hat{s}_1^2(x;h)\right]\\
&= \mathbb{E}\left[n^{-2}h^{-2}\sum_{i=1}^n\sum_{j=1}^n (X_i-x)^2 K_h(X_i-x)K_h(X_j-x)\right]\\
&\quad -h^2\mathbb{E}\left[n^{-2}h^{-4}\sum_{i=1}^n\sum_{j=1}^n (X_i-x)K_h(X_i-x)(X_j-x)K_h(X_j-x)\right]\\
&= \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)K_h(X_2-x)\right]\\
&\quad -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K_h(X_1-x)\left(\frac{X_2-x}{h}\right)K_h(X_2-x)\right]\\
&= \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)\right]\mathbb{E}\left[K_h(X_2-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K_h(X_1-x)\right]\mathbb{E}\left[\left(\frac{X_2-x}{h}\right)K_h(X_2-x)\right] \\
= \; & \frac{n-1}{n}h^{-2}\int (y-x)^2\, K_h(y-x)f(y)\,dy\int K_h(y-x)f(y)\,dy \\
& -\frac{n-1}{n}h^{-2}\left(\int (y-x)\,K_h(y-x)f(y)\,dy\right)^2 \\
= \; & \frac{n-1}{n}h^{-2}\left(\left[K_h * d_x^2\right](x)\left[K_h * d_x^0\right](x) - \left[K_h * d_x^1\right]^2(x)\right). \qquad \text{(C.37)}
\end{aligned}
$$

Following similar steps as for the previous expression, it follows:

$$
\begin{aligned}
\mathbb{E}\left[\hat{\Theta}_h^1\right] \; = \; & \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h) - \hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)\right] \\
= \; & \mathbb{E}\left[\hat{\Psi}_h^0(x;h)\hat{s}_2(x;h)\right] - \mathbb{E}\left[\hat{s}_1(x;h)\hat{\Psi}_h^1(x;h)\right] \\
= \; & \mathbb{E}\left[n^{-2}h^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-x)^2K_h(X_i-x)K_h(X_j-x)Y_j\right] \\
& -\mathbb{E}\left[n^{-2}h^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-x)K_h(X_i-x)(X_j-x)K_h(X_j-x)Y_j\right] \\
= \; & \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)K_h(X_2-x)Y_2\right] \\
& -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K_h(X_1-x)\left(\frac{X_2-x}{h}\right)K_h(X_2-x)Y_2\right] \\
= \; & \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)\right]\mathbb{E}\left[K_h(X_2-x)Y_2\right] \\
& -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K_h(X_1-x)\right]\mathbb{E}\left[\left(\frac{X_2-x}{h}\right)K_h(X_2-x)Y_2\right] \\
= \; & \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)\right]\mathbb{E}\left[\mathbb{E}\left[K_h(X_2-x)Y_2|_{X_2}\right]\right] \\
& -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)K_h(X_1-x)\right] \\
& \mathbb{E}\left[\mathbb{E}\left[\left(\frac{X_2-x}{h}\right)K_h(X_2-x)Y_2\bigg|_{X_2}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)\right]\mathbb{E}\left[K_h(X_2-x)\mathbb{E}\left[Y_2|_{X_2}\right]\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right) K_h(X_1-x)\right]\mathbb{E}\left[\left(\frac{X_2-x}{h}\right) K_h(X_2-x)\mathbb{E}\left[Y_2\,|_{X_2}\right]\right] \\
&= \frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right)^2 K_h(X_1-x)\right]\mathbb{E}\left[K_h(X_2-x)m(X_2)\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}\left[\left(\frac{X_1-x}{h}\right) K_h(X_1-x)\right]\mathbb{E}\left[\left(\frac{X_2-x}{h}\right) K_h(X_2-x)m(X_2)\right] \\
&= \frac{n-1}{n}h^{-2}\int (y-x)^2 K_h(y-x)f(y)\,dy \int K_h(y-x)m(y)f(y)\,dy \\
&\quad -\frac{n-1}{n}h^{-2}\int (y-x)\,K_h(y-x)f(y)\,dy \int (y-x)\,K_h(y-x)m(y)f(y)\,dy \\
&= \frac{n-1}{n}h^{-2}\left(\left[K_h * d_x^2\right](x)\left[K_h * b_x^0\right](x) - \left[K_h * d_x^1\right](x)\left[K_h * b_x^1\right](x)\right). \qquad \text{(C.38)}
\end{aligned}
$$

Collecting terms (C.28)-(C.36) and plugging them in (C.27), the proof of Theorem 13 is concluded.

Theorem 14 *Given x an interior point of the support of X^0, K a symmetric bounded density function and $\{(X_1,Y_1),\dots,(X_n,Y_n)\}$ a simple random sample, the bootstrap version of the MSE of the proxy estimator given in (4.17) admits the following representation:*

$$
\begin{aligned}
MSE_x^*(h) \;=\;& \left(\hat{\Theta}_g^0\right)^{-2} h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}_{x,g}^0\right]^2(x)\left[K_h * \hat{d}_{x,g}^2\right]^2(x)\right. \\
&+\frac{(n-1)(n-2)}{n^3}\left(\left[(K_h)^2 * \hat{a}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^2\right]^2(x)\right. \\
&+\left[(K_h)^2 * \hat{d}_{x,g}^4\right](x)\left[K_h * \hat{b}_{x,g}^0\right]^2(x) \\
&\left.+2\left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x)\right) \\
&\left.+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{a}_{x,g}^2\right](x)\left[(K_h)^2 * \hat{d}_{x,g}^4\right](x)\right] \\
&-2\left(\hat{\Theta}_g^0\right)^{-2} h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}_{x,g}^0\right](x)\right.
\end{aligned}
$$

$$\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\right.$$
$$+\left[(K_h)^2 * \hat{a}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)$$
$$+\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)$$
$$\left.+\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\right]$$
$$+\frac{n-1}{n^3}\left(\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\right.$$
$$\left.\left.+\left[(K_h)^2 * \hat{a}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\right)\right] + \left(\hat{\Theta}^0_g\right)^{-2} h^{-4}$$
$$\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^2(x)\left[K_h * \hat{b}^1_{x,g}\right]^2(x)\right.$$
$$+\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * \hat{a}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x)\right.$$
$$+\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right]^2(x)$$
$$\left.+2\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\right] + \frac{n-1}{n^3}$$
$$\left.\left(2\left[(K_h)^2 * \hat{b}^2_{x,g}\right]^2(x) + \left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{a}^2_{x,g}\right](x)\right)\right]$$
$$-2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x)\right.$$
$$\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\left[K_h * \hat{d}^0_{x,g}\right](x) + \frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\right.$$
$$\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) + \left[K_h * \hat{b}^0_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)$$
$$\left[K_h * \hat{d}^0_{x,g}\right](x) + \left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)$$
$$\left.\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) + \left[(K_h)^2 * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\right]$$
$$\left.+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}^0_g\right](x)\right]$$
$$+2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\right.$$
$$\left[K_h * \hat{d}^1_{x,g}\right]^2(x) + \frac{(n-1)(n-2)}{n^3}\left[3\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\right.$$

$$\left[K_h * \hat{d}^1_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\Big]$$
$$+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\Big]$$
$$+2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\right.$$
$$\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)$$
$$+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)$$
$$+\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)$$
$$+\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\Big]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\Big]$$
$$-2\left(\hat{\Theta}^0_g\right)^{-3}\hat{\Theta}^1_g h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^3(x)\left[K_h * \hat{b}^1_{x,g}\right](x)\right.$$
$$+2\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)$$
$$+\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x}\right]^2(x)\Big]$$
$$+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\Big]$$
$$+\left(\hat{\Theta}^1_g\right)^2\left(\hat{\Theta}^0_g\right)^{-4}h^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\left[K_h * \hat{d}^0_{x,g}\right]^2(x)\right.$$
$$+\frac{(n-1)(n-2)}{n^3}\Big[\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right]^2(x)$$
$$+\left[(K_h)^2 * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)$$
$$+2\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\Big]$$
$$+\frac{n-1}{n^3}\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}^0_g\right](x)\Big]$$
$$+h^{-4}\left(\hat{\Theta}^1_g\right)^2\left(\hat{\Theta}^0_g\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^4(x)\right.$$
$$+4\frac{(n-1)(n-2)}{n^3}\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x)$$
$$+3\frac{n-1}{n^3}\left[(K_h)^2 * \hat{d}^2_{x,g}\right]^2(x)\Big]$$

$$-2\,h^{-4}\left(\hat{\Theta}_g^1\right)^2\left(\hat{\Theta}_g^0\right)^{-4}\left[\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h*\hat{d}_{x,g}^2\right](x)\left[K_h*\hat{d}_{x,g}^0\right](x)\right.$$

$$\left[K_h*\hat{d}_{x,g}^1\right]^2(x)+\frac{(n-1)(n-2)}{n^3}\left[2\left[(K_h)^2*\hat{d}_{x,g}^3\right](x)\left[K_h*\hat{d}_{x,g}^0\right](x)\right.$$

$$\left.\left[K_h*\hat{d}_{x,g}^1\right](x)+2\left[(K_h)^2*\hat{d}_{x,g}^1\right](x)\left[K_h*\hat{d}_{x,g}^2\right](x)\left[K_h*\hat{d}_{x,g}^1\right](x)\right]$$

$$\left.+2\frac{n-1}{n^3}\left[(K_h)^2*\hat{d}_{x,g}^3\right](x)\left[(K_h)^2*\hat{d}_{x,g}^1\right](x)\right],$$

where $\hat{a}_{x,g}^j(y)=(y-x)^j\left(\hat{\sigma}_g^2(y)+g^2\mu_2(K)+\hat{m}_g^{LL}(y)^2\right)\hat{f}_g(y)$, $\hat{b}_{x,g}^j(y)=(y-x)^j\hat{m}_g^{LL}(y)\hat{f}_g(y)$ *and* $\hat{d}_{x,g}^j(y)=(y-x)^j\hat{f}_g(y)$, *with* $j\in\mathbb{N}$.

Proof of Theorem 14 As in the proof of Theorem 13, by means of splitting a four-tuple sum into different cases according to ties in the indices, computing expectations via conditional expectations and performing changes of variables in the integrals, it follows that:

$$\begin{aligned}
&\mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)^2\hat{s}_2^*(x;h)^2\right]\\
=\;&\mathbb{E}^*\left[\left(n^{-1}\sum_{i=1}^n K_h(X_i^*-x)Y_i^*\right)^2\left(n^{-1}h^{-2}\sum_{k=1}^n(X_k^*-x)^2K_h(X_k^*-x)\right)^2\right]\\
=\;&n^{-4}h^{-4}\mathbb{E}^*\left[\sum_{i=1}^n K_h(X_i^*-x)Y_i^*\sum_{j=1}^n K_h(X_j^*-x)Y_j^*\sum_{k=1}^n(X_k^*-x)^2K_h(X_k^*-x)\right.\\
&\left.\sum_{\ell=1}^n(X_\ell-x)^2K_h(X_\ell-x)\right]\\
=\;&n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n\mathbb{E}^*\left[K_h(X_i^*-x)Y_i^*K_h(X_j^*-x)Y_j^*(X_k^*-x)^2\right.\\
&\left.K_h(X_k^*-x)(X_\ell-x)^2K_h(X_\ell-x)\right]\\
=\;&h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*K_h(X_2^*-x)Y_2^*K_h(X_3^*-x)\right.\\
&\left.(X_3^*-x)^2K_h(X_4^*-x)(X_4^*-x)^2\right]+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2\left(Y_1^*\right)^2\right.
\end{aligned}$$

$$
\begin{aligned}
& (X_2^* - x)^2 K_h(X_2^* - x)(X_3^* - x)^2 K_h(X_3^* - x)\big] \\
& + h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 (X_1^* - x)^4 K_h(X_2^* - x) Y_2^* K_h(X_3^* - x) Y_3^*\right] \\
& + 4\, h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\big[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2 K_h(X_2^* - x) \\
& Y_2^* (X_3^* - x)^2 K_h(X_3^* - x)\big] \\
& + 2\frac{n-1}{n^3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^3 \left(Y_1^*\right)^2 (X_1^* - x)^2 K_h(X_2^* - x)(X_2^* - x)^2\right] \\
& + 2\frac{n-1}{n^3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^3 Y_1^* (X_1^* - x)^4 K_h(X_2^* - x) Y_2^*\right] \\
& + h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h\left(X_1^* - x\right)^2 \left(Y_1^*\right)^2 K_h\left(X_2^* - x\right)^2 \left(X_2^* - x\right)^4\right] \\
& + 2\, h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2 K_h(X_2^* - x)^2 Y_2^* (X_2^* - x)^2\right] \\
& + n^{-3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^4 \left(Y_1^*\right)^2 (X_1^* - x)^4\right] \\
= \;& h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x) Y_1^*\right]\mathbb{E}^*\left[K_h(X_2^* - x) Y_2^*\right] \\
& \mathbb{E}^*\left[K_h(X_3^* - x)(X_3^* - x)^2\right]\mathbb{E}^*\left[K_h(X_4^* - x)(X_4^* - x)^2\right] \\
& + h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 \left(Y_1^*\right)^2\right]\mathbb{E}^*\left[(X_2^* - x)^2 K_h(X_2^* - x)\right] \\
& \mathbb{E}^*\left[(X_3^* - x)^2 K_h(X_3^* - x)\right] \\
& + h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 (X_1^* - x)^4\right]\mathbb{E}^*\left[K_h(X_2^* - x) Y_2^*\right] \\
& \mathbb{E}^*\left[K_h(X_3^* - x) Y_3^*\right] \\
& + 4\, h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2\right]\mathbb{E}^*\left[K_h(X_2^* - x) Y_2^*\right] \\
& \mathbb{E}^*\left[(X_3^* - x)^2 K_h(X_3^* - x)\right] \\
& + 2\frac{n-1}{n^3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^3 \left(Y_1^*\right)^2 (X_1^* - x)^2\right]\mathbb{E}^*\left[K_h(X_2^* - x)(X_2^* - x)^2\right] \\
& + 2\frac{n-1}{n^3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^3 Y_1^* (X_1^* - x)^4\right]\mathbb{E}^*\left[K_h(X_2^* - x) Y_2^*\right] \\
& + n^{-3} h^{-4}\mathbb{E}^*\left[K_h(X_1^* - x)^4 \left(Y_1^*\right)^2 (X_1^* - x)^4\right] \\
& + h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h\left(X_1^* - x\right)^2 \left(Y_1^*\right)^2\right]\mathbb{E}^*\left[K_h\left(X_2^* - x\right)^2 \left(X_2^* - x\right)^4\right] \\
& + 2\, h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2\right]\mathbb{E}^*\left[K_h(X_2^* - x)^2 Y_2^* (X_2^* - x)^2\right] \\
= \;& h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^* - x) Y_1^* \mid_{X_1^*}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^* \mid_{X_2^*}\right]\right]\mathbb{E}^*\left[K_h(X_3^*-x)(X_3^*-x)^2\right] \\
& \mathbb{E}^*\left[K_h(X_4^*-x)(X_4^*-x)^2\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2\left(Y_1^*\right)^2 \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^4\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^* \mid_{X_2^*}\right]\right] \\
& \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_3^*-x)Y_3^* \mid_{X_3^*}\right]\right] \\
& +4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_1^*-x)^2 \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^* \mid_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3\left(Y_1^*\right)^2\left(X_1^*-x\right)^2 \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)^2\right] \\
& +2\frac{n-1}{n^3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3Y_1^*(X_1^*-x)^4 \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^* \mid_{X_2^*}\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^4\left(Y_1^*\right)^2\left(X_1^*-x\right)^4 \mid_{X_1^*}\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h\left(X_1^*-x\right)^2\left(Y_1^*\right)^2\Big|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h\left(X_2^*-x\right)^2\left(X_2^*-x\right)^4\right] \\
& +2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_1^*-x)^2\Big|_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)^2Y_2^*(X_2^*-x)^2\Big|_{X_2^*}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^* \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\right]\mathbb{E}^*\left[K_h(X_3^*-x)(X_3^*-x)^2\right] \\
& \mathbb{E}^*\left[K_h(X_4^*-x)(X_4^*-x)^2\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2\mathbb{E}^*\left[\left(Y_1^*\right)^2 \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^4\right]\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}^*\left[K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\,|_{X_3^*}\right]\right]\\
&+4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\,|_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]+2\frac{n-1}{n^3}h^{-4}\\
&\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^2\mathbb{E}^*\left[(Y_1^*)^2\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)^2\right]\\
&+2\frac{n-1}{n^3}h^{-4}\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^4\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\,|_{X_2^*}\right]\right]\\
&+n^{-3}h^{-4}\mathbb{E}^*\left[K_h(X_1^*-x)^4(X_1^*-x)^4\mathbb{E}^*\left[(Y_1^*)^2\,|_{X_1^*}\right]\right]\\
&+h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h\left(X_1^*-x\right)^2\mathbb{E}^*\left[(Y_1^*)^2\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h\left(X_2^*-x\right)^2\left(X_2^*-x\right)^4\right]\\
&+2\,h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)^2(X_2^*-x)^2\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]\\
=\;& h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\mathbb{E}^*\left[K_h(X_3^*-x)(X_3^*-x)^2\right]\\
&\mathbb{E}^*\left[K_h(X_4^*-x)(X_4^*-x)^2\right]\\
&+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\\
&+h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^4\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\\
&\mathbb{E}^*\left[K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\\
&+4\,h^{-4}\frac{(n-1)(n-2)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\\
&+2\frac{n-1}{n^3}h^{-4}\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^2(\hat{\sigma}_g^2(X_1^*)\right.\\
&\left.+\hat{m}_g^{LL}(X_1^*)^2)\right]\mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)^2\right]+2\frac{n-1}{n^3}h^{-4}\\
&\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\\
&+h^{-4}\frac{n-1}{n^3}\mathbb{E}^*\left[K_h\left(X_1^*-x\right)^2\left(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}^* \left[K_h \left(X_2^* - x\right)^2 \left(X_2^* - x\right)^4\right] + 2\, h^{-4} \frac{n-1}{n^3} \\
& \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^2 \hat{m}_g^{LL}(X_1^*)\right] \mathbb{E}^* \left[K_h(X_2^* - x)^2 (X_2^* - x)^2 \hat{m}_g^{LL}(X_2^*)\right] \\
& + n^{-3} h^{-4} \mathbb{E}^* \left[K_h(X_1^* - x)^4 (X_1^* - x)^4 \left(\hat{\sigma}_g^2(X_1^*) + \hat{m}_g^{LL}(X_1^*)^2\right)\right] \\
= \; & h^{-4} \frac{(n-1)(n-2)(n-3)}{n^3} \left(\int K_h(y-x) \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy\right)^2 \\
& \cdot \left(\int K_h(y-x)(y-x)^2 \hat{f}_g(y)\, dy\right)^2 \\
& + h^{-4} \frac{(n-1)(n-2)}{n^3} \int K_h(y-x)^2 (\hat{\sigma}_g^2(y) + \hat{m}_g^{LL}(y)^2) \hat{f}_g(y)\, dy \\
& \cdot \left(\int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy\right)^2 + h^{-4} \frac{(n-1)(n-2)}{n^3} \\
& \int K_h(y-x)^2 (y-x)^4 \hat{f}_g(y)\, dy \left(\int K_h(y-x) \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy\right)^2 \\
& + 4\, h^{-4} \frac{(n-1)(n-2)}{n^3} \int K_h(y-x)^2 (y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \\
& \int K_h(y-x) \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy \\
& + 2 \frac{n-1}{n^3} h^{-4} \int K_h(y-x)^3 (y-x)^2 (\hat{\sigma}_g^2(y) + \hat{m}_g^{LL}(y)^2) \hat{f}_g(y)\, dy \\
& \int K_h(y-x)(y-x)^2 \hat{f}_g(y)\, dy + 2 \frac{n-1}{n^3} h^{-4} \\
& \int K_h(y-x)^3 (y-x)^4 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int K_h(y-x) \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \\
& + h^{-4} \frac{n-1}{n^3} \int K_h \left(y-x\right)^2 \left(\hat{\sigma}_g^2(y) + \hat{m}_g^{LL}(y)^2\right) \hat{f}_g(y)\, dy \\
& \int K_h \left(y-x\right)^2 \left(y-x\right)^4 \hat{f}_g(y)\, dy + 2\, h^{-4} \frac{n-1}{n^3} \\
& \int K_h(y-x)^2 (y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int K_h(y-x)^2 (y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \\
& + n^{-3} h^{-4} \int K_h(y-x)^4 (y-x)^4 \left(\hat{\sigma}_g^2(y) + \hat{m}_g^{LL}(y)^2\right) \hat{f}_g(y)\, dy \\
= \; & h^{-4} \frac{(n-1)(n-2)(n-3)}{n^3} \left[K_h * \hat{b}_{x,g}^0\right]^2 (x) \left[K_h * d_x^2\right]^2 (x)
\end{aligned}
$$

$$+h^{-4}\frac{(n-1)(n-2)}{n^3}\Bigg(\left[(K_h)^2 * \hat{a}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)$$
$$+\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right]^2(x) + 4\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)$$
$$\left[K_h * \hat{d}^2_{x,g}\right](x)\Bigg) + h^{-4}\frac{n-1}{n^3}\Big(2\left[(K_h)^3 * \hat{a}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)$$
$$+2\left[(K_h)^3 * \hat{b}^4_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x) + \left[(K_h)^2 * \hat{a}^0_{x,g}\right](x)$$
$$\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{b}^2_{x,g}\right]^2(x)\Big) + n^{-3}h^{-4}\left[(K_h)^4 * \hat{a}^4_{x,g}\right](x), \quad \text{(C.39)}$$

where $\hat{a}^j_{x,g}(y) = (y-x)^j\left(\hat{\sigma}^2_g(y) + g^2\mu_2(K) + \hat{m}^{LL}_g(y)^2\right)\hat{f}_g(y)$, $\hat{b}^j_{x,g}(y) = (y-x)^j\hat{m}^{LL}_g(y)\hat{f}_g(y)$ and $\hat{d}^j_{x,g}(y) = (y-x)^j\hat{f}_g(y)$, with $j \in \mathbb{N}$.

Similar computations lead to:

$$\begin{aligned}
&\mathbb{E}^*\left[\hat{\Psi}^{0*}_h(x;h)\hat{s}^*_2(x;h)\hat{s}^*_1(x;h)\hat{\Psi}^{1*}_h(x;h)\right] \\
= \; &\mathbb{E}^*\Bigg[\left(n^{-1}\sum_{i=1}^n K_h(X^*_i - x)Y^*_i\right)\left(n^{-1}h^{-2}\sum_{j=1}^n (X^*_j - x)^2 K_h(X^*_j - x)\right) \\
&\left(n^{-1}h^{-2}\sum_{k=1}^n (X^*_k - x)K_h(X^*_k - x)\right)\left(n^{-1}\sum_{\ell=1}^n K_h(X^*_\ell - x)Y^*_\ell\right)\Bigg] \\
= \; &n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\big[K_h(X^*_i - x)Y^*_i(X^*_j - x)^2K_h(X^*_j - x) \\
&(X^*_k - x)K_h(X^*_k - x)K_h(X^*_\ell - x)Y^*_\ell\big] \\
= \; &h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\big[K_h(X^*_1 - x)Y^*_1(X^*_2 - x)^2K_h(X^*_2 - x) \\
&(X^*_3 - x)K_h(X^*_3 - x)(X^*_4 - x)K_h(X^*_4 - x)Y^*_4\big] \\
&+h^{-4}\frac{(n-1)(n-2)}{n^3}\big[\mathbb{E}^*\big[K_h(X^*_1 - x)^2Y^*_1(X^*_1 - x)^2(X^*_2 - x) \\
&K_h(X^*_2 - x)(X^*_3 - x)K_h(X^*_3 - x)Y^*_3\big] \\
&+\mathbb{E}^*\big[K_h(X^*_1 - x)^2(X^*_1 - x)Y^*_1(X^*_2 - x)^2K_h(X^*_2 - x) \\
&(X^*_3 - x)K_h(X^*_3 - x)Y^*_3\big] + \mathbb{E}^*\big[K_h(X^*_1 - x)^2(X^*_1 - x)\left(Y^*_1\right)^2
\end{aligned}$$

$$
\begin{aligned}
& (X_2^* - x)^2 K_h(X_2^* - x)(X_3^* - x) K_h(X_3^* - x)\big] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^3 K_h(X_2^* - x) Y_2^* (X_3^* - x) K_h(X_3^* - x) Y_3^* \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^3 Y_1^* K_h(X_2^* - x) Y_2^* (X_3^* - x) K_h(X_3^* - x) \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^2 Y_1^* K_h(X_2^* - x) Y_2^* (X_3^* - x)^2 K_h(X_3^* - x) \right] \big] \\
& + h^{-4} \frac{n-1}{n^3} \left[\mathbb{E}^* \left[K_h(X_1^* - x)^3 (X_1^* - x)^3 Y_1^* (X_2^* - x) K_h(X_2^* - x) Y_2^* \right] \right. \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^3 (X_1^* - x)^2 \left(Y_1^*\right)^2 (X_2^* - x)^2 K_h(X_2^* - x) \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^3 (X_1^* - x)^4 Y_1^* K_h(X_2^* - x) Y_2^* \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^3 \left(Y_1^*\right)^2 (X_1^* - x)^3 K_h(X_2^* - x)(X_2^* - x) \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2 \right]^2 \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x) \left(Y_1^*\right)^2 (X_2^* - x)^3 K_h(X_2^* - x)^2 \right] \\
& \left. + \mathbb{E}^* \left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x) K_h(X_2^* - x)^2 (X_2^* - x)^3 Y_2^* \right] \right] \\
& + n^{-3} h^{-4} \mathbb{E}^* \left[K_h(X_1^* - x)^4 \left(Y_1^*\right)^2 (X_1^* - x)^4 \right] \\
= \;& h^{-4} \frac{(n-1)(n-2)(n-3)}{n^3} \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_1^* - x) Y_1^* \mid_{X_1^*} \right] \right] \\
& \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \mathbb{E}^* \left[(X_3^* - x) K_h(X_3^* - x) \right] \\
& \mathbb{E}^* \left[\mathbb{E}^* \left[(X_4^* - x) K_h(X_4^* - x) Y_4^* \mid_{X_4^*} \right] \right] \\
& + h^{-4} \frac{(n-1)(n-2)}{n^3} \left[\mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_1^* - x)^2 Y_1^* (X_1^* - x)^2 \mid_{X_1^*} \right] \right] \right. \\
& \mathbb{E}^* \left[(X_2^* - x) K_h(X_2^* - x) \right] \mathbb{E}^* \left[\mathbb{E}^* \left[(X_3^* - x) K_h(X_3^* - x) Y_3^* \mid_{X_3^*} \right] \right] \\
& + \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x) Y_1^* \mid_{X_1^*} \right] \right] \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \\
& \mathbb{E}^* \left[\mathbb{E}^* \left[(X_3^* - x) K_h(X_3^* - x) Y_3^* \mid_{X_3^*} \right] \right] \\
& + \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x) \left(Y_1^*\right)^2 \mid_{X_1^*} \right] \right] \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \\
& \mathbb{E}^* \left[\mathbb{E}^* \left[(X_3^* - x) K_h(X_3^* - x) \mid_{X_3^*} \right] \right] \\
& + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^3 \right] \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_2^* - x) Y_2^* \mid_{X_2^*} \right] \right] \\
& \mathbb{E}^* \left[\mathbb{E}^* \left[(X_3^* - x) K_h(X_3^* - x) Y_3^* \mid_{X_3^*} \right] \right] \\
& + \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_1^* - x)^3 Y_1^* \mid_{X_1^*} \right] \right] \mathbb{E}^* \left[\mathbb{E}^* \left[K_h(X_2^* - x) Y_2^* \mid_{X_2^*} \right] \right]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2Y_1^*\,|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\,|_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\Big]\\
&+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^3Y_1^*\,|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*\,|_{X_2^*}\right]\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^2\left(Y_1^*\right)^2\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^4Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\,|_{X_2^*}\right]\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^3\left(Y_1^*\right)^2(X_1^*-x)^3\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_1^*-x)^2\,|_{X_1^*}\right]\right]^2\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\left(Y_1^*\right)^2\Big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_1^*-x)\Big|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)^2(X_2^*-x)^3Y_2^*\Big|_{X_2^*}\right]\right]\Big]\\
&+n^{-3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^4\left(Y_1^*\right)^2(X_1^*-x)^4\,|_{X_1^*}\right]\right]\\
= &\ h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\mathbb{E}^*\left[Y_4^*\,|_{X_4^*}\right]\right]\\
&+h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\,|_{X_3^*}\right]\right]\\
&+\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\right.\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\,|_{X_3^*}\right]\right]\Big]\Big]\\
&+\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[\left(Y_1^*\right)^2\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^3\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\,|_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\,|_{X_3^*}\right]\right]\\
&+\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right.
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\Big]\, \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] + \mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\right. \\
& \mathbb{E}^*\left[Y_1^* \mid_{X_1^*}\right]\Big]\, \mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\right] \mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\Big] \\
& +h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}^*\Big[K_h(X_1^*-x)^3(X_1^*-x)^3\mathbb{E}^*\left[Y_1^* \mid_{X_1^*}\right]\Big] \\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^2\mathbb{E}^*\left[(Y_1^*)^2 \mid_{X_1^*}\right]\right] \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^4\mathbb{E}^*\left[Y_1^* \mid_{X_1^*}\right]\right] \mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^* \mid_{X_2^*}\right]\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^3\mathbb{E}^*\left[(Y_1^*)^2 \mid_{X_1^*}\right]\right] \mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[Y_1^* \mid_{X_1^*}\right]\right]^2 \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[(Y_1^*)^2\big|_{X_1^*}\right]\right]\, \mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)^2(X_2^*-x)^3\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]\Big] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[K_h(X_1^*-x)^4(X_1^*-x)^4\mathbb{E}^*\left[(Y_1^*)^2 \mid_{X_1^*}\right]\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& \mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\hat{m}_g^{LL}(X_4^*)\right] \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\Big[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right] \\
& +\mathbb{E}^*\Big[K_h(X_1^*-x)^2(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\Big] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^3\right] \mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right] + \mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right] \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right] \mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\right] + h^{-4}\frac{n-1}{n^3} \\
& \left[\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\right. \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^2(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^3(X_1^*-x)^3(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)(X_2^*-x)\right] + \mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\right. \\
& \left.\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]^2 + \mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[(Y_1^*)^2\big|_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right] + \mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right] \\
& \left.\mathbb{E}^*\left[K_h(X_2^*-x)^2(X_2^*-x)^3\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[K_h(X_1^*-x)^4(X_1^*-x)^4(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right] \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \int (y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& +h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\int K_h(y-x)^2(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& +\int K_h(y-x)^2(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int (y-x)^2K_h(y-x)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& +\int K_h(y-x)^2(y-x)(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy \\
& \int (y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& +\int K_h(y-x)^2(y-x)^3\hat{f}_g(y)\,dy\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy + \int K_h(y-x)^2(y-x)^3\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
& \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& + \int K_h(y-x)^2(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \left.\int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy\right] + h^{-4}\frac{n-1}{n^3} \\
& \left[\int K_h(y-x)^3(y-x)^3\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& + \int K_h(y-x)^3(y-x)^4\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& + \int K_h(y-x)^3(y-x)^3(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy \\
& \int K_h(y-x)(y-x)\hat{f}_g(y)\,dy + \left(\int K_h(y-x)^2(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right)^2 \\
& + \int K_h(y-x)^2(y-x)\left(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2\right)\hat{f}_g(y)\,dy \\
& \int (y-x)^3 K_h(y-x)^2\hat{f}_g(y)\,dy + \int K_h(y-x)^2(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \left.\int K_h(y-x)^2(y-x)^3\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right] \\
& + n^{-3}h^{-4}\int K_h(y-x)^4(y-x)^4(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy \\
= \; & h^{-4}\frac{(n-1)(n-2)(n-3)}{n^3}\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x) \\
& \left[K_h * \hat{d}_{x,g}^1\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x) \\
& + h^{-4}\frac{(n-1)(n-2)}{n^3}\left[\left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x)\right. \\
& + \left[(K_h)^2 * \hat{b}_{x,g}^1\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x) + \left[(K_h)^2 * \hat{a}_{x,g}^1\right](x) \\
& \left[K_h * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) + \left[(K_h)^2 * \hat{d}_{x,g}^3\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x) \\
& \left[K_h * \hat{b}_{x,g}^0\right](x) + \left[(K_h)^2 * \hat{b}_{x,g}^3\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) \\
& \left. + \left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x)\right] + h^{-4}\frac{n-1}{n^3} \\
& \left[\left[(K_h)^3 * \hat{b}_{x,g}^3\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x) + \left[(K_h)^3 * \hat{a}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x)\right. \\
& + \left[(K_h)^3 * \hat{b}_{x,g}^4\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x) + \left[(K_h)^3 * \hat{a}_{x,g}^3\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x)
\end{aligned}
$$

$$+\left[(K_h)^2 * \hat{b}^2_{x,g}\right]^2(x) + \left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)$$
$$+\left[(K_h)^2 * \hat{a}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\Big] + n^{-3}h^{-4}\left[(K_h)^4 * \hat{a}^4_{x,g}\right](x). \quad \text{(C.40)}$$

Carrying on with similar computations as in the previous expressions, it follows that:

$$\begin{aligned}
&\mathbb{E}^*\left[\hat{\Psi}_h^{1*}(x;h)^2\hat{s}_1^*(x;h)^2\right] \\
=\;& \mathbb{E}^*\left[\left(n^{-1}\sum_{i=1}^n (X_i^*-x)K_h(X_i^*-x)Y_i^*\right)^2\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k^*-x)K_h(X_k^*-x)\right)^2\right] \\
=\;& n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[(X_i^*-x)K_h(X_i^*-x)Y_i^*(X_j^*-x)K_h(X_j^*-x)Y_j^*\right. \\
& \left.(X_k^*-x)K_h(X_k^*-x)(X_\ell^*-x)K_h(X_\ell^*-x)\right] \\
=\;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)(X_2^*-x)K_h(X_2^*-x)\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)Y_3^*(X_4^*-x)K_h(X_4^*-x)Y_4^*\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\left(Y_1^*\right)^2(X_2^*-x)K_h(X_2^*-x)\right.\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)\right] + \mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)K_h(X_2^*-x)Y_2^*\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)Y_3^*\right] + 4\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2Y_1^*\right. \\
& \left.\left.(X_2^*-x)K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)Y_3^*\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*(X_2^*-x)K_h(X_2^*-x)Y_2^*\right]\right. \\
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\left(Y_1^*\right)^2(X_2^*-x)\right. \\
& \left.K_h(X_2^*-x)\right] + 2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2Y_2^*\right] \\
& \left.+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)^2K_h(X_2^*-x)^2\left(Y_2^*\right)^2\right]\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\left(Y_1^*\right)^2\right] \\
=\;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^* \mid_{X_3^*}\right]\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)Y_4^* \mid_{X_4^*}\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\left(Y_1^*\right)^2|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*|_{X_2^*}\right]\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^*|_{X_3^*}\right]\right]\\
&+4\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&\left.\mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^*|_{X_3^*}\right]\right]\right]\\
&+h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*|_{X_2^*}\right]\right]\\
&+2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\left(Y_1^*\right)^2|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&+2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2Y_2^*|_{X_2^*}\right]\right]\\
&\left.+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\left(Y_2^*\right)^2|_{X_2^*}\right]\right]\right]\\
&+n^{-3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\left(Y_1^*\right)^2|_{X_1^*}\right]\right]\\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*|_{X_3^*}\right]\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\mathbb{E}^*\left[Y_4^*|_{X_4^*}\right]\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[\left(Y_1^*\right)^2|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*|_{X_3^*}\right]\right]\\
&+4\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\\
&\left.\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*|_{X_3^*}\right]\right]\right]\\
&+h^{-4}\frac{n-1}{n^3}\left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\mathbb{E}^*\left[(Y_1^*)^2\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right]\\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\mathbb{E}^*\left[(Y_2^*)^2\big|_{X_2^*}\right]\right]\Big]\\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\mathbb{E}^*\left[(Y_1^*)^2\mid_{X_1^*}\right]\right]\\
= \; & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\hat{m}_g^{LL}(X_4^*)\right]\\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right]\right.\\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\\
& +4\mathbb{E}^*\left[K_h(X_1^*-x)^2(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
& \left.\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\right]+\;h^{-4}\frac{n-1}{n^3}\\
& \left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\right.\\
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right]\\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\hat{m}_g^{LL}(X_2^*)\right]\\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\\
& \left.\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\left(\hat{\sigma}_g^2(X_2^*)+\hat{m}_g^{LL}(X_2^*)^2\right)\right]\right]\\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4(\hat{\sigma}_g^2(X_1^*)+\hat{m}_g^{LL}(X_1^*)^2)\right]\\
= \; & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2\\
& \cdot\left(\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right)^2\\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\int K_h(y-x)^2(y-x)^2(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy\right.
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\left(\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2 \\
& +\int (y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy\left(\int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right)^2 \\
& +4\int (y-x)^2K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& \left.\int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right] + h^{-4}\frac{n-1}{n^3} \\
& \left[2\int (y-x)^3K_h(y-x)^3\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int (y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& +2\int (y-x)^3K_h(y-x)^3(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy + 2\int (y-x)^2K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \int (y-x)^2K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy + \int (y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy \\
& \left.\int (y-x)^2K_h(y-x)^2\left(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2\right)\hat{f}_g(y)\,dy\right] \\
& +n^{-3}h^{-4}\int (y-x)^4K_h(y-x)^4(\hat{\sigma}_g^2(y)+\hat{m}_g^{LL}(y)^2)\hat{f}_g(y)\,dy \\
= \; & n^{-3}h^{-4}\left[(K_h)^4 * \hat{a}_{x,g}^4\right](x) + h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{d}_{x,g}^1\right]^2(x) \\
& \left[K_h * \hat{b}_{x,g}^1\right]^2(x) + h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * \hat{a}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right]^2(x)\right. \\
& +\left[(K_h)^2 * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{b}_{x,g}^1\right]^2(x) + 4\left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) \\
& \left.\left[K_h * \hat{b}_{x,g}^1\right](x)\right] + h^{-4}\frac{n-1}{n^3}\left[2\left[(K_h)^3 * \hat{b}_{x,g}^3\right](x)\left[K_h * \hat{b}_{x,g}^1\right](x)\right. \\
& +2\left[(K_h)^3 * \hat{a}_{x,g}^3\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) + 2\left[(K_h)^2 * \hat{b}_{x,g}^2\right]^2(x) \\
& \left.+\left[(K_h)^2 * \hat{d}_{x,g}^2\right](x)\left[(K_h)^2 * \hat{a}_{x,g}^2\right](x)\right]. \qquad \text{(C.41)}
\end{aligned}
$$

Following similar steps as in the last expressions, it leads to:

$$\mathbb{E}^*\left[\hat{s}_0^*(x;h)^2\hat{s}_2^*(x;h)^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}^* \left[\left(n^{-1} \sum_{i=1}^n K_h(X_i^* - x) \right)^2 \left(n^{-1} h^{-2} \sum_{k=1}^n (X_k^* - x)^2 K_h(X_k^* - x) \right)^2 \right] \\
&= n^{-4} h^{-4} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{\ell=1}^n \mathbb{E}^* \left[K_h(X_i^* - x) K_h(X_j^* - x)(X_k^* - x)^2 K_h(X_k^* - x) \right. \\
&\quad \left. (X_\ell^* - x)^2 K_h(X_\ell^* - x) \right] \\
&= h^{-4} \frac{(n-3)(n-2)(n-1)}{n^3} \mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)(X_2^* - x)^2 K_h(X_2^* - x) \right. \\
&\quad \left. K_h(X_3^* - x) K_h(X_4^* - x) \right] + n^{-3} h^{-4} \mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^4 \right] \\
&\quad + h^{-4} \frac{(n-2)(n-1)}{n^3} \left[\mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^2 K_h(X_2^* - x) K_h(X_3^* - x) \right] \right. \\
&\quad + \mathbb{E}^* \left[K_h(X_1^* - x)^2 (X_2^* - x)^2 K_h(X_2^* - x)(X_3^* - x)^2 K_h(X_3^* - x) \right] + \\
&\quad \left. + 4\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^2 (X_2^* - x)^2 K_h(X_2^* - x) K_h(X_3^* - x) \right] \right] \\
&\quad + h^{-4} \frac{n-1}{n^3} \left[2\mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^3 K_h(X_2^* - x) \right] \right. \\
&\quad + 2\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^3 (X_2^* - x)^2 K_h(X_2^* - x) \right] \\
&\quad + 2\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^2 (X_2^* - x)^2 K_h(X_2^* - x)^2 \right] \\
&\quad \left. + \mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^2 K_h(X_2^* - x)^2 \right] \right] \\
&= h^{-4} \frac{(n-3)(n-2)(n-1)}{n^3} \mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x) \right] \\
&\quad \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \mathbb{E}^* \left[K_h(X_3^* - x) \right] \mathbb{E}^* \left[K_h(X_4^* - x) \right] \\
&\quad + n^{-3} h^{-4} \mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^4 \right] \\
&\quad + h^{-4} \frac{(n-2)(n-1)}{n^3} \left[\mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^2 \right] \mathbb{E}^* \left[K_h(X_2^* - x) \right] \right. \\
&\quad \mathbb{E}^* \left[K_h(X_3^* - x) \right] + \mathbb{E}^* \left[K_h(X_1^* - x)^2 \right] \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \\
&\quad \mathbb{E}^* \left[(X_3^* - x)^2 K_h(X_3^* - x) \right] + 4\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^2 \right] \\
&\quad \left. \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \mathbb{E}^* \left[K_h(X_3^* - x) \right] \right] \\
&\quad + h^{-4} \frac{n-1}{n^3} \left[2\mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^3 \right] \mathbb{E}^* \left[K_h(X_2^* - x) \right] \right. \\
&\quad + 2\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^3 \right] \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x) \right] \\
&\quad + 2\mathbb{E}^* \left[(X_1^* - x)^2 K_h(X_1^* - x)^2 \right] \mathbb{E}^* \left[(X_2^* - x)^2 K_h(X_2^* - x)^2 \right] \\
&\quad \left. + \mathbb{E}^* \left[(X_1^* - x)^4 K_h(X_1^* - x)^2 \right] \mathbb{E}^* \left[K_h(X_2^* - x)^2 \right] \right] \\
&= h^{-4} \frac{(n-3)(n-2)(n-1)}{n^3} \left(\int (y - x)^2 K_h(y - x) \hat{f}_g(y) \, dy \right)^2 \\
&\quad \cdot \left(\int K_h(y - x) \hat{f}_g(y) \, dy \right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+n^{-3}h^{-4}\int (y-x)^4 K_h(y-x)^4 \hat{f}_g(y)\,dy + h^{-4}\frac{(n-2)(n-1)}{n^3} \\
&\left[\int (y-x)^4 K_h(y-x)^2 \hat{f}_g(y)\,dy \left(\int K_h(y-x)\hat{f}_g(y)\,dy\right)^2\right. \\
&+\int K_h(y-x)^2 \hat{f}_g(y)\,dy \left(\int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy\right)^2 \\
&+4\int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
&\left.\int K_h(y-x)\hat{f}_g(y)\,dy\right] \\
&+h^{-4}\frac{n-1}{n^3}\left[2\int (y-x)^4 K_h(y-x)^3 \hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy\right. \\
&+2\int (y-x)^2 K_h(y-x)^3 \hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
&+2\int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
&\left.+\int (y-x)^4 K_h(y-x)^2 \hat{f}_g(y)\,dy \int K_h(y-x)^2 \hat{f}_g(y)\,dy\right] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\left[K_h * \hat{d}^0_{x,g}\right]^2(x) \\
&+n^{-3}h^{-4}\left[(K_h)^4 * \hat{d}^4_{x,g}\right](x) + h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\right. \\
&\left[K_h * \hat{d}^0_{x,g}\right]^2(x) + \left[(K_h)^2 * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x) \\
&\left.+4\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\right] \\
&+h^{-4}\frac{n-1}{n^3}\left[2\left[(K_h)^3 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\right. \\
&+2\left[(K_h)^3 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{d}^2_{x,g})\right]^2(x) \\
&\left.+\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}_g\right](x)\right]. \qquad \text{(C.42)}
\end{aligned}
$$

Carrying on with similar calculations as in the previous expressions, it follows that:

$$
\begin{aligned}
& \mathbb{E}^*\left[\hat{s}_1^*(x;h)^4\right] \\
= \; & \mathbb{E}^*\left[\left(n^{-1}h^{-2}\sum_{i=1}^n (X_i^*-x)K_h(X_i^*-x)\right)^4\right] \\
= \; & n^{-4}h^{-8}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[(X_i^*-x)K_h(X_i^*-x)(X_j^*-x)K_h(X_j^*-x)\right. \\
& \left.(X_k^*-x)K_h(X_k^*-x)(X_\ell^*-x)K_h(X_\ell^*-x)\right] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)(X_2^*-x)K_h(X_2^*-x)\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)(X_4^*-x)K_h(X_4^*-x)\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\right]+6\,h^{-8}\frac{(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)\right] \\
& +h^{-8}\frac{n-1}{n^3}\left[4\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3(X_2^*-x)K_h(X_2^*-x)\right]\right. \\
& \left.+3\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& \mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\right]+6\,h^{-8}\frac{(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& +h^{-8}\frac{n-1}{n^3}\left[4\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\right. \\
& \left.+3\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right] \\
= \; & h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\left(\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^4 \\
& +n^{-3}h^{-4}\int (y-x)^4K_h(y-x)^4\hat{f}_g(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+6\,h^{-8}\frac{(n-2)(n-1)}{n^3}\int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
&\left(\int (y-x) K_h(y-x)\hat{f}_g(y)\,dy\right)^2 \\
&+h^{-8}\frac{n-1}{n^3}\left[4\int (y-x)^3 K_h(y-x)^3 \hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right. \\
&\left.+3\int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy\right] \\
= \;& h^{-8}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{d}^1_{x,g}\right]^4(x) + n^{-3}h^{-4}\left[(K_h)^4 * \hat{d}^4_{x,g}\right](x) \\
&+6\,h^{-8}\frac{(n-2)(n-1)}{n^3}\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x) + \; h^{-8}\frac{n-1}{n^3} \\
&\left[4\left[(K_h)^3 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x) + 3\left[(K_h)^2 * \hat{d}^2_{x,g}\right]^2(x)\right]. \qquad \text{(C.43)}
\end{aligned}
$$

Similarly as in the last expressions, it follows that:

$$
\begin{aligned}
&\mathbb{E}^*\left[\hat{s}^*_1(x;h)^2 \hat{s}^*_0(x;h)\hat{s}^*_2(x;h)\right] \\
= \;& \mathbb{E}^*\left[\left(n^{-1}h^{-2}\sum_{i=1}^n (X^*_i - x)K_h(X^*_i - x)\right)^2 n^{-1}\sum_{k=1}^n K_h(X^*_k - x)\right. \\
&\left. n^{-1}h^{-2}\sum_{\ell=1}^n (X^*_\ell - x)^2 K_h(X^*_\ell - x)\right] \\
= \;& n^{-4}h^{-6}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[(X^*_i - x)K_h(X^*_i - x)(X^*_j - x)\right. \\
&\left. K_h(X^*_j - x)K_h(X^*_k - x)(X^*_\ell - x)^2 K_h(X^*_\ell - x)\right] \\
= \;& h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X^*_1 - x)^2 K_h(X^*_1 - x)K_h(X^*_2 - x)\right. \\
&\left.(X^*_3 - x)K_h(X^*_3 - x)(X^*_4 - x)K_h(X^*_4 - x)\right] \\
&+n^{-3}h^{-6}\mathbb{E}^*\left[(X^*_1 - x)^4 K_h(X^*_1 - x)^4\right] + h^{-6}\frac{(n-2)(n-1)}{n^3} \\
&\left[\mathbb{E}^*\left[(X^*_1 - x)^2 K_h(X^*_1 - x)^2 (X^*_2 - x)K_h(X^*_2 - x)(X^*_3 - x)K_h(X^*_3 - x)\right]\right. \\
&+2\mathbb{E}^*\left[(X^*_1 - x)^3 K_h(X^*_1 - x)^2 K_h(X^*_2 - x)(X^*_3 - x)K_h(X^*_3 - x)\right] \\
&+2\mathbb{E}^*\left[(X^*_1 - x)K_h(X^*_1 - x)^2 (X^*_2 - x)^2 K_h(X^*_2 - x)(X^*_3 - x)K_h(X^*_3 - x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)^2K_h(X_2^*-x)K_h(X_3^*-x)\right]\Big] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3(X_2^*-x)^2K_h(X_2^*-x)\right]\right. \\
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3(X_2^*-x)K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2(X_2^*-x)K_h(X_2^*-x)^2\right]\right] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\right] \\
& +n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\right]+h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \Big[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right] \\
& \mathbb{E}^*\left[K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[K_h(X_3^*-x)\right]\Big] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\right. \\
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)^2\right]\right] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\int K_h(y-x)\hat{f}_g(y)\,dy \\
& \left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2+n^{-3}h^{-6}\int(y-x)^4K_h(y-x)^4\hat{f}_g(y)\,dy \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\int(y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy\right. \\
& \left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2
\end{aligned}
$$

$$
\begin{aligned}
& +2\int (y-x)^3 K_h(y-x)^2 \hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy + 2\int (y-x)K_h(y-x)^2\hat{f}_g(y)\,dy \\
& \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& + \int (y-x)^2 K_h(y-x)^2\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
& \left.\int K_h(y-x)\hat{f}_g(y)\,dy\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\int (y-x)^2 K_h(y-x)^3\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy\right. \\
& +2\int (y-x)^3 K_h(y-x)^3\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& + \int (y-x)^4 K_h(y-x)^3\hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy \\
& + \int (y-x)^2 K_h(y-x)^2\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^2\hat{f}_g(y)\,dy \\
& \left.+2\int (y-x)^3 K_h(y-x)^2\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)^2\hat{f}_g(y)\,dy\right] \\
= \; & h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right]^2(x) \\
& +n^{-3}h^{-6}\left[(K_h)^4 * \hat{d}^4_{x,g}\right](x) + h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\right. \\
& \left[K_h * \hat{d}^1_{x,g}\right]^2(x) + 2\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x) \\
& +2\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x) \\
& \left.+\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\left[(K_h)^3 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) + 2\left[(K_h)^3 * \hat{d}^3_{x,g}\right](x)\right. \\
& \left[K_h * \hat{d}^1_{x,g}\right](x) + \left[(K_h)^3 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) \\
& \left.+\left[(K_h)^2 * \hat{d}^2_{x,g}\right]^2(x) + 2\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\right]. \qquad \text{(C.44)}
\end{aligned}
$$

Considering analogous steps as in previous expressions, it leads to:

$$
\begin{aligned}
& \mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)^2\hat{s}_0^*(x;h)\right] \\
= \;& \mathbb{E}^*\left[n^{-1}\sum_{i=1}^n K_h(X_i^*-x)Y_i^*\left(n^{-1}h^{-2}\sum_{j=1}^n (X_j^*-x)^2K_h(X_j^*-x)\right)^2\right. \\
& \left. n^{-1}\sum_{\ell=1}^n K_h(X_\ell^*-x)\right] \\
= \;& n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[K_h(X_i^*-x)Y_i^*(X_j^*-x)^2\right. \\
& \left. K_h(X_j^*-x)(X_k^*-x)^2K_h(X_k^*-x)K_h(X_\ell^*-x)\right] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*(X_2^*-x)^2K_h(X_2^*-x)\right. \\
& \left.(X_3^*-x)^2K_h(X_3^*-x)K_h(X_4^*-x)\right] \\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\right] \\
& +h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\right.\right. \\
& \left.(X_2^*-x)^2K_h(X_2^*-x)K_h(X_3^*-x)\right] \\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2K_h(X_2^*-x)Y_2^*(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& \left.+\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*(X_2^*-x)^4K_h(X_2^*-x)^2K_h(X_3^*-x)\right]\right] \\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3Y_1^*K_h(X_2^*-x)\right]\right. \\
& +\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3K_h(X_2^*-x)Y_2^*\right] \\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3Y_1^*(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^4K_h(X_2^*-x)^2\right]\right] \\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^* \mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\mathbb{E}^*\left[K_h(X_4^*-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+n^{-3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\mid_{X_1^*}\right]\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\mid_{X_1^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[K_h(X_3^*-x)\right]\\
&+2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]\\
&\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*\mid_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\\
&\left.\mathbb{E}^*\left[K_h(X_3^*-x)\right]\right]\\
&+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\right.\\
&+\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]\\
&+2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&\left.+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)^2Y_1^*\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\right]\\
=\;&h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\mathbb{E}^*\left[K_h(X_4^*-x)\right]\\
&+n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]+h^{-4}\frac{(n-2)(n-1)}{n^3}\\
&\left[2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\right.\\
&\mathbb{E}^*\left[K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right].+\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\\
&\left.\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\mathbb{E}^*\left[K_h(X_3^*-x)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\right.\\
& +\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
& \left.+\mathbb{E}^*\left[K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\right]\\
= & \ h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\mathbb{E}^*\left[K_h(X_4^*-x)\right]\\
& +n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right]+h^{-4}\frac{(n-2)(n-1)}{n^3}\\
& \left[2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\right.\\
& \mathbb{E}^*\left[K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\\
& \mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]\\
& +\mathbb{E}^*\left[K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
& \mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]+\mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\\
& \left.\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\mathbb{E}^*\left[K_h(X_3^*-x)\right]\right]\\
& +h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\right.\\
& +\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
& +2\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
& \left.+\mathbb{E}^*\left[K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^2\right]\right]\\
= & \ h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\\
& \cdot\left(\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\right)^2
\end{aligned}
$$

$$
\begin{aligned}
& \int K_h(y-x)\hat{f}_g(y)\,dy + n^{-3}h^{-4}\int (y-x)^4 K_h(y-x)^4 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& + h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\int (y-x)^2 K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy \\
& + 2\int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy + \int K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \cdot\left(\int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy\right)^2 + \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& \left.\int (y-x)^4 K_h(y-x)^2 \hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy\right] \\
& + h^{-4}\frac{n-1}{n^3}\left[\int (y-x)^4 K_h(y-x)^3 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy\right. \\
& + \int (y-x)^4 K_h(y-x)^3 \hat{f}_g(y)\,dy \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& + 2\int (y-x)^2 K_h(y-x)^3 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
& + 2\int (y-x)^2 K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
& \left.+ \int K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^4 K_h(y-x)^2 \hat{f}_g(y)\,dy\right] \\
= \; & h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\left[K_h * \hat{d}^0_{x,g}\right](x) \\
& + n^{-3}h^{-4}\left[(K_h)^4 * \hat{b}^4_{x,g}\right](x) + h^{-4}\frac{(n-2)(n-1)}{n^3}\left[2\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\right. \\
& \left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) + \left[K_h * \hat{b}^0_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^4_{x,g}\right](x) \\
& \left[K_h * \hat{d}^0_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) \\
& \left.+ \left[(K_h)^2 * \hat{b}^0_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right]^2(x)\right] + h^{-4}\frac{n-1}{n^3}\left[\left[(K_h)^3 * \hat{b}^4_{x,g}\right](x)\right. \\
& \left[K_h * \hat{d}^0_{x,g}\right](x) + \left[(K_h)^3 * \hat{d}^4_{x,g}\right](x)\left[K_h * \hat{b}^0_{x,g}\right](x) \\
& + 2\left[(K_h)^3 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x) + 2\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x) \\
& \left.+ \left[(K_h)^2 * \hat{b}^4_{x,g}\right](x)\left[(K_h)^2 * \hat{f}_g\right](x)\right]. \qquad \text{(C.45)}
\end{aligned}
$$

Splitting a four-tuple sum into different cases according to ties in the indices, computing expectations via conditional expectations and performing changes of variables in the integrals can be used to prove:

$$
\begin{aligned}
& \mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)\hat{s}_1^*(x;h)^2\right] \\
= \; & \mathbb{E}^*\left[n^{-1}\sum_{i=1}^n K_h(X_i^*-x)Y_i^*\, n^{-1}h^{-2}\sum_{j=1}^n (X_j^*-x)^2K_h(X_j^*-x)\right. \\
& \left.\left(n^{-1}h^{-2}\sum_{k=1}^n (X_k^*-x)K_h(X_k^*-x)\right)^2\right] \\
= \; & n^{-4}h^{-6}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[K_h(X_i^*-x)Y_i^*(X_j^*-x)^2K_h(X_j^*-x)\right. \\
& \left.(X_k^*-x)K_h(X_k^*-x)(X_\ell^*-x)K_h(X_\ell^*-x)\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*(X_2^*-x)^2K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)\right. \\
& \left.(X_4^*-x)K_h(X_4^*-x)\right] \\
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)K_h(X_2^*-x)\right.\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2K_h(X_2^*-x)Y_2^*(X_3^*-x)^2K_h(X_3^*-x)\right] \\
& +2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)\right] \\
& \left.+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2K_h(X_2^*-x)Y_2^*(X_3^*-x)K_h(X_3^*-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*(X_2^*-x)K_h(X_2^*-x)\right]\right. \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)(X_2^*-x)^2K_h(X_2^*-x)^3Y_2^*\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*(X_2^*-x)^4K_h(X_2^*-x)^3\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^3K_h(X_2^*-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\,|_{X_1^*}\right]\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}
\end{aligned}
$$

$$\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\right]$$
$$+h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\mid_{X_1^*}\right]\right]\right.$$
$$\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]$$
$$\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*\mid_{X_1^*}\right]\right]$$
$$\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]$$
$$+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]$$
$$\left.\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-6}\frac{n-1}{n^3}$$
$$\left[2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\right.$$
$$+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^3Y_2^*\mid_{X_2^*}\right]\right]$$
$$+\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_1^*-x)Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^3\right]$$
$$+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]$$
$$\left.+2\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*\big|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\right]$$
$$=\quad n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}$$
$$\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\right]+h^{-6}\frac{(n-2)(n-1)}{n^3}$$
$$\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\right.$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]$$
$$\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]$$
$$+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]$$
$$\left.\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]$$

$$
\begin{aligned}
& +h^{-6}\frac{n-1}{n^3}\left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\right. \\
& \mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^3\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]+\mathbb{E}^*\left[K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^3\right]\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right] \\
& \left.\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\right]+h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)\right.\right. \\
& \left.K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right] \\
& \mathbb{E}^*\left[(X_3^*-x)^2K_h(X_3^*-x)\right]+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right] \\
& \mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& +2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right] \\
& \left.\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-6}\frac{n-1}{n^3} \\
& \left[2\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\right. \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^3\hat{m}_g^{LL}(X_2^*)\right] \\
& +\mathbb{E}^*\left[K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^4K_h(X_2^*-x)^3\right]\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+2\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\int(y-x)^4K_h(y-x)^4\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy \\
& \cdot\left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2
\end{aligned}
$$

$$
\begin{aligned}
& +h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\int (y-x)^2 K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& \cdot\left(\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2 + \int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
& \int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
& +2\int (y-x)K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \\
& \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy + 2\int (y-x)^3 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
& \left.\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right] + h^{-6}\frac{n-1}{n^3} \\
& \left[2\int (y-x)^3 K_h(y-x)^3 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right. \\
& +\int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^3 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& +\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^4 K_h(y-x)^3 \hat{f}_g(y)\,dy \\
& +\int (y-x)^2 K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^2 K_h(y-x)^2 \hat{f}_g(y)\,dy \\
& \left.+2\int (y-x)K_h(y-x)^2 \hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \int (y-x)^3 K_h(y-x)^2 \hat{f}_g(y)\,dy\right] \\
= \; & n^{-3}h^{-6}\left[(K_h)^4 * \hat{b}_{x,g}^4\right](x) + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h * \hat{b}_{x,g}^0\right](x) \\
& \left[K_h * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right]^2(x) + h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\right. \\
& \left[K_h * \hat{d}_{x,g}^1\right]^2(x) + \left[(K_h)^2 * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x) \\
& +2\left[(K_h)^2 * \hat{b}_{x,g}^1\right](x)\left[K_h * \hat{d}_{x,g}^2\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) \\
& \left.+2\left[(K_h)^2 * \hat{d}_{x,g}^3\right](x)\left[K_h * \hat{b}_{x,g}^0\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x)\right] + h^{-6}\frac{n-1}{n^3} \\
& \left[2\left[(K_h)^3 * \hat{b}_{x,g}^3\right](x)\left[K_h * \hat{d}_{x,g}^1\right](x) + \left[K_h * \hat{d}_{x,g}^2\right](x)\left[(K_h)^3 * \hat{b}_{x,g}^2\right](x)\right. \\
& +\left[K_h * \hat{b}_{x,g}^0\right](x)\left[(K_h)^3 * \hat{d}_{x,g}^4\right](x) + \left[(K_h)^2 * \hat{b}_{x,g}^2\right](x)\left[(K_h)^2 * \hat{d}_{x,g}^2\right](x) \\
& \left.+2\left[(K_h)^2 * \hat{b}_{x,g}^1\right](x)\left[(K_h)^2 * \hat{d}_{x,g}^3\right](x)\right]. \qquad \text{(C.46)}
\end{aligned}
$$

Carrying out similar computations as in previous expressions, it leads to:

$$
\begin{aligned}
&\mathbb{E}^*\left[\hat{\Psi}_h^{1*}(x;h)\hat{s}_0^*(x;h)\hat{s}_1^*(x;h)\hat{s}_2^*(x;h)\right]\\
=\;&\mathbb{E}^*\Bigg[n^{-1}\sum_{i=1}^n(X_i^*-x)K_h(X_i^*-x)Y_i^*\,n^{-1}\sum_{j=1}^n K_h(X_j^*-x)\\
&n^{-1}h^{-2}\sum_{k=1}^n(X_k^*-x)K_h(X_k^*-x)n^{-1}h^{-2}\sum_{\ell=1}^n(X_\ell^*-x)^2K_h(X_\ell^*-x)\Bigg]\\
=\;&n^{-4}h^{-4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n\mathbb{E}^*\left[(X_i^*-x)K_h(X_i^*-x)Y_i^*K_h(X_j^*-x)\right.\\
&\left.(X_k^*-x)K_h(X_k^*-x)(X_\ell^*-x)^2K_h(X_\ell^*-x)\right]\\
=\;&n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\right]\\
&+h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)Y_1^*K_h(X_2^*-x)\right.\\
&\left.(X_3^*-x)K_h(X_3^*-x)(X_4^*-x)^2K_h(X_4^*-x)\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2K_h(X_2^*-x)\right.\right.\\
&\left.(X_3^*-x)K_h(X_3^*-x)Y_3^*\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)Y_3^*\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2Y_1^*(X_2^*-x)K_h(X_2^*-x)K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2(X_2^*-x)^2K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)Y_3^*\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)K_h(X_3^*-x)\right]\\
&\left.+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)\right]\right]\\
&+h^{-4}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3(X_2^*-x)K_h(X_2^*-x)Y_2^*\right]\right.\\
&+\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3Y_1^*K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*(X_2^*-x)K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3Y_1^*(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^3K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&\left.+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2Y_1^*(X_2^*-x)K_h(X_2^*-x)^2\right]\right]\\
=\;&h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)Y_1^*\mid_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}^*\left[(X_4^*-x)^2K_h(X_4^*-x)\right]+n^{-3}h^{-4}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\,|_{X_1^*}\right]\right]\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\right.\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^*\,|_{X_3^*}\right]\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^*\,|_{X_3^*}\right]\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)Y_3^*\,|_{X_3^*}\right]\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\left.\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-4}\frac{n-1}{n^3}\\
&\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*\,|_{X_2^*}\right]\right]\right.\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3Y_1^*\,|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\Big]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&\left.+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)^2\right]\right]\\
=\;&h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]\\
&\mathbb{E}^*\left[K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)^2K_h(X_4^*-x)\right]\\
&+n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\mathbb{E}^*\left[Y_1^*\,|_{X_1^*}\right]\right]
\end{aligned}
$$

$$+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\right.$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\mid_{X_3^*}\right]\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]$$
$$\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\mid_{X_3^*}\right]\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\right]$$
$$\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\mathbb{E}^*\left[Y_3^*\mid_{X_3^*}\right]\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]$$
$$\left.\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-4}\frac{n-1}{n^3}$$
$$\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\right.$$
$$+\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]$$
$$+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)^2\right]$$
$$\left.+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right]$$
$$= h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]$$
$$\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\mathbb{E}^*\left[(X_4^*-x)^2K_h(X_4^*-x)\right]$$
$$+n^{-3}h^{-4}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right]+h^{-4}\frac{(n-2)(n-1)}{n^3}$$
$$\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\right.$$

$$
\begin{aligned}
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\hat{m}_g^{LL}(X_3^*)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&\mathbb{E}^*\left[K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\\
&\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\Big]\\
&+h^{-4}\frac{n-1}{n^3}\Big[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\Big]\\
&+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^3K_h(X_2^*-x)^2\right]\\
= \;& h^{-4}\frac{(n-3)(n-2)(n-1)}{n^3}\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\\
&\int K_h(y-x)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\\
&\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy+n^{-3}h^{-4}\int(y-x)^4K_h(y-x)^4\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\\
&+h^{-4}\frac{(n-2)(n-1)}{n^3}\left[\int(y-x)^3K_h(y-x)^2\hat{f}_g(y)\,dy\int K_h(y-x)\hat{f}_g(y)\,dy\right.\\
&\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy+\int(y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy\\
&\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\\
&+\int(y-x)^3K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\\
&\int K_h(y-x)\hat{f}_g(y)\,dy+\int(y-x)K_h(y-x)^2\hat{f}_g(y)\,dy\\
&\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
& + \int (y-x)^2 K_h(y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy \\
& \int K_h(y-x) \hat{f}_g(y)\, dy + \int (y-x) K_h(y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \\
& \left. \int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy \int (y-x) K_h(y-x) \hat{f}_g(y)\, dy \right] + h^{-4} \frac{n-1}{n^3} \\
& \left[\int (y-x)^3 K_h(y-x)^3 \hat{f}_g(y)\, dy \int (y-x) K_h(y-x) \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \right. \\
& + \int (y-x)^4 K_h(y-x)^3 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int K_h(y-x) \hat{f}_g(y)\, dy \\
& + \int (y-x)^3 K_h(y-x)^3 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x) K_h(y-x) \hat{f}_g(y)\, dy \\
& + \int (y-x)^2 K_h(y-x)^3 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy \\
& + \int (y-x) K_h(y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^3 K_h(y-x)^2 \hat{f}_g(y)\, dy \\
& + \int (y-x)^2 K_h(y-x)^3 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^2 K_h(y-x) \hat{f}_g(y)\, dy \\
& \left. + \int (y-x) K_h(y-x)^2 \hat{m}_g^{LL}(y) \hat{f}_g(y)\, dy \int (y-x)^3 K_h(y-x)^2 \hat{f}_g(y)\, dy \right] \\
= \; & h^{-4} \frac{(n-3)(n-2)(n-1)}{n^3} \left[K_h * \hat{b}_{x,g}^1 \right](x) \left[K_h * \hat{d}_{x,g}^0 \right](x) \left[K_h * \hat{d}_{x,g}^1 \right](x) \\
& \left[K_h * \hat{d}_{x,g}^2 \right](x) + n^{-3} h^{-4} \left[(K_h)^4 * \hat{b}_{x,g}^4 \right](x) + h^{-4} \frac{(n-2)(n-1)}{n^3} \\
& \left[\left[(K_h)^2 * \hat{d}_{x,g}^3 \right](x) \left[K_h * \hat{d}_{x,g}^0 \right](x) \left[K_h * \hat{b}_{x,g}^1 \right](x) \right. \\
& + \left[(K_h)^2 * \hat{d}_{x,g}^2 \right](x) \left[K_h * \hat{d}_{x,g}^1 \right](x) \left[K_h * \hat{b}_{x,g}^1 \right](x) \\
& + \left[(K_h)^2 * \hat{b}_{x,g}^3 \right](x) \left[K_h * \hat{d}_{x,g}^1 \right](x) \left[K_h * \hat{d}_{x,g}^0 \right](x) \\
& + \left[(K_h)^2 * \hat{d}_{x,g}^1 \right](x) \left[K_h * \hat{d}_{x,g}^2 \right](x) \left[K_h * \hat{b}_{x,g}^1 \right](x) \\
& + \left[(K_h)^2 * \hat{b}_{x,g}^2 \right](x) \left[K_h * \hat{d}_{x,g}^2 \right](x) \left[K_h * \hat{d}_{x,g}^0 \right](x) \\
& \left. + \left[(K_h)^2 * \hat{b}_{x,g}^1 \right](x) \left[K_h * \hat{d}_{x,g}^2 \right](x) \left[K_h * \hat{d}_{x,g}^1 \right](x) \right]
\end{aligned}
$$

$$+h^{-4}\frac{n-1}{n^3}\left[\left[(K_h)^3 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x)\right.$$
$$+\left[(K_h)^3 * \hat{b}^4_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x)+\left[(K_h)^3 * \hat{b}^3_{x,g}\right](x)\left[K_h * \hat{d}^1_{x,g}\right](x)$$
$$+\left[(K_h)^3 * \hat{b}^2_{x,g}\right](x)\left[K_h * \hat{d}^2_{x,g}\right](x)+\mathbb{E}^*\left[(K_h)^2 * \hat{b}^1_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^3_{x,g}\right](x)$$
$$\left.+\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)+\left[(K_h)^2 * \hat{b}^3_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^1_{x,g}\right](x)\right]. \tag{C.47}$$

Following similar steps as in previous expressions, we have:

$$\begin{aligned}
& \mathbb{E}^*\left[\hat{\Psi}^{1*}_h(x;h)\hat{s}^*_1(x;h)^3\right] \\
= \; & \mathbb{E}^*\left[n^{-1}\sum_{i=1}^n (X_i^*-x)K_h(X_i^*-x)Y_i^*\left(n^{-1}h^{-2}\sum_{j=1}^n (X_j^*-x)K_h(X_j^*-x)\right)^3\right] \\
= \; & n^{-4}h^{-6}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \mathbb{E}^*\left[(X_i^*-x)K_h(X_i^*-x)Y_i^*(X_j^*-x)K_h(X_j^*-x)\right. \\
& \left.(X_k^*-x)K_h(X_k^*-x)(X_\ell^*-x)K_h(X_\ell^*-x)\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)(X_2^*-x)K_h(X_2^*-x)\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)(X_4^*-x)K_h(X_4^*-x)Y_4^*\right] \\
& +3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2(X_2^*-x)K_h(X_2^*-x)Y_2^*\right.\right. \\
& \left.(X_3^*-x)K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\right. \\
& \left.\left.(X_2^*-x)K_h(X_2^*-x)(X_3^*-x)K_h(X_3^*-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3(X_2^*-x)K_h(X_2^*-x)Y_2^*\right]\right. \\
& +3\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*(X_2^*-x)K_h(X_2^*-x)\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& +\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right] \\
& \left.+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right] \\
= \; & n^{-3}h^{-6}\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4Y_1^*\mid_{X_1^*}\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\\
&\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)Y_4^*\mid_{X_4^*}\right]\right]\\
&+3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\right.\\
&\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&\left.\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-6}\frac{n-1}{n^3}\\
&\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)Y_2^*\mid_{X_2^*}\right]\right]\right.\\
&+3\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&\left.+\mathbb{E}^*\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right]\\
=\;&n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]+h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\\
&\mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\\
&\mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\mathbb{E}^*\left[Y_4^*\mid_{X_4^*}\right]\right]+3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\\
&\left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\right.\\
&\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\\
&\left.\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right]+h^{-6}\frac{n-1}{n^3}\\
&\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\mid_{X_2^*}\right]\right]\right.\\
&+3\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\mathbb{E}^*\left[Y_1^*\mid_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\\
&\left.+\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\mathbb{E}^*\left[Y_1^*|_{X_1^*}\right]\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& n^{-3}h^{-6}\mathbb{E}^*\left[(X_1^*-x)^4K_h(X_1^*-x)^4\hat{m}_g^{LL}(X_1^*)\right] + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \mathbb{E}^*\left[(X_1^*-x)K_h(X_1^*-x)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] \\
& \mathbb{E}^*\left[(X_4^*-x)K_h(X_4^*-x)\hat{m}_g^{LL}(X_4^*)\right] + 3\,h^{-6}\frac{(n-2)(n-1)}{n^3} \\
& \left[\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\right. \\
& \mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right] + \mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right] \\
& \left.\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right]\mathbb{E}^*\left[(X_3^*-x)K_h(X_3^*-x)\right]\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2^*)\right]\right. \\
& +3\mathbb{E}^*\left[(X_1^*-x)^3K_h(X_1^*-x)^3\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)K_h(X_2^*-x)\right] \\
& \left.+3\mathbb{E}^*\left[(X_1^*-x)^2K_h(X_1^*-x)^2\hat{m}_g^{LL}(X_1^*)\right]\mathbb{E}^*\left[(X_2^*-x)^2K_h(X_2^*-x)^2\right]\right] \\
= \;& n^{-3}h^{-6}\int(y-x)^4K_h(y-x)^4\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3} \\
& \left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^3\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
& +3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\int(y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy\right. \\
& \int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& \left.+\int(y-x)^2K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\left(\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2\right] \\
& +h^{-6}\frac{n-1}{n^3}\left[\int(y-x)^3K_h(y-x)^3\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\right. \\
& +3\int(y-x)^3K_h(y-x)^3\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy \\
& \left.+3\int(y-x)^2K_h(y-x)^2\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy\int(y-x)^2K_h(y-x)^2\hat{f}_g(y)\,dy\right] \\
= \;& n^{-3}h^{-6}\left[(K_h)^4*\hat{b}_{x,g}^4\right](x) + h^{-6}\frac{(n-3)(n-2)(n-1)}{n^3}\left[K_h*\hat{d}_{x,g}^1\right]^3(x) \\
& \left[K_h*\hat{b}_{x,g}^1\right](x) + 3\,h^{-6}\frac{(n-2)(n-1)}{n^3}\left[\left[(K_h)^2*\hat{d}_{x,g}^2\right](x)\left[K_h*\hat{b}_{x,g}^1\right](x)\right. \\
& \left.\left[K_h*\hat{d}_{x,g}^1\right](x) + \left[(K_h)^2*\hat{b}_{x,g}^2\right](x)\left[K_h*\hat{d}_{x,g}^1\right]^2(x)\right]
\end{aligned}
$$

$$+h^{-6}\frac{n-1}{n^3}\left[\left[(K_h)^3 * \hat{d}^3_{x,g}\right](x)\left[K_h * \hat{b}^1_{x,g}\right](x) + 3\left[(K_h)^3 * \hat{b}^3_{x,g}\right](x)\right.$$
$$\left.\left[K_h * \hat{d}^1_{x,g}\right](x) + 3\left[(K_h)^2 * \hat{b}^2_{x,g}\right](x)\left[(K_h)^2 * \hat{d}^2_{x,g}\right](x)\right]. \tag{C.48}$$

Splitting a double sum into two cases and computing a change of variable, it follows that:

$$\begin{aligned}
\mathbb{E}^*\left[\hat{\Theta}^{0*}_h\right] &= \mathbb{E}^*\left[\hat{s}^*_2(x;h)\hat{s}^*_0(x;h) - h^2\hat{s}^*_1(x;h)^2\right] \\
&= \mathbb{E}^*\left[\hat{s}^*_2(x;h)\hat{s}^*_0(x;h)\right] - h^2\mathbb{E}^*\left[\hat{s}^*_1(x;h)^2\right] \\
&= \mathbb{E}^*\left[n^{-2}h^{-2}\sum_{i=1}^n\sum_{j=1}^n (X^*_i - x)^2 K_h(X^*_i - x)K_h(X^*_j - x)\right] \\
&\quad -h^2\mathbb{E}^*\left[n^{-2}h^{-4}\sum_{i=1}^n\sum_{j=1}^n (X^*_i - x)K_h(X^*_i - x)(X^*_j - x)K_h(X^*_j - x)\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X^*_1 - x}{h}\right)^2 K_h(X^*_1 - x)K_h(X^*_2 - x)\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X^*_1 - x}{h}\right)K_h(X^*_1 - x)\left(\frac{X^*_2 - x}{h}\right)K_h(X^*_2 - x)\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X^*_1 - x}{h}\right)^2 K_h(X^*_1 - x)\right]\mathbb{E}^*\left[K_h(X^*_2 - x)\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X^*_1 - x}{h}\right)K_h(X^*_1 - x)\right]\mathbb{E}^*\left[\left(\frac{X^*_2 - x}{h}\right)K_h(X^*_2 - x)\right] \\
&= \frac{n-1}{n}h^{-2}\int (y-x)^2 K_h(y-x)\hat{f}_g(y)\,dy \int K_h(y-x)\hat{f}_g(y)\,dy \\
&\quad -\frac{n-1}{n}h^{-2}\left(\int (y-x)K_h(y-x)\hat{f}_g(y)\,dy\right)^2 \\
&= \frac{n-1}{n}h^{-2}\left(\left[K_h * \hat{d}^2_{x,g}\right](x)\left[K_h * \hat{d}^0_{x,g}\right](x) - \left[K_h * \hat{d}^1_{x,g}\right]^2(x)\right).
\end{aligned} \tag{C.49}$$

Carrying out analogous computations as in the expression above, we have:

$$\mathbb{E}^*\left[\hat{\Theta}^{1*}_h\right] = \mathbb{E}^*\left[\hat{\Psi}^{0*}_h(x;h)\hat{s}^*_2(x;h) - \hat{s}^*_1(x;h)\hat{\Psi}^{1*}_h(x;h)\right]$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[\hat{\Psi}_h^{0*}(x;h)\hat{s}_2^*(x;h)\right] - \mathbb{E}^*\left[\hat{s}_1^*(x;h)\hat{\Psi}_h^{1*}(x;h)\right] \\
&= \mathbb{E}^*\left[n^{-2}h^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i^*-x)^2K_h(X_i^*-x)K_h(X_j^*-x)Y_j^*\right] \\
&\quad -\mathbb{E}^*\left[n^{-2}h^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i^*-x)K_h(X_i^*-x)(X_j^*-x)K_h(X_j^*-x)Y_j^*\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)^2K_h(X_1^*-x)K_h(X_2^*-x)Y_2^*\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)K_h(X_1^*-x)\left(\frac{X_2^*-x}{h}\right)K_h(X_2^*-x)Y_2^*\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)K_h(X_1^*-x)\right]\mathbb{E}^*\left[\left(\frac{X_2^*-x}{h}\right)K_h(X_2^*-x)Y_2^*\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(X_2^*-x)Y_2^*|_{X_2^*}\right]\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)K_h(X_1^*-x)\right]\mathbb{E}^*\left[\mathbb{E}^*\left[\left(\frac{X_2^*-x}{h}\right)K_h(X_2^*-x)Y_2^*\Big|_{X_2^*}\right]\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*|_{X_2^*}\right]\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)K_h(X_1^*-x)\right]\mathbb{E}^*\left[\left(\frac{X_2^*-x}{h}\right)K_h(X_2^*-x)\mathbb{E}^*\left[Y_2^*\,|_{X_2^*}\right]\right] \\
&= \frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)^2K_h(X_1^*-x)\right]\mathbb{E}^*\left[K_h(X_2^*-x)\hat{m}_g^{LL}(X_2)\right] \\
&\quad -\frac{n-1}{n}\mathbb{E}^*\left[\left(\frac{X_1^*-x}{h}\right)K_h(X_1^*-x)\right]\mathbb{E}^*\left[\left(\frac{X_2^*-x}{h}\right)K_h(X_2^*-x)\hat{m}_g^{LL}(X_2)\right] \\
&= \frac{n-1}{n}h^{-2}\int(y-x)^2K_h(y-x)\hat{f}_g(y)\,dy\int K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
&\quad -\frac{n-1}{n}h^{-2}\int(y-x)K_h(y-x)\hat{f}_g(y)\,dy\int(y-x)K_h(y-x)\hat{m}_g^{LL}(y)\hat{f}_g(y)\,dy \\
&= \frac{n-1}{n}h^{-2}\left(\left[K_h*\hat{d}_{x,g}^2\right](x)\left[K_h*\hat{b}_{x,g}^0\right](x)-\left[K_h*\hat{d}_{x,g}^1\right](x)\left[K_h*\hat{b}_{x,g}^1\right](x)\right).
\end{aligned}
\tag{C.50}
$$

Collecting terms (C.39)-(C.48) and plugging them in (4.20), Theorem 14 holds.

Theorem 15 *Given K a symmetric bounded density function, a simple random sample known as source sample, $\left\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\right\}$, and $(X_1^1, \ldots, X_{n_1}^1)$ a simple random sample coming from the target population, then the MASE of the proxy estimator given in (4.17) is given by:*

$$
\begin{aligned}
MASE_{\tilde{m}_h^{LL}, X^1}(h) \;=\; & \frac{1}{n_1} \sum_{j=1}^{n_1} \left[\left(\Theta_j^0\right)^{-2} h^{-4} \left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3} \right.\right. \\
& \left[K_h * b^0_{X_j^1}\right]^2 (X_j^1) \left[K_h * d^2_{X_j^1}\right]^2 (X_j^1) \\
& + \frac{(n_0-1)(n_0-2)}{n_0^3} \left(\left[(K_h)^2 * a^0_{X_j^1}\right] (X_j^1) \left[K_h * d^2_{X_j^1}\right]^2 (X_j^1) \right. \\
& + \left[(K_h)^2 * d^4_{X_j^1}\right] (X_j^1) \left[K_h * b^0_{X_j^1}\right]^2 (X_j^1) \\
& \left. + 2 \left[(K_h)^2 * b^2_{X_j^1}\right] (X_j^1) \left[K_h * b^0_{X_j^1}\right] (X_j^1) \left[K_h * d^2_{X_j^1}\right] (X_j^1) \right) \\
& \left. + \frac{n_0-1}{n_0^3} \left[(K_h)^2 * a^2_{X_j^1}\right] \left(X_j^1\right) \left[(K_h)^2 * d^4_{X_j^1}\right] \left(X_j^1\right) \right] \\
& - 2 \left(\Theta_j^0\right)^{-2} h^{-4} \left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3} \left[K_h * b^0_{X_j^1}\right] (X_j^1) \right. \\
& \left[K_h * d^2_{X_j^1}\right] (X_j^1) \left[K_h * d^1_{X_j^1}\right] (X_j^1) \left[K_h * b^1_{X_j^1}\right] (X_j^1) \\
& + \frac{(n_0-1)(n_0-2)}{n_0^3} \left[\left[(K_h)^2 * b^1_{X_j^1}\right] (X_j^1) \left[K_h * d^2_{X_j^1}\right] (X_j^1) \right. \\
& \left[K_h * b^1_{X_j^1}\right] (X_j^1) \\
& + \left[(K_h)^2 * a^1_{X_j^1}\right] (X_j^1) \left[K_h * d^2_{X_j^1}\right] (X_j^1) \left[K_h * d^1_{X_j^1}\right] (X_j^1) \\
& + \left[(K_h)^2 * d^3_{X_j^1}\right] (X_j^1) \left[K_h * b^1_{X_j^1}\right] (X_j^1) \left[K_h * b^0_{X_j^1}\right] (X_j^1) \\
& \left. + \left[(K_h)^2 * b^3_{X_j^1}\right] (X_j^1) \left[K_h * b^0_{X_j^1}\right] (X_j^1) \left[K_h * d^1_{X_j^1}\right] (X_j^1) \right] \\
& + \frac{n_0-1}{n_0^3} \left(\left[(K_h)^2 * b^1_{X_j^1}\right] \left(X_j^1\right) \left[(K_h)^2 * b^3_{X_j^1}\right] \left(X_j^1\right) \right. \\
& \left.\left. + \left[(K_h)^2 * a^1_{X_j^1}\right] \left(X_j^1\right) \left[(K_h)^2 * d^3_{X_j^1}\right] \left(X_j^1\right) \right) \right] + \left(\Theta_j^0\right)^{-2} h^{-4} \\
& \left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3} \left[K_h * d^1_{X_j^1}\right]^2 (X_j^1) \left[K_h * b^1_{X_j^1}\right]^2 (X_j^1) \right.
\end{aligned}
$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * a^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)\right.$$

$$+\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right]^2(X^1_j)$$

$$\left.+2\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\right]$$

$$+\frac{n_0-1}{n_0^3}\left(2\left[(K_h)^2 * b^2_{X^1_j}\right]^2\left(X^1_j\right)\right.$$

$$\left.\left.+\left[(K_h)^2 * d^2_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * a^2_{X^1_j}\right]\left(X^1_j\right)\right)\right]$$

$$-2\left(\Theta^0_j\right)^{-3}\Theta^1_j\,h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^0_{X^1_j}\right](X^1_j)\right.$$

$$\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\right.$$

$$+\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[(K_h)^2 * d^4_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)+\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)$$

$$\left.\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)+\left[(K_h)^2 * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)\right]$$

$$\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^4_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * \hat{f}^0_g\right]\left(X^1_j\right)\right]$$

$$+2\left(\Theta^0_j\right)^{-3}\Theta^1_j h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\right.$$

$$\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[3\left[(K_h)^2 * b^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\right.$$

$$\left.\left[K_h * d^1_{X^1_j}\right](X^1_j)+2\left[(K_h)^2 * d^3_{X^1_j}\right](X^1_j)\left[K_h * b^0_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^1_{X^1_j}\right]\left(X^1_j\right)\left[(K_h)^2 * d^3_{X^1_j}\right]\left(X^1_j\right)\right]$$

$$+2\left(\Theta^0_j\right)^{-3}\Theta^1_j h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * b^1_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\right.$$

$$\left[K_h * d^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * d^3_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\right.$$

$$+\left[(K_h)^2 * b^3_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)$$

$$+\left[(K_h)^2 * d^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\Big]$$

$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^3_{X^1_j}\right](X^1_j)\left[(K_h)^2 * d^1_{X^1_j}\right](X^1_j)\Big]$$

$$-2\left(\Theta^0_j\right)^{-3}\Theta^1_j h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^1_{X^1_j}\right]^3(X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)$$

$$+2\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\left[K_h * b^1_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)$$

$$+\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j}\right]^2(X^1_j)\Big]$$

$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * b^2_{X^1_j}\right](X^1_j)\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\Bigg]+\left(\Theta^1_j\right)^2\left(\Theta^0_j\right)^{-4}h^{-4}$$

$$\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)\left[K_h * d^0_{X^1_j}\right]^2(X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[\left[(K_h)^2 * d^4_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right]^2(X^1_j)$$

$$+\left[(K_h)^2 * d^0_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right]^2(X^1_j)$$

$$+2\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\Big]$$

$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^4_{X^1_j}\right](X^1_j)\left[(K_h)^2 * \hat{f}^0_g\right](X^1_j)\Bigg]$$

$$+h^{-4}\left(\Theta^1_j\right)^2\left(\Theta^0_j\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^1_{X^1_j}\right]^4(X^1_j)$$

$$+4\frac{(n_0-1)(n_0-2)}{n_0^3}\left[(K_h)^2 * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)$$

$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^2_{X^1_j}\right]^2(X^1_j)\Bigg]$$

$$-2\,h^{-4}\left(\Theta^1_j\right)^2\left(\Theta^0_j\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * d^2_{X^1_j}\right](X^1_j)$$

$$\left[K_h * d^0_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right]^2(X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[2\left[(K_h)^2 * d^3_{X^1_j}\right](X^1_j)\left[K_h * d^0_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)$$

$$+2\left[(K_h)^2 * d^1_{X^1_j}\right](X^1_j)\left[K_h * d^2_{X^1_j}\right](X^1_j)\left[K_h * d^1_{X^1_j}\right](X^1_j)\Big]$$

$$+2\frac{n_0-1}{n_0^3}\left[(K_h)^2 * d^3_{X_j^1}\right](X_j^1)\left[(K_h)^2 * d^1_{X_j^1}\right](X_j^1)\Big]\Big],$$

where $a_x^j(y) = (y-x)^j\left(\sigma_0^2(y) + m^2(y)\right)f^0(y)$, $b_x^j(y) = (y-x)^j m(y) f^0(y)$ *and* $d_x^j(y) = (y-x)^j f^0(y)$, *with* $j \in \mathbb{N}$.

Proof of Theorem 15 Similar computations as those used to prove Theorem 13 would lead to conclude the proof of Theorem 15.

Theorem 16 *Consider* K*, a symmetric bounded density function, a simple random sample known as source sample,* $\left\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\right\}$*, a simple random sample coming from the target population,* $(X_1^1, \ldots, X_{n_1}^1)$*, and* $g > 0$ *a pilot bandwidth, then the MASE of the proxy estimator given in (4.17) is given by:*

$$
\begin{aligned}
MASE^*_{\tilde{m}_h^{LL}, X^1}(h) \;=\; & \frac{1}{n_1}\sum_{j=1}^{n_1}\Bigg[\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3} \\
& \left[K_h * \hat{b}^0_{X_j^1,g}\right]^2(X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right]^2(X_j^1) \\
& +\frac{(n_0-1)(n_0-2)}{n_0^3}\Bigg(\left[(K_h)^2 * \hat{a}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right]^2(X_j^1) \\
& +\left[(K_h)^2 * \hat{d}^4_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^0_{X_j^1,g}\right]^2(X_j^1) \\
& +2\left[(K_h)^2 * \hat{b}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\Bigg) \\
& +\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{a}^2_{X_j^1,g}\right]\left(X_j^1\right)\left[(K_h)^2 * \hat{d}^4_{X_j^1,g}\right]\left(X_j^1\right)\Bigg] \\
& -2\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{b}^0_{X_j^1,g}\right](X_j^1) \\
& \left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1) \\
& +\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[\left[(K_h)^2 * \hat{b}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1) \\
& \left[K_h * \hat{b}^1_{X_j^1,g}\right](X_j^1) \\
& +\left[(K_h)^2 * \hat{a}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)
\end{aligned}
$$

$$+\left[(K_h)^2 * \hat{d}^3_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^1_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)$$

$$+\left[(K_h)^2 * \hat{b}^3_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right](X^1_j)\Big]$$

$$+\frac{n_0-1}{n_0^3}\left(\left[\left[(K_h)^2 * \hat{b}^1_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2 * \hat{b}^3_{X^1_j,g}\right]\left(X^1_j\right)\right.\right.$$

$$\left.\left.+\left[(K_h)^2 * \hat{a}^1_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2 * \hat{d}^3_{X^1_j,g}\right]\left(X^1_j\right)\right)\right]$$

$$+\left(\hat{\Theta}^0_{g,j}\right)^{-2} h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{d}^1_{X^1_j,g}\right]^2(X^1_j)\right.$$

$$\left[K_h * \hat{b}^1_{X^1_j,g}\right]^2(X^1_j)+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * \hat{a}^2_{X^1_j,g}\right](X^1_j)\right.$$

$$\left[K_h * \hat{d}^1_{X^1_j,g}\right]^2(X^1_j)+\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^1_{X^1_j,g}\right]^2(X^1_j)$$

$$\left.+2\left[(K_h)^2 * \hat{b}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^1_{X^1_j,g}\right](X^1_j)\right]+\frac{n_0-1}{n_0^3}$$

$$\left.\cdot\left(2\left[(K_h)^2 * \hat{b}^2_{X^1_j,g}\right]^2\left(X^1_j\right)+\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2 * \hat{a}^2_{X^1_j,g}\right]\left(X^1_j\right)\right)\right]$$

$$-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}\, h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)\right.$$

$$\left[K_h * \hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2 * \hat{b}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)\right.$$

$$+\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)\left[(K_h)^2 * \hat{d}^4_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^0_{X^1_j,g}\right](X^1_j)$$

$$+\left[(K_h)^2 * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)$$

$$\left.+\left[(K_h)^2 * \hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right]^2(X^1_j)\right]$$

$$\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{b}^4_{X^1_j,g}\right]\left(X^1_j\right)\left[(K_h)^2 * \hat{f}^0_g\right]\left(X^1_j\right)\right]$$

$$+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h * \hat{b}^0_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^2_{X^1_j,g}\right](X^1_j)\left[K_h * \hat{d}^1_{X^1_j,g}\right]^2(X^1_j)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[3\left[(K_h)^2*\hat{b}^1_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^2_{X_j^1,g}\right](X_j^1)\right.$$

$$\left[K_h*\hat{d}^1_{X_j^1,g}\right](X_j^1)+2\left[(K_h)^2*\hat{d}^3_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{b}^0_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^1_{X_j^1,g}\right](X_j^1)\Big]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^1_{X_j^1,g}\right]\left(X_j^1\right)\left[(K_h)^2*\hat{d}^3_{X_j^1,g}\right]\left(X_j^1\right)\right]$$

$$+2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h*\hat{b}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$\left[K_h*\hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^2_{X_j^1,g}\right](X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{d}^3_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{b}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$+\left[(K_h)^2*\hat{b}^3_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^0_{X_j^1,g}\right](X_j^1)$$

$$\left.+\left[(K_h)^2*\hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{b}^1_{X_j^1,g}\right](X_j^1)\right]$$

$$\left.+\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^3_{X_j^1,g}\right]\left(X_j^1\right)\left[(K_h)^2*\hat{d}^1_{X_j^1,g}\right]\left(X_j^1\right)\right]$$

$$-2\left(\hat{\Theta}^0_{g,j}\right)^{-3}\hat{\Theta}^1_{g,j}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h*\hat{d}^1_{X_j^1,g}\right]^3(X_j^1)\left[K_h*\hat{b}^1_{X_j^1,g}\right](X_j^1)$$

$$+2\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{b}^1_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^1_{X_j^1,g}\right](X_j^1)\right.$$

$$\left.+\left[(K_h)^2*\hat{b}^2_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^1_{X_j^1}\right]^2(X_j^1)\right]$$

$$\left.+3\frac{n_0-1}{n_0^3}\left[(K_h)^2*\hat{b}^2_{X_j^1,g}\right](X_j^1)\left[(K_h)^2*\hat{d}^2_{X_j^1,g}\right](X_j^1)\right]$$

$$+\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}h^{-4}\left[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\right.$$

$$\left[K_h*\hat{d}^2_{X_j^1,g}\right]^2(X_j^1)\left[K_h*\hat{d}^0_{X_j^1,g}\right]^2(X_j^1)$$

$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\left[\left[(K_h)^2*\hat{d}^4_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^0_{X_j^1,g}\right]^2(X_j^1)\right.$$

$$+\left[(K_h)^2*\hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h*\hat{d}^2_{X_j^1,g}\right]^2(X_j^1)$$

$$+2\left[(K_h)^2 * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)\Big]$$
$$+\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^4_{X_j^1,g}\right](X_j^1)\left[(K_h)^2 * \hat{f}^0_g\right](X_j^1)\Big]$$
$$+h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}\left[K_h * \hat{d}^1_{X_j^1,g}\right]^4(X_j^1)$$
$$+4\frac{(n_0-1)(n_0-2)}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right]^2(X_j^1)$$
$$+3\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^2_{X_j^1,g}\right]^2(X_j^1)\Bigg]$$
$$-2\,h^{-4}\left(\hat{\Theta}^1_{g,j}\right)^2\left(\hat{\Theta}^0_{g,j}\right)^{-4}\Bigg[\frac{(n_0-1)(n_0-2)(n_0-3)}{n_0^3}$$
$$\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right]^2(X_j^1)$$
$$+\frac{(n_0-1)(n_0-2)}{n_0^3}\Big[2\left[(K_h)^2 * \hat{d}^3_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^0_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)$$
$$+2\left[(K_h)^2 * \hat{d}^1_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^2_{X_j^1,g}\right](X_j^1)\left[K_h * \hat{d}^1_{X_j^1,g}\right](X_j^1)\Big]$$
$$+2\frac{n_0-1}{n_0^3}\left[(K_h)^2 * \hat{d}^3_{X_j^1,g}\right](X_j^1)\left[(K_h)^2 * \hat{d}^1_{X_j^1,g}\right](X_j^1)\Big]\Bigg],$$

where $\hat{a}^j_{x,g}(y) = (y-x)^j\left(\hat{\sigma}^2_g(y) + g^2\mu_2(K) + \hat{m}^{LL}_g(y)^2\right)\hat{f}_g(y)$, $\hat{d}^j_{x,g}(y) = (y-x)^j\hat{f}_g(y)$ *and* $\hat{b}^j_{x,g}(y) = (y-x)^j\hat{m}^{LL}_g(y)\hat{f}_g(y)$, *with* $j \in \mathbb{N}$.

Proof of Theorem 16 Analogous computations as those used to prove Theorem 14 would lead to conclude the proof of Theorem 16.

Lemma 1 *Under regularity conditions (K1), (D1), (M1) and (V1), the function* $MISE^a := \mathbb{E}\left[\int\left(\tilde{m}^{NW}_h(x) - m(x)\right)^2 dF_1(x)\right]$ *admits the following representation:*

$$\begin{aligned} MISE^a(h) &= \frac{R(K)}{n_0\,h}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \\ &+\frac{h^4}{4}\mu_2(K)^2\int\Bigg[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \\ &+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\Bigg]f^1(x)\,dx + \mathcal{O}(h^6) + \mathcal{O}\left(\frac{h}{n_0}\right). \end{aligned} \tag{C.51}$$

Moreover, the asymptotic version of expression (C.51), namely $AMISE^a(h)$, *is given* *by:*

$$
\begin{aligned}
AMISE^a(h) &= \frac{R(K)}{n_0\,h}\int \frac{\sigma^2(x) f^1(x)}{f^0(x)}dx \\
&\quad + \frac{h^4}{4}\mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right. \\
&\quad \left. + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx.
\end{aligned}
$$

Proof of Lemma 1 Using a Taylor expansion and a change of variable, we obtain:

$$
\begin{aligned}
& \mathbb{E}\left[\int \left(\tilde{m}_h^{NW}(x) - m(x)\right)^2 dF_1(x)\right] \\
=\; & \mathbb{E}\left[\int \frac{1}{n_0 f^0(x)} \sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]^2 dF_1(x) \\
=\; & \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]^2 dF_1(x) \\
=\; & \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[Var\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right. \\
& \left. + \left(\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right)^2\right] dF_1(x) \\
=\; & \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[\sum_{i=1}^{n_0} Var\left[K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right. \\
& \left. + \left(\sum_{i=1}^{n_0} \mathbb{E}\left[K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right)^2\right] dF_1(x) \\
=\; & \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[n_0 Var\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right. \\
& \left. + n_0^2\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2\right] dF_1(x) \\
=\; & \frac{1}{n_0}\int \frac{1}{f^0(x)^2} Var\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right] dF_1(x)
\end{aligned}
$$

$$
\begin{aligned}
& + \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(Y_1^0 - m(x)\right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \left[\mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(Y_1^0 - m(x)\right)^2 \right] \right. \\
& \left. - \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(Y_1^0 - m(x)\right) \right] \right)^2 \right] dF_1(x) \\
& + \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(Y_1^0 - m(x)\right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(Y_1^0 - m(x)\right)^2 \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(Y_1^0 - m(x)\right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[\mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(Y_1^0 - m(x)\right)^2 \mid_{X_1^0} \right] \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(Y_1^0 - m(x)\right) \mid_{X_1^0} \right] \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \mathbb{E}\left[\left(Y_1^0 - m(x)\right)^2 \mid_{X_1^0} \right] \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(\mathbb{E}\left[Y_1^0 \mid_{X_1^0} \right] - m(x) \right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(Var\left(Y_1^0 - m(x) \mid_{X_1^0} \right) \right. \right. \\
& \left. \left. + \left(\mathbb{E}\left[Y_1^0 - m(x) \mid_{X_1^0} \right] \right)^2 \right) \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(m(X_1^0) - m(x) \right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(Var\left(Y_1^0 \mid_{X_1^0} \right) + \left(\mathbb{E}\left[Y_1^0 \mid_{X_1^0} \right] - m(x) \right)^2 \right) \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(m(X_1^0) - m(x) \right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \left(\sigma^2(X_1^0) + \left(m(X_1^0) - m(x) \right)^2 \right) \right] dF_1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\mathbb{E}\left[K_h\left(x - X_1^0\right) \left(m(X_1^0) - m(x) \right) \right] \right)^2 dF_1(x) \\
= \; & \frac{1}{n_0 h^2} \int \frac{1}{f^0(x)^2} \mathbb{E}\left[K\left(\frac{x - X_1^0}{h} \right)^2 \left(\sigma^2(X_1^0) + \left(m(X_1^0) - m(x) \right)^2 \right) \right] dF_1(x)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{n_0-1}{n_0\,h^2}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K\left(\frac{x-X_1^0}{h}\right)\left(m(X_1^0)-m(x)\right)\right]\right)^2 dF_1(x) \\
= \; & \frac{1}{n_0\,h^2}\int \frac{1}{f^0(x)^2}\int K\left(\frac{y-x}{h}\right)^2\left(\sigma^2(y)+(m(y)-m(x))^2\right) f^0(y)\,dy\,dF_1(x) \\
& +\frac{n_0-1}{n_0\,h^2}\int \frac{1}{f^0(x)^2}\left(\int K\left(\frac{y-x}{h}\right)(m(y)-m(x))\,f^0(y)\,dy\right)^2 dF_1(x) \\
= \; & \frac{1}{n_0\,h}\int \frac{1}{f^0(x)^2}\int K\left(u\right)^2\left(\sigma^2(x+hu)+(m(x+hu)-m(x))^2\right) \\
& f^0(x+hu)\,du\,dF_1(x)+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2} \\
& \left(\int K\left(u\right)(m(x+hu)-m(x))\,f^0(x+hu)\,du\right)^2 dF_1(x).
\end{aligned}
\tag{C.52}
$$

Let us denote $\psi(y,x)=(m(y)-m(x))\,f^0(y)$. Then, $\psi(x,x)=0$; $\frac{\partial\psi}{\partial y}(y,x)=m'(y)f(y)+(m(y)-m(x))\left(f^0\right)'(y)$; $\frac{\partial^2\psi}{\partial y^2}(y,x)=m''(y)f^0(y)+2m'(y)\left(f^0\right)'(y)+(m(y)-m(x))\left(f^0\right)''(y)$. Using Taylor expansion:

$$
\begin{aligned}
\int K(u)\psi(x,x+hu)\,du \; &= \; \int K(u)\psi(x,x)\,du+h\int u\,K(u)\,\frac{\partial\psi}{\partial y}(x,x)\,du+\frac{h^2}{2}\int u^2 \\
& \quad K(u)\frac{\partial^2\psi}{\partial y^2}(x,x)\,du+\frac{h^3}{3!}\int u^3K(u)\frac{\partial^3\psi}{\partial y^3}(x,x)\,du+\mathcal{O}(h^4) \\
&= \; \frac{h^2}{2}\frac{\partial^2\psi}{\partial y^2}(x,x)\mu_2(K)+\mathcal{O}(h^4),
\end{aligned}
$$

which leads to the second term in (C.52):

$$
\begin{aligned}
& \frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\int K\left(u\right)\psi(x,x+hu)\,du\right)^2 dF_1(x) \\
= \; & \frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\frac{h^4}{4}\left(\frac{\partial^2\psi}{\partial y^2}(x,x)\right)^2\mu_2(K)^2dF_1(x)+\mathcal{O}(h^6) \\
= \; & \frac{(n_0-1)h^4}{n_0}\frac{1}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.
\end{aligned}
$$

$$+\frac{4m'(x)^2\,(f^0)'\,(x)^2}{f^0(x)^2}\Bigg]\, f^1(x)dx+\mathcal{O}(h^6). \tag{C.53}$$

On the other hand, denote $\phi(y)=\left[\sigma^2(y)+(m(y)-m(x))^2\right]f^0(y)$. Then, $\phi(x)=\sigma^2(x)f^0(x)$ and using Taylor expansion, the first term in (C.52) is:

$$\begin{aligned}
&\frac{1}{n_0\,h}\int\frac{1}{f^0(x)^2}\int K(u)^2\phi(x+hu)\,du\,dF_1(x)\\
=\;&\frac{R(K)}{n_0\,h}\int\frac{1}{f^0(x)^2}\sigma^2(x)f^0(x)f^1(x)\,dx+\mathcal{O}\left(\frac{h}{n_0}\right)\\
=\;&\frac{R(K)}{n_0\,h}\int\frac{\sigma^2(x)\,f^1(x)}{f^0(x)}dx+\mathcal{O}\left(\frac{h}{n_0}\right).
\end{aligned}\tag{C.54}$$

Assembling terms (C.53) and (C.54), and inserting them in expression (C.52), we have that Lemma 1 holds.

Lemma 6 *Consider a sequence of bandwidths* $c\,n_0^{-1/5}, c>0$ *and the function* $MISE^a(h)$, *given in (4.26) it turns out:*

$$\begin{aligned}
MISE^a\left(c\,n_0^{-1/5}\right)\;=\;&R(K)n_0^{-4/5}c^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx\\
&+\frac{c^4\,n_0^{-4/5}}{4}\mu_2(K)^2\int\Bigg[m''(x)^2+\frac{4m'(x)m''(x)\,(f^0)'\,(x)}{f^0(x)}\\
&+\frac{4m'(x)^2\,(f^0)'\,(x)^2}{f^0(x)^2}\Bigg]\,f^1(x)\,dx+\mathcal{O}(n_0^{-6/5}),
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{n_0\to\infty}n_0^{4/5}MISE^a\left(c\,n_0^{-1/5}\right)\;=\;&R(K)c^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx\\
&+\frac{c^4}{4}\mu_2(K)^2\int\Bigg[m''(x)^2+\frac{4m'(x)m''(x)\,(f^0)'\,(x)}{f^0(x)}\\
&+\frac{4m'(x)^2\,(f^0)'\,(x)^2}{f^0(x)^2}\Bigg]\,f^1(x)\,dx.
\end{aligned}\tag{C.55}$$

Then,

$$\lim_{n_0\to\infty}n_0^{1/5}\cdot h_{MISE^a}\;=\;c_0.$$

Proof of Lemma 6 Given that h_{MISE^a} is the value which minimizes the function $MISE^a$, it turns out:

$$n_0^{4/5} MISE^a\left(c_0 \cdot n_0^{-1/5}\right) \geq n_0^{4/5} MISE^a\left(h_{MISE^a}\right), \forall n_0 \in \mathbb{N},$$

which implies that

$$\begin{aligned} & R(K) c_0^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\ & + \frac{c_0^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \\ \geq & \lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \end{aligned} \tag{C.56}$$

Bringing together expressions (4.26) and (C.56), we can conclude that:

$$\lim_{n_0 \to \infty} \sup n_0^{1/5} h_{MISE^a} < \infty.$$

Indeed,

$$\begin{aligned} & \lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \\ \geq & \lim_{n_0 \to \infty} \sup \left[\frac{\left(n_0^{1/5}\, h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right.\right. \\ & \left.\left. + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \right], \end{aligned}$$

and together with expression (C.56) lead to conclude that

$$\begin{aligned} k \geq & \lim_{n_0 \to \infty} \sup \left[\frac{\left(n_0^{1/5}\, h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right.\right. \\ & \left.\left. + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \right], \end{aligned}$$

being k a real positive number, which implies that:

$$\lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} \, h_{MISE^a} \right)^4 \leq k \quad \Leftrightarrow \quad \lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} \, h_{MISE^a} \right) \leq k.$$

By means of similar calculations, we can prove that $\lim\limits_{n_0 \rightarrow \infty} \inf n_0^{1/5} h_{MISE^a} > 0$. As a matter of fact,

$$\lim_{n_0 \rightarrow \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \quad \geq \quad \lim_{n_0 \rightarrow \infty} \sup \left[\frac{R(K)}{n_0^{1/5} \, h_{MISE^a}} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \right],$$

which implies, together with expression (C.56), that

$$\lim_{n_0 \rightarrow \infty} \sup a_{n_0} \quad = \quad \lim_{n_0 \rightarrow \infty} \sup \left[\frac{1}{n_0^{1/5} \, h_{MISE^a}} \right] = k, \forall k \in \mathbb{R}^+. \tag{C.57}$$

Given that a_{n_0} is a positive sequence and using expression (C.57), we can conclude that $\lim\limits_{n_0 \rightarrow \infty} \inf n_0^{1/5} h_{MISE^a} > 0$.

From both limit conditions, we can state that there exist two numbers $L, U \in \mathbb{R}^+, L < U$, satisfying:

$$L \leq n_0^{1/5} h_{MISE^a} \leq U, \text{ for almost all } n_0 \in \mathbb{N}. \tag{C.58}$$

Using expressions (4.26) and (C.58), we obtain:

$$\begin{aligned}
& n_0^{4/5} MISE^a(h_{MISE^a}) \\
= \quad & R(K) \left(n_0^{1/5} h_{MISE^a} \right)^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx + \frac{\left(n_0^{1/5} h_{MISE^a} \right)^4}{4} \mu_2(K)^2 \\
& \int \left[m''(x)^2 + \frac{4m'(x)m''(x) \left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x) \, dx
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}\left(\left(n_0^{1/5}h_{MISE^a}\right)^4 h_{MISE^a}^2\right)+\mathcal{O}\left(n_0^{-1/5}h_{MISE^a}\right)\\
=\quad & R(K)\left(n_0^{1/5}h_{MISE^a}\right)^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{\left(n_0^{1/5}h_{MISE^a}\right)^4}{4}\mu_2(K)^2\\
&\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx\\
&+\mathcal{O}\left(h_{MISE^a}^2\right)+\mathcal{O}\left(n_0^{-2/5}\right). \qquad \text{(C.59)}
\end{aligned}
$$

Consider a subsequence of

$$
\left\{n_0^{1/5}h_{MISE^a}\right\}_{n_0\in\mathbb{N}}, \qquad \text{(C.60)}
$$

which converges to a real number l (which is positive as a consequence of expression (C.58)). Furthermore, expression (C.59) assures that the corresponding subsequence of

$$
\left\{n_0^{4/5}MISE^a\left(h_{MISE^a}\right)\right\}_{n_0\in\mathbb{N}},
$$

converges to

$$
\begin{aligned}
&R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx+\mathcal{O}\left(h_{MISE^a}^2\right)+\mathcal{O}\left(n_0^{-2/5}\right).
\end{aligned}
$$

Moreover, expression (C.56) guarantees that

$$
\begin{aligned}
&R(K)c_0^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx\\
&+\frac{c_0^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx\\
\geq\quad & R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx.
\end{aligned}
$$

As a consequence of c_0 being a strict absolute minimum (in $c > 0$) of the second term of expression (C.55), the previous inequality can only be satisfied when $l = c_0$. This argument establishes that c_0 is the unique adherent point of the sequence given in (C.60). Accordingly, considering expression (C.58), the proof is concluded.

Up to now, we have obtained an asymptotic result concerning a first order bandwidth which minimizes $MISE^a$. From now on, we investigate the second order term further.

Lemma 7 *Consider the sequence of functions defined below and L, U satisfying expression (C.58),*

$$\Gamma_{n_0}(z) := MISE^a(z\, n_0^{-1/5}), z > 0, n_0 \in \mathbb{N},$$

and

$$\gamma_{n_0} := \arg \min_{L \leq z \leq U} \Gamma_{n_0}(z).$$

It turns out that the relation among these two functions and expression (4.26) is as follows:

$$h_{MISE^a} = \gamma_{n_0} n_0^{-1/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{C.61}$$

Proof of Lemma 7 Expression (4.26) guarantees the existence of L, U verifying expression (C.58). In addition, the continuity of the kernel function K assures the continuity of $MISE^a(h)$. Therefore, a point exists, namely γ_{n_0}, within the interval $[L, U]$, in which the function Γ_{n_0} attains its minimum, whether it is unique or not.

Nevertheless, given that inequalities in expression (C.58) are fulfilled, any minimizer of Γ_{n_0} is attained within the interval $[L, U]$. Indeed, if it were not the case, there would exist a minimizer z_0 outside $[L, U]$ verifying

$$\Gamma_{n_0}(z_0) \leq \Gamma_{n_0}(z), \forall z > 0, \tag{C.62}$$

and therefore,

$$MISE^a(z_0\, n_0^{-1/5}) \leq MISE^a(h), \forall h > 0, \tag{C.63}$$

so that $z_0\, n_0^{-1/5}$ would be a minimizer of $MISE^a$, but it would not satisfy inequalities given in (C.58) for that particular $n_0 \in \mathbb{N}$. Finally, the equivalence between expressions (C.62) and (C.63) provides (C.61) and the proof is concluded.

In particular, the function Γ_{n_0} satisfies

$$\begin{aligned}\Gamma_{n_0}(z) \;&=\; R(K) n_0^{-4/5} z^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\ &+ \frac{z^4\, n_0^{-4/5}}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\,(f^0)'(x)}{f^0(x)} \right. \\ &\left. + \frac{4m'(x)^2\,(f^0)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + \mathcal{O}(z^6\, n_0^{-6/5}) + \mathcal{O}\left(z\, n_0^{-6/5} \right), z > 0\end{aligned}$$

If we define the function Λ_{n_0} such that

$$\Lambda_{n_0}(z) := n_0^{4/5} \Gamma_{n_0}(z), z > 0,$$

then,

$$\begin{aligned}\Lambda_{n_0}(z) \;&=\; R(K) z^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx + \frac{z^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\,(f^0)'(x)}{f^0(x)} \right. \\ &\left. + \frac{4m'(x)^2\,(f^0)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + \mathcal{O}(z^6\, n_0^{-6/5}) + \mathcal{O}\left(z\, n_0^{-2/5} \right), z > 0.\end{aligned}$$

If we restrict to those z within the interval $[L, U]$, then

$$\begin{aligned}\Lambda_{n_0}(z) \;&=\; R(K) z^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx + \frac{z^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\,(f^0)'(x)}{f^0(x)} \right. \\ &\left. + \frac{4m'(x)^2\,(f^0)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + \mathcal{O}\left(n_0^{-2/5} \right),\end{aligned}$$

uniformly in $z \in [L, U]$.

Define now the function

$$T(z) := R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx, \tag{C.64}$$

then we have,

$$\sup_{L\leq z\leq U} |\Lambda_{n_0}(z) - T(z)| = \mathcal{O}\left(n_0^{-2/5}\right). \tag{C.65}$$

Denote, for the sake of brevity,

$$\begin{aligned} a &:= \frac{1}{4}\mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx \\ &= \frac{1}{4}\mu_2(K)^2 \int \left[m''(x)f^0(x) + 2\,m'(x)\left(f^0\right)'(x)\right]^2 f^1(x)\left(f^0(x)\right)^{-1} dx, \text{ and} \\ b &:= R(K)\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx. \end{aligned}$$

Then, $T(z) = b\,z^{-1} + z^4\,a$. Due to the fact that the kernel K is a nonnegative function and $\int K(u)\,du = 1$, then $R(K) > 0$ and $\mu_2(K) > 0$. Moreover, assume that K is bounded and $\int u^2K(u)\,du < \infty$, $R(K)$ and $\mu_2(K)$ are finite numbers. In addition, assume that the density function, f^0, is one time differentiable and its first derivative is bounded and continuous in the point x. As for the regression function, m, assume it is two times differentiable and its first and second derivatives are bounded and continous in the point x. Then, we can conclude that $a > 0$ and $b > 0$.

Lemma 8 *Consider T the function defined in (C.64), it turns out that T attains a strict relative minimum in*

$$z_0 := \left(\frac{b}{4\,a}\right)^{1/5}. \tag{C.66}$$

Furthermore, fix $\delta > 0$ *a real number,*

$$T(z_0 + \delta) - T(z_0) \;\; > \;\; 6\, a\, z_0^2\, \delta^2. \tag{C.67}$$

If $\delta < z_0$*, then we have*

$$T(z_0 - \delta) - T(z_0) \;\; > \;\; 6\, a\, z_0^2\, \delta^2 - 4\, a\, z_0\, \delta^3. \tag{C.68}$$

Proof of Lemma 8 Function T is differentiable in $(0, +\infty)$ and its derivative is given by:

$$T'(z) \;\; = \;\; 4\, a\, z^3 + b\, z^{-2}, \forall z > 0.$$

It is straightforward that z_0 is the unique point in $(0, +\infty)$ where T' is zero. Moreover, T' takes negative values within the interval $(0, z_0)$ and positive values within the interval $(z_0, +\infty)$. Then, T is strictly decreasing in $(0, z_0)$ and strictly increasing in $(z_0, +\infty)$, leading to (C.66).

Now, so as to prove expression (C.67), fix $\delta > 0$ and consider

$$\begin{aligned}
T(z_0 + \delta) - T(z_0) \;\; &= \;\; b(z_0 + \delta)^{-1} + (z_0 + \delta)^4\, a - b\, z_0^{-1} - a\, z_0^4 \\
&= \;\; b(z_0 + \delta)^{-1} + a(z_0^4 + 4\, \delta\, z_0^3 + 6\, \delta^2\, z_0^2 + 4\, \delta^3\, z_0 + \delta^4) - b\, z_0^{-1} - a\, z_0^4 \\
&= \;\; b\left((z_0 + \delta)^{-1} - \; z_0^{-1}\right) + a(4\, \delta\, z_0^3 + 6\, \delta^2\, z_0^2 + 4\, \delta^3\, z_0 + \delta^4) \\
&= \;\; a(4\, \delta\, z_0^3 + 6\, \delta^2\, z_0^2 + 4\, \delta^3\, z_0 + \delta^4) - b\, \delta\, ((z_0 + \delta)\, z_0)^{-1} \\
&= \;\; \delta(4\, a\, z_0^3 - b(\, z_0^2 + \delta z_0)^{-1}) + 6\, a\, \delta^2 z_0^2 + 4\, a\, \delta^3 z_0 + a\, \delta^4 \\
&> \;\; \delta(4\, a\, z_0^3 - b\, z_0^{-2}) + 6\, a\, \delta^2 z_0^2 + 4\, a\, \delta^3 z_0 + a\, \delta^4 \\
&> \;\; 6\, a\, \delta^2 z_0^2.
\end{aligned}$$

Similarly, in order to prove expression (C.68), consider $\delta < z_0$ and,

$$\begin{aligned} T(z_0-\delta) - T(z_0) &= b(z_0-\delta)^{-1} + (z_0-\delta)^4\, a - b\, z_0^{-1} - a\, z_0^4 \\ &= b(z_0-\delta)^{-1} + a(z_0^4 - 4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) - b\, z_0^{-1} - a\, z_0^4 \\ &= b\left((z_0-\delta)^{-1} - z_0^{-1}\right) + a(-4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) \end{aligned}$$

$$\begin{aligned} &= a(-4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) - b\,\delta\,((z_0-\delta)\, z_0)^{-1} \\ &= \delta(-4\,a\, z_0^3 + b(\, z_0^2 - \delta z_0)^{-1}) + 6\,a\,\delta^2 z_0^2 - 4\,a\,\delta^3 z_0 + a\,\delta^4 \\ &> \delta(-4\,a\, z_0^3 + b\, z_0^{-2}) + 6\,a\,\delta^2 z_0^2 - 4\,a\,\delta^3 z_0 + a\,\delta^4 \\ &> 6\,a\,\delta^2 z_0^2 - 4\,a\,\delta^3 z_0. \end{aligned}$$

These two last inequalities prove expressions (C.67) and (C.68).

Remark 21 *The minimizer of T, z_0 is precisely c_0. Indeed, T happens to be the dominant part of Λ_{n_0}, the minimization of which is (under certain conditions) equivalent to the minimization of $MISE^a$.*

Theorem 17 *Under regularity conditions (K1), (D1), (M1), the bandwidth parameter selector which minimizes the function $MISE^a$ happens to be:*

$$h_{MISE^a} \;=\; c_0\, n_0^{-1/5} + \mathcal{O}\left(n_0^{-2/5}\right).$$

Proof of Theorem 17 Firstly, assume that $z_0 \in (L,U)$. Indeed, if $z_0 \notin (L,U)$, we would consider a wider interval which would fulfil that condition. Furthermore, expression (C.65) guarantees the existence of some constant $C > 0$ such that

$$\sup_{L\le z\le U} |\Lambda_{n_0}(z) - T(z)| \;\le\; C\, n_0^{-2/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{C.69}$$

Consider the following sequence of real positive numbers,

$$\delta_{n_0} \;:=\; \left(C\,(2\,a)^{-1}\right)^{1/2}\, z_0^{-1}\, n_0^{-1/5}, n_0 \in \mathbb{N}. \tag{C.70}$$

Using expression (C.67), we have

$$T(z_0+\delta_{n_0})-T(z_0) \;\;>\;\; 6\,a\,z_0^2\,\delta_{n_0}^2 = 3\,C\,n_0^{-2/5}, \forall n_0\in\mathbb{N}. \tag{C.71}$$

On the other hand, given that $\delta_{n_0} < z_0$, for almost all natural n_0,

$$\begin{aligned} T(z_0-\delta_{n_0})-T(z_0) \;\;&>\;\; 6\,a\,z_0^2\,\delta_{n_0}^2 - 4\,a\,z_0\,\delta_{n_0}^3 \\ &= 3\,C\,n_0^{-2/5} - 2\,C^{3/2}\,z_0^{-2}(2a)^{-1/2}n^{-3/5}. \end{aligned}$$

As a consequence,

$$T(z_0-\delta_{n_0})-T(z_0) \;\;>\;\; 2\,C\,n_0^{-2/5},\text{ for almost all } n_0\in\mathbb{N}.$$

Providing that z_0 is an interior point of the interval $[L,U]$ and $\{\delta_{n_0}\}$ tends to zero as n_0 tends to ∞, the sets given below are well defined for almost all $n_0 \in \mathbb{N}$

$$A_{n_0} \;\;:=\;\; [L, z_0-\delta_{n_0}] \cup [z_0+\delta_{n_0}, U]\,.$$

Consider now $z \in [L,U]$ and expressions (C.66), (C.69), (C.70) and (C.71). For almost $n_0 \in \mathbb{N}$, we have

$$\begin{aligned} z\in[L,z_0-\delta_{n_0}] \;\;&\Rightarrow\;\; z \le z_0-\delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C\,n_0^{-2/5} \ge T(z) \ge T(z_0-\delta_{n_0}) \\ &> T(z_0) + 2\,C\,n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + C\,n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \\ z\in[z_0+\delta_{n_0},U] \;\;&\Rightarrow\;\; z \ge z_0+\delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C\,n_0^{-2/5} \ge T(z) \ge T(z_0+\delta_{n_0}) \\ &> T(z_0) + 3\,C\,n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + 2\,C\,n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \end{aligned}$$

Consequently,

$$z \in A_{n_0} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0).$$

Therefore, for all $n_0 \in \mathbb{N}$ except for a finite number of them (at most), any minimizer γ_{n_0} of Λ_{n_0} (and Γ_{n_0}) verifies $\gamma_{n_0} \in [L,U] \setminus A_{n_0}$. In other words, $|\gamma_{n_0} - z_0| \le \delta_{n_0}$.

According to expression (C.61), for almost all $n_0 \in \mathbb{N}$, every minimizer h_{MISE^a} of $MISE^a$ satisfies:

$$\left|h_{MISE^a} - c_0\, n_0^{-1/5}\right| \leq \delta_{n_0} n_0^{-1/5} = (C\,(2\,a)^{-1})^{1/2}\, z_0^{-1}\, n_0^{-2/5},$$

for almost all $n_0 \in \mathbb{N}$, which implies expression (4.29).

Proposition 2 *Suppose conditions (C1), (C2) and (C3) are fulfilled. Consider* h_{n_0} *is a sequence of bandwidhts such that* $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$. *Assume* $(n_0\, h)^{-1/2} \log(h^{-1}) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\, h \longrightarrow \infty$. *Then,*

$$\begin{aligned} ISE(h) &= ISE^a(h) + \mathcal{O}_P\left(h^6\right) + \mathcal{O}_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathcal{O}_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) \\ &\quad + \mathcal{O}_P\left(\frac{\log\frac{1}{h}}{n_0^{3/2} h^{3/2}}\right), \end{aligned}$$

where $ISE(h) = \int \left(\hat{m}_h^{NW}(x) - m(x)\right)^2 dF^1(x)$ *and, on the other hand,* $ISE^a(h) = \int \left(\tilde{m}_h^{NW}(x) - m(x)\right)^2 dF^1(x)$.

Proof of Proposition 2 Thanks to results proven by Silverman (1978), under conditions (C1), (C2) and assuming that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\, h \longrightarrow \infty$, we have:

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| = \mathcal{O}_P\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).$$

Additionally, thanks to results proven by Mack and Silverman (1982), we have:

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| \;=\; \mathcal{O}_P\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).$$

Consider now $\{\xi_{n_0}\}$ a sequence of random variables defined in the probability space (Ω, A, P) such that $\xi_{n_0} \geq 0$ and $\xi_{n_0} = \mathcal{O}_P\left(a_{n_0} + b_{n_0}\right)$, where $\left(a_{n_0}\right), \left(b_{n_0}\right)$ are sequences of real positive numbers. Let's see that

$$\xi_{n_0} = \mathcal{O}_P\left(a_{n_0} + b_{n_0}\right) \Leftrightarrow \xi_{n_0} = \mathcal{O}_P\left(\max\{a_{n_0}, b_{n_0}\}\right) \tag{C.72}$$

'$\Leftarrow$' $a_{n_0} + b_{n_0} \geq \max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{\xi_{n_0}}{a_{n_0} + b_{n_0}} \leq \dfrac{\xi_{n_0}}{\max\{a_{n_0}, b_{n_0}\}}$.

'$\Rightarrow$' $a_{n_0} + b_{n_0} \leq 2\max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{2\xi_{n_0}}{a_{n_0} + b_{n_0}} \geq \dfrac{\xi_{n_0}}{\max\{a_{n_0,b_{n_0}}\}}$.

It is straightforward that

$$MISE(h) \;=\; MISE^a(h) + \mathbb{E}\left[\int 2\,A_1\,A_2\,dF^1(x)\right] + \mathbb{E}\left[\int A_2^2\,dF^1(x)\right],$$

where

$$\begin{aligned} A_1 &= \frac{\hat{\Psi}_h(x)}{f^0(x)} - \frac{m(x)\hat{f}_h^0(x)}{f^0(x)} = \frac{1}{n_0\,f^0(x)}\sum_{i=1}^{n_0} K_h(x - X_i^0)(Y_i^0 - m(x)), \\ A_2 &= \left(\hat{m}_h^{NW}(x) - m(x)\right)\frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)}, \end{aligned}$$

and $MISE^a(h)$ is given in (4.26).

On the one hand, using expression (C.72) as well as Silverman (1978) and Mack and Silverman (1982) statements, it turns out:

$$\int_J A_2^2\,dF^1(x) \;=\; \left|\int_J A_2^2\,dF^1(x)\right| = \left|\int_J \left(\hat{m}_h^{NW}(x) - m(x)\right)\frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)}f^1(x)\,dx\right|$$

$$
\begin{aligned}
&\leq \sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right|^2 \sup_J \left|\hat{f}_h^0(x) - f^0(x)\right|^2 \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \left(\mathcal{O}_P\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \left(\mathcal{O}_P\left(\max\left\{h^2, n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \left(\mathcal{O}_P\left(\max\left\{h^2, n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \left(\mathcal{O}_P\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\,dx \left(\mathcal{O}_P\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right) \\
&= \mathcal{O}_P\left(h^8\right) + \mathcal{O}_P\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right). \qquad \text{(C.73)}
\end{aligned}
$$

The first term in expression (C.73) is negligible in comparison to the second term in expression (4.26). Similarly, the second term in expression (C.73) becomes insignificant compared to the first term in expression (4.26).

In particular,

$$
\begin{aligned}
\left(\int_J A_2^2\,dF^1(x)\right)^{1/2} &= \left(\mathcal{O}_P\left(h^8\right) + \mathcal{O}_P\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right)^{1/2} \\
&= \left(\mathcal{O}_P\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right)^{1/2} \\
&= \left(\mathcal{O}_P\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}\right)\right)^{1/2} \\
&= \mathcal{O}_P\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}^{1/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{O}_P\left(\max\left\{h^4, n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right\}\right)\\
&= \mathcal{O}_P\left(h^4 + n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right)\\
&= \mathcal{O}_P\left(h^4\right) + \mathcal{O}_P\left(n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right). \qquad \text{(C.74)}
\end{aligned}
$$

Moreover, thanks to expression (4.26), we know that

$$
\mathbb{E}\left[\int_J A_1^2\, dF_1(x)\right] \;=\; \mathcal{O}\left(n_0^{-1}h^{-1} + h^4\right). \qquad \text{(C.75)}
$$

Applying Markov's inequality to expression (C.75), it turns out:

$$
\int_J A_1^2\, dF^1(x) \;=\; \mathcal{O}_P\left(n_0^{-1}h^{-1} + h^4\right).
$$

Thus,

$$
\begin{aligned}
\left(\int_J A_1^2\, dF^1(x)\right)^{1/2} &= \left(\mathcal{O}_P\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = \mathcal{O}_P\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right)\\
&= \mathcal{O}_P\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = \mathcal{O}_P\left(n_0^{-1/2}h^{-1/2} + h^2\right)\\
&= \mathcal{O}_P\left(n_0^{-1/2}h^{-1/2}\right) + \mathcal{O}_P\left(h^2\right). \qquad \text{(C.76)}
\end{aligned}
$$

Bringing together expressions (C.76) and (C.74) and using Cauchy-Swchartz inequality, we compute:

$$
\begin{aligned}
\left|\int_J A_1\, A_2\, dF^1(x)\right| &\le 2\left(\int_J A_1^2\, dF^1(x)\right)^{1/2}\left(\int_J A_2^2\, dF^1(x)\right)^{1/2}\\
&= 2\left(\mathcal{O}_p\left(h^4\right) + \mathcal{O}_P\left(n_0^{-1}h^{-1}\log\frac{1}{h}\right)\right)\\
&\quad \cdot\left(\mathcal{O}_p\left(h^2\right) + \mathcal{O}_P\left(n_0^{-1/2}h^{-1/2}\log\frac{1}{h}\right)\right)
\end{aligned}
$$

$$= \mathcal{O}_P\left(h^6\right) + \mathcal{O}_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathcal{O}_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathcal{O}_P\left(\frac{\log\frac{1}{h}}{(n_0\,h)^{3/2}}\right). \tag{C.77}$$

The first term in expression (C.77) is negligible as compared to the second term in expression (4.26). Furthermore, the fourth term in expression (C.77) becomes insignificant in comparison to the first term in expression (4.26) if $\left(\frac{\log\frac{1}{h}}{n_0^{1/2}h^{1/2}}\right) \longrightarrow 0$ as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0\,h \longrightarrow \infty$.

It remains to be seen what happens with the second and third terms in expression (C.77). We begin with the second one. Given that $\left(n_0^{-1}h^{-1} + h^4\right) \cdot n_0 h = 1 + h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1}h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0\,h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2}h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

which is true providing that $n_0 h^5 \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2}h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h^{7/2}}{n_0^{1/2}} = o\left(\dfrac{1}{n_0\, h} + h^4\right)$.

Finally, as for the third term in (C.77),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0} \log \frac{1}{h} \sim n_0^{-6/5} \log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{\frac{1}{n_0\, h}} = h^2 \log \frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

which is true providing that $h \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{h^4} = \frac{\log \frac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h}{n_0} \log \dfrac{1}{h} = o\left(\dfrac{1}{n_0\, h} + h^4\right)$.

Considering this last argument and collecting terms (C.76), (C.74) and (C.77), expression (4.30) is proven.

Lemma 2 *Under regularity conditions (K1), (D1), (M1) and (V1), the function $MISE^{a*}$ admits the following representation:*

$$\begin{aligned} MISE^{a*}(h) \;\; = \;\; & \frac{R(K)}{n_0\, h} \hat{A}_g + \frac{h^4}{4} \mu_2(K)^2 \hat{B}_g \\ & + \mathcal{O}_P\left(h^6\, n_1^{-1}\, g^{-7}\left(g^{-2} + g^{-1} + 1\right)\right) + \mathcal{O}_P(h^8\, n_1^{-1} g^{-9}) \\ & + \mathcal{O}_P\left(h^{-1}\, g^2\, n_1^{-1}\right) + \mathcal{O}_P\left(h\, n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right). \end{aligned}$$

Besides, the asymptotic version of expression (4.32), namely $AMISE^{a*}(h)$, is given by:

$$AMISE^{a*}(h) \;=\; \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g.$$

Proof of Lemma 2 Consider the smoothed bootstrap version of $MISE^a$. We start by computing $MISE^{a*}(h) := \mathbb{E}^*\left[\int \left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x)\,dx\right]$, which is the theoretical analogue to $MASE^*_{\tilde{m}_h^{NW},X^1}$ given in (4.13). Using a Taylor expansion and a change of variable, we obtain:

$$\begin{aligned}
& \mathbb{E}^*\left[\int \left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x)\right] \\
= \;& \mathbb{E}^*\left[\int \left[\frac{1}{n_0\hat{f}_g^0(x)}\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x)\right] \\
= \;& \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x) \\
= \;& \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[Var^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right. \\
& \left. + \left(\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
= \;& \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[\sum_{i=1}^{n_0} Var^*\left[K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right. \\
& \left. + \left(\sum_{i=1}^{n_0}\mathbb{E}^*\left[K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
= \;& \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[n_0 Var^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right. \\
& \left. + n_0^2\left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
= \;& \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2} Var^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right] d\hat{F}_g^1(x)
\end{aligned}$$

$$
\begin{aligned}
& + \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left[\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(Y_1^{0*} - \hat{m}_g(x)\right)^2\right]\right. \\
& \left. - \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
& + \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(Y_1^{0*} - \hat{m}_g(x)\right)^2\right] d\hat{F}_g^1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(Y_1^{0*} - \hat{m}_g(x)\right)^2 |_{X_1^{0*}}\right]\right] d\hat{F}_g^1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right) |_{X_1^{0*}}\right]\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \mathbb{E}^* \left[\left(Y_1^{0*} - \hat{m}_g(x)\right)^2 |_{X_1^{0*}}\right]\right] d\hat{F}_g^1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(\mathbb{E}^* \left[Y_1^{0*} |_{X_1^{0*}}\right] - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(Var^* \left(Y_1^{0*} - \hat{m}_g(x) |_{X_1^{0*}}\right)\right.\right. \\
& \left.\left. + \left(\mathbb{E}^* \left[Y_1^{0*} - \hat{m}_g(x) |_{X_1^{0*}}\right]\right)^2\right)\right] d\hat{F}_g^1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(Var^* \left(Y_1^{0*} |_{X_1^{0*}}\right)\right.\right. \\
& \left.\left. + \left(\mathbb{E}^* \left[Y_1^{0*} |_{X_1^{0*}}\right] - \hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x) \\
& + \frac{n_0 - 1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
= \; & \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h\left(x - X_1^{0*}\right)^2 \left(\hat{\sigma}_g^2(X_1^{0*}) + g^2 \mu_2(K)\right.\right. \\
& \left.\left. + \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
= &\ \frac{1}{n_0\,h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K\left(\frac{x-X_1^{0*}}{h}\right)^2\left(\hat{\sigma}_g^2(X_1^{0*})+g^2\,\mu_2(K)\right.\right.\\
&\left.\left.+\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x)\\
&+\frac{n_0-1}{n_0\,h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K\left(\frac{x-X_1^{0*}}{h}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
= &\ \frac{1}{n_0\,h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\int K\left(\frac{y-x}{h}\right)^2\\
&\left(\hat{\sigma}_g^2(y)+g^2\,\mu_2(K)+\left(\hat{m}_g(y)-\hat{m}_g(x)\right)^2\right)\hat{f}_g^0(y)\,dy\,d\hat{F}_g^1(x)\\
&+\frac{n_0-1}{n_0\,h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\int K\left(\frac{y-x}{h}\right)\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\hat{f}_g^0(y)\,dy\right)^2 d\hat{F}_g^1(x)\\
= &\ \frac{1}{n_0\,h}\int \frac{1}{\hat{f}_g^0(x)^2}\int K\left(u\right)^2\left(\hat{\sigma}_g^2(x+hu)+g^2\,\mu_2(K)+\left(\hat{m}_g(x+hu)-\hat{m}_g(x)\right)^2\right)\\
&\hat{f}_g^0(x+hu)\,du\,d\hat{F}_g^1(x)+\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\\
&\left(\int K\left(u\right)\left(\hat{m}_g(x+hu)-\hat{m}_g(x)\right)\hat{f}_g^0(x+hu)\,du\right)^2 d\hat{F}_g^1(x). \qquad \text{(C.78)}
\end{aligned}
$$

Let us denote $\hat{\psi}_g(y,x)=\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\hat{f}_g^0(y)$. Then, $\hat{\psi}_g(x,x)=0$; $\frac{\partial\hat{\psi}_g}{\partial y}(y,x)=\hat{m}_g'(y)\hat{f}_g^0(y)+\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\left(\hat{f}_g^0\right)'(y)$; $\frac{\partial^2\hat{\psi}_g}{\partial y^2}(y)=\hat{m}_g''(y)\hat{f}_g^0(y)+2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)+\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\left(\hat{f}_g^0\right)''(y)$; $\frac{\partial^3\hat{\psi}_g}{\partial y^3}(y,x)=\hat{m}_g'''(y)\hat{f}_g^0(y)+3\hat{m}_g''(y)\left(\hat{f}_g^0\right)'(y)+3\hat{m}_g'(y)\left(\hat{f}_g^0\right)''(y)+\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\left(\hat{f}_g^0\right)'''(y)$; $\frac{\partial^4\hat{\psi}_g}{\partial y^4}(y,x)=\hat{m}_g^{(4)}(y)\hat{f}_g^0(y)+4\hat{m}_g'''(y)\left(\hat{f}_g^0\right)'(y)+6\hat{m}_g''(y)\left(\hat{f}_g^0\right)''(y)+4\hat{m}_g'(y)\left(\hat{f}_g^0\right)'''(y)+\left(\hat{m}_g(y)-\hat{m}_g(x)\right)\left(\hat{f}_g^0\right)^{(4)}(y)$; $\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)=\hat{m}_g''(x)\hat{f}_g^0(x)+2\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)$;$\frac{\partial\hat{\psi}_g}{\partial^4 y^4}(x,x)=\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)+4\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x)+6\hat{m}_g''(x)$

$\left(\hat{f}_g^0\right)''(x) + 4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x)$. Using Taylor expansion:

$$
\begin{aligned}
& \int K(u)\hat{\psi}_g(x+hu,x)\,du \\
= \; & \int K(u)\hat{\psi}_g(x,x)\,du + h\int u\,K(u)\,\frac{\partial\hat{\psi}_g}{\partial y}(x,x)\,du + \frac{h^2}{2}\int u^2K(u)\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\,du \\
& + \frac{h^3}{3!}\int u^3K(u)\frac{\partial^3\hat{\psi}_g}{\partial y^3}(x,x)\,du + \frac{h^4}{4!}\int u^4K(u)\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x)\,du \\
& + \frac{h^5}{5!}\int u^5K(u)\frac{\partial^5\hat{\psi}_g}{\partial y^5}(x,x)\,du + \mathcal{O}_P(h^6) \\
= \; & \frac{h^2}{2}\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\mu_2(K) + \frac{h^4}{24}\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x)\mu_4(K) + \mathcal{O}_P(h^6), \text{ and} \\
& \left(\int K(u)\hat{\psi}_g(x+hu,x)\,du\right)^2 \\
= \; & \frac{h^4}{4}\left(\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\right)^2\mu_2(K)^2 + \frac{h^6}{24}\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x)\mu_2(K)\mu_4(K) \\
& + \frac{h^8}{576}\left(\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x)\right)^2\mu_4^2(K) + \mathcal{O}_P(h^{12}) \\
= \; & \frac{h^4}{4}\left(\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\right)^2\mu_2(K)^2 + \mathcal{O}_P(h^6n_0^{-2}g^{-8}\left(g^{-2}+g^{-1}+1\right)) + \mathcal{O}_P(h^8n_0^{-2}g^{-10}),
\end{aligned}
$$

where

$$
\begin{aligned}
\left(\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\right)^2 &= \hat{m}_g''(x)^2\hat{f}_g^0(x)^2 \\
&\quad + 4\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2 + 4\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x), \\
\left(\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x)\right)^2 &= \hat{m}_g^{(4)}(x)^2\hat{f}_g^0(x)^2 + 8\hat{m}_g^{(4)}(x)\hat{m}_g'''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&\quad + 12\hat{m}_g^{(4)}(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)''(x) \\
&\quad + 8\hat{m}_g^{(4)}(x)\hat{m}_g'(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x) + 16\hat{m}_g'''(x)^2\left(\hat{f}_g^0\right)'(x)^2
\end{aligned}
$$

$$+48\hat{m}_g'''(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x)$$
$$+32\hat{m}_g'''(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x)\left(\hat{f}_g^0\right)'(x)+36\hat{m}_g''(x)^2\left(\hat{f}_g^0\right)''(x)^2$$
$$+48\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)''(x)\left(\hat{f}_g^0\right)'''(x)+16\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'''(x)^2,$$

$$\begin{aligned}\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\frac{\partial^4\hat{\psi}_g}{\partial y^4}(x,x) &= \hat{m}_g''(x)^2\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)^2+4\hat{m}_g'''(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x)\\ &\quad +6\hat{m}_g''(x)^2\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x)+2\hat{m}_g'(x)\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)\\ &\quad \left(\hat{f}_g^0\right)'(x)+8\hat{m}_g'(x)\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x)^2+12\hat{m}_g'(x)\hat{m}_g''(x)\\ &\quad \left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x)+8\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)'''(x),\end{aligned}$$

$$\left(\hat{f}_g^0\right)^{(r)}(x) = n_0^{-1}\,g^{-r-1}\sum_{i=1}^{n_0}K^{(r)}\left(\frac{x-X_i^0}{g}\right),$$

$$\begin{aligned}\hat{m}_g'(x) &= g^{-1}\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^{-2}\cdot\left(\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)Y_i^0\right.\\ &\quad \left.\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)-\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)Y_i^0\right),\end{aligned}$$

$$\hat{m}_g''(x) = \left[\frac{\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)}+\frac{2\left(\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right.$$

$$- \frac{\sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2}$$

$$\left. - \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} \right] \cdot g^{-2},$$

$$\begin{aligned}
\hat{m}_g'''(x) \quad = \quad & g^{-3} \left[\frac{\sum_{i=1}^{n_0} K'''\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)} - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} \right. \\
& + \frac{4\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^3} \\
& - \frac{3\sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} \\
& - \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} \\
& + \frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^3} \\
& \left. + \frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\right)^3 \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^4} \right], \text{ and}
\end{aligned}$$

$$
\begin{aligned}
\hat{m}_g^{(4)}(x) &= g^{-4}\left[\frac{\sum_{i=1}^{n_0} K^{(4)}\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}\right. \\
&+\frac{22\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&+\frac{4\left(\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&+\frac{12\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&+\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&-\frac{3\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&-\frac{6\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&-\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&-\frac{12\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^3\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}
\end{aligned}
$$

$$-\frac{24\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^4 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5}$$
$$\left.+\frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}\right].$$

Carrying on with calculations for the second term in (C.78) leads to:

$$\begin{aligned}
& \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\int K(u)\,\hat{\psi}_g(x+hu,x)\,du\right)^2 d\hat{F}_g^1(x) \\
= \;& \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\frac{h^4}{4}\left(\frac{\partial^2\hat{\psi}_g}{\partial y^2}(x,x)\right)^2 \mu_2(K)^2 d\hat{F}_g^1(x) \\
& +\mathcal{O}_P(h^6\, n_1^{-1}\, g^{-7}\left(g^{-2}+g^{-1}+1\right)) + \mathcal{O}_P(h^8\, n_1^{-1} g^{-9}) \\
= \;& \frac{n_0-1}{n_0}\frac{h^4}{4}\mu_2(K)^2 \\
& \int \left[\hat{m}_g''(x)^2 + \frac{4\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)} + \frac{4\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^2}\right]\hat{f}_g^1(x)dx \\
& +\mathcal{O}_P(h^6\, n_1^{-1}\, g^{-7}\left(g^{-2}+g^{-1}+1\right)) + \mathcal{O}_P(h^8\, n_1^{-1} g^{-9}). \qquad \text{(C.79)}
\end{aligned}$$

On the other hand, denote $\hat{\phi}_g(y,x) = \left[\hat{\sigma}_g^2(y) + g^2\,\mu_2(K) + (\hat{m}_g(y)-\hat{m}_g(x))^2\right]\hat{f}_g^0(y)$. Then,

$$\begin{aligned}
\hat{\phi}_g(x,x) &= \hat{\sigma}_g^2(x)\hat{f}_g^0(x) + g^2\,\mu_2(K)\,\hat{f}_g^0(x), \\
\frac{\partial\hat{\phi}_g}{\partial y}(y,x) &= \left(\hat{\sigma}_g^2\right)'(y)\hat{f}_g^0(y) + \hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)'(y) + g^2\mu_2(K)\left(\hat{f}_g^0\right)'(y), \\
& +2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y) + \hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y) + \hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(y) \\
& -2\hat{m}_g'(y)\hat{m}_g(x)\hat{f}_g^0(y) - 2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \hat{\phi}_g}{\partial y^2}(y,x) &= \left(\hat{\sigma}_g^2\right)''(y)\hat{f}_g^0(y) + 2\left(\hat{\sigma}_g^2\right)'(y)\left(\hat{f}_g^0\right)'(y) + \hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)''(y) \\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(y) + 2\hat{m}_g'(y)^2\hat{f}_g^0(y) + 2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y) \\
&+2\hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y) + 2\hat{m}_g(y)\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y) \\
&+\hat{m}_g(x)^2(y)\left(\hat{f}_g^0\right)''(y) + \hat{m}_g^2(x)\left(\hat{f}_g^0\right)''(y) - 2\hat{m}_g''(y)\hat{f}_g^0(y)\hat{m}_g(x) \\
&-2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)\hat{m}_g(x) - 2\hat{m}_g'(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y) \\
&-2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)''(y), \text{ and} \\
\frac{\partial^2 \hat{\phi}_g}{\partial y^2}(x,x) &= \left(\hat{\sigma}_g^2\right)''(x)\hat{f}_g^0(x) + 2\left(\hat{\sigma}_g^2\right)'(x)\left(\hat{f}_g^0\right)'(x) + \hat{\sigma}_g^2(x)\left(\hat{f}_g^0\right)''(x) \\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(x) + 2\hat{m}_g'(x)^2\hat{f}_g^0(x) + 2\hat{m}_g(x)\hat{m}_g'(x)\hat{f}_g^0(x) \\
&+2\hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(x) + 2\hat{m}_g(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x) \\
&-2\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g(x) - 4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)\hat{m}_g(x),
\end{aligned}$$

where

$$\begin{aligned}
\hat{\sigma}_g^2(x) &= \frac{1}{n_0\hat{f}_g^0(x)}\sum_{i=1}^{n_0} K_g(x-X_i^0)\left(Y_i^0\right)^2 - \left[\frac{1}{n_0\hat{f}_g^0(x)}\sum_{i=1}^{n_0} K_g(x-X_i^0)Y_i^0\right]^2 \\
&= \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}\right] - \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}\right]^2, \\
\left(\hat{\sigma}_g^2\right)'(x) &= g^{-1}\left(\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.\right. \\
&\left. - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right]
\end{aligned}$$

$$-2\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.$$

$$\left.\left.-\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right]\right), \text{ and}$$

$$\begin{aligned}\left(\hat{\sigma}_g^2\right)''(x) \quad = \quad & g^{-2}\left(\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) (Y_i^0)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right. \\ & +\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) (Y_i^0)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\ & -\frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) (Y_i^0)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\ & +\frac{2\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) (Y_i^0)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\ & -2\left[\frac{\left(\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right.}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.\end{aligned}$$

$$+\frac{\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$-\frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$-\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$+\frac{3\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5}$$

$$\left.\left.-\frac{2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right]\right).$$

Applying a Taylor expansion:

$$\begin{aligned}\int K(u)^2\hat{\phi}_g(x+hu,x)\,du &= R(K)\hat{\phi}_g(x,x)+\frac{h^2}{2}\frac{\partial^2\hat{\phi}_g}{\partial y^2}(x,x)\int u^2K(u)^2\,du+\mathcal{O}_P\left(h^4\right)\\ &= R(K)\hat{\phi}_g(x,x)+\mathcal{O}_P\left(h^2\,n_0^{-1}g^{-1}\left(1+g^{-1}+g^{-2}\right)\right).\end{aligned}$$

Carrying on with calculations leads to:

$$\begin{aligned}&\frac{1}{n_0\,h}\int\frac{1}{\hat{f}_g^0(x)^2}\int K(u)^2\hat{\phi}_g(x+hu,x)\,du\,d\hat{F}_g^1(x)\\ =\ &\frac{R(K)}{n_0\,h}\int\frac{\left(\hat{\sigma}_g^2(x)+g^2\,\mu_2(K)\right)\,\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx+\mathcal{O}_P\left(h\,n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{R(K)}{n_0\, h} \int \frac{\hat{\sigma}_g^2(x)\, \hat{f}_g^1(x)}{\hat{f}_g^0(x)} dx + \frac{R(K)\, g^2\, \mu_2(K)}{n_0\, h} \int \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)} dx \\
&\quad + \mathcal{O}_P\left(h\, n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right) \\
&= \frac{R(K)}{n_0\, h} \int \frac{\hat{\sigma}_g^2(x)\, \hat{f}_g^1(x)}{\hat{f}_g^0(x)} dx \\
&\quad + \mathcal{O}_P\left(h\, n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right) + \mathcal{O}_P\left(h^{-1}\, g^2\, n_1^{-1}\right). \qquad \text{(C.80)}
\end{aligned}
$$

Assembling terms (C.79) and (C.80), and inserting them in expression (C.78), Lemma 2 holds.

Lemma 3 *Consider the approximation given in (4.39) and expressions for $\hat{A}_g$ and A, given in (4.31). Then,*

$$
\begin{aligned}
\hat{A}_g - A \;&=\; \int \frac{\sigma^2(x)}{f^0(x)} \left[\hat{f}_g^1(x) - f^1(x)\right] dx - \int \frac{\sigma^2(x) f^1(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&\quad + \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right] dx - \int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&\quad - 2 \int \frac{f^1(x)\Psi_1(x)}{f^0(x)^3} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&\quad - 2 \int \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&\quad + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right)
\end{aligned}
$$

$$+\mathcal{O}\left(\int\left(\hat{\sigma}_g^2(x)-\sigma^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right).$$

In other terms,

$$\hat{A}_g - A = \sum_{i=1}^{k_0} a_i C^{[s]_i}_{\nu_i,\ell_i,r_i} + A_1,$$

where $k_0 = 6$, $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, a_5 = -2, a_6 = -2$, $\nu_1(x) = \frac{\sigma^2(x)}{f^0(x)}$, $\nu_2(x) = \frac{\sigma^2(x) f^1(x)}{f^0(x)^2}$, $\nu_3(x) = \frac{f^1(x)}{f^0(x)^2}$, $\nu_4(x) = \frac{f^1(x)\Psi_2(x)}{f^0(x)^3}$, $\nu_5(x) = \frac{f^1(x)\Psi_1(x)}{f^0(x)^3}$, $\nu_6(x) = \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4}$, $\ell_1 = 0, \ell_2 = 0, \ell_3 = 2, \ell_4 = 0, \ell_5 = 1, \ell_6 = 0$, $r_1 = 0, r_2 = 0, r_3 = 0$, $r_4 = 0, r_5 = 0, r_6 = 0$, $[s]_1 = 1$, $[s]_2 = 0$, $[s]_3 = 0$, $[s]_4 = 0$, $[s]_5 = 0$, $[s]_6 = 0$ *and* $A_1 = \mathcal{O}\left(0, r_{n_0}\right)$, *with*

$$\begin{aligned}
r_{0,n_0} &= \int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx + \int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right) dx \\
&+ \int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x)-\sigma^2(x)f^1(x)\right) dx \\
&+ \int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right) dx \\
&+ \int\left(\hat{\sigma}_g^2(x)-\sigma^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx + \int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx \\
&+ \int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2 dx.
\end{aligned}$$

Proof of Lemma 3 Our aim is to obtain upper bounds for $\hat{A}_g - A$, in expression (4.38), using approximation given in (4.39). On the one hand,

$$\begin{aligned}
&\hat{A}_g - A \\
&= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}\,dx - \int\frac{\sigma^2(x)f^1(x)}{f^0(x)}\,dx \\
&= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)-\sigma^2(x)f^1(x)}{f^0(x)}\,dx - \int\frac{\sigma^2(x)f^1(x)\left(\hat{f}_g^0(x)-f^0(x)\right)}{f^0(x)^2}\,dx
\end{aligned}$$

$$
\begin{aligned}
& +\mathcal{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 \, dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) \, dx\Bigg) \\
= \; & \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \hat{f}_g^1(x)}{f^0(x)} \, dx + \int \frac{\sigma^2(x)}{f^0(x)} \left[\hat{f}_g^1(x) - f^1(x)\right] \, dx \\
& - \int \frac{\sigma^2(x) f^1(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x)\right] \, dx \\
& +\mathcal{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 \, dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) \, dx\Bigg) \\
= \; & \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)}{f^0(x)} \, dx + \int \frac{f^1(x)}{f^0(x)} \left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] \, dx \\
& + \int \frac{\sigma^2(x)}{f^0(x)} \left[\hat{f}_g^1(x) - f^1(x)\right] \, dx - \int \frac{\sigma^2(x) f^1(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x)\right] \, dx \\
& +\mathcal{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 \, dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) \, dx\Bigg) \\
= \; & \int \frac{f^1(x)}{f^0(x)} \left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] \, dx + \int \frac{\sigma^2(x)}{f^0(x)} \left[\hat{f}_g^1(x) - f^1(x)\right] \, dx \\
& - \int \frac{\sigma^2(x) f^1(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x)\right] \, dx \\
& +\mathcal{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 \, dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) \, dx\Bigg) \\
& +\mathcal{O}\Bigg(\int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) \, dx\Bigg).
\end{aligned}
\tag{C.81}
$$

Furthermore,

$$
\int \frac{f^1(x)}{f^0(x)} \left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] \, dx
$$

$$
\begin{aligned}
= & \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{2,g}(x)}{\hat{f}^0_g(x)} - \frac{\Psi_2(x)}{f^0(x)} \right] dx - \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}^2_{1,g}(x)}{\hat{f}^0_g(x)^2} - \frac{\Psi^2_1(x)}{f^0(x)^2} \right] dx \\
= & \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{2,g}(x) - \Psi_2(x)}{f^0(x)} - \frac{\Psi_2(x)\left(\hat{f}^0_g(x) - f^0(x)\right)}{f^0(x)^2} \right] dx \\
& - \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)}{f^0(x)^2} - \frac{\Psi^2_1(x)\left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)}{f^0(x)^4} \right] dx \\
& + \mathcal{O}\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x) - f^0(x)\right) \right. \\
& \left. \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx \right) \\
& + \mathcal{O}\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) \right. \\
& \left. \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right) dx \right) \\
= & \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right] dx - \int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
& - \int \frac{f^1(x)}{f^0(x)^3} \left[\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right] dx - \int \frac{f^1(x)\Psi^2_1(x)}{f^0(x)^5} \left[\hat{f}^0_g(x)^2 - f^0(x)^2\right] dx \\
& + \mathcal{O}\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx \right. \\
& \left. + \int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx \right) \\
& + \mathcal{O}\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2 dx \right. \\
& \left. + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right) dx \right) \\
= & \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right] dx - \int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
& - 2 \int \frac{f^1(x)\Psi_1(x)}{f^0(x)^3} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
& - 2 \int \frac{f^1(x)\Psi^2_1(x)}{f^0(x)^4} \left[\hat{f}^0_g(x) - f^0(x)\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx \right.\\
&\left. + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right). \qquad \text{(C.82)}
\end{aligned}
$$

Combining expressions (C.82) and (C.81), Lemma 3 is concluded.

In the following remark (expression (C.83)) is collected the proof used in last equality of (C.82).

Remark 22 *Let $\hat{\Psi}_2$ be the estimator of Ψ_2. Then:*

$$
\begin{aligned}
\hat{\Psi}_2^2 - \Psi_2^2 &= \left(\hat{\Psi}_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) = \left(\hat{\Psi}_2 - \Psi_2 + \Psi_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 2\Psi_2 \left[\hat{\Psi}_2 - \Psi_2\right] + \mathcal{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Psi}_2^3 - \Psi_2^3 &= \left(\hat{\Psi}_2^2 + \hat{\Psi}_2\Psi_2 + \Psi_2^2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\hat{\Psi}_2^2 - \Psi_2^2 + \Psi_2^2 + \hat{\Psi}_2\Psi_2 - \Psi_2^2 + \Psi_2^2 + \Psi_2^2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(3\Psi_2^2 + (\hat{\Psi}_2 + \Psi_2) \cdot (\hat{\Psi}_2 - \Psi_2) + \Psi_2(\hat{\Psi}_2 - \Psi_2)\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 3\Psi_2^2 \left[\hat{\Psi}_2 - \Psi_2\right] + \mathcal{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Psi}_2^4 - \Psi_2^4 &= \left(\hat{\Psi}_2^3 + \hat{\Psi}_2^2\Psi_2 + \Psi_2^3 + \Psi_2^2\hat{\Psi}_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\left(\hat{\Psi}_2^3 - \Psi_2^3\right) + \Psi_2^3 + \left(\hat{\Psi}_2^2\Psi_2 - \Psi_2^3\right) + \Psi_2^3 + \Psi_2^3 \right.\\
&\quad \left. + \left(\Psi_2^2\hat{\Psi}_2 - \Psi_2^3\right) + \Psi_2^3\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(4\Psi_2^3 + \left(\hat{\Psi}_2^3 - \Psi_2^3\right) + \Psi_2\left(\hat{\Psi}_2^2 - \Psi_2^2\right) + \Psi_2^2\left(\hat{\Psi}_2 - \Psi_2\right)\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right)
\end{aligned}
$$

$$= 4\Psi_2^3\left[\hat{\Psi}_2-\Psi_2\right]+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2 + \left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^2-\Psi_2^2\right)+\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right).$$

$$\begin{aligned}
\hat{\Psi}_2^5-\Psi_2^5 &= \left(\hat{\Psi}_2^4+\Psi_2^4+\hat{\Psi}_2\Psi_2^3+\Psi_2\hat{\Psi}_2^3\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&= \Big(\left(\hat{\Psi}_2^4-\Psi_2^4\right)+\Psi_2^4\\
&\quad+\left(\hat{\Psi}_2\Psi_2^3-\Psi_2^4\right)+\Psi_2^4+\Psi_2^4+\left(\Psi_2\hat{\Psi}_2^3-\Psi_2^4\right)+\Psi_2^4\Big)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&= \left(\left(\hat{\Psi}_2^4-\Psi_2^4\right)+4\Psi_2^4+\Psi_2^3\left(\hat{\Psi}_2-\Psi_2\right)+\Psi_2\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&= 4\Psi_2^4\left[\hat{\Psi}_2-\Psi_2\right]+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^4-\Psi_2^4\right)+\left(\hat{\Psi}_2-\Psi_2\right)^2\right)\\
&\quad+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right).
\end{aligned}$$

Then,

$$\hat{\Psi}_2^2-\Psi_2^2 = 2\Psi_2\left[\hat{\Psi}_2-\Psi_2\right]+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2\right), \tag{C.83}$$

$$\hat{\Psi}_2^3-\Psi_2^3 = 3\Psi_2^2\left[\hat{\Psi}_2-\Psi_2\right] +\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2+\left(\hat{\Psi}_2+\Psi_2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\right), \tag{C.84}$$

$$\hat{\Psi}_2^4-\Psi_2^4 = 4\Psi_2^3\left[\hat{\Psi}_2-\Psi_2\right]+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2+\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^2-\Psi_2^2\right)\right) +\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right),\ \textit{and} \tag{C.85}$$

$$\hat{\Psi}_2^5-\Psi_2^5 = 4\Psi_2^4\left[\hat{\Psi}_2-\Psi_2\right]+\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^4-\Psi_2^4\right)+\left(\hat{\Psi}_2-\Psi_2\right)^2\right) +\mathcal{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right). \tag{C.86}$$

Lemma 4 *Given the approximation in (4.39) and expressions for $\hat{B}_g$ and B in (4.31), $\hat{B}_g - B$ consists of a sum of 60 terms similar to those in expression (4.41).*

Specifically,

$$\hat{B}_g - B \;=\; \sum_{i=7}^{k_1} a_i C^{[s]_i}_{\nu_i,\ell_i,r_i} + B_1,$$

where $k_1 = 66$. *Values of* $\nu(x)$, r, ℓ, a *and* $[s]$ *are collected in Tables 4.1, 4.2, 4.3 and 4.4,*

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
7	$\frac{(f^0)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}$	0	1	1	8
8	$\frac{(f^0)'(x)^2 f^1(x) m'(x) \Psi_1'(x)}{f^0(x)^4}$	0	0	0	-8
9	$\frac{(f^0)'(x)^2 f^1(x) m'(x) (f^0)'(x)}{f^0(x)^4}$	0	1	0	-8
10	$\frac{m'(x)^2 (f^0)'(x)^2}{f^0(x)^2}$	1	0	0	4
11	$\frac{m'(x)^2 f^1(x) (f^0)'(x)}{f^0(x)^2}$	0	0	1	8
12	$\frac{(f^0)'(x)^2 f^1(x) m'(x) (f^0)'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	0	-16
13	$\frac{(f^0)'(x)^2 f^1(x) m'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	1	-8
14	$\frac{m'(x)^2 (f^0)'(x)^2 f^1(x)}{f^0(x)^3}$	0	0	0	-8
15	$\frac{f^1(x) m''(x)}{f^0(x)}$	0	1	2	2
16	$\frac{f^1(x) m''(x) \Psi_1''(x)}{f^0(x)^2}$	0	0	0	-2
17	$\frac{f^1(x) m''(x) \Psi_1(x)}{f^0(x)^2}$	0	0	2	-2
i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
18	$\frac{f^1(x) m''(x) (f^0)''(x)}{f^0(x)^2}$	0	1	0	-2

Table C.1: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{7, \ldots, 18\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
19	$\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}$	0	0	0	8
20	$\frac{f^1(x)m''(x)\Psi_1'(x)}{f^0(x)^2}$	0	0	1	-4
21	$\frac{f^1(x)m''(x)\left(f^0\right)'(x)}{f^0(x)^2}$	0	1	1	-4
22	$\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}$	0	0	0	8
23	$\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}$	0	0	1	8
24	$\frac{f^1(x)m''(x)\left(f^0\right)'(x)^2}{f^0(x)^3}$	0	1	0	4
25	$\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}$	0	0	0	-12
26	$m''(x)^2$	1	0	0	1
27	$\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}$	1	0	0	4
28	$\frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}$	0	0	1	4
29	$\frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	2	4
30	$\frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	1	4
31	$\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}$	0	0	0	-4
32	$\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	0	-8
33	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}$	1	0	0	8
34	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}$	0	1	0	8
35	$\frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}$	0	0	1	24
36	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8

Table C.2: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{19, \ldots, 36\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
37	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	0	0	-32
38	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	1	0	0	-4
39	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	0	1	-4
40	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
41	$\frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	2	-4
42	$\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	1	-4
43	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}$	0	0	0	-12
44	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}$	1	0	0	-8
45	$\frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-16
46	$\frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}$	0	1	1	-16
47	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}$	0	0	0	24
48	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}$	1	0	0	-4
49	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	1	2	-4
50	$\frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
51	$\frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-8
52	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}$	0	0	0	12
53	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}$	1	0	0	-8
54	$\frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}$	0	1	0	-16

Table C.3: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{37, \ldots, 54\}$.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
55	$\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}$	0	0	1	−32
56	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}$	0	0	0	32
57	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}$	1	0	0	4
58	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}$	0	0	2	4
59	$\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}$	0	1	0	8
60	$\frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}$	0	0	1	8
61	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$	0	0	0	−16
62	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}$	1	0	0	8
63	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
64	$\frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}$	0	1	0	8
65	$\frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}$	0	0	1	24
66	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}$	0	0	0	−32

Table C.4: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (4.43), $i \in \{55, \ldots, 66\}$.

Proof of Lemma 4 Focusing now on expression $\hat{B}_g - B$ in (4.31),

$$\hat{B}_g - B = B_1 + B_2 + B_3, \tag{C.87}$$

where

$$\begin{aligned}
B_1 &:= \int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx, \\
B_2 &:= 4\int \left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right] dx, \text{ and} \\
B_3 &:= 4\int \left[\frac{\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right] dx.
\end{aligned}$$

We will focus on term B_1 in the first place. Carrying on with calculations, considering expression (C.83), it leads to:

$$\begin{aligned}
B_1 &:= \int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx \\
&= \int \hat{m}_g''(x)^2 \hat{f}_g^1(x)\,dx - \int m''(x)^2 f^1(x)\,dx \\
&= \int \hat{m}_g''(x)^2 \left(\hat{f}_g^1(x) - f^1(x) + f^1(x)\right) dx - \int m''(x)^2 f^1(x)\,dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int \hat{m}_g''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&\quad + \int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&\quad + \mathcal{O}\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&= 2\int f^1(x)m''(x)\left[\hat{m}_g''(x) - m''(x)\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&\quad + \mathcal{O}\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&\quad + \mathcal{O}\left(\int \left(\hat{m}_g''(x) - m''(x)\right)^2 dx\right). \qquad \text{(C.88)}
\end{aligned}$$

Furthermore, considering expressions (C.83), (C.84) and (C.85), it turns out that the first term in (C.88) is:

$$
\begin{aligned}
& 2\int f^1(x)m''(x)\left[\hat{m}_g''(x) - m''(x)\right] dx \\
= \; & 2\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}''(x)}{\hat{f}_g^0(x)} - \frac{\Psi_1''(x)}{f^0(x)}\right] dx \\
& -2\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^2}\right] dx \\
& -4\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^2}\right] dx \\
& +4\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^3} - \frac{\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\right] dx \\
= \; & 2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right] dx \\
& -2\int \frac{f^1(x)m''(x)\Psi_1''(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +2\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)^4 - f^0(x)^4\right] dx \\
& +4\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)^2 - f^0(x)^2\right] dx \\
& -2\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x) - \Psi_1(x)\left(f^0\right)''(x)\right] dx \\
& -4\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x) - \Psi_1'(x)\left(f^0\right)'(x)\right] dx \\
& +4\int \frac{f^1(x)m''(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2 - \Psi_1(x)\left(f^0\right)'(x)^2\right] dx \\
& -4\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^6}\left[\hat{f}_g^0(x)^3 - f^0(x)^3\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{''}(x)-\Psi_1(x)\left(f^0\right)^{''}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)-\Psi_1^{'}(x)\left(f^0\right)^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)^2-\Psi_1(x)\left(f^0\right)^{'}(x)^2\right)dx\right)\\
= \;& 2\int\frac{f^1(x)m^{''}(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right]dx\\
&-2\int\frac{f^1(x)m^{''}(x)\Psi_1^{''}(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-2\int\frac{f^1(x)m^{''}(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right]dx\\
&-2\int\frac{f^1(x)m^{''}(x)}{f^0(x)^2}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)^{''}(x)\right]dx\\
&+8\int\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{''}(x)}{f^0(x)}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-4\int\frac{f^1(x)m^{''}(x)\Psi_1^{'}(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx\\
&-4\int\frac{f^1(x)m^{''}(x)}{f^0(x)^2}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^0\right)^{'}(x)\right]dx\\
&+8\int\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+4\int\frac{f^1(x)m^{''}(x)\Psi_1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right]dx\\
&+4\int\frac{f^1(x)m^{''}(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)^{'}(x)^2\right]dx\\
&-12\int\frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{''}(x) - \Psi_1(x)\left(f^0\right)^{''}(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x) - \Psi_1^{'}(x)\left(f^0\right)^{'}(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)^2 - \Psi_1(x)\left(f^0\right)^{'}(x)^2\right)\, dx\right) \\
= \quad & 2\int \frac{f^1(x)m^{''}(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right] dx \\
&-2\int \frac{f^1(x)m^{''}(x)\Psi_1^{''}(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&-2\int \frac{f^1(x)m^{''}(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right] dx \\
&-2\int \frac{f^1(x)m^{''}(x)\left(f^0\right)^{''}(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+8\int \frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{''}(x)}{f^0(x)}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&-4\int \frac{f^1(x)m^{''}(x)\Psi_1^{'}(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right] dx \\
&-4\int \frac{f^1(x)m^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right] dx \\
&+8\int \frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+8\int \frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+4\int \frac{f^1(x)m^{''}(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]\,dx\\
&-12\int \frac{f^1(x)m^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx\\
&+\mathcal{O}\left(\int \hat{f}_g^0(x)^2-f^0(x)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{''}(x)-\Psi_1(x)\left(f^0\right)^{''}(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)-\Psi_1^{'}(x)\left(f^0\right)^{'}(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)^2-\Psi_1(x)\left(f^0\right)^{'}(x)^2\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\,dx\right).
\end{aligned}
\tag{C.89}
$$

Plugging expression (C.89) in (C.88) leads to the following expression for B_1:

$$
\begin{aligned}
B_1 := \; & 2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx + \int m''(x)^2\left[\hat{f}^1_g(x)-f^1(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi''_1(x)}{f^0(x)^2}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\left(f^0\right)''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\Psi'_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& +4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& -12\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)^2dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)
\end{aligned}
$$

$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x) - \Psi_1'(x)\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2 - \Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{m}_g''(x) - m''(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)dx\right) + \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)dx\right). \tag{C.90}$$

We now move on to achieve a deeper insight of term B_2, considering expressions (C.83), (C.84), (C.85) and (C.86), it follows that:

$$\begin{aligned}
B_2 \;&:=\; 4\int\left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right]dx \\
&=\; 4\int\frac{1}{f^0(x)}\left[\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right]dx \\
&\quad -4\int\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right]dx \\
&\quad +\mathcal{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)\right. \\
&\quad \left.\cdot\left(\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right).
\end{aligned} \tag{C.91}$$

Computing further calculations with the first term in expression (C.91) using (C.83), (C.84), (C.85) and (C.86), we have:

$$4\int \frac{1}{f^0(x)}\left[\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x) - m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)\right]dx$$

$$= 4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\hat{\Psi}_{1,g}^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^2}\right]dx$$

$$+8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4} - \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx$$

$$-4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{''}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.$$

$$\left.-\frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\right]dx$$

$$-8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3} - \frac{\Psi_1^{'}(x)^2\left(f^0\right)^{'}(x)^2f^1(x)}{f^0(x)^3}\right]dx$$

$$-4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{''}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3} - \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^3}\right]dx$$

$$-8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)^{'}(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^5} - \frac{\left(f^0\right)^{'}(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^5}\right]dx$$

$$+4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}\right.$$

$$\left.-\frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^4}\right]dx$$

$$+8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^4}\right]dx. \tag{C.92}$$

The first term in expression (C.92), using (C.83), (C.84), (C.85) and (C.86), happens to be:

$$\begin{aligned}
& 4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}-\frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\right]dx \\
= \; & 4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx \\
& -4\int \frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
& \left.\cdot\left(\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right) \\
= \; & 4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx \\
& -8\int \frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
& \left.\cdot\left(\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right).
\end{aligned} \tag{C.93}$$

Working out calculations with the first term in expression (C.93) leads to:

$$4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx$$

$$
\begin{aligned}
= \;& 4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right]dx \\
& +4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
= \;& 4\int \frac{\Psi_1'(x)\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)f^1(x)\right]dx \\
& +4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
= \;& 4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x) - f^1(x)\right]dx \\
& +4\int \frac{\Psi_1'(x)\Psi_1''(x)}{f^0(x)^3}\left[\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\hat{f}_g^1(x)\right]dx \\
& +4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat{f}_g^1(x)\right]dx \\
= \;& 4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x) - f^1(x)\right]dx
\end{aligned}
$$

$$+4\int \frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$

$$+4\int \frac{\Psi_1'(x)\Psi_1''(x)}{f^0(x)^3}\left[\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx$$

$$+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$

$$+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$
\begin{aligned}
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx\\
= \quad & 4\int \frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&+4\int \frac{\Psi'_1(x)\Psi''_1(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+4\int \frac{\Psi'_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx\\
&+4\int \frac{\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)
\end{aligned}
$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right). \tag{C.94}$$

Plugging expression (C.94) in (C.93), the first term in expression (C.92) happens to be:

$$4\int\frac{\Psi_{1}^{'}(x)\Psi_{1}^{''}(x)\left(f^{0}\right)^{'}(x)}{f^{0}(x)^{3}}\left[\hat{f}_{g}^{1}(x)-f^{1}(x)\right]\,dx$$
$$+4\int\frac{\Psi_{1}^{'}(x)\Psi_{1}^{''}(x)f^{1}(x)}{f^{0}(x)^{3}}\left[\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right]\,dx$$
$$+4\int\frac{\Psi_{1}^{'}(x)\left(f^{0}\right)^{'}(x)f^{1}(x)}{f^{0}(x)^{3}}\left[\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right]\,dx$$
$$+4\int\frac{\Psi_{1}^{''}(x)\left(f^{0}\right)^{'}(x)f^{1}(x)}{f^{0}(x)^{3}}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right]\,dx$$
$$-8\int\frac{\Psi_{1}^{'}(x)\Psi_{1}^{''}(x)\left(f^{0}\right)^{'}(x)f^{1}(x)}{f^{0}(x)^{4}}\left[\hat{f}_{g}^{0}(x)-f^{0}(x)\right]\,dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{2}-f^{0}(x)^{2}\right)^{2}\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{2}-f^{0}(x)^{2}\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}^{'}(x)\hat{\Psi}_{1,g}^{''}(x)\left(\hat{f}_{g}^{0}\right)^{'}(x)\hat{f}_{g}^{1}(x)-\Psi_{1}^{'}(x)\Psi_{1}^{''}(x)\left(f^{0}\right)^{'}(x)f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)-f^0(x)\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right). \qquad \text{(C.95)}
\end{aligned}
$$

The second term in expression (C.92) turns out to be:

$$
\begin{aligned}
& 8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)}{\hat{f}^0_g(x)^4}-\frac{\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx\\
= \;& 8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)-\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx\\
& -8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^9}\left[\hat{f}^0_g(x)^4-f^0(x)^4\right]dx\\
& +\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \left(\hat{\Psi}^{'}_{1,g}(x) \left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x) \right) \, dx \right) \\
= \; & 8 \int \frac{1}{f^0(x)^5} \left[\hat{\Psi}^{'}_{1,g}(x) \left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x) \right] dx \\
& -32 \int \frac{\Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5} \left[\hat{f}^0_g(x) - f^0(x) \right] dx \\
& + \mathcal{O} \left(\int \left(\hat{f}^0_g(x)^4 - f^0(x)^4 \right)^2 dx \right) + \mathcal{O} \left(\int \left(\hat{f}^0_g(x)^4 - f^0(x)^4 \right) \right. \\
& \cdot \left(\hat{\Psi}^{'}_{1,g}(x) \left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x) \right) \, dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{f}^0_g(x) - f^0(x) \right) \cdot \left(\hat{f}^0_g(x)^4 - f^0(x)^4 \right) dx + \int \left(\hat{f}^0_g(x) - f^0(x) \right)^2 dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{f}^0_g(x) - f^0(x) \right) \cdot \left(\hat{f}^0_g(x)^3 - f^0(x)^3 \right) dx \right) . \qquad \text{(C.96)}
\end{aligned}
$$

Carrying on with calculations considering the first term in expression (C.96) leads to:

$$
\begin{aligned}
& 8 \int \frac{1}{f^0(x)^5} \left[\hat{\Psi}^{'}_{1,g}(x) \left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x) \right] dx \\
= \; & 8 \int \frac{\Psi^{'}_1(x)}{f^0(x)^5} \left[\left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \left(f^0 \right)^{'} (x)^3 \Psi_1(x) f^1(x) \right] dx \\
& +8 \int \frac{1}{f^0(x)^5} \left[\left(\hat{\Psi}^{'}_{1,g}(x) - \Psi^{'}_1(x) \right) \cdot \left(\hat{f}^0_g \right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) \right] dx \\
= \; & 8 \int \frac{\Psi^{'}_1(x) \left(f^0 \right)^{'} (x)^3}{f^0(x)^5} \left[\hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) - \Psi_1(x) f^1(x) \right] dx \\
& +8 \int \frac{\Psi^{'}_1(x)}{f^0(x)^5} \left[\left(\left(\hat{f}^0_g \right)^{'} (x)^3 - \left(f^0 \right)^{'} (x)^3 \right) \cdot \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) \right] dx \\
& +8 \int \frac{\left(f^0 \right)^{'} (x)^3}{f^0(x)^5} \left[\left(\hat{\Psi}^{'}_{1,g}(x) - \Psi^{'}_1(x) \right) \cdot \hat{\Psi}_{1,g}(x) \hat{f}^1_g(x) \right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)\right]dx\\
=\ & 8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
=\ & 8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right]dx\\
&+8\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx+8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& +8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int\frac{\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx \\
& +8\int\frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int\frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right]dx \\
& +8\int\frac{\Psi_1(x)}{f^0(x)^5}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right) \\
& \cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx \\
& +8\int\frac{f^1(x)}{f^0(x)^5}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right) \\
& \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\Big]\,dx \\
& +8\int\frac{1}{f^0(x)^5}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right) \\
& \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx \\
= \; & 8\int\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& +8\int\frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +24\int\frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& +8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right). \qquad \text{(C.97)}
\end{aligned}
$$

Plugging expression (C.97) in (C.96), the second term of expression (C.92) turns out ot be:

$$
\begin{aligned}
&8 \int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3 \Psi_1(x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+8 \int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3 f^1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+24 \int \frac{\Psi'_1(x)\Psi_1(x) f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx
\end{aligned}
$$

$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]\,dx$$
$$-32\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]\,dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)-\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)+\left(\hat{f}^0_g(x)-f^0(x)\right)^2\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)^2\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\,dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{\prime}(x)-\Psi_{1}^{\prime}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{\prime}(x)^{3}-\left(f^{0}\right)^{\prime}(x)^{3}\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right). \tag{C.98}$$

Carrying on with calculations with the third term in expression (C.92), using (C.83), (C.84), (C.85) and (C.86), it leads to:

$$\begin{aligned}
& -4\int\frac{1}{f^{0}(x)}\left[\frac{\hat{\Psi}_{1,g}^{\prime}(x)\left(\hat{f}_{g}^{0}\right)^{\prime\prime}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_{g}^{0}\right)^{\prime}(x)\hat{f}_{g}^{1}(x)}{\hat{f}_{g}^{0}(x)^{3}}\right.\\
& \left.-\frac{\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)}{f^{0}(x)^{3}}\right]dx\\
= \; & -4\int\frac{1}{f^{0}(x)^{4}}\left[\hat{\Psi}_{1,g}^{\prime}(x)\left(\hat{f}_{g}^{0}\right)^{\prime\prime}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_{g}^{0}\right)^{\prime}(x)\hat{f}_{g}^{1}(x)\right.\\
& \left.-\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)\right]dx\\
& -4\int\frac{\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)}{f^{0}(x)^{7}}\left[\hat{f}_{g}^{0}(x)^{3}-f^{0}(x)^{3}\right]dx\\
& +\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{3}-f^{0}(x)^{3}\right)^{2}dx\right)\\
& +\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{3}-f^{0}(x)^{3}\right)\cdot\left(\hat{\Psi}_{1,g}^{\prime}(x)\left(\hat{f}_{g}^{0}\right)^{\prime\prime}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_{g}^{0}\right)^{\prime}(x)\hat{f}_{g}^{1}(x)\right.\right.\\
& \left.\left.-\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)\right)dx\right)\\
= \; & -4\int\frac{1}{f^{0}(x)^{4}}\left[\hat{\Psi}_{1,g}^{\prime}(x)\left(\hat{f}_{g}^{0}\right)^{\prime\prime}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_{g}^{0}\right)^{\prime}(x)\hat{f}_{g}^{1}(x)\right.\\
& \left.-\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)\right]dx\\
& -12\int\frac{\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)f^{1}(x)}{f^{0}(x)^{5}}\left[\hat{f}_{g}^{0}(x)-f^{0}(x)\right]dx\\
& +\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{3}-f^{0}(x)^{3}\right)^{2}dx\right)+\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)^{3}-f^{0}(x)^{3}\right)\right.\\
& \cdot\left(\hat{\Psi}_{1,g}^{\prime}(x)\left(\hat{f}_{g}^{0}\right)^{\prime\prime}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_{g}^{0}\right)^{\prime}(x)\hat{f}_{g}^{1}(x)-\Psi_{1}^{\prime}(x)\left(f^{0}\right)^{\prime\prime}(x)\Psi_{1}(x)\left(f^{0}\right)^{\prime}(x)\right.
\end{aligned}$$

$$f^1(x)\big)\ dx\big) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 + \left(\hat{f}_g^0(x) - f^0(x)\right)\right.$$
$$\left.\cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\ dx\right). \tag{C.99}$$

Working out further calculation with the first term in expression (C.99), using (C.83), (C.84), (C.85) and (C.86), it turns out:

$$\begin{aligned}
& -4\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right. \\
& \left.-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]\ dx \\
= \ & -4\int \frac{\Psi'_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right. \\
& \left.-\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]\ dx \\
& -4\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx \\
= \ & -4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]\ dx \\
& -4\int \frac{\Psi'_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx \\
& -4\int \frac{\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right)\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx \\
& -4\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right. \\
& \left.\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx \\
= \ & -4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)f^1(x)\right]\ dx \\
& -4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx \\
& -4\int \frac{\Psi'_1(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]\ dx
\end{aligned}$$

$$-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx$$
$$= -4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx-4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^4}$$

$$
\begin{aligned}
& \left[\left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\Psi_1^{'}(x)}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{''}(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \cdot \hat{f}_g^1(x)\right] dx - 4 \int \frac{\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4} \\
& \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\Psi_1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right)\right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \cdot \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{'}(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right)\right. \\
& \left. \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}_g^1(x)\right] dx - 4 \int \frac{1}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)\right. \\
& \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \\
& \left. \cdot \hat{f}_g^1(x)\right] dx \\
= \; & -4 \int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x)\right] dx
\end{aligned}
$$

$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx-4\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^4}$$
$$\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-4\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$
$$-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Bigg]\,dx$$

$$-4\int\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx$$

$$-4\int\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int\frac{\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int\frac{\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\Bigg[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)$$

$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Bigg]\,dx$$

$$-4\int\frac{\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$-4\int\frac{\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\Bigg[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$

$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Bigg]\,dx-4\int\frac{\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}$$

$$\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int\frac{\left(f^0\right)''(x)}{f^0(x)^4}\Bigg[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$

$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Bigg]\,dx-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$$

$$\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx$$

$$-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\Bigg[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$

$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Bigg]\,dx-4\int\frac{\Psi_1(x)f^1(x)}{f^0(x)^4}$$

$$\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int\frac{\Psi_1(x)}{f^0(x)^4}\Bigg[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$

$$
\begin{aligned}
& \cdot \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) \Big] \, dx - 4 \int \frac{\left(f^0\right)^{'}(x) f^1(x)}{f^0(x)^4} \\
& \Big[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{''} (x) - \left(f^0 \right)^{''} (x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \Big] \, dx \\
& -4 \int \frac{\left(f^0\right)^{'}(x)}{f^0(x)^4} \Big[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{''} (x) - \left(f^0 \right)^{''} (x) \right) \\
& \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) \Big] \, dx - 4 \int \frac{f^1(x)}{f^0(x)^4} \\
& \Big[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{''} (x) - \left(f^0 \right)^{''} (x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \\
& \cdot \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) \Big] \, dx \\
& -4 \int \frac{1}{f^0(x)^4} \Big[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{''} (x) - \left(f^0 \right)^{''} (x) \right) \\
& \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) \Big] \, dx \\
= \; & -4 \int \frac{\Psi_1^{'}(x) \left(f^0\right)^{''}(x) \Psi_1(x) \left(f^0\right)^{'}(x)}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x) \right] dx \\
& -4 \int \frac{\Psi_1^{'}(x) \left(f^0\right)^{''}(x) \Psi_1(x) f^1(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right] dx \\
& -4 \int \frac{\Psi_1^{'}(x) \left(f^0\right)^{''}(x) \left(f^0\right)^{'}(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right] dx \\
& -4 \int \frac{\Psi_1^{'}(x) \Psi_1(x) \left(f^0\right)^{'}(x) f^1(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0 \right)^{''} (x) - \left(f^0 \right)^{''} (x) \right] dx \\
& -4 \int \frac{\left(f^0\right)^{''}(x) \Psi_1(x) \left(f^0\right)^{'}(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right] dx \\
& + \mathcal{O} \left(\int \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) dx \right) \\
& + \mathcal{O} \left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)^{'} (x) - \left(f^0 \right)^{'} (x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right)
\end{aligned}
$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\,dx\right)$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right).
\end{aligned}
\tag{C.100}
$$

Plugging expression (C.100) in (C.99), the third term in expression (C.92) happens to be:

$$
\begin{aligned}
&-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int\frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx
\end{aligned}
$$

$$-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx$$
$$-12\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)^{''}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)\right.\right.$$
$$\left.\left.-\Psi'_1(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right). \tag{C.101}$$

The fourth term in expression (C.92), using (C.83), (C.84), (C.85) and (C.86), happens to be:

$$-8\int\frac{1}{f^0(x)}\left[\frac{\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)}{\hat{f}^0_g(x)^3}-\frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\right]dx$$
$$=-8\int\frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)\right]dx$$
$$+8\int\frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^7}\left[\hat{f}^0_g(x)^3-f^0(x)^3\right]dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)$$
$$=-8\int\frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)\right]dx$$
$$+24\int\frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)$$

$$
\begin{aligned}
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
= \; & -8\int \frac{\Psi_1'(x)^2}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 f^1(x)\right] dx \\
& -8\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x)\right] dx \\
& +24\int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 + \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right) \\
= \; & -8\int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
& -8\int \frac{\Psi_1'(x)^2}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right] dx \\
& -8\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right)\cdot \hat{f}_g^1(x)\right] dx - 8\int \frac{1}{f^0(x)^4} \\
& \left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right] dx \\
& +24\int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
= \; & -8\int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
& -8\int \frac{\Psi_1'(x)^2 f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right] dx
\end{aligned}
$$

$$
\begin{aligned}
& -8\int \frac{\Psi_1'(x)^2}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& -8\int \frac{\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right]dx \\
& -8\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& -8\int \frac{f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right]dx \\
& -8\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right) \\
= \; & -8\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& -16\int \frac{\Psi_1'(x)^2 f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& -16\int \frac{\left(f^0\right)'(x)^2 f^1(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& +24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right).
\end{aligned}
\tag{C.102}
$$

The fifth term in expression (C.92) turns out to be, considering (C.83), (C.84), (C.85) and (C.86):

$$
\begin{aligned}
&-4\int\frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\right]dx\\
=\;&-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
=\;&-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
& +12 \int \frac{\left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
& \left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
= \; & -4 \int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \Psi_1(x) \Psi_1^{''}(x) f^1(x)\right] dx \\
& -4 \int \frac{1}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x)\right] dx \\
& +12 \int \frac{\left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
& \left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
= \; & -4 \int \frac{\left(f^0\right)^{'}(x)^2 \Psi_1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \Psi_1^{''}(x) f^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x)\right] dx \\
& -4 \int \frac{\Psi_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x)\right] dx - 4 \int \frac{1}{f^0(x)^4} \\
& \left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x)\right] dx \\
& +12 \int \frac{\left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
& \left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& -4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
& +12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right) \\
= \quad & -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& -4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& -4\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx
\end{aligned}
$$

$$-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx$$
$$-4\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right]\,dx$$
$$-4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx$$
$$-4\int \frac{\Psi_1(x)\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx$$
$$-4\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right]\,dx$$
$$-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx$$
$$-4\int \frac{\Psi_1^{''}(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]\,dx$$
$$-4\int \frac{\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx-4\int \frac{f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right]\,dx$$
$$-4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx$$
$$+12\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right)$$

$$
\begin{aligned}
= \quad & -4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x) \Psi_1''(x)}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right] dx \\
& -4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1''(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
& -8 \int \frac{\Psi_1(x) \Psi_1''(x) f^1(x) \left(f^0\right)'(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
& +12 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x) \Psi_1''(x) f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) \\
& +\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right. \\
& \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) dx + \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right) \\
& \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\Big) + \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right. \\
& \left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right)\right. \\
&\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\, dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{''}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 \Psi_1(x) \Psi_1^{''}(x) f^1(x)\right)\, dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\, dx\right). \qquad \text{(C.103)}
\end{aligned}
$$

Working out calculations with the sixth term in expression (C.92), using (C.83), (C.84), (C.85) and (C.86), it leads to:

$$
\begin{aligned}
& -8\int \frac{1}{f^0(x)} \left[\frac{\left(\hat{f}_g^0\right)^{'}(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x)}{\hat{f}_g^0(x)^5} - \frac{\left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^5}\right] dx \\
= \; & -8\int \frac{1}{f^0(x)^6} \left[\left(\hat{f}_g^0\right)^{'}(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)\right] dx \\
& +8\int \frac{\left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^{11}} \left[\hat{f}_g^0(x)^5 - f^0(x)^5\right] dx \\
& +\mathcal{O}\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)\right. \\
& \left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)\right) dx\right) \\
= \; & -8\int \frac{1}{f^0(x)^6} \left[\left(\hat{f}_g^0\right)^{'}(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)\right] dx \\
& +32\int \frac{\left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)\right. \\
& \left.\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^4 \Psi_1^2(x) f^1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\,dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)\\
= \quad &-8\int\frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\Psi_1^2(x)f^1(x)\right]\,dx\\
&-8\int\frac{1}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)\right]\,dx\\
&+32\int\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\,dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)\\
= \quad &-8\int\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]\,dx\\
&-8\int\frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\hat{f}_g^1(x)\right]\,dx\\
&-8\int\frac{\Psi_1^2(x)}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\hat{f}_g^1(x)\right]\,dx\\
&-8\int\frac{1}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\hat{f}_g^1(x)\right]\,dx\\
&+32\int\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\,dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\, dx\right) \\
= \; & -8 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x)}{f^0(x)^6} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
& -8 \int \frac{\left(f^0\right)'(x)^4 f^1(x)}{f^0(x)^6} \left[\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right] dx \\
& -8 \int \frac{\Psi_1^2(x) f^1(x)}{f^0(x)^6} \left[\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right] dx \\
& +32 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& -8 \int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6} \left[\left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx \\
& -8 \int \frac{\Psi_1^2(x)}{f^0(x)^6} \left[\left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx \\
& -8 \int \frac{f^1(x)}{f^0(x)^6} \left[\left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right)\right] dx \\
& -8 \int \frac{1}{f^0(x)^6} \left[\left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right)\right. \\
& \left. \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)\right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right) dx \\
& + \int \mathcal{O}\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)^2 dx\right) \\
= \; & -8 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x)}{f^0(x)^6} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
& -16 \int \frac{\left(f^0\right)'(x)^4 f^1(x) \Psi_1(x)}{f^0(x)^6} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
& -32 \int \frac{\Psi_1^2(x) f^1(x)}{f^0(x)^3} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\right.\\
&+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)\\
&\left.\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right). \qquad \text{(C.104)}
\end{aligned}
$$

Carrying on with calculations with the seventh term in expression (C.92) following similar steps as in the previous expressions, it turns out:

$$
\begin{aligned}
&4\int\frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\right]dx
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 4\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)\right. \\
& \left. -\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right] dx \\
& -4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4 - f^0(x)^4\right] dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right) dx\right) \\
= \quad & 4\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)\right. \\
& \left. -\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right] dx \\
& -16\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right) dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) \\
= \quad & 4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x) - \Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right] dx \\
& +4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right)\cdot\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)\right] dx \\
& -16\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
= \;& 4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)''(x)f^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx\\
&+4\int\frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx\\
&+4\int\frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx-16\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\\
&\left[\hat{f}_g^0(x)-f^0(x)\right]dx+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
= \;& 4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+4\int\frac{\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
& +4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
& \left.\cdot\hat{f}_g^1(x)\right]\,dx\\
& +4\int \frac{\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
& \left.\cdot\hat{f}_g^1(x)\right]\,dx+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
& \left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]\,dx-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\\
& \left[\hat{f}_g^0(x)-f^0(x)\right]\,dx+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2\,dx\right)\\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
& \left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)\,dx\right)\\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right.\\
& \left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)\\
= \; & 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]\,dx\\
& +4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]\,dx\\
& +4\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right]\,dx\\
& +4\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]\,dx\\
& +4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx\\
& +4\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]\,dx\\
& +4\int \frac{\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right]dx\\
&+4\int \frac{\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+4\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx\\
&+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& +4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right]dx \\
& +8\int \frac{\left(f^0\right)^{'}(x)^2\left(f^0\right)^{''}(x)f^1(x)\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +8\int \frac{\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\left(f^0\right)^{'}(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx \\
& -16\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right) \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2 dx\right) \\
& +\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\Big)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\cdot\left.\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\cdot\left.\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\cdot\left.\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)\,dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right.\\
&+\left.\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right). \qquad \text{(C.105)}
\end{aligned}
$$

Finally, following analogous steps as in previous expressions, the eigthth term in expression (C.92) happens to be:

$$
\begin{aligned}
&8\int\frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^4}\right]dx\\
=\;&8\int\frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx\\
&-8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\cdot\left.\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)\,dx\right)\\
=\;&8\int\frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx\\
&-32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right. \\
& \left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right) \\
= \; & 8\int\frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx \\
& +8\int\frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx \\
& -32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right. \\
& \left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right. \\
& \left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right) \\
= \; & 8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\Psi_1'(x)f^1(x)\right]dx \\
& +8\int\frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx \\
& +8\int\frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx+8\int\frac{1}{f^0(x)^5} \\
& \left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx \\
& -32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \left(\left(\left(\hat{f}_g^0\right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{'}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'} (x)^3 \Psi_1(x) \Psi_1^{'}(x) f^1(x) \right) dx \right) \\
& + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) dx \right. \\
& \left. + \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx \right) \\
= \; & 8 \int \frac{\left(f^0\right)^{'} (x)^3 \Psi_1(x) \Psi_1^{'}(x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x) \right] dx \\
& + 8 \int \frac{\left(f^0\right)^{'} (x)^3 \Psi_1(x)}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{\left(f^0\right)^{'} (x)^3 \Psi_1^{'}(x)}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{\left(f^0\right)^{'} (x)^3}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{\Psi_1(x) \Psi_1^{'}(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)^{'} (x)^3 - \left(f^0\right)^{'} (x)^3 \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{\Psi_1(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)^{'} (x)^3 - \left(f^0\right)^{'} (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{\Psi_1^{'}(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)^{'} (x)^3 - \left(f^0\right)^{'} (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \hat{f}_g^1(x) \right] dx \\
& + 8 \int \frac{1}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)^{'} (x)^3 - \left(f^0\right)^{'} (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \right. \\
& \left. \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x) \right) \cdot \hat{f}_g^1(x) \right] dx - 32 \int \frac{\left(f^0\right)^{'} (x)^3 \Psi_1(x) \Psi_1^{'}(x) f^1(x)}{f^0(x)^6} \\
& \left[\hat{f}_g^0(x) - f^0(x) \right] dx + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right)^2 dx \right) \\
& + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right) \right. \\
& \left. \cdot \left(\left(\left(\hat{f}_g^0\right)^{'} (x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{'}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'} (x)^3 \Psi_1(x) \Psi_1^{'}(x) f^1(x) \right) dx \right) \right. \\
& + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) dx \right. \\
& \left. + \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx \right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +8\int \frac{\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3\Psi_1'(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int \frac{\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
& +8\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx \\
& +8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx \\
& +8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx \\
& +8\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\Big]\,dx\\
&-32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^6}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}^1_g(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)^2dx+\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)dx\right)\\
=\;&8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)}{f^0(x)^5}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi'_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+24\int\frac{\Psi_1(x)\Psi'_1(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^6}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)^2+\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)
\end{aligned}
$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right). \tag{C.106}$$

Collecting terms (C.95)-(C.106) and plugging them in (C.92), we obtain an expression for term B_2 (C.91), given by:

$$\begin{aligned}B_2 \;:=\; & 4\int\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\ & +4\int\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx\\ & +4\int\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right]dx\end{aligned}$$

$$
\begin{aligned}
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-4\int \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+24\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-32\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-12\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx
\end{aligned}
$$

$$-16\int \frac{\left(f^0\right)'(x)^2 f^1(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]\,dx$$

$$+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx$$

$$-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]\,dx$$

$$-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]\,dx$$

$$-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]\,dx$$

$$-8\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]\,dx$$

$$+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx$$

$$-8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]\,dx$$

$$-16\int \frac{\left(f^0\right)'(x)^4 f^1(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]\,dx$$

$$-32\int \frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]\,dx$$

$$+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]\,dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]\,dx$$

$$+8\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]\,dx$$

$$+8\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]\,dx$$

$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$
$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$
$$+24\int \frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$
$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\,dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$

$$\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + \mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)$$
$$\cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)$$
$$\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\right.\right.$$
$$\left.\left.\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left.\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) + \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{''}(x)-\left(f^{0}\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_{1}(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 + \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x)^2 - \Psi_1^{'}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x)^2 - \Psi_1^{'}(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x)^2 - \Psi_1^{'}(x)^2\right)\right.$$
$$\left. \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right.$$
$$\left. \cdot \left(\hat{\Psi}_{1,g}^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - \Psi_1^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 + \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right)^2 dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^4-\left(f^0\right)^{'}(x)^4\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^4-\left(f^0\right)^{'}(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^4-\left(f^0\right)^{'}(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^4\Psi_1^2(x)f^1(x)\right)\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right) dx\right)+\mathcal{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2\right)$$
$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2 dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$
$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right) dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx+\int\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\,dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)dx\right)$$

$$+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$

$$
\begin{aligned}
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\Big) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right. \\
&\cdot \left.\left(\left(\hat{f}_g^0\right)'(x)^3 \hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)^3 \Psi_1(x)\Psi_1'(x) f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\right. \\
&\cdot \left.\left(\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - m'(x)m''(x)\left(f^0\right)'(x) f^1(x)\right) dx\right). \quad (C.107)
\end{aligned}
$$

Finally, further calculations with term B_3 remain to be worked out. Considering expression (C.83), it turns out:

$$
\begin{aligned}
B_3 \;:=\; & 4\int \left[\frac{\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right] dx \\
= \; & 4\int \frac{1}{f^0(x)^2}\left[\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right] dx \\
& -4\int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)^2 - f^0(x)^2\right] dx \\
& +\mathcal{O}\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right. \\
& \cdot \left.\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
= \; & 4\int \frac{1}{f^0(x)^2}\left[\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right] dx \\
& -8\int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
& +\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right. \\
& \cdot \left.\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 4\int \frac{1}{f^0(x)^2}\left[\hat{m}_g^{'}(x)^2\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)+\left(f^0\right)^{'}(x)^2 f^1(x)-\left(f^0\right)^{'}(x)^2 f^1(x)\right)\right. \\
&\quad \left.-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right]\,dx-8\int \frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx \\
&\quad +\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right) \\
&\quad +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\quad \left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right)\,dx\right) \\
&= 4\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^2}\left[\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right]\,dx \\
&\quad +4\int \frac{1}{f^0(x)^2}\left[\hat{m}_g^{'}(x)^2\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2 f^1(x)\right)\right]\,dx \\
&\quad -8\int \frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx \\
&\quad +\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right) \\
&\quad +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\quad \left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right)\,dx\right) \\
&= 8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)}{f^0(x)^2}\left[\hat{m}_g^{'}(x)-m^{'}(x)\right]\,dx \\
&\quad +4\int \frac{m^{'}(x)^2}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2 f^1(x)\right]\,dx \\
&\quad +4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2 f^1(x)\right)\right]\,dx \\
&\quad -8\int \frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]\,dx \\
&\quad +\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\right) \\
&\quad +\mathcal{O}\left(\int\left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2\,dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\quad \left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right)\,dx\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^2}\left[\frac{\hat{\Psi}'_{1,g}(x)}{\hat{f}^0_g(x)} - \frac{\Psi'_1(x)}{f^0(x)}\right] dx \\
& -8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^2}\left[\frac{\left(\hat{f}^0_g\right)'(x)\hat{\Psi}_{1,g}(x)}{\hat{f}^0_g(x)^2} - \frac{\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^2}\right] dx \\
& +4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}^1_g(x) - f^1(x)\right] dx \\
& +4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot \hat{f}^1_g(x)\right] dx \\
& +4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}'_g(x)^2 - m'(x)^2\right)\cdot\left(\hat{f}^1_g(x) - f^1(x)\right)\right] dx + 4\int \frac{1}{f^0(x)^2} \\
& \left[\left(\hat{m}'_g(x)^2 - m'(x)^2\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot \hat{f}^1_g(x)\right] dx \\
& -8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
& +\mathcal{O}\left(\left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{m}'_g(x) - m'(x)\right)^2 dx\right) + \mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\right. \\
& \left.\cdot\left(\hat{m}'_g(x)^2\left(\hat{f}^0_g\right)'(x)^2 \hat{f}^1_g(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
= \quad & 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}\left[\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right] dx \\
& -8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi'_1(x)}{f^0(x)^4}\left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
& -8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^4}\left[\left(\hat{f}^0_g\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right] dx \\
& -8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{f}^0_g(x)^2 - f^0(x)^2\right] dx \\
& +4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}^1_g(x) - f^1(x)\right] dx \\
& +4\int \frac{m'(x)^2 f^1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+4\int \frac{m^{'}(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right]dx\\
&+4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-8\int \frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)^{'}(x)\Psi_1(x)\right)dx\right)\\
=\;&8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx\\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)\Psi_1^{'}(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-16\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)\left(f^0\right)^{'}(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+4\int \frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int \frac{m^{'}(x)^2 f^1(x)\left(f^0\right)^{'}(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx\\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)m^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{\Psi}_{1,g}(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
&+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
&+4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right]dx \\
&+4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right. \\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{m}_g'(x)-m'(x)\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right)dx\right) \\
=\;& 8\int \frac{\left(f^0\right)'(x)^2f^1(x)m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
&-8\int \frac{\left(f^0\right)'(x)^2f^1(x)m'(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&-8\int \frac{\left(f^0\right)'(x)^2f^1(x)m'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
&-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&+4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
&+8\int \frac{m'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-16\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)\left(f^0\right)^{'}(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx \\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx \\
&+4\int \frac{m^{'}(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
&+4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx \\
&+4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right]dx \\
&+4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right. \\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2 f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)^{'}(x)\Psi_1(x)\right)dx\right) \\
=\;& 8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx \\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)\Psi_1^{'}(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&-8\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-8\int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
&+4\int \frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right] dx \\
&+8\int \frac{m'(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx \\
&-16\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{m}_g'(x)-m'(x)\right)^2 dx\right)+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\right)+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2 dx\right)+\mathcal{O}\left(\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right). \qquad \text{(C.108)}
\end{aligned}
$$

Bringing together terms (C.90), (C.107) and (C.108), and afterwards plugging them in (C.87), Lemma 4 is proven.

Furthermore, term r_{1,n_0} in Lemma 4 is given by:

$$
\begin{aligned}
r_{1,n_0} \;=\; & \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx \\
& + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \\
& \cdot \left(\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)
\end{aligned}
$$

$$\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)$$

$$\cdot \left(\hat{\Psi}_{1,g}^{'}(x) \left(\hat{f}_g^0\right)^{'}(x)^3 \hat{\Psi}_{1,g}(x) \hat{f}_g^1(x) - \Psi_1^{'}(x) \left(f^0\right)^{'}(x)^3 \Psi_1(x) f^1(x)\right) dx$$

$$+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) + \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx$$

$$+ \int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right)^2 dx + \int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx$$

$$+ \int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) dx$$

$$+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx$$

$$+ \int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)$$

$$\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)$$

$$\cdot \left(\hat{\Psi}_{1,g}^{'}(x) \left(\hat{f}_g^0\right)^{''}(x) \hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)^{'}(x) \hat{f}_g^1(x)\right.$$

$$\left. - \Psi_1^{'}(x) \left(f^0\right)^{''}(x) \Psi_1(x) \left(f^0\right)^{'}(x) f^1(x)\right) dx$$

$$+\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx$$
$$+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx+\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx$$

$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx$$
$$+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$
$$\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)^2dx+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)$$
$$\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx$$
$$+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx$$
$$+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx$$

$$+\int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx + \int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right)^2 dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\, dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right)$$

$$\cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx + \int \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right) + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx$$

$$\cdot \left(\left(\hat{f}^0_g\right)'(x)^2 \hat{\Psi}_{1,g}(x)\hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x) - \left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x) f^1(x)\right)\, dx$$

$$+\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\, dx$$

$$+\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\, dx$$

$$+\int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right)\, dx$$

$$+\int \left(\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\, dx$$

$$+\int \left(\left(\hat{f}^0_g\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right)\, dx$$

$$
\begin{aligned}
&+\int \left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx+\int \left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2\,dx\\
&+\int \left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2\,dx+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2\,dx+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\\
&\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\,dx+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\\
&\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx
\end{aligned}
$$

$$+\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx$$

$$\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{\Psi}_{1,g}(x)^2 \left(\hat{f}_g^0\right)^{''}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 \Psi_1(x)^2 \left(f^0\right)^{''}(x) f^1(x)\right) dx$$

$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx$$

$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right)^2 dx + \int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) dx$$

$$+\int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) dx$$

$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) dx$$

$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)$$

$$\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)$$

$$\cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}^{'}(x) \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^3 \Psi_1(x) \Psi_1^{'}(x) f^1(x)\right) dx$$

$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx + \int \left(\hat{m}_g^{'}(x) - m^{'}(x)\right)^2 dx$$
$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)$$
$$\cdot \left(\hat{m}_g^{'}(x)\hat{m}_g^{''}(x) \left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x) - m^{'}(x)m^{''}(x) \left(f^0\right)^{'}(x)f^1(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{''}(x) - \Psi_1^{''}(x)\right) dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx$$
$$+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)^{''}(x) - \Psi_1(x) \left(f^0\right)^{''}(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx$$
$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx$$
$$+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) \left(\hat{f}_g^0\right)^{'}(x) - \Psi_1^{'}(x) \left(f^0\right)^{'}(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)^{'}(x)^2 - \Psi_1(x) \left(f^0\right)^{'}(x)^2\right) dx$$
$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{''}(x) - \left(f^0\right)^{''}(x)\right) dx$$
$$+\int \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right) dx$$
$$+\int \left(\hat{m}_g^{''}(x)^2 - m^{''}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{m}_g^{''}(x) - m^{''}(x)\right)^2 dx$$
$$+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) dx$$
$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right)^2 dx$$
$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x) - \left(f^0\right)^{'}(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx$$
$$+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$
$$+\int \left(\hat{m}_g^{'}(x)^2 - m^{'}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$
$$+\int \left(\hat{m}_g^{'}(x)^2 - m^{'}(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 - \left(f^0\right)^{'}(x)^2\right) dx$$

$$+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right) dx$$
$$+\int \left(\hat{m}_g'(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx$$
$$+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx.$$

Remark 23 *Consider two random variables η, ξ. Thanks to Cauchy-Schwartz inequality, we have:*

$$\mathbb{E}\left[(\eta+\xi)^2\right] \leq 2\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right). \quad \text{(C.109)}$$

Similarly, given an additional random variable ζ, it follows that:

$$\mathbb{E}\left[(\eta+\xi+\zeta)^2\right] \leq 3\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right]\right). \quad \text{(C.110)}$$

Proof of Remark 23 Using that $2\mathbb{E}[\eta]\mathbb{E}[\xi] \leq \mathbb{E}[\eta]^2 + \mathbb{E}[\xi]^2$, since $0 \leq (\mathbb{E}[\eta] - \mathbb{E}[\xi])^2 = \mathbb{E}[\eta]^2 + \mathbb{E}[\xi]^2 - 2\mathbb{E}[\eta]\mathbb{E}[\xi]$. Then,

$$\begin{aligned} 0 &\leq \mathbb{E}\left[(\eta+\xi)^2\right] = \left|\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + 2\mathbb{E}\left[\eta\xi\right]\right| \\ &\leq \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] \\ &= 2\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &\leq \mathbb{E}\left[(\eta+\xi+\zeta)^2\right] = \left|\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right]\right. \\ &\quad \left. +2\mathbb{E}\left[\eta\xi\right] + 2\mathbb{E}\left[\eta\zeta\right] + 2\mathbb{E}\left[\zeta\xi\right]\right| \\ &\leq \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right] + \left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right) \\ &\quad + \left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\zeta^2\right]\right) + \left(\mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right]\right) \\ &= 3\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right). \end{aligned}$$

Corollary 3 *Consider a pilot bandwidth $g > 0$ of exact order $n_0^{-1/2}$ and expressions*

(4.31). Assume that f^0 and its derivatives tend to zero as $x \to \infty$ and m_ℓ, ν and their derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Then:

$$\begin{aligned} MISE^{a*}(h) &= \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g \\ &+\mathcal{O}_P\left(h^6\, n_1^{-1}\left(n_0^{7/2}+n_0^4+n_0^{9/2}\right)\right)+\mathcal{O}_P\left(h\, n_1^{-1}\left(1+n_0^{1/2}+n_0\right)\right) \\ &+\mathcal{O}_P\left(h^{-1}\, n_0^{-1}\, n_1^{-1}\right). \end{aligned}$$

Proof of Corollary 3 Recall now expression (4.40). From now on, $C^{[s]}_{\nu,\ell,r}$, $\Psi^{(r)}_{s,\ell}$ will be denoted just by C and $\Psi^{(r)}_\ell$, respectively, for the sake of brevity, assuming $s = 0$. We begin by anaylizing the expectation of expression (4.40),

$$\begin{aligned} \mathbb{E}\left[\hat{\Psi}^{(r)}_\ell(x)\right] &= \frac{1}{n_0\, g^{r+1}}\sum_{i=1}^{n_0}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_i^0}{g}\right)Y_i^{0\ell}\right] \\ &= \frac{1}{g^{r+1}}\mathbb{E}\left[\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)Y_1^{0\ell}\bigg|_{X_1^0}\right]\right] \\ &= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)\mathbb{E}\left[Y_1^{0\ell}\big|_{X_1^0}\right]\right] \\ &= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)m_\ell(X_1^0)\right] \\ &= \int K_g^{(r)}(x-y)\, m_\ell(y) f^0(y)\, dy = K_g^{(r)} * \phi_\ell(x), \end{aligned} \tag{C.111}$$

where $\phi_\ell(u) = m_\ell(u)f^0(u)$. Considering now expression (C.111), we can compute further calculations with expression (4.40). We denote by $\hat{\mu}_{i,g}(x) := K_g^{(r)}(x - X_i^0)\, Y_i^{0\ell} - K_g^{(r)} * \phi_\ell(x)$ and $\xi_i := \int \nu(x)\hat{\mu}_{i,g}(x)$. Then,

$$\begin{aligned} C &= \int \nu(x)\left(\frac{1}{n_0\, g^{r+1}}\sum_{i=1}^{n_0}K^{(r)}\left(\frac{x-X_i^0}{g}\right)Y_i^{0\ell} - K_g^{(r)} * \phi_\ell(x) + K_g^{(r)} * \phi_\ell(x)\right. \\ &\left. -\Psi_\ell^{(r)}\right) dx \\ &= \int \nu(x)\left(\frac{1}{n_0}\sum_{i=1}^{n_0}\hat{\mu}_{i,g}(x) + K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\int \nu(x)\hat{\mu}_{i,g}(x)\,dx + \int \nu(x)\left[K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right]\,dx \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\xi_i + \int \nu(x)\left[K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right]\,dx. \qquad \text{(C.112)}
\end{aligned}
$$

It follows that the MSE of $\hat{\Psi}_\ell^{(r)}(x)$ turns out to be, considering expression (C.112) and taking into account the fact that $\mathbb{E}[\xi_1] = 0$:

$$
\begin{aligned}
\mathbb{E}\left[C^2\right] &= Var\,[C] + \mathbb{E}\,[C]^2 = \frac{Var\,[\xi_1]}{n_0} + \mathbb{E}\,[C]^2 = \frac{\mathbb{E}\,[\xi_1^2]}{n_0} + \mathbb{E}\,[C]^2 \\
&= \frac{\mathbb{E}\,[\xi_1^2]}{n_0} + \left(\int \nu(x)\left(K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}(x)\right)\,dx\right)^2. \qquad \text{(C.113)}
\end{aligned}
$$

From now on, assume that f^0 and its derivatives tend to zero as $x \to \infty$ and m_ℓ and its derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Focusing now on the second term of expression (C.113) leads to:

$$
\begin{aligned}
\int \nu(x) K_g^{(r)} * \phi_\ell(x)\,dx &= \int \nu(x)\int K_g^{(r)}(x-y)\phi_\ell(y)\,dy\,dx \\
&\overset{(1)}{=} \int \nu(x)\int K_g(x-y)\phi_\ell^{(r)}(y)\,dy\,dx \\
&= \int \phi_\ell^{(r)}(y)\int \nu(x)\frac{1}{g}K\left(\frac{x-y}{g}\right)\,dy\,dx \\
&= \int \phi_\ell^{(r)}(y)\int \nu(y+g\,u)K\,(u)\,du\,dy \\
&= \int \nu(x)\phi_\ell^{(r)}(x)\,dx + \frac{g^2}{2}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \\
&\quad +\mathcal{O}\left(g^4\right). \qquad \text{(C.114)}
\end{aligned}
$$

It is straightforward to prove equality (1) just by applying integration by parts to:

$$\begin{aligned} I &:= \int K_g^{(r)}(x-y)\phi_\ell(y)\,dy \\ &= \left[-K_g^{(r-1)}(x-y)\phi_\ell(y)\right]_{y=-\infty}^{y=+\infty} + \int_{-\infty}^{\infty} K_g^{(r-1)}(x-y)\phi_\ell'(y)\,dy \\ &= \int_{-\infty}^{\infty} K_g^{(r-1)}(x-y)\phi_\ell'(y)\,dy = \ldots = \int_{-\infty}^{\infty} K_g(x-y)\phi_\ell^{(r)}(y)\,dy. \end{aligned}$$

Combining the second term in (C.113) and expression (C.114) leads to conclude that:

$$\begin{aligned} \left(\int \nu(x)\left(K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}(x)\right)dx\right)^2 &= \left(\frac{g^2}{2}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \right. \\ &\quad \left. +\mathcal{O}\left(g^4\right)\right)^2 \\ &= \frac{g^4}{4}\mu_2(K)^2\left(\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\ &\quad +\mathcal{O}\left(g^6\right). \end{aligned} \tag{C.115}$$

Further computations with the first term of expression (C.113) remain to be carried out:

$$\begin{aligned} \frac{\mathbb{E}\left[\xi_1^2\right]}{n_0} &= \frac{1}{n_0}\mathbb{E}\left[\int \nu(x)\hat{\mu}_{1,g}(x)\,dx \int \nu(y)\hat{\mu}_{1,g}(y)\,dy\right] \\ &= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\hat{\mu}_{1,g}(x)\hat{\mu}_{1,g}(y)\right]dx\,dy \\ &= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\left(K_g^{(r)}\left(x-X_1^0\right)Y_1^{0\ell} - K_g^{(r)} * \phi_\ell(x)\right)\right. \\ &\quad \left. \cdot\left(K_g^{(r)}\left(y-X_1^0\right)Y_1^{0\ell} - K_g^{(r)} * \phi_\ell(y)\right)\right]dx\,dy \\ &= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\right]dx\,dy \\ &\quad -\frac{1}{n_0}\left(\int \nu(x)K_g^{(r)} * \phi_\ell(x)\,dx\right)^2. \end{aligned} \tag{C.116}$$

The second term in (C.116) has already been analyzed in (C.114), turning out:

$$
\begin{aligned}
\frac{1}{n_0}\left(\int \nu(x) K_g^{(r)} * \phi_\ell(x)\, dx\right)^2 \quad = \quad & \frac{1}{n_0}\left(\int \nu(x)\phi_\ell^{(r)}(x)\, dx\right)^2 \\
& + \frac{g^2}{n_0}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\, dx \qquad \text{(C.117)} \\
& \int \nu(x)\phi_\ell^{(r)}(x)\, dx + \mathcal{O}\left(\frac{g^4}{n_0}\right).
\end{aligned}
$$

Finally, what remains to be worked out are further computations with:

$$
\begin{aligned}
& \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x - X_1^0\right) K_g^{(r)}\left(y - X_1^0\right)\left(Y_1^{0\ell}\right)^2\right] dx\, dy \\
= \; & \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\mathbb{E}\left[K_g^{(r)}\left(x - X_1^0\right) K_g^{(r)}\left(y - X_1^0\right)\left(Y_1^{0\ell}\right)^2\Big|_{X_1^0}\right]\right] dx\, dy \\
= \; & \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x - X_1^0\right) K_g^{(r)}\left(y - X_1^0\right)\mathbb{E}\left[\left(Y_1^{0\ell}\right)^2\Big|_{X_1^0}\right]\right] dx\, dy \\
= \; & \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x - X_1^0\right) K_g^{(r)}\left(y - X_1^0\right) m_{2\ell}(X_1^0)\right] dx\, dy \\
= \; & \frac{1}{n_0}\int\int \nu(x)\nu(y)\int K_g^{(r)}\left(x - z\right) K_g^{(r)}\left(y - z\right) m_{2\ell}(z) f^0(z)\, dz\, dx\, dy \\
= \; & \frac{1}{n_0\, g^{2r+2}}\int\int \nu(x)\nu(y)\int K^{(r)}\left(\frac{x-z}{g}\right) K^{(r)}\left(\frac{y-z}{g}\right) m_{2\ell}(z) f^0(z)\, dz\, dx\, dy \\
= \; & \frac{1}{n_0\, g^{2r}}\int m_{2\ell}(z) f^0(z)\int\int \nu(z + g\,u)\nu(z + g\,v) K^{(r)}\left(u\right) K^{(r)}\left(v\right)\, du\, dv\, dz \\
= \; & \frac{2\, g^2}{r!\,(r+2)!\, n_0}\int u^r K^{(r)}(u)\, du \int u^{r+2} K^{(r)}(u)\, du \int \nu^{(r)}(z)\nu^{(r+2)}(z) m_{2\ell}(z) f^0(z)\, dz \\
& + \frac{1}{(r!)^2\, n_0}\left(\int u^r K^{(r)}(u)\, du\right)^2 \int \nu^{(r)}(z)^2 m_{2\ell}(z) f^0(z)\, dz + \mathcal{O}\left(\frac{g^4}{n_0}\right). \qquad \text{(C.118)}
\end{aligned}
$$

Assume K is a symmetric function with zero mean. Therefore, K' and K'' are antisymmetric and symmetric functions, respectively. Thus, it is easy to prove by integration by parts that:

- If $r = 0$,

$$
\int K(u)\, du = 1, \int uK(u)\, du = 0, \int u^2 K(u)\, du = \mu_2(K).
$$

- If $r = 1$,

$$\int K^{'}(u)\,du = 0, \int uK^{'}(u)\,du = -1, \int u^2K^{'}(u)\,du = 0,$$
$$\int u^3K^{'}(u)\,du = -3\,\mu_2(K).$$

- If $r = 2$,

$$\int K^{''}(u)\,du = 0, \int uK^{''}(u)\,du = 0, \int u^2K^{''}(u)\,du = 2,$$
$$\int u^3K^{''}(u)\,du = 0, \int u^4K^{''}(u)\,du = 12\,\mu_2(K).$$

In general, $\int u^r K^{(r)}(u)\,du = (-1)^r\, r!$ and $\int u^{r+2}K^{(r)}(u)\,du = \frac{1}{2!}\,(-1)^r\,(r+2)!\mu_2(K)$.

Hence, expression (C.118) results in:

$$\frac{1}{n_0}\int \nu^{(r)}(z)^2 m_{2\ell}(z) f^0(z)\,dz \tag{C.119}$$
$$+\frac{g^2}{n_0}\mu_2(K)\int \nu^{(r)}(z)\nu^{(r+2)}(z) m_{2\ell}(z) f^0(z)\,dz + \mathcal{O}\left(\frac{g^4}{n_0}\right).$$

Using expression (C.109), a deeper insight can be achieved for expression (4.38). Indeed, our aim is to work out further computations for:

$$\begin{aligned} & \mathbb{E}\left\{\left[\left(\hat{A}_g - A\right) + \frac{A}{4B}\left(\hat{B}_g - B\right)\right]^2\right\} \\ = \ & \mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{16B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right] + \frac{A}{2B}\mathbb{E}\left[\left(\hat{A}_g - A\right)\cdot\left(\hat{B}_g - B\right)\right] \\ \leq \ & 2\mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{8B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right]. \end{aligned} \tag{C.120}$$

Finally, considering expression (C.110), collecting terms (C.115), (C.117) and (C.118), and afterwards plugging them in (C.113), it turns out that, $\forall \ell \in \{0, 1, 2\}, \forall r \in$

$\{0,1,2\}$:

$$
\begin{aligned}
\mathbb{E}\left[C^2\right] \;=\;& \frac{g^4}{4}\mu_2(K)^2\left(\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
& -\frac{g^2}{n_0}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\int \nu(x)\phi_\ell^{(r)}(x)\,dx-\frac{1}{n_0}\left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
& +\frac{2\,g^2}{r!\,(r+2)!\,n_0}\int u^r K^{(r)}(u)\,du\int u^{r+2}K^{(r)}(u)\,du \\
& \int \nu^{(r)}(x)\nu^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx \\
& +\frac{1}{(r!)^2\,n_0}\left(\int u^r K^{(r)}(u)\,du\right)^2\int \nu^{(r)}(x)^2 m_{2\ell}(x)f^0(x)\,dx \\
& +\mathcal{O}\left(g^6\right)+\mathcal{O}\left(\frac{g^4}{n_0}\right).
\end{aligned}
$$

In other words, considering expression (C.119) instead of (C.118):

$$
\begin{aligned}
\mathbb{E}\left[C^2\right] \;=\;& \frac{g^4}{4}\left(\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
& +\frac{1}{n_0}\left[\int \nu^{(r)}(x)^2 m_{2\ell}(x)f^0(x)\,dx-\left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2\right] \\
& +\frac{g^2}{n_0}\mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx\right. \\
& \left.-\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right] \\
& +\mathcal{O}\left(g^6\right)+\mathcal{O}\left(\frac{g^4}{n_0}\right), \forall \ell\in\{0,1,2\}, \forall r\in\{0,1,2\}.
\end{aligned}
$$

Similarly, $\forall \ell\in\{0,1,2\}, \forall r\in\{0,1,2\}$,

$$
\mathbb{E}\left[C^2\right] \;=\; g^4\,C_1+\frac{g^2}{n_0}C_2+\frac{1}{n_0}C_3+\mathcal{O}\left(g^6\right)+\mathcal{O}\left(\frac{g^4}{n_0}\right),
$$

where

$$C_1 := \left(\frac{\mu_2(K)}{2}\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2, \tag{C.121}$$

$$C_2 := \mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx - \int \nu''(x)\phi_\ell^{(r)}(x)\,dx \int \nu(x)\phi_\ell^{(r)}(x)\,dx\right], \tag{C.122}$$

$$C_3 := \int \nu^{(r)}(x)^2 m_{2\ell}(x)f^0(x)\,dx - \left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2. \tag{C.123}$$

It is clear that C_1 in (C.121) is a non negative quantity. Let's focus now on C_2 in (C.122). Assume that f^0 and its derivatives tend to zero as $x \to \infty$; m_ℓ and ν and their derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Thus, by applying integration by parts to the second term in (C.122), it turns out:

$$\begin{aligned}
\int \nu''(x)\phi_\ell^{(r)}(x)\,dx &= \left[\nu''(x)\phi_\ell^{(r-1)}(x)\right]_{-\infty}^{+\infty} - \int \nu'''(x)\phi_\ell^{(r-1)}(x)\,dx \\
&= -\int \nu'''(x)\phi_\ell^{(r-1)}(x)\,dx \\
&= -\left[\nu'''(x)\phi_\ell^{(r-2)}(x)\right]_{-\infty}^{+\infty} + \int \nu^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\
&= \int \nu^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\
&= \cdots = (-1)^r \int \nu^{(r+2)}(x)\phi_\ell(x)\,dx.
\end{aligned} \tag{C.124}$$

Similarly,

$$\int \nu(x)\phi_\ell^{(r)}(x)\,dx = (-1)^r \int \nu^{(r)}(x)\phi_\ell(x)\,dx. \tag{C.125}$$

Hence, expression (C.122) can be rewritten as:

$$C_2 := \mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx\right.$$

$$
\begin{aligned}
& -(-1)^{2r}\int \nu^{(r+2)}(x)\phi_\ell(x)\,dx \int \nu^{(r)}(x)\phi_\ell(x)\,dx\Big] \\
= \;& \mu_2(K)\left[\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx\right.\right. \\
& \left.-\int \nu^{(r+2)}(x)m_\ell(x)f^0(x)\,dx \int \nu^{(r)}(x)m_\ell(x)f^0(x)\,dx\right] \\
= \;& Cov\left[\nu^{(r)}(X^0)Y^{0\ell}, \nu^{(r+2)}(X^0)Y^{0\ell}\right]. \qquad \text{(C.126)}
\end{aligned}
$$

Considering now expressions (C.121) and (C.124), it follows that:

$$
C_1 \;:=\; \left(\frac{\mu_2(K)}{2}\int \nu^{(r+2)}(x)m_\ell(x)f^0(x)\,dx\right)^2 = \left(\frac{\mu_2(K)}{2}\mathbb{E}\left[\nu^{(r+2)}(X^0)Y^{0\ell}\right]\right)^2.
$$

Finally, combining expressions (C.123) and (C.125) leads to:

$$
\begin{aligned}
C_3 \;:=\; & \int \nu^{(r)}(x)^2 m_{2\ell}(x)f^0(x)\,dx - \left(\int \nu^{(r)}(x)m_\ell(x)f^0(x)\,dx\right)^2 \\
= \;& Var\left[\nu^{(r)}\left(X^0\right)Y^{0\ell}\right], \qquad \text{(C.127)}
\end{aligned}
$$

turning out that C_3 (C.123) is, as well, greater than 0.

As a consequence of expressions (C.121), (C.126) and (C.127), the optimal g will be determined by the positive or negative sign accompanying term C_2 in (C.126). In particular, denoting by $\vartheta(g) := \mathbb{E}\left[C^2\right]$ and considering expressions (4.42), (4.43) and (C.110), expression (C.120) could be rewritten as:

$$
\begin{aligned}
& \sum_{i=1}^{k_0} a_{0,i}^2\vartheta_i(g) + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2\vartheta_i(g) + \sum_{i=1}^{k_2} a_{2,i}^2\vartheta_i(g) + \sum_{i=1}^{k_3} a_{3,i}^2\vartheta_i(g)\right) \\
= \;& \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right]\cdot \vartheta_i(g) \\
= \;& \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right]\cdot \left(g^4\, C_{1,i} + \frac{g^2}{n_0}C_{2,i} + \frac{1}{n_0}C_{3,i}\right) \\
= \;& g^4\, C_1^0 + \frac{g^2}{n_0}\,C_2^0 + \frac{1}{n_0}\,C_3^0,
\end{aligned}
$$

where

$$\begin{aligned}
C_1^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{1,i} \geq 0 \\
C_3^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{3,i} \geq 0, \text{ and} \\
C_2^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{2,i} \\
&= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] \\
&\quad \cdot Cov\left[\nu_i^{(r)}(X^0)Y^{0\ell}, \nu_i^{(r+2)}(X^0)Y^{0\ell}\right].
\end{aligned}$$

If $C_2^0 \geq 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (4.41) and (4.43), respectively) is as close as 0 as possible. On the other hand, if $C_2^0 < 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (4.41) and (4.43), respectively) happens to be

$$\begin{aligned}
g_{OPT} &\approx \arg\min_{g>0} \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] \cdot \vartheta_i(g) \\
&= \left(-\frac{C_2^0}{2\,C_1^0}\right)^{1/2} \cdot n_0^{-1/2}. \qquad \text{(C.128)}
\end{aligned}$$

Moreover, it is straightforward that:

$$\left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] \vartheta_i(g_{OPT}) = n_0^{-1}\, C_3^0 - n_0^{-2}\, \frac{\left(C_2^0\right)^2}{4\, C_1^0}.$$

Combining expressions (4.32) and (C.128), Corollary 3 holds.

Proposition 3 *Assume conditions (C1), (C2), (C3). Suppose, additionally, that* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\, h \longrightarrow \infty$. *Consider* h_{n_0} *a sequence of bandwidths such that* $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$. *Assume, moreover, that* $(n_0\, h)^{-1/2} \log(h^{-1}) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\, h \longrightarrow \infty$.

Then,

$$\begin{aligned} ISE^*(h) &= ISE^{a*}(h) + \mathcal{O}_{P^*}\left(h^6\right) + \mathcal{O}_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathcal{O}_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) \\ &\quad + \mathcal{O}_{P^*}\left(\frac{\log\frac{1}{h}}{n_0^{3/2}h^{3/2}}\right), \end{aligned}$$

almost sure with respect to P, where

$$\begin{aligned} ISE^*(h) &= \int \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x), \text{ and} \\ ISE^{a*}(h) &= \int \left(\tilde{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x). \end{aligned}$$

Proof of Proposition 3 Given that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\, h \longrightarrow \infty$, we have:

$$\sup_x \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty,$$

and

$$\sup_x \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right| = \mathcal{O}_{P^*}\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).$$

Moreover,

$$\sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty,$$

and

$$\sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right| = \mathcal{O}_{P^*}\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).$$

Focusing now on $MISE^*(h)$, it is straightforward that

$$MISE^*(h) = MISE^{a*}(h) + \mathbb{E}^*\left[\int 2\, A_1^*\, A_2^*\, d\hat{F}_g^1(x)\right] + \mathbb{E}^*\left[\int (A_2^*)^2\, d\hat{F}_g^1(x)\right],$$

where

$$A_1^* = \frac{\hat{\Psi}_h^*(x)}{\hat{f}_g^0(x)} - \frac{\hat{m}_g(x)\hat{f}_h^{0*}(x)}{\hat{f}_g^0(x)} = \frac{1}{n_0\, \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_h(x - X_i^{0*})(Y_i^{0*} - \hat{m}_g(x)),$$

$$A_2^* = \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right) \frac{\left(\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)},$$

and $MISE^{a*}(h)$ is given in (4.32).

On the one hand, using expression (C.72) as well as Silverman (1978) and Mack and Silverman (1982) statements, it turns out:

$$\begin{aligned}
\int_J (A_2^*)^2\, d\hat{F}_g^1(x) &= \left|\int_J (A_2^*)^2\, d\hat{F}_g^1(x)\right| \\
&= \left|\int_J \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right) \frac{\left(\hat{f}_h^0(x) - \hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)} \hat{f}_g^1(x)\, dx\right| \\
&\leq \sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right|^2 \sup_J \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right|^2 \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(\mathcal{O}_{P^*}\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4 \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(\mathcal{O}_{P^*}\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4 \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(\mathcal{O}_{P^*}\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right)
\end{aligned}$$

$$= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx \left(\mathcal{O}_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right)$$
$$= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx \left(\mathcal{O}_{P^*}\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)$$
$$= \mathcal{O}_{P^*}\left(h^8\right) + \mathcal{O}_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right). \tag{C.129}$$

The first term in expression (C.129) is negligible in comparison to the second term in expression (4.32). Similarly, the second term in expression (C.129) becomes insignificant compared to the first term in expression (4.32).

In particular,

$$\left(\int_J (A_2^*)^2\, d\hat{F}_g^1(x)\right)^{1/2} = \left(\mathcal{O}_{P^*}\left(h^8\right) + \mathcal{O}_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2}$$
$$= \left(\mathcal{O}_{P^*}\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2}$$
$$= \left(\mathcal{O}_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right)^{1/2}$$
$$= \mathcal{O}_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}^{1/2}\right)$$
$$= \mathcal{O}_{P^*}\left(\max\left\{h^4, n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right\}\right)$$
$$= \mathcal{O}_{P^*}\left(h^4 + n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right)$$
$$= \mathcal{O}_{P^*}\left(h^4\right) + \mathcal{O}_{P^*}\left(n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right).$$

Moreover, thanks to expression (4.32), we have

$$\mathbb{E}\left[\int_J (A_1^*)^2\, dF_1(x)\right] = \mathcal{O}\left(n_0^{-1}h^{-1} + h^4\right). \tag{C.130}$$

Applying Markov's inequality to expression (C.130), it turns out:

$$\int_J (A_1^*)^2 \, d\hat{F}_g^1(x) \;=\; \mathcal{O}_{P^*}\left(n_0^{-1}h^{-1} + h^4\right).$$

Thus,

$$\begin{aligned}\left(\int_J (A_1^*)^2 \, d\hat{F}_g^1(x)\right)^{1/2} &= \left(\mathcal{O}_{P^*}\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = \mathcal{O}_{P^*}\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right) \\ &= \mathcal{O}_{P^*}\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = \mathcal{O}_{P^*}\left(n_0^{-1/2}h^{-1/2} + h^2\right) \\ &= \mathcal{O}_{P^*}\left(n_0^{-1/2}h^{-1/2}\right) + \mathcal{O}_{P^*}\left(h^2\right). \qquad \text{(C.131)}\end{aligned}$$

Bringing together expressions (C.131) and (C.130) and using Cauchy-Swchartz inequality, we compute:

$$\begin{aligned}\left|\int_J A_1 \, A_2 \, d\hat{F}_g^1(x)\right| &\leq 2\left(\int_J (A_1^*)^2 \, d\hat{F}_g^1(x)\right)^{1/2}\left(\int_J (A_2^*)^2 \, d\hat{F}_g^1(x)\right)^{1/2} \\ &= 2\left(\mathcal{O}_p\left(h^4\right) + \mathcal{O}_{P^*}\left(n_0^{-1}h^{-1}\log\frac{1}{h}\right)\right) \\ &\quad \cdot\left(\mathcal{O}_p\left(h^2\right) + \mathcal{O}_{P^*}\left(n_0^{-1/2}h^{-1/2}\log\frac{1}{h}\right)\right) \\ &= \mathcal{O}_{P^*}\left(h^6\right) + \mathcal{O}_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathcal{O}_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right) \\ &\quad + \mathcal{O}_{P^*}\left(\frac{\log\frac{1}{h}}{(n_0 \, h)^{3/2}}\right). \qquad \text{(C.132)}\end{aligned}$$

The first term in expression (C.132) is negligible as compared to the second term in expression (4.32). Furthermore, the fourth term in expression (C.132) becomes insignificant in comparison to the first term in expression (4.32) if

$$\left(\frac{\log\frac{1}{h}}{n_0^{1/2}h^{1/2}}\right) \longrightarrow 0,$$

as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0\, h \longrightarrow \infty$.

It remains to be seen what happens with the second and third terms in expression (C.132). We begin with the second one. Given that $\left(n_0^{-1}h^{-1} + h^4\right) \cdot n_0 h = 1 + h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1}h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0\, h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2} h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $n_0 h^5 \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2} h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h^{7/2}}{n_0^{1/2}} = o\left(\dfrac{1}{n_0\, h} + h^4\right)$.

Finally, as for the third term in (C.132),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0} \log \frac{1}{h} \sim n_0^{-6/5} \log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h}{n_0}\log\frac{1}{h}}{\frac{1}{n_0\,h}} = h^2\log\frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $h \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h}{n_0}\log\frac{1}{h}}{h^4} = \frac{\log\frac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h}{n_0}\log\frac{1}{h} = o\left(\frac{1}{n_0\,h} + h^4\right)$.

Considering this last reasoning and collecting terms (C.131), (C.130) and (C.132), expression (4.46) is proven.

Theorem 18 *Consider h_{MISE^a} and its bootstrap version, $h^*_{MISE^a}$, which are the minimizers of expressions (4.26) and (4.32), respectively. Under regularity conditions (K1), (D1), (M1) and (V1), it turns out*

$$h^*_{MISE^a} - h_{MISE^a} \;=\; \mathcal{O}_P\left(n_0^{-7/10}\right),$$

or rather,

$$\frac{h^*_{MISE^a} - h_{MISE^a}}{h_{MISE^a}} \;=\; \mathcal{O}_P\left(n_0^{-1/2}\right).$$

Proof of Theorem 18 Consider h_{MISE^a} and its bootstrap version, $h^*_{MISE^a}$, which are the minimizers of expressions (4.26) and (4.32), respectively. Hence, it is obvious that $AMISE^{a*'}(h^*_{AMISE^a}) = 0$ as well as $MISE^{a'}(h_{MISE^a}) = 0$. Moreover, applying

Taylor expansion to $MISE^{a*'}(h^*_{MISE^a})$ towards the point h_{MISE^a}, leads to:

$$\begin{aligned} MISE^{a*'}(h^*_{MISE^a}) &= MISE^{a*'}(h_{MISE^a}) - AMISE^{a'}(h_{AMISE^a}) \\ &\quad + (h^*_{MISE^a} - h_{MISE^a}) \cdot MISE^{a*''}(h_{MISE^a}) + \\ &\quad + \frac{1}{2}(h^*_{MISE^a} - h_{MISE^a})^2 \\ &\quad \cdot MISE^{a*'''}(\widetilde{h_{MISE^a}}), \end{aligned} \qquad \text{(C.133)}$$

where $\widetilde{h_{MISE^a}}$ is an intermediate value between $h^*_{MISE^a}$ and h_{MISE^a}.

As a consequence of expression (C.133), it follows that:

$$\begin{aligned} h^*_{MISE^a} - h_{MISE^a} &= -\frac{MISE^{a*'}(h_{MISE^a}) - AMISE^{a'}(h_{AMISE^a})}{MISE^{a*''}(h_{MISE^a})} \\ &\quad \cdot (1 + o_P(1)) \\ &= -\frac{MISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{MISE^{a*''}(h_{AMISE^a})} \\ &\quad \cdot (1 + o_P(1)) \\ &= -\frac{AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{AMISE^{a*''}(h_{AMISE^a})} \\ &\quad \cdot (1 + o_P(1)). \end{aligned} \qquad \text{(C.134)}$$

Thanks to expressions (4.28) and (4.45), we have:

$$\begin{aligned} AMISE^{a''}(h_{AMISE^a}) &= \frac{R(K)}{n_0\, h^3_{AMISE^a}} A + 3\, h^2_{AMISE^a}\, \mu_2(K)^2\, B \\ &= n_0^{-2/5} \cdot \left[R(K)\, \tilde{c}_0\, A + 3\, \mu_2(K)^2 c_0^2\, B \right], \end{aligned} \qquad \text{(C.135)}$$

where $\tilde{c}_0 = 1/c_0^3$ and,

$$\begin{aligned} & AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a}) \\ &= -\frac{R(K)}{n_0\, h^2_{AMISE^a}} \left(\hat{A}_g - A \right) + h^3_{AMISE^a}\, \mu_2(K)^2 \left(\hat{B}_g - B \right) \\ &= -n_0^{-3/5} \cdot \left[R(K) \tilde{\tilde{c}}_0 \left(\hat{A}_g - A \right) + c_0^3 \mu_2(K)^2 \left(\hat{B}_g - B \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= -n_0^{-3/5} \cdot n_0^{-1/2} \cdot \left[\tilde{\tilde{c}}_0 \, R(K) + c_0^3 \mu_2(K)^2\right] \\
&= -n_0^{-11/10} \cdot \left[\tilde{\tilde{c}}_0 \, R(K) + c_0^3 \mu_2(K)^2\right], \qquad \text{(C.136)}
\end{aligned}
$$

where $\tilde{\tilde{c}}_0 = 1/c_0^2$.

Considering now expressions (4.45) and (C.135), it turns out:

$$
\begin{aligned}
AMISE^{a*''}(h_{AMISE^a}) &= AMISE^{a*''}(h_{AMISE^a}) - AMISE^{a''}(h_{AMISE^a}) \\
&\quad + AMISE^{a''}(h_{AMISE^a}) \\
&= AMISE^{a''}(h_{AMISE^a}) + \mathcal{O}_P\left(n_0^{-9/10}\right). \qquad \text{(C.137)}
\end{aligned}
$$

Therefore, combining expressions (C.134) and (C.137),

$$
\begin{aligned}
h^*_{MISE^a} - h_{MISE^a} &= -\frac{AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{AMISE^{a''}(h_{AMISE^a})} \\
&\quad \cdot (1 + o_P(1)). \qquad \text{(C.138)}
\end{aligned}
$$

Finally, collecting terms (C.136) and (C.135), and plugging them in (C.138) leads to prove the result in Theorem 18.

Appendix D

Resumo en galego

Nesta tese preténdense incluir os estudos realizados ao longo do período de doutoramento. Nela, desenvólvense expresións pechadas para criterios de erro e a súa versión bootstrap para a estimación non paramétrica da curva da densidade (baixo dependencia), da razón de fallo e, por último, para a predición e o *matching* estatístico. Estas expresións pechadas son moi útiles, xa que deste xeito, a aproximación por Monte Carlo xa non é necesaria. Ademais, propóñense novos selectores de ventá bootstrap nos contextos mencionados. En concreto, defínese o selector de ventá como aquel valor que fai mínima a expresión pechada bootstrap para o criterio de erro considerado para a estimación non paramétrica de cada curva. En todos os casos, compróbase o bo comportamento empírico destes métodos por medio de estudos de simulación, ademais de comparalos coas alternativas xa existentes na literatura. Estes métodos son ilustrados por medio da aplicación a distintos conxuntos de datos reais. Ademais, obtéñense resultados da teoría asintótica no caso da selección da ventá para a predición e *matching* estatístico nun dos contextos considerados no Capítulo 4.

Capítulo 1: Introdución

Durante as últimas décadas, a estimación non paramétrica de curvas é un campo de investigación moi activo na área de Estatística. En concreto, se o que queremos é estimar non paramétricamente a función de densidade, Parzen (1962) e Rosenblatt

(1956) introduciron o coñecido estimador tipo núcleo da densidade (KDE), que depende de dous elementos a escoller polo usuario: o núcleo e o selector de ventá. Se ben é certo que a escolla da función núcleo non repercute de xeito notable na eficiencia do estimador, si o fai o selector de ventá, h. En particular, se o selector de ventá é moi pequeno, o nesgo do KDE será pequeno, pero pagará o prezo a súa varianza, que crecerá. Sen embargo, se o valor do selector do ventá é grande, a varianza do estimador é pequena, pero o nesgo verase incrementado. Por tanto, para seleccionar un parámetro de suavizado apropiado, trátase de atopar un equilibrio entre o nesgo e a varianza do estimador. Para iso, estableceranse criterios de erro (como o erro cadrático medio ou o erro cadrático medio integrado, entre outros), dos cales se busca o seu mínimo en h, para poder atopar un selector de ventá axeitado. Tamén pode ser que o obxecto de estudo sexan outras funcións como a regresión, a función de razón de fallo ou, mesmo, a densidade condicional, entre outras, para as cales tamén existen na literatura actual estimadores tipo núcleo propostos.

Neste capítulo realízase unha revisión das principais condicións de dependencia, tanto paramétrica como non paramétrica na Sección 1.2 (condicións fortemente mixing ou α-mixing, condicións uniformemente mixing ou ϕ-mixing, m-dependencia e ψ-dependencia). Realízase, tamén, unha revisión dos principais algoritmos bootstrap dende a súa introdución en Efron (1979) e Efron and Tibishirani (1993) na Sección 1.3. En particular, preséntanse, no contexto de independencia, o bootstrap uniforme, suavizado, así como o método de submostraxe e, no contexto de dependencia, o bootstrap por bloques, o bootstrap estacionario e o método de submostraxe para datos dependentes. Finalmente, faise unha revisión da literatura existente sobre selección do parámetro de suavizado na estimación non paramétrica de curvas, centrándose naqueles escollidos mediante bootstrap e considerando distinta tipoloxía de datos (independencia, dependencia, censura, truncamento, datos agrupados, procesos espaciais, datos contaminados por ruído aleatorio, datos nesgados por lonxitude, datos faltantes e datos direccionais) e distintas curvas (densidade, distribución, regresión, razón de fallo, estimación de parámetros, intensidade e latencia e incidencia en modelos de curación).

Descríbense, tamén, os algoritmos bootstrap existentes na literatura en diferentes contextos (datos independentes, dependentes, para datos nesgados por lonxitude, datos censurados, datos agrupados, ...). Para obter as remostras bootstrap, moitas veces recúrrese á aproximación por Monte Carlo. Sen embargo, este método pode ser computacionalmente moi custoso, o que leva a que nesta tese tratemos de atopar expresións pechadas para a versión bootstrap dalgún criterio de erro, de xeito que a aproximación de Monte Carlo xa non sexa necesaria. Non obstante, hai veces que atopar unha expresión pechada bootstrap é complicado. Introdúcese, para tratar de solventar isto, o concepto de estimador aproximado e xustifícase a súa necesidade neste tipo de contextos, nos que non é posible obter unha expresión pechada bootstrap para o criterio de erro considerado. Esta revisión está recollida en Barbeito and Cao (2019a).

Cabe mencionar que o bootstrap tamén é moi útil noutros contextos nos que o obxectivo non é seleccionar o parámetro ventá, tales como son, por exemplo, construir contrastes de hipóteses ou establecer intervalos de confianza (ver Barbeito et al. (2019), entre outros).

Capítulo 2: Seleción da ventá para a estimación da densidade con datos dependentes

A principal motivación deste capítulo é estender as ideas de Cao (1993) ao contexto de dependencia. En efecto, Cao (1993) desenvolve no seu artigo unha expresión pechada para o erro cadrático medio integrado (MISE), e a súa versión bootstrap suavizada, do estimador tipo núcleo da función de densidade. A finalidade é tratar de minimizar esta expresión pechada bootstrap, en h, e así poder definir un selector de ventá para a estimación non paramétrica da densidade.

No contexto de dependencia, existen xa algoritmos bootstrap como o bootstrap estacionario (ver Politis and Romano (1994b)), o bootstrap por bloques (ver Künsch (1989) e Liu and Singh (1992)) ou o método de submostraxe (ver Politis and Romano (1994a)). Sen embargo, ningún deles recolle e emprega a información de que, ao tratar de estimar a densidade, estamos considerando unha poboación continua. É por iso que, neste capítulo, propóñense dous novos algoritmos bootstrap: o bootstrap estacionario suavizado (SSB) e o bootstrap por bloques suavizado (SMBB).

En ámbolos dous casos desenvólvense expresións pechadas para a versión bootstrap do MISE, propoñendo, ademais, dous selectores de ventá para a estimación da densidade. O primeiro selector de ventá está baseado no algoritmo SSB (ver Barbeito and Cao (2016)), mentres que o segundo está baseado no algoritmo SMBB (ver Barbeito and Cao (2017)).

Lévase a cabo, tanto para o SSB como para o SMBB, un estudo de simulación para comprobar o bo comportamento empírico destes dous novos selectores de ventá bootstrap. Compáranse, tamén, cos selectores de ventá xa existentes na literatura, ou coa súa extensión ao contexto que se está a tratar. En concreto, compáranse co selector de ventá de validación cruzada para dependencia (ver Hart and Vieu (1990)), co plug-in para dependencia (ver Hall et al. (1995)), coa extensión do selector de validación cruzada modificada de Stute (1992) ao caso de dependencia e, finalmente, coa extensión do selector de Estévez-Pérez et al. (2002) para a razón de fallo con dependencia ao contexto de densidade. Os resultados de simulación indican que o selector de ventá baseado no algoritmo SSB é o que mellores resultados aporta, ademais de ser computacionalmente rápido, seguido moi de preto polo selector de ventá SMBB.

Por último, ilústrase esta metodoloxía por medio dunha aplicación a datos reais. Os conxuntos de datos considerados están dispoñibles na librería `TSA` do software estatístico `R`. O primeiro deles, recolle o número de linces canadienses atrapados no río Mackenzie, entre os anos 1821 e 1934. O segundo deles, recolle o número de manchas solares anuais entre 1700 e 1988. En ámbalas situacións, semella que os selectores de ventá baseados nos algoritmos SSB e SMBB producen estimacións precisas da función de densidade.

Capítulo 3: Seleción da ventá para a estimación da razón de fallo

Este capítulo céntrase no contexto da función de razón de fallo. Os estudos realizados ao longo do mesmo recóllense en Barbeito and Cao (2019b). A idea principal é tratar de atopar expresións pechadas para o erro cadrático medio integrado (MISE), e a súa versión bootstrap suavizada, do estimador tipo núcleo da función de razón de fallo. Sen embargo, debido á aleatoriedade do denominador do estimador clásico da función de razón de fallo (ver Watson and Leadbetter (1964a) e Watson and Leadbetter (1964b)), é preciso definir un estimador aproximado do mesmo, de xeito que sexa posible desenvolver unha expresión pechada para o MISE^*.

Así, nas Seccións 3.3 e 3.4 propóñense dúas aproximacións do estimador da razón de fallo, onde a segunda delas ten en conta termos de segunda orde, co que é máis precisa. Desenvólvense, para os dous casos, expresións exactas para o MISE e o seu análogo bootstrap. Posteriormente, mediante a minimización das dúas expresións pechadas para o MISE^* en h, defínense dous novos selectores de ventá bootstrap. Na Sección 3.5 establécese unha simplicación das fórmulas exactas propostas para o MISE^*, cando o núcleo é Gaussiano.

Na Sección 3.6 realízase un estudo de simulación para comprobar o bo comportamento na práctica destes novos selectores bootstrap, ademais de comparalos cos xa existentes na literatura. En concreto, compáranse co selector de validación cruzada de Patil (1993a), co selector bootstrap orixinalmente pensado para datos censurados e baseado nunha expresión pechada para a versión asintótica do erro cadrático medio integrado (denominada AMISE) de González-Manteiga et al. (1996) e co selector DO-validation de Gámiz et al. (2016) proposto para o estimador local linear da función de razón de fallo. Os resultados de simulación amosaron que os dous novos selectores de ventá bootstrap son os que presentan mellores resultados empíricos, seguidos polo selector bootstrap de González-Manteiga et al. (1996).

Aplícanse tamén estes selectores a un conxunto de datos reais, que se poden atopar na librería `MASS` do software estatístico `R`. Estes datos recollen o índice de masa corporal de mulleres que viven en Phoenix e que foron testeadas de diabetes. Consideráronse tres poboacións: mulleres que foron diagnosticadas de diabetes, mulleres cuxo test resultou negativo e, finalmente, toda a poboación que foi analizada desta enfermidade. En conclusión, tanto os dous novos selectores bootstrap como o proposto por González-Manteiga et al. (1996) semellan producir estimadores axeitados da función de razón de fallo.

Capítulo 4: Seleción da ventá para o *matching* estatístico e a predición

Neste capítulo, considérase o seguinte plantexamento: sexa unha poboación, á que denominaremos source, da que observamos tanto a variable explicativa, X^0, como a variable resposta, Y^0. Ademais, sexa unha poboación target, da que observamos soamente a variable explicativa, X^1. Asumimos que a función de regresión, m, é común para ambas poboacións. Neste contexto, interésanos obter unha estimación do valor esperado da variable resposta da poboación target, Y^1, que é inobservable. Para iso, precisaremos considerar estimadores da función de regresión, que forman parte do estimador da esperanza da variable Y^1. Así, tratarase de atopar expresións pechadas para o erro cadrático medio (MSE) e o erro cadrático medio promediado (MASE), ademais das súas versións bootstrap suavizadas (ver Cao and González-Manteiga (1993)). A finalidade é definir selectores de ventá locais (co MSE) e globais (co MASE). Parte do traballo deste caítulo está recollido en Barbeito et al. (2020).

Na Sección 4.2 considérase o estimador clásico da regresión, o estimador de Nadaraya-Watson (ver Nadaraya (1964) e Watson (1964)). Como xa foi mencionado, proponse un estimador aproximado do mesmo, de xeito que permita desenvolver unha expresión pechada para o MSE e MASE bootstrap. Posteriormente, proponse unha ventá local e outra global, sendo estas o valor de h que fai mínimo o MSE^* e o MASE^*, respectivamente. A teoría asintótica relativa aos resultados propostos nesta sección preséntase na Sección 4.4.

Por outra banda, na Sección 4.3 considérase o estimador local linear da regresión (ver Fan and Gijbels (1992)). De xeito paralelo a como se procede na Sección 4.2 para o estimador de Nadaraya-Watson, proponse un estimador aproximado baseándose no estimador local linear. Desta maneira, é posible propoñer unha expresión pechada para o MSE e o MASE bootstrap, que serán empregadas para definir un selector de ventá local e outro global.

O bo comportamento empírico destes novos selectores de ventá é analizado mediante un estudo de simulación na Sección 4.5. En concreto, tanto para o caso do estimador de Nadaraya-Watson, como para o estimador local linear, deséñase o mesmo estudo de simulación, no que se compara o selector de ventá global proposto nas Seccións 4.2 e 4.3, h_1, con outro selector de ventá pensado para a estimación, h_0, en lugar de para a predición. Deste xeito, h_0 non ten en conta a información que provén da mostra target e, simplemente, ten en conta a mostra provinte da poboación source. En todos os casos considerados, h_1 é claramente mellor que h_0. Ademais, o selector de ventá baseado nunha aproximación do estimador de Nadaraya-Watson é mellor que o baseado nunha aproximación do estimador local linear por dúas razóns: o tempo de cómputo é notablemente menor e, ademais, non presenta restricións para evitar problemas numéricos na súa implementación (como si é o caso do estimador local linear).

Finalmente, o selector de ventá global baseado nunha aproximación do estimador de Nadaraya-Watson é ilustrado mediante unha aplicación a datos reais. Os datos proceden da Enquisa de Poboación Activa levada a cabo polo INE (Instituto Nacional de Estadística) e recollen os salarios españois no ano 2014. Considéranse aquelas persoas que traballan a tempo completo e as variables de estudo: salario, CNAE, nivel de estudos e experiencia laboral. Neste contexto, considéranse dúas poboacións: homes (poboación source, onde X^0 é o nivel de estudos, CNAE e experiencia laboral dos homes e, Y^0 é o salario dos mesmos) e mulleres (poboación target, onde X^1 o nivel de estudos, CNAE e experiencia laboral das mulleres). Asúmese, ademais, que ambas poboacións teñen función de regresión común. A finalidade desta aplicación a datos reais é, precisamente, empregar a mostra dos homes para, supoñendo que as mulleres están igualmente pagadas, predicir cal é o salario delas. Posteriormente, comparar esta predición co salario real das mesmas durante o ano 2014. Ademais, proponse nesta sección unha versión adaptada do estimador da media de Y^1, sendo Y^1 o salario das mulleres, para poder incluir varias variables (unha cualitativa e as demais cuantitativas) en dita estimación. Os resultados da aplicación a datos reais mostran que, en efecto, existe unha diferencia salarial entre homes e mulleres no mercado laboral. Ademais, a media salarial das mulleres é entre o 10% e o 30% menor do que debería ser se estas estivesen igual pagadas que os homes, como función da experiencia, estudos e sector de actividade. A raíz da análise de datos reais levada a cabo nesta sección, a mencionada desigualdade salarial manifféstase incluso en sectores maioritariamente públicos como é o da Educación.

Capítulo 5: Conclusións e traballo futuro

Nas últimas décadas, propuxéronse numerosos métodos bootstrap para a selección da ventá en estimación non paramétrica de curvas. Un xeito de proceder á hora de aproximar a versión bootstrap dalgún criterio de erro é mediante Monte Carlo. Outra alternativa sería tratar de desenvolver expresións exactas para dito criterio de erro, así como para a súa versión bootstrap. Esta tese enmárcase precisamente neste contexto, definindo, ademais, estimadores aproximados. A finalidade destes estimadores aproximados é obter fórmulas bootstrap exactas dos criterios de erro, de xeito que a aproximación de Monte Carlo xa non é necesaria.

Como traballo futuro, considérase estender a metodoloxía desenvolta nesta tese ao caso doutras funcións como a densidade multivariante, a densidade condicional, a regresión ou a función de distribución. Ademais, considérase a estensión das ideas do Capítulo 3 ao caso de datos censurados ou, tamén, as ideas do Capítulo 4 á estimación da función de distribución da resposta da poboación target (que é inobservable), supoñendo que a función de distribución condicional é común na poboación target e source. Para estes dous últimos casos, ademais de para establecer expresións bootstrap pechadas para algún criterio de erro do estimador da función de regresión e densidade condicional, sería preciso propoñer estimadores aproximados.

Apéndices

No Apéndice A recóllense as demostracións dos resultados presentados ao longo do Capítulo 2, que trata sobre a estimación non paramétrica da función de densidade baixo dependencia dos datos. Ademais, as probas dos resultados recollidos no Capítulo 3 atópanse no Apéndice B. Este capítulo trata sobre a estimación non paramétrica da función de razón de fallo. Finalmente, o Apéndice C recolle as demostracións dos resultados presentados ao longo do Capítulo 4, que trata sobre a selección da ventá para a predición e o *matching* estatístico dende un punto de vista non paramétrico.

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