UNIVERSIDADE DA CORUÑA

PHD. THESIS

New Approaches to Quantification and Management of Model Risk

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El abajo firmante hace constar que es director de la Tesis Doctoral titulada **"New Approaches to Quantification and Management of Model Risk"** desarrollada por Zuzana Krajčovičová, cuya firma también se incluye, dentro del programa de doctorado **"Métodos Matemáticos y Simulación Numérica en Ingeniería y Ciencias Aplicadas"** en el Departamento de Matemáticas (Universidade da Coruña), dando su consentimiento para su presentación y posterior defensa.

The undersigned hereby certifies that he is supervisor of the Thesis entitled **"New Approaches to Quantification and Management of Model Risk"** developed by Zuzana Krajčovičová, whose signature is also included, inside the Ph.D Program **"Mathematical Modelling and Numerical Methods in Applied Sciences and Engineering"** at the Department of Mathematics (University of A Coruña), consenting to its presentation and posterior defense.

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"The secret to creativity is knowing how to hide your sources." (Albert Einstein)

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I find that a great part of what I have gained from the project, was acquired by expecting something and finding something else, much more valuable, on the way.

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List of Abbreviations

ACV .... Asset Value Correlation
AIRB .... Advanced Internal Rating Based
ALCO .... Asset Liability Committee
ALM .... Asset and Liability Management
AMA .... Advanced Measurement Approach
ASRF .... Asymptotic Risk Factor
AT1 .... Additional Tier 1
AVA .... Additional Valuation Adjustment
BCBS .... Basel Committee on Banking Supervision
BIS .... Bank for International Settlement
CAR .... Capital Adequacy Ratio
CCAR .... Comprehensive Capital Analysis and Review
CCF .... Credit Conversion Factor
CCP .... Central Counterparty
CCR .... Counterparty Credit Risk
CDO .... Collateralised Debt Obligations
CDS .... Credit Default Swap
CET1 .... Common Equity Tier 1
CEV .... Constant Elasticity of Variance
CollVA .... Collateral Valuation Adjustment
CR .... Capital Ratio
CRAR .... Risk Based Total Capital
CRM .... Comprehensive Risk Measure
CVA .... Credit Valuation Adjustment
CVaR .... Conditional Value–at–Risk
DDE .... Discounted Expected Exposure
DF .... Discount Factor
DFAST .... Dodd–Frank Act Stress Testing
DR .... Default Rate
DVA .... Debit Valuation Adjustment
EAD .... Exposure at Default
EaR .... Earnings–at–Risk
EBA .... European Banking Authority
EC .... Economic Capital
ECL .... Expected Credit Loss
EE .... Expected Exposure
EEPE .... Effective Expected Positive Exposure
EFE .... Expected Future Exposure
EL .... Expected Loss
ENE .... Expected Negative Exposure
EPE .... Expected Positive Exposure
ES .... Expected Shortfall
EV .... Economic Value
EVE .... Economic Value of Equity
FASB .... Financial Accounting Standards Board
FIRB .... Foundation Internal Rating Based
FVA .... Funding Valuation Adjustment
FX .... Foreign Exchange
HVA .... Hedging Valuation Adjustment
IAS 39 .... International Accounting Standards 39
IASB .... International Accounting Standards Board
ICAAP .... Internal Capital Adequacy Assessment Process
ICL .... Incurred Credit Loss
ICR .... Issuer Credit Risk
IDR .... Incremental Default Risk
IFRS .... International Financial Reporting Standards
IL .... Incurred Loss
IM .... Initial Margin
IMA .... Internal Models Approach
IMM .... Internal Model Method
IR .... Interest Rate
IRB .... Internal Rating Based approach
IRC .... Incremental Risk Charge
IRF .... Interest Rate Futures
IRR .... Interest Rate Risk
IRRBB .... Interest Rate Risk in the Banking Book
IRS .... Interest Rate Swaps
KVA .... Capital Valuation Adjustment
LIP .... Loss Identification Point

LR .... Leverage Ratio

LCR .... Liquidity Coverage Ratio

LDP .... Low Default Portfolio

LGD .... Loss Given Default

LTV .... Loan–to–Value

LVA .... Liquidity Valuation Adjustment

MC .... Monte Carlo

MVA .... Margin Valuation Adjustment

NEV .... Net Economic Value

NII .... Net Interest Income

NIM .... Net Interest Margin

NMD .... Nonmaturity Deposit

NPV .... Net Present Value

NS .... Netting Set

NSFR .... Net Stable Funding Ratio

M .... Maturity

RR .... Recovery Rate

MtM .... Mark–to–Market

MRM .... Model Risk Management

OOC .... Office of the Comptroller of the Currency

OTC .... Over–the–Counter

PD .... Probability of Default

P&L .... Profit and Loss

PFE .... Potential Future Exposure

PIT .... Point In Time

PPNR .... Pre–Provision Net Revenue

RAMP .... Risk–Adjustment Performance Measurement

RAROC .... Risk–Adjusted Return on Capital

RC .... Replacement Cost

RNIV .... Risk Not In Value–at–Risk

RoRWA .... Return on Risk Weighted Assets

RTS .... Regulatory Technical Standards

RWA .... Risk–Weighted Assets

SA .... Standardised Approach

SCAP .... Supervisory Capital Assessment Program

SIFI .... Systematically Important Financial Institutions

SMA .... Standard Measurement Approach

SMM .... Standardized Measurement Approach

SREP .... Supervisory Review and Evaluation Process

sVaR .... Stressed Value–at–Risk

TTC .... Through the Cycle

UL .... Unexpected Loss

VaR .... Value–at–Risk

VC .... Value Creation

WWR .... Wrong–Way Risk

XVA .... X–Valuation Adjustments

Abstract

The present contribution focuses on the problem of an objective assessment of model risk in practice. In spite of the awareness of model risk significance and the regulatory requirements for its proper management, there are no globally defined industry or market standards on its exact definition and quantification. The main objective of this dissertation is to address this issue by designing a general framework for the quantification of model risk, taking into account both internal policies and regulatory issues, applicable to most modelling techniques currently under usage in financial institutions.

We address the quantification of model risk through differential geometry and information theory, by the calculation of the norm of an appropriate function defined on a Riemannian manifold endowed with a proper Riemannian metric. Pulling back the model manifold structure, we further introduce a consistent Riemannian structure on the sample space that allows us to investigate and quantify model risk by working merely with the samples. This offers primarily practical advantages such as a computational alternative, easier application of business intuition, and easier way to assign the uncertainty in the data. Additionally, one gains the insight on model risk from both the data and the model perspective.

The proposed framework has the following properties: provides a systematic and repeatable procedure to identify and assess model risk, allows for the quantification of risk materiality, incorporates most of the relevant aspects of model risk management, such as usage, model performance, mathematical foundations, data and model calibration, and facilitate establishing a control environment around the use of models. The theoretical analysis is completed with practical applications to a credit risk model used for capital calculation, currently employed in the financial industry.

As another application of the proposed framework, we emphasize the importance of the geometry of the underlying space in financial models and apply curvature not only to control and reduce the inherent model risk but also to improve the overall performance of a model. These ideas are exemplified through the P&L explanation of digital options with the Black–Scholes model and demonstrate the improvement by comparing results under Euclidean and non–Euclidean geometries.

The results of this thesis are addressed to both practitioners and scientists. With regard to the academic society, this thesis should contribute to the scientific analysis of the complex problem of model risk and introduce differential geometry and information theory into financial modelling. On the other hand, the proposed approach gives direct benefits in practice, for the management and the use of models inside financial institutions: The confidence in the model can be quantified, model limits, weaknesses and gaps can be assessed quantitatively and so managed constructively and proportionally. The model risk stemming from usage of a model can be communicated

transparently and consistently to users, managers and regulators, communicating model credibility, setting controls systematically, and focusing on model management. As such, a strong model risk management with objective assessment of model risk can act as a competitive advantage for an institution.

Resumen

La presente contribución se centra en el problema de una evaluación objetiva del riesgo modelo en la práctica. A pesar de ser conscientes de la importancia del riesgo del modelo y los requisitos regulatorios para su gestión adecuada, no existen normas del sector o de mercado establecidas globalmente sobre su definición y cuantificación exactas. El objetivo principal de esta tesis es abordar esta cuestión mediante el diseño de un marco general para la cuantificación del riesgo modelo, teniendo en cuenta tanto las políticas internas como las cuestiones regulatorias, que sea aplicable a la mayoría de las técnicas de modelado actualmente en uso en las instituciones financieras.

Abordamos la cuantificación del riesgo del modelo a través de la geometría diferencial y la teoría de la información, mediante el cálculo de la norma de una función apropiada definida en una variedad riemanniana dotada de una métrica adecuada. Además, haciendo un *pull back* de la variedad que contiene los modelos, introducimos una estructura riemanniana consistente en el espacio muestral (espacio de los datos) que nos permite investigar y cuantificar el riesgo del modelo trabajando meramente con los datos. Esto ofrece ventajas principalmente prácticas, como una alternativa computacional, una aplicación más fácil de la intuición empresarial y una manera más fácil de asignar la incertidumbre en los datos. Además, uno obtiene la información sobre el riesgo del modelo tanto desde el punto de vista de los datos como desde el del modelo.

El marco propuesto tiene las siguientes propiedades: proporciona un procedimiento sistemático y repetible para identificar y evaluar el riesgo del modelo, permite la cuantificación de la materialidad del riesgo, incorpora la mayoría de los aspectos relevantes de la gestión del riesgo del modelo: como uso, rendimiento del modelo, fundamentos matemáticos, calibración de datos y modelos, y facilitar el establecimiento de un entorno de control en torno al uso de modelos. El análisis teórico se completa con aplicaciones prácticas para un modelo de riesgo de crédito utilizado para el cálculo de capital, actualmente empleado en la industria financiera.

Como otra aplicación del marco propuesto, resaltamos la importancia de la geometría del espacio subyacente en los modelos financieros y aplicamos la curvatura no solo para controlar y reducir el riesgo del modelo inherente, sino también para mejorar el rendimiento general de un modelo. Estas ideas se ilustran a través de la aplicación al P &L de las opciones digitales con el modelo Black–Scholes y demuestran la mejora al comparar los resultados bajo geometrías euclídeas y no euclídeas.

Los resultados de esta tesis pueden ser de utilidad tanto a profesionales como a investigadores. Con respecto al ámbito académico, esta tesis pretende contribuir al análisis científico del complejo problema del riesgo del modelo e introducir la geometría diferencial y la teoría de la información en la modelización financiera. Por otro lado, el enfoque propuesto pretende aportar beneficios directos en la práctica, para la gestión y el uso de modelos dentro de las instituciones financieras: la confianza en el modelo puede cuantificarse, los límites del modelo, las debilidades

y las brechas pueden evaluarse cuantitativamente y así manejarse de manera constructiva y proporcionalmente. El riesgo del modelo derivado del uso de un modelo se puede comunicar de forma transparente y consistente a los usuarios, gerentes y reguladores, comunicando la credibilidad del modelo, estableciendo controles sistemáticamente y centrándose en la gestión del modelo. Como tal, una gestión sólida del riesgo del modelo con una evaluación objetiva del mismo puede representar una ventaja competitiva para una institución financiera.

RESUMO

A presente contribución céntrase no problema dunha avaliación obxectiva do risco modelo na práctica. A pesar de ser conscientes da importancia do risco do modelo e os requisitos regulatorios para a súa xestión axeitada, non existen normas do sector ou de mercado establecidas globalmente sobre a súa definición e cuantificación exactas. O obxectivo principal desta tese é abordar esta cuestión mediante o deseõ dun marco xeral para a cuantificación do risco de modelo, tendo en conta tanto as políticas internas como as cuestións regulatorias, que sexa aplicable á maioría das técnicas de modelado actualmente en uso nas institucións financeiras.

Abordamos a cuantificación do risco do modelo a través da xeometría diferencial e a teoría da información, mediante o cálculo da norma dunha función apropiada definida nunha variedade riemanniana dotada dunha métrica adecuada. Ademais, facendo un *pull back* da variedade que contén os modelos, introducimos unha estrutura riemanniana consistente no espazo muestral (espazo dos datos) que nos permite investigar e cuantificar o risco do modelo traballando meramente cos datos. Isto ofrece vantaxes principalmente prácticas, como unha alternativa computacional, unha aplicación máis sinxela da intuición empresarial e unha maneira máis fácil de asignar a incerteza nos datos. Ademais, un obtén a información sobre o risco do modelo tanto desde o punto de vista dos datos como desde o do modelo.

O marco proposto ten as seguintes propiedades: proporciona un procedemento sistemático e repetible para identificar e avaliar o risco do modelo, permite a cuantificación da materialidad do risco, incorpora a maioría dos aspectos relevantes da xestión do risco do modelo: como uso, rendemento do modelo, fundamentos matemáticos, calibración de datos e modelos, e facilitar o establecemento dunha contorna de control en torno ao uso de modelos. A análise teórica complétase con aplicacións prácticas para un modelo de risco de crédito utilizado para o cálculo de capital, actualmente empregado na industria financeira.

Como outra aplicación do marco proposto, resaltamos a importancia da xeometría do espazo subxacente nos modelos financeiros e aplicamos a curvatura non só para controlar e reducir o risco do modelo inherente, senón tamén para mellorar o rendemento xeral dun modelo. Estas ideas ilústranse a través da aplicación ao P&L das opcións dixitais co modelo Black–Scholes e demostran a mellora ao comparar os resultados baixo xeometrías euclídeas e non euclídeas.

Os resultados desta tese poden ser de utilidade tanto a profesionais como a investigadores. Con respecto ao ámbito académico, esta tese pretende contribuír á análise científica do complexo problema do risco do modelo e introducir a xeometría diferencial e a teoría da información na modelización financeira. Doutra banda, o enfoque proposto pretende achegar beneficios directos na práctica, para a xestión e o uso de modelos dentro das institucións financeiras: a confianza no modelo pode cuantificarse, os límites do modelo, as debilidades e as brechas

poden avaliarse cuantitativamente e así manexarse de maneira construtiva e proporcionalmente. O risco do modelo derivado do uso dun modelo pódese comunicar de forma transparente e consistente aos usuarios, xerentes e reguladores, comunicando a credibilidade do modelo, establecendo controis sistematicamente e centrándose na xestión do modelo. Como tal, unha xestión sólida do risco do modelo cunha avaliación obxectiva do mesmo pode representar unha vantaxe competitiva para unha institución financeira.

INTRODUCTION:

MODEL RISK OVERVIEW

"It is better to be vaguely right than exactly wrong." (Carveth Read)

The current practice of finance heavily relies on a high level of sophistication, quantitative analysis and models to assist with decision making, thereby improving efficiency, enabling the ability to better understand, manage and oversee various risks as well as the capability to synthesize complex issues and centralize modelling infrastructure. As a consequence of that very sophistication and complexity, risks have emerged that are unpredictable, global and difficult to hedge and measure. Model risk is an eminent example of such risks since it arises as the result of technological progress, innovations or attempt at managing other risks in a more effective way; it is difficult to manage and account for; it is hard to measure and often difficult to comprehend.

According to the Board of Governors of the Federal Reserve System (Fed) and the Office of the Comptroller of the Currency (OCC), [32], model risk is defined as

> *"[...] the potential for adverse consequences from decisions based on incorrect or misused model outputs and reports. "*

The definition explicitly points to a model intrinsic accuracy on one side and to its various uses on the other. The latter aspect implies not only that no model can be judged out of context, as its performance may completely differ depending on what asset or portfolio it is applied to, but besides, its particular usage, be it capital, pricing or hedging, will determine the ultimate financial impact of any inaccuracy or error. Model error encompasses phenomena such as simplifications of or approximations to reality, inadequate data, incorrect or missing assumptions, incorrect design process, and measurement or estimation error, among many other. On the other hand, model misuse includes applying models outside the use for which they were intended.

The European Banking Authority (EBA) [57] differentiates between two main types of model risk. First, the risk related to the underestimation of own funds requirements by regulatory approved models (e.g., Internal rating–based (IRB) models), and second, the risk of losses associated with the development, implementation or improper use of any other model employed for decision–making (e.g., evaluation of financial instruments, monitoring risk limits). Model risk, in a broader business and regulatory context, incorporates the exposure from making poor decisions based on inaccurate model analyses or forecasts and, in either context, can arise from any model in active use [85]. Decisions based on incorrect or misleading model outputs may lead to financial losses to the bank and its customers, inferior business decisions or ultimately reputation damage. Model risk may be particularly high, especially under stressed conditions or combined with other interrelated trigger events. Future changes and evolution of model risk are to be expected in the mid term both by regulatory demands and institutions best practices.

Any source of model risk of an individual model may in addition be propagated, cumulated or amplified as model outputs are used both directly and as inputs into other models. In other words, a model may provide input to, or use the output from, other models. For example, the interest rate model using yield curve models often serves as a sub–model in a larger module, used to simulate terms structures in order to price caps/floors, swaptions or the hedge cost; the use of segmentation while assessing operational loss;[1] projection assumptions in stochastic modelling are typically created by means of scenario generators which follow a particular probability structure or distribution; credit rating models are used as inputs to credit pricing models; the transformation of the data including cleaning, digitizing, sampling, summarizing, normalizing, smoothing, reshaping of by imposing any assumptions in order to transform them into a usable form, e.g. for performing next steps of model development; process of stress testing creates sequential model dependencies; more complicated models such as derivative valuation models, ALM models, concentration risk models or economic capital models[2] are usually composed of several sub–models. In such a chain of models, understanding the interconnections (upstream and downstream dependencies to other models) between different models and clear identification of potential sources of model risk are crucial and an important mitigant of model risk.

Financial institutions use a wide range of models designed to meet regulatory requirements and to achieve business needs that are subject to governance and model risk management (see Figure 1). Quantitative analysis and models are central to the operation within and across financial institutions; they are employed for a variety of purposes including exposure calculations, trading (e.g. algorithmic trading), instrument and position valuation, risk measurement and management, calculation of regulatory and economic capital adequacy for all exposures, subject to different risk types (e.g., market, credit, operational, Asset and Liability Management (ALM)) through their individual components, the installation of compliance measures, the application of stress and scenario testing and macroeconomic forecasting. They are integral to financial reporting (e.g., to value asset and liabilities, in particular derivative products and loan provisions), to prudential requirements determination, or to decision making that directly affects customer outcomes, such as customer acquisition decisions, establishing customer loyalty and engagement actions in all stages of the relationship with the institution and at any time in the customer life cycle, or account fraud or compliance screening. Furthermore, more models are being developed in order to comply with

[1] Loss severity and frequency distributions are typically undertaken on a more granular basis than at the total level and so it is common to segment losses into as many homogeneous categories as is practical, given the constraints on data availability, data quality and the resources available to undertake the assessment process. Segmentation approaches can range from expert judgement, statistical methods and data mining techniques, including logistic and linear regression, decision trees to advanced cluster modelling, factor segmentation, machine learning or neural networks. An appropriate segmentation is fundamental to achieving a thorough identification of risks.

[2] For an illustration, modelling a bank portfolio credit risk requires a specification of credit loss. The expected loss (EL) is defined as

$$EL = PD \cdot LGD \cdot EAD,$$

where PD refers to the Probability of Default, LGD is the expected Loss Given Default and EAD denotes the expected Exposure Amount at Default (see Section 2.1). Each of these metrics is quantified using a variety of stochastic, market comparative and statistical approaches, and in some instances by deriving volatility from historical or implied means. For instance, modelling LGD requires consideration of the seniority of the assets, the industry, the issuer, historical recoveries and trends, and an evaluation of the current climate, as well as the assets and liabilities of the issuer; estimation of EAD involves the modelling of the investment value that in turn depends on the evolution of the market factors upon which the value of the investment depends.

parallel regulations, such as IFRS 9, FRTB, IRB models and TRIM, PRA and ECB Regular Stress Tests[3], or with legal, compliance, or economic research, as a key competitive advantage in leveraging opportunities in the market, to name just few.

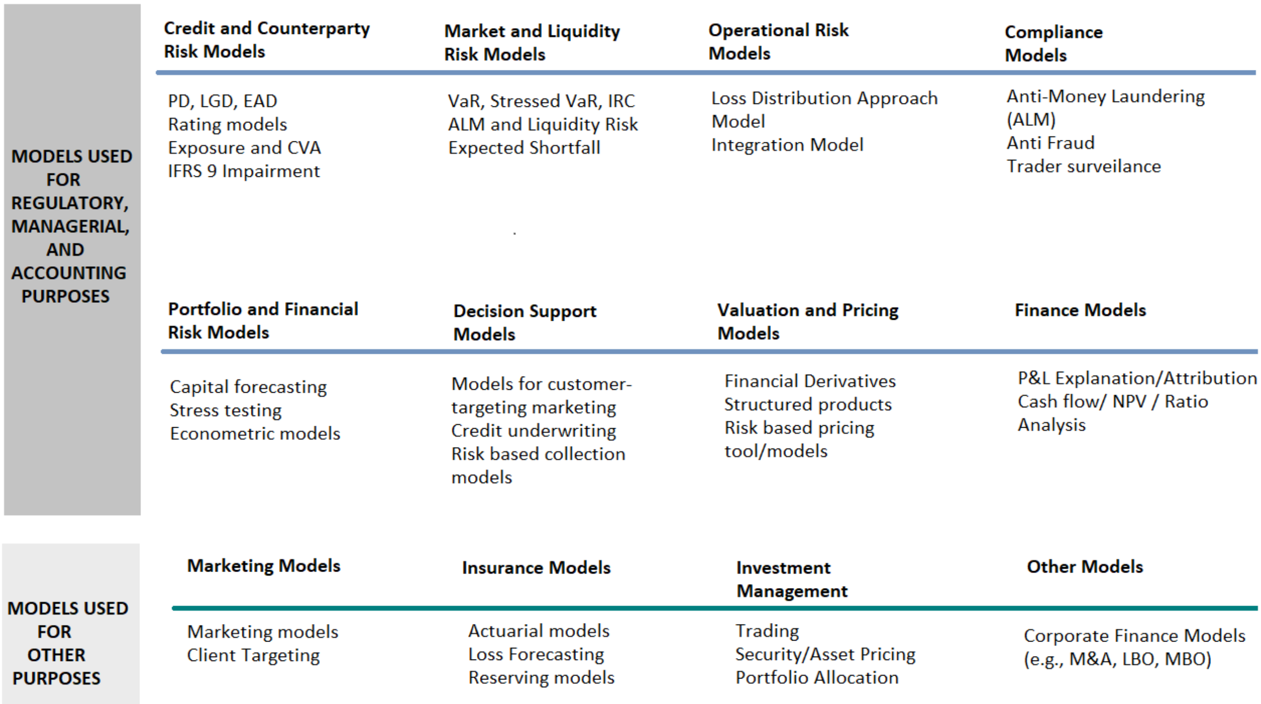

Figure 1: Examples of models subject to governance and model risk management of a large global bank.

Models range from simple formulae to complex models that require simulations and optimization routines, some are internally developed and some may be sourced through consultants and vendors. They may be employed in spreadsheets or prototyped first in 4^{th} generation languages such Python, MATLAB, or R and then translated to C/C# for performance. Models may run standalone on desktops or deployed onto serves.

In general, financial models are simplifying mappings of reality to serve a specific purpose aimed at applying mathematical, financial and economic theories to available data. They deliberately focus on specific aspects of the reality and degrade or ignore the rest, certainly, within the level of precision required by the specific usage and application. Most of the models are quantitative approaches, including the complex manipulations of expert judgements, or systems that apply mathematical techniques and assumptions to process input information— often containing distributional information— into quantitative estimates that drive decisions. Input data may be of economic, financial or statistical nature, partially or entirely qualitative or based on expert judgement, but in all cases, the model outputs are quantitative and subject to interpretation.

[3]The International Financial Reporting Standards (IFRS 9) demands banks to use a new set of credit risk models that doubles the number of risk parameters models to manage; the Fundamental review of the trading book (FRTB) includes updates to both the advanced and standardized models as well as stricter disclosure requirements and validation standards; EBA Guidelines on PD, LGD estimation and treatment of defaulted asset as well as new default definition, conservatism margins, NPL assessment, rating process and the new stress testing methodology and principles defined by the Prudential Regulation Authority (PRA) and EBA.

Most of the models currently used within financial industry and virtually all of their basic underlying assumptions can find their roots in measure theory. For instance, econometric models specify a probability measure in order to capture the historical evolution of market price, pricing models use a risk–neutral probability measure to specify a pricing rule, or risk models aim at estimating the probability distribution of future losses. The risk of using such models comes, among many others, from the potential failure in devising a realistic and plausible representation of the factors influencing the value of security, the inaccuracy in the estimation of the relevant probability distribution that translates into smaller or larger losses depending on the way such distribution is used and the subsequent incorrect decisions this may entail. For example, a credit or market risk measure may be used for computing economic and regulatory capital, pricing, provisioning, assessing the eligibility of a transaction or a counterpart, for establishing a portfolio, counterpart or other limits, among others. Subsequent imprecision in each of these estimates, in turn, may have financial consequences and so, financial models require a certain fundamental prudence in their interpretation and usage.

The main underlying mathematical theories include closed–form formulas; regressors of any kind and their combinations: transformations (e.g., logit), extensions (e.g., multi–linear, polynomial) as well as linear dynamic methods; optimizations, looking for local or global maxima and minima; interpolations such as splines, linear or SABR; extrapolation; distributions adjustments including the Loss Distribution Approach, extreme value theory, methodology of copulá; stochastic analysis encompassing unifactor and multifactor, with mean reversion, various volatilities ranging from constant, local to stochastic; Monte Carlo method that is employed for distribution generation and numerical integration, or convolutions; error–correction models used for variables forecasting; and Partial Differential Equations with Dirichlet and Von Neuman boundary conditions, simple integration algorithms such as explicit or implicit Euler.

Luis Jean–Baptiste Alphonse Bachelier, in his dissertation on the modelling of financial markets, states that [42]

"[. . .] the influences which determine the movements of the Stock Exchange are innumerable. Events past, present or even anticipated, often showing no apparent connection with its fluctuations, yet have repercussions on its course. Beside fluctuations from, as it were, natural causes, artificial causes are also involved. The Stock Exchange acts upon itself and its current movement is a function not only of earlier fluctuations, but also of the present market position. The determination of these fluctuations is subject to an infinite number of factors: it is therefore impossible to expect a mathematically exact forecast. "

Though not certain or accurate, financial models help us to narrow down the choices we have in terms of decision–making, as a model may fall short and still provide some valuable information to certain degree.

To mitigate risks stemming from the usage of financial models, regulators are expecting institutions to have a proper and well–conceived model risk management (MRM) framework in place that features model development and validation criteria, promotes careful model use, sets criteria to assess model performance, and defines policy governance and applicable standards of documentation. A clearly defined MRM framework with a strong management insight on monitoring models and their risks allows institutions to strengthen their decision making process, improves their overall profitability, add value to the enterprise as well as reduce risk, improves earnings through cost reduction, loss avoidance, and capital improvement.

Both financial institutions and regulators propose a variety of approaches ranging from model risk mitigation via model validation to the establishment of comprehensive frameworks for active MRM that sets out the guidelines for the entire design, development, implementation, validation and inventory and use process (see e.g., Bulletin 2011–12 or BOG–FRB SR 11–7 [32]) while requiring to manage model risk with the same severity as any other type of risks, comprising the well–known steps of identification, assessment, measurement, mitigation, monitoring, control and reporting.

Developing a comprehensive approach to MRM is accompanied by many challenges. This is partially due to the focus of regulators and academic literature mainly on the analytics and accuracy, with very little emphasis on governance, organizational structure or human behaviour and with almost no stress on the usage of a model. Next, the main focus has been on the computational component of model development and deployment, while paying little to no attention on data analytics or controls.

Naturally, reviewing models for accuracy is a necessary, but not always a sufficient, condition for an effective MRM. Models are as good as the data and assumptions that feed them. If the decision is being made based on reasonably accurate model that is nevertheless not suited for the purpose, the decision is bound to remain questionable and the model should be rejected as inappropriate.[4] Data quality, assumptions inventories and the mapping between the data, assumptions and outputs, and the particular usage are a crucial component of MRM.

Data ~ Inputs	**Model (foundations)** ~ Computations	**Usage** ~ Outputs
• Availability	• Data selection	• Model use
• Quality	• Methodology	• Controls
• IT infrastructure	• Drivers and Segmentation	• Governance
	• Computational Accuracy	• Provisions
	• Documentation and Controls	
	• Model Performance	
	• Sensitivity and Scenario Analysis	

Figure 2: The main areas and subareas of an effective MRM.

Model risk is or should be related to the quantitative nature of the particular institution process and the way this is actually accomplished. A comprehensive MRM framework should therefore comprise data (inputs), model foundation (computations) and usage (outputs). Figure 2 highlights the main areas of an effective MRM.

To render objective judgement on model suitability for use requires not only qualitative assessment, but also a proper measurement of model risk. The quantification is a significant component for comprehensive MRM required for effective and objective communication of model weaknesses and limitations to decision makers and users and to assess model risk in the context of the overall position of the organization. Furthermore, it enables to impartially determine the credibility of models in an ongoing risk management, to properly prioritize the issues, to effectively allocate resources and to establish adequate cushions against losses.

[4]Validating models that may be technically or theoretically correct but have been misused is, in general, challenging. Organizational or human behavioural issues related to the decision framework based on favourite models that are outdated or irrelevant to the decision being made, may expose an institution to financial, regulatory, legal or reputational risk.

In spite of the awareness of model risk importance, to the best of our knowledge, there are no globally defined industry and market standards on its exact definition and quantification.[5] The regulatory guidance on MRM [32, 57], while covering several dimensions of model risk management, fall short in providing a systematic way of formally quantifying it and in introducing a measure of model risk that can be computed and reported by model and by transaction thereby becoming an internal part of the overall risk measure for the institution portfolio.

The complexity that surrounds the quantitative treatment of model risk measurement and management comes simply from the high diversity of models, the wide range of techniques and methodologies employed, and the different uses of models, among others. Model risk may arise in any stage of model development and deployment process, i.e. it originates at model conception, prototyping, testing, documentation, review and validation, production, ongoing monitoring and model enhancements (or retirement if it has met its purpose). Some model outputs drive decisions; other model outputs provide one source of management information, some outputs are further used as an inputs in other models. Additionally, the model outputs may be completely overridden by expert judgement, not to mention that in order to quantify model risk requires some kind of modelling itself, which is again prone to model risk, i.e. replacing one kind of risk with another, possibly even more elusive.

The aim of the dissertation is to address these particular issues, focusing on the development of a new approach for quantifying model risk within the framework of differential geometry, see [89], and information theory, see [6]. The main objective is twofold: introduce a sufficiently general and sound mathematical framework to cover the main areas of model risk management and illustrate how a practitioner may identify the relevant abstract concepts and put them to work.

In the present contribution we introduce a measure of model risk on a statistical manifold where models are represented by a probability distribution function. Differences between models are determined by the geodesic distance under a suitable Riemannian metric. This metric allows us to utilize the intrinsic structure of the manifold of densities and to respect the geometry of the space we are working on, i.e. it accounts for the non–linearities of the underlying space. The quantification of model risk is then determined as the calculation of the norm of some appropriate function defined over a weighted Riemannian manifold endowed with the Riemannian metric (see Chapter 6).

By pulling back the manifold structure we further define a consistent Riemannian structure on the sample space that allows us to investigate and quantify model risk and to examine data uncertainty by working merely with the samples (see Chapter 8). This provides a novel way to examine the data for insight through the data intrinsic distance induced by the pullback metric, and to objectively assess the propagations of the perturbations. Furthermore, it has substantial practical relevance as it offers a computational alternative, easier application of business intuition, and an easier way to assign the uncertainty in the data. In addition to the quantification of model risk, our proposed framework is further suitable for sensitivity analysis, stress testing (i.e. for regulatory and planification exercises), or for checking the validity of the approximations done through the modelling process, and for testing model stability and validity.

[5] An exception to this are rather specialized situations, such as the valuation of certain instruments. In these cases, there may even be requirement for Asset Valuation Adjustment (AVA) which may result in a large capital requirement or in the possible use of a capital buffer for model risk as a mitigating factor, without its calculation being specified [73].

Finally, we emphasize the importance of differential geometry within the financial modelling and usage process, and highlight the importance of curvature in the presence of model risk (see Chapter 9). We propose that the measure of curvature may be used in two contexts. First, it can be seen as a mean to control and reduce the inherent model risk. Second, even in the case of a flat underlying model variety, different geometries may better fit the particular usage of a model and so increase the overall performance.

The theoretical framework is completed by the applications to some of the models currently used by the financial institution, namely we quantify model risk for the capital allocation of a credit risk portfolio arising from the selected segmentation (see Chapter 7) and we apply differential geometry into the daily P&L analysis for a digital options under the Black–Scholes model and demonstrate the improvement by comparing results under Euclidean and non–Euclidean geometries (see Chapter 9).

OVERVIEW AND CONTRIBUTIONS OF THE DISSERTATION

Apart from this introductory chapter, the dissertation is essentially divided into two main parts, i.e. a general introduction to the topic and the development of a framework for the quantification of model risk and its application to some of the models currently under use— framed by a short introduction and the final conclusions.

The main intention of this dissertation is the objective assessment of model risk. In order to understand the relevance of this issue, one needs to comprehend the governance and organizational structure of financial institutions, the high diversity of models employed, the wide range of techniques, the different uses of models within the financial industry, limitations and weaknesses of these models, among others. Part I is therefore dedicated to a brief introduction to the Risk and Capital Management of financial institutions with the aim to present and explain how leading banks use quantitative models in their daily operations for capital planning and budgeting, measuring risks, pricing derivatives or loans, provisioning and stress testing, and to emphasize their underlying nature based on measure theory. Furthermore, we point out and discuss some of the potential sources of model risk associated with the use of these models and look at the many things that can go wrong (from original conception to final use).

Part I consists of five chapters, particularly, Chapter 1 describes models used for calculating economic and regulatory capital. In Chapter 2 we set out management and control models used for credit and market risks such as credit risk (see Section 2.1), market (trading) risk (see Section 2.2), risks associated with financial derivatives (see Section 2.3) and structural and balance sheet risks (see Section 2.4). Chapter 3 is dedicated to non–financial risks and is divided into three Sections, Section 3.1 covers operational risk, Section 3.2 represents a brief introduction to other non–financial risks, such as IT and cyber risk, legal risk, compliance risk, risk emerging from macro forecasts and third–party risk. Chapter 4 covers models used for stress testing (see Section 4.1) and assessment of reserves (see Section 4.2). Finally, in the last Section 4.3 we identify the potential sources of model risk within each sector and point out the inter–relationships between different types of risks.

Part II is dedicated to Model Risk with the objective first to introduce the complexity of the issue and the challenges with its management, and to design a general framework for the quantification of model risk that accounts for different sources of model risk during the entire lifecycle of a model and is applicable to most of the financial models currently under usage. This part comprises 6 chapters accompanied by an introduction to model risk and the final conclusions.

Chapter 5 highlights the potential sources of model risk that may occur during the entire life–cycle of a model and discusses the possibilities of how to mitigate, control or reduce some of these sources. As the complete elimination of model risk is not possible, implementing an approach that combines rigorous risk management structure with prudent detailed quantification is of high importance. Therefore, in Chapter 6 a general approach for the quantification of model risk within the framework of differential geometry and information theory is proposed. We introduce a measure of model risk on a statistical manifold where models are represented by a probability distribution function and that is capable of coping with relevant aspects of model risk management and has the potential to assess many of the mathematical approaches currently used in financial institutions: credit risk, market risk, derivatives pricing and hedging, operational risk, capital allocation, provisioning, stress–testing or XVA (valuation adjustments), among others. Chapter 7 is dedicated to an empirical example of the model risk calculation in the case of a credit risk model used for capital assessment with detailed illustration how a practitioner may identify the relevant abstract concepts and put them to work.

The main objective of Chapter 8 is to deepen in the influence of data uncertainty in model risk by relating the data uncertainties with the model structure. Pulling back the model manifold metric introduces a consistent Riemannian structure on the sample space that allows us to quantify model risk, working with samples. In practice it offers a computational alternative, eases application of business intuition and assignment of data uncertainty, as well as insight on model risk from the data and the model perspective. Chapter 9 discusses the possible deployment of the principles discussed before to improve on the usage of a model and to reduce the inherent model risk with the application to the P&L explanation of the digital options.

Ultimately, the financial conclusions in Chapter 10 summarize the main results and emphasize the advantages and benefits of the proposed framework, and provide directions for future work.

Part I

Capital and Risk Management Models

Introduction

"There are no accidents; there is only some purpose that we haven't yet understood." (Deepak Chopra)

The objective of the present Part I is to provide a brief overview of the Risk and Capital Management of financial institutions with the aim to present and explain how leading banks use quantitative models in their daily operations for capital planning and budgeting, measuring and managing risks, stress testing, pricing derivatives or loans and to emphasize their underlying nature based on measure theory. Furthermore, we point out and discuss some potential sources of model risk associated with the use of these models arising in any stage of the model development and deployment process, i.e. from original conception to final use. Although the focus of this Part I is on models used for risk and capital management, many of the issues discussed here apply to models used for other business purposes as well.

Managing risk and capital are at the core of financial institutions activities and fundamental to their long–term profitability and stability. Risk and capital management is not only required by regulators,[6] e.g. see [9], [49], but having an effective risk and capital management in place is fundamental to the business activities of leading financial institutions as it represents an integral part of the long–term strategic and business planning process.

Risk is a measure of adverse deviation from the expectation, expressed at a level of uncertainty (probability), while capital is the value of the net assets of the owners of an institution with the primary purpose, from a bank perspective, to absorb risk. In such a way, taking risk is closely related to business activities, development, and customer needs,[7] and it directly depends upon the ability of an institution to evaluate, manage and price risks while maintaining adequate capital and liquidity to meet unforeseen events.

Risk and capital are managed via a framework of principles, organizational structures, measurement and monitoring processes that are closely aligned with the activities of the divisions and business units. Capital is managed using regulatory and economic capital metrics, at both business line and legal entity level. Risks are controlled at individual exposure levels as well as in aggregate within and across all business lines, legal entities and risk types. By understanding what risk means, the spectrum of different attitudes towards risk, and a basic process framework for managing risk, institutions can not only mitigate unwanted risks but turn challenges to opportunities.

Although financial institutions place varying emphasis on different aspects of risk governance, there is a common theme to employ the role of the *three lines of defence governance model*, as developed by the Institute

[6] The ICAAP within the Pillar 2 of the Basel framework [9] requires credit institutions to have in place an internal risk and capital management which is adapted to institution specific risk profile; to establish procedures to calculate and safeguard their risk–bearing capacity and to manage their risks.

[7] Risk has traditionally been viewed as something to be minimized or avoided, with significant effort spent on protecting value. However, risk is also a creator of value and, approached in the right way, can play a unique role in driving business performance.

of Internal Auditors in 2013 [61]. The objective of this model is to provide a framework for managing enterprise risk at the strategic, tactical and operational levels and to set out how risks can be manage effectively. The model distinguishes between functions that own and manage risks, functions overseeing risks and functions providing independent assurance. It is used to manage uncertainty and mitigate downside risk, and it enhances understanding of risk management and control. The responsibilities of each of the three lines are:

1. **The first line — front line management:** directly responsible for identifying and managing risks, i.e. risk owners, developers and users. Their responsibilities are defining, developing, implementing, and operating the model, monitoring its performance and managing changes.

2. **The second line — risk management and compliance functions:** considers the management of implementation, which involves oversight and effective execution of the risk management framework by the various senior risk management and compliance committees. This line is responsible for establishing policies and standards, performing model risk assessment, managing and inventory of models, independent monitoring of model performance, model usage, and adherence to management policies, reporting to board/senior management.

3. **The third line — internal audit and other independent assurance providers:** this line of defence has an assurance function that remains independent and objective, and so provides an independent assessment of the adherence of the first and second line of defence to risk policies. They operate in coordination with one another in order to maximize their efficiency and strengthen their effectiveness.

All three lines of defence should be independent of each another and accountable for maintaining structures that ensure adherence to the design principles at all levels. In addition, the Financial Stability Institute (FSI) [13] suggests to consider also a fourth line representing **the external audit and supervisors**.

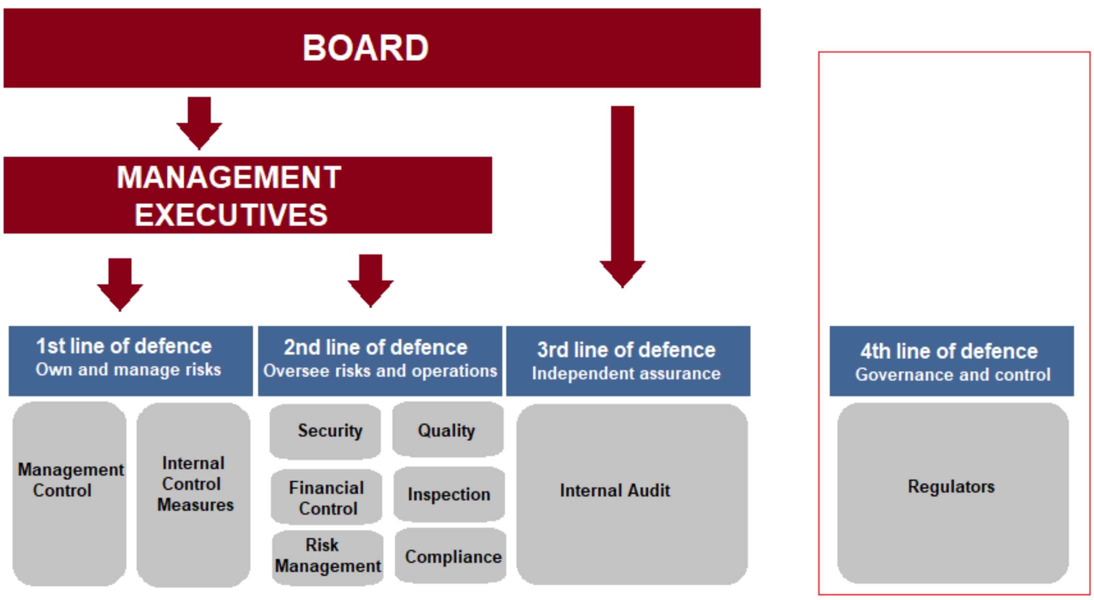

Figure 3: The structure of the *four lines of defence model* as proposed by FSI.

In most institutions, over and above the defence lines, the board committees and the executive risk committees at both corporate level and in the units are responsible for adequate management and control of risks from the highest level of the organization (see Figure 3). The main objective of all processes is to provide adequate transparency and understanding of existing but also emerging risk issues, and to ensure a holistic cross–risk perspective.

The diversity of business operations require an effective risk management that is an on–going iterative process based, in general, on four key processes:

- **Risk Identification**: considers both internal and external risks, factors, and events that affect or could adversely affect the achievement of the institution objectives and their potential impact on capital. Risk identification typically involves three types of activities: defining and categorizing risks; conducting internal qualitative surveys on the frequency and severity of each risk; and scanning the external environment for emerging risks.
- **Risk Measurement**: includes quantification of individual risks and their combinations. In general, risk is measured in terms of probability of occurrences and severity (consequences), i.e. the evaluation of a risk provides a defensible information on the frequency and the impact. Risk assessment is one of the most difficult steps as it is not always possible to quantify all material risks.
- **Risk Control**: refers to a system of independent, on–going assessment of the risk management processes. The results should then be communicated to the board of directors and senior management.
- **Risk Monitoring and Reporting**: represents the system for monitoring and reporting the condition of individual risks, including determining the adequate provisions, reserves and capital.

Thus, risk management system comprises risk management strategy and policies, as well as procedures for risk identification, measurement and monitoring; appropriate internal organization; effective and efficient risk management process covering all risks an institution is exposed to or may potentially be exposed to in its operations; adequate internal controls system; appropriated information system, and adequate process of internal capital adequacy assessment.

Identifying and managing all risks relevant to the organization, based on a strong risk taxonomy with different levels of aggregation that comprehensively covers financial and non–financial risks is a prerequisite for the implementation of a sound risk management framework. Risk identification processes have traditionally centred on the key risk types of credit, market, business, operational, liquidity and reputational risk, however, in the course of banks activities new risks emerged, such as concentration risk, investment risks, IT risks including cyber risk, model risk stemming from the usage of models, risks related to the country of origin of an entity to which a bank is exposed (country risk), or risk of money laundering and terrorist financing. An example of a high–level of aggregation into risk classes is provided in Figure 4.[8]

A robust risk taxonomy allows for a reduction in complexity, provides a bank–wide standardized language, allows responsibilities to be assigned across the three lines of defence and is necessary for implementing a methodology for monitoring and measurement. Besides, a governed risk identification process can help to inform and

[8]Note that there is no single or definitive way to categorize risk. The process should be tailored to the size of the institution and the complexity of the environment it faces. The main point, however, is to ensure that the chosen taxonomy covers each type of risk an institution faces and is understood by all those using the outputs.

FINANCIAL RISKS	NON-FINANCIAL RISKS	EXTERNAL MARKET RISKS
Credit Risk	Operational Risk	Strategic Risk
Market Risk	Compliance Risk	Systematic Risk
Interest rate Risk in Banking Book	Conduct Risk	
Liquidity Risk	Model Risk	
	Cyber Risk	
	Third-party Risk	
	IT Risk	
	Reputational Risk	
	Legal Risk	
	Pension and Insurance Risks	
	DTA Risk	
	Goodwill Risk	

Figure 4: Risk Taxonomy summarizing the main risk classes, including financial, non–financial risks and external market risks. For more details, we refer the reader to Chapters 2 for credit and market risks, to Chapter 3 for operational risks and other relevant economic capital inputs, and to Chapter 4 for other types of non–financial risk.

enhance capital adequacy, strategic planning, stress testing and other downstream risk management processes and capabilities. On the other hand, risk managers have to bear in mind that by setting arbitrary scales and classifications of risk that facilitate the current definition or the perception of what risk is, they might constrain their ability to comprehend and manage risk by casting aside information that conflicts their vision of reality.

Assessment of risk and capital is a key step towards the management and mitigation of risk. Financial institutions use a wide range of quantitative and qualitative methodologies for measuring and managing risk and capital. Some of these approaches are common to a number of risk categories, while others are tailored to the particular features of specific risk classes. Many of these approaches continue to evolve and at times converge under MiFID II/MiFIR [110], Solvency II [111], CRR/CRD–IV [49] or Basel III [9], becoming more standardized across the US, within the U.K., across the EU and within local regulatory regimes.

Some of the key examples of common tools and metrics that are currently used across financial institutions to measure, manage and report risks include the risk–weighted assets (RWAs) (see Section 1.1), Economic capital (see Section 1.2), stress testing (see Section 4.2), sensitivity analysis, scenario analysis, Value–at–Risk and Expected Shortfall (see Chapter 2), Loss Distribution Approach (see Section 3.1), or the valuation of positions.

Capital models include, for example, approaches to optimization, capital forecasting, stress testing, econometric models, proxy models, asset volatility methodologies, regulatory capital approaches, aggregation approaches, risk–adjusted models, risk–adjusted performance measure (RAPM), risk–adjusted return on assets (RAROA), risk–

adjusted return on capital (RAROC), earnings volatility models, or earnings–at–risk model (see Chapter 1 for more details).

Other models and techniques include simulation methodologies (historical or Monte Carlo), risk rating models, models for segmentation, pricing models, correlation and covariance modelling, cash–flow approach, arbitrage models, equilibrium models, duration models, interest rate models, rate shocks, macro–forecasting, pricing models, just to name a few. For more details on the usage and limitations of these models we refer the reader to Chapters $1-4$.

Risk and capital control models support expanding areas of decision–making with increasing sophistication and interdependencies, enabled by technology developments, such as automation and Big Data analytics. The use of these models covers solvency assessment (i.e., capital adequacy), performance attribution, risk pricing, risk identification and monitoring, strategic business planning, provisioning, asset allocation, overall risk profile, diversification strategy, capital allocation and management, product development, evaluating financial hedges, insurance or matching. In addition, they offer the advantage of combining all relevant operations of an institution, such as underwritting, investment, pricing, taxes, assets, liabilities, into an integrated framework that further provides an insight into future operations and capital requirements.

The final step in the risk management process is control, monitoring and reporting that represent an imperative precondition to ensure sustainable and reliable creation of value and to protect the financial reputation and soundness of an institution. They are based on the following principles:

- **Protection of financial strength:** by controlling risk exposures and avoiding potential risk concentrations at individual exposure levels, at specific portfolio levels and at an aggregate firm–wide level across all risk types.

- **Protection of reputation:** through a sound risk culture characterized by a holistic and integrated view of risk, performance and reward.

- **Business management accountability:** ensuring management accountability, whereby business management, as opposed to risk control, owns all risks assumed throughout the firm and is responsible for the continuous and active management of all risk exposures to ensure that risk and return are balanced.

- **Independent controls:** independent control functions to monitor the effectiveness of the business risk management and oversee risk–taking activities.

- **Risk disclosure:** disclosure of risks to senior management, the Board of Directors, investors, regulators, credit rating agencies and other stakeholders with an appropriate level of comprehensiveness and transparency.

Their function is to assure that the risk profile stays within the levels set for the risk appetite and other limits.

While risk and capital models are central to the day–to–day operations and certainly provide competitive benefits, a key to leveraging the power of these models is thorough understanding of their fundamental assumptions, approximations and limitations these impose on a model use. Financial institutions need to consider the inherent risk of their models and to understand the circumstances in which the key model assumptions and dependencies might break down. If such understanding is not in place, models can be misused with potentially severe consequences such as financial loss, underestimation of provisions or capital requirements, poor business and strategic decision making, or reputation damage.

Model risk refers to the potential financial loss or suboptimal decisions due to models inadequately or improperly performing the function for which they are being used. It can arise in any stage of the model life cycle, i.e. data, development, implementation and use, and in various forms that tend to easily overlap, co–occur, or co–vary. Wrong design or implementation, incomplete or insufficient data, improper calibration or estimation approach, misuse or inadequate knowledge regarding the model development and usage, or inadequate controls over the model use, may expose financial institutions to additional risks. Besides, a major source of model risk represents flaws in the nested models where inputs to one model are the outputs from another model with moderately incompatible assumptions, or built for slightly incompatible purposes or somewhat different markets. Even appropriately applied and correctly used models can produce model errors, and perfectly accurate model outputs can be misused or misapplied.

Although the elimination of model risk is not possible,[9] implementing an approach that combines rigorous model risk management structures, as prescribed by the supervisory guidance [65], with a prudent detailed quantification may be an effective strategy to mitigate and control it. The quantification, as an essential part of a proper MRM, is required for an objective assessment of model risk and an effective communication of model credibility to decision makers and users. The proposed approach introduced in Chapters 6 – 9 to measure model risk within the framework of differential geometry [89] and information theory [6] offers a reliable way to objectively quantify model risk inherent in models used also for the capital and risk management purposes, without the need to refit or rebuild models and with the prioritizing analysis (immediate assurance on shifts that are immaterial).

In this framework, a model is represented by a particular probability distribution that refers to a point on a statistical manifold, i.e. a family of probability distributions, in which similar distributions are mapped to nearby points. This manifold can be further equipped with the geometric structure that, among other things, allows us to quantify variations and dissimilarities between different models. The measure of model risk is then addressed by the calculation of the norm of an appropriate function defied on a statistical manifold endowed with a proper Riemannian metric.

Decision making in capital and risk management often requires quantities and extreme percentiles that can only be obtained as the outputs of a quantitative models. The core to quantitative approaches constitutes measure theory with the objective to generate realistic conditional forecasts of the distribution of returns to a financial institution. For example, to calculate capital requirements via a Value-at-Risk (VaR) principle, a probability distribution of future values of a relevant portfolio quantity (e.g. net asset position) needs to be determined. Based on the loss distribution, an institution can make informed decisions about its portfolio, risk and capital structure, and design internal incentive and control systems to ensure proper implementation.[10] As such, the proposed framework for the quantification of model risk is well suited also for these models.

The remainder of this Part I is structured as follows. Chapter 1 provides insights into capital management from regulatory and institution perspective. This includes the purpose and the main principles of capital risk man-

[9] Though, certain risk factors may be mitigated at source by proper validation process, internal audit, monitoring, controls, skilled employees, underwriting, operational practices or by application of expert judgement, model risk can never be eliminated.

[10] The reason for focusing on such loss rates is that their distribution is crucially important for maximization of an institution value as the distribution of tail losses directly impacts the probability of financial institution distress.

agement, a brief discussion on the regulatory capital with the short overview of the history of the development of Basel regulatory capital frameworks, while focusing primarily on the current assessments. Next, the management and assessment of the economic capital is considered.

The second chapter is dedicated to credit and market risks including credit risk(see Section 2.1), market (trading) risk (see Section 2.2), risk in financial derivatives (see Section 2.3), structural and balance sheet risks (see Section 2.4). Chapter 3 describes non–financial risk with the main focus on operational risk methodologies (see Section 3.1) while the last Chapter 4 delves into provision calculation (see Section 4.1), stress testing (see Section 4.2) and the aggregation methodology for assessing total economic capital (see Section 4.3).

Chapter 1

CAPITAL MANAGEMENT

Capital management, although complex and subject to regulatory changes, represents a critical factor in the value creation for financial institutions, with the aim to absorb risk, measure a sufficient level of capital, assess the internal capital adequacy and calculate the capital adequacy ratio. Managing capital is an ongoing process of determining and maintaining the quantity and quality of capital appropriate for an institution through regulatory and economic capital metrics, at both business line and legal entity level, while optimizing capital structure and the overall performance of an institution. For more details on regulatory and economic capital see Sections 1.1 and 1.2, respectively.

Bank capital is an intricate concept with various meanings and uses, and can be defined as the value of the net assets of the owners of the firm. For shareholders, capital stands for a fixed wealth (measured as market value), requiring an adequate return as risk compensation while at the same time being an instrument for controlling the institution. From a regulatory perspective, capital is a buffer to absorb unexpected losses, to protect depositors and to ensure the ongoing viability of the financial system. From the standpoint of a risk manager, capital likewise represents a buffer to absorb losses that, however, should reflect the risk tolerance and risk appetite of an institution, to prevent financial distress and to measure performance on a risk–adjusted basis.

The challenge for banks and regulators is to determine an appropriate amount of capital that would serve as a cushion against all types of risk a company is exposed to or may potentially be exposed in the future. Risk of an institution is estimated based on a loss distribution and quantified by the expected and unexpected loss that are metrics used in a statistical context.

Expected loss (EL) refers to

"[...] the statistically established loss expected to occur on an exposure."

Specifically, it represents the average loss, or the probability weighted mean of a loss distribution, due to specific event or combination of events over a specific time horizon. It is calculated by multiplying each exposure by its probability of default and loss given default estimates. EL, seen as a cost of doing business, is managed through pricing, business revenue, loss provisions or in the incident of a shortfall by deductions from capital.

Unexpected loss (UL) relates to

"[...] the risk of greater than expected losses arising due to uncertainties in the estimates"

and it is calculated as the difference between the total exposure at the target risk predetermined tolerance level and the expected loss.[1] The UL therefore measures the level of risk at the predetermined probability level. For an illustration, total exposure at the 99.9% probability level within one–year time horizon represents the level of loss where a larger loss is expected to occur only once in thousand years or has only a 0.1% chance of occurring in any given year. Some components of UL may be absorbed by interest rates, including premia, charged on credit exposures, however most of UL is covered by capital given that these represent peak losses exceeding the expected levels.

A loss distribution, which refers to the cumulative loss exposure for the specified horizon, associates all potential future losses with their estimated probabilities and is typically displayed as analytical loss distributions or derived through Monte Carlo simulations (frequency diagram). Analytical loss distribution can be expressed through a closed–form descriptions, while Monte Carlo simulations are based on future loss scenarios that come from an underlying distributional and statistical assumptions. As Figure 1.1 illustrates, the aggregate EL and aggregate UL

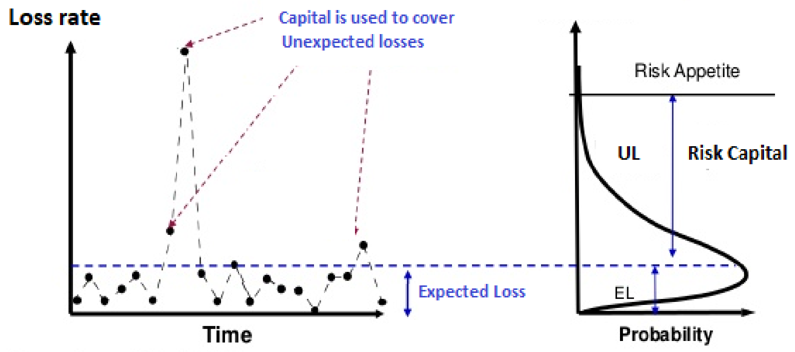

Figure 1.1: Figure captures how variation in realised losses over time leads to a distribution of losses for a financial institution. Note that EL does not necessarily equal to the historical loss experience as the portfolio components may change.

are calculated by combining individual frequency and severity distributions. The frequency distribution shows the probability of events occurring based on a one–year time horizon. The severity distribution represents the probability associated with loss magnitude and has no time element.

Capital acts as a cushion to absorb unanticipated losses within the 12–month period and declines in asset values that could otherwise cause a bank failure. Besides, it provides protection to uninsured depositors and debt holders in the event of liquidation. From a structural point of view, capital is composed by a pool of balance sheet and income items. These assets can be grounded into two capital components: the **regulatory capital** and the **economic capital**. Regulatory capital refers to the minimum capital required by the regulator which is expected to reduce the risk of default to a predetermined level, while economic capital is determined internally in accordance with macroeconomic conditions to support risks associated with individual lines of business, portfolios, or transactions within the financial institution.

The objective of these two capital concepts differs, as the ultimate intention of regulators is to protect depositors, ensure the soundness of financial institutions, and prevent financial crisis, whereas the aim of financial

[1] Unexpected loss is often described as the volatility around the average over time.

institutions is to maximize the return to their shareholders through the optimal allocation of capital across different business lines.[2] Thus, economic capital is concerned with the internal risk management of an institution, while regulatory capital is about ensuring the solvency of the institution and of the financial sector as a whole. A more detailed description of regulatory and economic capital is provided in Sections 1.1 and 1.2, respectively.

1.1 Regulatory capital

Regulatory capital is the amount of capital a financial institution has to hold as required by its financial regulator. The Basel Committee on Banking Supervision (BCBS)[3] provides international regulatory capital standards through a number of capital accords and related publications, which have collectively been in effect since 1988. The BCBS aims at strengthening the regulation, supervision and practices of banks worldwide with the purpose of enhancing global financial stability.

The primary objective of the regulatory capital standards it to set a minimum capital that would absorb losses and withstand economic downturns thereby strengthening the soundness and stability of the international banking system, and to ensure a fair and high degree of consistency in the application of capital standards to banks in different countries with a view on reducing a source of competitive inequality among internationally active banks.

The first publication about capital adequacy issued by BCBS, known as Basel Capital Accord (or Basel I) was introduced in 1988. Since then, the BCBS has significantly strengthen its capital adequacy framework by issuing the documents known as Basel 2.5 [27] in 2009 and Basel III [24] in 2010. These build on the so–called Basel II [9], which contains many of the central principles of the BCBS current approach to capital adequacy.

In general, the Basel framework [26] is based on three **Pillars**:

Pillar 1 (Minimum requirements):

- **Capital Requirements**: framework for banks on how to handle their capital in relation to their assets. BCBS introduced the first international level capital requirements in 1988. The current framework of capital requirements is called Basel III.
- **Reserve Requirements**: set the level of minimum amount of funds that banks must hold, normally as cash or as deposits with a central bank.

Pillar 2 (Supervisory review): sets out the institution responsibility for assessing risks other than those described under Pillar 1. It relates the capital requirements to address risks and risk management processes, licensing by regulators, giving directions, acquiring undertakings, assigning penalties or voiding the bank license.

[2] As noted by [5], the two concepts reflect the needs of different primary stakeholders. For economic capital, the primary stakeholders are the bank shareholders, and the objective is the maximization of (their) wealth. For regulatory capital, the primary stakeholders are the bank (depositors), and the objective is to minimize the possibility of loss.

[3] The Basel Committee on Banking Supervision (BCBS) was established in 1974 by the central banks of the G–10 countries. The Group of Ten (G–10) is composed by eleven industrial countries (Belgium, Canada, France, Germany, Italy, Japan, the Netherlands, Sweden, Switzerland, the United Kingdom and the United States) which consult and co–operate on economic, monetary and financial matters; the United States serves as a participating member.

Pillar 3 (**Market discipline**): focuses on minimum disclosure requirements, covering the key elements of information required to assess the capital adequacy of a credit institution. It aims to promote greater market discipline by enhancing transparency of information disclosure, i.e. more information on risks, risk management practices and capital adequacy are required to be publicly available.

Basel I Accord

The first international regulatory standards for setting bank capital requirements were developed by BCBS, known as the Basel Capital Accord (Basel I), for credit risk in 1988 and for market risk in 1996. Although, the scope of application of these standards was limited to the internationally active banks, many national regulators adopted Basel I as the formal national regulatory capital requirement.

Basel I introduced two key concepts, the segregated capital and the risk–weighted assets. Capital was categorized into **Tier 1** and **Tier 2** according to the loss–absorbing or creditor–protecting characteristics. **Tier 1** capital, consisting of common stocks and retained earnings, served to absorb unexpected losses to a certain level without a significant disruption to trading. On the other hand, **Tier 2** capital, comprised by subordinated debts, provided certain protection to depositors in the event of bank failure, i.e. the last layer of debt that would be repaid in a bank insolvency.

The minimum capital requirement (Tier 1 plus Tier 2) was set to be not less than 8% of the total risk–weighted asset amount (RWA),[4] at least half of which had to be Tier 1. The Capital Adequacy Ratio (CAR) was then given by [109]:

$$\begin{aligned} \text{CAR} &= \frac{\text{Capital (Tier 1 + Tier 2)}}{\text{Assets (weighted by credit type) + credit risk equivalents}} \\ &= \frac{\text{Total Capital}}{\text{RWA}} \geq 8\%. \end{aligned}$$

Basel I was widely accepted and became the best practice for capital adequacy within more than 100 countries. It was successful in raising capital levels, however, it failed mainly due to the capital rules that missed to adopt important differences in risk exposures across banks, and to keep pace with the innovations in the banking industry. Moreover, Basel I used unsophisticated measurement of a bank credit risk exposure ([109], [48]) that did not distinguish between various levels of risk thus creating opportunities for arbitrage [74].

Basel II Accord

In 2004, the BCBS released the second Basel Accord [9], known as Basel II, which was developed as a response to perceived shortcomings and deficiencies of Basel I. Basel II was built significantly on Basel I with increasing capital sensitivity to the key types of risk, ranging from the traditional risks associated with financial intermediation, the day–to–day risks of operating a business, to the risks associated with the ups and downs of the local and international

[4]RWA were the result of multiplying exposures by weighting coefficients. Higher weights were assigned to riskier assets such as corporate loans, and lower weights to less risky assets, such as exposures to government. One of the following four weighting factors was applied to each type of exposure: $0\%, 20\%, 50\%$ *or* 100%.

economies. As a result, the new framework more explicitly associates capital requirements with the particular categories of material risks that banks face.

In addition to the first accord, Basel II accounts for operational risk, disclosure requirements and supervisory review process [20]. A major development of Basel II is in allowing banks, under certain conditions, to use their own internal models and techniques to measure the key material risks that they face, internal rating mechanism, the probability of loss, and to manage capital. Large internationally active banks use approaches to risk measurement and management typically based on measure theory.

The Basel II retains key concepts of the Basel I Accord, such as the definition of eligible capital and the general requirement for banks to hold total capital equivalent to at least 8% of their risk–weighted assets, i.e.

$$\text{CAR} = \frac{\text{Total Capital}}{\text{Credit risk + Market Risk + Operational Risk}} \geq 8\%$$

Although the definition of eligible capital was almost kept unchanged, Basel II introduced significant changes in the RWA calculation. Specifically, the total RWA refers to the sum of the risk–weighted assets for credit risk and a 12.5 multiple of the capital requirements for market and operational risk where the individual RWA are computed by applying a weight to each exposure, not set for a very broad categories of risks as in Basel I.

In line with Basel I, Basel II reinforced the three pillars: capital requirements, supervisory review process and market discipline that became closely interlinked, interrelated and mutually reinforcing in safeguarding against the operational risk. Pillar 1, however, expands the regulatory minimum capital requirements proposed by Basel I by introducing a system that is based on external credit ratings. It covers the minimum requirements for credit, market, and operational risks, and it promotes to measure these risks by Value–at—Risk (VaR).[5] The minimum requirements for capital for these risks is determined using one of the following approaches:

- **Credit risk**: can be calculated in three different ways of varying degree of sophistication and risk sensitivity, namely Standardised Approach (SA), Foundation Internal Rating–Based (IRB) Approach and Advanced IRB Approach. SA is akin to that of Basel I, but it is more risk–sensitive. IRB approach allows banks to use their internal models for estimations of borrower creditworthiness to determine credit risk in their portfolios.
- **Market risk**: Standardised and Internal Ratings-Based approaches are available. The preferred approach is VaR.
- **Operational risk**: there are three different approaches: Basic Indicator Approach (BIA), Standardized Approach (STA) and Internal Measurement Approach, an advanced form of which is the Advanced Measurement Approach (AMA)

Likewise, Pillar 2 enhances the link between an institution risk profile, risk management and risk mitigation systems, and its capital planning. It consists of two complementary processes [55]:

- the **Internal Capital Adequacy Assessment Process (ICAAP)** aimed at institutions that are expected to establish sound, effective and complete strategies and processes to evaluate and maintain, on an ongoing basis, the amounts, types and distribution of internal capital proportional to their risk profiles, as well as robust governance and internal control arrangements, and

[5] Value–at–Risk measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval.

- the **Supervisory Review and Evaluation Process (SREP)** with the objective to ensure that institutions have adequate arrangements, strategies, processes and mechanisms as well as capital and liquidity to ensure a sound management and coverage of their risks, to which they are or might be exposed, including those revealed by stress testing.

Finally, Pillar 3 aims to promote market discipline through regulatory disclosure requirements and to improve comparability and consistency [10]. It enforces the market discipline with disclosures of objectives and policies of risk management for each risk type including strategies, processes, structure and organization of the relevant risk management functions, extent and content of risk reporting and/or measurement systems, risk management/risk mitigation strategies and processes for monitoring the efficiency of risk mitigation strategies.

Basel II represents a more coherent relationship between how supervisors determine regulatory capital and how financial institutions manage risks. The essential achievement of Basel II is the improvement of the financial system by encouraging continuous enhancements in risk measurement and risk management in financial institutions. Moreover, the allocation of bank capital is better balanced to specific categories of risk.

Basel III Accord

The failure of Basel II in preventing the global financial crisis of 2008 led to the development of a new accord, Basel III ([26], [25]), that extends the existing Basel II framework by introducing new capital, leverage and liquidity standards to strengthen the regulation, supervision, and risk management of the banking and financial sector. [6]

The objective of the new regulations is to create a strong and impregnable banking sector and to improve the ability to absorb shocks arising from financial and economic stress. The key elements of the proposals are:

- increases the level and quality of capital,
- enhances risk coverage (notably counterparty risk),
- constrains and reduces bank leverage through the introduction of a backstop leverage ratio in order to supplement risk–based capital requirements,
- improves bank liquidity, and
- limits procyclicality through counter–cyclical buffers. [7]

Besides, Basel III imposes more restrictive measures for calculating counterparty credit risk, as compared with Basel II. Unlike to the previous standards, where historical data were used to estimate volatility and correlation assumptions, Basel III requires banks to include a period of economic and market stress when making model assumptions. Additionally, historical observations should be multiplied by 1.25 when calculating the correlation between financial firm assets value and the economy. Basel III includes three complementary pillars, as defined in the context of Basel II.

[6] Basel III began to be implemented on January 1, 2013 and is phased in over a 12–year period. The full implementation of most of the rules should be by the end of 2018, though phase out of non–qualifying capital instruments will go longer to 2023.

[7] Counter–cyclical buffers entail banks, during the periods of stability, to hold capital greater than the regulatory minimum in order to sufficiently maintain themselves during a sudden downward spiral.

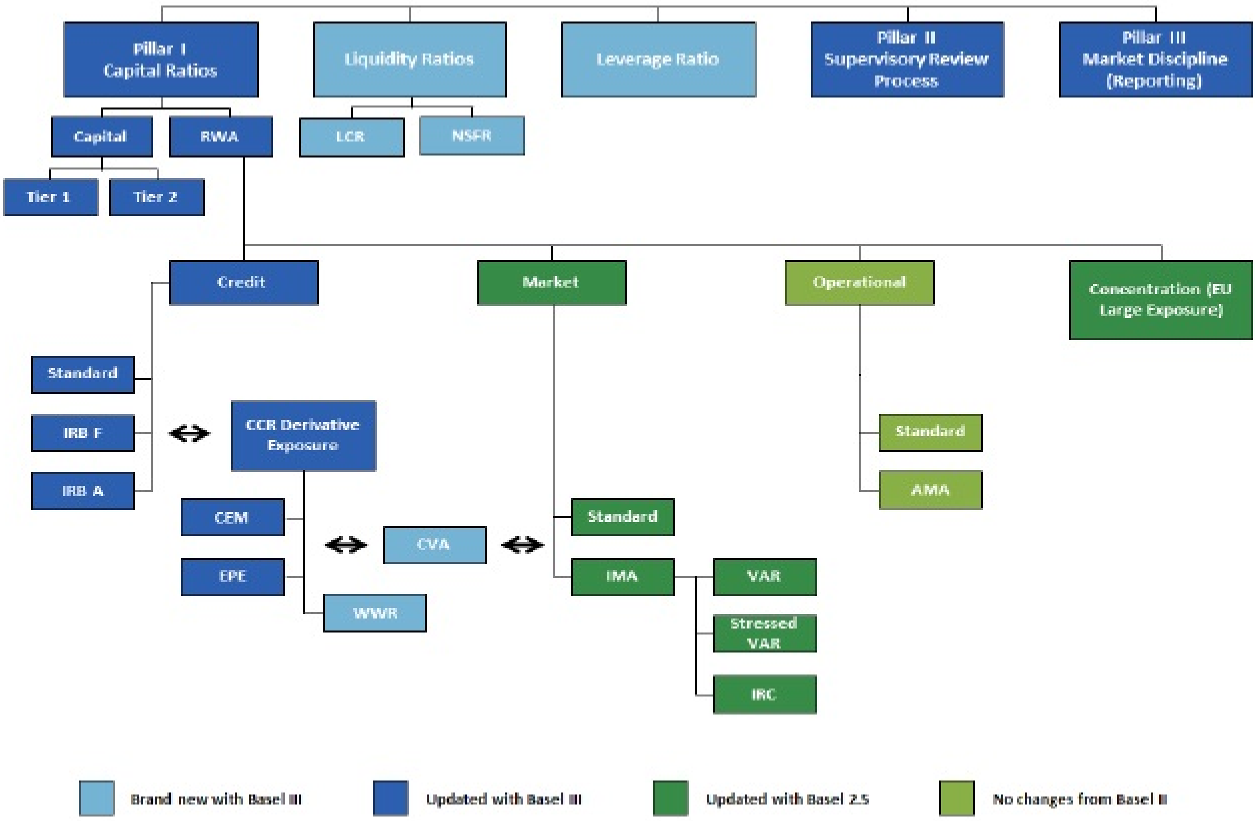

Figure 1.2: Structure of Basel III Accord. Source: Moody's Analytics

In what follows, we briefly describe the main changes in Basel III.

1. **Increased quality and quantity of capital:** The new capital standards and additional capital buffers require banks to hold more capital, and higher quality of capital through changes in the risk–weighting rules for credit and market risk, the definition of the capital, and the minimum level of capital adequacy ratio.

   The CAR, under Basel III, is given by

$$CAR \leq \frac{Capital\,(Tier1 + Tier2) - Adjustments}{RWA_{Credit\,Risk} + RWA_{Market\,Risk} + RWA_{Operational\,Risk} + RWA_{Concentration\,Risk}}.$$

   The regulation increases capital requirements for counterparty credit with the objective of expanding the coverage of the capital base.

   Capital is composed of Tier 1 and Tier 2, where Tier 1 is composed of two different classes:

   (a) **Common Equity Tier** 1 **(CET1)**: best quality capital

   (b) **Additional Tier** 1 **(AT1)**: non–CET1 capital instruments with strict requirements in terms of subordination and loss absorption

   Tier 1 ratio increases from 4% to 6%. Tier 2 capital is harmonized and simplified, and there is no longer requirement for Tier 3 capital (see Subsection 1.1.1 for more details).[8] Table 1.1 describes the main changes made compared to Basel II. Besides, two additional buffers [9] were introduced:

Capital class	Basel II	Basel III
	Tier 1	
Retained earnings	Common equity	Common equity
Common shares	Common equity	Common equity (includes member certificates)
Innovative capital instruments	Additional	Excluded and grandfathered due to incentive to redeem
Non–innovative capital instruments	Additional	Included, but will probably need some restructuring
	Tier 2	
Subordinate debt	Tier 2	included in Tier 2 only if it is loss–absorbent in the case of stress (gone–concern)

Table 1.1: Comparison between Basel II and Basel III capital classification.

[8] Tier 3 capital is short–term subordinated debt and was used under Basel II to support market risk from trading activities.
[9] A third buffer is under discussion for Systematically Important Financial Institutions (SIFI).

- **Capital Conservation Buffer**: buffer for outside periods of stress that comprises common equity of 2.5% of RWA (implemented by 2019). It gives regulators the ability to control banks' earnings distribution. In case the capital of the bank falls into the buffer range and approaches the minimum requirement, the bank would be subject to increasing restrictions on earnings distribution.
- **Countercyclical buffer**: extra buffer in case when regulator estimates that there is an excess credit growth in the market and expects an crisis. Imposed within a range of $0 - 2.5\%$ and consists of either Tier 1 common equity or other fully loss absorbing capital instruments.

These buffers are designed to restrict the bank ability to distribute its earnings until the buffers are rebuilt.[10] Including the additional buffers, regulatory capital requirements for banks rise from a minimum of $8 - -10.5\%$ depending on the size of the countercyclical buffer (including the countercyclical buffer it could go up to 13%). According to BCBS guidance, the requirements are to be phased in by 2019 [23]. Yet, in common with previous regulations, the final implementation schedule is determined by national supervisors.

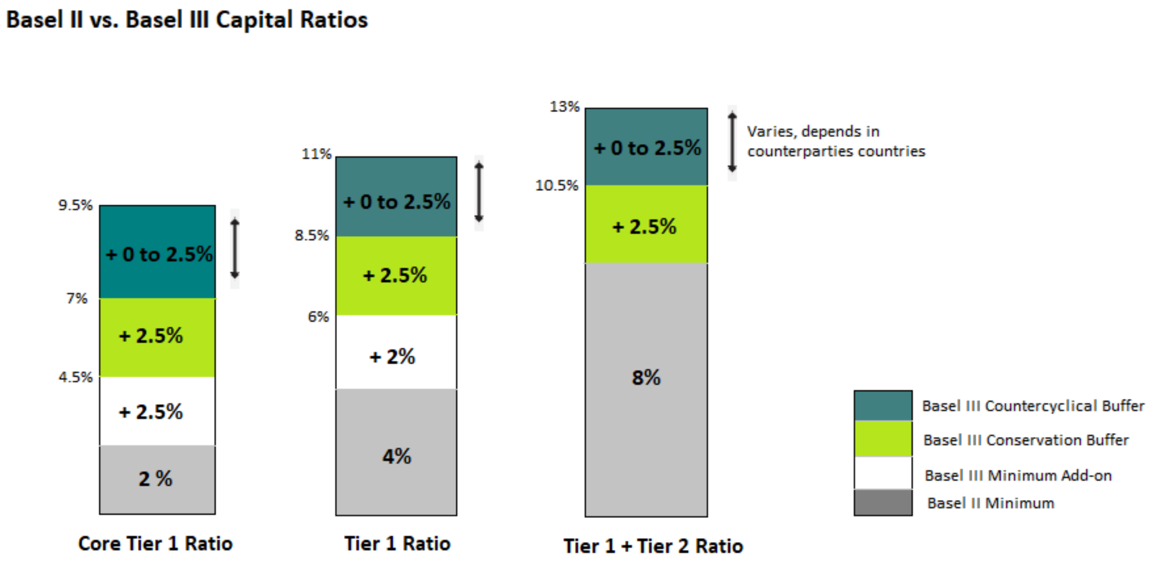

Figure 1.3: The diagram outlines how the Basel III minimum add–on, conservation buffer and counter–cyclical buffer affect the core, Tier 1 and Tier 1 + 2 ratios in comparison to Basel II.

2. **Risk coverage:**

   Basel III [25] revises the standard approaches for calculating

   - credit risk
   - market risk
   - credit valuation adjustment risk
   - operational risk

   that represent greater risk–sensitivity and comparability. The capital requirements for credit risk and counterparty credit risk are strengthen through:

[10]Note that these buffers are not strictly additional minimum capital requirements and may be drawn down during periods of stress.

- **Asset Value Correlation (AVC)**: captures risks of certain financial institutions that are highly correlated to the financial system, moreover (reflecting experience from past crisis events) Basel III increases the asset value correlation by 25% (multiplier of 1.25 to the correlation coefficient) which relates to all financial exposures under the internal rating based (IRB) approach of regulated financial firms with assets of at least $\$100$ billion, and to all unregulated financial firms, regardless of their size.

- **Wrong–Way risk (WWR)**: Basel II introduced an explicit Pillar 1 capital charge for WWR that is defined as the risk arising when derivative exposure increases as the credit quality counterparty declines [19]. Banks are exposed to specific WWR when future exposure to a specific counterparty is highly, positively correlated with the counterparty probability of default. Under Basel III, banks must implement a new capital charge to WWR that is achieved by adjusting the multiplier applied to the exposure amount identified as wrong way risk.

- **Credit Valuation Adjustment (CVA)**: capital charge on counterparty credit risk for derivatives. CVA has two main purposes:

  (a) to reflect how much the default–free price of a derivative should be adjusted to account for the possibility of a counterparty default (hence, a component of the theoretical value of a derivative);

  (b) in capital terms, it accounts for the amount of reserves to cover the risk of changes in the Mark–to–Market (MtM) values of Over–the–Counter (OTC) derivatives (as a result, OTCs are only subject to CVA). CVA is linked to other components of the trading book capital such as market risk, liquidity risk and operational risk.

  Different units of the bank view the market risk differently. For example the impact of risky MtM value is different for accounting (unrealised losses on book need to be added), funding (traders need to source additional funds to address the liquidity deficit), liquidity risk (manage and monitor the funding gap), credit risk department (manage and monitor the risky MtM value), etc. These different views have to be implemented into the framework of CVA. CVA capital should be integrated with other components of the trading book capital. Since CVA is used for the pricing adjustment of OTC derivatives and as an additional component of reserves calculations, the deal assessment, the hedging techniques and other strategic decisions of a bank are strongly impacted.

- **Central Counterparty (CCP)**: Enhancing incentives for clearing instruments through CCPs by applying lower own funds requirements relative to bilateral OTC transactions. Also, the additional CVA capital charge does not apply to exposures towards eligible CCPs. BCBS supports to use central exchanges and standardized derivative contracts which is done by determining a modest risk weight $(1-3\%)$ to evaluate regulatory capital when these exchanges conform with several criteria.

- **Calibration of capital charge to stressed period**: Calibration of counterparty credit risk modelling like Internal Model Methods to stressed period, determine capital charges for counterparty credit risk using stressed inputs (stressed Effective Expected Positive Exposure), similar to the approach used for determining stressed VaR for market risk [21].

- **Other**: like revised model validation framework, supervisory guidance for sound back-testing practices of counterparty risk. Besides, higher capital buffer for positions in the trading book and complex securitization transactions are required.

3. **LEVERAGE RATIO**

The leverage ratio is an alternative measure to the risk–weighting process and aims to guard against the build–up of excessive leverage in the banking system. A non–risk–based leverage ratio including off–balance sheet exposures is meant to serve as a backstop to the risk–based capital requirements. It is defined as the ratio of Tier 1 Capital to Total Leverage Exposure [28],

$$Leverage\,Ratio\,(\%) = \frac{Tier\,I\,Capital\,\,(after\,\,related\,\,deduction)}{Total\,Leverage\,Exposure\,\,(after\,\,related\,\,deduction)} \geq 3\%$$

which takes into account both on– and off–balance sheet items and secularizations and the minimum is set to 3%. It also serves as a 'safety net', to guard against any inaccuracies or unforeseen problems with risk weightings.

4. **LIQUIDITY RATIOS**

Basel III introduces two internally consistent regulatory standards for liquidity risk supervision [26]: (NSFR).

- **Liquidity Coverage Ratio (LCR)** requires banks to have sufficient high–quality assets to withstand a 30–day stressed funding scenario that is specified by supervisors, i.e. LCR is the amount of liquid assets needed to survive a 30 day stress period. LCR is defined by

$$LCR = \frac{Stock\,of\,\,high-quality\,liquid\,assets}{Net\,cash\,outflows\,over\,30\,days\,period} \geq 100\%$$

  The 30–day stressed period assumes certain institution–specific and system wide liquidity shocks.

- **Net Stable Funding Ratio (NSFR)** measures the amount of longer–term, stable sources of funding employed by an institution relative to the liquidity profiles of the assets, and is given by

$$NSFR = \frac{Available\,amount\,of\,\,stable\,funding}{Required\,amount\,of\,\,stable\,funding} \geq 100\%$$

  It is the amount of stable funding needed to ensure that banks balance their long–term assets with long–term liabilities, so that, under structural economic pressure, a financial institution can provide stable financing sources to provide funding for its operation over one year. NSFR covers the entire balance sheet and provides incentives for banks to use stable sources of funding.

The main objectives of the new capital treatment of credit risk are to strengthen global capital regulations with the goal of promoting a more resilient banking sector, improve the ability to absorb shocks arising from financial and economic stress which, in turn, would reduce the risk of a spillover from the financial sector to the real economy, increase capital buffer for systemic banks/derivatives, and to provide additional incentives to move OTC derivative contracts to CCPs and central clearing houses.

1.1.1 CAPITAL ADEQUACY REQUIREMENTS

The final rule of Basel III contains two types of capital ratio requirements: the risk–based capital ratio and the leverage capital ratio.

The risk–based capital ration refers to the quotient of regulatory capital to RWA. A bank should compute Basel III capital ratios (CR) in the following manner

$$\begin{aligned}
\text{Common Equity Tier 1 CR } (\%) &= \frac{\text{Common Equity Tier 1 Capital}}{\text{Credit risk RWA + Market Risk RWA + Operational Risk RWA}} \geq \mathbf{4.5\%} \\
\text{Tier 1 CR } (\%) &= \frac{\text{Eligible Tier 1 Capital}}{\text{Credit risk RWA + Market Risk RWA + Operational Risk RWA}} \geq \mathbf{6\%} \\
\text{Risk Based Total Capital (CRAR) } (\%) &= \frac{\text{Regulatory Total Capital}}{\text{Credit risk RWA + Market Risk RWA + Operational Risk RWA}} \geq \mathbf{8\%}
\end{aligned}$$

where

- **Regulatory capital:** according to Basel III, the capital consists of three categories, namely Tier 1 capital (at least 6% of RWAs) compounded from Common Equity Tier 1 (at least 4.5% of RWAs) and Additional Tier 1, and Tier 2 capital.[11] Total Capital must be at least 8% of RWA at all times and Tier 2 capital is limited to 100% of Tier 1 capital
- **Risk–Weighted Assets (RWA):** are a financial institution's assets or off–balance–sheet exposures weighted according to the risk of the asset. At a high level, the total RWA equals the sum of the RWA amounts. The calculation of RWA amounts involves multiplying the amount of an asset or exposure by the standardized risk weight (%) associated with that type of asset or exposure

  $$RWA = Exposure\ at\ Default \times Risk\ Weight$$

  where

  · **Risk weights** are function of Probability of Default (PD) [12], Loss Given Default (LGD) [13], Maturity and annual sales[14], they are prescribed in the bank capital rules and reflect regulatory judgement regarding the riskiness of a type of asset or exposure. For more details on how to calculate these concepts we refer the reader to Chapter 2 and Section 2.1.

  · **Exposure at default** is the sum of Current exposure and Potential Future Exposure (PFE) (see Section 2.1).

In general, banks are required to calculate their RWA on the basis of two risk types: financial and non–financial risk. The financial risk can then further be decomposed into credit risk and market risk where market risk consists of three other components, namely Value–at–Risk (VaR)[15] or Expected Shortfall (ES)[16],

[11] Tier 1 capital is intended to ensure that each bank remains a 'going–concern'. It is the highest quality form of a bank's capital as it can be used to write off losses. Tier 2 capital is meant to protect depositors in the event of insolvency, and is thus re–categorised as a 'gone–concern' reserve.

[12] Probability of Default (PD) is a measure of credit rating that is assigned internally to a customer or a contract with the aim of estimating the probability of non-compliance within a year. It is obtained through a process using scoring and rating tools.

[13] Loss given default (LGD) is defined as the percentage exposure at risk that is not expected to be recovered in the event of default.

[14] Firms with annual sales of EUR50M or less are entitled to smaller RWA.

[15] The VaR at the $100(1-\alpha)\%$ confidence level is the upper 100α percentile of the loss distribution. $VaR_\alpha(X)$ denotes the VaR at the $100(1-\alpha)\%$ confidence level and X is the random variable of loss.

[16] Expected shortfall (ES) is an alternative measure, more consistent mathematically, to complement/substitute VaR. It combines aspects of the VaR methodology with more information about the distribution of returns on the tail. It measures how much one has lost on average in states beyond the VaR level. $ES_\alpha(X)$ is defined by the following equation:

$$ES_\alpha(X) = \mathbb{E}[X|X \geq VaR_\alpha(X)]$$

Incremental Risk Charge (IRC) and Credit Valuation Adjustment–VaR (CVA–VaR) (see Section 2.2). The calculation of credit risk can be divided into two subsequent components: Counterparty Credit Risk (CCR) for financial derivatives and Issuer Credit Risk (ICR) for debt products like loans or bonds (see Section 2.1). The non–financial risks is referred to operational risk, strategic risk, reputation risk, capital risk, earnings risk and outsourcing risk. The total RWAs is the summation of all components

$$RWA = RWA_{VaR} + RWA_{IRC} + RWA_{CVA} + RWA_{ICR} + RWA_{CCR} + RWA_{OR}$$

Due to the significantly different size and sophistication of global banks, capital requirements do not tend to follow a single methodology but rather allow different choices of varying sophistication. Basel framework provides at least two different sets of rules, the *advanced* and *standardized* approach, that may be used depending on the existing approvals that a particular bank has. A bank with internal model method and specific risk approvals has to use the advanced approach whilst other banks would use the more basic standardised approach [25].

The Leverage Ratio (LR) as defined on p. 49. LR is calibrated to act as a credible supplementary measure to the risk based capital requirements. The main intend of LR is to constrain the build–up of leverage in the banking sector which can damage the broader financial system and the economy and to reinforce the risk based requirements with an easy to understand and a non–risk based measure.

In summary, Basel III focuses on enhancing the stability of the financial system by increasing both the quantity and quality of regulatory capital and liquidity through making some forms of capital ineligible for regulatory purposes, adding higher capital charges for market risk, counterparty credit risk, and securitizations, financial institutions carry a stronger capital buffer, over and above regulatory minimums. Realizing certain weaknesses in the regulatory capital model, such as their limitation in moderating risk–based decisions, limitations in reflecting the economic reality and real risks an entity faces, or insufficiency in the time period for reporting and/or monitoring, many financial institutions and companies use economic capital that can facilitate risk–based decisions and implement risk management controls. It is important to note that the minimum regulatory capital, even under Basel III, will not be a substitute for an economic capital. See Section 1.2 for more details on economic capital.

Further Regulations

After the Basel III framework was introduced, financial institutions have to deal with further, ongoing, regulatory adjustments and discussions, such as the revision of the standard approach (SA) for credit (BCBS 347) **??** to increase comparability of capital requirements under the SA and the IRB approaches and to reduce reliance on external ratings; the proposal of a new standardized approach (SA) for operational risk in 2017 [93] including a revised business indicator, new size–based risk coefficients instead of segment–based risk coefficients, and a loss components that accounts for observed operational losses; risk data aggregation and IT (BCBS 239) [12]; revision to the advanced IRB and the F–IRB approaches to reduce of variation in RWA (BCBS 362) [17]; the fundamental review of the trading book (BCBS 352) becoming effective by 2019 [16]; the revised interest rate risk in the banking book standards (BCBS 368) [15]; and the introduction of IFRS 9 accounting standards [71] consisting of a new framework for classification, impairments, and hedge accounting, introduction of lifetime expected loss and earlier provisioning; just to name a few.

In addition, banks and other financial institutions are subject to stress testing requirements that directly impacts capital. The stress testing started with the 2009 Supervisory Capital Assessment Program (SCAP) [60]. The success of the SCAP led to the stress testing requirements under the Dodd–Frank Wall Street Reform and Consumer Protection Act (the Dodd–Frank Act) [116], Comprehensive Capital Analysis and Review (CCAR) exercises [59] conducted by the Federal Reserve Board, and the European Central Bank Comprehensive Assessment, as well as others, that provide guidance to create what–if scenarios to test capital sufficiency through stress testing.

Supervisors provide guidance on modelling and define a small number of scenarios through economic and financial variables, banks and their supervisors need to evaluate losses resulting from these scenarios. For example, CCAR is an annual exercise to assess whether the largest bank holding companies operating have sufficient capital to continue operations throughout times of economic and financial stress and that they have robust, forward–looking capital planning processes that account for their unique risks. Within this exercise, the institutions capital adequacy, internal capital adequacy assessment processes, and their individual plans to make capital distributions, such as dividend payments or stock repurchases are evaluated. The Dodd–Frank Act stress testing is a forward–looking complementary exercise to CCAR that helps to assess whether institutions have sufficient capital to absorb losses and support operations during adverse economic conditions.

1.2 Economic capital

Economic capital (EC) is a forward–looking measure of risk that provides an actual economic quantification of the amount of capital that an institution needs to hold to cover losses at a certain risk tolerance level, over a specified time horizon. The overall EC of a financial institution is typically calculated by first assessing the individual risk components and then considering possible techniques for their aggregation in order to determine an integrated picture, i.e. an aggregate risk and an overall capital. Assessing EC, thus, involves analyses of various risks, dependencies and complexities to which the firm is exposed and is often used as an important tool for risk–based decision making.

EC is addressed in the Basel II regulatory framework as a central requirement of Pillar 2 [9], under the Internal Capital Adequacy Process (ICAAP) that represents a financial institution own assessment of the capital needed to run the business. The ICAAP requires institutions to have in place a process for assessing their overall capital adequacy in relation to their risk profile and a strategy for maintaining their capital levels through which all the relevant risks are identified and quantified, and appropriate capital limits and targets are set. It should include a strategic plan, outlining economic and regulatory capital requirements, anticipated capital expenditures as well as related sources of capital.

Financial institutions are expected to have in place effective plans and procedures in order to determine on a regular basis the amount, the composition and the distribution of capital available for the quantitative and qualitative coverage of all material risks from banking transactions and banking operations and to hold capital in the amount necessary [30], nevertheless regulators do not determine how the ICAAP must be conceived or implemented.

Besides, institutions are free to choose the methodology for assessing capital adequacy. They may employ complex models allowing for correlations between risks as well ass simpler approaches that calculate the necessary capital requirements under Pillar 1 and account for additional risks to which the respective institution is exposed (i.e., interest rate risk in the banking book, liquidity risks) and external factors (i.e., the impact of the economic cycle) in light of the fact that methods described under Pillar 1 cannot fully cover all risks (for instance concentration risk is not integrated into the assessment of credit risk under Pillar 1).

The EC complements the regulatory approach by including in its measurements all the material risks in bank operations (see Fig. 1.4). The major risk categories under Basel framework include: *Credit Risk, Market Risk (Non trading equity, Structural FX, Trading), Operational Risk, Liquidity Risk, Structural Interest Rate Risk, Concentration Risk, Business Risk (regulatory, political, tax, legislative, economic, overall market security), Pension risk, Goodwill and Material Risk* and other risks outside the Pillar 1. Furthermore, SR $11-7$[65], issued by The Federal Reserve Board and the Office of the Comptroller of Currency, requires *Model Risk* to be managed from the time the models are developed. The BCBS recognizes that some of these risks cannot always be measured precisely, but it requires banks to at least have a procedure for estimating them. Apart from the major categories, some institutions include also other risks, such as transfer risk, separate account risk, governance or audit risk. All the relevant risk inputs to economic capital are further discussed in more detail within Chapters 2 to 4, namely Chapter 2 for credit and market risks, Chapter 3 for operational risks and other non–financial risks, and Chapter 4 for stress testing, reserves and aggregation of risks.

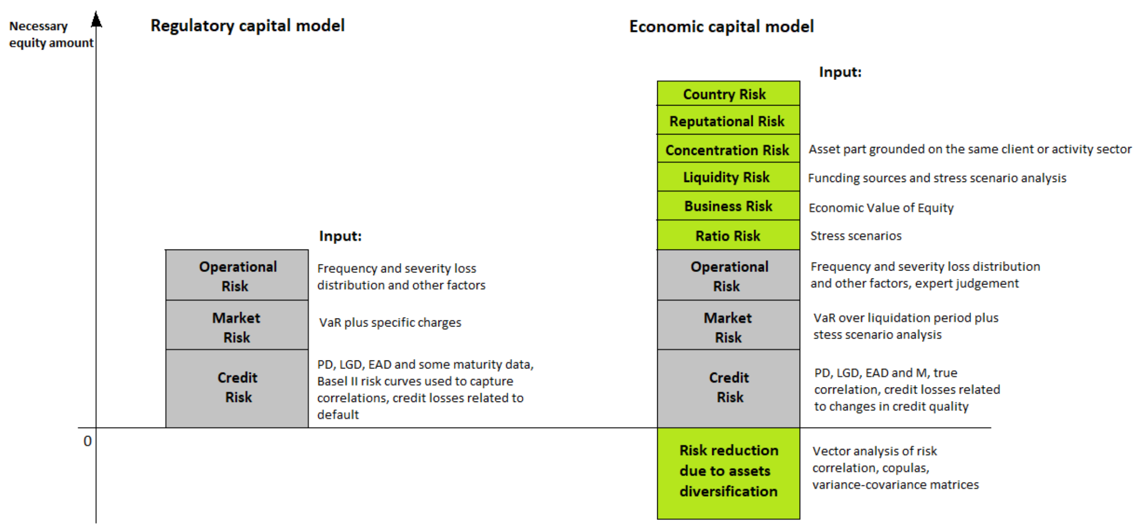

Figure 1.4: Types of risks included in Regulatory and Economic capital under Basel framework.

The structure of the approaches and models used for measurements depend on the nature, scale and complexity of the activities of the institution as well as on the potential risks posed by its activities. According to BCBS [30], EC models are

"[. . .] methods or practices that allow banks to consistently assess risk and attribute capital to cover the economic effects of risk–taking activities."

As such, EC is distinct from familiar accounting and regulatory capital measures. Whereas traditional measures of capital adequacy focus on some ratios such as present capital levels to assets or some forms of adjusted balance sheet items, EC fundamentally concentrates on capital, various types of risk and their relations without considering the existence of assets and its basic structure. In essence, EC models are efficient risk management tools that help in identifying risk exposures and optimizing profitability in all of the business activities.

There are two main approach used to derive the risk profile of an organization. First approach is developed by modelling all components of risk and value together in a holistic manner, and the second approach is based on a silo approach where each risk category or type is analysed separately (see Figure 1.5), and then allowing for dependencies between risk categories. The fundamental difference in these approaches is: while a silo approach attempts to estimate capital required for each risk category first, before combining in some way to give an overall capital requirement, a holistic approach estimates overall capital required first before allocating back to risk category or line of business (or other classification) if required.

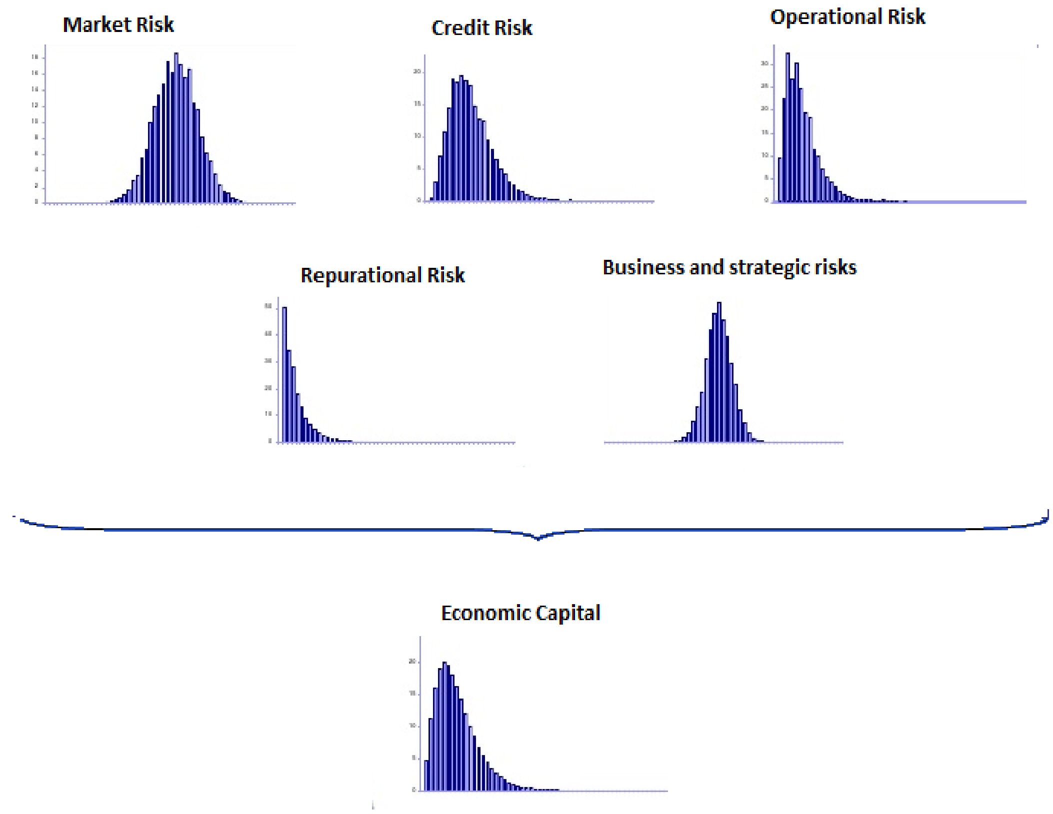

Figure 1.5: One of the approaches to derive EC (silo approach) is based on estimating a risk profile for each risk category and then combined to give an aggregate risk profile, from which an overall capital requirement is obtained by applying a risk metric to the risk profile.

In general, the EC modelling has three key elements:

- **A comprehensive identification of risks**: Risk identifications can be employed, for example, through process mapping that is the key preliminary step to risk mapping, or scenario analysis as it allows risk managers to go from an abstract representation of generic events to concrete examples of what can go wrong and how.
- **The selection of suitable (and comparable) risk measures**: After the estimation a loss distribution, risk is typically quantified using standard measures of risk, i.e. Value–at–Risk (VaR) and Expected Shortfall (ES). VaR refers to the quantile of the loss distribution and thus denotes the loss amount that will not be exceeded

within a given confidence level, while ES is the expected value of all losses greater than this quantile (see Fig. 1.6). The quantile used to determine VaR is generally derived from the bank target external rating or from the supervisory provisions for Pillar 1 minimum capital ratios (typically 99.9% for credit risk and operational risk and 99% for market risk, but this varies across institutions).

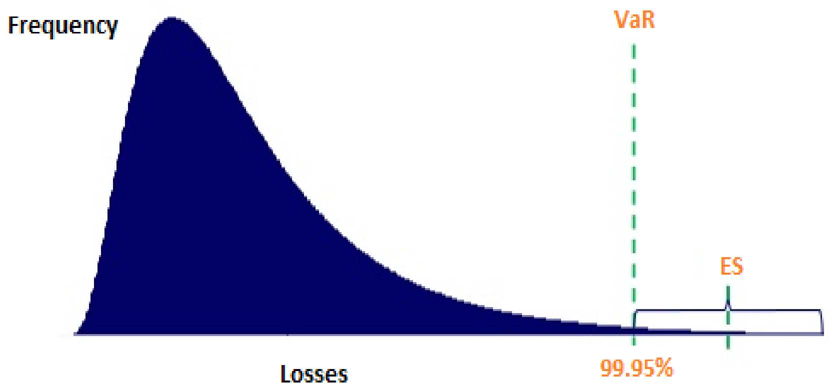

Figure 1.6: Comparison of two standard risk measures: Value–at–risk (VaR) and Expected Shortfall (ES).

These statistical measures are difficult to apply in situations when the potential losses cannot, either analytically or empirically, be easily modelled through a probability distribution over a given time horizon (e.g., model risk, reputational or strategic risk).

- **The development of the aggregation methodology**: The aggregation methodology should account for relations and diversification while at the same time producing a capital charge that can be reliably allocated depending, in the first place, on the selected risk classification structure. This structure may follow the economic nature of the risks or the organizational structure of the institution. The aggregation methodology is then be applied across the different risk types, across business lines or both. The aggregation requires first that measures are comparable. As risk measures are typically expressed through probabilities at a given confidence level over a certain time horizon, comparability means that risk measures need to be consistent in terms of all these features. Examples of the aggregation methodologies include summation, constant diversification, variance–covariance, copulas or full modelling/simulation of the common risk drivers on all risk components and construction of the joint distribution of losses. The two most common methods used to model dependency structure are correlation coefficients and copula approach.[17] Modelling dependency structures and interrelations between different risk types are further discussed in Section 4.3.

As such, EC modelling requires a risk profile (distribution of outcomes of a relevant statistic), a risk measure, and a risk tolerance or appetite (such as loss no more than in every 250 modelled outcomes). A risk profile can be either a large number of simulated outcomes, or a distribution fitted around some statistical measures, such as the first and second moments of an assumed distribution.

There are several different approaches to calculate economic capital that combine both qualitative and quantitative aspects, however, they all have similar theoretical support based on measure theory and may be viewed as

[17] Applying correlation coefficient is equivalent to applying a Gaussian copula.

different paths to accomplish the same goal. Economic capital is estimated either deterministically or using Monte Carlo simulation, where key variables are randomly selected from statistical distributions. Investment scenarios are often modelled as time series to allow for temporal dependencies, with mean reversion as the underlying assumption. Typically, investment scenarios are modelled separately to other risk categories using an economic scenario generator, the output of which serves as inputs into the capital model.

To take an example: an approach for ES may be based on the evaluation exposition of the bank in terms of losses through different scenarios which take into account cyclical conjectural effects, as well as the projected evolution of several macro–economic indicators. Afterwards, to each scenario an expected probability is assigned to weight its relative importance. Scenarios are modelled using the Monte Carlo methods (Figure 1.7), and based on the risk appetite and the mission statement defined for the banks, the confidence level acceptable to cover exceptional losses is prescribed.

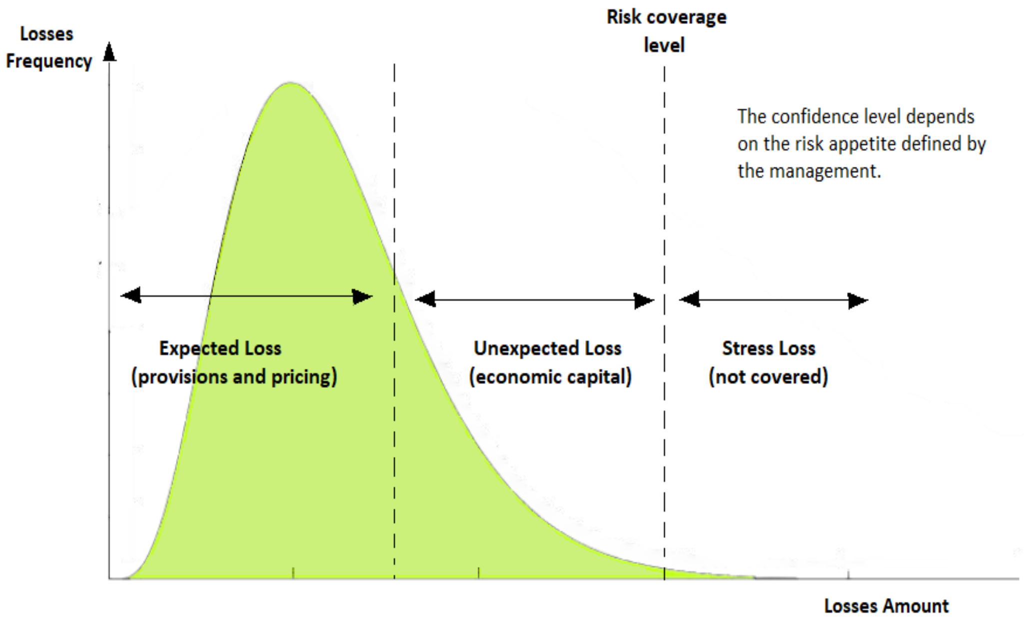

Figure 1.7: Loss distribution: Expected and Unexpected Loss.

Another viable methods are for example stress test method, factor tables, stochastic models (univariate or multivariate stochastic model), statistical method (mean–variance–covariance model) or frequency and severity/recovery models. It should however be taken into consideration that all these models are nothing more than methodologies, and cannot be regarded as autonomous decision–making processes.

To aggregate the different risk measures in order to derive a view of risk for a whole financial institution, has proven to be challenging. Financial institutions find it difficult to aggregate firm–wide risks for mainly the following reasons:

- The shape of individual distributions differs for various types of risk. While distributions for market risks are typically close to symmetric but with fat tails, distributions for credit risks and for operational risks are extremely skewed. Such differences in risk distributions usually make it inappropriate to use simple portfolio formulas to aggregate these individual risk measures as means, variances, and covariances are not sufficient statistics for these risk distributions.

- Conditional correlations of different types of risk are difficult to measure with confidence. For example, the historical evidence suggests that bad–tail market and credit risk events are correlated, even though not perfectly, and historical data do not cover enough potential states of the world. In addition, correlation may not be the appropriate measure of dependence between various types of risks due to their fat tails as tail outcomes of different types of risk could be more correlated than other outcomes.

- Certain risks are estimated over different time horizons, but to aggregate risks at the firm level they need to be forecasted over comparable periods. For example, the focus in market risk is generally on days; for credit risk and operational risk is often on one budget year.

- When risk categories are modelled separately and then combined to give an aggregate scenario, the assumptions underlying each risk category in any one scenario may not always be consistent.

The standard aggregation approaches are Monte Carlos simulations with copulas that also more accurately reflect diversification benefits.

EC models are crucial to pursue all business activities for any financial institutions as they reflect the risks specific to the institutions. The usage of these models is not limited only to the risk management purposes as they are used in a number of different business contexts. Generally, they are used as:

1. **Risk management tool**: EC models provide a quantitative amount (money) for the risk that is used by risk management to absorb unexpected losses in different business transactions and activities, portfolios, and institution–wide asset. EC enables to assess the overall capital adequacy regarding the risk profile of the financial institutions.

2. **Indicator in decision and strategy making:** The calculation of EC involves risk–return analysis, i.e. explaining and quantifying the circumstances that a financial institution assesses the risk–return results, and as such, EC models can be used to determine the profit margins for a transactions, portfolio or asset over the costs arising from business activities and market conditions.

3. **Economic valuation of assets and liabilities**: mark–to–market, mark–to–model rather than book values.

4. **Performance measurement:** EC models are used to determine risk adjusted performance measures called the Risk–Adjusted Return on Capital (RAROC), the Return on Risk–Adjusted Capital (RORAC) and the Risk–Adjusted Return on Risk–Adjusted Capital (RAROCAC). For example, RAROC is defined as

$$RAROC = \frac{Profit\, -\, Privision}{EC}.$$

The RORAC methodology is often used for calculating the consumption of economic capital and the return on it, for budgeting the capital consumption and for analysing and setting prices in the decision–taking process for operations and clients [107]. The assessment of the RAROC measure, the most general of these measures, involves calculation of the overall required capital position of the company which is then allocated to the various product lines and business units. The total capital is then aligned with the allocated division amounts.[18]

When properly designed and employed, EC models can improve risk management and overall position of a financial institution. A robust EC framework provides the management of an institution with an effective tool with respect to decisions, support, and pricing at the product level, a performance measurement at the business unit level, a strategic development in helping to identify creation of shareholder value within each business activity, and it contributes to the assessment of the overall capital adequacy of the organization. Nevertheless, the use of these models exposes an institution to model risk. These models may suffer from data limitations in terms of both quantity and quality, erroneous assumptions, other weaknesses in methodologies, inappropriate aggregation methodology, inability to sufficiently quantify risks, or potential misuse or misunderstanding of model outputs.

For example, the EC models are typically calibrated on the basis of historical data and experience which may lead to situations in which the model is unable to accurately map individual risks owing to an unprecedented market disruption; or the correlation between types of risks, which leads to reduction in the amount of overall capital, may not be properly factored.

Besides, parameter uncertainty arises as economic capital models typically contain variables which are selected both deterministically and stochastically based on analysis of relevant data and the application of expert judgement.[19] This uncertainty can be allowed within models in a distinct ways, from explicitly allowing for some random variability in the parameters, to choosing parameters with conservative margins embedded, or applying loadings to results. Some measures of uncertainty may be at time undefined; for example, a number of statistical distributions appropriate for insurance modelling have infinite second moments.

Understanding the capabilities, weaknesses and limitations of the underlying assumptions is key when dealing with these models and their outputs (see Part II for more details on this topic). As most of the financial institutions recognize the imprecision inherent in such estimation and their need to be responsive to potential changes in condition, banks generally operate with a capital cushion above the level of risk measured by the economic capital model.[20] The extent to which held capital exceeds economic capital can be considered to reflect the skepticism regarding capital modelling, allowance for model risk (see Chapter 6 on how to objectively quantify model risk) and/or capital to support modelled risk.

[18]The RAROC is also used for calculation of the level and evolution of value creation (VC),

$$VC = Profit - Average\ economic\ capital \times Cost\ of\ capital$$

which is the profit generated above the cost of the economic capital employed [107]. The profit used is then obtained by making the required adjustments by accounting profit in order to reflect only the recurring profit obtained by each unit from its business activity.

[19]For example, in the case of stochastic modelling, the exercise of judgement falls not to the selection of assumptions but to the choice of the statistical distribution from which random selections of assumptions are generated.

[20]Several factors can be considered in determining the appropriate cushion, including the robustness of the economic capital methodologies, the quality of data, volatility of the business model, the composition of capital, and external factors, such as business cycle effects and the macroeconomic environment.

1.3 CONCLUSIONS

Capital adequacy is a fundamental consideration in terms of both regulatory and economic capital. The concept of bank capital comprises the processes and methodologies through which a financial institution assesses the risks generated by its activities and estimates the necessary amount of capital against those risks. These processes and methodologies do not typically emerge into a single, well–identified mathematical or statistical model, but rather consist of a combination of techniques whose outputs are aggregated on the basis of assumptions that can be both complex and strong.

Decision making in capital management process often requires quantiles and extreme scenarios that can only be obtained as outputs of a quantitative model; for example, to calculate capital requirement via a VaR principle, a probability distribution of future values of a relevant portfolio quantity needs to be determined. These models inevitably employ a large number of simplifying assumptions and approximations, and their performance is constrained in several ways. EC models, therefore, need a well conceived model risk management with an objective assessment of model risk. As EC assessment is based on measure theory, the framework developed in Chapters $6-9$ for the quantification of model risk, data uncertainty assessment and mitigation of model risk can be employed and so the credibility of these models can be objectively assessed.

Capital requirements are highly important and useful. However, one should keep in mind that capital alone cannot prevent a crisis, not even with the combination of effective risk management. After all, financial institutions have intermediary role in society and in order to accomplish this role they need to preserve reliable capital ratios by keeping up their earnings which in turn can only be generated by constant serving their clients.

Chapter 2

Credit and Market Risks

This chapter briefly introduces the principles of credit and market risks, discusses the main methodologies and approaches for their measurements and for the calculation of capital while relating these concepts to model risk and the proposed framework for its quantification considered in this dissertation (Chapters $6-9$). Furthermore, the nature and the scope of the considered problems is clarified, without attention to details.

2.1 Credit Risk

Credit risk refers to the potential loss arising from the failure of a client or counterparty to meet its contractual obligations in accordance with the agreed terms [11]. In a broader sense, credit risk can be defined as the potential loss emerging either from a default of the borrower/issuer or a decrease of the market value of a financial obligation due to a deterioration in its credit quality. Credit risk management, meanwhile, is the practice of mitigating those losses by understanding the adequacy of both an institution capital and loan loss reserves at any given point in time — a process that has long been a challenge for financial institutions.

In banking industry, credit risk is a predominant issue as it is accounted throughout the activities of a bank, both in the banking book and the trading book, and on and off the balance sheet. It arises from all transactions where actual, contingent or potential claims against any counterparties (counterparty, borrower, obligor or issuer) exist, including those claims that the institution plan to distribute. In addition, it results from traded bonds and debt securities of its direct trading activities with clients (e.g., Over–the–Counter (OTC) derivatives like foreign exchange forwards). In general, there are two types of transactions that lead to credit risk:

- **Lending transactions**, giving rise to *counterparty risk*, that is usually defined as the risk that the value of a portfolio changes due to unexpected changes in the credit quality of trading counterparts, i.e. outright default or down–trading.

- **Trading transactions**, leading to *issuer* and *settlement risk*. Issuer risk is the risk that payments due from the issuer of a financial instrument will not be received. Settlement risk is the risk that settlement of a transaction does not take place as expected, with one party effecting settlement as they fall due but not receiving settlements to which they are entitled.

Credit risk may also include *industry risk, country risk* and *product risk*:

- *Industry risk* refers to the risk of adverse developments in the operating environment for a specific industry segment leading to deterioration in the financial profile of counterparties operating in that segment and resulting in increased credit risk across this portfolio of counterparties.
- *Country risk* is the risk that the institution may experience unexpected default or settlement risk and subsequent losses, in a given country, due to a range of macro–economic or social events primarily affecting counterparties in that jurisdiction including: a material deterioration of economic conditions, political and social upheaval, nationalization and expropriation of assets, government repudiation of indebtedness, or disruptive currency depreciation or devaluation. Country risk also includes transfer risk which arises when debtors are unable to meet their obligations owing to an inability to transfer assets to non–residents due to direct sovereign intervention.
- *Product risk* captures product–specific credit risk of transactions that could arise with respect to specific borrowers or group of borrowers. It takes into account whether obligations have a similar risk characteristics and market place behaviours.

For most banks, loans are the largest and most obvious source of credit risk; however, financial institutions are increasingly facing credit risk (or counterparty risk) in various financial instruments other than loans, including acceptances, interbank transactions, trade financing, foreign exchange transactions, financial futures, swaps, bonds, equities, options, and in the extension of commitments and guarantees, and the settlement of transactions.

The first step to model credit risk is the segmentation process. Within the proposed regulatory framework, the Internal Rating Based (IRB) approach, in both Foundation or Advanced version, obligors are assigned into classes based on the ratings, i.e. assigning debtors to classes according to their creditworthiness. This approach methodology represents an expert–based approach. Alternative classification framework refers to the scoring system, a statistically based methodology, that is of more quantitative nature. Banks typically follow some kind of combination of the two, whereby a mix of quantitative and qualitative components are compound within a statistical framework that also incorporates expert inputs or adjustments. These approaches identify classes, so called segments, to which the various debtors belong, in such a way that debtors to the same class shall have, in principle, the same credit risk (see Subsection 2.1.1 for more details).

Financial institutions typically model credit risk based on the estimation of the key parameters as required under the Basel Accord:

- **The probability of default (PD)** - the probability that a particular company will default within one year (see Subsection 2.1.1),
- **The loss given default (LGD)** - the losses incurred on a defaulted loan as a percentage of the outstanding balance at time of default (see Subsection 2.1.2), and
- **The exposure to default (ED)** - the expected outstanding balance if the institution defaults that is equal to the expected utilization plus a percentage of the unused commitments (see Subsection 2.1.3).

The risk measures to be used in credit risk models are therefore both accepted and standardized. However, related definitions and estimation approaches are not homogeneous across financial industry and are surrounding with many challenges.

Definition of what constitutes a default varies across industry. According to the BCBS, a default event is defined as occurring when the obligor is either more than 90 days late in its payments or is unlikely to pay its obligations. Theoretically, while the first part of their definition may or may not imply losses, the second part implies a rather subjective judgement which may or may not turn to be correct. Other default criteria used inside the industry are, for example, a firm is considered to be in default if the value of its assets falls below that of its liabilities (e.g., the Merton model); actual bankruptcy. Regardless the default criterion used, the definition of default needs to allow for the fact that obligors may in the end pay in full while technically being in bankrupt.

LGD is the loss on a credit exposure when the counterparty defaults. Such loss, however, can be estimated in various ways. One way consists in considering the recovered part over the debt workout period discounted to the default date, which requires the estimation of a cash flow that is not, in principle, known in advance as well as the selection of discount rate. Other alternative is to estimate LGD from the market price of bonds or other tradable debt after the default has occurred[1], or to use the market value of risky, but not defaulted bonds using theoretical asset pricing models that, however, come with their own risk (see Section 2.3).

EAD represents the actual exposure at the time of default. Its computation requires a facility–specific approach, i.e. distinguishing between fully drawn lines, secured loans, undrawn lines, derivatives, guarantees and other off balance sheet items. The approach to EAD estimation should consider simultaneous variations between PD and EAD which is especially important in case of derivative instruments, where credit exposure are particularly market–driven.

There may likely be dependence between EAD and LGD. The LGD is often modelled as fixed percentage of EAD, with actual percentage depending on the seniority of claim, but in practice, LGD is not constant and should be modelled as a random variable or as dependent on other factors.

There are broadly two main types of credit risk models: *reduced–form* and *structural models*. *Reduced–form models* view default as a part of random process, i.e. default events are assumed to occur unexpectedly due to one or more exogenous events, independent of the borrower asset value, while LGD is determined exogenously. These models are data–driven and model the PD separately from the corresponding loss and then produce an aggregate measure of risk through numerical methods.

A *structural model*, also known as asset value model or option–theoretical model, considers default risk to be a European put option on the value of the institution asset. When these fall below the level of the institution debt obligations at maturity, a default occurs (see Fig. 2.1). Credit risk, as well its key components like PD and LGD, is a function of the structural characteristics of an institution, such as leverage or asset volatility. Key instances of structural models are the Merton and the KMV model.

The parameters of the models are determined either empirically, by applying mathematical methods to known data, or on the basis of expert judgement. In case where an institution does not have sufficient data to build a model statistically, a set of risk factors are selected by experts. These factors may include some financial

[1] This approach can be employed when the counterparty is a large organization whose bonds are traded on the market even after a bankruptcy.

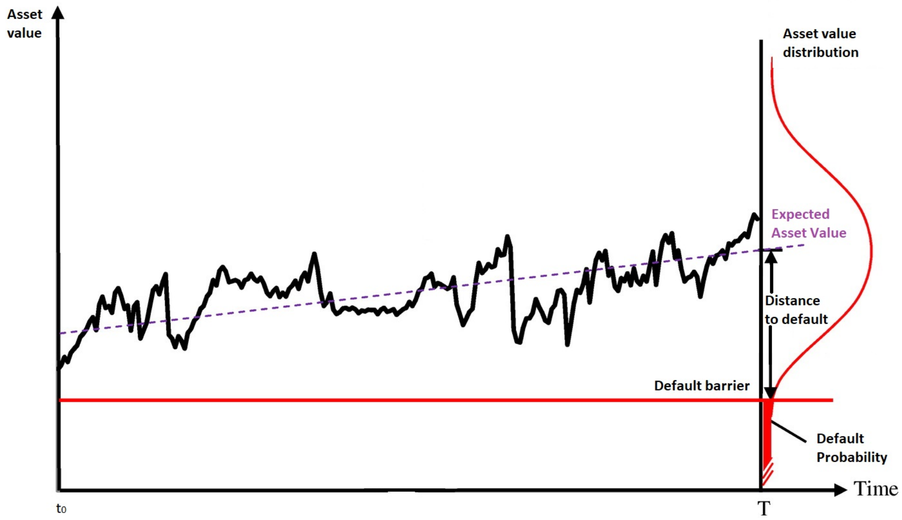

Figure 2.1: Asset value process in the Black–Scholes–Merton model. In this model, a default occurs whenever the asset value settles below the default barrier at horizon.

factors (e.g., balance sheet ratio), non–financial factors (e.g., financial flexibility), or behavioural factors (e.g., credit utilization).

2.1.1 Probability of Default Models

The PD is the probability of obligor defaulting on its contractual obligations within a predetermined period of time.[2] PD ratings are used for credit risk measurement and are an important and key input for determining credit risk approval authorities. For the calculation of RWA, a 3 basis–points PD floor is applied to banks, corporates and retail exposures as required under the Basel III framework.

PD is assessed using rating or scoring tools tailored to the various categories of counterparties. Where available, market data may also be used to derive the PD for large corporates. For low default portfolios, where available, relevant external default data should be included in the rating tool development.

Rating systems can be either statistical models or expert–based approaches, or the combination of both, that classify obligors in different rating categories. They may be aimed at measuring the creditworthiness of a borrower either in its current state in its current economic environment or over a complete economic cycle. The former approach is called Point–in–Time (PIT), while the latter is called Through–the–Cycle (TTC) approach.

A fully TTC model should display close to a fixed distribution of PDs per rating grade (conditional probabilities per rating) over time, and obligors would migrate from grade to grade only for idiosyncratic reasons. This means that grade migrations are assumed to be uncorrelated to the changes due to the economic cycle. A fully PIT model, on the other hand, explains changes in default rates as arising from changes in the ratings and conditional PDs of many obligors or, alternatively, that the same rating grade corresponds to different levels of creditworthiness, depending on the state of economy. In other words, cyclical changes will be reflected as a large–scale ratings

[2]Currently, 12 months period is required.

migrations while PDs per rating grade will be more stable over time. In practice, rating models are a combination of TTC and PIT characteristics to reflect both cyclical and idiosyncratic risk factors.

The Basel Accord description that PDs be based on a long–run average of one–year default rates should in fact be a feature of both PIT and TTC models. Both estimations should be done by averaging over many years of data[3], and a model would be considered as PIT to the extent that it includes explanatory variables that track the credit cycle. This means that it is the combination of risk factors that determines where on the PIT/TTC scale the model is located, and the difference between the two is very much a matter of model design rather than of mere PD calibration.

There are several problems that may arise during the estimation of PD including wrong asset correlation level and/or structure, bias from small sample size, flaws in the segmentation (e.g., rating class is not homogeneous in terms of PDs), or wrong assumption about normality of asset returns [98], just to name a few. Besides, due to the various definitions of default event, one of the main challenges faced by banks is to verify historic loan performance information against the definition of default used by the regulator. Estimation of PD depends above all on the macroeconomic forecasts (e.g., unemployment, GDP growth rate, interest rate) that are outputs of other models with their own limitations and weaknesses, and obligor specific informations (e.g., financial ratios/growth, demographic information). Each estimate of PD represents a conservative view of a long–run average PD for each obligor grade and banks need to justify their PD estimates with sufficient historical experience and empirical evidence.

2.1.2 Loss Given Default Models

Loss given default (LGD) is the magnitude of the potential loss in case of default. BCBS defines LGD as a percentage of the exposure of default (EAD) while taking into account all relevant factors, such as material discount effects, material direct and indirect costs associated with collecting on the exposure. The LGD estimate should reflect economic downturn conditions where necessary to capture the relevant risks. The Basel Accord requires tha t the estimation of LGD is based on observed loss rate estimates and in any case be not less than the long–run default–weighted average loss rate given default that was determined based on the average economic loss of all observed defaults within the data source for that type of facility.

A LGD model can be constructed on the bases of a regression analysis of historical default data against a number of potential drivers of LGD based information of defaulted borrowers available in external, internal, pooled data sources, or a combination of the three. The required data observation period consists of at least five years for retail exposures and at least seven years for corporate, sovereigns and bank exposure so to ensure that they cover at least one whole economic cycle.

A LGD estimate can be build by performing a statistics of the empirical distribution or by regression–based modelling of a relationship between realized LGDs and other variables present in the reference dataset, among which are interest rate, GDP, or loan to value ration.

In a like manner to PD estimation, there are several modelling approaches to determine LGD that can be classified into explicit and implicit approaches. Explicit approaches estimate LGD directly from the available loss

[3]The minimum length of underlying data for this analysis is five years, however if more relevant data is available it must be used.

data[4], while implicit approaches infer LGD from data that contain relevant information.[5] In both cases, models may be based either on market data and/or internal data.

The authors in [46] list the following approaches for the estimation of LGD:

- **Parametric methods:** ordinary least squares regression, ridge regression, fractional response regression, Tobit model or decision three model.
- **Transformation regressions:** Inverse Gaussian regression, Box–Cox transformation/OLS, Beta transformation/OLS, Fractional logit transformation and Log transform.
- **Non–parametric approaches:** Regression tree, Neutral networks, Multivariate adaptive regression spline, Least square support vector machine.
- **Semi–parametric:** Joint Beta Additive model.

Regardless of the approach used, models may rely on statistical analysis of data or on expert judgements. Although expert judgment is present in any modelling approach, expert–based methods are applied when data is unreliable or unavailable, as is the case of low default portfolios. They may consists of a combination of interviews, peer reviews, expert panels, scenario analysis, reviews of industry as well as technical literature. The reference data needs to cover at least a complete business cycle, contain all the defaults produced within the considered time frame, include all the relevant information to estimate the risk parameters, and to include data on the relevant drivers of loss. The sources of external data, used when internal data are insufficient or absent, typically include external rating agencies and consortia of banks created for pooling of default and loss, e.g. Moody's or Standard and Poors.

2.1.3 Exposure at Default Models

BCBS [92] defines Exposure at Default (EAD) as the size of the credit risk exposure that the bank expects to face on a facility assuming that economic downturn conditions occur within a one–year time horizon and the associated borrower defaults on its obligations within that horizon. As such, EAD does not only refer to the current exposure that an institution faces today but it also reflects an estimate of the total future exposure that an institution would face in case of a default event under downturn conditions within the predetermined time horizon.

EAD models comprise wide range of approaches depending on the type of product considered. The estimation differs for credit risk and counterparty risk, for both on–balance sheet (drawn amount) and off–balance sheet (undrawn amount). In what follows we focus on EAD estimation for credit risk under the A–IRB approach. For the EAD calculation in case of financial derivatives see Section 2.3.

Credit risk is present in many types of facilities that either produce **fixed** or **variable exposure**. A fixed credit exposure is one where an institution has not made any commitment to provide credit in the future, while a variable credit exposure is one where such commitment, in addition to the current credit, was made. The EAD on a

[4] Explicit approaches include market LGD which is based on comparing market prices of defaulted bonds or of tradable loans shortly after default with their par values, and workout LGD, where all post–default recoveries are discounted in order to estimate the value of the defaulted deal to be compared with the associated EAD.

[5] Implicit approaches comprise implied market LGD based on credit spreads on non–defaulted bonds and implied historical LGD, as suggested by Basel II for retail exposure, that use realized losses and an internal long–run PD estimates.

fixed credit exposure is the EAD on the on–balance sheet exposure which is constrained from above by the sum of the amount by which a regulatory capital would be reduced if the exposure were written–off fully and any specific provisions and partial write–offs [92]. A variable exposure, on the other hand, may default when additional money has been drawn and thus cause a loss higher than the current drawn amount.

An EAD estimation consists of two components, namely the exposure on the drawn amount on the facility at the time of estimation and the potential exposure that may materialize in the future due to further withdrawals, subject to any relevant contractual covenant, on the facility by the borrower until it defaults, if it ever does, within a one–year time horizon.

The first component, on–balance sheet exposure, is equal to the book value of the drawn amount on the facility and is known with certainty. The drawn EAD on a facility is considered to be equivalent to the sum of *the current drawn amount on the facility* and *the interest and the fees that are expected to be collected from the borrower within the one–year horizon.*

The second component, off balance sheet exposure, is conditional on the default event and economic downturn conditions. It is not known a priori for the facilities related to a non–defaulted borrower and has to be, therefore, estimated. The undrawn EAD is a product of two factors: *committed but as–yet unpayed amount on the facility* and *a Credit Conversion Factor (CCF)* that represents an estimated ratio of the unpayed amount that will be converted into an effective on–balance sheet amount by the time the borrower defaults, if it ever happens, under economic downturn conditions within a one–year time horizon. For an A–IRB bank, the CCF is the key input for the EAD estimation, and it needs to be expressed as a percentage of the off–balance sheet amount of the commitment.[6] CCF should take into account drawdowns following a default when these are not reflected in the LGD estimation.

There are no regulatory requirements as to the design of the model for the CCF estimation, however, there are number of regulatory requirements focusing on how to define CCF, how to assign CCF to facilities and how to build the reference dataset used for estimation. Figure 2.2 summarizes the estimation process. Nevertheless, a

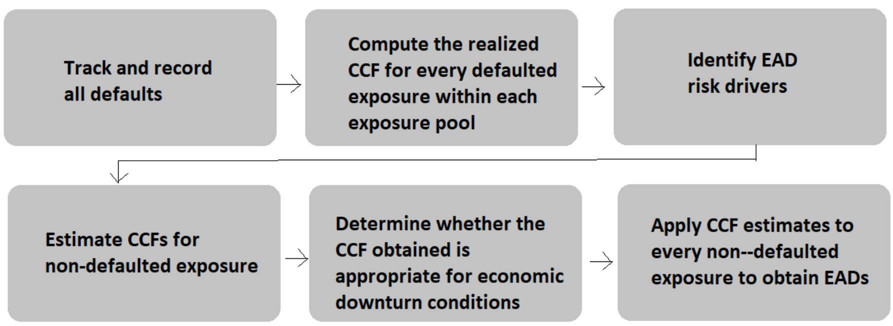

Figure 2.2: The key steps of the EAD estimation process.;

number of approaches exist that banks can use, depending on the effectiveness and efficiency of their processes, default data management especially. The standard methodology for a CCF estimation is based empirical approach using a dataset of observed EAD of defaulted facilities at the date preceding the default. One way how to compute

[6]The CCF shall be non–negative or, alternatively, that the EAD of an exposure cannot be less than the current outstanding amount.

the CCF at the reference time t_r is simply by

$$CCF(t_r) = \frac{EAD(t_d)}{ExpMax(t_r)}$$

where $EAD(t_d)$ is the observed exposure at time of default t_d and $ExpMax(t_r)$ is the limit of the facility at the reference time t_r.

Other alternatives include fixed–time horizon, variable–time horizon and cohort approaches. In case of severe lack of data and/or highly specialized portfolios, the CCF may be estimated entirely based on expert judgement, external information on defaults and recoveries, the expected cash flow on the facility or combined with the direct estimates. The complexity of EAD or CCF models is largely caused by their dependence on the contractual details of the underlying products and the specific relation with the borrower.

2.1.4 MAIN SOURCES OF MODEL RISK

There are various sources of model risk embedded in the different components of the credit risk modelling process: data, model design that includes the choice of model, the key assumptions and the documentation as the relevant factors, and various usages of credit risk models.

Data. Reliability of the results heavily depend on the availability and quality of relevant data. Main sources of model risk are incomplete databases, low numbers of observations, the lack of sufficient statistical data, the resulting difficulty in backtesting risk parameters, problems with external data - not reliable all the time.

In absence of reliable and sufficient data, a bank needs to use an external sources of data that may be in several forms: external ratings, data pooling or internal assessment. In this case, one needs to assess to what extent these data are relevant to the portfolios and the contained information is representative, the consistency of default definitions as external providers and rating agencies will not necessarily use the same definition as the one internally used in the bank.

The predictive power of the model may depend heavily on the size of the sample of loans, as the observed distribution of losses may have a larger variance than the model shows. Thus the model could underestimate the corresponding exposure.

Model design. There is number of different modelling approaches that range from expert–based to more complex and sophisticated ones, all having their limitations, advantages and disadvantages that need to be taken into account. More sophisticated approaches involve formal modelling using regressions or more complicated techniques such as neural networks or machine learning. Highly complex models are often prone to data–mining and over–fitting, meaning that field or out–of–sample performance can be quite poor relative to the model fit (or in–sample performance). Basic regression models tend to be more robust than complex approaches but at the cost of lower accuracy.

For example, in the case of PD estimation, probabilities of default are not observable and so need to be inferred from observed default rates by applying the law of large numbers. However, for each exposure, such observations will typically happen once in a given year. Besides, the law of large numbers assumes that events are independent and identically distributed; two conditions that are in general not fulfilled. The first since defaults are always correlated to some extent, and especially so in times of crisis. The later due to the fact defaults will always, at least in part, depend on idiosyncratic causes and therefore be generated by different probability distributions.

Next, segmentation approach and adopted definition of default will of course significantly impact the results of any exercise.

Usage. The credit models are used not only for the capital assessment, but as well for pricing, stress testing, fair value accounting as well as for risk management and reporting. It is, therefore, important to be aware whether a particular model is used for the intended purposes, but also if the model output itself is appropriate for the different uses, where different estimates (like downturn or through–the–cycle), different time horizons, different regulatory and accounting rules may apply, or distinct pricing practices and default definitions may be required. For example, when employed for accounting purposes, the credit risk models are subject to the International Financial Reporting Standards (IFRS) that differ from Basel framework, among other things, in definition of default, time horizon for estimates, and discount rates (for more details see Section and Table 4.1 for the differences between Basel Accord requirements and IFRS 9).

Documentation. The importance of completeness and quality in the documentation should not be underestimates as it reflects both the diligence around model development and maintenance, and the level of understanding of the model complexity. It is especially important when expert–based and external sources are employed.

Needless to say, all these challenges are magnified for those portfolios that typically exhibit low number of defaults due to scarcity of default observations and subsequent need for numerous assumptions.

2.2 Market Risk (Trading)

Market risk is defined as the potential loss due to changes in the market value of both trading and invested positions, i.e. the risk of loss resulting from adverse movements in market variables. Risk can emerge from changes in interest rates, credit spreads, foreign exchange rates, equity prices, commodity prices and other relevant parameters, such as market volatility and market implied default probabilities.

In general, we can distinguish between three substantially different types of market risk:

- **Trading market risk**: results primarily from the market–making activities and client facilitations such as taking positions in debt, equity, FX, other securities and commodities as well as in equivalent derivatives.
- **Traded default risk**: arises from defaults and rating migrations relating to trading instruments. See Section 2.4 for more details.
- **Nontrading market risk**: arises from market movements, predominantly outside the activities of the trading units, in the banking book and from off–balance sheet items. This includes interest rate risk, credit spread risk, investment risk and FX risk. as well as risk from the modelling of client deposits savings and loan products.

Market risk, thus, encompasses the interest rate risk (the risk of adverse movements in interest rates), the foreign exchange risk (changes in currency exchange rates and foreign interest rates affect foreign currency denominated instruments), credit spread risk (changes in credit spreads affect bond pricing and loan pricing), volatility risk (variations in volatility impact financial instruments with embedded optionality, such as mortgage–backed securities), commodity risk (changes in crude oil prices affect both credit spreads of oil producing companies and commodity swap valuations), just to name a few [2]. The following Table 2.1 summarizes risk factors for trading and banking book[7]:

BANKING BOOK		TRADING BOOK
Funding	**Investment**	Interest rate risk
Interest rate risk	Interest rate risk	Credit spread risk
Foreign exchange risk	Credit spread risk	Foreign exchange risk
	Foreign exchange risk	Equity risk
	Equity risk	Commodity risk

Table 2.1: Risk factors divided according to banking and trading book.

Market risk management (MRM) is aimed to define and implement a framework to systematically identify, assess, monitor and report the market risk. Managers identify market risks through active portfolio analysis and engagement with the business area. All types of market risks should be measured by a comprehensive set of risk metrics reflecting economic and regulatory requirements, through internally developed key risk metrics and regulatory defined market risk approaches.[8] The primary objective of MRM is to ensure that the risk exposure is within the approved appetite of the institution proportionate to its defined strategy.

2.2.1 Data, Models and Methodologies

Market risk is the risk of an increase or decrease in the market price of a financial instrument or portfolio, due to changes, e.g., in stock prices, interest rates, credit spreads, or implied volatilities. Market risk measurement aims therefore at predicting the distribution of potential future changes in value of a financial instrument over a given time horizon. Market risk is usually measured and managed through a comprehensive framework that may range from statistical to non–statistical measures and related limits. Statistical approaches may vary from distribution moments like volatility, skewness, kurtosis to Value–at–Risk (VaR), Conditional VaR (CVaR), Tail VaR to Expected Shortfall (ES). Risks that are not quantified in a VaR methodology such as proxy risks, basis risks, risks from calibration parameter errors, and other higher order risks are usually scoped out separately using a Risks not in VaR (RNIV) framework. Non–statistical approaches include series of stress tests and scenario analyses.

In general, market risk models use mark–to–market approach in which market information is used as input. The quality of the market risk figures obtained from the model is highly dependent on the valuation models employed in the risk calculation. Hence, using appropriate valuation methods for all financial instruments is a prerequisite for

[7]The trading book consists of all positions in financial instruments and commodities held either with trading intent or in order to hedge other elements of the trading book. Positions held with trading aim are those held intentionally for short–term resale and/or with the intention of benefiting from actual or expected short–term price differences or locking in risk–free yields [113]. All positions other than those included in the trading book are regarded as banking book positions, or sometimes called non–trading book positions. These positions consist of positions relating to lending and long–term investments, normal assets and investments relating to business activities.

[8]To determine market risk for regulatory capital requirements, a bank may choose between two broad methodologies: the standardised approach and the advanced approach, subject to the approval of the banking authorities. Under the advanced approach, banks may further choose between the Internal models approach (IMA) and the Standardized measurement method (SMM).

the adequate measurement of market risk. These models include, for example, interest rate models, valuation models for a synthetic CDO, including derivation of the default intensities from bond spreads, implied volatility or volatility surface, financial derivative models such as Black–Scholes model, Heston model, Variance Gamma, SABR, or Hull White model (financial derivatives are described in more detail in Section 2.3), among others. These models often rely on strong assumptions and thus are prone to model risk.

Development of a market risk model, typically, comprises the identification of the relevant risk factors, the estimation of a suitable multivariate distribution of the changes in the risk factors, the establishment of a functional relationship between changes in risk factors and changes in portfolio values. The output of a model is an univariate predictive distribution of the changes in portfolio value conditional on the risk factor variations. Measures are then extracted from the resulting distribution like α–quantile of VaR, or weighted average for the ES.

When assessing market risk exposure, banks use measures that capture losses under normal market conditions as well as measures that focus on extreme, stressed market situations. Since no methodology can cover all risks at all times, several approaches are used, and the outputs are assessed based on expert judgement and experience. Examples of the internally developed market risk models, also required by regulators, include VaR, Stressed VaR (SVaR), Incremental Risk Charge (IRC) and Comprehensive Risk Measure (CRM).

VaR can be estimated based on, for example, variance–covariance matrices, historical simulations, or Monte Carlo simulations. The minimum regulatory standards (currently Basel III [25]) require banks to compute VaR on their trading portfolios with a 99% confidence using an instantaneous price shock equivalent to a 10–day movement in prices, meaning that the minimum holding period equals ten trading days. Shorter periods may be used and then scaled up to 10 days by the square root of time. The minimum length of sample periods has to be at least one year, updated at least quarterly, unless otherwise states (e.g., in periods of higher volatility). Besides, VaR models used for internal purposes have to capture non–linearities in option positions and apply a full 10–day price shock to options positions or positions that display option–like characteristics.

In addition, banks are required to calculate a Stressed VaR (SVaR) measure that adopts broadly the same methodology as VaR, i.e. 10–day holding period, 99th percentile, one–tailed confidence level, but calibrated to historical data and observed correlations from a continuous 12–months period of significant financial stress (e.g. a 12–month period relating to significant losses in 2007/2008), that is however subject to regulatory approval. The BCBS suggests to estimate SVaR, for example, by using antithetic data or employment of absolute rather than relative volatilities. The capital requirement is then calculated as

$$K = \{VaR_{t-1}; m_c VaR_a rg\} + \{SVaR_{t-1}; m_s SVaR_a rg\}$$

where VaR_{t-1} is the previous day VaR, VaR_{arg} is the average of the daily VaR over the preceding 60 business days, $SVaR_{t-1}$ is the previous day Stressed VaR, $SVaR_{arg}$ is the average of the daily SVaR over the preceding 60 business days, and m_c and m_s are multiplicative factors determined by the supervisor.[9]

[9] The multiplication factors m_c and m_s are both set at a minimum level of 3, subject to the decision of local supervisory authorities who can raise them up to 4 on the basis of their assessment of the quality of the risk management system. The assessment will take into account only back–testing of VaR and not of SVaR.

VaR and VaR based models

VaR is a quantitative measure of the potential loss (in value) of fair value positions due to market movements that will not be exceeded over a predefined time period (holding period) at the established confidence level. VaR at confidence level $\alpha \in (0, 1)$ for a portfolio of financial instruments, can be defined as the inverse cumulative density function of the changes in portfolio values. Formally, let V_t be the portfolio value at time t, $\triangle V_{t+1} = V_{t+1} - V_t$ and $F(x) = \mathbb{P}(\triangle V_{t+1} \leq x)$ be the cumulative density function of $\triangle V$. Then VaR is given by

$$VaR_\alpha = F_t^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_t(x) = \alpha\}$$

VaR is thus the α–quantile of the inverse cumulative density function of $\triangle V$, representing the largest loss that is expected to occur with probability $1 - \alpha$. It is typically calculated on a daily bases using 99% confidence level. For regulatory purposes, the holding period is ten days at a 99% confidence level, subject to qualitative and quantitative requirements.

Note that technically, the level of confidence $(1 - \alpha)$ is not a property of the interval, but of the methodology used to obtain the interval itself. This means, that if we use one particular methodology over and over in order to estimate VaR, about $100(1 - \alpha)\%$ times the true value of VaR will fall within this interval.

Stressed VaR (SVaR) adopts broadly the same methodology as regulatory VaR and is calibrated to a continuous 12–month period of financial stress relevant to an institution portfolio. The selected stress period is the most material element determining the output of the model and is therefore subject to approval by the competent authorities [56]. The SVaR calculation utilizes the same systems, trade information and processes as those used for the calculation of VaR. The difference is that historical market data and observed correlations from a period of significant financial stress is used as an input for the Monte Carlo Simulation.

Even though by far the most widely used measure of financial risk, VaR is less than ideal both for theoretical (not coherent measure) and practical reasons (it is defined in terms of a parameter that is by and large arbitrary). From theoretical perspective, VaR is not a coherent risk measure,[10] failing the sub–additivity condition, i.e. the total risk of two portfolios may be larger than the sum of the risks of the individual portfolios. Alternative measures of risk are, for example, the Expected Shortfall (ES), a coherent measure defined as the average of the worst $\alpha\%$ of the cases and is given, for a continuous loss distribution, by:

$$ES_\alpha = \frac{1}{\alpha} \int_0^\alpha VaR_p dp;$$

or the Spectral Risk Measure (SRM) defined as a weighted average where worse outcomes are included with larger weights (for more details see e.g. [3]).

Selection of the relevant risk factors, as the first modelling step, requires knowledge of the portfolio and products in order to maximize not just their explanatory power, but also the operational efficiency of the resulting model. The trading business is typically subject to multiple market risk limits that take into account the extent of market liquidity and volatility, available operational capacity, valuation uncertainty and might include the credit quality of issuers.

In general, we can distinguish two VaR approaches: **parametric approaches** and **nonparametric approaches**. Parametric approaches make specific assumptions about the functional form of the probability distribu-

[10] A risk measure is called coherent if it satisfies following conditions [**?**]: normalization, translation invariance, sub–additivity, positive homogeneity and monotonicity.

tion of risk factors while nonparametric do not. The most common parametric approach is the variance–covariance approach, which is based on the assumption that changes in risk factors and portfolio values are normally distributed. VaR can then be easily computed from variances and covariances of the portfolio positions. The assumption of normality is, however, in contrast with the evidence that most asset returns are fat–tailed and may cause substantial underestimation of the true risk of a portfolio, especially if it contains positions that are nonlinear in the risk factors.

Another parametric family relies on Taylor expansion using sensitivities to approximate the changes in portfolio values with different order of approximation. For example, the Delta approximation containing the first–order derivatives with respect to risk factors and time is given by

$$\triangle V(\triangle F_{t+1}, X_t) \approx \frac{\partial V(\triangle F_{t+1}, X_t, t)}{\partial \triangle F_{t+1}} \triangle F_{t+1} + \frac{\partial V(\triangle F_{t+1}, X_t, t)}{\partial t} \triangle t$$

and the Delta–Gamma of the second–order is given by

$$\triangle V(\triangle F_{t+1}, X_t) \approx \frac{\partial V(\triangle F_{t+1}, X_t, t)}{\partial \triangle F_{t+1}} \triangle F_{t+1} + \frac{1}{2} \frac{\partial^2 V(\triangle F_{t+1}, X_t, t)}{\partial \triangle F_{t+1} \partial \triangle F_{t+1}} \triangle F_{t+1} + \frac{\partial V(\triangle F_{t+1}, X_t, t)}{\partial t} \triangle t$$

Non–parametric approaches rely on simulation techniques based either on a given distribution of $\triangle F$ or from historical data. Historical simulations are both simple and flexible, not making any assumptions about the shape of the distribution, however, fundamentally dependent on both relevance and availability of historical data. It is thus important to have a long enough historical data series to account for potentially severe movements, but at the same time, a too–long estimation period may cause recent, and presumably more relevant data, to be assigned insufficient weight in the computation.

A standard non–parametric method is Monte Carlo simulation where market factors are assumed to follow specific stochastic processes whose parameters are estimated on the basis of historical data. The output of Monte Carlo simulation is the entire distribution of changes in portfolio values without making any assumptions. With this approach, one can also account for any non–linearities in the portfolio to the extent that positions are fully re–evaluated for each stochastic draw of risk factors. Though, this is also a source of limitation in the approach in terms of computational time, the use of complex pricing models and the use of specific risk factor distributions.

Market VaR models are widely used across the financial industry, especially by institutions that actively trade. They are employed to assess portfolio risk, allocate capital, and evaluate alternative investment strategies. They are also part of internal control systems designed to detect excessive risk taking by individual units or traders, and often are part of incentive systems designed to optimize the level of risk taken by individual units or traders.

The output of a VaR model is affected by the intrinsic limitation of VaR as a risk measure, but also by the assumptions and approximation required to make it feasible for practical purposes and by the fact risk predictions are based on historical data that do not necessarily reflect the potential future evolution of the markets. However, unlike the case of credit risk models discussed in Section 2.1, VaR can be backtested by systematically comparing the model estimates to the actual profit and loss. Backtesting is performed in order to check the accuracy of the VaR calculation model, i.e. for accepting or rejecting the model used for estimating the maximum loss of a portfolio for a given confidence level and time horizon. Backtesting exercise consist of comparing the VaR forecasts, given a certain confidence level and time horizon, with the actual losses incurred over a time horizon equal to the VaR time horizon [8]. A lower confidence level, for instance 95%, instead of 99% is often used, i.e. the higher the confidence level the smaller the number of exceptions and the less reliable as a result is the test. This ensures a lower rate of type 2 error, i.e. accepting an incorrect model, for a given rate of type 1 error, i.e. rejecting a correct model.

Other market risk models

VaR is by far the most popular risk measurement approach also because its usage is a key regulatory requirement. Yet, VaR approaches have many shortcomings and limitations. Besides theoretical shortcomings, VaR models rely on historical data on one side and on a combination of approximations and assumptions on the other, thereby making the systematic assessment of their accuracy a fundamental component of modern risk management practice. A best practice risk management should, therefore, rely on a range of risk measures including, but not limited to VaR. Examples of market risk models used by banks are Incremental Risk Charge (IRC) or Comprehensive Risk Measure (CRM).

IRC is intended to cover market risk from defaults and credit migrations of the trading book positions that are incremental to the risks captured by the VaR model, such as bonds and credit default swaps. It is applied to credit products over a one–year capital horizon at a 99.9% confidence level (at least weakly computation), employing a constant position approach, rating–based simulation or direct simulation of spreads with migration/default barriers. Monte Carlos is used to calculate the IRC as the 99.9% quantile of the portfolio loss distribution and for allocating contributory incremental risk charge to individual positions. Important parameters for the IRC estimations are exposures, recovery rates, maturity ratings with corresponding default and migration probabilities and parameters specifying issuer correlations. In the near future, IRC is going to be replaced by the Incremental Default Risk (IDR) which only captures the default risk of issuer.

More complex instruments like collateralized debt obligations (CDO) are subject to the Comprehensive Risk Measure (CRM). CRM captures an incremental risk for the correlation trading portfolio subject to the floor of 8% of the equivalent capital charge under the standardized approach securitization framework. of the standard charge. Similarly to IRC, CRM is based on 99.9% loss quantile at one–year capital horizon. The CRM RWA calculations include two regulatory–prescribed add–ons, namely stressing the implied correlation within nth–to–default baskets and any stress test loss in excess of the internal model spot value.

Future market risk–related regulatory capital developments

In December 2017, the BCBS extended the implementation data of the revised minimum capital requirements for market risk to 1 January 2022. The extension aligns implementation with the Basel III revisions to credit risk and operational risk and recognizes that some of the market risk–related rules are still being finalized by the Basel Committee.

Main elements of the revised market risk framework includes:

- Changes in the internal model–based approach inclusive of changes to the model approval and performance measurement, replacement of the VaR based IMA with the one based on the ES design to better capture the tail risk and market liquidity risk.

- Changes to the standardized approach towards more risk–sensitive approach consisting of three main components: a sensitivities–based method for capturing risk sensitivities, a standardized default risk charge and a residual risk add–on.

- Revision of the boundary between the trading and banking book in order to reduce incentives for arbitrage of regulatory capital requirements between the two books.

2.2.2 MAIN SOURCES OF MODEL RISK

The process of measuring market risk requires an accurate modelling of statistical properties of future price variations in financial assets. The standard approach involves statical inference of the distribution of future changes from historical data. The steps involve suggesting a model for price changes, fitting the parameters to historical data and then using the model to make inference about the future. To quantify risk, a risk measure is applied to the distribution of future price changes. There are several sources of model risk that may arise within this process. Some of the examples are:

Data. The reliability of outputs heavily depends on the quality and quantity of relevant data and inputs. The quality of the market risk figures obtained from the model is highly dependent on the valuation models employed in the risk calculation. Any error or flaw may propagate and amplify causing market risk models to fail, i.e. chain of models. Hence, using appropriate valuation methods for all financial instruments is a prerequisite for the adequate measurement of market risk.

To calculate a VaR, one needs two elements: a set of positions in financial instruments (the portfolio) and their prospective returns (or price changes). Typically, the positions are taken as fixed for the time horizon under consideration, and so the critical assumption concerns the distribution of expected price changes. For example, in order to construct VaR one may consider exponentially weighted moving average or historical simulation. In each method, the proportions of the assets in the portfolio are considered fixed and the price changes (return) variances and their covariances across the assets take on different assumptions. These different assumptions give rise to alternative measurements of VaR.

In case when there is not long enough historical data series, an institution uses proxies that need to produce conservative results under relevant market scenarios that, however, may not always be the case.

Model design. The VaR metric does not adequately capture tail–risk events, credit risk events as well as market illiquidity. Two different portfolios could have the same VaR, but have entirely different expected levels of loss. VaR is build upon assumptions that do not reflect the real world in stressed financial markets (assuming market liquidity and large diversification effects across asset classes, etc.) and so it is particularly vulnerable to statistical error. Other source of model risk might be a wrong assumption about the correlation between risk classes or the historical bias as possible future changes in market rates, such as interest and foreign exchange rates, are implied by observed historical market movements.

Usage. The market risk models are widely used by banks, as well as by many hedge funds for various purposes, such as risk measuring, capital allocation, risk control, the calculation of the RWAs of individual positions or individual desks, and so model risk inherent to models may vary depending on each particular context of use.

Most of the financial institutions recognize that VaR does not adequately capture periods of extreme volatility and market illiquidity, but the alternatives are typically data intensive, difficult to verify through backtesting, and hard to explain to senior management. And so, more sophisticated tail modelling often leads to market risk being estimated with a higher degree of uncertainty. Therefore, financial institutions tend to complement VaR measures with more straightforward stress tests to assess the impact of tail events and ensure proper model risk management over these models. As market risk models are based on measure theory through the estimation of the distribution of future changes, the quantification of model risk of these models can, therefore, be objectively assessed through the proposed framework introduced in Chapters $6-9$.

2.3 Financial Derivatives

Derivatives are financial instruments with a value contingent upon the value or price of an underlying asset, set of assets, other financial instruments including bonds, market benchmarks such as interest rates, index prices, or on certain events. Derivatives may be settled either by delivery of the underlying asset(s) or by cash settlement based upon a formula with contract rules determined at the maturity date. Derivatives find applications in credit risk, market risk both in the trading and banking book and investment risk. They are used for hedging, speculating, arbitrage, accessing remote markets, or distribution of risk as securities (securitisations).

Financial derivatives come in various forms and can be categorized along three main dimensions:

- **Traded on** or **off an exchange** and are known as:
  - **Exchange–Traded Derivatives (ETD)**: are standardized contracts traded on a recognized exchange with the counterparties being the holder and the exchange. The contract terms are non–negotiable and their prices are publicly available.
  - **Over–the–Counter Derivatives (OTC)**: refer to bespoke contracts traded off–exchange with specific terms and conditions determined and agreed by the counterparties (the buyer and the seller). As a result OTC derivatives are more liquid, e.g., forward contracts or swaps. The booking of transactions between contracting parties means that OTC derivatives involve direct exposures between the parties, thus leading to counterparty credit risk (see Subsection 2.3.2 for more details).
  - **Cleared OTC Derivatives**: involve the bilateral trading of standardized transactions that are privately negotiated but booked with a central counterparty. This means that dealers do not have direct counterparty credit risk to each other but to the clearinghouse.
- **Type of the underlying** that can either be a financial instrument, physical asset, or any measurable risk factor. Typical examples are fixed–income, foreign exchange, credit risk, equities and equity indices or commodities.
- **Type of the instrument** that differs in terms of the dependence on the value of the underlying:
  - **Forward–based**: entail symmetrical rights and obligations between the parties and have the effect of locking in a specified price or rate, with delivery and settlement at a specified future date. They include three main product types: *forward contracts* between two parties to purchase and sell a specified quantity of an underlying (e.g., financial instrument, foreign currency, or commodity) at a specified price, with delivery and settlement at a predetermined future date; *future contracts* are standardized contracts to make or take delivery of a financial instrument (e.g, interest rate instrument, currency and certain stock indices) at a future date and are traded on an exchange; and *swaps* that are OTC agreements between parties to exchange sets of cash flows based on a predetermined notional principal, e.g., interest

rate swaps, foreign currency swaps, fixed–rate currency swaps, basis swaps, equity swaps, commodity swaps, mortgage swaps.

· **Option–based**: provide one party by the right, not the obligation, to buy (or sell) a specified instruments, such as currencies, interest rate products, or futures. They also impose the obligation on the seller to deliver (or take delivery of) the instrument to the buyer of the option. Option–based derivative contracts, such as caps, floors, collars and swaptions, can be combined to transfer risks from one entity to another.

2.3.1 Data, Models and Methodologies

Financial derivatives are measured at a fair value through the profit and loss, i.e. a market–based measurement [71].The fair value, in terms of derivatives, refers to the estimated amount that an institution would receive or pay to terminate the instrument at the reporting date under market conditions. The objective and common reference for the fair value is the price that would be paid for a financial instrument on an organized, transparent and deep market.

The fair value measurement is classified at three levels of the IFRS framework [71] fair value hierarchy reflecting the significance of the inputs used in the calculation:

- **Level 1: Valuation based on quoted prices in an active market.**

  The fair value of derivatives traded in the active markets[11] is determined using the market price, e.g. futures or options.

- **Level 2: Valuation based on observable market data.**

  The fair value of financial investments that are not traded in an active market (e.g., some OTC derivatives) is determined by valuation methodology. The fair value is calculated using the information that does not consist in quoted prices, but where the prices are directly (e.g., prices) or indirectly (e.g., derived from prices) observable, and which also include quoted prices in non–active markets. Examples of these derivatives include many OTC derivatives, currency futures, interest rate and exchange rate swaps, equity derivatives,.

- **Level 3: Valuation based on other than observable data.** Valuation is based on unobservable inputs that are supported by little or no market activity and that are significant to the fair value of the derivative. Certain financial instruments are classified within Level 3 of the fair value hierarchy because they trade infrequently and therefore have little or no price transparency. Such instruments include private equity, less liquid corporate debt securities and certain asset-backed securities. Certain OTC derivatives trade in less liquid markets with limited pricing information, and the determination their fair value is inherently more difficult.

Exchange–traded derivative instruments typically fall within Level 1 or Level 2 of the fair value hierarchy depending on whether they are considered to be actively traded or not. The valuation of financial instruments categorized as Level 3 involves one or more significant inputs that are not directly observable on the market using pricing models, discounted cash flow methodologies, or similar techniques, as well as instruments for which the determination of a fair value requires significant expert judgement or estimation. The estimates used in such models take into consideration the specific features of the derivative to be measured and, in particular, the various types of

[11] A market is considered active if market prices are easily and regularly available from a stock exchange, dealer, industry group, price–setting service or regulatory authority. This category also includes quoted shares and Treasury bills.

risk associated with them. However, the limitations inherent in the measurement models and possible inaccuracies in the assumptions and parameters required by these models may mean that the estimated fair value of a derivative does not exactly match the price for which it could be exchanged or settled on the date of its measurement. We refer the reader to Chapters 6 to 9 on how to quantify this risk.

Valuation of financial derivatives is based on *the specific terms of the derivative*, i.e. the mix of rights and obligations of both parties; *the likely range of payoffs* to both parties and how they relate to the value of the underlying asset (or assets or index price) at some future date or dates; and *the potential distribution of probable future outcomes* that would affect the realized value of the derivative. The right model should reflect a compromise between considering the relevant complexities while being simple enough to be tractable and periodically replicable.

Valuations are prepared by discounting future cashflows to arrive at a current value. Discounting cash flows at different times is a key part of the valuation process. Discount rates for distinct time periods can be implemented, based on pure interest rate instruments[12], forward curves but also probabilistic models, simulation models or mathematical pricing models in order to determine the expected value of future cash flows before discounting. In these models, volatility levels, forward curves, along with other factors such as mid–market price, foreign exchange spot or forward rates and correlations may be used as inputs. These inputs are based on observable market inputs or other estimates, depending on the asset class and availability.

The standard valuation models include discounting cash flow models, option models of the Black–Scholes- and Hull–White–type, hazard rate models as well as Monte Carlo simulations for more complex instruments. Models are calibrated on quoted market data (including implied volatilities), data obtained from less frequent transactions or by the use of extrapolation techniques. Inputs to valuation models are determined from observable market data wherever possible, including prices available from exchanges, dealers, brokers or providers of consensus pricing. Unobservable inputs cane be determined from observable prices via model calibration procedures or estimated from historical data or other sources. Examples of inputs that may be unobservable include volatility surfaces, in whole or in part, for less commonly traded option products, and correlations between market factors such as foreign exchange rates, interest rates and equity prices.

Black–Scholes–Merton model

The most widely used approach is to employ the risk neutral framework based on the Black–Scholes–Merton (BSM) model and its various extensions. The BSM model requires application of a stochastic differential equation and an Ito process. Generally, it determines the price of an option through the calculation of the return an investor gets less the amount that investor has to pay, using lognormal distribution probabilities to account for volatility of the underlying asset, with transactions that may be made continuously over time at no cost to investors, and with no discrete events. The log–normal distribution of returns is based on the assumptions that movements in the underlying follow a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

[12] Interest rate instrument include, for example, government bonds, interest rate futures, interest rate swaps, Treasury bills, or money market deposits

where S_t refers to the price of the underlying asset at time t with the drift μ and the volatility σ, and W is a standard Brownian motion [69]. This implies that S_t has a lognormal distribution with a mean

$$\mathbb{E}[S_t] = S_t e^{\mu(T-t)}$$

and a variance

$$Var[S_t] = S_t^2 e^{2\mu(T-t)} \left[e^{\sigma^2(T-t)} - 1 \right].$$

The lognormal property of asset prices can be then used to provide information on the probability distribution of the continuously compounded rate of return earned on an asset between times t and T, i.e. $\eta = \dfrac{1}{T-t} \log \dfrac{S_T}{S}$, and so,

$$\eta \sim \mathcal{N}\Big(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T-t}} \Big).$$

The price of the underlying asset and its distribution is not contingent on other possible future events.

The advantage of the risk neutral framework is that it allows to price derivative contracts without having to estimate certain parameters that are often required in the absence of this framework. For example, under the risk neutral framework one avoids the estimation of the expected appreciation in the value of the underlying asset dynamically over time and the calculation of the risk averse discount rate required by the investor as the expected appreciation in the value of the derivative security at any point in time (minus a risk–free rate of return) is offset by the risk premium over the risk–free interest rate in the valuation process.

Although the BSM model is often reasonable approximation and its usage is justified for pricing short–term and even some medium options on well–established and relatively stable publicly traded securities, for options on certain commodities, or for option on currency exchange rates (under certain circumstances), the assumptions behind this model are often violated even by publicly observed trades on regulated exchanges [72]. For instance, the implied volatility varies over time and is often dependent on the state of the development of the company, its products and the amount of leverage; the assumption that the underlying asset follows a random walk means that at any given point in time, its price can go up or down with the same probability which is, however, usually not true as stock prices are determined by many economic factors that happen with different probabilities as those assigned to the movements of stock price; normally distributed stock return do not fit the observed financial data accurately, as financial data have a finite variance and semi–heavy tails contrary to the log normal distribution with infinite variance and heavy tails. Other limitations of the model include: treasury rates that are often used as 'risk–free' rates can change in period of increased volatility, stock brokers often charge rates based on spreads and other criteria, markets are not perfectly liquid and investors are limited by the amount of money they can invest or policies of their companies.

The valuation methodologies are often merely approximations requiring very specific assumptions regarding past, present and future market conditions, the characteristics of the underlying asset and the likely distributions over time, and often do not match the exact terms of the derivative of interest. The formulas and even the simulations typically assume well–behaved distributions for prices and values that are continuous and do not allow for discrete events or discontinuities, e.g. in case of barrier options or rest features. Furthermore, these models do not consider the existence of multiple potentially dilutive securities and the effect that they might have on volatility over the time period being modelled in the context of different instruments. Next, the risk of default or bankruptcy (financial distress) is often material to the valuation of the institution securities. In this case, the assumption of lognormal stock price no longer holds since it implies zero probability of the asset having zero value or price in the future, and so the distribution of the future asset value or price may need to be modelled in the context of the risk of failure.

Therefore, the assumptions underlying the standard options valuation models cannot and should not be used without correction to the value based on a considerable expert judgement and experience.

2.3.2 Risks associated with Financial Derivatives

Although financial derivatives are also used for managing risks, they involve simultaneous risks in several different markets. The principal risks connected with derivatives include market risk, liquidity risk, credit risk (in form of settlement and pre–settlement risk), counterparty risk, hedge (or basis) risk[13], operational risk, and legal risk. For example:

- **Market Risk** related to derivatives is based on their price behaviour with respect to changes in market conditions, i.e. it is defined as the sensitivity of the price to changes in factors that affect an asset value (also called greeks). Market risk arises through the holding of any financial instrument, physical or derivatives, which creates exposure to movements in interest rates, prices of a security or market.

  The market risk associated with the use of derivatives is typically assessed in the context of the risk profile of the total portfolio. The effective exposure is achieved through a derivative position that reflects the equivalent amount of the underlying security that would provide the same profit or loss as the derivative position, given an incremental changes in the factors that contribute to the movements in price of an options. The fundamental market risks include *the absolute price or rate risk (Delta)*, *convexity risk (Gamma)*, *Volatility risk (Vega)*, *Time decay risk (Theta)*, *Basis or correlation risk*, and *Discount rate risk (Rho)*.

- **Liquidity Risk.** There are two types of liquidity risks related to derivatives, namely market liquidity risk and funding liquidity risk. Market liquidity risk is the risk that an investment manager may not be able to, or cannot easily unwind or offset a particular position due to inadequate market depth or disruptions in the market place, while funding liquidity risk is the risk that an investment manager may not be able to meet the future cash flow obligations from the derivative activities such as meeting margin calls on futures contracts.

- **Operational Risk.** Operational risk is the risk that deficiencies in the effectiveness and accuracy of the information systems or internal controls will result in a material loss.

- **Counterparty Credit Risk Model**

  Counterparty credit risk (CCR) can be defined as the risk that the value of a portfolio changes due to unexpected changes in the credit quality of trading counterparts, i.e. outright default or downgrading.

  CCR is important in the context of over–the–counter (OTC) derivatives and securities financing transactions (SFT).[14] Note that the CCR for exchange traded derivatives is guaranteed by the exchange institution. The

[13] A hedge risk refers to the risk that market prices of an instrument used as a hedge may not move in line with the market prices of the position being hedged.

[14] OTC derivatives are tailor–made derivatives created through bilateral negotiations. SFTs are repurchase agreement transactions and securities lending transactions.

BCBS in [25] requires institutions to calculate a capital associated with CCR for both OTC and SFT. The model for CCR is described in more detail in the text below.

In addition to the technical risks, there is also the risk that the use of derivative instruments are misaligned with institution risk management strategy and risk profiles. Given the lack of transparency and the leverage effect, the risks associated to derivatives can manifest themselves differently as in other financial instruments, thereby requiring a more precise assessment and more frequent monitoring.

Counterparty Credit Risk Model

CCR models are complex from both a conceptual and a practical implementation point of view, mainly due to the estimation of the exposure at default (EAD). In derivative contracts, the EAD depends on the future level of market factors, even if credit quality does not change. This means, that rather than the current exposure, i.e. the loss that would be expected if the derivative cash flows were known with certainty or if it the counterparty defaulted today, one needs also to consider the potential exposure, i.e. the loss that would be expected given a range of potential future market scenarios and if the counterparty defaulted at some time before the contract maturity.

The Basel III Accord introduced a capital requirement to cover losses from changes in the market value of counterparty risk, while proposing different approaches with increasing level of complexity and sophistication. First step in the calculation is the estimation of EAD using one the following methodologies: the original exposure method (the estimation is based on nominal value and time to maturity) that is allowed only for institutions without a trading book, the current exposure method (the estimation is based on market value, instrument type and time to maturity), the Standardized Method and the Internal Model Method (IMM) based on internally based models.

The IMM is the most advanced and risk–sensitive method to determine EAD where the dependencies between PD, LGD and EAD need to be taken into account, as well as the so–called wrong–way risk (see Section 2.1). The later is the risk originating from the fact that exposure may be negatively correlated to the counterparty creditworthiness.[15] The estimation of EAD requires to determine the following risk measures:

- **The Potential Future Exposure (PFE)** refers to the maximum positive exposure estimated to occur on a future date at a high level of statistical confidence. PFE is used in monitoring counterparty eligibility and credit limits with respect to CCR.
- **The Expected Exposure (EE)** is the probability–weighted average exposure estimated to exist on a future date.
- **The Expected Positive Exposure (EPE)** is defined ad the time–weighted average of individual expected exposures estimated for given forecasting horizons.
- **The Effective Expected Exposure (EEE)** at a specific date refers to the maximum expected exposure that occurs on or before that date.
- **The Effective EPE** is the average of the EEE over one year or until the maturity of the longest–maturity contract in the set of netting contracts, whichever is smaller.

[15] For example, a rise in interest rates may have a negative effect on the counterparty credit rating while at the same time increasing EAD.

The EAD of the counterparty is then given by

$$EAD = \alpha \cdot \textit{Effective EPE}$$

where α stands for the ratio between the total economic capital required for CCR and the capital that would be required if counterparty exposure were deterministic and equal to EPE. The Basel Accord prescribes a value of 1.4 for α, but allows financial institutions to perform their own estimates subject to the floor of 1.2.

Finally, the Current Exposure (CE) represents the value of the contract if the were to be sold on the market. CE is defined as the larger of zero, or the market value of a transaction or portfolio of transactions within a netting set with a counterparty that would be lost, assuming no recovery, upon the default of the counterparty.

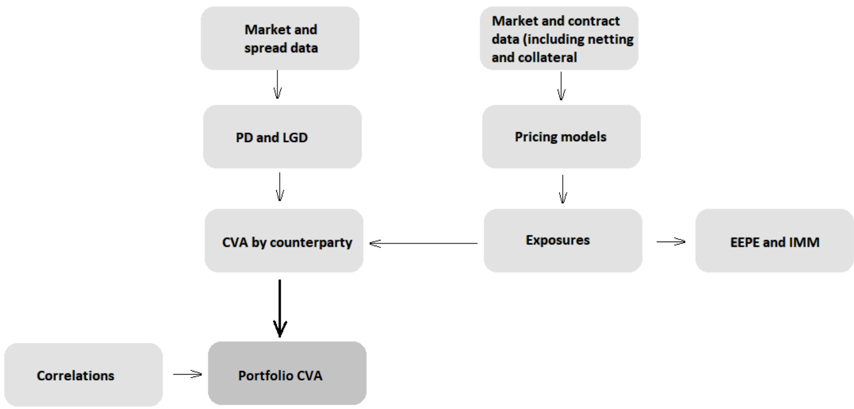

Figure 2.3: The process of CCR estimation.

One of the key purposes of the CCR estimation is to incorporate the credit risk of a counterparty into the price of a bank derivative exposure. The difference between the credit risk–free price of an exposure and the price that takes into account the credit risk is the market value of CCR, also called Credit Value Adjustment (CVA). CVA is the risk neutral expectation of the discounted loss and is given by

$$CVA = (1-R)\int_0^T \mathbb{E}^Q\left[\frac{DF_0}{DF_t}\mathbb{E}(\tau)|\tau = t\right] dPD(0,t)$$

where R is the recovery rate, Q risk neutral measure, DF_t discount factor, τ time to default and PD the Probability of Default. This risk measure is required for both regulatory and accounting purposes, but is also considered in risk management practices, such as economic capital calculations, stress testing or active portfolio management.

The modelling of EAD for CCR is especially complex, as one needs first to estimate the distribution of the exposure in the future (see Figure 2.3). This calculation is done by re–valuing the instruments and so involves simulation models to generate combination of market factors at future dates, as well as of pricing models to produce the future models. Scenarios for the evolution of the risk factors, which will typically include interest and FX rates, equity and commodity prices as well as credit spreads, can be generated either through path–dependent or direct jump simulations. Such generated exposure is then used to determine the EAD for CCR capital charge and is combined with estimates of PDs and LGDs bootstrapped from CDS spreads in order to compute the market price of CCR (i.e., CVA).

2.3.3 MAIN SOURCES OF MODEL RISK

An important component of a financial institutions model risk results from the valuation of derivative instruments. With the increased use of and dependence on complex models for pricing financial derivatives, model risk is emerging as a prominent type of risk in its own right.

Investment bank typically take substantial positions in long–term or exotic OTC derivative instruments that involve simultaneous risks in several different markets. These positions are usually marked and hedged by means of sophisticated and complex financial models, implemented by software and then embedded in front–office risk management system. Models derive prices from market parameters (such as volatilities, correlations, prepayment rates and default probabilities) that are forward–looking and should ideally be implemented from market prices of traded securities.

In general, pricing financial derivatives requires choice of a stochastic model for the dynamics of the underlying variables, calibration of the chosen model using market data and the pricing of a derivative with numerical methods (e.g., Monte Carlos methods, numerical integration). All these steps have a considerable impact on the final outcome and the reliance on this output thus exposes a financial institution to model risk. Some examples of potential sources are:

Data: Critical source of model risk is related to the estimation of the required input parameters to value a derivative, especially the volatility. Next, the market data may not be able to sufficiently characterize the distribution of the dynamics of the underlying.

Model design: The valuation of financial derivatives is often provided through simplified models and assumptions that employ inappropriate volatility modelling and that fail to consider the interactions of the value of different instruments, changes in leverage or developments within the company over time. Some sources of model risk include errors in the analytical solution, misspecification of the underlying stochastic process, missing risk factors, missing considerations, misclassifying or misidentifying the underlying asset, changing market conditions, length of sampling period, errors in variables.

Model fitting: Competing statistical estimation techniques, estimation errors, outlier problems, estimation intervals, calibration and revision of estimated parameters.

Usage. Financial derivatives are used for several purposes, including hedging, speculating, arbitrage, accessing remote markets, distribution of risk as securities but also as risk management tools. Therefore, the overall performance of derivative models highly depend on the usage.

Every aspect of the market cannot be considered in any given model, as every factor affecting the price of a financial security cannot be captured mathematically. Therefore, an objective measurement of model risk is a crucial question faced by financial institutions that would provide significant benefits not only in trading but also in the risk management process (see Chapters 6 to 9 on a proposed framework on how to quantify it).

2.4 Structural and Balance sheet risk

Structural and Balance sheet risks emerge primarily from the activities in the banking book and from certain off–balance sheet items. The significant risk factors include interest rate risk, credit spread risk, FX risk, equity risk, market risk from off–balance sheet items such as pension schemes or structural FX risk and equity compensation risk.

A bank balance sheet, a statement of financial position, refers to a financial report at a specific point in time that displays the bank total assets, and how these assets are financed, through either debt or equity. Its main objective is to showcase an accurate trade–off between bank profit and risk. It consists of three parts (see Figure 2.4): assets, liabilities and ownership equity or capital such that

$$Total\ Assets = Liabilities + Equity.$$

The assets part includes cash and cash equivalents (funds deposits in the central bank that can be immediately converted to cash), earning assets (total of all the credit and loans granted, bank securities portfolio such as stocks, public or private debt, derivatives, etc.) and non–earning assets (all the necessary infrastructure for a bank to function), while the liabilities include the different ways a bank finances its activities, through customer deposits and issuing debt. The equity primarily include the bank own resources, i.e. shareholders contributed capital and retained earnings.

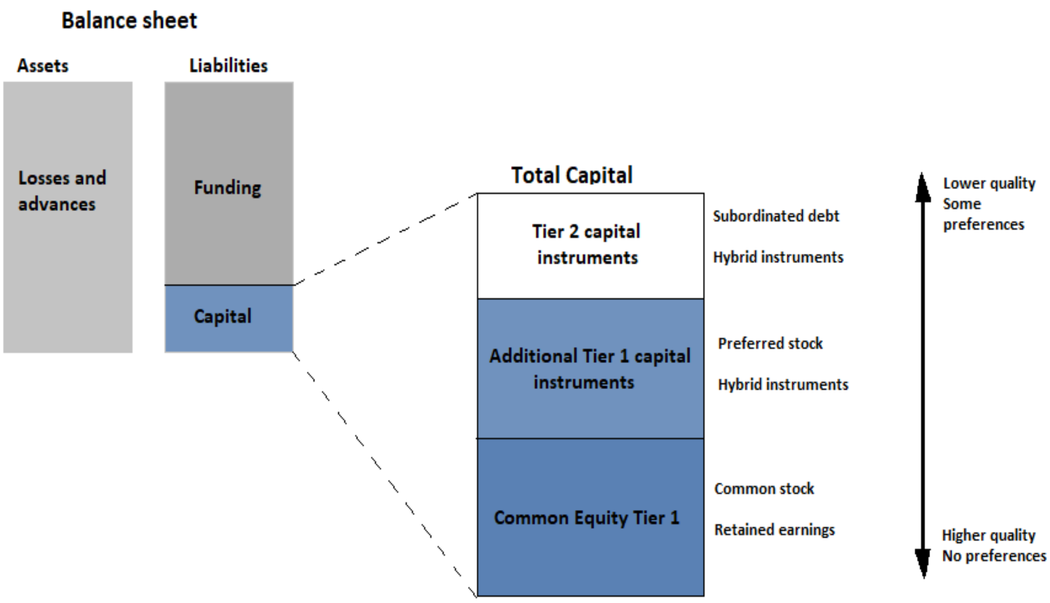

Figure 2.4: A structure of a bank balance sheet.

A financial institution banking book is composed of all assets that are not actively traded and that are meant to be held until they mature, while those on the trading book are traded on the market and valued accordingly (see Figure 2.5).

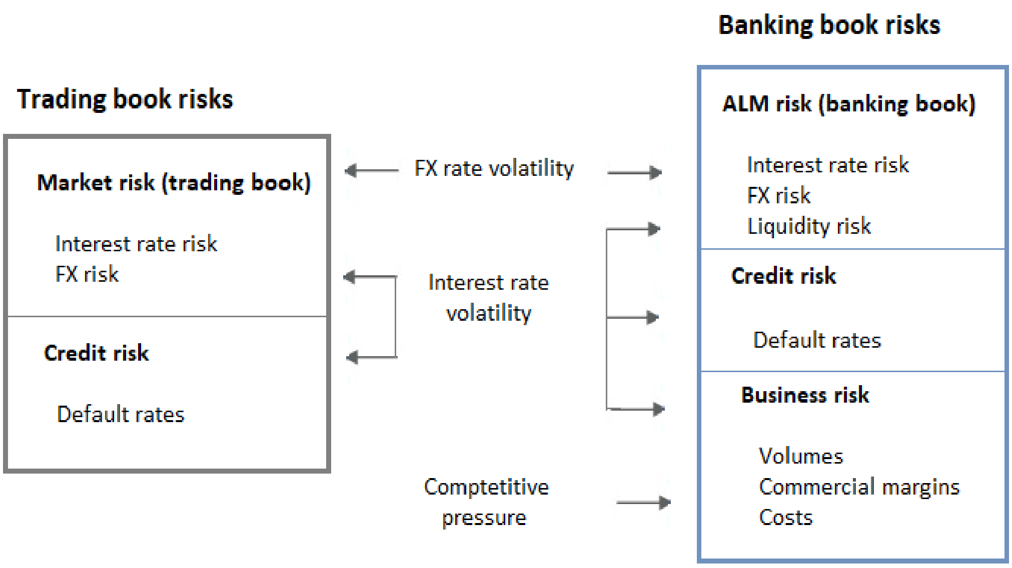

Figure 2.5: A structure of a bank balance sheet.

The different regulatory and accounting treatment of the banking and trading book is considered as a results of the 2008-09 financial crisis as financial institutions shifted substantial assets, namely the collateralized debt obligations, from the banking to the trading book to take advantage of lower capital requirements. As a consequence, national and international regulators have reviewed and modified the processing of the trading and banking book, with the later mainly through the capital requirements under Pillar 1 of the Basel framework.

2.4.1 Liquidity Risk

Liquidity, according to the BCBS, is the capacity of a financial institution to meet its cash and collateral obligations without incurring unacceptable losses. Liquidity risk refers, therefore, to the risk arising from a bank inability to meet its contractual obligations, expected or unexpected, when they come due or only being able to meet these obligations at excessive costs [26]. Liquidity risk has often a low probability of occurrence, and at the same time can potentially have extreme consequences for an institution stability. The classical definition of the BCBS distinguishes between two specific types of risk:

- **Funding liquidity risk** refers to the inability to obtain the necessary funding at a reasonable cost. In other words, it reflects the probability that an entity incurs losses or have to give up new businesses or growth of existing, unable to meet the commitments normally due or being unable to finance the additional needs costs to market.

- **Market liquidity risk** represents the inability to liquidate assets at an acceptable price. It is the risk that an institution cannot easily offset or eliminate a position at the market price because of an inadequate market depth or market downturn.

The objective of the liquidity risk management is to assess the ability of a bank to meet its cash flow and collateral needs (under both normal and stressed conditions) without having a negative impact on day–to–day operations or its overall financial position, and to mitigate that risk by developing strategies and taking appropriate actions designed to ensure that necessary funds and collateral are available when needed.

Data, Models and Methodologies

Liquidity is dynamic and can change according to both business and market conditions that can be both expected or unexpected. Liquidity risk can arise from a number of areas within the business, including seasonal fluctuations, business disruption, unplanned reduction in revenue, unplanned capital expenditure, increase in operational costs, inadequate management of working capital, future debt repayments, not matching the maturity profile of debts to the assets which they are funding, inadequate or non-existent financing facilities, inadequate cash flow management, among others. And as such, all balance sheet items need to be taken into account; for liabilities the main risk is early withdrawal, and for assets the main risk is that a loan is extended (rolled over) after the maturity data or prepaid before the maturity data.

The models and methodologies used to measure liquidity risk are often simple but rather sophisticated, mainly due to the broad range of heterogeneous parameters that might impact the liquidity position of a bank. The funding liquidity risk needs to be measured by forecasting potential inflows and outflows, that may be influence by other factors, such as credit risk, market risk or new business assumptions. The market liquidity risk, in contrast, may be assesses by bid–ask spreads or the amount of traded volumes.

There are several methodologies used across the industry, most of them forward–looking incorporating the quantification of the structure of the balance sheet, cash flows, liquidity positions and risks in off–balance sheet items. For example, a regular cash flow forecast where cash flows are estimated after the settlement of total assets, liabilities and off–balance sheet times and then allocated to the various horizons in which they are expected to occur. The net cash flow for each horizon is calculated as the sum of the expected cash outflows minus the sum of the expected cash inflows. The accumulated cash flow across all horizons is then used to determine the period with a positive cash flow. Based on these figures, internal limits are establishes, imposing constraints on treasury regarding exposure to maturity mismatches. The reliability of these methods is highly dependent on the reliability of the inputs; effective and timely flow of information is essential for an accurate estimation of liquidity exposure.[16]

Survival approaches are other standard methods for liquidity risk assessment that estimate the time horizon through which the bank can fund potential withdrawals out of their counterbalancing capacity, i.e. the volume of all potential sources of liquidity that might be tapped in a particular scenario. Depending on the type of the product, models range from (multinomial) logistic regressions, in case of maturing products, to combinations of logistic regressions (survival) and an ARDL (average balance) model for non–maturing products (see Figure 2.6).

Liquidity stress testing and scenario analysis, required also by regulators, are fundamental tools for measuring liquidity risk and for the evaluation of bank short–term positions. They evaluate the impact of sudden and severe stress events on the liquidity position. The methods employed range from deterministic to stochastic ones and their combinations. The deterministic approach is based on an user–defined what–if analysis resulting in a single estimate (perspective) of the future. The quality of the outputs, therefore, depends on the imposed assumption. For instance,

[16]When cash flows are initiated by third parties, predictions are difficult due to the non–contractual maturities or variant customer behaviour.

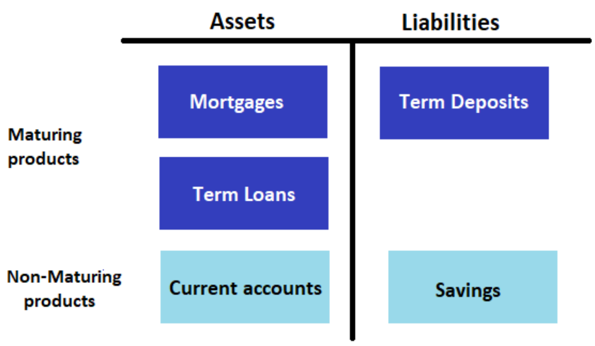

Figure 2.6: Models for assessing liquidity risk differ according to the type of a product.

a common procedure is to use a parallel shift scenario, where the exposure is re–evaluated after application of a uniform shock to a risk factor. This, however, is often unrealistic and the conclusions drawn remain unpractical. Other approaches are based on historical events, where the historical shocks are replicated with current market curves and values. Indications from history may lead to misleading results, as extreme events, even appearing similar to each other, often occur in different market contexts and have various impacts on market variables. Besides, the deterministic approach provides no information about the probability that a certain scenario will prevail.

On the other hand, stochastic approaches are based on the assessment of probability distributions to stress scenarios and have higher predictive ability. The more comprehensive the scope of scenario–based analysis, the more effective the cash flow projections and risk assessment can be. Thus, as many significant risk factors as possible should be subject to stochastic scenario generation and analysis. Typical risk factors include future developments in market rates, customer behaviour, credit spreads, or other market characteristics. Analysis of the resulting probability distributions can produce a comprehensive view of the bank exposure to liquidity risk.[17] A typical way is to examine the tails of the probability distributions obtained through scenario generation; the analysis of the left tail yields an understanding of which events and risk factors present the greatest potentially adverse impact. The outputs depend, however, on the imposed parametrization and the historical data that means that events representing discontinuities in market behaviour are not take into account. The most effective approach is to incorporate stochastic modelling into specific deterministic events where scenarios are compounded by combining stochastic and shock deterministic scenarios.

BCBS in the Basel III requires banks to manage liquidity through two regulatory liquidity ratios: the **Nest Stable Funding ratio (NSFR)** and the **Liquidity Coverage Ratio (LCR)**, to measure and improve the structural health and the short–term liquidity risk profile, respectively. Formally, they are defined as follows. The LCR, the primary short–term metric, is given by

$$LCR = \frac{Stock\,of\,high\,quality\,liquidity\,assers}{Net\,cash\,outflows\,over\,30 - day\,horizon} \geq 100\%$$

and it determines the level of liquidity buffer to be held for covering short–term funding gaps under severe liquidity stress. A minimum LCR is set to 3% of Tier 1 capital as percentage of total assets (and some off–balance sheet items) [26].

[17]The current market is one example of a stress scenario.

The NSFR, the long–term metric, is given by

$$NSFR = \frac{Available\, amount\, of\, stable\, funding}{Required\, amount\, of\, stable\, funding} \geq 100\%$$

and it defines the minimum acceptable amount of stable funding in an extended firm–specific stress scenario over a one–year horizon to cover the liquidity required to finance assets and off–balance sheet exposures.

The lack of standardized quantitative approaches to liquidity risk relates in part to its fundamental nature. Liquidity risk is difficult to quantify in a single number representing an accurate and comprehensive perspective.

2.4.2 Interest Rate Risk on The Banking Book

Interest rate risk on the banking book (IRRBB) is defined as the current or perspective risk, to both the capital and earnings, arising from adverse movements in interest rate affecting the banking book exposure [15]. IRRBB arises from gaps in maturity and repricing dates of both assets and liabilities (*depreciation risk*), from imperfect hedging, for example differences in repricing of otherwise mirroring instruments (*basis risk*); from option derivative positions or from embedded optionality in the on– and off–balance sheet items, such as prepayment options on mortgages, withdrawing options on deposits, caps and floors, on variable rate loans (*option risk*); from asymmetric movements interest rates at different maturities, i.e. when unanticipated shifts of the curve have adverse effects on the income of a bank or underlying economic value (*yield curve risk*).

Excessive level of IRRBB can pose a substantial threat to an institution earnings and capital. Hence, an effective risk management that maintains interest rate risks at reasonable levels is crucial to the safety and soundness of banks.

The nature of IRRBB is complex as it materializes along different dimensions. For many transactions, such as fixed rate loans, bonds and derivatives, the interest rate characteristics are entirely contained within the contractual conditions; for other, such as prepayable loans or saving deposits, interest rate risk arises primarily from behaviour of customers in light of market–changing conditions; for items without a contractual maturity like non–maturity deposits or building and equipment interest rate characteristics are conventionally assigned for modelling purposes.

Measures of interest rate risk can be broadly classified in **earnings measures** and **economic value measures** [58], depending on whether they focus on the short–term impact of interest rate changes on profitability and solvency, or the long–term impact of such changes on the net present value of the current balance sheet.

The basis earnings measure of interest rate risk is a gap analysis which is based on the evaluation of the absolute difference between the nominal amounts of assets and liabilities that are sensitive to the variations of the interest rate. A negative gap indicates a negative relationship between income and asset value, while a positive refers to the opposite relationship. Product of a change in interest rate and the gap that is usually computed on different time intervals corresponding to maturity or repricing dates denotes the resulting change in interest income. Methodologies range from statistic gap analysis that takes into accounting only the existing portfolio and assumes contracts with fixed interest rates to more sophisticated analysis based on assumptions on prolongations, prepayment and new business as well as a forecast of market conditions.

The advanced earning measure refers to the Earnings at Risk (EaR) that estimates the loss of net interest income (NII) over a given time horizon and is assessed as the difference between NII in a base scenario and in an alternative one. When computed over multiple alternative scenarios, EaRs raise a statistical measure of maximum losses at a given confidence level similar to a VaR. EaR is a comprehensive dynamic method that considers all components of the interest rate sensitivity and provides a good indication of the short–term effects of convexity and yield curve risk. Though, the outputs of the modelling are highly sensitive to assumptions about customer behaviour and management responses to different scenarios. In addition, it covers only a short horizon, so changes in earnings outside the observation period are ignored.

The main economic value measure of IRBB is known as Economic Value of Equity (EVE) that is given by

$$EVE = Fair\,Value\,of\,Asset - Fair\,Value\,of\,Liabilities.$$

The estimation can be either obtained statistically, i.e. assuming that cash flow do not change under the alternative scenarios, or dynamically, re–evaluating cash flows on the basis of movement in behaviour of customers induced by the different scenarios. The assumptions imposed during the estimation process, the timing of cash flows and the chosen discount rate may have a major impact on the accuracy of this measure.

Duration measures and PV01 (basis point value) measures are used to evaluate the change in market value of a financial instrument due to a one basis point parallel shift of the yield curve. The modified duration of equity is given by the average (by exposure in each class) of the modified durations of instrument classes based on their repricing date. The PV01 of equity can be computed by multiplying the modified duration by the value of equity and dividing by 10^{-4}. A shift of one basis point in the zero rates is approximately equal to

$$BPV \approx -\sum_{i=1}^{n} CF_i t_i d(t_i) 10^{-4} = -D \cdot PV \cdot 10^{-4}$$

where $d(t_i)$ are discount factors, CF_i is the interest cash flow for an instrument class i given by

$$CF_i(d(t_1), \cdots, d(t_n)) = \frac{\partial}{\partial d(t_i)} PV(d(t_1), \ldots, d(t-n)),$$

PV is the net present value of a set of interest rate sensitive cash flow given by $PV = \sum_{i=1} CF_i d(t_i)$, i.e. the impact of a parallel shift in interest rate of one basis point ($1\,bp = 0.01\%$) and D is computed as

$$D = \frac{\sum_{i=1}^{n} CF_i t_i d(t_i)}{CF_i d(t_i)}$$

and it approximates the maturity, for a fixed rate instrument, and the time until the next rate fixing for a floating rate instrument. Both these measures, PV01 and BPV, are simple measures limited to the analysis of small, parallel shifts in the yield curve. They do not account for large movements, optionality, convexity, nor they can measure yield curve or basis risk.

At last, a VaR measure applied to the banking book estimates the maximum loss in the market value of equity that can be incurred over a given time horizon with a specified confidence level. Time horizon should be consistent with the intended holding of positions, and is often fixed to one year. Most regulatory guidances (see for example [58], [62]) suggest to estimate VaR for the banking book using historical simulation, variance–covariance or Monte Carlo simulation approaches (for more details on VaR see Section 2.2). Unlike other measures, a VaR for IRRBB comprises both convexity and optional effects, as well as diversification effect due to less–then–perfect correlations between balance sheet positions. Similarly to the case of the trading book, VaR assumes normal conditions, and its estimation is largely driven by historical data on price levels and volatility.[18] Besides, VaR estimation over a long horizon incorporates challenges driven by long–term forecasting of returns and volatilities (for more details see for example [76], [52], [117]).

Besides, similarly to EVE, VaR requires estimation of future cash flows that depend on market rates, contractual terms and behaviour of borrowers and depositors. Modelling such cash flows is a fundamental feature of IRRBB measures, however, it comes with challenges as these cash flows are either partially or even fully unknown priori, and even when it is specified in the contract, the customer may behave in an unforeseen manner.

Another issue arises from the way the IRRBB models are used. Following the current regulatory framework, most banks compute interest rate risk by using a parallel shift of the yield curve [58], parallel shocks on the short end or the long end of the curve, as well as different types of inversions like steeper or flatter shocks [62] and comparing it to a base case scenario. Interest rate movements, nevertheless, do not in general occur in parallel shifts of the entire yield curve, or it remains unlikely that any of these scenarios will exactly occur in practice.

IRRBB arises from many different types of transactions and its modelling requires assumptions, hypothesis and forecasts of many variables. Due to the long–term nature of many transactions, IRBBB measures have to be computed on long time frames which rises the requirements for data, IT, the complexity and reliability of projections as well as the interpretation of outputs. Moreover, earnings or economic value are represented through rather simple arithmetic relationships between these measures and cash flows, where in reality cash flow depends on rates in complex and very nonlinear ways.

2.4.3 FX Structural risk

The FX risk arises from balance sheet items denominated in a currency different from the one used in the reporting, and it is driven by the volatility of exchange rates of all currencies on the balance sheet. Therefore, it affects shareholders equity and Common Equity Tier 1 capital. The changes in the FX structural rate arise from the bank's operations in currencies, mainly related to permanent financial investments, and the results of these movements. The volatile nature of FX markets poses a great risk of sudden and drastic FX rate movements, which may cause significant financial losses from otherwise profitable export sales.

FX derivatives typically appear in both the trading and the banking books, however derivatives used for managing Asset and Liability management (ALM) positions in the non–trading book investment securities portfolio is considered within the banking book market risk. ALM estimates the maximum amount of permissible unhedged foreign exchange risk and requires that this be monitored closely. The open currency position is part of the banking book and is limited as a percentage of equity by the regulator. The exchange risk management is dynamic and seeks to limit the impact on equity of currency depreciations and optimize the financial cost of hedging. The primary

[18] Both historical VaR and variance–covariance VaR are backward–looking methods where history is used to indicate the future and therefore more likely not to capture the tail risks. The Monte Carlo simulation method is very demanding in terms of technology and computation.

objective of a FX risk management is to minimize potential currency losses and to make a profit from unpredictable and frequent FX rate movements.

2.4.4 Main Sources of Model Risk

We have examined the different issues surrounding the identification and assessment of structural and balance sheet risks. These risks are one of the oldest risks in banking, as they are inherent in some of the fundamental roles of a financial institution. The nature of these risks is very complex as they materialize along different dimension, i.e. they emerge from a wide combination of activities, and do not always manifest themselves in outright losses. On the contrary, most of the times they will rather translate in an opportunity cost to a bank.

Due to the structure of the operations involved, the requirements on data, the complexity and reliability of predictions as well as interpretation of results are rather high. Furthermore, given the simple methods and models used, correlations between different asset classes and different business lines or countries and nonlinear relationships might not necessarily be taken into consideration.

Given that these risks arise from many different transactions and bank activities requires to impose rather strong hypothesis, assumptions and forecasts of many variables. Thus, one need to be careful in separating the actual impact of the interest rate or liquidity in the various risk measures from the impacts of other variables and of the various assumptions imposed. This further complicates the task of back–testing while all other predictions and assumptions will rarely occur in reality in the way they were formulated in the model.

The limitations of each quantitative tool and model used should be fully understood by the institution, and these limitations should be taken into account in the risk management process of these risks. As most of these methodologies are of statistical nature, model risk emerging from their usage can be quantified using the proposed framework based on the information theory and differential geometry introduced in Chapters $6-9$.

Chapter 3

NON–FINANCIAL RISKS

"Time transforms risk, and the nature of risk is shaped by the time horizon: the future is the playing field."
(Peter Bernstein)

In the present chapter, we provide a brief introduction into non–financial risks (NFR), whether related to compliance failures, misconduct, technology, damage of a reputation, or operational challenges, as they make a significant contribution to bank losses and capital requirements.

The importance of the NFR management has increased after a number of high–severity losses derived from isolated and often interconnected risks, such as conduct, cyber, compliance, IT or model risk, leading to direct financial losses and reputational impacts suffered by financial institutions. Besides, the regulator scrutiny surrounding NFR management has been significantly increased by the recent regulatory actions related to concepts such as money laundering, sales practices, market manipulation and model risk through the ICAAP/SREP process and Pillar 2 add–ons. Regulators are expecting to see financial institutions adopt a common definition and understanding of these risks to help distinguish risks traditionally included under operational risk.

Non–financial risks are sometimes referred to as non–portfolio or non–traditional risks, however, a common industry understanding and terminology is yet to be agreed. The ability to monitor non–financial risks has been generally limited to aggregated operational risk metrics bound to Advanced Measurement Approach (AMA) models. Due to the increasing focus on NFR, managing these risks in a more effective way has become a prominent issue for many banks that shift from the combined loss–driven approach to employing various methodologies for different types of NFR.

NFR can be difficult to isolate and can better be defined by exclusion. For the purpose of this thesis we understand non–financial risks as those risk that are not directly associated to the primary business and revenue generating activities reflected in the balance sheet, but can nevertheless have adverse strategic, business, economic or reputational consequences, i.e. NFR comprise all risks that are not credit, market, counterparty, interest rate or liquidity risk. They include operational, compliance, reputational, strategic and business risks, as well as disturbances in the management of financial risks. These risks refer to disturbances in the regular processes of running a financial institution, and risks that arise from reacting to internal or external changes or from the failure to do so.

Effectively managing these risks or estimating the total loss is challenging as these risks are often global, complex, unpredictable and difficult to measure and account for. A common trend is to focus on forward–looking risk assessments and prevention versus after–the–fact analysis of a risk failure, and enhancement of scenario processes and tools aimed at more effective assessment of forward risk. The most common approaches include risk indicators, exposure–based modelling, stress–testing models and advanced analytics (including the use of big data).

Advanced analytics and high speed computing combined with the analysis of a broader range of data than traditional loss databases offer new potential tools to risk managers. The ability of machine learning models or neural networks to analyse large amount of data both financial and non–financial, with higher granularity, can improve analytical capabilities in risk management and compliance. These models can be applied, for example, for anti money laundering and terrorist financing to increase the predictive power of bank transaction alert models.

Whether measuring and monitoring conduct or operational risk, enforcing compliance or dealing with the emerging threat of cybersecurity, banks lack efficient tools and techniques that would help them to better understand and track the intrinsic non–financial risks more effectively. Institutions will, therefore, need to address a set of challenges stemming from the prevailing framework. While it is important to identify potential risks, it is just as important to measure and prioritize risks so that the institution is prepared to respond to them in an appropriate manner.

This chapter contains two sections with the main focus on operation risk that is addressed in Section 3.1. Section 3.2 provides very brief introduction to other non–financial risks, such as business risk, insurance risk, reputation risk, compliance risk, risk in intangible assets and goodwill, IT risks, third–party risk and macroeconomic risk.

3.1 Operational Risk

Operational risk is very central to the financial industry owing to the immediate and very direct impact on the economy and the business. It is, to an extent, inherent in all bank operations and support activities, including revenue–generating businesses (e.g. sales and trading) and support and control groups (e.g. information technology and trade processing), and can result in a financial loss, regulatory sanctions and damage of the reputation of a financial institution. Besides, its sources could be highly diverse (processes, internal and external fraud, technology, human resources, commercial practices, disasters and suppliers).

At this point, there is no generally accepted definition of operational risk. Some practitioners consider operational risk all the risk that are not market and credit risks. Other follow the Basel III definition of operational risk where it refers to the potential of loss, direct or indirect, to which a financial institution is exposed by internal factors, such as inadequate or failed internal processes and systems (e.g., breakdown of IT system), from inadequate human behaviour or human error (e.g., fraud), or from external events such as disruptive events (e.g. fire, flood, earthquakes, terrorist actions, vandalism), [92]. This broad definition covers a myriad of non–financial risks, including conduct risk, fraud, cyber, vendor risk, privacy, but excludes strategic and reputational risk.

As operational risk is a heterogeneous category covering all risk associated with people, processes and systems. In order to make the measurement meaningful, there is a need to break this risk down into more homogeneous classes to enable exposures to be properly understood and modelled. Segmentation to an appropriate level of granularity is key when incorporating forward looking information, as using too heterogeneous population would not provide loss distribution that is responsive to particular risks. A fundamental element of the operational framework is, therefore, a segmentation model that is often internally developed.

BCBS delineates seven categories of operational risks and for each one there are a number of contributing factors. These classes include internal fraud (e.g., credit fraud, theft, robbery, bribery, unauthorized transactions), external fraud (e.g., robbery, theft or forgery), employment practices and workplace safety (e.g., employment discrimination, compensation, benefit, termination issues), clients, products and business practices (e.g., misuse of confidential information, disclosure issues, lender liability), execution, delivery and process management (e.g., failed or inaccurate mandatory reporting, transaction execution and maintenance), damage to physical assets (e.g., terrorism, vandalism) and business disruption and system failure (e.g., system or hardware failure).[1] Though, this approach may lead to a common danger encountered by operational risk managers which have the tendency to stick to this classification and ignore the individual risk components.

In December 2017, the BCBS has introduced a single non–model based method for the calculation of operational risk capital, Standardised Approach (SA) [93], that is expected to be fully employed by 1 January 2022. This approach replaces all three current approaches under Pillar 1: the Basic Indicator Approach (BIA), the (Alternative) Standardized Approach (TSA/ASA) and the Advanced Measurement Approach (AMA). The capital requirements under the SA methodology are based on three components:

1. **The Business Indicator (BI)** which is a simple financial–statement–based proxy for operational risk exposure and it refers to a sum of three components: the interest, leases and dividends component; the services component and the financial component,

2. **The Business Indicator Component (BIC)** that is calculated by multiplying the BI by a set of regulatory determined marginal coefficients based on three buckets ($12\%, 15\%$ and 18%), and

3. **The Internal Loss Multiplier (ILM)** represents a measure of a bank historical losses.

The minimum (Pillar 1) operational risk capital requirement is the product of the BIC and the ILM, with risk–weighted assets for operational risk being this capital requirement multiplied by 12.5. The required capital is thus a function of a bank income and historical losses. Conceptually, it assumes that operational risk increases at an increasing rate with an income, and banks that have experienced larger operational risk losses historically are assumed to be more likely to experience operational risk losses in the future.

The ILM is derived from a Loss Component (LC) which is equal to 15 times average annual operational risk losses incurred over the previous ten years. The calculation of average losses in the LC has to be based on a high–quality annual loss data with a minimum of five years when 10 years history is not available. The SA is essentially a backward looking approach as the BI is based on the last three years of financial performance and the ILM utilizes the last ten years of loss data, [93]. Moreover, the new SA is much simpler than the current AMA but as a result no longer as risk–sensitive.

Irrespective to the changes to the minimum capital requirements, banks are required to use a model based approaches for assessing their economic capital and their Pillar 2 capital requirements. Financial institutions, typically, manage operational risk through a framework that encompasses elements of identification (risk register,

[1] A more detail categorization was introduced by the IFoA Risk Classification Working Party which identifies 350 sub–types of operational risk.

internal losses, scenarios), assessment (modelling, quantification, scenario analysis, economic capital), mitigation (insurance, control), reporting (MI, KRIs), risk appetite, among others. Having a better view of operational risks can allow a company to act pro–actively and lead to competitive advantage.

3.1.1 Data, Models and Methodologies

Although the value of effectively managing operational risk is significant, it can be difficult to clearly quantify its exact value, i.e. the value of losses that did not occur. A key responsibility of the risk management function should be to measure, control, challenge and oversee risks in the business. Financial institutions are free to choose methodology and specific quantitative model for the calculation of their economic capital. Under the current AMA approach, irrespective of the specific approach employed, a quantitative estimate at a 99.95% confidence level (depending on the institution, different percentiles are suggested) needs to be provided, classified and mapped according to the Basel Accord loss and business categories [92]. Such an approach should derive the quantitative measure of operational risk based on a reasonable combination of internal data, external data, scenario analysis and factors, reflecting the business environment and internal control system.

DATA

The data used are fundamental issue with respect to operational risk. No matter how sophisticated and trusted the statistical methodology and testing, without reliable input data, the estimation is bound to remain questionable and unconvincing. The AMA models are typically based on data coming from different sources including internal data, external data, data coming from scenario analysis, and factors of the business environment and internal control system. As the loss data will become a direct input under the new SA, and given enhanced disclosure requirements, the quality of internal loss data collection has become even more relevant.

The value of data standards is especially high as data for operational risk is difficult to collect and to classify. Capturing all material losses is one of the key requirements, since it affects directly the reliability and the outputs, but also the most difficult one accompanying by several challenges. The identification and collection of operational losses throughout the institution is often based on a reporting process relying on the assumption that in each business unit losses are accurately measured and completely reported to the management. As in any process that depends on a chain of human interventions, the potential for errors and omissions (intentional or otherwise) is rather large. In addition, to report a loss, one must be aware of it, as it may not always be the case.

Internal data are often biased towards low severity and low frequency, the length of sample is not always sufficient for reliable estimation, or the dataset may be of poor quality. External data are used to supplement the internal source and in general are coming from two sources: databases including publicly available losses and those including losses which are only privately released. Publicly available losses tend to be strongly biased towards high–severity losses, i.e. they are truncated above a very high threshold, while the private database reflects the quality of that particular institution's collection procedure. Another issues while using external data include reporting bias (not all loss events reach the public domain), scale bias or survival bias (data available for banks that did not bankrupt). Besides, using external data in order to supplement internal data involves assumptions that external data are drawn from the same distribution as internal ones.

The key to operational risk measurement is the assessment of the likelihood and impact of individual events through scenario analysis, almost regardless of what quantification methodology is chosen. A typical scenario analysis methodology consists of a systematic and consistent process for the collection of the estimates and a statistical approach to model frequency and severity of each potential loss event under analysis. The aim of scenario data is to supplement internal loss data with expert assessment of the prospect of higher severity losses that are typically of lower frequency. Another source of data come from business and internal control environment factors that capture actual and future changes. The main objective of a comprehensive set of key risk indicators is the early detection of any operational issue.

Thus, operational risk is typically assessed on the basis of more than just one kind of data, including the existing information on past losses, events and near misses (i.e., those operational failures that did not result in financial loss), on the expert knowledge of the business, on comparison with similar units within or outside the institution and/or on external data. The lack of sufficient data means that a purely data–driven approach is not feasible, leading towards a more scenario–based approach which, on the other, places a large reliance on expert judgement.

MODELS AND METHODOLOGIES

As operational risk is, in essence, a idiosyncratic form of risk depending on the inner organization and activities of a particular institution, be in terms of procedures, people, top management culture and so on, its assessment is not straight forward. For large financial institutions, a strong data–driven approach provides the most apparent means to assess capital for operational risk. With sufficient data, the Pillar 2 capital models have to follow a set of key principles: they should be empirically based, using historical figures where they can be justified; risk–sensitive; include internal controls and business factors together with external factors, such as economic forecasts; and they should be benchmarked with scenarios.

The common business practice is to use the Loss Distribution Approach (LDA) as the essential modelling technique employed within the quantification process or back–testing combined with the robust statistical methods, such as forecasting the uncertainty in parameter estimation. This approach is empirically driven, risk sensitive and allows for the application of internal and external scenarios and stress testing.

The LDA approach involves a separate modelling of loss frequency (through which the frequency of occurrence of events is expressed) and severity distributions (through which the amounts of losses resulting from risk events are represented) through appropriate continuous and discrete distributions. The distributions are then convoluted to derive an Aggregate Loss Distribution at various confidence levels over a one–year period. Moreover, this approach is performed separately either for each of the risk category, a specific scenario or even for an individual risk level. The results are then aggregated, taking into account diversification, to determine the required capital. Typical aggregation processes are Gaussian Copula, Simple Summation, Sum of Squares or Rank Copula [114] and the common aggregation techniques is Monte Carlo simulation. The aggregated loss distribution is then used to allocate capital, for assessing appetite or stress testing. Further steps involve for example the allocation of capital per geographical units.

Note that the selected categorization has a high impact on what is actually being modelled and on the level of diversification. Besides, it determines dependencies across different units or risk levels. The selected segmentation represent a model by itself that is subject to separate validation.

The LDA approach is based on the estimation of the loss distribution function which involves the distribution fitting process, the techniques used to estimate parameters and the tools used for evaluating the quality of the fit of the distribution to the data. The estimation of the loss distribution is typically performed through parametric methods. These approaches are based on the assumption that the relevant loss distribution takes a predetermined parametric form, i.e. the loss distribution belongs to a family of probability distributions, i.e. statistical manifold given by

$$\mathcal{M} = \{p(x, \theta) \,|\, \theta \in \Theta, x \in \mathbb{R}^n\}$$

where θ is the vector of parameters belonging to the set of available parameters Θ (see Chapter 6 for more details). In general, financial institutions use multiple types of distribution for the frequency and the severity distributions depending on the operational risk being modelled and the availability of meaningful loss data. The estimation of the parameters can be done through several methods including, but not limited to Maximum Likelihood Estimation (MLE), Least squares, Method of moments or Bayesian methods.

Parametric methods are particularly suited when the distributions are known which is not often the case for operational risk, where only small samples are generally available. The specification of the model, given that the underlying loss–generating process varies widely from case to case and so it may be very difficult to formalize with a reasonably limited number of variables. Another issue with the calculation of the capital charge for operational risk lies in the required holding time frame (usually one–year horizon), thus worsening the issue of obtaining a reliable fit of distributions and parameter estimates. Likewise, this requirement makes it more difficult to collect sufficient historical data. Other options are non–parametric methods or stochastic (Monte Carlo) methods, coming with their own limitations, that, however, are challenging to apply in practice while ensuring reliable estimates.

Distribution fitting

The key paradigm in operational risk modelling consists in determining the likelihood of the operational events (frequency) and that of the probability distribution of operational losses (severity) given an operational event.

The common distributions used for modelling the frequency are the Poisson distribution, the Negative Binomial distribution and Cox processes. The choice of a particular distribution may be important as the intensity parameter is deterministic in the first case and stochastic in the second. The most relevant distributions for modelling severity include Chi–square, Exponential, Gamma, Lognormal, LogLogistic, Pareto, Releygh or Weibull [103]. The process of selection of a particular distribution combines graphical (visual) representation, algorithmic tests and at times some expert judgement.

Due to lack of sufficient data on high–severity, low–frequency losses, a single parametric distribution may not sufficiently describe the probabilistic behaviour of severity over the entire range. It is known that the extreme losses in the tail, [45], behave differently than losses in the body. The suggested solution to this problem is to model the tail and the body with different distributions, heavy–tail or fat–tail parametric distributions for the tail estimation.

Parameter estimation

Methods for the parameter estimation can be broadly classified into two main categories: those that are based on solving a system of equations (number of equations equals number of parameter) such as methods of moments (MM), and those based on optimization of some relevant criteria such as MLE.

MM is based on solving a system of equations, where the population parameters are estimated by equating sample moments with unobservable population moments. On the other hand, MLE is based on the maximization of the probability (likelihood) of the sample data. Formally, consider x being a continuous random variable with probability density function $p(x, \theta)$ where $\theta = (\theta_1, \ldots, \theta_n)$ is a vector of unknown parameters. The log–likelihood function is given by

$$\ell = \sum_{i=1}^{n} \log p(x_i, \theta).$$

The MLE estimates of the parameters θ_i are then obtained by solving

$$\frac{\partial \ell}{\partial \theta_i} = 0.$$

This method yields an asymptotically unbiased, efficient and functionally invariant estimates for large enough sample. However, MLE has several limitations. It is not robust, i.e. it may produce wrong results when there are errors or outliers in the data or when the distribution is not correctly identified. It may fail when there is a range restriction parameter or inequality constraints on the parameter estimates. If the maximum likelihood function occurs on the boundary of the parameter space, this may have adverse implications for the MLE estimation and associated tests. Though, thanks to invariance property of MLE, in many situations this may be overcome by an appropriate re–parametrization. Finally, MLE techniques should be used with caution when the volume of data available for parameter estimation is not sufficiently large and when the data show a high level of kurtosis.

Another possibility is to use bootstrapping that consists in evaluating estimators against the population from which the samples are taken, generated via simulations.

The aggregate loss distribution

The aggregate loss distribution can be derived in a number of ways both analytically and numerically. The most simple method assuming that the aggregate loss (AL) is a sum of n individual losses, determines the aggregated mean and variance as follows

$$\begin{aligned} \mathbb{E}[AL] &= \mathbb{E}[Loss/Failure] \cdot \mathbb{E}[Failure] \\ \sigma^2[AL] &= \mathbb{E}[NumberofFailures]\sigma^2(Loss/Failure) + \sigma^2(NumberofFailures)\mathbb{E}[NumberofFailures]^2. \end{aligned}$$

With complex combinations of distributions, alternative methods include convolution integrals and Monte Carlo simulations.

Financial institutions use operational risk models to quantify and better understand the risks they are facing. They are used for decision making in assessing capital adequacy level, appetite or in stress–testing, and so should serve as a reliable decision–making tool.

The Traditional Operational risk is inherent in all banking products, activities, processes and systems, and the effective management of operational risk has always been a fundamental element of a financial institution's risk management program. As a result, sound operational risk management is a reflection of the effectiveness of the board and a senior management in administrating its portfolio of products, activities, processes and systems.

3.1.2 Main Sources of Model Risk

Any operational risk model will only be as good as the institution underlying framework for the identification, assessment and management of operational risk losses. Without an appropriate framework, losses may not be captured while scenario analysis is likely to miss important risks. Priority should be given to addressing any deficiencies and modelling restrictions in the operational risk control framework. Some potential sources of model risk follow.

Inputs. Due to the high severity and low frequency of operational events, operational risk is typically assessed using different sources of data, including internal loss data, scenario analysis, external loss data and expert judgement which is prevailing through the whole modelling process. Internal and external data are subject to the relevance of historical data that may not be sufficient to capture low frequency, high severity loss events. What often happens with high impact operational risks is that they hit multiple institutions around the same time rather than being spread across a prolonged period.

Other inputs constitute correlation and dependency assumptions, and segmentation. Wrong segmentation may lead to losses being reported in one category when they are meant to be classed and modelled under another, or a common scenario analysis failing is where the same risk is considered under two separate categories due to confusion as to which category it should belong, or the risk may be missed altogether.

Model design. The right selection of the loss distribution and its estimation constitute a crucial part in operational risk modelling. Since each approach is based on a different set of assumptions, different probability models emerge. Yet, due to the lack of sufficient data on high–severity, low–frequency losses, it may be difficult to ensure reliability and stability of the results. In particular, the credibility of such estimates depends on a number of very strong assumptions which are often very difficult to prove.

Usage. The operational risk models are used for several purposes, as such every limitation and potential model risk cannot be judged out of context as the final usage will determine the impact.

As the approaches to operational risk are based on a variety of assumptions, simplifications, different sources of data and different methodologies and techniques are applied during the model development process and deployment, their usage subjects financial institutions to model risk. Since an accepted industry standard is to use the LDA–based approaches that, as we have shown, are based on measure theory, model risk can be quantified using our proposed framework defined in Chapter 6 and Chapter 8, and so can be properly and objectively managed.

3.2 Other Non–Financial Risks

This section briefly introduces the main concepts of other non–financial risks that are typically relevant and considered material through the ICAAP. Material risks refer to risks that either alone or in combination with other risks may have a significant impact on the financial institution either be it through financial loss, reputational loss or the bank capacity to maintain appropriate level of capital.

BUSINESS RISK

Business (strategic) risk represents the potential that can adversely affect the fulfilment of business and strategic objectives to the extent that the viability of a business is compromised. It may arise from poor strategic positioning, failure to execute strategy or lack of effective responses to material negative plan deviations caused by either external or internal factors, such as macro, financial and idiosyncratic drivers.

Business risk can be assessed using scenario–based or statistical methods where the capital is measured on the basis of fluctuations in income and expenses that cannot be linked to any other risk categories. An alternative to this approach is to relate general economic conditions such as GDP to the financial performance for instance by regression techniques [51]. A probability distribution is assigned based on the volatility of earnings and costs.

INSURANCE RISK

Insurance risk, typically, refers to the risk related to non–trading market risk that has been classified as material risk. It can be classified into life insurance risk and non–life insurance risk. The first refers to the potential that future claims and expenses will cause an adverse change in the value of long–term life insurance contracts through the realisation of a loss, or the change in insurance liabilities. The value of life insurance contracts is the expectation in the pricing and/or liability of the underlying contract where insurance liabilities are determined using an economic boundary. Hence, it relates to the following risk exposures: mortality, morbidity or disability, retrenchment, longevity, life catastrophes, lapse and persistence, expenses and business volumes.

The non–life insurance or short–term insurance risk relates to the potential of unexpected underwriting losses in respect of existing business as well as new business expected to be written over the following twelve months. These losses could results from adverse claims, expenses, insufficient pricing, inadequate reserving, or through ineffective mitigation strategies like inadequate or non–adherence to underwriting guidelines. It covers premium, reserve, lapse and catastrophe risk exposures.

REPUTATION RISK

Reputation risk refers to the potential of adverse impact on earnings, liquidity or capital arising from negative stakeholder (customers, regulators, shareholders, employees) perception or opinion of the bank business operations, activities and financial condition. The most prevalent drivers are risks related to ethics and integrity, physical and cyber security, products and services, or third-party relationship risk. So as those risks proliferate, reputation risk heightens as well.

Financial institutions typically address reputation risk as an ongoing strategic issue while using data analytics to identify potential threats to their reputation. Reputation risk can measured either through content harvesting, content analytics, sentiment (reputation) score, correlation model or sentiment distribution fit with simulation model.

COMPLIANCE RISK

Compliance risk refers to the potential of legal or regulatory sanctions, material financial loss or loss to reputation that a financial institution may suffer as a result of its failure to comply with legislation, regulation, rules, related self–regulatory organisation standards or codes of conduct applicable to the activities of the entity.

INTANGIBLE ASSETS AND GOODWILL

An intangible asset is an identifiable asset, non–monetary, without physical substance, e.g. licences, patents, software, intellectual property rights or trademarks. The risk emerges from adverse changes in the book value of intangible assets and deferred tax assets. It is a material risk hedged through processes, reporting and monitoring.

Goodwill plays a significant role in determining the value of an entity and is defined, according to IFRS 3 Business Combinations [70], as the future economic benefit arising from assets acquired that are not individually defined and separately recognized. The goodwill is calculated as the difference between the cost of the investment and the investor share of the net fair value of the investee identifiable assets and liabilities. Goodwill arises when the overall value of the acquired credit union in total is greater than the fair value of its assets, liabilities, and intangible assets. In other words, goodwill arises when the overall value of the credit union in total is greater than the sum of its parts.

As such, goodwill is a set of assets that encourage customers to use the products and services of the given company. Negative goodwill (asset value) may be the lack of stable customers, poor product quality, lack of marketing skills, low-skilled personnel, lawsuits, precedent to the manufacture of counterfeit products. Currently existing methods for determining the value of goodwill of the institution are not universal.

INFORMATION TECHNOLOGY RISKS

Banks are moving to fully digitized, increasingly automated operations with the goal of improving analytics, customer service, and operational efficiency. Technology has huge benefits, but it also presents pervasive, potentially high-impact risk. Cyber risk in the form of data theft, compromised accounts, destroyed files, or disabled or degraded systems. Other IT risks that institution face include, for example, the risk from misalignment between business and IT strategies, management decisions that increase the cost and complexity of the IT environment, IT resilience and continuity risk, data management risk, incident response risk, insufficient or mismatched talent, or technologies may become obsolete, disrupted, or uncompetitive. Technology risk may lead to strategic, financial, operational, regulatory, and reputational implications.

THIRD–PARTY RISK

Financial institutions rely on third parties in their day–to–day operations with the objective to enhance organizational agility and scalability, and to introduce product or service innovation by leveraging specialized knowledge or skills. Despite the benefits these relationships promise, however, reliance on third parties also exposes organizations to higher levels of risk. This risk ranges the threat of high–profile customer service disruption to the risk of regulatory violation, from reputational damage and supply chain breakdown to exposure to financial fraud.

A standard methodology to manage this risk is through a silo approach where individual risks are approached separately. For example, the focus might be on the IT department and the data protection issues and risks of sharing data with third parties, or on risks to product quality and safety.

MACROECONOMIC RISK

Macroeconomic risk refers to the potential loss due to changes in macroeconomic factors and their impact on the minimum capital requirements. Macroeconomic models are important and valuable tools for prediction, understanding and analysis. When properly designed and employed, they allow to aggregate a wide range of diverse economic data and provide the information contained therein into a specific set of predictions for different purposes, including forecasting, insight and policy (or scenario) evaluation.

The selection of an accurate macroeconomic model is typically a trade–off between being vaguely correct and precisely wrong. As an example, consider a single regression where one of the modelling assumptions is the selection of casual variables. The more variables are chosen the less the potential of bias in the forecast. Though, this represent an increased collinearity and reduced degree of freedom, and so to less reliable estimates.

There are several approaches currently used in forecasting and policy analysis ranging from the structural models such Dynamic Factor models (DFM) or Dynamic Stochastic General Equilibrium (DSGE) models to other widely used econometric models such as vector autoregressions (VAR) or large–scale econometric models. The DSGE models that are based on a complex, forward–looking dynamic optimization problems with the aim to explain business cycle features of the data applicable for quantitative policy analysis [83]. These models are effective tools for evaluating specific equations, yet they are purpose–built and rely on fairly strong assumptions to allow computational tractability. Thus, they do not represent a reliable forecasting tools rather predictive tools for policy evaluation.

On the other hand, the VAR models are data–driven ([106]) where endogenous variables are regressed on their own lagged values and those of all other endogenous variables. These models lead to superior forecasting results within very short term, but their performance decreases with increasing time horizon.

In general, macroeconomic models rely on rather strong assumptions and simplifications of the reality that exposes financial institutions to high level of model risk and their reliability has to be carefully examined.

3.2.1 CONCLUSIONS

Non–financial risk covers a broad range of issues and cuts across traditional categories. Improving NFR metrics, indicators, and reporting has to be addressed and embedded into the risk appetites and risk culture of institutions, although it has to be understood that perfect quantification and aggregation of some risks is impossible.

Chapter 4

Provisions, Stress Testing and Dependencies

The present chapter represents an introduction to the standard calculation of provisions (Section 4.1) and to the stress testing (Section 4.2) within financial industry. The last Section 4.3 is dedicated to the interrelationships and dependencies between different types of risk and the methodologies used for the risk aggregation.

4.1 Provisions

Provisions, in contrast to the capital that is expected to cover the unexpected loss, are designed to control the expected credit losses (ECL), i.e. the average credit loss rates anticipated over time.[1] From an accounting perspective, provisions reflect the reductions in the carrying amount of a loan, or a group of loans, based on the evidence of impairment.

Under the International Financial Reporting Standard (IFRS) 9 [71], issued by the International Accounting Standards Board (IASB), banks are required to estimate provisions on all loans in their banking book based on forward–looking expected credit loss (ECL) impairment model that requires early recognition of credit losses in financial reporting.[2] The expected loss is defined as the estimated present value of all expected cash shortfalls over the expected life of the asset (debt instruments, loan commitments, lease receivables, contract assets).

Under IFRS 9, financial assets are classified into three groups (see Figure 4.1):

- **Stage 1 (performing assets)** including financial instruments with no significant increase in credit risk since initial recognition, or those with low credit risk at the reporting date.

[1] Credit losses are defined as the difference between all the contractual cash flows that are due to an institution and the cash flows that it actually expected to receive. This difference is discounted at the original effective interest rate or credit–adjusted effective interest rate for purchased or originated credit-impaired financial assets.

[2] IFRS 9 serves also as a single accounting standard that includes requirements for the classification and measurement of financial instruments, and general hedge accounting (aligns hedge accounting more closely with risk management).

- **Stage 2 (underpeforming assets)** comprising assets with significant deterioration in credit quality but with no objective evidence of impairment.

- **Stage 3 (non–performing assets)** including non–performing financial instruments, i.e. instruments with objective evidence of impairment at the reporting data (classified as doubtful or default).

The asset classification is based on the business model for managing the asset (assessed at portfolio/business unit) and the asset contractual cash flow characteristics (assessed at instrument level). The transfer of financial assets among levels is symmetric, i.e. any asset can move back to the first stage if there is a significant improvement in the credit quality [71].

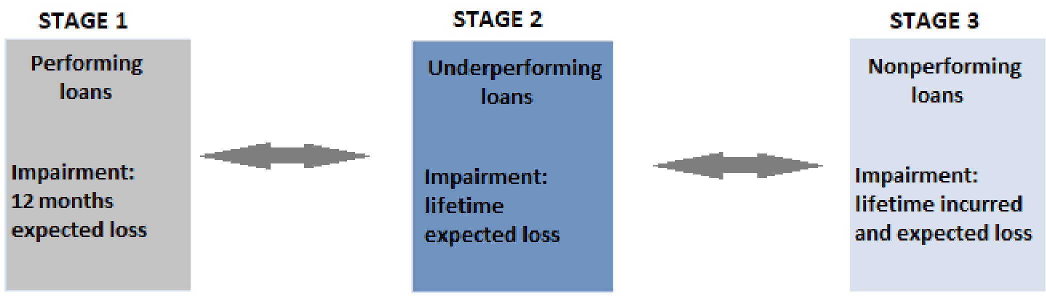

Figure 4.1: Three categories of financial instruments under IFRS 9. As indicated by the arrows, the staging of assets can move in both directions. For example, a loan from Stage 2 can move back to Stage 1 if it is no longer considered to be subject to a significant increase in credit risk.

4.1.1 Data, Models and Methodologies

IFRS 9 does not prescribe a particular method for measuring expected credit losses as it acknowledges that measurements might vary based on the type of instrument and the information that is available. It does however requires to measure expected credit losses in a way that reflects [71]:

- an unbiased and probability–weighted amount that is determined by evaluating a range of possible outcomes;

- the time value of money; and

- reasonable and supportable information that is available without undue cost and effort at the reporting data, historical events, current conditions and forecast of future economic conditions.

The objective of an unbiased and probability–weighted amount is neither to estimate a worst–case scenario nor to estimate the best–case scenario. An estimate should however reflect the possibility that no credit loss occurs even if the most likely outcome is no credit loss [**?**].

Financial institutions are free to select an appropriate methodology for the ECL estimation as well ass the type of macro–economic variables considered. A standard practice is to employ either **econometric models** that are developed to quantify the historical relationship between macroeconomic factors and generate some number of forward–looking scenarios, or **scenario–based models** that are developed to quantify the impact of a given scenario on credit losses. The estimated ECLs need to be discounted to the reporting date using the effective interest rate determined at initial recognition or an approximation of it.

Expected loss is estimated for all loans using a wide range of inputs. Suitable input information refers to the information that is reasonably available at the reporting date without undue cost or effort, including information about past events, current conditions and forecasts of future economic conditions. The information used is required to reflect factors that are specific to the borrower, general economic conditions and an assessment of both the current as well as the forecast direction of conditions at the reporting date. Possible data sources comprise internal historical credit loss experience, internal ratings, credit loss experience of other entities, external ratings, reports and statistics. In case an institution does not have sufficient sources of institution–specific data of its own, it may use peer group experiences for comparable financial instruments.

The estimation of the expected credit loss differs across the three categories of instruments (see Figure 4.1). For Stage 1, provisions are estimated based on 12–month expected losses, i.e. the probability of default over the next 12 months multiplied by the total expected credit loss if they default.[3] Provisioning for Stage 2 assets is being recognized on lifetime expected losses, i.e. the expected shortfalls in contractual cash flows, taking into account the potential of default at any point during the life of the financial instrument, usually assessed collectively for similar contracts. The interest revenue for Stage 1 and Stage 2 assets is recognized based on effective interest rate method on gross carrying amount, so no loss allowance is taken into account. Provisions for Stage 3 are estimated based on lifetime losses, though the probability of default, assessed usually individually contract by contract. The interest revenue is calculated and recognized based on the amortized costs, i.e. the gross carrying amount minus loss allowance.

In order to calculate the expected loss, banks should apply **credit risk models** (PD, EAD and LGD), **balance sheet forecast** (prepayments, facility withdraws) and **interest rates** (discount factors). Even though, the estimation of the EL, under both accounting impairment and regulatory capital standards, is based on the same key input credit parameters, their estimation differs as their different objectives (prudential vs. neutral). For example, PD used for capital calculation is typically based on a 12–month time horizon through the economic cycle, and the LGD is estimated based on a downturn scenario. On the other hand, to calculate provisions, the PD is estimated on the basis of a 12–month or lifetime horizon at a specific point–in–time of an economic cycle, reflecting current economic circumstances (i.e., it is a best estimate rather than a conservative estimate) whereas the LGD is calculated based on a neutral scenario reflecting current economic circumstances (see Figure 4.2). These parameters represent forward–looking estimates, factoring in macroeconomic forecast variables with their probability of occurrence for the years ahead. The key differences between requirements on capital and provision calculations are shown in Table 4.1.

The estimation of the lifetime PD term structure can be based on the cohort analysis (e.g., conditional for survival, Kaplan–Meier estimate of hazard functions to remove potential biases in the data), the regression modelling (e.g., relationship between historical PDs and behavioural factors using linear regression with logistic transform or Cox regression for survival function), or in case when cohort data is not available on transition matrices where extrapolation assumes a memoryless process. The estimation of the lifetime EAD can be performed through amortizing products such as term loans and mortgages, revolving products such as credit card or line of credit,

[3]Note that 12–month expected credit losses do not represent the expected cash shortfalls over the next 12 months, nor the credit losses on financial instruments that are forecast to actually default in the next 12 months

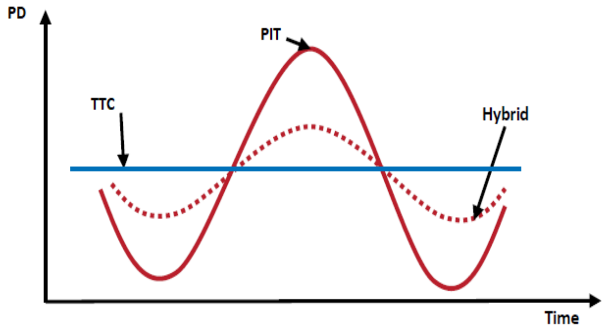

Figure 4.2: The difference between Point–in–Time (PIT) and Through–the–Cycle (TTC) estimation of PD. The PD estimate using the PIT approach will be lower during a good time and higher during an economic downturn, compared to the TTC or hybrid approaches.

Calculation of provisions using IFRS 9 methods is rather simple and straightforward once the credit parameters and cash flows have been computed. Commonly used methods rely on either estimates of forward–balances or discounted value of expected cash flows. The key challenges constitute in managing large volumes of data coming from multiple sources and the estimation of the lifetime credit parameters.

4.1.2 Main Sources of Model Risk

The implementation of IFRS 9 represents a significant challenge to the risk and finance functions across banks. IFRS 9 introduces new concepts and measures such as significant increase in credit risk and lifetime expected credit losses. The impairment models vary in complexity and inputs depending on the size of the portfolio, the amount of data available and the sophistication of the market concerned. The ECL is required to be estimated over a range of economic scenarios and then weighted according to some probability based scheme. This poses substantial challenges theoretically as well as operationally.

The calculation of provisions involves a high degree of expert judgements in several areas. For example, the requirement of a long data history (loan lifetime; segmentation and a full economic cycle) may lead to data issues, e.g. a lack of sufficient or missing data, and as a consequence to documented business assumptions.

The three–stage process for estimating provisions poses a further challenges. The migration of assets between the three stages is a function of quantitative risk measures as well as individual banks own policies on what is considered a significant deterioration in loan quality as the concept of ’a significant increase’ in credit risk is not explicitly defined. Thus, uncertainties in estimations can result from the definition of migration from Stage 1 to Stage 2. To make this assessment an institution compares the risk of a default occurring on the financial instrument as at the reporting date with the same risk as at the date of initial recognition, considering sufficient information. The definition can be based either on quantitative or qualitative factors and should be consistent with the definition used for internal credit risk management purposes. Quantitative factors include changes in credit ratings, changes in internal price indicators of credit risk or changes in external market indicators, whole qualitative factors comprise changes in business, financial or economic conditions, changes in operating results, or other qualitative inputs.

Finally, there are no industry standards on what type of forward–looking information should be used, the time horizon or how the information should be integrated in setting provisions.

	Key risk parameter	Basel III	IFRS 9
PD	Measurement standard	Average of default within the next 12 months	Depending on the asset, the PD measures either for the next 12 months or for the remaining life of the financial instrument
	Period of measurement	Estimates based on long–run average default rate, ranging from PIT to TTC	Estimates based on PIT measures, at the reporting data, of current and expected future conditions reflecting future economic cycles
LGD	Intention of estimate	Downturn LGD to reflect adverse economic scenarios	Current and forward–looking LGD to reflect impact of economic scenarios
	Collection cost	Considers both direct and indirect cost associated with collection of the exposure	Considers only cost directly attributed to the collection of recoveries
	Discount rate	Based on weighted average cost of capital or risk–free rate	Depends on the type of financial instrument but is broadly based on effective interest rate
	Period of observation	Minimum of five years for retail exposures, seven years for sovereign, corporate, and bank exposures	No specific requirements about observation period or collection of historical data used
EAD	Intention of estimate	Downturn EAD to reflect what would be expected during a period of economic downturn	Considers all the contractual terms over the lifetime of the instrument
	Period of estimation	Minimum of five years for retail exposures, seven years for sovereign, corporate, and bank exposures	No specific requirements about observation period or collection of historical data used
Expected Loss	Calculation	$PD \times LGD(loss\,rate)$ is applied to EAD	$PD \times PV$ of cash shortfalls refers to a probability–weighted estimate of credit loss
	Economic assumptions	Reflect downturn LGD and EAD (factoring in macroeconomic stress conditions)	Reflects an unbiased probability–weighted amount, determined by evaluating a range of possible outcome

Table 4.1: Comparison between Basel III and IFRS 9 requirements.

4.2 STRESS TESTING

Stress testing is a critical component of both the capital adequacy assessment process (ICAAP) and the risk and capital management process. The concept of stress testing refers to several different techniques used to assess the impact of events or combinations of events that, while plausible, are far from business as usual, and so not only comprises the mechanics of applying specific individual tests, but also the wider environment within which the tests are developed, evaluated and used during the decision–making process. The BCBS defines a stress test as the evaluation of a financial position of an institution under a severe but plausible scenario to assist in decision–making [21].

Stress testing is required by regulators, such as the Bank of International Settlement (BIS), the Basel Committee on banking Supervision (BCBS), the International Accounting Standards Board (IASB) or the European Banking Authority (EBA) that provide an extensive and detailed guidance on a stress testing exercise and establish responsibilities with a systematic link to the bank risk appetite and strategy. Financial institutions are free to choose the specific methodology but it has to encompass the following key components: **sensitivity analysis** for specific portfolios or risks, e.g. the 200bp shift in interest rates required under the Basel Accord, and the **forward–looking scenario analysis** incorporating system–wide interactions and simultaneous occurrences of events or crisis, especially if they are likely to recur, as well on hypothetical combinations of exceptional but plausible events, namely considering downturn scenarios, and be meaningfully translated into a risk parameters while considering the impact of concentration risk. Finally, the stress testing framework should include the **reserve stress tests** through which scenarios and the related level of losses that would potentially cause a bank failure are identified.

4.2.1 DATA, MODELS AND METHODOLOGIES

Approaches to stress testing vary from basic sensitivity testing (the impact of a move in a particular risk factor on a portfolio or business unit; single–factor tests) to scenario analysis (the impact of a simultaneous move in a number of risk factors). In general, one might broadly distinguish between **structural** and **reduced–form** methodologies. Structural approaches establish a casual relationship between asset values and probability of default which posit a particular state of the economy (macroeconomic scenarios) while reduced–from approaches regard default event as a surprise. Reduced form models depend on assumptions about distribution and may have very little connection with macro variables. Whereas they are adequate for random, uncorrelated price movements or volatilities, they are less suitable for modelling systematic events or highly correlated changes. Eventually, some risk factors may be considered in either structural or a reduced–form, e.g. interest rate either directly affecting asset value or endogenously determined for instance as a function of inflation, economic growth or unemployment.

The recent requirements of specific regulatory exercises mainly focus on the macroeconomic scenarios where macroeconomic indicators such as GDP, unemployment rate, inflation, interest rate or property prices reflect severe shocks to the economy. The impact is modelled in terms of risk parameters, and then risk measures either on the individual level such as impairments, provisions or losses, or aggregate level such as regulatory or economic capital, or other metrics of the ICAAP framework. Forecast models are used to simulate paths of macroeconomic variables under alternative scenarios that are further used for regulatory stress–testing and accounting purposes. The quality of these forecasts will directly impact the results from stress–testing exercise. Thus, it is of high importance that macro models provide accurate forecasts but also appropriate shock responses in an efficient and transparent manner.

For example, in market risk, models need to consider key indicators of financial markets like their ability to anticipate economic developments, and the consequent potential of a non–synchronize translation of a recession scenario to a associated impact on prices. For OTC derivatives, a model needs to stress the underlying trades, the consequent derivative exposure, the corresponding credit spreads as well as the resulting credit losses. Within credit risk, stress testing is typically focusing on PD, LGD and CCF estimates either for individual counterparties or on the basis of sector portfolios and countries. Finally, operational risk scenario analysis is the most challenging, mainly due to the difficulties in data collection and the complexity in the aggregation of the results.

The selection of a stress testing model may be based on expert judgement or on a mathematically based set of rules. The most common practice is to model the relationships between the macroeconomic factors and the bank internal risk parameters through a multivariate regression model that uses these factors as independent variables and the internal risk parameters as dependent ones. Most stress testing models use a combination of GDP growth rate, unemployment rates, property prices, interest rates, inflation rates as well as other financial indicators including share price indexes, inventory levels, government debt, just to name a few. These indicators need to be measurable, with long enough data history at a desired level of granularity to allow for extrapolations over the relevant time horizon (especially for IFRS 9 models that are subject to long time horizon requirements).

A particular model is often chosen through some selection criteria, like the Akaike information criteria, the Bayesian information criteria, the Deviance information criteria, the Focused information criteria or the Hannan–Quinn information criteria [103], that sort potential statistical models by their effectiveness. However, many regressions often assume that data are stationary, i.e. their statistical properties remain constant over time, or that the residuals are uncorrelated or normally distributed, and that no heteroscedasticity occurs, which in general is not always true in reality.

Data must be available on macroeconomic factors as well as for internal risk parameters, but the nature and structure of the dataset varies in terms of their relevance to the specific stress testing exercise, the length of time horizon, the granularity, just to name a few. Conducting a proper stress testing may be challenging due to the limited availability of recession data which are the most relevant data for stress testing purposes. This data is often limited to few occurrences at most and so the reliability of the model is no less then doubtful. This problem is often addressed through back–testing both in its in–sample and out–of–sample version. The objective of back–testing is to assess the predictive power of the model by comparing its forecasts. The limitation of the in–sample approach is in the usage of historical data while the main weakness of the out–of–sample version is in many cases a requirement of internal data from an earlier recession period that may be of lesser quality or simply unavailable.

Expert judgement in the form of scenario construction and selection, constitutes another crucial element of stress testing models. An excellent model will be of no use if the scenario used as input makes no sense or is completely implausible. However, there are no simple ways for designing scenarios and it is often more an art than science. Scenarios can be developed by drawing on relevant historical experience or by analysis the impact of exceptional, but plausible events. Other techniques used to determine exposure to extreme events include a maximum loss approach, in which risk managers estimate the combination of market moves that would be most damaging to a portfolio, and extreme value theory which is essentially a statistical methodology based on the evaluation of the tails of a distribution of market returns [94]. Macroeconomic forecasting in an essence inaccurate activity, and so even if one believes that a scenario is plausible and internally consistent, there is still uncertainty on how likely the combination of the events is. Next, the probability of a scenario is in general not in any functional relationship with the probability of losses, as the latter depends on the severity of the scenario and not on its frequency.

Conducting stress testing is a requirement as a way to see how economic shocks affect banks capitalization. Scenario analysis, as already widely adopted in the industry, can be successfully applied in combination with management assessment of elementary scenarios corresponding to the unsystematic shocks of individual variables, aiming to identify and analyze complex and extreme combinations of events based on complex and entirely new scenarios from simpler historical evidence and variations. Even though this exercise comes with the lack of specific data and the difficulty to determine and formulate such extreme scenarios, it provides a means to systematically address the prevention or avoidance of financial distress.

4.3 Interrelationships Between Different Types of Risk

Although analysing each type of risk in isolation allows measures to be customized to suit the properties of that particular risk type, and thus improves their quality as stand–alone measures, to give a view of risk for a whole financial institution and the total capital these measures need to be combined. However, as risks do not occur in isolation but are interconnected and may even mutually reinforce each other, the aggregation process is surrounded by many challenges.

The concept of economic capital comprises the processes and methodologies whereby a bank determines the risks generated by its activities and estimates the amount of capital it should hold against those risks. The overall economic capital is typically assessed first by calculating the individual risk components (marginal distributions) that are further aggregated into a integrated picture, i.e. an aggregated risk and an overall capital. The assessment and management of individual risk components involves a vast range of qualitative and quantitative methodologies, some of them common to a number of risk categories, others adjusted to the particular characteristics of a specific type of risk. The assessment of the total capital is then, primarily, an exercise in risk integration across different products, business lines and areas, geographies, and so on. The validation techniques, aside from ensuring the quantitative soundness of the individual components, have to focus on the quality of data and procedures as well as on the soundness of the theoretical assumptions underpinning the integration process.

The development of an allocation methodology capable of modelling relationships and diversification while providing a capital charge that can be reliably allocated is a crucial step when assessing the total economic capital. The aggregation methodologies vary both in complexity and sophistication, and depend, primarily, on the risk classification structure chosen. This structure may follow the economic nature of the risks or the organizational structure of the firm. A typical classification of risk types may include: market risk, credit risk, counterparty credit risk, interest rate risk on the banking book, operational risk (including legal risk), business risk (risk to future earnings, dividends and equity price), reputation risk and strategic risk. The aggregation methodology is then applied across the different risk types, across business lines or both, under the assumption that individual risk measures are comparable.[4]

The fundamental issues related to the aggregation process are:

- **Differences in risk distributions:** The shapes of distributions differ for various types of risk and the use of simple portfolio risk formulas for aggregation becomes inappropriate. For example, distributions for operational risks and credit risks are extremely skewed, market risk distributions are typically more symmetric but with fat tails.

- **Dependencies** between different types of risk are difficult to measure with confidence. For example, correlation may not be the appropriate measure of dependence between different types of risk with fat tails, as tail outcomes of diverse risks might be more highly correlated than other outcomes.

- **Different time horizons:** Risk tends to be measured over different horizons for diverse types of risk, though to aggregate risks at the firm level forecasts over comparable periods are required.

A key modelling challenge in this regard is the specification of a dependency structure for the different risk segments. The BCBS distinguishes between several approaches to risk aggregation and describes benefits and drawbacks thereof. They consider following approaches:

- **Summation** involves adding together individual capital components, i.e. assuming perfect dependency between risks (equal weighting assumption); does not discriminate across risk types and does not capture non–linearities.

- **Constant diversification** is similar to summation but it assumes a fixed percentage deduction from the overall capital figure; the fixed diversification effect is not sensitive to underlying interactions between components and it does not capture non–linearities.

- **Variance–Covariance** entails weighted sum of components on bases of bilateral correlation between risks; estimates of inter–risk correlations are, however, difficult to obtain and it does not capture non–linearities.

- **Copulas** combine marginal distributions into a joint risk distribution through copula functions. It is more flexible than covariance matrix while allowing for non–linearities and higher order dependencies. Yet, it is complex and the parametrization is very difficult to validate.

- **Structural modelling/simulation** refers to a process where the impact of common risk drivers on all risk components are simulated and then the joint distribution of losses is constructed. This method is practically the most demanding in terms of inputs, IT, time requirements and due to its complexity may provide false sense of accuracy. Nevertheless, it is theoretically the most appealing method and potentially the most accurate and intuitive.

Apart form the structural modelling, all of the other approaches, mentioned above, consist of combining the outputs of individual, potentially very different, risk models using correlations and dependencies estimated on the basis of

[4] As risk measures are typically expressed through probabilities at a given confidence level over a certain time horizon, comparability means that risk measures need to be consistent in terms of all these features.

a number of assumptions that rather strict and often assigned ad hoc[5]. For example, the output of an economic capital may just be a weighted average of distribution quantiles like VaR measures or the quantile of a multivariate distribution, in case of copula, of individual risks which are scaled in terms of time horizon and confidence interval. In practice, financial institutions often do not rely on a single measure and use a combination of various methods within their decision–making process.

The realistic measurement and modelling of a dependency structure that links the individual risks is one of the most challenging aspects of economic capital modelling. Wrong assumptions may result in a specious optimistic view of an institution, despite the fact that the individual capital components themselves may be quite reasonable. In general, however, little is known about interrelations, dependencies and partial overlaps between different risk types, e.g. between credit and market risk, as well as within a specific risk type, e.g. between interest rate risk and equity risk. Some of the examples of the relationships between the key risk types are presented in Table 4.2. For

	TYPES OF RISKS						
	Credit (AIRB)	*Credit (LDP)*	*Market*	*Derivative*	*XVA*	*Operational*	*ALM*
Credit (AIRB)	X		X	X		X	X
Credit (LDP)		X	X			X	
Market			X			X	
Derivatives			X	X	X	X	
XVA			X	X	X	X	
Operational	X	X	X	X	X	X	X
ALM	X	X	X	X	X	X	X

Table 4.2: Interactions across different types of risk where Credit (LDP) stands for Low default portfolios, XVA refers to the X-Value Adjustment pricing and ALM is the Asset and Liability Management.

instance, there is a dependence between

- credit and liquidity risk considering the probability of negative impact of degenerated financial condition of the borrower on the bank cash flows and respectively the ability of the bank to timely meet its commitments,

- credit risk and risks in the trading book considering the possibility of negative impact of fluctuations of such market variables like exchange rates, interest rate, commodity price, etc., as well as financial condition of the counterparty, factors of the risk of settlements, large exposures in the trading book on the ability of the bank to timely meet its commitments,

[5] An example is the National Association of Insurance Commissioners (NAIC) risk–based capital regime for insurance companies, in which risks are aggregated by a formula based on relatively simple but rather arbitrary correlation assumptions. Another possibility is to stop short of aggregating bottom-up risk measures, turning instead to alternative measures of risk at the whole-institution level.

- operational and credit risk considering the possibility of negative impact of operational risk events related with fraud, occupational safety, damage to physical assets, interruption of business and systems, as well as application of credit risk mitigation measures and credibility of provider of credit risk mitigation measured on performance of the bank, and so on.

Next, the parametrization of the variables used to model dependency structure is often very complex besides the fact that these parameters evolve over time as a result of changes in economic indicators, business cycle or underwriting cycle. The changing feature of their relationship as well as the tail dependencies among them should be considered.

The estimation may be based on the historical time series data of underlying risks, either internal, external or simulated data; and/or on the expert judgement (for example parameters may be based on the consensus of risk officers, underwriters, business managers, and other specialists in the institution who understand the nature of risks being modelled) which is frequently complemented with an input from external consultants and industry benchmarks. As the economic models comprise different components, it makes them also very demanding in terms of data. Data need to include the input information, the functional logic and the related outputs of all the models whose outputs are relevant for the computation of economic capital.

Within the context of interrelations across risk types, it is similarly important to avoid any double counting of risks, i.e. overlapping risks. The BCBS in Basel III [25] requires banks to subtract from the highest quality form of capital (CET1) a significant investment to prevent double–counting, i.e. to prevent that the bank is not sustaining its own capital with capital that is also used to support the risks of insurance subsidiary. Another double–counting problem arises in the calculation of the loss absorbing resources between the accounting standards (forward–looking loss provisions) and prudential regulations (regulatory capital). Thus, the impact on capital ratios resulting from the accounting standards IFRS 9 should be taken into account in the overall calibration of the capital framework to avoid double counting.

4.3.1 MAIN SOURCES OF MODEL RISK

The concept of economic capital is very complex comprising the processes and methodologies through which a bank assesses the risks generated by its activities and estimates the amount of capital it should hold against those risks. While the sophistication and employment of the data, models, and software systems for individual risk types has become widespread, the methods and tools to consistently measuring integrated risk and deriving the comprehensive picture of the economic capital lag behind. There is a wide margin of potential error in aggregation methods and measured diversification effects (determined primarily by dependencies). Model risk is a prominent in the area of risk aggregation.

Data. As the allocation of capital is based on the tail events that by definition are rarely experienced, the quality and quantity of availability data represents an issue.

Expert judgement plays a major role in many approaches to integration, mainly due to the complexity of the problem and scarcity of the available data. For example, determining realistic or credible dependencies generally is a matter of informed judgement exercised outside of the models themselves.

Model design. The aggregation approaches range from very simple summations to more complex methodologies, such as copulas or structural modelling. These approaches very often rely on both complex and strong assumptions, and are typically limited in their ability to describe the dependency effects of various risk factors that drive the loss. Other issue arises from the inability to capture certain material risks, such as reputation risk or liquidity risk that are difficult to accurately quantify.

Furthermore, these challenges become evident with increasing number of risks considered and the particular aggregation methodologies attempt at modelling the complex relationships amongst those risks.

Financial institutions that recognize the high level of model risk inherent in the aggregation modelling typically use multiple methods of risk aggregation, to highlight the potential range of results and to generate benchmarks. Beside, model risk has to be adequately managed, ideally, through identification, assessment and by setting a proper control over the use of these models. To objectively assess model risk, to assess the credibility and reliability of outputs, model risk should be quantified. We refer the reader to Chapters $6-9$ on how to measure model risk inherent in these models.

CONCLUSIONS – PART I

Financial models are quantitative methods (including the complex manipulations of expert judgements) or systems that apply theories and methodologies to process input data into quantitative estimates for decision making. They are designed to meet regulatory requirements and to achieve business needs. Quantitative analysis and models are employed for a variety of purposes including capital allocation, risk management and assessment, exposure calculation, trading, budgeting, instrument and position valuation, financial reporting, stress and scenario testing, macro forecasting, prudential requirements determination, reserves calculation, establishing customer loyalty, among many others.

Financial models vary in complexity and sophistication, ranging from simple formulae to complex models that require simulations and optimization routines. Some are internally developed and some may be sourced through third–party; some are used for a variety of purposes, while others are tailored to the particular features of a specific use. Some model outputs drive decisions; other outputs provide one source of management information, some outputs are further used as inputs in other models. Additionally, the model outputs may be completely overridden by expert judgement.

The use of these models brings undoubted benefits, such as automated decision–making that in turn improves efficiency by reducing analysis and costs related to manual decision, objective decision–making or ability to synthesize complex issues such as a firm–wide risk. However, the potential failures and misuse of these models can present large risks to the institution through financial loss, poor business or strategic decisions, or reputational damage. Model error encompasses, for example, simplifications, approximations, data deficiencies in terms of both quality and quantity, including missing values, lack of critical variables, lack of historical depth, the use of unobservable parameters, the lack of market consensus on the model functional form, wrong or missing assumptions, improper fitting process, macroeconomic stress, computational difficulties, among others. On the other hand, model misuse comprises applying model outside its design objective and intended use, for example a rating model based on a specific portfolio and applying it to a different portfolio from another country, or absent re–estimation and re–calibration for a long time, lack of understanding of model limitations or assumptions, just to name a few. See Chapter 5 for more potential sources of model risk.

Regardless of the source, model risk can have profound financial and reputational implications and may lead to missed opportunities, costly errors or unforeseen exposures, i.e. it may substantially impair business decision making. For instance, a fault in a hedging model may lead directly to inappropriate trade, causing unintended risk and realized losses; misinterpretation of key financial metrics can negatively impact policyholder benefits and sales force morale; or errors in the estimation of the loss distribution may result in the underestimation of capital, risk or reserves.

Regulators require financial institutions to have a proper MRM in place and provide a clearly defined supervisory guidance on its structure, e.g. BCBS directive 21, US Fed SR 7–11 and Solvency II. They fail, however, to discuss model risk quantification aspects in detail nor provide a general framework for its assessment, except in very specific cases related to the valuation of certain products where model risk is assessed through valuation adjustments (model risk AVAs).

Against this background, the aim of the next Part II is to introduce model risk by providing the definition, analysing its nature and sources, designing a general approach within the framework of differential geometry [89] and information theory [6] for the quantification of model risk (see Chapters 6, 8) and illustrating its potential practical applications through examples on credit risk (see Chapter 7) and financial derivatives (see Chapter 9). In our framework, models are represented by particular probability distributions that belong to the set of probability measures, so called statistical manifold. As such, this framework has the potential to asses many of the mathematical approaches currently used in financial institutions: credit risk, market risk, operational risk, derivatives pricing and hedging, XVAs (valuation adjustments), models for capital calculation, stress–testing or reserves. For example, the essence of a risk quantification is the estimation of the uncertainty of the variables that govern the events, i.e. probability distribution (e.g. to quantify market risk, a risk measure is applied to the distribution of future price changes), capital management is assessed by examining the loss distribution, a fair value of a derivative is calculated as an (conditional) expectation, simulation techniques produce distribution of possible outcome values.

The potential impacts of model risk, such as an institution reputation, credit rating or financial loss, may more than offset the cost of implementing a sound MRM encompassing identification, quantification, mitigation and control. Combined with a strong management insight on monitoring models and their risks will allow institutions to strengthen their decision making processes, improve their profitability, and add value to the enterprise as well as reduce risk.

Part II

MODEL RISK: MANAGEMENT AND QUANTIFICATION

INTRODUCTION

"If the map were the territory, we could run our models and predict the future." (Sergio Scandizzo)

Financial models, quantitative methods (including the complex manipulations of expert judgements) or systems, are simplifying mappings of reality to serve a specific purpose aimed at applying mathematical, financial and economic theories to process the available data into quantitative estimates for decision making. They deliberately focus on specific aspects of the reality and degrade or ignore the rest. Understanding the capabilities and limitations of the underlying assumptions and the materiality of their consequences are central when dealing with a model and its outputs. Wrong design or implementation, misuse or inadequate knowledge regarding the model development and usage, misunderstanding of the uncertainty or risk ignorance, expose a bank to additional risks that might be unpredictable, global, difficult to hedge and ensure, and potentially fatal. Therefore, the implied objective of the model risk management should be to determine where models are useful 'enough' while understanding and managing the integral limitations implied by and inherent to model boundary conditions as wilful abstractions.

Model risk emerges from the model deployment and usage, it is difficult to measure and account for, it is challenging to manage and control for, it has no event[6] and no simple connection with financial loss, it lies in the knowledge of the business, the applicability and limitations of the model, the mathematics and numerical analysis used to solve it, the computer science used to implement and present it, and in the transmission of information and knowledge accurately from one part of the model to the next. It is precisely the kind of risk that may have a significant impact on the soundness and the stability of a financial institution or the financial system as a whole.

The wider application of models brought into focus the need for an efficient model risk management function, to ensure the development and validation of high–quality models across the whole institution, eventually beyond risk itself. The significance of managing model risk to modern banking have been recognized by the regulatory community that provides several approaches ranging from mitigation via model validation to the establishment of comprehensive framework for active Model Risk Management (MRM). The extensive guidance for the entire design, development, implementation, validation and inventory and use process was developed and issued jointly by the Office of the Comptroller of the Currency's (OCC) and the US Board of Governors of the Federal Reserve System (Fed) [33]. The guidelines were adopted by the OCC as Bulletin 2011–12a and by the Fed as FRB SR 11–7 and apply to all US institutions as well as foreign banks operating in the US while serving as starting point for the industry. This document provided an early definition of model risk that subsequently became standard in industry:

[6] Unless we only consider catastrophic events wit a complete failure of a model (e.g., due to fundamental design flaw), a model may be ordinarily affected by all sorts of issues ranging from inadequate data to improper usage and decision making, without these issues being recognized or even observed.

"[...] The use of models invariably present model risk, which is the potential for adverse consequences from decisions based on incorrect or misused model outputs and reports. Model risk can lead to financial loss, poor business and strategic decision making, or damage to bank's reputation."

Besides, they identify two main reasons of model risk:

- *" The model may have fundamental errors and may produce inaccurate outputs when viewing against the design objective and intended business uses. The mathematical calculation and quantification exercise underlying any model generally involves application of theory, choice of sample design and numerical routines, selection of inputs and estimation, and implementation in information systems. Errors can occur at any point from design through implementation. In addition, shortcuts, simplifications, or approximations used to manage complicated problems could compromise the integrity and reliability of outputs from those calculations. Finally, the quality of model outputs depends on the quality of input data and assumptions, and errors in inputs or incorrect assumptions will lead to inaccurate outputs.*
- *The model may be used incorrectly or inappropriately. Even a fundamentally sound model producing accurate outputs consistent with the design objective of the model may exhibit high model risk if it is misapplied or misused.... Limitations come in part from weaknesses in the model due to its various shortcomings, approximations, and uncertainties. Limitations are also a consequence of assumptions underlying a model that may restrict the scope to a limited set of specific circumstances and situations."*

SR 11–7 explicitly addresses incorrect model outputs, taking into account all errors at any point from design through implementation. It also requires that decision makers understand the limitations of a model and avoid using it in ways inconsistent with the original intend. Furthermore, they state that model risk should be managed and addressed with the same severity as any type of risk and that banks should identify the sources of model risk and assess their magnitude. They also emphasizes that expert modelling, robust model validation and a properly justified approach are necessary elements in model risk moderation, though they are not sufficient and should not be used as an excuse for not improving models.

In addition to the guidance issued by OCC and Fed there are different references and standards typically found in specific guidances related to areas such as valuation, and market and credit risk management. The main regulatory references in the EU include:

- The EU capital requirements rules CRR / CRD IV (2013), [112], that define model risk and the process by which the institutions should manage and implement policies and processes to evaluate the exposure to model risk in the context of operational risk as a requirement for consideration in the ICAAP (Pillar 2).
- The Guidelines on common procedures and methodologies for the supervisory review and evaluation process (2016),[57], requires to consider to what extent, and for which purposes, the institution uses models to make decisions and its level of awareness of and how it manages model risk.
- Furthermore, in the Stress Testing Guidance (2018), [18], the Bank of England request specific information on the key items that make up a typical Model Risk Framework, in particular documentation on: the firm's existing stress–testing policies, methodologies, and overall framework across all risk types including roles, responsibilities, governance arrangements, and coverage of portfolios.

Regardless of the institution size and structure, world–wide regulators require banks to implement a MRM comprehensive framework incorporating all relevant aspects of the MRM life–cycle, i.e. model risk identification and assessment, model risk measurement and mitigation, and model risk monitoring and reporting, with clearly assigned roles and responsibilities. Model risk is assessed as a material risk to capital, and institutions are asked to quantify it accordingly. If the institution is unable to calculate capital needs for a specific risk, then a buffer must be set.

The value of sophisticated MRM extends well beyond the satisfaction of the regulatory regime. Effective MRM can improve an institution earnings through cost reduction, loss avoidance or capital improvement. The first two come mainly from increased operational and process efficiency in model development and validation, including the elimination of defective models. On the other hand, capital improvement comes mainly from the reduction of undue capital buffer and add–ons. MRM reduces rising modelling costs, addressing fragmented model ownership and processes caused by high number of complex models, thereby improving in profit and loss (P&L).

To manage P&L, capital and regulatory challenges to their institution advantage, banks are moving towards a robust MRM framework that deploys all available tools to capture efficiencies and value. The standard practice for the governance of models is to adopt the **three lines of defence model** (BCBS, 2012). The first line of defence refers to **model development and use** consisting of the model owners, developers, and users within the line of business. Their responsibilities encompass defining, developing, implementing, and operating the model, monitoring its performance, and managing changes. The second line of defence represents the **model validation and control** that establishes policies and standards, performs model risk assessment, manages an inventory models, provides independent monitoring of model performance, model usage, and adherence to management policies, and reports to board/senior management. The third line is typically performed by **Internal Audit** and its objective is, among other things, to review activities against documented procedures and to set tests on the available records.

The aim of the model validation is to manage and, if possible, minimize model risk. The structure of validation can be described as encompassing the validation of the model itself as well as the process in place around it. Validation, therefore, covers the conceptual design of the model, assumptions, data and analytics, the model outputs, usage and performance. It refers to the first step in assessing whether the model is fit for purpose or adequate for fulfilling both the intended objective and the specified requirements. Like any other risk management activity, validation comprises the well–known steps of identification, assessment, monitoring, control and reporting.

Effective model risk management allows institutions to reduce the risk of potential losses and underestimation of own funds requirements as a result of flaws in the development, implementation or use of the models. To mitigate these risks, institutions should have a model risk management framework in place that allows them to identify, understand and manage their model risk for internal models across the group. In spite of the rise of awareness of model risk and understanding its significant impact, there are no globally defined industry nor market standards on its exact definition, management and quantification, even though as aforementioned a proper model risk management is required by regulators. Currently, the supervisory guidance on MRM is limited at the quantitative level. Likewise, the academic and technical practitioner literature on quantitative approaches to MRM is rather deficient and often restricted to a specific set of models. This is why, in the following pages, we will try to develop a methodology to assess model risk comprehensively and in a manner that is, as much as possible, practically achievable, focusing on identification, quantification and mitigation of model risk.

The objective of the Part II of this dissertation is twofold: introduce a sufficiently general and sound mathematical framework to cover the main sources of model risk and illustrate how a practitioner may identify the relevant abstract concepts and put them to work.

Chapter 5

IDENTIFICATION OF POTENTIAL SOURCES OF MODEL RISK

"It ain't what you don't know that gets you into trouble. It's what you know for sure that just ain't so." (Mark Twain)

Models are scale–down versions of the reality, and as such, all models have model risk. There are always some areas that are deliberately left out of the model, as they are viewed as being irrelevant for the specific model purpose. Simplifying assumptions and approximations, which might not correspond to empirical evidence (such as normality, independence, absence of tail events, or constant volatility) are, however, often used either because they allow substantial simplifications with obvious advantages on costs and transparency of the model itself or because, more rarely, they are thought not to cause substantial differences in the outputs, at the level of precision required by the particular usage. Abandoning such assumptions does not eliminate model risk, albeit potentially leading to more accurate models, as new assumptions needs to be added.

Model risk is therefore marginally a problem of inappropriate choice or incorrect assumptions or method used. The model risk drivers are permeated throughout the whole life–cycle of the model deployment and usage, i.e. it originates at model's conception (requirements drawn from a methodology), prototyping, testing, documentation, review and validation, production, ongoing monitoring and model enhancements (or retirement if it has met its purpose). And as such, no model can be judged out of context, as its performance may be completely different depending on what asset or portfolio it is applied to, on level of quality and completeness of data used for its development and usage, but also the nature of the particular usage will determine the ultimate financial impact of any inaccuracy or error.

Next, model risk can be a special case of risk ignorance, as our misunderstanding of the uncertainty inherent in financial transactions or operations can lead us to incorrect estimates or even into product, markets and deals that we should better stay out of. Furthermore, risk for each model also develops and changes over time based on the design, implementation and usage of the model and the validation results.

Hence in seeking to manage model risk we need to identify and implicitly understand the drivers of model risk associated with each model, the intrinsic uncertainty introduced in choosing a particular model, risks associated with the design and implementation of the model and the risks in emerging from its usage in practice. Although model risk cannot be entirely eliminated, many sources of model risk can be effectively mitigated and reduced through addressing of the issues or by putting controls in place. It is crucial that the advantages, limitations, and key assumptions are well understood not only by the users and the modellers but also by the top management.

To follow the classical approach to risk management, one starts with a taxonomy to identify and describe the sources of model risk, covering the model assumptions, data, calibration, infrastructure weaknesses and limitations, methodology weaknesses, review issues, and so on. In what follows we offer a list of some of the potential sources of model risk, clearly not comprehensive, that might occur during the whole life–cycle of a model.

- **DATA AND CALIBRATION ERRORS** - arising from deviations in the quality of data input compared to that suitable for the model; no matter how sophisticated the model we use, it will fail if the data used for modelling are inadequate
  - **Raw data:**
    - ◇ sparse/missing or incomplete data
    - ◇ inaccurate/wrong, outdated or misinterpreted data or assumptions, e.g. assumptions set prior to 2007–2008 financial crisis, insufficient data feeding frequency
    - ◇ the frequency of the data depends on the nature of the underlying, e.g. asset related time series data is available at least daily whereas loss data may typically only be available on either a quarterly or annual basis
    - ◇ error in data definition, error in the mapping of data with the data sources
    - ◇ lack of critical variables
    - ◇ the assumption that the data for all assets was taken at exactly the same point of time does not always fulfil the requirement since it comes from different sources and agencies
    - ◇ errors in the database are sometimes very hard to find and thus it is impossible to check the data for correctness
    - ◇ mistakes in the course of classification in credit scoring models, e.g. we might reject a potentially good applicant, causing the bank to lose the profit that might have been made on that client; or the system might classify a potentially bad client as good, resulting in the bank granting the loan (if the client defaults and fails to fulfill their payment obligations, the bank will suffer actual losses)
  - **Augmented data:** data created by users; where there are gaps in the existing data which are attempted to be filled by duplication or other simple rules:
    - ◇ missing values, e.g. substituting missing data with mode or mean which can lead to over/underestimation of some factors
    - ◇ duplication, e.g. in case when a data cannot be obtained on a day, then use the value from previous day
    - ◇ handling outliers, e.g. data points that differ a lot from the others may distort the real parameters if the outliers were just a mistake, otherwise correct outliers that are neglected cause that some important information gets lost

- ◇ scenario generation: expert generation of data to fill in gaps in the sample, usually in tails of distributions. Expert judgement can easily under/overestimate scenario properties, e.g. severity or frequency of occurrence. [1]
- ◇ limited expertise pools (e.g. by lacking diverse backgrounds or proper feedback, and hence tend to make similar or naive assumptions)

· **Sub–model data:** when models use output of other models, either internal or external

- ◇ not checking the source models for issues, e.g. when estimating Expected Loss models derive input from PD, EAD and LGD models, if there is some issue with such upstream models, they may impact the downstream models critically

· discrepancy when combining multiple sources of data since different databases may have varying formats, data layouts and record identification

· sampling error, e.g. understanding of sampling error for tail estimates

· interpolation and extrapolation. The algorithm of extrapolation may have a big impact, e.g. volatility surface in derivatives pricing

· lack of sufficient history, or strong dependence of the model on historical data, e.g. risk that data do not represent the current reality, the model is based on insubstantial real data or economic theory

· aggregation of 'good' and 'bad' data

· wrong, inappropriate and/or inaccurate parameters, sensitivity to particular parameters

· high sensitivity of expert adjustments, the impact of expert adjustments to the model should be carefully analyzed since it increases the risk of poor model predictions/measurements and unsupported model estimates, which may lead to inappropriate business decisions

· potential errors with the level of work-around data and/or shoehorning required to reconstruct the underlying features into the prescribed data format for modelling

- ◇ unsuitable, inaccurate proxies (margin of error)
- ◇ divers objects may be bundled together into an apparently homogeneous type
- ◇ objects may be mislabelled as other types
- ◇ deviations of the quality of data presented to the model compared to that expected by the model, e.g. data at higher levels (as locations, states) may provide poor inputs or make it impossible for assessment against data at low levels

· data translation: this refers to the level of work around and/or shoehorning required to translate the underlying characteristics into the prescribed modelling data format. There is a risk of a systematic distortion that is invisible at later stages, making people overconfident in the calibration of the model, e.g. if it is not possible to encode the type of coverage because the model was originally designed for a more specific domain, disparate objects may be bundled together into an apparently homogeneous type or mislabelled as other types; this data will then potentially distort model estimates of the content of the portfolio.

[1] Depending on the situation, a balance between missing data and potentially polluted data should be sought.

- wrong interpretation of ratings, e.g. risk ratings require subjective interpretation, and different users may obtain opposite ratings of the same quantitative risks (careful explanations of embedded judgement is required)
- inappropriate calibration due to lack of data, wrong selection of some relevant subset, limitations of the optimization algorithm, complexity of the problem, computational limitations and difficulties, use of unobservable parameters, lack of market consensus, etc.

Possible mitigation: parameter/assumption review (review all assumptions and their changes for reasonableness and use explicit rather than implicit assumptions); data policy; data validation and compliance report; process automation; data storage, sufficient documentation, data linkage system, tracking model dependencies; subject the judgemental assumptions to oversight by the governance committee that will review owner's support for the assumptions on e.g., a monthly basis pr to benchmark the model output to the output from the alternative model

- **MODEL SELECTION**
  - inadequate or incorrect assumptions such as missing or incorrect modelling of certain factors; regulatory actions (trading rules); assumption on independence for observations in standard logistic regression (is it really true in retail credit?); poor assumptions of the nature of the markets, e.g assumption of market liquidity and efficiency; wrong assumption of the distribution function of some parameters; appropriateness; regime change; wrong set of variables (misidentify or miss variables), ignoring of trading cost; wrong assumptions of the underlying assets, etc.
  - model misspecification, e.g. over–simplification (e.g. simplified modelling of asset portfolio leading overstatement of risk), too many variables which may cause multicollinearity (where predictor variables are themselves highly correlated, this can lead to inefficient or wrong estimates)
  - regulatory actions and limitations — institutions do not necessarily have final control over which models they use and how they use them. Regulators with their wide reach and influence may have potential to create or mitigate systematic risk.
  - outdated model due to parameter decalibration, expert adjustments that are not updated, variables with no discriminating capacity, etc.
  - introduction of arbitrage
  - trade–off between model complexity and model accuracy
  - model inefficiency — statistical methods often have an uncertainty in their estimations. This uncertainty depends on the input that are often estimated itself what makes things even worse (Every model is inaccurate and every estimate is just an estimate.)
  - use of new methodology not supported by academic research
  - misuse of the model, e.g. due to the lack of comprehensive documentation; having one good model for a derivative might mislead to use it for a similar derivative ignoring that the similar one includes more or slightly different features; or using a default model as the basis of a strategy on customer retention
  - hidden risk, e.g. re–weighting, stratification, errors in the analytical solution

- model credibility — the extent of the knowledge (or uncertainty) of the real underlying phenomena captured in the model, and how much to trust it (e.g. estimating extreme but rare events, model risk may become much greater than the underlying phenomenon)
- systematic risk — different models fitted to the same historical data and insurers using the same models will have correlated model errors, causing systematic risk if they are not countered by good model management, expertise and individual view of risk
- human factor is also a source of model risk, e.g. quants like to build models but further maintenance is less attractive; poor communication between modellers and IT, management, end-users; poor (or non-existent) training and change management; or excessively focusing on inconsequential technical issues rather than key business decisions
- choosing wrong accuracy over model simplicity and parsimony ('being precisely wrong vs. approximately right')

Possible mitigation: backtesting; statistical tests; assumptions inventory; access to multiple models (comparing multiple models, especially as they get more complex, choosing false accuracy over model simplicity and parsimony (being precisely wrong vs. approximately right)); limitations of models in context of application; complimentary approaches; test variation in all assumptions and scenario elements, rather than over-emphasizing one or two; sufficient documentation; adequate understanding about the environment and intended use can help determine if modelling is fit for the purpose; control, monitoring, definition of indicators such as KPI, KRI and KCI, comprehensive documentation of entire model development, testing and validation process, establish appropriate limitations on model use to avoid using it in ways not consistent with their original intent.

- **IMPLEMENTATION AND INFRASTRUCTURE**
  - lack of communication between modellers and users, inadequate, outdated or lack of comprehensive documentation, a discrepancy may exist between database developers and modellers since modellers requirements may not be understood completely by IT personnel; on the other hand, modellers may not be aware of challenges that developers face in fulfilling their requirements
  - large, complex projects — implementation over months/years
  - using of third party libraries and e.g. not having appropriate documentation to validate it (black boxes)
  - human error during the process, especially when manipulating raw data, inputs and outputs
  - model inefficiency, e.g. due to inappropriate software, incorrect extensions (covering unintended products and features)
  - unnoticed IT failure, e.g. during Monte Carlo simulations, outages
  - inappropriate numerical methods often cause approximation errors, e.g. even if program may run under normal conditions, errors under extreme conditions may occur
  - tractability vs. accuracy, e.g making approximations and ignoring intermediate errors and warnings
  - poor efficiency vs. accuracy trade–off

- manual processes or interventions, e.g. codes are manually written, so they introduce the risk of errors in the codes and these may be catastrophic for the financial institution, which uses the outputs of these flawed models for their day to day decision making
- unwillingness to change, e.g.,
  - because of different results or governance process, regulators (business accepts need to change model, but change process is prohibitive)
  - the assumptions underlying models and the set of data used at the time of model development or implementation may no longer be relevant as banks move through the product life cycle and the economic peaks and troughs — banks often continue using the same models with reduced performance, simply to avoid changing/building a new model
- implementation model risk increases with the model complexity, e.g. even relatively simple VaR calculations may differ from one expert to another (because of different understanding of the specifications, errors, rounding, etc.)

Possible mitigation: *model ownership; independent testing and validation; knowledge transfer; software architecture; detailed documentation, discussions and tests can help to scale the risks down; a well-documented structure of circumstances under which to effect a model change might help to avoid losses due to the continued usage of sub-par or outdated models.*

- **USAGE, GOVERNANCE AND CONTROL**
  - lack of complete understanding regarding models, their content and limitations, may lead to inefficient use across the business cycle
  - over time, models tend to deteriorate in performance, however, due to the lack of comprehensive model documentation (e.g., supporting model assumptions and limitations, an ongoing monitoring plan, evidence reflecting when models under development go into production, and model changes) institutions are unable to either track or address these changes
  - data quality may vary greatly across different databases, this variation in quality may be caused by database users as well as by the database design, itself
  - integration and reliance of model usage — to what extent are model users reliant on the model, and to what extent is the model integrated into a more well–rounded risk assessment framework
  - overconfidence and over–reliance, users must fully acknowledge that all models provide only a limited view of reality and should not in any circumstance substitute for sound managerial judgement
  - intuition behind model dynamics, e.g. misinterpretation of results
  - model fitness for purpose — using model for different purpose for which it was developed or using models beyond their original intent
  - differences between regulatory and management practices, differences in model definition and uses between the commercial and the risk area
  - extending the model's use beyond its original scope (new products, markets, segments, etc.)
  - model not re–estimated or re–calibrated in a long time

- inadequate information flow between people in an institution - information flow can be as important as how data is transferred between parts of a model for the eventual outcome, e.g. if transparency, correctness checks, incentives or feedback are problematic the institution may make its members use the model in a wrong way
- dependence on an automated system introduced to replace a human act, thus transforming the function of the human into an overseer. It dictates the outcome of decisions, which in turn can either reduce or introduce model risk (it is like, when models turn on, brains train off)
- not sufficient ability of users to understand the content of models and their limitations — this may be based on their inadequate experience with the models in case when they work as they should, but also when/why unexpected real world situations occur; this may lead to systemic risks when users have too limited expertise pools, for example by lacking diverse backgrounds or proper feedback, and hence tend to make similar or naive assumptions
- model risk considered inside operational risk [2]:
  - using simplistic modeling, despite markets are moving on to a newer model ('regime change')
  - cognitive bias, e.g. tendency to estimate the likelihood of an event based on how mentally available similar examples are; fixing estimates of a given quantity on a single piece of information, even when that information is known to be unrelated to the value of that quantity

<u>Possible mitigation</u>: *frequent feedback, hone skills and knowledge of model users, understanding enough about the environment and intended use can help determine if modelling is fit for the purpose, risk diversification - model errors and structural uncertainty might mean that certain risks are underestimated. By diversifying risk, one can ensure not only that a single disaster would not bankrupt the company, but also that a single set of model errors or poor estimates would not either, cooperation between data users, contributions and developers*

These risk should be expressed quantitatively and then combined based on whether the individual model risks are independent, or some amplify each other, or counteract or absorb each other. Aggregate model risk is affected by the interaction of dependencies among models, reliance on common assumptions, data, or methodologies, and any other factors that could adversely affect several models and their outputs as the same time (OCC and BOG–FRB SR 11–7, 2011 [33]). In fact, it is not just the vast number of different models, diverse in their objectives as well as in their conceptual structure, but also the growing complexity of the many interconnections between data, models and uses, that make the model risk relevant.

Model risk can be also a special case of risk ignorance, i.e. our misunderstanding of the uncertainty may leads us to failure in identifying the risks we are exposed to. Furthermore, validators should never forget that the limits of model development apply to their work as well and that they should consider their results with the same scepticism with which they consider the results of any model they examine. This, in itself, is a source of risk, it is difficult to control somebody's activities when our understanding of those activities is inferior to the one that we are supposed to control.

[2]Currently, operational risk concerning models and model risk are mixed at times. Errors are normally related to operational risk, e.g. replicating a model with the role intention of finding coding bugs

Although model risk cannot be entirely eliminated, proficient modelling by competent practitioners together with rigorous validation, ongoing monitoring of model performance under various conditions, limiting model use, frequent revision of assumptions and recalibration of input parameters can significantly reduce it.

Chapter 6

A New Approach to Quantification of Model Risk for Practitioners

"Whenever you find yourself on the side of the majority, it is time to pause and reflect." (Mark Twain)

In spite of the rise of awareness of model risk and understanding its significant impact, there are no globally defined industry nor market standards on its exact definition and quantification, even though as aforementioned a proper model risk management is required by regulators. Although regulators provide theoretical references and practical methodologies for model validation, they fall short to introduce a measure of model risk that can be computed and reported by model and by transaction thereby becoming an internal part of the overall risk measure for the institution portfolio.

Within the finance literature, some authors have defined model risk as the uncertainty about the risk factor distribution as in [63], the misspecified underlying model in [41], the deviation of a model from a 'true' dynamic process in [35], the discrepancy relative to a benchmark model in [68], and the inaccuracy in risk forecasting that arises from the estimation error and the use of an incorrect model in [34]. Model risk has been previously classified in all asset classes, see [86] for interest rate products and credit products, [39] for portfolio applications, [102] for asset backed securities, and [34] for relation to measuring market risk.

The existing contributions to the problem of measuring model risk range from value and price approaches, Bayesian model averaging and worst–case approaches, to approaches based on divergence measures. For example, [105] suggest two approaches for the measurement of model risk: the worst–case and the Bayesian approach. The worst–case approach measures model risk as the difference between the risk measure of a model and the risk measure of a model under the worst–case scenario, while the Bayesian approach is based on the average of the risk measures computed by all candidate models, by using proper prior distributions on both the parameters and the probability that a specific model is the best one.

Kerkhof et al. [77] suggest to measure model risk by computing the worst–case risk measure over a tolerance set of models, i.e. over a set of alternative models (or distributions) around a reference one. They focus in particular on model risk associated to uncertainties in econometric modelling which leads in a natural way to division of total model risk into three components: estimation risk, misspecification risk and identification risk. This classification is base on the different levels of ambiguity which are reflected by model classes that are typically used in econometric modelling of financial markets.

Another contribution was provided by Glasserman and Xu [64] where model risk is assessed using an entropy–based measure of the worst–case change in the probability law effecting the relevant risk measure. They account for model risk by acknowledging that the probability law for a stochastic element of a model might be misspecified. This robust approach begin from baseline model and finds the worst–case error in risk measurement that would be incurred through a deviation.

In addition, it is worth mentioning the work of Abasto and Kust [1] which is based on a local sensitivity approach, where model risk refers to the price spread among neighbouring models at a fixed ”1 basis point” distance from the reference model, or that the authors in [82] who study how different divergence measures affect the degree of uncertainty about model tail behaviour included in the worst–case analysis. However, as other divergence approaches, the classical expected values are replaced by worst–case expectations over all models within a ball around the reference model, i.e. within some fixed radius from a given reference model defined by some divergence measure.

The vast majority of the existing literature focuses on the examination of the output of the model without giving any consideration to the model design, the data, the usage, or the implementation process. Though, the output of a model is always the result of an imperfect implementation of an inaccurate model. In addition, most of the contributions are limited to specific models, such pricing specific derivative products, and their practical application face considerable challenges in specifying the alternatives and benchmarks of the approaches required. Since the model risk differs in nature from all other risk types, i.e. it can be observed beyond all types of risks, a special framework across all models and risks should be established.

The quantification, as an essential part of model risk management, is required for a consistent management and effective communication of model weaknesses and limitations to decision makers and users and to assess model risk in the context of the overall position of the organization. The quantification of model risk should consider the uncertainty stemming from the selection of the mathematical techniques (e.g. focusing on fitting a normal distribution hence leaving aside other distribution families), the calibration methodology (e.g. different optimization algorithms may derive different parameter values), and from the limitations on the sample data (e.g. sparse or incomplete database).

Model risk quantification poses many challenges that come from the high diversity of models, the wide range of techniques, the different use of models, among others. Some model outputs drive decisions; other model outputs provide one source of management information, some outputs are further used as an inputs in other models. Additionally, the model outputs may be completely overridden by expert judgement, not to mention that in order to quantify model risk you need another model, which is again prone to model risk. Next, although we can identify a set of factors driving model risk, it is difficult to postulate a structural relationship between these factors and either the corresponding impact on the model output or a level of model risk. Besides, the impact itself is elusive, as it depends on what the model output is used for and, typically, on additional external circumstances. For instance, underestimation of probability of default of a segment may result in an underestimation of exposure against limits, of risk pricing and of regulatory capital that may translate into losses, but only if default actually happens. Thus, model risk may be particularly high, especially under stressed conditions or combined with other interrelated trigger events.

It is common practice to consider the most relevant areas of analysis for the quantification of model risk to be data and calibration, model foundations, model performance, IT infrastructure, model use, controls and governance, and model sensitivity. The model may be fundamentally wrong due to errors in its theoretical foundation or conceptual design that emerge from incorrect assumptions, model misspecification or omission of variables. Data quality issues, inadequate sample sizes and outdated data contribute to model performance issues such as instability, inaccuracy or bias in model forecasts. Model risk also arises from inadequate controls over the model use. Flawed test procedures or failure to perform consistent and comprehensive user acceptance tests can lead to material model risk. To name just a few. The risk driven by these factors will manifest itself in the model output, affecting its performance and accuracy.

The focus of this chapter is on developing a new approach for quantifying model risk within the framework of differential geometry, see [89], and information theory, see [6]. In this work we introduce a measure of model risk on a statistical manifold where models are represented by a probability distribution function. Differences between models are determined by the geodesic distance under the Fisher–Rao metric. This metric allows us to utilize the intrinsic structure of the manifold of densities and to respect the geometry of the space we are working on, i.e. it accounts for the non–linearities of the underlying space. Besides, every time a model is used there is a certain probability that the model will experience one or more model events driven by the risk factors mentioned above and that the impact of such failure on the overall performance will, in turn, be a random variable drawn from a certain probability distribution.

The rest of this chapter is structured as follows. In Section 2 we summarize the central concepts from Riemannian geometry and introduce the terminology used throughout the paper. Modelling process steps and a general description of our proposed method for quantification of model risk are presented in Section 3, which is followed by a detailed discussion on the main quantification steps. Section 4 and Section 5 describe the construction of the neighbourhood containing material variations of the model, and the definition of the weight function, respectively. The model risk measure is then defined and explained in Section 6. Section 7 is dedicated to an empirical example of the model risk calculation of a credit risk model. Section 8 provides some final conclusions and directions for future work, and finally, the Appendix contains the construction of the weight function and the proofs of the main results.

This chapter constitutes an adapted version of [80], with only minor corrections and modifications.

6.1 Background on Riemannian Geometry

In this section we introduce some required concepts from differential geometry and information theory, and fix the notation and terminology used throughout the article. For more details we refer the reader to e.g. [6] or [89].

Let $\mathcal{M}$ be a statistical manifold consisting of probability density functions $p(x|\theta)$ of random variable $x \in \mathcal{X}$ with respect to a measure μ on $\mathcal{X}$, such that every distribution is uniquely parametrized by an n–dimensional vector

$\theta = (\theta^i) = (\theta^1, \ldots, \theta^n)$.[1] Specifically, let

$$\mathcal{M} = \{p(x, \theta) \mid \theta \in \Theta \subset \mathbb{R}^n\},$$

with a one–to–one mapping between θ and $p(x, \theta)$. Additionally, under the assumptions that the parametrization of $\mathcal{M}$ is differentiable and C^∞ diffeomorphism, the parametrization θ forms a coordinate system of $\mathcal{M}$, [6]. The local coordinate system $\theta = (\theta^1, \cdots, \theta^n)$ then induces a basis $\frac{\partial}{\partial \theta} = (\partial_1, \cdots, \partial_n)$ of the tangent spaces (∂_i is a shorter notation for $\partial/\partial\theta^i$).

The structure of $\mathcal{M}$ is specified by a Riemannian metric, $g = (g_{ij})$, that is defined by a local product on tangent vectors at each point $p \in \mathcal{M}$ denoted by $g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$, that is symmetric, bi-linear, positive definite and C^∞ differentiable in p. By the bi–linearity of the inner product of g, for any two tangent vectors $u, v \in T_p\mathcal{M}$

$$g(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i u_j g(\partial_i, \partial_j)$$

where $\{\partial_i\}_{i=1}^n$ form the basis elements of $T_p\mathcal{M}$. For a statistical manifold, the Fisher–Rao information metric is given by

$$I_{ij}(p) = g_{ij}(p) = \mathbb{E}\Big[\frac{\partial \log(p)}{\partial \theta^i} \frac{\partial \log(p)}{\partial \theta^j}\Big] = \int p \frac{\partial \log(p)}{\partial \theta^i} \frac{\partial \log(p)}{\partial \theta^j} d\theta. \tag{6.1}$$

where $I(p) = I\big(p(x|\theta)\big)$ can be considered as a Riemannian metric. Note that $det\,(I(p(x|\theta)))$ represents the amount of information a sample point conveys with respect to the problem of estimating the parameter θ, and so $I(\cdot)$ can be used to determine the dissimilarities between distributions. It measures the ability of the random variable x to discriminate the values of the parameter θ' from θ for θ' close to θ.

Example (statistical manifold of normal distributions): The normal distribution with mean μ and variance σ^2 is given by

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\},\ x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0.$$

Set $\Theta = \{(\theta^1, \theta^2) \in \mathbb{R}^2 | \theta^2 > 0\}, \mathcal{X} = \mathbb{R}$ and $p(x|\theta) = \mathcal{N}(x|\sqrt{2}\theta^1, (\theta^2)^2)$. Then the statistical manifold with respect to $\{p(\cdot|\theta)|\theta \in \Theta\}$ has a Riemannian metric $g = 2(\theta^2)^{-2} \sum d\theta^i d\theta^i$ of constant curvature $-1/2$.

The Riemannian metric encodes how to measure distances, angles, areas and curvature on the manifold by specifying the scalar product between tangent vectors at a particular point. If we consider a curve $\gamma : [a, b] \subset \mathbb{R} \to \mathcal{M}$ on the manifold, its length $\ell(\gamma)$ can be defined as

$$\ell(\gamma) = \int_a^b \Big\| \frac{d\gamma}{dt} \Big\| dt = \int_a^b \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j} dt = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}.$$

where $\dot{\gamma}^i$ is the derivative of $\dot{\gamma}^i = \frac{\partial \gamma}{\partial \theta^i}$. Then, the distance between two points $p, q \in \mathcal{M}$ is defined by the infimum of the length of all smooth curves between these two points and is give by

$$d(p, q) = \int_{\gamma \in \Gamma} \ell(\gamma)$$

where Γ is a set of all smooth curves between these two points. The locally length–minimizing smooth curve $\gamma(t) : [0, 1] \to \mathcal{M}$ is called geodesic and is characterized by the fact that it is autoparallel, e.g. the field of tangent

[1] We describe only the case for continuum on the set $\mathcal{X}$, however if $\mathcal{X}$ were discrete, the given framework will still apply by switching $\int(\cdot)$ with $\sum(\cdot)$.

vectors $\dot{\gamma}(t)$ stays parallel along γ (velocity is constant along the geodesic $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ on γ). In local coordinates, a curves is a geodesic *iff* it is the solution of the system of n second order Euler–Lagrange equations:

$$\frac{d^2\theta^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{\partial\theta^i}{dt}\frac{\partial\theta^j}{dt} = 0 \quad \forall k = 1, \ldots, n,$$

where Γ_{ij}^k are the Christoffel symbols of the second kind.

From the result of existence and uniqueness of solutions of differential equations, for each $p \in \mathcal{M}$ and for each tangent vector $v \in T_p\mathcal{M}$, there exists an open interval I_v with $0 \in I_v$ and a unique geodesic $\gamma_v : I_v \to \mathcal{M}$, such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. Therefore, the exponential mapping $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is defined by $\exp_p(v) = \gamma_v(1) = \gamma_{v_1}(\|v\|)$, with $v_1 = v/\|v\|$. For each $p \in \mathcal{M}$, there exists a neighbourhood $\mathcal{V}$ of the origin in $T_p\mathcal{M}$, such that $\exp_p$ is a diffeomorphism from $\mathcal{V}$ onto a neighbourhood V of p. The neighbourhood $\mathcal{V}$ is star–shaped, i.e. for any point belonging to $\mathcal{V}$, the line joining the point to the origin is contained in $\mathcal{V}$. The image of a star–shaped neighbourhood of the origin under the exponential map is a neighbourhood of p on the manifold (also called normal neighborhood).

A notion of connection ∇ defines a map between any neighbouring tangent spaces. The canonical affine connection on a Riemannian manifold is the Levi–Civita connection and it is directly defined from the covariant derivative, i.e. orthogonal projection of the usual derivative on the vector fields onto tangent space. It parallel transports a tangent vector along a curve while preserving its inner product (it is compatible with the metric, i.e. the covariant derivative of the metric is zero). The Levi–Civita coefficients are defined, in each local chart by the Christoffel symbols of the second kind Γ_{ij}^k given by

$$\nabla_{ij}^k = \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial\theta^i} + \frac{\partial g_{il}}{\partial\theta^j} - \frac{\partial g_{ij}}{\partial\theta^l} \right)$$

$\forall i, j, k, l = 1, \ldots, n$ is used the Einstein summation convention, g^{kl} is the metric inverse. For this choice of connection, the shortest paths are geodesics. When the connection coefficients of ∇ with respect to a coordinate system of $\mathcal{M}$ are all identically 0, then ∇ is said to be flat, or alternatively, $\mathcal{M}$ is flat with respect to ∇.

6.2 Modelling Process Steps and Quantification of Model Risk

There are different types and aspects of model risk that tend to easily overlap, co–occur or co–vary. In this context, we propose four rough model creation steps: Data, Calibration, Model Selection and Testing, and Implementation and Usage. These steps may occur in an iterative fashion, although they result in a general linear flow that ends with institutional use (implementation and maintenance) to direct decision making (often encoded into an IT system). Limitations in any of these areas can impair reliance on model results.

1. **Data** refers to the definition of the purpose for modelling, the specification of the modelling scope, human and financial resources, the specification of data and other prior knowledge, their interpretation and preparation. The data may be obtained from both internal and external sources, and they are further prepared by cleaning and reshaping. Model risk may arise from data deficiencies in terms of both quality and availability, including, among others, error in data definition, insufficient sample, inaccurate proxies, sensitivity to expert judgements, or misinterpretation.

2. **Calibration** includes the selection of the types of variables and the nature of their treatment, the tuning of free parameters and links between system components and processes. Estimation uncertainty may occur due to simplifications, approximations, flawed assumptions, inappropriate calibration, errors in statistical estimation or in market benchmarks, computational or algorithmic limitations, or use of unobservable parameters.

3. **Model Selection and Testing** involves the choice of the estimation performance criteria and techniques, the identification of model structure and parameters, which is generally an iterative process with the underlying aim to balance sensitivity to system variables against complexity of representation. Furthermore, it is related to the conditional verification which includes checking the sensitivity to changes in the data and to possible deviations from the initial assumptions. In this step, model risk stems from, e.g., inadequate and incorrect modelling assumptions, outdated model due to parameter decalibration, model instability or model misspecification.

4. **Implementation and Usage** refers to the deployment of the model into production which is followed by a regular maintenance and monitoring. Sources of model risk in this step include using the model for unintended purposes, lack of recalibration, IT failures, lack of communication between modellers and users, lack of understanding on model limitations.

Quantification of model risk, from a best practice perspective, should be quick and reliable, without refitting or building models, without reference to particular structure and methodologies, and with prioritizing analysis (getting immediate assurance on shifts that are immaterial).

In this article we aim to show that differential geometry and information theory offer a base for such an approach. For this purpose, in our setting a model is represented by a particular probability distribution, $p : \mathcal{X} \to \mathbb{R}_+$, that belongs to a manifold of probability measures $\mathcal{M}$, so called statistical manifold, available for modelling. The manifold $\mathcal{M}$ can be further equipped with the information–theoretic geometric structure that allows us to quantify variations and dissimilarities between probability distribution functions (models), among other things.

The set of probability measures may be further parametrized in a canonical way by a parameter space Θ, $\mathcal{M} = \{p(x;\theta) \mid \theta \in \Theta\}$. This set forms a smooth Riemannian manifold $\mathcal{M}$. Every distribution is a point in this space, and the collection of points created by varying the parameters of the model, $p \in \mathcal{M}$, gives rise to a hypersurface (a parametric family of distributions) in which similar distributions are mapped to nearby points (see Figure 6 for illustration).The natural Riemannian metric is shown to be the Fisher–Rao metric (see [95]), which is the unique intrinsic metric on the statistical manifold. It is the only metric that is invariant under re–parametrization, see [6] for example.

Let us consider a given model p_0 which can be uniquely parametrized using the vector $\theta_0 = (\theta_0^1, \ldots, \theta_0^n)$ over the sample space $\mathcal{X}$ and which can be described by the probability distribution $p_0 = p(x;\theta_0)$. This probability distribution belongs to a set (family) of distributions $\mathcal{M} = \{p(x;\theta) \mid \theta \in \Theta \subset \mathbb{R}^n\}$ that forms a model manifold. We assume that for each $x \in \mathcal{X}$ the function $\theta \mapsto p(x;\theta)$ is C^∞. Thus, $\mathcal{M}$ forms a differentiable manifold and we can identify models in the family with points on this manifold. Thus, choosing a particular model is equivalent to fixing a parameter vector $\theta \in \Theta$.

We define the model risk for a given model p_0 at the scale of an open neighbourhood around p_0 that contains alternative models that are not too far in a sense quantified by the relevance to (missing) properties and limitations

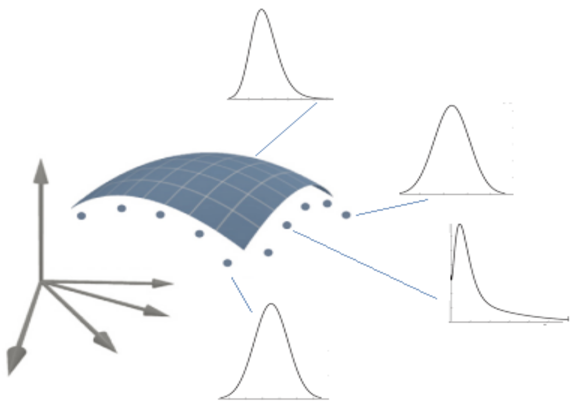

Figure 6.1: Illustration of statistical manifold, i.e. space of probability distributions.

of the model. The model risk is then measured with respect to all models inside this neighbourhood as a norm of an appropriate function of the output differences over a weighted Riemannian manifold endowed with the Fisher–Rao metric and the Levi–Civita connection. The analysis consists of five steps:

1. Embedding the model manifold into one that considers missing properties[2] in the given model p_0.
2. Choosing a proper neighbourhood around the given model.
3. Choosing an appropriate weight function, that assigns relative relevance to the different models inside the neighbourhood.
4. Calculating the measure of model risk with respect to all models inside the neighbourhood, through the corresponding norm.
5. Interpretation of the measure with respect to the specific use of the model risk quantification.

Each step addresses and aligns different limitations of the model and the uncertainty in various areas related to the model.[3] In the following sections we further develop these steps and describe the intuition behind.

6.3 Neighbourhood Around the Model

Recall that the given model p_0 belongs to a n–dimensional manifold $\mathcal{M}$, where each dimension represents different aspects of information inherited in p_0. In order to incorporate missing properties, the uncertainty surrounding the data and the calibration, the additional information about the limitations of the model, or wrong underlying assumptions, we may need to add new dimensions to $\mathcal{M}$, and thus consider a higher–dimensional space into which $\mathcal{M}$ is embedded.[4]

[2] Or properties not appropriately modelled, for which there is no consensus, cannot be adequately calibrated, among many others.

[3] Such as data, calibration, model selection, model performance, model sensitivity and scenario analysis, and most importantly the usage of the model.

[4] Consider for example a case when the underlying model space represents the family of normal distributions, i.e. 2–dimensional manifold $\mathcal{M}_0$, we may want to consider the family of skew normal distributions, i.e. 3–dimensional manifold $\mathcal{M}$, for which $\mathcal{M}_0 \subset \mathcal{M}$, in order to also examine skewness.

We define a neighbourhood[5] around p_0 with the help of the tangent space $T_{p_0}\mathcal{M}$. Note that $T_{p_0}\mathcal{M}$ is a vector space that describes at first order approximation, infinitesimal displacements or deformations on the manifold at the point p_0. From a practical point of view, not all perturbations are relevant, thus taking into account the materiality with respect to the intended purpose of the model, its usage, business and market, we consider only a small subset of the tangent bundle.

Let $\mathcal{U}$ be the open set around p_0 of normal neighbourhood V such that

$$\mathcal{U} = \{tv \in V \subset T_{p_0}\mathcal{M} \mid 0 < t \leq \alpha(v),\ v \in \mathcal{S}(p_0, 1) \text{ and normal coordinates are defined}\},$$

where $\mathcal{S}(p_0, 1) = \{v \in T_{p_0}\mathcal{M}, \|v\| = 1\}$ is the unit sphere on $T_{p_0}\mathcal{M}$.

The set $\mathcal{U}$ includes the directions of all relevant perturbations of the model p_0 up to a certain level $\alpha(v)$in direction v. The level $\alpha(v)$ depends on the tangent vectors, since the degree of our uncertainty on p_0 might not be constant across the canonical parameter space; for instance we could assume more uncertainty in the tails of the distribution p_0 than in its body, or higher sensitivity to one of the parameters than to others. The level $\alpha(v)$ can be interpreted as a means to control uncertainty regarding the choice of the model p_0, the uncertainty surrounding the data and calibration, and it is appropriately chosen based on the usage of the model.

Since $\mathcal{U}$ is a subset of the normal neighbourhood around p_0, the exponential map is well defined and we can construct a corresponding set of models close to p_0:

$$U = \exp_{p_0}(\mathcal{U}) = \{p \in \mathcal{M} \mid d(p_0, p) \leq \alpha(v)\}, \tag{6.2}$$

From now on, we shall require the boundary $\partial U = \{\exp_{p_0}(\alpha(v)\, v) \mid v \in \mathcal{S}(p_0, 1)\}$ to be continuous and piecewise regular. Moreover, U shall be a compact *star–shaped set with respect to* p_0 that is defined as follows:

Definition 6.3.1 *A compact subset U of a Riemannian manifold $\mathcal{M}$ is called star–shaped with respect to $p_0 \in U$ if for all $p \in U, p \neq p_0$ there exists a geodesic segment γ with $\gamma(0) = p_0$ connecting p_0 and p such that $\gamma(t) \in U$ for all $t \in [0, \alpha(v)]$, where $\alpha(v) > 0$, with $v \in \mathcal{U}$.*

One advantage of the exponential map in this setting is that we can avoid calibration of different alternative models inside U. For each unit vector $v \in \mathcal{U}$ there exists a unique geodesic connecting points on the boundary of U with the point p_0. This geodesic is given by $\gamma(t) = \exp_{p_0}(tv)$ for $t \in [0, \alpha(v)]$.

6.4 Weight Function Definition

Variations of the chosen model are not equally material and they all might take place with different probabilities. By introducing a non–linear weight function (kernel), K, over the set U we can easily place relative relevance to each alternative model, and assign the credibility of the underlying assumptions that would make alternative models partially or relatively preferable to the nominal one p_0. The particular choice of the structure of the kernel depends on various factors, such as usage of the model, distance from p_0 or sensitivity to different changes.

In what follows we define a general weight function K and show that under certain conditions it is well defined and unique. In general, we consider K to be a non–negative and continuous function that depends on the

[5]Note that throughout the paper we do not refer to the neighbourhood as to a strictly topological neighbourhood.

local geometry of $\mathcal{M}$ by incorporating a Riemannian volume associated to the Fisher–Rao information metric, which given by $dv(p) = \sqrt{det(I(\theta))}d\theta$. The volume measure is the unique Borel measure on $\mathcal{M}$ ([**?**]). With respect to a coordinates system, the information volume of p represents the amount of information the single model possesses with respect to the parameters. For example, a small $dv(p)$ means that the model contains much uncertainty and requires many observations to learn.

As the underlying factors[6] that influence the perturbations of the given model happen with some likelihood, we treat all models inside $\mathcal{M}$ as random objects. As a consequence, we require K to be a probability density with respect to the Riemannian volume, i.e. $\int_{\mathcal{M}} K dv(p) = 1$. Additionally, we state that the right model does not exist and that the choice of p_0 was to some extent a subjective preference.

Definition 6.4.1 *An admissible weight function K defined on $\mathcal{M}$ satisfies the following properties:*

($K1'$) *K is continuous on $\mathcal{M}$,*

($K2'$) *$K \geq 0$ for all $p \in \mathcal{M}$,*

($K3$) $\int_{\mathcal{M}} K dv(p) = 1$.

Recall that to compute the n–dimensional volume of the objects in $\mathcal{M}$, one considers a metric tensor on the tangent space $T_p\mathcal{M}$. In particular, the Fisher–Rao information metric I on $\mathcal{M}$ maps each $p \in \mathcal{M}$ to a volume $dv(p)$ which is a symmetric and n–form that further defines an n–dimensional volume measure on any measurable subset $U \subset \mathcal{M}$ by $Vol(U) = \int_U dv(p)$. A smooth probability density K over $\mathcal{M}$ with respect to the Riemannian measure induces a new absolutely continuous probability measure ζ with respect to Vol

$$\zeta(U) = \int_U d\zeta = \int_U K dv(p) \tag{6.3}$$

for all measurable $U \subset \mathcal{M}$ and $\zeta(\mathcal{M}) = 1$. The pair $(\mathcal{M}, \zeta)$ is then called a weighted manifold, or a Riemannian metric–measure space and is proved to be a nontrivial generalization of Riemannian manifolds ([**?**]).

The weight function K of the Definition 6.4.1 represents a general characterization of a probability density over the Riemannian manifold $\mathcal{M}$. In order to tune K for proper analysis of model risk, we need to impose additional properties which are connected with the specific uncertainties surrounding the given model.

From a practitioner point of view, models that do not belong to the chosen neighbourhood U are not relevant from the perspective of model risk, and so they do not add any uncertainty. Therefore, we assume the weight function K to be non–negative only over the neighbourhood U and zero elsewhere. Moreover, translation of the changes in various underlying assumptions, data or calibration into the changes in output and further usage of the model are going to vary with respect to the direction of the change. Hence, we require K to be continuous along the geodesic curves γ uniquely determined by $v \in \mathcal{S}(p_0, 1) \subset T_{p_0}\mathcal{M}$ starting at p_0 and ending at the points on ∂U. These additional properties motivate the following respective modifications of $(K1')$ and $(K2')$ in previous Definition 6.4.1:

($K1$) *K is continuous along all geodesics γ starting at p_0 for all unit vectors on $\mathcal{S}(p_0, 1)$ and ending at the points on ∂U*

[6] For example the uncertainty surrounding data, calibration or model selection.

$(K2)$ $K > 0$ *for all* $p \in U \backslash \{\partial U\}$, $K \geq 0$ *for all* $p \in \partial U$, *and* $K = 0$ *for all* $p \in \mathcal{M} \backslash \{U\}$

A weight function satisfying properties $(K1) - (K3)$ takes into consideration and is adjusted according to the different directions of the changes, i.e. prescribes different sensitivities to different underlying factors.

The construction of a weight function on a given Riemannian manifold may become technically difficult since it requires precise knowledge of the intrinsic geometry and the structure of the manifold. In order to overcome this difficulty and to determine a weight function K that satisfies all the required properties, we introduce a continuous mapping from a manifold endowed with an Euclidean geometry to the model manifold endowed with a Riemannian geometry that preserves the local properties. In summary, we construct three mappings: the exponential map $\exp_{p_0}$, the polar transform P and a further coordinate transform Λ_ρ. Euclidean geometry is well understood and intuitive, and thus a construction of a function on this space is considerably easier and more intuitive. The steps of the construction and the associated proofs are presented in the next Subsection.

6.4.1 Weight Function Construction

Every Riemannian manifold $\mathcal{M}$ is locally diffeomorphic to the Euclidean space $\mathbb{R}^n$, and so in a small neighbourhood of any point the geometry of $\mathcal{M}$ is approximately Euclidean. All inner product spaces of the same dimension are isometric, therefore, all the tangent spaces $T_p\mathcal{M}$ on a Riemannian manifold $\mathcal{M}$ are isometric to the n–dimensional Euclidean space $\mathbb{R}^n$ endowed with its canonical inner product. Hence, all Riemannian manifolds have the same infinitesimal structure not only as manifolds but also as Riemannian manifolds.

The weight function is defined with respect to the neighbourhood U and is continuous on the geodesic curves γ connecting p_0 to the points on the boundary ∂U. All material perturbations, i.e. alternative models inside U, are uniquely described by the distances from p_0 and by the vectors tangent to the unique geodesics γ that pass through them. In order to maintain these properties, we consider an n–dimensional cylinder

$$C^n = [0,1] \times \mathbb{S}^{n-1} = \{(t,\nu) : t \in [0,1], \nu \in \mathbb{S}^{n-1}\} \subset \mathbb{R}^{n+1}, \tag{6.4}$$

where the parameter t stands for the normalized distance of geodesics, and where $\mathbb{S}^{n-1}$ denotes the $(n-1)$–dimensional unit sphere on $\mathbb{R}^n$ containing all the unit tangent vectors of $T_{p_0}\mathcal{M}$. The boundaries of C^n are

$$\partial_1 C^n = \{(0,\nu) \mid \nu \in \mathbb{S}^{n-1}\}, \quad \partial_2 C^n = \{(1,\nu) \mid \nu \in \mathbb{S}^{n-1}\},$$

and represent the end points of the geodesics, i.e. $\partial_1 C^n$ will be transformed into p_0 and $\partial_2 C^n$ into ∂U.

The Riemannian structure on C^n is given by the restriction of the Euclidean metric in $\mathbb{R}^{n+1}$ to C^n. Hence, C^n is a compact smooth Riemannian manifold with a canonical measure given by the product measure $dt \times d\nu$. This manifold allows us to construct an appropriate function on C^n, and then obtain a weight function satisfying all required properties $(K1) - (K3)$.

As a first step to obtain a mapping from C^n to $\mathcal{M}$, we consider the exponential map from the tangent space at the point p_0 onto the neighbourhood U. Since U is a subset of the normal neighbourhood of p_0, the exponential map is well–defined and it defines a local diffeomorphism from $T_{p_0}U$ to U. The geodesics γ are then given in these coordinates by rays emanating from the origin.

Next, we introduce a polar coordinate transformation on $T_{p_0}\mathcal{M}$. For the sake of distinction, we denote by $\mathcal{S}(p_0,1)$ the $(n-1)$–dimensional unit sphere in $T_{p_0}\mathcal{M}$, and by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$:

$$\begin{aligned}
&P: \ [0,\infty)\times\mathcal{S}(p_0,1)\to T_{p_0}\mathcal{M}: \ \ P(t,v)=tv,\\
&P^{-1}: \ T_{p_0}\mathcal{M}\backslash\{0\}\to(0,\infty)\times\mathcal{S}(p_0,1): \ \ P^{-1}(v)=\left(\|v\|, \frac{v}{\|v\|}\right).
\end{aligned} \tag{6.5}$$

In order to properly describe the neighbourhood U, we define the distance function to the points on the boundary of the neighbourhood along the geodesic $\gamma_v(t)$. For $p_0\in\mathcal{M}$ and a unit vector $v\in\mathcal{S}(p_0,1)\subset T_{p_0}\mathcal{M}$, we define the distance $\rho(v)$ by

$$\rho(v)=\sup\{t>0\mid tv\in\mathcal{U},\ d(p_0,\gamma_v(t))=t\}, \tag{6.6}$$

as the distance from the origin to the boundary $\partial\mathcal{U}$ in direction $v\in\mathcal{S}(p_0,1)$, i.e. $\mathcal{U}=\{tv: 0\le t\le\rho(v), v\in S(p_0,1)\}$. In other words, $\rho(v)$ is the maximal distance in direction v to the boundary ∂U. The point $\gamma_v(\rho(v))$ is the boundary point of U along the geodesic $\gamma_v(t)$. The geodesic $\gamma_v(t)$ minimizes the distance between p_0 and $\exp_{p_0}(tv)$ for all $t\in[0,\rho(v)]$. The distance ρ, considered as a real valued function on $\mathcal{S}(p_0,1)$, is strictly positive and Lipschitz continuous on $\mathcal{S}(p_0,1)$.

We now define the coordinate transformation

$$\Lambda_\rho:(t,\nu)\mapsto\Big(\rho(v)t,v\Big). \tag{6.7}$$

The mapping $\mathbb{S}^{n-1}\to\mathcal{S}(p_0,1)$, $\nu\mapsto v$ is well defined in the sense that there exists a canonical identification between a unit vector $\nu\in\mathbb{S}^{n-1}\subset\mathbb{R}^n$ and the element $v\in\mathcal{S}(p_0,1)\subset T_{p_0}\mathcal{M}$. Since the distance $v\mapsto\rho(v)$ is strictly positive and Lipschitz continuous on $\mathcal{S}(p_0,1)$, so is the inverse $v\mapsto\dfrac{1}{\rho(v)}$. Therefore, the mapping Λ_ρ defines a bi–Lipschitz mapping from $[0,1]\times\mathbb{S}^{n-1}$ onto the subset $[0,\rho(v)]\times\mathcal{S}(p_0,1)$.

The composition $\exp_{p_0}\circ P\Lambda_\rho$ defines a mapping from C^n onto U that maps $\partial_1 C^n$ onto the point $\{p_0\}$ and the right hand side boundary onto ∂U. Thus, the image of a continuous function f on U is also continuous on C^n, but not every function h on C^n is the image of a continuous function on U under the pull back operator $\Lambda_\rho^* P^* exp_{p_0}^*$. In order to identify a continuous function h on C^n with a continuous function f on U, the function h has to satisfy additional consistency conditions.

Definition 6.4.2 *A continuous function h defined on a cylinder C^n is called consistent with a continuous function f on U under the mapping $\exp_{p_0}\circ P\Lambda_\rho$ if $h=f(exp_{p_0}\circ P\Lambda_\rho)$. In this case, h satisfies the following consistency conditions:*

$$\begin{aligned}
(i)\ h(0,\nu_1) &= h(0,\nu_2)\ \forall\nu_1,\nu_2\in\mathbb{S}^{n-1},\\
(ii)\ h(1,\nu_1) &= h(1,\nu_2)\ \textit{if}\ \exp_{p_0}P\Lambda_\rho(1,\nu_1)=\exp_{p_0}P\Lambda_\rho(1,\nu_2)\ \textit{on}\ \mathcal{M}.
\end{aligned}$$

The first condition (i) implies that h is constant on the boundary $\partial_1 C^n$. When the function h on C^n is consistent with f, the constant value at $\partial_1 C^n$ corresponds exactly with the value $f(p_0)$. The second condition ensures compatibility of function h with function f at the points of the boundary ∂U, i.e. if $\exp_p\circ P\Lambda_\rho$ maps two different points $(1,\nu_1)$ and $(1,\nu_2)$ in C^n onto the same point $p\in\partial U$, then $h(1,\nu_1)=h(1,\nu_2)=f(p)$.

Lemma 6.4.3 *The existence of the weight function K satisfying assumptions $(K1)-(K3)$ is equivalent to assuming the existence of a consistent function h defined on C^n with codomain $\mathbb{R}$ that satisfies the following properties:*

(H1) *h is a continuous function on the compact manifold C^n,*

(H2) $h(t,\nu) > 0, \ (t,\nu) \in [0,1) \times \mathbb{S}^{n-1}$,

(H3) $h(1,\nu) = \kappa(\nu)$ *for all* $\nu \in \mathbb{S}^{n-1}$*, where* κ *is some non–negative function of* ν*,*

(H4) $h(0,\nu_1) = h(0,\nu_2) = const.$ *for all* $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$*,*

(H5) $\int_{C^n} h(t,\nu)d\varsigma = 1$*, where* $d\varsigma = dt \times d\mu$*.*

Proof of Lemma 6.4.3. In order to prove the stated equivalence, we need to show that the function h defined in Lemma 6.4.3 preserves the required properties of K under the continuous mapping $\exp_{p_0} \circ P\Lambda_\rho$. First, we show that the composition of three different mappings is well defined.

First note that since we consider U to be a subset of the normal neighbourhood with respect to p_0 the exponential map is isometric. The assumption that the distance function ρ given by Eq.6.6 is Lipschitz continuous on $S(p_0,1) \subset T_{p_0}\mathcal{M}$ is equivalent to the assumption of continuity and piecewise regularity of ∂U.

We define an $n-$dimensional subset of the cylinder C^n (Eq.6.4) by

$$C^n_\rho := \{(t,v) \mid t \in [0,\rho(v)], v \in \mathbb{S}^{n-1}\} \subset [0,1] \times \mathbb{S}^{n-1},$$

with boundaries

$$\partial_1 C^n_\rho := \{(0,v) \mid v \in \mathbb{S}^{n-1}\}, \ \ \partial_2 C^n_\rho := \{(\rho(v),v) \mid v \in \mathbb{S}^{n-1}\}.$$

The new set C^n_ρ is a compact subset of C^n and C^n is mapped onto C^n_ρ through the coordinate transform Λ_ρ defined by Eq.6.7.

Since the distance function $v \mapsto \rho(v)$ is strictly positive and Lipschitz continuous on $\mathcal{S}(p_0,1)$, so it is the inverse function $v \mapsto \dfrac{1}{\rho(v)}$. Therefore, the mapping Λ_ρ defines a bi–Lipschitz mapping from C^n onto C^n_ρ. The Jacobian determinant of Λ_ρ equals ρ almost everywhere on C^n.

Next we consider the polar transformation P given by Eq.6.5 which is well defined by continuity in $T_{p_0}\mathcal{M}$, and maps C^n_ρ onto $\mathcal{U} \subset T_{p_0}\mathcal{M}$. Moreover, the transformation P defines a diffeomorphism from $C^n_\rho \backslash \{\partial_1 C^n_\rho, \partial_2 C^n_\rho\}$ onto the open set $\mathcal{U}\backslash\{0, \partial\mathcal{U}\}$. Combining P with the exponential map $\exp_{p_0}$, we have

$$\exp_{p_0} \circ P(C^n_\rho) = U.$$

The composition $\exp_{p_0} P$ defines a diffeomorphism from $C^n_\rho \backslash \{\partial_1 C^n_\rho, \partial_2 C^n_\rho\}$ onto $U\backslash\{p_0, \partial U\}$. Furthermore, the boundary $\partial_1 C^n_\rho$ is mapped onto $\{p_0\}$ and the boundary $\partial_2 C^n_\rho$ onto the boundary ∂U [7]. Then the points $(t,v) \in C^n_\rho$ induce geodesic polar coordinates on $\mathbb{R}^n$.

We have introduced three mappings

$$C^n \xrightarrow{\Lambda_\rho} C^n_\rho \xrightarrow{P} \mathcal{U} \xrightarrow{\exp_{p_0}} U \subset \mathcal{M}$$

[7] When $p_0 \in \partial U$, the boundary $\partial_2 C^n_\rho$ is mapped onto $\partial U \backslash \{p_0\}$

The composition $\exp_{p_0} \circ P\Lambda_\rho$ is a continuous mapping from C^n onto U. Moreover, $\exp_{p_0} \circ P\Lambda_\rho$ maps the boundary $\partial_1 C^n$ of the cylinder C^n onto the point p_0 and the boundary $\partial_2 C^n$ onto ∂U.

Now we prove that a consistent function satisfying properties $(H1) - (H5)$ uniquely determines the weight function satisfying $(K1) - (K3)$:

- It is straightforward to see, that properties $(K1) - (K2)$ are satisfied by construction. The composition $\exp_{p_0} \circ P\Lambda_\rho$ preserves connectedness and compactness and it a continuous mapping from C^n onto $\mathcal{M}$. Moreover, $\exp_{p_0} \circ P\Lambda_\rho$ maps the left hand boundary $\partial_1 C^n$ onto the point p_0 and the right hand boundary $\partial_2 C^n$ onto the boundary of U. Hence, the image $f(\exp_{p_0} \circ P\Lambda_\rho)$ of a continuous function f on $\mathcal{M}$ is also continuous on the cylinder C^n, and every function g defined on C^n satisfying consistency properties $(i) - (ii)$ of Definition 6.4.2 is the image of a continuous function on $\mathcal{M}$ under the pull–back operator $\Lambda_\rho^* P^* \exp_{p_0}^*$.

  The composition $\exp_{p_0} \circ P\Lambda_\rho$ applied to a function h that is continuous on C^n and satisfies the consistency conditions $(i) - (ii)$ of Def. 6.4.2 will give us a continuous function K on $\mathcal{M}$ that by construction is continuous along the geodesics starting at p_0 and ending at the points of the boundary. That means, property $(K1)$ is satisfied. The same argument applies to any non–negative function h on C^n. Thus, properties $(H1) - (H2)$ ensures $(K1) - (K2)$ under the composition $\exp_{p0} \circ P\Lambda_{\rho(v)}$.

- Further, it remains to prove that the weight function K is indeed a probability density on $\mathcal{M}$ with respect to the measure $dv(p)$, i.e. to show that $\int_{\mathcal{M}} d\zeta = 1$.

  $$\int_{\mathcal{M}} d\zeta = \int_{\mathcal{M}} K(p)dv(p) = \int_{T_{p_0}\mathcal{M}} K(\exp_{p_0}(v))\eta_{p_0}(v)d\xi,$$

  where $d\xi$ is the standard Lebesgue measure on the Euclidean space $T_{p_0}\mathcal{M}$ and $\eta_{p_0}(v) = det((d\exp_{p_0})_v)$ is the Jacobian determinant of the exponential map. Note that $\eta_{p_0}(v)$ represents the density function that is a positive and continuously differentiable function on $U \subset T_{p_0}\mathcal{M}$. Further, we have

  $$\int_{T_{p_0}\mathcal{M}} K(\exp_{p_0}(v))\eta_{p_0}(v)d\xi = \int_{\mathcal{S}(p_0,1)} \int_0^{\rho(v)} t^{n-1} K(\exp_{p_0}(tv), t)\eta_{p_0}(tv)dtd\mu(v),$$

  where t^{n-1} is the Jacobian determinant of the polar coordinate transformation and $d\mu(v)$ is the standard Riemannian measure on the unit sphere $\mathcal{S}(p_0, 1)$. The last step is the mapping from C_ρ to C^n:

  $$\begin{aligned}&\int_{\mathcal{S}(p_0,1)} \int_0^{\rho(v)} t^{n-1} K(\exp_{p_0}(tv), t)\eta_{p_0}(tv)dtd\mu(v)\\ =\;& \int_{S^{n-1}} \int_0^1 \frac{1}{\rho(\nu)} K\Big(\exp_{p_0}(\rho(\nu)t\nu, t)\Big)\big(\rho(\nu)t\big)^{n-1}\eta_{p_0}(\rho(\nu)t\nu)dtd\mu(\nu),\end{aligned}$$

  where the Jacobian determinant is $\dfrac{1}{\rho(\nu)}$. Then using the expression for K we have that the expression above is equal to:

  $$\begin{aligned}&\int_{S^{n-1}} \int_0^1 \frac{1}{\rho(\nu)} \frac{1}{\eta_{p_0}(\rho(\nu)t\nu)} t^{1-n}\rho(\nu)^{2-n} h\left(\frac{t}{\rho(\nu)}, \nu\right)\big(\rho(\nu)t\big)^{n-1}\eta_{p_0}(\rho(\nu)t\nu)dtd\mu(\nu)\\ =\;& \int_{S^{n-1}} \int_0^1 h\Big(t, \nu\Big)dtd\mu(\nu) = 1.\end{aligned}$$

□

Using this result, the construction of the weight function becomes easier and more intuitive. One chooses the appropriate function h defined on C^n with respect to the particular model and the uncertainty surrounding it. Then, applying the above transformation one obtains an appropriate weight function K defined on U satisfying properties $(K1)-(K3)$ relevant for model risk analysis. Besides, for a chosen function h the weight function K is unique and well defined.

Theorem 6.4.4 *A continuous function h defined on C^n satisfying conditions $(H1)-(H5)$, determines a unique and well–defined weight function K satisfying $(K1)-(K3)$ on U. More precisely, the map $K: U \to \mathbb{R}$, which to each point p, associates the value $K(p)$ given by*

$$K(p) = \frac{1}{\eta_{p_0}(p)} t^{1-n} \rho(v)^{2-n} h\left(\frac{t}{\rho(v)}, v\right), \tag{6.8}$$

where $\eta_{p_0}(p)$ is the volume density with respect to p_0 [8], v is the tangent vector, $t \in [0, \rho(v)]$ is a scaling parameter, and $\rho(v)$ is the distance function defined above.

Proof of Theorem 6.4.4. Note that the composition $\exp_{p_0} \circ P\Lambda_\rho$ induces a change of variables for integrable function f that yields to the following formula:

$$\begin{aligned}
\int_{\mathcal{M}} f(p)d\zeta &= \int_U f(p)d\zeta = \int_{\mathcal{U}} f(\exp_{p_0}(v))\eta_{p_0}(v)dv \\
&= \int_{\mathcal{S}(p_0,1)} \int_0^{\rho(v)} t^{n-1} f(\exp_{p_0}(tv))\eta_{p_0}(tv)dtd\mu(v) \\
&= \int_{S^{n-1}} \int_0^1 \frac{1}{\rho(\nu)} f(\exp_{p_0}(\rho(\nu)t\nu))(\rho(\nu)t)^{n-1}\eta_{p_0}(t\rho(\nu)\nu)dtd\mu(\nu).
\end{aligned}$$

The volume density η_{p_0} is a well–defined, non–negative function in a neighbourhood of p_0 (globally it may be defined using the Jacobi fields ([118], p.219)). Besides, η_{p_0} is a continuous and differentiable function on $\mathcal{M}$[9]. The distance function ρ is a well defined, strictly positive and Lipschitz continuous function on $\mathcal{S}(p_0, 1)$, and thus is the inverse $1/\rho(\nu)$. Therefore, the mapping Λ_ρ defines a bi–Lipschitzian mapping from C^n to C^n_ρ. Moreover, the composition $\exp_{p_0} \circ P$ defines a diffeomorphism from C^n_ρ $\{\partial_1 C^n_\rho, \partial_2 C^n_\rho\}$. Then, using the fact that the point set $\{p_0\}$ and the boundary of U are subsets of $t\nu-$measure zero, we can conclude that the mapping $\exp_{p_0} \circ P\Lambda_\rho$ is an isomorphism. Then, for any h defined on C^n satisfying conditions $(i)-(ii)$ of Definition 6.4.2, the associated weight function K is well defined and continuous on U. The uniqueness of K follows after specifying a function h that satisfies properties $(H1)-(H5)$. □

6.5 Measure of Model Risk

In this section we introduce a definition of the quantification of model risk, relate it to the previously introduced concepts and study some realistic applications.

Recall that we have so far focused on a weighted Riemannian manifold $(\mathcal{M}, I, \zeta)$ with I the Fisher–Rao metric and ζ as in Eq. (6.3). The model was assumed to be identified with a distribution $p \in \mathcal{M}$. More likely,

[8] In terms of geodesic normal coordinates at p_0, $\eta_{p_0}(p)$ equals the square–root of the determinant of the metric g expressed in these coordinates at $\exp_{p_0}^{-1}(p)$. For p and q in a normal neighbourhood U of $\mathcal{M}$, we have $\eta_p(q) = \eta_q(p)$ [118].

[9] Note that when $\mathcal{M}$ is $\mathbb{R}^n$ with the canonical metric, then $\eta_{p_0}(p) = 1$ for all $p \in \mathbb{R}^n$.

a practitioner would define the model as some mapping $f : \mathcal{M} \to \mathbb{R}$ with $p \mapsto f(p)$, i.e. a model outputs some quantity.[10]

We introduce the normed space $(\mathcal{F}, \|\cdot\|)$ such that $f \in \mathcal{F}$. Though not strictly necessary at this stage we shall assume completeness, so that $(\mathcal{F}, \|\cdot\|)$ is a Banach space.

Definition 6.5.1 *Let $(\mathcal{F}, \|\cdot\|)$ be a Banach space of measurable functions with respect to ζ. The* model risk *Z of $f \in \mathcal{F}$ and p_0 is given by*

$$Z(f, p_0) = \|f - f(p_0)\| . \tag{6.9}$$

Note that the measure represents the standard distance. All outcomes are constrained by the assumptions used in the model itself and so, the model risk is related to the changes in the output while relaxing and changing them.

The quantification of model risk itself can be thought of as a model with a purpose such as provisions, capital calculation or comparison of modelling approaches. Possibilities are endless, so we might have started with some $T : \mathcal{F} \to \mathcal{F}$ and set $Z(f, p_0) = \|T \circ f\|$;[11] however, we think Eq. (8.2) is general enough for our present purposes.

In what follows we consider four examples of Def. 8.0.1. Their suitability very much depends among other factors on the purpose of the quantification, as we shall see later.

1. $Z^1(f, p_0)$ for $f \in L^1(\mathcal{M})$ represents the total relative change in the outputs across all relevant models:
$$Z^1(f, p_0) = \|f - f(p_0)\|_1 = \int_{\mathcal{M}} \Big| f - f(p_0) \Big| d\zeta.$$
2. $Z^2(f, p_0)$ for $f \in L^2(\mathcal{M})$ puts more importance on big changes in the outputs (big gets bigger and small smaller). It would allow to keep consistency with some calibration processes such as the maximum likelihood or least square algorithms:
$$Z^2(f, p_0) = \|f - f(p_0)\|_2 = \left(\int_{\mathcal{M}} \Big(f - f(p_0) \Big)^2 d\zeta \right)^{1/2}.$$
3. $Z^\infty(f, p_0)$ for $f \in L^\infty(\mathcal{M})$ finds the relative worst–case error with respect to p_0:
$$Z^\infty(f, p_0) = \|f - f(p_0)\|_\infty = \operatorname*{ess\,sup}_{\mathcal{M}} \Big| f - f(p_0) \Big|.$$
Furthermore, it can point to the sources of the largest deviances: Using $\exp_{p_0}^{-1}$ we can detect the corresponding direction and size of the change in the underlying assumptions.

   In addition, one might as well benefit from finding the relation between the level of uncertainty, direction of the change and the worst–case error, which involves comparing $Z^\infty(f, p_0)$ for different values of $\alpha(v)$ and $v \in \mathcal{S}(p_0, 1)$. This might be of interest, especially, when the value of model risk under Z^1 or Z^2 is relatively big with respect to the size of the selected neighbourhood. Large values may mean high sensitivity of the model output to the changes in the underlying assumptions or parameters, the result of arbitrage behavior or the usage of the model in areas for which it does not longer fit.

[10] This is not always the case but we can proceed along these lines depending on the usage to be given to the quantification itself. For example, an inter(extra)polation methodology on a volatility surface is a model whose output is another volatility surface, not a number. If we want to quantify the model risk of that particular approach for Bermudan derivatives we might consider its impact on their pricing.

[11] For example, another possibility is to use $\left\| \frac{f}{f(p_0)} \right\|$ or $\left\| \frac{f - f(p_0)}{f(p_0)} \right\|$. These functional forms would allow us to obtain a dimensionless number which might be a desirable property.

(a) The relation between the level of uncertainty and the worst case: the neighbourhood around p_0 is defined for different $T \in [0, \alpha(v)]$, such that

$$U_T = \{p \in \mathcal{M} \mid d(p_0, p) \leq T\}, \tag{6.10}$$

and then Z^∞ can be evaluated for every T

$$Z_T^\infty(f, p_0) = \operatorname*{ess\,sup}_{U_T} \Big| f - f(p_0) \Big|.$$

(b) The relation between the worst case and the direction of the change: as there exists a unique geodesics for every $v \in T_{p_0}\mathcal{M}$ such that $\gamma(0) = p_0$ and $\gamma'(0) = v$, the behaviour and the response of the model to a particular change can be examined along the geodesic, i.e. response analysis.

4. $Z^{s,p}(f, p_0)$ for $f \in W^{s,p}(\mathcal{M})$ is a Sobolev norm that can be of interest in those cases when not only f is relevant but its rate of change:[12]

$$Z^{s,p}(f, p_0) = \|f - f(p_0)\|_{s,p} = \left(\sum_{|k| \leq s} \int_{\mathcal{M}} \Big| \partial^k \Big(f - f(p_0) \Big) \Big|^p d\zeta \right)^{1/p}.$$

A sound methodology for model risk quantification should at least consider the data used for building the model, the model foundation, the IT infrastructure, overall performance, model sensitivity, scenario analysis and, most importantly, usage. Within our framework we address and measure the uncertainty associated with the aforementioned areas and the information contained in the models. The choice of the embedding and proper neighbourhood of the given model takes into account the knowledge and the uncertainty of the underlying assumptions, the data and the model foundation. The weight function that assigns relative relevance to the different models inside the neighbourhood considers the model sensitivity, scenario analysis, the importance of the outcomes with connection to decision making, the business, the intended purpose, and also addresses the uncertainty surrounding the model foundation. Besides, every particular choice of the norm provides different information of the model. Last and most important, the model risk measure considers the usage of the model represented by the mapping f.[13]

In order to explore the further financial implications of model risk, it is necessary to know the explicit link between models and decisions. The potential impact of model risk will depend on the very specific features of the business use that the model is put into, for example pricing, financial planning, hedging, or capital management. The plausible size and direction of such financial impact provides us with a measure of the materiality of model risk, in the context of a specific application.

6.6 Conclusions and Further Research

In this chapter we introduce a general framework for the quantification of model risk using differential geometry and information theory. We also provide a sound mathematical definition of model risk using weighted Riemannian manifolds, applicable to most modelling techniques using statistics as a starting point.

[12] An example can be a derivatives model used not only for pricing but also for hedging.
[13] Or equivalently by any possible transformation $T : \mathcal{F} \to \mathcal{F}$.

Our proposed mathematical definition is to some extent comprehensive in two complementary ways. First, it is capable of coping with relevant aspects of model risk management, such as model usage, performance, mathematical foundations, model calibration or data. Second, it has the potential to asses many of the mathematical approaches currently used in financial institutions: Credit risk, market risk, derivatives pricing and hedging, operational risk or XVA (valuation adjustments).

It is worth noticing that the approaches in the literature, to our very best knowledge, are specific in these same two ways: They consider very particular mathematical techniques and are usually very focused on selected aspects of model risk management.

There are many directions for further research, all of which we find to be both of theoretical and of practical interest. We shall finish by naming just a few of them:

Banach spaces are very well known and have been deeply studied in the realms of for example functional analysis. On the other hand, weighted Riemannian manifolds are non–trivial extensions of Riemannian manifolds, one of the building blocks of differential geometry. The study of Banach spaces over weighted Riemannian manifolds shall broaden our understanding of the properties of these spaces as well as their application to the quantification of model risk.

Our framework can include data uncertainties by studying perturbations and metrics defined on the sample, which are then transmitted to the weighted Riemannian manifold through the calibration process.

The general methodology can be tailored and made more efficient for specific risks and methodologies. For example, one may interpret the local volatility model for derivatives pricing as an implicit definition of certain family of distributions, extending the Black–Scholes stochastic differential equation (which would be a means to define the lognormal family).

Related to the previous paragraph, and despite the fact that there is literature on the topic, the calculation of the Fisher–Rao metric itself deserves further numerical research in order to derive more efficient algorithms.

Chapter 7

Application to Capital Calculation

"Information is the resolution of uncertainty." (Claude Shannon)

In this chapter we contextualize the proposed framework by applying it to a credit risk model used by a commercial bank. More specifically, we employ the proposed methodology for the quantification of model risk to the probability of default (PD) model of high default portfolios used for capital calculation.[1] We analyse different scenarios and risk parameter assumptions in order to assess how these scenarios affect the economic capital based on methodologies commonly applied by IRB institutions.

In general, the purpose of the credit risk model is to estimate the PD of future credit losses on a bank portfolio. For a given time horizon, the model generates a distribution—a probability density function— of future losses that can be used to calculate the losses associated with any given percentile of the distribution. In practice, banks concentrate on two such loss components: expected loss (EL) and unexpected loss (UL). EL is the mean of the loss distribution and represents the amount that a bank expects to lose on average on its credit portfolio over a given time horizon. In contrast, UL refers to the risk of the portfolio that is computed as the losses associated with some high percentile of the loss distribution (e.g., the $99.99th$ percentile), thus covering all but most extreme events. For more details about credit risk, see for example [31].

Capital models are usually based on three risk parameters: the PD, the loss given default (LGD) and the exposure at default (EAD). Under the Basel II IRB framework, the PD per rating grade is the average percentage of obligors that will default over a one–year period, EAD gives an estimate of the amount outstanding if the borrower defaults, and LGD represents the proportion of the exposure that will not be recovered after default. These parameters are aggregated from obligor level (risk bucket) to portfolio level with the correlation set by the regulator or the financial entity.

Let N be the number of borrowers in a given loan portfolio. Assuming a uniform value of LGD, the aggregated expected loss amount, L, can be calculated as the sum of individual Ls in the portfolio, i.e.,

$$L = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot PD_i. \tag{7.1}$$

[1]Note that the purpose of the analysis is only to illustrate the proposed framework, and so it does not fully comprehend all potential risks inherent in PD calculation.

Based on the loss distribution, the capital requirement for a bank under the IRB approach at confidence level α can be calculated simply as the difference between EL and the percentile for the level being considered:

$$C_\alpha(L) = q_L(\alpha) - EL \tag{7.2}$$

where $q_L(\alpha)$ is the α-quantile of L defined by $\mathbb{P}(L \leq q_L(\alpha)) = \alpha$. In practice, the portfolio is categorized into homogeneous risk buckets, $j = 1, \ldots, M$, and the capital calculation is done on the risk–bucket level. The same default probability PD_j and LGD_j is then assigned for all borrowers in each of these buckets.

When modeling credit risk losses, several sources of model risk may arise. Examples are the scarcity of default events, lack of data driving calibration and backtesting, correlations between failures, wrong way exposure (growing utilization of credit lines in case of an increase in PD), or independence of PD and LGD.

For the purpose of illustration, we only focus on the quantification of the model risk arising from the PD estimation on the level of capital, and so we assume both LGD and EAD be known and independent of PD.

This chapter appears as the application part of [80] including small adjustments.

7.1 PD MODEL (p_0)

PD is the likelihood that an obligor will default within one–year given all currently available information. We consider a particular internal model for the long–run PD estimation that serves the regulatory purposes. The PD model is built on internal behavioural data and bureau information. Each customer account is scored and the portfolio is categorized, with respect to the scoring, into M homogeneous risk buckets. The number of defaults in each bucket is assumed to follow a binomial random variable, in which the defaults are independent across customers and over time, and defaults occur with common probability. The underlying Point–in–time (PIT) PD model is calibrated to the latest observed PD, i.e. observed default frequency (ODF), that is just the approximation of the maximum likelihood estimator of the parameter of the binomial distribution. PIT PD is then adjusted by the Central Tendency (CT) to generate the long–run PD based on a combination of historical misalignment of the underlying model and expert judgment. The Through–the–Cycle (TTC) PD, i.e. pooled PD for each risk bucket, is given by the formula:

$$PD_i = ODF_i \cdot \frac{CT}{ODF}, \tag{7.3}$$

where:

- ODF_i is the observed default frequency obtained for each risk bucket i over the most recent quarter of the calibration sample.
- CT refers to the Central Tendency used for the TTC adjustment that is based on external macroeconomic data series in order to extend the internal ODF series.[2]
- ODF is the average ODF over the given segment portfolio.

[2] In case there is not enough data history in order to cover the whole economic cycle.

Thus, the long–run observed default frequencies are calculated for each risk bucket, and are adjusted to the average PD observed for each portfolio over a complete economic cycle. The PD therefore gives the likelihood for obligors with a particular rating grade at the start of a given time period defaulting within this period. The distribution of defaults under our simplifying assumptions would equal the loss distribution of a portfolio for which all the borrowers had an EAD of 1, an LGD of 0.45 and a maturity of 1 year.

The empirical analysis is based on a hypothetical portfolio that was elaborated in order to represent the behavior of one particular segment consisting of $N = 9860$ clients. All customers are categorized by risk scoring into buckets that are heterogeneous at a defined confidence level and represent the grade of credit quality. We assume a portfolio consisting of ten buckets ($M = 10$) with a monotonous PD related to the model scores: higher scores imply lower ODFs.

All customers within the same risk bucket are assigned the same "pooled" PD, which can be thought of as the average of individual PDs. This means that the pooled PD assigned to a risk bucket is a measure of the average value of the PDs of customers in that bucket.

Risk buckets	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
PD	0.276	0.198	0.181	0.090	0.085	0.065	0.037	0.034	0.023	0.011
N(frequency)	0.007	0.012	0.021	0.055	0.066	0.088	0.176	0.280	0.241	0.056

Table 7.1: Normalized PD and frequency of accounts in the portfolio across risk buckets.

The normalized PDs vary between 0.01143 and 0.27638 and decrease from lower to higher risk ratings, although this decrease is moderate (Table 1). The distribution is highly skewed to the right (as expected) with mean and variance equal to 0.4177 and 0.08768, respectively. The frequency of accounts across buckets ranges from 0.70% to 27.96% (Table 1). The low PD buckets account for the majority of the portfolio, where approximately 75% of the total accounts have PDs lower than 0.04.

There are many factors that influence the direction and the extent of the relations between the developments in defaults and the risk factors influencing solvency. These comprise the macro indicators, market factors and idiosyncratic factors. For example, foreign clients tend to be more risky than domestic; the risk of default is expected to grow with increasing interest rates, unemployment rate or percentage deviation between the exchange rate level at which the individual loan was granted from the actual exchange rates.

7.2 Quantification of Model Risk

The question of understanding the impact of the uncertainty surrounding PD on the capital calculation can be approached by obtaining a mathematical representation of the underlying statistical distribution over the portfolio

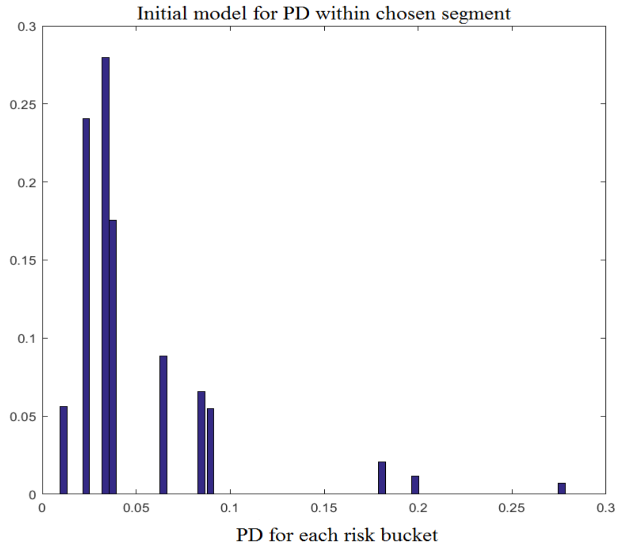

Figure 7.1: Distribution of the PD within the portfolio

(see Figure 1). A change in the model assumptions or segmentation would represent a shift in the parameters defining the statistical distribution.[3] We propose, based on the goodness of fit, to predict the best future values of our given portfolio by using the 2–parameter Inverse Gaussian distribution (ING) given by

$$p(x, \lambda, \mu) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \cdot \exp\left(-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right)$$

with mean μ and variance $\sigma^2 = \dfrac{\mu^3}{\lambda}$.[4] This means that we consider our model space to be a manifold of the ING distribution family, $\mathcal{M}$, parametrized by (λ, μ).

The initial model, $p_0 \in \mathcal{M}$, was estimated via Maximum Likelihood approach (MLE) with parameters $\lambda_0 = 0.5418$ and $\mu_0 = 4.8075$, i.e. $p_0 = ING(\lambda_0, \mu_0)$. With this representation we take into consideration how specific risk buckets perform within the default portfolio or, alternatively, the weight of default of each grade within the portfolio.

The PD model allows one to estimate the probability of default for a particular segment across different risk–buckets within the limits of defined parameters. The model represents a probability distribution of several correlated and mutually interacting events. Note that the parameters of the distribution are non–linear functions of the individual PD and N for each risk bucket, i.e. $\mu(PD, N)$ and $\lambda(PD, N)$. This means, that for each value of PD there is a set of coordinates, expressing characteristics of the portfolio. The set of PDs then defines the 2–dimensional statistical model manifold, $\mathcal{M}$.

For the quantification of model risk, it is important to understand how the model integrates the information provided by the sample and how sensitive it is to changes in the sample characteristics (see Figure 2). The way to recognize details about the transition from one state to another, either by comparing sample data or models for

[3] Note that we can equally well work with empirical distributions (histograms) and use the Fisher–Rao distance between them [90]. However, to properly illustrate the aforementioned framework we approximate the portfolio by a parametric probability distribution.

[4] The Inverse Gaussian density function represents a wide class of distributions, ranging from highly skewed distribution to a symmetrical one as λ/μ varies from 0 to ∞. For more details about the Inverse Gaussian distributions see for example [115].

a given portfolio (see Chapters 8 or 6, respectively), is entirely based on examining changes in the corresponding probability distribution.

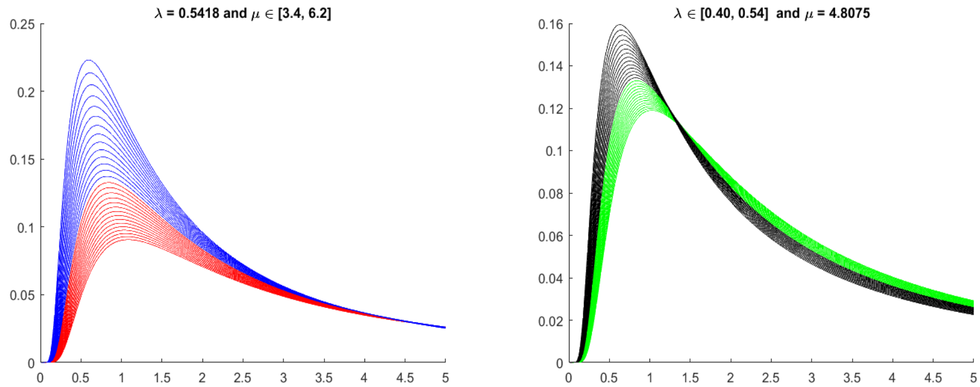

Figure 7.2: Probability density functions of Inverse Gaussian distributions: The left panel shows densities for different μ with $\lambda = 0.5418$. The right panel shows densities for different λ for $\mu = 4.8075$. The densities are unimodal with mode between 0 and μ. As μ/λ increases the distribution becomes more skewed to the right and the mode decreases relative to the mean.

Remark: This approach is particularly beneficial when dealing with missing or insufficient data, or any other issues or errors, as examining the degree of uncertainty might be adaptable by means of measuring the evolution of an entropy in the model space. This means, that the evolution or transformation of the input data, can be studied by the distance between the corresponding probability distributions. The problem of quantifying the uncertainty between two models, such as characteristics of the portfolio in our specific example, is described by a distance of the geodesic connecting the corresponding probability distributions.

7.2.1 Identification of potential sources of model risk

To properly assess the model risk, we first need to identify and describe the potential sources of model risk, covering the model assumptions, weaknesses and arbitrariness in the development process. For the given PD model, the main risk may arise from the applied segmentation and risk scoring, bias toward historical experience, unexpected moves in the exchange rate, relevant changes in macro variables, missing values, incorrect structure and methodology, inappropriate factors used for the TTC adjustment, or due to the subjective selection of some parameter values.

The impact of these risk sources should be expressed quantitatively and then combined based on whether the separate model risks are independent, or some amplify each other, or counteract or absorb each other. In what follows and for the sake of illustration, we concentrate on the risk inherent in the applied segmentation arising from the PD estimation. The given model, p_0, is only one possible description of the data incorporating the expert knowledge and assumptions directed by the expected use, as well as implementation constraints. Any uncertainty and errors in the estimation of PD influence the required capital.

7.2.2 Choosing a suitable Riemannian metric

We develop our example on the assumption that the model space should resemble the metric properties of a Riemannian manifold of negative curvature equipped with the Fisher–Rao information metric. Since there are many possible choices of metrics on a given differentiable manifold, it is important to consider the additional motivating properties of the Fisher–Rao information metric:

1. Fisher–Rao information metric is based on the notion of maximum entropy, which might be geometrically interpreted by the possible trajectory in a statistical manifold that describes its evolution. We can examine the degree of uncertainty by measuring the evolution of entropy in the model space.

2. The Fisher–Rao information may be thought of as the amount of information a sample supplies with respect to the problem of estimating the parameters. [5]

3. The Fisher–Rao information metric determines a constant negative Gaussian curvature for the ING distribution (for details see [115]). The metric is given by

$$g(\lambda,\mu) = \begin{bmatrix} \frac{1}{2\lambda^2} & 0 \\ 0 & \frac{\lambda}{\mu^3} \end{bmatrix}.$$

   Negative curvature guarantees that MLE estimates are well–defined and unique, [101].

In this metric, the model space has non–zero curvature and reflects model specification sensitivity accurately.

7.2.3 Neighbourhood selection

Following Section 4, defining a proper neighbourhood around the given model is a trade–off between plausibility and severity in order to ensure that no harmful but plausible risks are missed and the irrelevant risk factors are not included. As aforementioned, we focus on the sensitivity of the chosen segmentation and the inherent uncertainty. Namely, we analyze the model risk arising from the uncertainty about the default counts, and so, we ask how different could the model be if the default counts were undercounted or overcounted under the assumption of the fixed number of risk buckets.

The default rates are calculated on the risk bucket level, and so the miss–estimation in each bucket is going to contribute to the overall uncertainty. Therefore, we consider all possible deviations within $[-0.3, 0.3]$ standard deviation changes in each of the ODF_i, $i = 1, \ldots, 10$.[6] We take into account all possible shifts in each bucket separately as well as all of the relevant combinations. The limit on the deviation was set to preserve the heterogeneity among risk buckets in terms of PD and to restrict the variability of nearby risk buckets, which could be influenced by the presence of outliers, noise or variation in the density of the points on the manifold.[7]

for the given manifold $\mathcal{M}$ and the point $p_0 \in \mathcal{M}$, in order to build the neighbourhood U we start by fitting probability distributions to all of the combinations of the maximal ± 0.3 standard deviation changes in the

[5] Since the MLE is asymptotically unbiased, the inverse Fisher information represents the asymptotic fluctuations of the MLE around the true value.

[6] Recall that ODFs refer to the maximum likelihood estimators of the binomial distributions for each risk bucket.

[7] The given portfolio (sample) is going to constrain the specificity and sensitivity possible.

corresponding ODFs (i.e. 2^{10} distributions), and first to $1\,000\,000$ and then to $10\,000\,000$ random combinations within the interval $(-0.3, 0.3)$. The scale parameter varies in the interval $[\lambda_1, \lambda_2] = [0.517300, 0.565403]$ and the mean is in the interval $[\mu_1, \mu_2] = [4.58638, 4.95968]$. These fitted distributions form our chosen neighbourhood.[8] For illustration, Figure 3 represents the 3–dimensional embedding of U, the set of alternative models around p_0, determined by the estimated ING distributions.

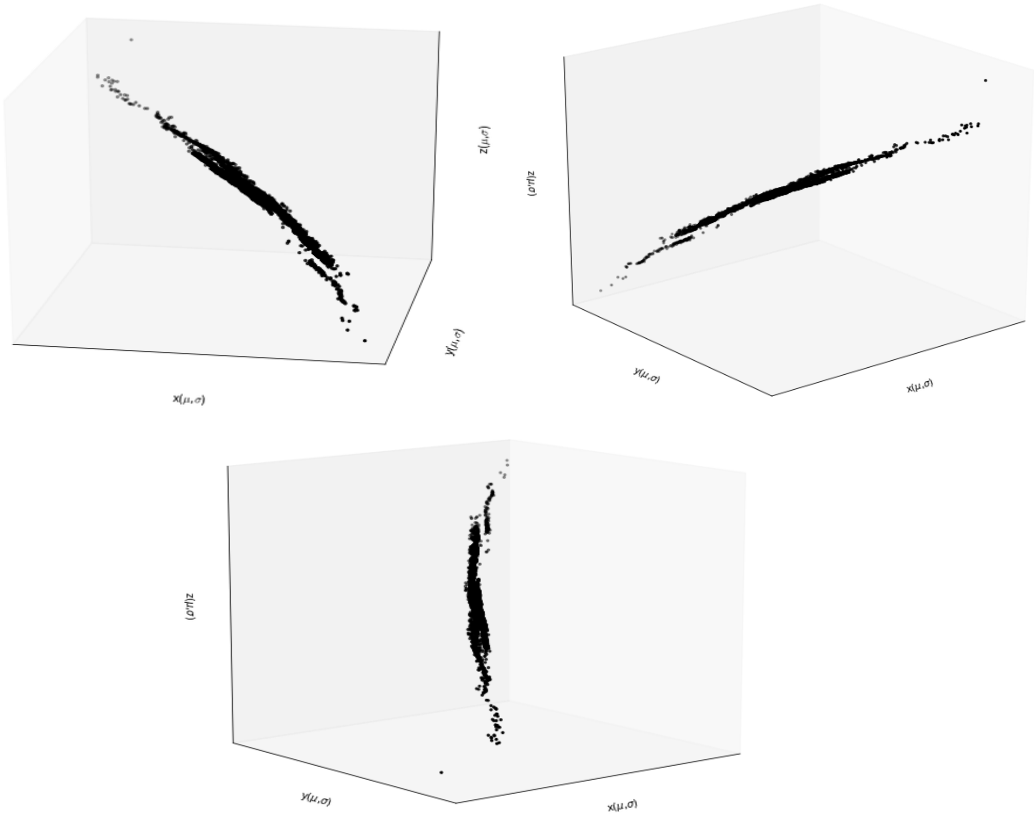

Figure 7.3: 3D multidimensional scaling embedding of U.

Within this chosen neighbourhood we can analyse and quantify the possible impact on the individual shifts but also all of their possible combinations on the model output that might be capable of causing material model movements. Notice the conservatism in the chosen neighbourhood.

Remark: Different adjusted data may result in the same distributions. Considering only movements in the distribution guarantees that the quantification of model risk will not depend on double count risk factors that are highly correlated with factors already included in the analysis.

7.2.4 Choosing an appropriate weight function

In what follows, we show how to estimate the weight function entirely from the data without imposing any additional constraint or particular structure. Roughly speaking, once we assume that the alternative models are random objects valued on the Riemannian manifold, the weight function estimation consists of having the alternative models

[8] The key idea is to leave the data to determine the neighbourhood, instead of imposing one. With increasing number of sampled distributions we can see convergence also in the weight function (see Subsection 7.6) but also in the overall calculated Model Risk (see Subsection 7.7).

within U "contribute" to the estimate at a given point according to their distances from p_0.

More precisely, let $(\mathcal{M}, g)$ be a Riemannian manifold and let us consider $p_1, \ldots, p_n$ independent and identically distributed random points on $\mathcal{M}$ with density function $K(p)$. The estimate of $K(p)$ is then a positive function of the geodesic distance in U, which is then normalized by the volume density of (U, g) to take into account for curvature. These estimators are an average of the weights depending on the distance between p_i and p_0. Formally, using alternative models $p_1, \ldots, p_n$ we propose to define the weight function by a map $K_i : p \in U \to K_i(p) \in \mathbb{R}$ given by

$$K_i(p) = \frac{1}{\eta_{p_0}(p)}\, w_i \quad \forall i = 1, \ldots, n \tag{7.4}$$

where $\eta_{p_0}(p)$ denotes the volume density function and w_i are assigned weights, such that $\sum_{i=1}^{n} w_i = 1$.

The weights are determined as follows. First, we calculate the geodesic distance of all of the estimated distributions inside U from p_0. Next, we determine the number of levels m with respect to the maximal distance from p_0, i.e.

$$L(\lambda_i) = \{p \in U \mid \lambda_{i-1} < d(p_0, p) \leq \lambda_i\} \tag{7.5}$$

for a sequence $\lambda_1, \ldots, \lambda_m$ with $\lambda_m = \max d(p_0, p)$ and $\lambda_i - \lambda_{i-1} = \lambda_m/m$. We set $m = 5\,000$ in case of $1\,001\,024$ distributions inside U and $m = 50\,000$ for $10\,001\,024$ alternative models inside U. Next, we examine the number of distributions within all of these m level sets and based on the concentration within these levels we calculate normalized weights, i.e. w_i. The normalized average frequency of alternative models within U for $1\,001\,024$ and $10\,001\,024$ with respect to the distance from p_0 are illustrated in Figs. 4 and 5, respectively. The resulting normalized weights are then multiplied by the associated values of the volume density at $p \in U$.

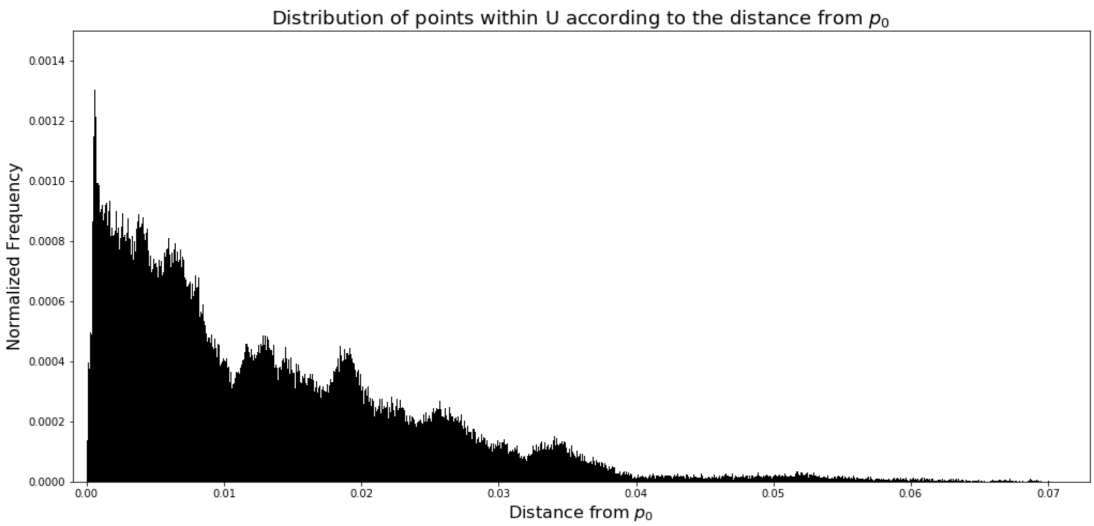

Figure 7.4: The figure represents the weight function given by the average concentration of the alternative models with respect to their geodesic distance from p_0 for $1\,000\,000 + 1024$ alternative models with $5\,000$ level sets.

Remark: The initial model and data single out a particular choice of a weight function through the concentration and variations of the fitted probability distributions on the manifold.

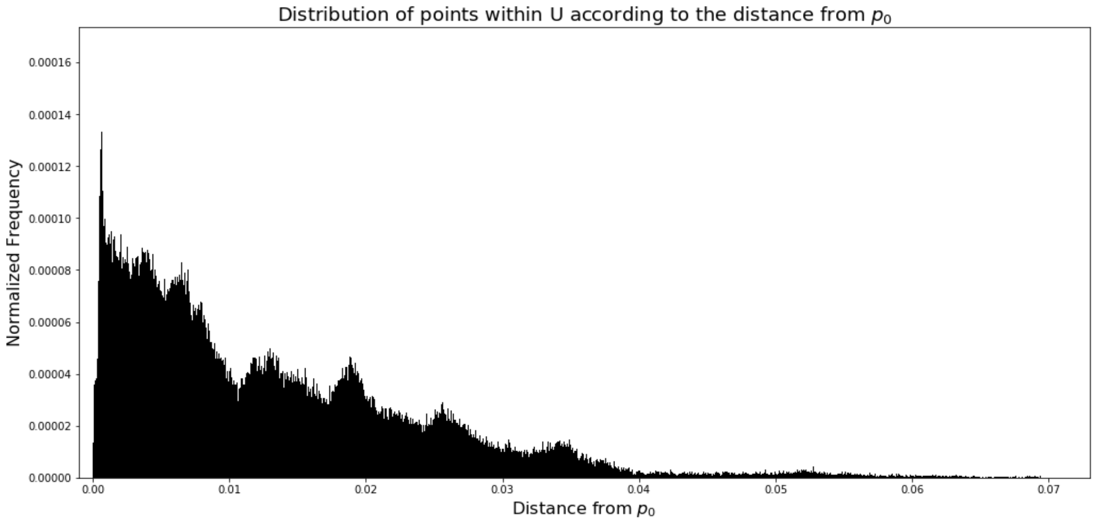

Figure 7.5: The figure represents the weight function given by the average concentration of the alternative models with respect to their geodesic distance from p_0 for $10\,001\,024$ alternative models with $50\,000$ level sets.

7.2.5 Measure of Model Risk

The last step in our proposed framework is the choice of a proper norm and the evaluation of the identified model risk described in Section 6. We suggest to use the $L^2(\mathcal{M})$ norm that in our particular example guarantees the consistency with the Maximum likelihood used for estimation and amplifies big changes in the outputs, i.e. capital level. Given that the PD is used for capital allocation, $C : \mathcal{M} \to \mathbb{R}$ with $p \mapsto C(p)$, and for the choice of the norm, we obtain[9]:

$$Z\big(C(p), p_0\big) = \|C(p) - C(p_0)\|_2 = 2.35 \times 10^{-4}$$

This means that the model risk from segmentation represents the 0.0235% variation of capital inside U.[10] The model risk is obtained in terms of capital, thus allowing an easy interpretation. As the next step, it can be used, for example, for allocating appropriate capital, reserves or provisions.

By dividing the model risk with the capital, we obtain a measure without units that is objective and enables us to compare segmentations of different credit risk portfolios but even different risk portfolios that are based on segmentations, for example compare segmentation between credit risk portfolio and operational risk portfolio.

[9]Model Risk calculated with $1\,001\,024$ distributions with the weight function based on $5\,000$ level sets equals $2.33499704056 \times 10^{-4}$ and Model Risk based on $10\,001\,024$ distributions inside U with $50\,000$ distance levels equals $2.35152164064 \times 10^{-4}$.

[10]Note that this refers only to one source of model risk inherent in the model for capital calculation.

Chapter 8

Quantification of Model Risk: Data Uncertainty

"There is a great difference between knowing and understanding: you can know a lot about something and not really understand it." (Charles Kattering)

The framework for the quantification of model risk defined in Chapter 6 through differential geometry and information theory is applicable to most of the techniques currently under usage in financial institutions. [1] Our framework is general enough to cope with relevant aspects of model risk management such as model usage, model performance, mathematical foundations, model calibration or data.

In the framework of Chapter 6, models are represented by particular probability distributions that belong to the set of probability measures $\mathcal{M}$ (the model manifold), available for modelling. We assume that the examined model p_0 is represented by a probability distribution that can be uniquely parametrized using the n–dimensional vector parameter $\theta_0 = (\theta_0^1, \ldots, \theta_0^n)$ and can be described by the probability distribution $p_0 = p(x, \theta_0)$, i.e.

$$p_0 \in \mathcal{M} = \{p(x, \theta) : \theta \in \Theta \subset \mathbb{R}^n\}.$$

Selecting a particular p_0 is then equivalent to fixing a parameter setting $\theta_0 \in \Theta$ and induces $\mathcal{M}$.

A nonlinear weight function K on $\mathcal{M}$ places a relative relevance to every alternative model, and assigns the credibility of the underlying assumptions that would make other models partially or relatively preferable to the nominal one, p_0.[2] Requiring K to be a smooth density function over $\mathcal{M}$ induces a new absolutely continuous probability measure ζ with respect to the Riemannian volume dv defined by

$$\zeta(U) = \int_U d\zeta = \int_U K dv(p), \tag{8.1}$$

where $U \subseteq \mathcal{M}$ is an open neighbourhood around p_0 containing alternate models that are not too far in a sense quantified by the relevance to (missing) properties and limitations of the model (i.e., the uncertainty of the model selection).

The model risk measure considers the usage of the model represented by some predefined mapping $f\colon \mathcal{M} \to \mathbb{R}$ with $p \mapsto f(p)$, the output function. Model risk is then measured as a norm of an appropriate function of the output

[1] For example credit risk, market risk, derivatives pricing and hedging, operational risk or XVA (valuation adjustments).

[2] The particular K is depends on the model sensitivity, scenario analysis, relevance of outcomes in decision making, business, intended purpose, or uncertainty of the model foundations.

differences over a weighted Riemannian manifold (K above) endowed with the Fisher–Rao metric and the Levi–Civita connection:

Definition 8.0.1 *Let* $(\mathcal{F}, \|\cdot\|)$ *be a Banach space of measurable functions* $f \in \mathcal{F}$ *with respect to* ζ *with* f *as above. The* model risk Z *of* f *and* p_0 *is given by*

$$Z(f, p_0) = \|f - f(p_0)\| \,. \tag{8.2}$$

This approach is used in Chapter 6 to quantify the model risk by embedding $\mathcal{M} \hookrightarrow \mathcal{M}'$ where $\mathcal{M}'$ is given by

$$\mathcal{M}' = \{p(x, \theta) : \theta \in \Theta;\, dim(\theta) \geq dim(\theta_0)\}$$

is a Riemannian manifold containing a collection of probability distributions arising from the variations of θ_0, adding properties or considering data and calibration uncertainty.

The main objective of this chapter is to deepen in the influence of data uncertainty in model risk. In practice, the data used for model development is rarely sufficient nor complete, coming from different sources and often in various forms. Any error or uncertainty in the development data propagates through calibration and model to the outputs, and hence usage. In the context of this work the goal is to relate the uncertainties surrounding data with the model structure and development. Pulling back the model manifold structure, we will introduce a consistent Riemannian structure on the sample space that allows us to investigate and quantify model risk by working merely with the samples. This has mainly practical importance, namely, it offers a computational alternative, easier application of business intuition, and easier way to assign the uncertainty in the data. Additionally, we gain the insight on model risk from two different perspectives, the data and the model perspective.

The rest of this chapter is organized as follows: Section 1 introduces the concept of input data, and describes the model risk that may arise from the data deficiencies. Likewise, it is devoted to the sampling and the fitting processes. In Section 2 a metric on the sample space is introduced, and the Riemannian structure is obtained. On this basis, quantification of model risk in the sample space is proposed in Section 3. Section 4 concludes this article.

This chapter constitute an adapted version of [81], with minor corrections and modifications.

8.1 Sample Space and Fitting Process

Model risk can arise from within any point of the model development process. Once a model has been selected or developed, there are typically four main sources of uncertainty: input data,[3] model structure (given data), estimation of the model parameters (given data and model structure) and model application (given the model structure).

Important components of model risk associated with the development and implementation of the model are flaws in the input data.[4] By data we understand not only sample inputs but any prior knowledge applied to them such

[3]Data deficiencies in terms of both quality and availability, including among others errors in data definition, lack of historical depth, lack of critical variables, insufficient sample, data migration, inaccurate proxies, high sensitivity of expert judgements or wrong interpretation.

[4]There are three main categories of financial data: time series, cross–sectional, and panel data. Time series data consist of information and variables collected over multiple time periods. Cross–sectional data consist of data collected at one point in time for many different subjects, and a panel data consist of cross–sectional data collected at different points in time.

as expert adjustments, smoothing or completion. There are various issues encountered when working with financial data that distort the consistency and affect the availability of data. For example, backfilling or restating data may influence the consistency, some data items may be available for only a subset of the cross–section of the subjects, some inputs may not be available for longer period of time, or simply they are not recorded at certain points in time and are only accessible in certain periodicities.

The uncertainty in the inputs leads to errors that may be amplified over time and space, irrespective of the model equations or models applied. The accuracy of the outputs depends, in part, on the scope and extent of data used in development, on the variability of factors within the population, and on the methods used to formulate and calibrate the model. Additional uncertainty is introduced from the small number of available measurements and the differences between the scale of the parameter measurements and appropriate parameter value at the model grid scale.

For the quantification of model risk, a practitioner is given the particular model $p_0 = p(x_0, \theta_0)$, the sample x_0 used for the model development of p_0, and the fitting process used to fit p_0 to x_0. We assume that p_0 is parametrized by an n–dimensional vector of parameters $\theta_0 \in \Theta$ where each dimension represents a different piece of information inherited in p_0. The quantification, as defined in Chapter 6, is performed over the model manifold

$$\mathcal{M} = \{p(x, \theta) : \theta \in \Theta; dim(\theta) \geq dim(\theta_0)\}$$

containing a collection of probability distributions created by varying θ_0, adding missing properties, changing wrong underlying assumptions, or by considering uncertainty surrounding the data and the calibration.

To assess and better control the uncertainty in the data we move the setting for quantification of model risk into a sample space that will represent perturbed inputs and thus alternative models in $\mathcal{M}$. Besides, the advantages of working with samples instead of probability distributions include ease of use, better analyst understanding and easier application of business intuition.

In practice, what is crucial for the quantification of model risk is to introduce a metric on the sample space. Nearness and distance between samples should be defined taking into account the information stored in the samples and their impact on the model usage through for example the choice of a particular weight function and a proper norm in eq. 8.2. It is quite common to implicitly presuppose an Euclidean metric within the sample in order to study e.g. the effect of perturbations in the data.

Let $\bar{\sigma} : \mathcal{M} \to \bar{S}$ be a sampling process where $\bar{S}$ is the sample space.[5] Note that a bunch of samples with different elements and varying sizes are associated to the same distributions via $\bar{\sigma}$. And conversely, some distributions may be connected with the same sample via the fitting process (estimation, calibration). Thus, we impose further conditions on $\bar{\sigma}$ (see eq. 8.3 below) that will be linked to the fitting process used for the development. Particularly, the sampled data need to fit back to models in $\mathcal{M}$ and models need to resemble the image in the sample space.

In general, the fitting process refers to setting the model parameter values so that the behaviour of the model matches features of the measured data in as many dimensions as unknown parameters. Therefore, $\bar{\pi} : \bar{S} \to \mathcal{M}$ which associates to each sample $x \in \bar{S}$ a probability distribution $p(x, \theta) = \bar{\pi}(x) \in \mathcal{M}$. We will assume that this

[5] Each sample, i.e. a collection of independent sample points of the probability distribution, is a point in $\bar{S}$. For example, each point in $\bar{S}$ is a particular instance of the portfolio of loans under consideration.

map is a smooth immersion, i.e. a local embedding[6]. Besides, if a *small* change was applied to the sample, we expect the change in the parameters to be *relatively small* as well, i.e. we want $\bar{\pi}$ to be stable in the samples.

The important remark is that a fitting process produces a set of estimated parameters, that is the best approximation of true parameters given the representation of the data. As a matter of fact, these best estimated values are both fixed values and random variables. Fixed values since historical data are fixed and known, and random variables since historical data are in fact a given realization (therefore the dependence with respect to the sample space cannot be neglected). Thus, a fitting process yields to parameters with uncertainties and these uncertainties have a random character.

Based on $\bar{\pi}$, we define a quotient space out of $\bar{S}$ with the equivalence classes containing samples that are equally valid with respect to the model manifold $\mathcal{M}$. Different samples, regardless of their sample size, will share the same relevant information of the models, for they give rise to the same probability distribution through $\bar{\pi}$.

Definition 8.1.1 *Define the equivalence relation $\sim$ on the sample space $\bar{S}$ by declaring $x \sim x'$ if $p(x, \theta(x)) = p(x', \theta'(x'))$ where $\theta(\cdot)$ is an estimator, i.e. a mapping from the space of random variables to $\mathcal{M}$. We denote $S = \bar{S}/\sim = \{[x] : x \in \bar{S}\}$ the corresponding quotient space and the projection by $\delta : \bar{S} \to S$ defined by $x \mapsto [x]$.*

Remark: The equivalence relation may refer to various limitations and constraints imposed by or on the practitioner, e.g. finite precision, floating point arithmetic, or equivalence with respect to the output function.

The mapping $\delta \colon \bar{S} \to S$ defined by $x \mapsto [x]$ is called the natural projection. So, $\delta(x) = \delta(x')$ if and only if $x \sim x'$. This means that for any sample $x' \in [x]$, the model $p(x', \theta) = p(x, \theta)$ and so any perturbation inside the equivalence class is uninformative for selecting among the possible models.

To guarantee consistency between samples and models, and to ensure uniqueness and compatibility with the sampling process, we require $\bar{\pi}$ and $\bar{\sigma}$ with $Dom(\bar{\sigma}) = \mathcal{M}$ to satisfy

$$\bar{\pi}\bar{\sigma}\bar{\pi} = \bar{\pi}. \tag{8.3}$$

In particular, this implies that not all samples are acceptable. For example, samples with too few elements would not ensure the fitting process to work as inverse of $\bar{\sigma}$[7].

Proposition 8.1.2 *Given the fitting process $\bar{\pi}$ satisfying the equation 8.3, the equivalence relation*

$$x \sim x' \text{ if } p(x, \theta(x)) = p(x', \theta'(x'))$$

is well defined.

Proof. The fitting process $\bar{\pi} : \bar{S} \to \mathcal{M}$ maps each sample $x \in \bar{S}$ to a model on $\mathcal{M}$. We assume $\bar{\pi}$ to be a regular embedding that satisfies $\bar{\pi}\bar{\sigma}\bar{\pi} = \bar{\pi}$. Let $\sim$ be the relation induced by $\bar{\pi}$: $x \sim y \Leftrightarrow \bar{\pi}(x) = p(x, \theta) = p(y, \theta) = \bar{\pi}(y)$. It is easy to check that $\sim$ is reflexive, symmetric and transitive, and so it induces an equivalence relation on S.

By contradiction we prove that the equivalence relation is well–defined. Let's assume that $x \in [y]$ and $x \in [z]$ but $[y] \neq [z]$. Then for any $y' \in [y]$ and $z' \in [z]$ we have $x \sim y'$ and $x \sim z'$. This implies that

[6] For any point $x \in S$ there is a neighbourhood $U \subset S$ of x such that $\bar{\pi} : U \to \mathcal{M}$ is an embedding.

[7] These requirements are not too restrictive since most of the estimation algorithms, such as the Maximum Likelihood, the Method of Moments, Least Square or Bayesian estimation, as well as many calibration processes fall into this class of mappings when appropriate restrictions are applied (if needed).

$\bar{\pi}(x) = \bar{\pi}(y') = p(x, \theta)$ and $\bar{\pi}(x) = \bar{\pi}(z') = p(x, \theta)$, so $\bar{\pi}(y') = \bar{\pi}(z')$. Since $\bar{\pi}\sigma\bar{\pi} = \bar{\pi}$ we have $[y] = [z]$. $\square$

The fitting process $\bar{\pi}$ is invariant under $\sim$ since $\bar{\pi}(x) = \bar{\pi}(x')$ whenever $x \sim x'$, and induces a unique function π on S, such that $\bar{\pi} = \pi \circ \delta$, and similarly for σ.

8.2 Choice of Metric on the Sample Space

To introduce a measure and a notion of distance intrinsic to the data we need to define a metric on S that binds the geometric and the measure–theoretic aspects of the underlying space. Whenever a specific metric is used to measure the distance between two points, it means that an assumption has been made about the geometry of the surface since a metric is unique to a geometry. Often an Euclidean geometry for the sample space is explicitly selected or implicitly assumed in the model. This assumption might be right at a local level but arguable at a global level. Empirically, many data and model spaces share two properties: they are smooth and locally Euclidean. The natural generalization of Euclidean spaces to locally Euclidean spaces are manifolds, being differentiable manifolds smooth counterparts.

Given Proposition 1 we can endow the sample space with the rich structure of the model manifold. The geometric structure on the model manifold $\mathcal{M}$ can be pulled back from $\mathcal{M}$ to S, and allow it to account for the non–linearities between different samples. In this way, the pulled back metric will induce a proper Riemannian structure on S.

Let g be the Riemannian metric on $\mathcal{M}$ and $\pi : S \to (\mathcal{M}, g)$ be a smooth immersion (as defined above). Then the definition of the pullback of the metric $\pi^* g = h$ acting on two tangent vectors u, v in $T_x S$ is

$$h(u, v) = (\pi^* g)_x(u, v) = g_{\pi(x)}(\pi_*(u), \pi_*(v))$$

where $\pi_* : T_x S \to T_{\pi(x)}\mathcal{M}$ is the tangent (derivative) map.

In a similar way we can define a connection ∇^* on the manifold S as the pullback of the Levi–Civita connection ∇ on $\mathcal{M}$. For two vector fields $X, Y \in \Gamma(TS)$ in the tangent bundle, the pullback connection is given by $\nabla^*_X Y = (\pi^*\nabla)_X Y$. The pullback connection exists and is unique, [54], therefore $\pi^*\nabla = \nabla^*$ for the pullback metric $h = \pi^* g$.

Theorem 8.2.1 *As above, let π be the fitting process and σ be the sampling process with domain $\mathcal{M}$ such that*

$$\pi\sigma\pi = \pi. \tag{8.4}$$

Then the sample space S becomes a weighted Riemannian manifold through the pullback of π.

Proof: The fitting process π is a smooth immersion that satisfies $\pi\sigma\pi = \pi$. From the definition of pullback, $\forall x \in S$ and $p = \pi(x) \in \mathcal{M}$, and $\forall v_1, v_2 \in T_x S$ tangent vectors,

$$\pi^* g(x)(v_1, v_2) = g(\pi(x))(D_x\pi \cdot v_1, D_x\pi \cdot v_2),$$

so that $\pi^* g$ is symmetric and positive semi–definite for any map π. Thus, for $v \in T_x S$, $v \neq 0$, $\pi^* g$ is a Riemannian metric iff $\pi^* g(x)(v, v) > 0$ iff $D_x\pi \cdot v \neq 0$ (since g is Riemannian) iff $ker\, T_x\pi = 0$. In this case, $\pi : (S, \pi^* g) \to (\mathcal{M}, g)$ is an isometric immersion.

Using π we can pull back all the extra structure defined on $\mathcal{M}$ required for the quantification of model risk, including the weight function and the Banach space of the output functions. $\square$

Since (S, h) and $(\mathcal{M}, g)$ are both Riemannian manifolds and the fitting process $\pi : S \to \mathcal{M}$ is a smooth immersion, for any $x \in S$ there is an open neighbourhood U of $x \in S$ such that $\pi(x)$ is a submanifold of $\mathcal{M}$ and $\pi\big|_U : U \to \pi(U)$ is a diffeomorphism. So every point $x \in S$ has a neighbourhood U such that $\pi\big|_U$ is an isometry of U onto an open subset of $\mathcal{M}$, and so π is a local diffeomorphism. As a consequence, (S, h) is locally isometric to $(\mathcal{M}, g)$.

Denote $D\pi(p)$ to be the tangent map at p. Then, given the isometry $\pi|_U : U \to V = \pi(U)$, for geodesics γ_u and γ with initial tangent vectors $u \in T_x U$ and $v = D\pi(p)u \in T_{\pi(p)}V$, respectively, we get

$$\pi(\gamma_u(t)) = \gamma, \quad D\pi(\gamma(t)) \circ P(\gamma_u)_t^s = P(\gamma)_t^s \circ D\pi(\gamma_u(s)),$$

where $P(\gamma)$ denotes the parallel transport. In fact, from the equation

$$\nabla_{D\pi(\gamma_u(t))\dot{\gamma}_u(t)} D\pi(\gamma_u(t))\dot{\gamma}_u(t) = D\pi(\gamma_u(t))(\nabla^*_{\dot{\gamma}(t)}\dot{\gamma}(t)) \equiv 0$$

we see that $t \mapsto \pi(\gamma(t))$ is a geodesic satisfying the initial conditions $\pi(\gamma_u(t)) = \pi(p)$ and $D\pi(\gamma_u(0))\dot{\gamma}(0) = v$. Consequently, it coincides with $\gamma(t)$, i.e. $\gamma(t) = \Phi(\gamma_u(t))$.

In the neighbourhood on which π is an isometry, the probability measure ζ defined on the weighted Riemannian manifold $\mathcal{M}$ given by eq. 10.1 can be pulled back to the sample space. The pullback measure of ζ, is a Riemannian measure $\pi^*\zeta$ on S with respect to the metric h induced on σ given by

$$\pi^*\zeta(f) := \zeta(f \circ \sigma), \quad f \in C^\infty(\mathcal{M})$$

Assuming $\mathcal{M}$ being oriented, we can endow S with the pullback orientation via the bundle isomorphism $TS \cong \pi^*(T\mathcal{M})$ over S. Then $\int_{\mathcal{M}} \zeta = \int_S \pi^*\zeta$.

Proposition 8.2.2 *Let $\mathcal{M}$ be oriented manifold and $\pi\big|_U : U \to \pi(U)$ be an isometry. For any integrable function f on $\pi(U) \subset \mathcal{M}$ we have*

$$\int_{\pi(U)} f d\zeta = \int_U (\pi^* f) d\pi^*\zeta. \tag{8.5}$$

Proof: It suffices to prove eq. (8.5) for functions with compact support. As $\pi(U)$ is endowed with a canonical parametrization we only consider f with *supp* f contained in a chart (apply partition of unity otherwise). Let ψ be a chart on $\pi(U)$ with the coordinates $(\theta^1, \ldots, \theta^n)$. We can assume that $\phi = \pi^{-1}(\psi)$ is a chart on S with coordinates $(y^1, \ldots, y^n)$. By pushing forward $(y^1, \ldots, y^n)$ to $\pi(U)$, we can consider $(y^1, \ldots, y^n)$ as new coordinates in ψ.

With this identification of ϕ and ψ, π^* becomes the identity. Hence, eq. 8.5 amounts to proving that $\pi^*\zeta$ and ζ coincide in ψ. Let g^θ_{ij} be the components of the metric g in ψ in coordinates $(\theta^1, \ldots, \theta^n)$, and g^x_{kl} the components of the metric g in ψ in coordinates $(y^1, \ldots, y^n)$. Then

$$g^x_{kl} = g^\theta_{ij} \cdot \partial\theta^i/\partial y^k \cdot \partial\theta^j/\partial y^l.$$

Let $\tilde{h}_{kl}$ be the components of the metric h in ϕ in the coordinates $(y^1, \ldots, y^n)$. Since $h = \pi^* g$, we have

$$\tilde{h}_{kl} = (\pi^* g)_{kl} = g^\theta_{ij} \cdot \partial\theta^i/\partial x^k \cdot \partial\theta^j/\partial x^l,$$

hence $\tilde{h}_{kl} = g^x_{kl}$. As measures $\pi^*\zeta$ and ζ have the same density function, say P,

$$d\zeta = P\sqrt{\det g^x}dx^1 \cdots dx^n = P\sqrt{\det \tilde{h}}dx^1 \cdots dx^n = d\pi^*\zeta,$$

which proves the identity of measures ζ and $\pi^*\zeta$. □

Theorem 8.2.1 and Proposition 8.2.2 show that in spite of the apparent differences between S and $\mathcal{M}$, one being a space of observations and the other being a statistical manifold, they can be both endowed with the same mathematical structure and become locally equivalent from a geometric point of view. One may choose where and with what objects to work, either with models on $\mathcal{M}$ or with samples on S. Moreover, any kind of analysis on S will not depend on which copy of $x \in [x]$ we choose.

8.3 Quantification of Model Risk on the Sample Space

After pulling back the Riemannian structure from $\mathcal{M}$ to S, we are in the position to quantify model risk directly on the quotient sample space S and introduce the sensitivity analysis of a model to different perturbations in the inputs.

The model in previous sections was assumed to be some probability distribution $p \in \mathcal{M}$, or the corresponding class of samples $x \in S$ after the pullback. More likely, a practitioner would define the model as some mapping $f : \mathcal{M} \to \mathbb{R}$ with $p \mapsto f(p)$, i.e. a model outputs some quantity. The associated pullback output function is then given by $F = \pi^* f,\ \ F : S \to \mathbb{R}$, with F belonging to a Banach space $F \in (\mathcal{G}, \|\cdot\|)$ with respect to $\pi^*\zeta$. For example, the Sobolev norms for $1 \leq q < \infty$ are defined as

$$\|F - F(x_0)\|_{s,q} = \left(\sum_{|k| \leq s} \int_S \left| \nabla^k \Big(F - F(x_0) \Big) \right|^q d(\pi^*\zeta) \right)^{1/q},$$

where ∇ denotes the associated connections on the vector bundles. The Sobolev space with respect to $\pi^*\zeta$ is then given by

$$W^{s,q}(S, d(\pi^*\zeta)) = \{F \,:\, \|F\|_{s,q} < \infty\}.$$

Theorem 8.3.1 *Let $\mathcal{M}$ be a model manifold containing all alternative models relevant for the quantification of model risk. Through the sampling process σ satisfying (8.4) and the natural projection $\delta : \bar{S} \to S$, we construct the sample space S that can be endowed with the Riemannian structure through the pullback of π. Letting $(\mathcal{G}, ||.||)$ be a Sobolev space of measurable functions with respect to the pullback measure $\pi^*\zeta$, we get an equivalent measure of model risk Z as defined in (8.0.1) for the functions defined on S, i.e. $F \in \mathcal{F}$, in the neighbourhood U of x_0 where π is a diffeomorphism. The measure is given by*

$$Z(F, x_0) = \Big|\Big|F - F(x_0)\Big|\Big|.$$

Proof: Recall that Z on $\mathcal{M}$ for p_0 is $Z(f, p_0) = \big|\big|f - f(p_0)\big|\big|$ where $f \in (\mathcal{F}, ||.||)$ is a measurable function belonging to a Banach space with respect to ζ. We want to show that Z is equivalent to $Z(F, x_0) = \big|\big|F - F(x_0)\big|\big|$

defined on S endowed with the pullback structure and measurable functions $F = \pi^* f$ belonging to a Sobolev space with respect to $\pi^*\zeta$.

The fitting process π represents a smooth map $\pi : S \to \mathcal{M}$, and so provides a pullback of differential forms from $\mathcal{M}$ to S. Namely, let $D_x\pi$ denote the tangent map of π at $x \in S$. The pullback of any tensor $\omega \in \mathcal{T}^k(\mathcal{M})$, where $\mathcal{T}^k(\mathcal{M})$ denotes the set of all $C^\infty-$covariant tensor fields of order k on $\mathcal{M}$ by π, is defined at $x \in S$ by $(\pi^*\omega)(x) = (D_x\pi)^*(\omega(x))$. The pullback π^* is a map from $\mathcal{T}^k(\mathcal{M})$ to $\mathcal{T}^k(S)$. The pullback respects exterior products and differentiation:

$$\pi^*(\omega \wedge \eta) = \pi^*\omega \wedge \pi^*\eta, \qquad \pi^*(d\omega) = d(\pi^*\omega), \quad \omega, \eta \in \mathcal{T}(\mathcal{M})$$

Besides being a smooth map, π is an immersion, i.e. for each $p \in \mathcal{M}$, there is a neighborhood U of p such that $\pi\Big|U : U \to \pi(U)$ is a diffeomorphism. Since S and $\mathcal{M}$ are Riemannian manifolds with respective volume forms ζ_S and ζ, the tangent map $D_x\pi : T_xS \to T_{\pi(x)}\mathcal{M}$ can be represented by an $n \times n$ matrix Φ independent of the choice of the orthonormal basis. Following [108], the matrix Φ of a local diffeomorphims π has n positive singular values, $\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0$. Similarly, the inverse map

$$T_{\pi(x)}(\pi^{-1}) : T_{\pi(x)}\mathcal{M} \to S$$

is represented by the inverse matrix Φ^{-1}, whose singular values are the reciprocals of those for Φ, i.e.

$$\beta_1(\pi(x)) \geq \cdots \geq \beta_n(\pi(x)) > 0$$

which satisfies $\beta_i(\pi(x)) = \alpha_{n-i+1}(x)^{-1}$, i.e. $\beta_i = \alpha^{-1}_{n-i+1} \circ \pi^{-1}$, for $i = 1, \ldots, n$. Then the pullback of the volume form on $\mathcal{M}$ is given

$$\pi^*\zeta = (det\, \Phi)\zeta_S = (\alpha_1 \ldots \alpha_n)\zeta_S.$$

Since π is a local isometry on U, the linear map $D_x\pi : T_xS \to T_{\pi(x)}\mathcal{M}$ at each point $x \in U \subset S$ is an orthogonal linear isomorphism and so $D_x\pi$ is invertible [99]. Then, the matrix Φ is orthogonal at every $x \in S$, which implies that the singular values are $\alpha_1 = \cdots = \alpha_n = 1$. So, π preserves the volume, i.e. $\zeta_S = \pi^*\zeta$, and the orientation on a neighbourhood U around each point through the bundle isomorphism $TS \cong \pi^*(T\mathcal{M})$.

In [108], the authors provide a general inequality for the L^q–norm of a pullback for an arbitrary k–form on Riemannian manifolds. Given $q, r \in [1, \infty]$ such that $1/q + 1/r = 1$, and some $k = 0, \ldots, n$, suppose that the product $(\alpha_1 \ldots \alpha_{n-k})^{1/q}(\alpha_{n-k+1} \ldots \alpha_n)^{-1/r}$ is uniformly bounded on S. Then, for any smooth k–form $\omega \in L^q\Lambda^k(S)$,

$$\left|\left|(\alpha_1 \ldots \alpha_k)^{1/r}(\alpha_{k+1} \ldots \alpha_n)^{-1/q}\right|\right|^{-1}_\infty ||\omega||_q \leq ||\phi_*\omega||_q \leq \left|\left|(\alpha_1 \ldots \alpha_{n-k})^{1/r}(\alpha_{n-k+1} \ldots \alpha_n)^{-1/q}\right|\right|_\infty ||\omega||_q$$

Similarly, for any $\eta \in L^q\Lambda^k(\mathcal{M})$,

$$\left|\left|(\beta_1 \ldots \beta_k)^{1/r}(\beta_{k+1} \ldots \beta_n)^{-1/q}\right|\right|^{-1}_\infty ||\eta||_q \leq ||\pi^*\eta||_q \leq \left|\left|(\beta_1 \ldots \beta_{n-k})^{1/r}(\beta_{n-k+1} \ldots \beta_n)^{-1/q}\right|\right|_\infty ||\eta||_q$$

For isometry, the singular values are $\alpha_1 = \cdots = \alpha_n = 1$, so that the above stated inequalities reduce to

$$||\omega||_q \leq ||\pi^*\omega||_q \leq ||\omega||_q \;\text{ and }\; ||\eta||_q \leq ||\pi^*\eta||_q \leq ||\eta||_q$$

for any $\omega \in L^q\Lambda^k(S)$ and for any $\eta \in L^q\Lambda^k(\mathcal{M})$, respectively. This means that π preserves the L^q norm for all $q \in [1, \infty]$, and consequently the Sobolev norm since this norm is a finite sum of L^q norms. □

Thus, the fitting process induces an isomorphism of the L^q–space on S and $\mathcal{M}$ for any arbitrary k–form. This means, for example, that for any function f that is k–form for $k = 0$, we have $f \in L^q(\mathcal{M})$ if and only if $\pi^* f \in L^q(S)$. Then, the norms of $L^q(\mathcal{M})$ and $L^q(S)$ are equivalent on a neighbourhood U around each point.

The choice of a specific norm depends among other factors on the purpose of the quantification. Two interesting examples are the L^q and Sobolev norms [4]:

1. $Z^q(F, p_0)$ for $F \in L^q(S, d(\pi^*\zeta))$ is the L^p norm that is given by

$$Z^q(F, x_0) = \|F - F(x_0)\|_q = \left(\int_S \Big(F - F(x_0)\Big)^q d(\pi^*\zeta)\right)^{1/q} \quad \text{for} \quad 1 \leq q \leq \infty$$

   Every particular choice of the norm provides different information and picture of the model with respect to the model risk it induces. For instance, the L^1 norm represents the total relative change in the outputs across all relevant sample classes, the L^2 norm puts more importance on big changes in the outputs (it would allow to keep consistency with some calibration processes such as the maximum likelihood or least square algorithms). The L^∞ norm for for $F \in L^\infty(\mathcal{S}, d(\pi^*\zeta))$ given by

$$Z^\infty(F, p_0) = \|F - F(p_0)\|_\infty = \operatorname*{ess\,sup}_{\mathcal{S}} \Big|F - F(p_0)\Big|,$$

   finds the relative worst–case error with respect to p_0 and besides, it can point to the sources of the largest deviances (using the inverse of the exponential map we can detect the corresponding direction and size of the change in the underlying assumptions).

2. $Z^{s,q}(F, x_0)$ for $F \in W^{s,q}(S, d(\pi^*\zeta))$ is a Sobolev norm that can be of interest in those cases when not only F is relevant but its rate of change:[8]

$$Z^{s,q}(F, x_0) = \|F - F(x_0)\|_{s,q} = \left(\sum_{|k| \leq s} \int_S \left|\nabla^k \Big(F - F(x_0)\Big)\right|^q d(\pi^*\zeta)\right)^{1/q}$$

   where ∇ denotes the associated connections on the vector bundles. Note that for $s = 0$ we have $L^q = W^{0,q}$.

The possibility of working with samples instead of models offers a particular computational alternative where the information about the model is summarized in a tractable manner, easier application of business intuition, easier way to assign the uncertainty in the data.

8.4 Conclusions and Further research

By pulling back the model manifold structure, we introduce a consistent Riemannian structure on the sample space that allows us to investigate and quantify model risk by working merely with the samples. The Riemannian structure on the sample space provides a novel way to investigate the data for insight through the data intrinsic distance induced by the pullback metric, and to assess the propagations of the perturbations.

There are many directions for further research, apart from the quantification of model risk, all of which might be both of theoretical and of practical interest. The framework can be applied, for example, to sensitivity analysis

[8] An example can be a derivatives model used not only for pricing but also for hedging.

by using the directional and total derivatives. The directional derivative would allow us to estimate the individual contribution to the output variance of each perturbation in the input, as well as to assess the sensitivity to missing properties, model assumptions or missing values. With the total derivative, determined by the pushforward map, we could measure the total contribution in a small neighborhood of a given sample x_0. Additionally, they both provide an indication of wrong model assumptions, bad model fitting or of influential observations that could somewhat distort the parameter estimates leading in some cases to inaccurate or wrong outputs.

The framework is further suitable for stress testing, i.e. for regulatory and planification exercises, or for checking the validity of the approximations done through the modeling process, testing model validity and stability, or applied in other tests and analysis used in the model risk management. The general methodology can be tailored and made more efficient for specific risks and algorithms. We may enlarge the neighborhood around the model or adjoin new dimension to $\mathcal{M}$ [9], i.e. considering a higher–dimensional space within which $\mathcal{M}$ is embedded, that would consider missing properties, the additional information about the limitations of the model, or wrong underlying assumptions, in order to verify the model robustness or stability. Additionally, the framework may be extended by the concept of data sub–manifold in the case of hidden variables and incomplete data.

[9] As an example, consider P&L that is modeled by the normal distribution $\mathcal{M} = \mathcal{N}(\mu, \sigma)$. To evaluate the impact of relaxing the assumption of symmetry we may introduce the skew in the model, and so embed $\mathcal{M}$ into a larger manifold of skew–normal distributions, $\bar{\mathcal{M}} = \{p(x, \mu, \sigma, s) : \mu \in \mathbb{R}, \sigma > 0, s \in \mathbb{R}\}$ where s is the shape parameter. For $s = 0$ we re–obtain the initial normal distribution.

Chapter 9

Model Risk and Differential Geometry Applied to Sensitivity Analysis

"Once we accept our limits, we go beyond them." (Albert Einstein)

The model development, implementation and usage often involve assumptions concerning the geometry of the data and model space, setting by default the Euclidean geometry. The assumption of Euclidean geometry is rarely questioned or empirically tested, e.g. inherent in many algorithms is differential calculus that implicitly assumes a metric structure. The computation of gradients, directional derivatives or Hessians of the output function is often required while performing various business applications such as risk and capital allocation, sensitivity analysis or optimization methods (e.g., quadratic programming, least–squared problems or unconstrained optimization). For example, in capital allocation one often considers the effect of a marginal change in the volume of a particular business line on the total capital, or the sensitivity analysis often involves the computation of derivatives of the output function with respect to one or several independent variables.

As was pointed out in Chapter 6, most financial models can be encoded as statistical varieties that capture all assumption regarding the likely nonlinear behavior of the system and the dependencies of the model on the development data. Statistical models can be thought of as a family of parametrized probability distributions that forms a geometric variety, known as the model manifold $\mathcal{M}$ in which model parameters form its coordinate system.

The natural geometric structure of the family of probability distributions is defined by a Riemannian manifold, $\mathcal{M}$, that is determined by two fundamental forms. A Riemannian metric tensor, g, can be viewed as the first fundamental form which aims to compute the intrinsic geometric structure of $\mathcal{M}$, such as geodesics, area and volume. The second fundamental form aims to uncover the extrinsic structure of the manifold relative to the ambient space and is determined by the Riemannian curvature tensor (the second order derivative of vector field on $\mathcal{M}$), where the directional derivative is the Riemannian connection ∇. The Riemannian structure generalizes the Euclidean counterparts by studying smooth manifolds endowed with a smoothly changing metric. When the underlying space of the given problem is nonlinear in nature, the assumption of Euclidean geometry may among other things decrease the accuracy and introduce a bias throughout the computation process.

We propose that the measure of curvature which, in a broad sense, is a measure by which a geometrical object deviates from begin flat, may be used in two contexts. First, it can be seen as a mean to control the inherent model risk.[1] Second, even in the case of a flat underlying model variety, different geometries may better fit the particular usage of a model and so increase the overall performance.

The aim of this chapter is to emphasize the importance of the geometry of the underlying space in financial models, focusing on differential computations in the more general context of a Riemannian manifold.The use of the Riemannian geometry allows us to define a measure of distance between sets of parameters in terms of changes in the model structure, rather than changes in the values of the parameters themselves. With some examples of the daily P&L explanation of digital options we show that when the underlying geometry of the space of interest is taken into account, the modeling results are not only more accurate and consistent, but also reduce the potential model risk[2] inherent in the given problem.

The rest of this chapter is structured as follows. In Section 2 we summarize the central concepts from Riemannian geometry of our interest and introduce the terminology used throughout the paper. Section 3 presents the main arguments for the consideration of the non–linear structure of the financial models. Section 4 and Section 5 describe the option price sensitivities and the covariant version of the Taylor expansion followed by the daily P&L analysis for the digital option. A choice of an appropriate Riemannian metric is discussed in Section 6 and applied to the example of the P&L introduced in previous section. Section 7 provides some final conclusions and directions for future work. Finally, the Appendix contains two more examples of the daily P&L of digital options for a synthetic underlying asset and a volatility equity index.

This chapter is based on [79], including small corrections and adjustments.

9.1 Geometrical Background

In what follows, we introduce some standard elements of the differential calculus on the manifold to emphasize which geometrical concepts are of our particular interest, and also to fix the terminology and notation. For a more detailed exposition we refer the reader to [50].

We assume that the manifold under consideration, $\mathcal{M}$, is always sufficiently smooth, and so can be endowed with a Riemanian metric, $g = (g_{ij})$, i.e. smoothly varying inner product on the tangent space. Riemannian manifolds locally resemble an Euclidean space meaning that up to first order they look like Euclidean spaces. Differences arise when we study the second order approximation to these spaces which suffices to recover all the information encoded in the Riemannian manifold, [91]. To proceed with a dynamic analysis, we need to be able to differentiate a vector field along a curve with the use of a notion of affine connection.

Given a Riemannian manifold $(\mathcal{M}, g)$, the fundamental theorem of Riemannian geometry states that there exists a unique linear connection ∇_g on $\mathcal{M}$, called the Levi–Civita connection (of g) that preserves the metric

[1] A Riemannian manifold is an n–dimensional space that near each of its points resembles an n–dimensional Euclidean space. Thus, while locally it looks Euclidean, globally it does not — precisely what happens in the presence of model risk (see Chapters 6, 8).

[2] In the present contribution risk is understood as uncertainty.

($\nabla_g g = 0$) and is torsion–free.[3] This connection is determined in a local coordinate system through the Christoffel symbols: $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$. With these conventions, the covariant derivative of the coordinates v^i of a vector field is $v^i_{;j} = (\nabla_j v)^i = \partial_j v^i + \Gamma^i_{jk} v^k$.

The connection defines the correspondence between vectors in different tangent spaces of the manifold. Note that the rate of change of the tangent space defines the geometry of the manifold. The actual rate of change of a vector field X on the manifold is the change of X along some coordinate curve γ, plus the change of the coordinate curve itself. Therefore, we have

$$\frac{DX^i}{d\gamma^i} = \frac{\partial X^i(\gamma)}{\partial \gamma^j} + \Gamma^i_{jk} X^k(\gamma).$$

If $(U, x^1, \ldots, x^n)$ is a coordinate chart on $\mathcal{M}$, then the Christofell symbols Γ^k_{ij} of the Levi–Civita connection are related to the functions g_{ij} by the formulas

$$\Gamma^k_{ij} = \sum_l \frac{1}{2} g^{kl} \Big(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \Big), \tag{9.1}$$

while the curvature R has components

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^l_{jr} - \Gamma^r_{ji}\Gamma^l_{kr} \ .$$

The curvature tensor is a function which describes the degree to which the manifold deviates from being a flat Euclidean space. The curvature is defined in terms of the intrinsic geometry of a surface and gives rise to changes in acceleration, which is mathematically described by the affine connection. For any smooth function $f : \mathcal{M} \to \mathbb{R}$ we then obtain the second covariant derivative (see [104]) given by

$$\nabla^2_g f = \Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \Big) dx^i \otimes dx^j.$$

Geometrical interpretation of covariant derivative. Let us first look at flat space. When we want to take a derivative of a vector, we consider two vectors $V(x^\alpha)$ and $V(x^\alpha + dx^\alpha)$ separated by an infinitesimal displacement dx^α along the direction of the derivative. Thus, to construct the derivative, we first transport the vector $V(x^\alpha + dx^\alpha)$ parallel to itself back to the point x^α, to give the vector $V_{||}(x^\alpha)$. Only then it is in the tangent space of x^α, and then at a second step, we subtract the vector $V(x^{alpha})$ from it, using the parallelogram rule. The key thing we do is parallel transport.

In the curved space we can perform parallel transport only in a local inertial frame which is equivalent to flat space. However, when we do that, the coordinates of a vector change. This results from the change in the angle the vector make with the basis vectors. This is demonstrated in Figure 9. This change is linear in the vector components. We therefore expect a term of the form

$$\Delta_\beta V^\alpha = \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} V^\gamma$$

where the first term arises from a change in the vector field V between x^α and $x^\alpha + dx^\alpha$, the second term arise from a change in the basis vectors between these two points. In Cartesian coordinates, the Christoffel symbols are going to vanish, because the coordinate directions do not change with respect to any direction. In other bases, we can just think of the second term as a corrective factor for the curvature of the coordinate frames.

[3] The torsion free condition on the connection is given by $[X, Y] = \nabla_X Y - \nabla_Y X$, where $[X, Y]$ is the Lie bracket of the vector fields X and Y.

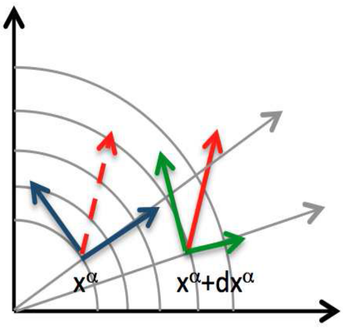

Figure 9.1: When parallel transporting a vector in non–Cartesian coordinates, the components of the vector change, due to change in the basis vectors: in this example, we use polar coordinates, and while the vector itself does not change when parallel-transported, its components do.

With notation as above, we next introduce the coordinate free version of the covariant Taylor series that can be further used to learn how vectors, 1–forms and higher rank tensors move along the manifold. For a function f defined on a Riemannian manifold, a second–order approximation of f around a point $p \in \mathcal{M}$ is given by

$$\begin{aligned} f(q) &= f(p) + \langle grad\, f, v \rangle_p + \frac{1}{2}\langle Hess\, f(p)[v], v \rangle_p + O(t^3) \\ &= f(p) + \langle grad\, f, v \rangle_p + \frac{1}{2}\langle \nabla_{X(p)} grad\, f(p), v \rangle_p + O(t^3) \end{aligned}$$

where $v \in T_p\mathcal{M}$. Note that every term of the Taylor expansion is a tensor. The covariant Taylor approximation is robust to local re–parametrization [88] of the model which comes from the fact that the Riemannian distance is a measure of how the probability density function changes, regardless on how it was parametrized.

9.2 Need to consider nonlinear structures

The principles of sensitivity analysis have a wide range of applications in finance, e.g. in various stages of the model development process, model usage, validation, or as a part of different methodologies and strategies. Sensitivity analysis is often computed either via the linearisation of the model parameters, by analysing the system mean behaviour, by using simulations for approximating finite differences of system outputs or by other sensitivity measures. The chosen method often depends upon assumptions and the amount of information required from the analysis, and they range in both complexity and fundamental understanding.

Performing a proper sensitivity analysis is challenging for many types of financial models, particularly for those that are highly dimensional and whose nonlinear dynamics induce a very strong nonlinear dependence structure in the underlying probability distribution. These models may exhibit widely varying parameter values that can even change depending on the dynamic behaviour of the model and, thus, can vary dramatically in different parts of the parameter space.

A possible explanation for this changing behaviour may be the underlying nonlinear manifold structure, in particular the mathematical relationship between the parameters and the states, but it may equally well be caused by the inherent model risk, for example, by measurement uncertainty of the data, due to inaccurate calibration strategies, model mis–specification or even improper usage. On top of that, the varying behaviour may just be a result of a mismatch between the underlying geometry of the space and the canonical Euclidean geometry that is often, implicitly or explicitly, assumed for the model. If this is the case, this source of model risk can be overcome by formulating the sensitivities directly on the underlying space with a proper inner product.

Furthermore, the covariant differentiation of section of a vector bundle which is a generalization of the standard differentiation of a function, can be used to encode properties of the financial market system, like fluctuations and to make the difference between points depend on the neighbourhood. This means that the relation between points would to some extend be described by the market behaviour.

Essential to sensitivity analysis is the vector calculus with notions such as gradient, divergence, partial derivative or Laplacian. In this case, the varying behaviour plays a key role, particularly when exploring larger regions of the nonlinear underlying model space, where the standard methods that rely on Euclidean geometry may fail or perform poorly at best. These concepts can be replaced by the generalized covariant differentiation defined on the Riemannian manifolds —-crucial when working in curved spaces— in order to consider the local geometry of the underlying Riemannian space.

Differential geometry seems particularly well suited to exploit the natural representation of the underlying model space as a Riemanian manifold, as it may make use of the local information for sensitivity analysis. This way the curvature of the model manifold, which is directly defined by the parameters of the underlying model describing its dynamic behaviour, can be taken into account. The effects of model risk can then be assessed by comparing the Taylor expansion with its covariant version. So, the covariant Taylor expansion will converge in the absence of model risk but also in its presence (provided that we work in sufficiently small region of the given variety).

9.3 Option Price Sensitivities

In this section we start by briefly reviewing the theory of option pricing, recalling the basic concepts and fixing notation in passing. For more information see [67, 69].

A contingent T–claim, for $T \geq 0$, is an $\mathcal{F}_T$ measurable random variable $Y \geq 0$, see [78]. The arbitrage free price V_t of Y at time $0 \leq t \leq T$ is given by[4]

$$V_t = \mathbb{E}\left[\exp\Big(-\int_t^T r_u du\Big)Y \,\Big|\, \mathcal{F}_t\right], \tag{9.2}$$

where the expectation is taken with respect to the risk neutral probability measure p of the discounted cash flow. Since the option price is an expectation of some random variable Y, where Y can be written as a function of the stock price at maturity, the sensitivities of interest are given by the partial derivatives with respect to the corresponding parameter, $\partial V/\partial\theta^i$.

For the rest of the chapter we consider that the price of the underlying security follows the Black–Scholes (BS) model, i.e. its price under the equivalent martingale measure is given by the geometric Brownian motion

$$S_t = S_0 \exp\left(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma\sqrt{t}W_t\right)$$

[4]To simplify notation we shall at times omit the subscript 't' in V_t.

at any time $t \geq 0$, where $S_0 > 0$ and $\sigma > 0$ denotes the initial price and the volatility, respectively, and W_t is a Brownian motion, see [69]. Thus, S_t follows a log–normal distribution with density

$$p(S|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(- \frac{(\log S - \mu)^2}{2\sigma^2} \right), \qquad \mu = \log S_0 + \left(r - \frac{\sigma^2}{2} \right) t. \tag{9.3}$$

The second order Taylor expansion of the option price V_t is then given by

$$dV \approx \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \sigma} d\sigma + \frac{\partial V}{\partial r} dr + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial^2 V}{\partial \sigma^2} d\sigma^2 + \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial t^2} dt^2 \right)$$
$$+ \frac{\partial^2 V}{\partial S \partial \sigma} dS d\sigma + \frac{\partial^2 V}{\partial S \partial r} dS dr + \frac{\partial^2 V}{\partial r \partial \sigma} dr d\sigma + \textbf{error}. \tag{9.4}$$

The Taylor expansion is mainly used to relate sensitivities to the market value for derivative instruments and to revalue the market price (profit and loss, P&L, further addressed in Sec. 5). From a geometrical perspective, the standard Taylor expansion is calculated under the Euclidean geometry of the data and the model. The key for our argument is to realize that, similarly as for many other financial models, the underlying hypothesis of the option pricing is encoded as a statistical model that captures all assumptions regarding the likely nonlinear behaviour of the system and the dependencies of the model on the development data, see Chapters 6, 8.

For example, in the case of BS, the statistical model represents a family of log–normal probability distributions that forms a curved statistical manifold $\mathcal{M}$ with coordinates μ and σ (refer to eq. 9.3). A point on $\mathcal{M}$, determined by (μ_i, σ_i), denotes a specific probability distribution $p(x|\mu_i, \sigma_i)$ defined on the Euclidean data space, $\mathcal{X}$. The option price V refers to a functional over $\mathcal{M}$, and so the underlying geometrical structure of $\mathcal{M}$ induces a Riemannian structure also on the corresponding option model space. For more details see Chapter 6.

A distinction can be made between two types of sensitivities. First, the sensitivities of the option price with respect to variations in market variables such as variations in time or in price of the underlying which measure the risk exposure (e.g., Theta, Delta, Gamma). Second, the sensitivities of the option value with respect to changes in the model parameters (e.g., Rho, Vega, Volga). Besides, these different types of sensitivities can be seen as various movements on two diverse underlying spaces. Derivatives with respect to the market factors refer to changes in the flat $\mathcal{X}$ for a point on $\mathcal{M}$, i.e. for a given $p(x|\mu, \sigma)$, whereas sensitivities with respect to model parameters, i.e. μ and σ, pertain to changes and movements across the potentially curved space of models $\mathcal{M}$. The curved geometry of the underlying model variety implies that the standard differential calculus should be modified according to the inherent local geometry by considering covariant differentiation.

The covariant differential of a function along a tangent vector v for a given connection ∇ that determines the curvature of the underlying space, written $\nabla_v V$, is simply $dV(v)$. In a specific coordinate system, the i–th component of ∇V is $\partial V / \partial \theta^i$ which is just the standard partial derivative. The ij–th component of the second differential $\nabla^2 V$ is given by

$$\left(\nabla^2 V \right)_{(ij)} = \frac{\partial^2 V}{\partial \theta^i \partial \theta^j} - \sum_k \Gamma^k_{ij} \frac{\partial V}{\partial \theta^k},$$

where Γ^k_{ij} are the Christoffel symbols of the corresponding connection defined as in eq. 9.1.

To reduce potential model risks we suggest to modify the second and higher–order greeks[5] with respect to model parameters by correction in the curvature of the underlying space. The second–order covariant Taylor

[5] Option price sensitivities or option factor sensitivities are approximations used to determine the change in price of the option due to a change in the value of one of the inputs or model parameters, commonly called greeks. Refer to [69] for more details.

expansion of the option price that accounts for the underlying model structure can then be written

$$\begin{aligned} dV &\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{\partial^2 V}{\partial S \partial \sigma}dSd\sigma + \frac{\partial^2 V}{\partial S \partial r}dSdr \\ &+ \frac{\partial^2 V}{\partial r \partial \sigma}drd\sigma + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \sigma^2}d\sigma^2 - \Gamma^{\sigma}_{\sigma\sigma}\frac{\partial V}{\partial \sigma} - \Gamma^{r}_{\sigma\sigma}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}dr^2 - \Gamma^{\sigma}_{rr}\frac{\partial V}{\partial \sigma} - \Gamma^{r}_{rr}\frac{\partial V}{\partial r}\right) + \text{error} \quad (9.5) \\ &\approx \text{Euclidean Taylor expansion} + \text{Curvature correction} + \text{residuals.} \end{aligned}$$

By comparing the Taylor expansion given by eq. 9.4 with its covariant alternative, we can notice that in the case of the Euclidean geometry the curvature terms vanish. On the other hand, if $\mathcal{M}$ is not flat the significance of the curvature corrections in eq. 9.5, depending on the specific application, may become relevant. Since a Riemannian manifold is locally Euclidean, within a small neighbourhood and for directions with a small curvature, the curvature terms may often be neglected. However, for larger curvatures or neighbourhoods, it is necessary to explicitly compute the curvature terms if only to show that they are indeed negligible.

The arguments outlined above and a consideration of the limitations of the flat geometry when applied to financial markets suggest the introduction of a Riemannian geometry in the model manifold $\mathcal{M}$. Considering the geometry of the problem may open new modelling possibilities due to the introduction of a new modelling degree of freedom.

9.4 Daily P&L analysis for Digital options under the Black–Scholes model

In this section we emphasize the importance of the curvature in the presence of model risk on the daily P&L analysis for a digital option under the BS model.

Digital options are among the most popular types of exotic options traded over–the–counter (OTC) on stocks, stock indices, foreign currencies, commodities and interest rates. Models used in banks to value these payoffs are more complex in order to better simulate the smile profile for each component of the underlying.

However, there is no generally accepted model for pricing exotic FX options to market. The use of not only different models but also of different methodologies, results in widely dispersed model–dependent exotic option prices for any given set of market and contract inputs. Model risk is especially acute in the case of OTC exotic options for which the daily traded closing mark–to–market prices do not exist, so they need to be mark–to–model instead. Given that models are used throughout the price–making process, from pricing to identifying hedging strategies, calculating daily P&L, defining risk limits, reporting to key stakeholders internally and externally, model risk is an important consideration in the OTC exotic option market.

To highlight model risk we choose the BS model due to its known limitations and drawbacks to price digital options, consequently when used for the P&L explanation.[6] The motivations for this are twofold. First, while using the BS model we can focus on the issue of model misspecification. Second, with the choice of this model we can address the link between curvature and the inherent model risk.

In general, more complex models are usually hard to calibrate. Some market practitioners, therefore, tend to stick to the BS constant–volatility model to price exotic options, but they also adopt some rules of thumb, based

[6]Note that the same approach outlined in this paper can be used for other models such as local or stochastic volatility models.

on hedging arguments, to include the volatility smile/skew or market frictions into the pricing and hedging. We show that enhancements of the BS model approach can be performed through changes in the curvature of $\mathcal{M}$ for a properly selected Riemannian metric. In the end, the choice of a particular model is always a trade–off between accuracy and simplicity.

For the sake of completion, we first briefly review some material on the valuation of a standard digital option. Full details can be found e.g. in [84]. For a digital option with European exercise, pricing is relatively straightforward as a closed–form expression is available in the BS framework. For the European digital call option that pays off 1 at expiration T if the underlying S_T is over the strike K, the payoff at maturity is

$$V_T^D = \begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K, \end{cases}$$

where S_T is the underlying value at maturity. In a risk neutral world, the price of an option can be determined by integrating over the range of possible price outcomes as given by eq. 9.2, and so for the digital option we have

$$V_0^D = e^{-rT} \int_K^{\infty} p(S) dS,$$

where r is the risk free rate and $p(S)$ is the probability density function. With log–normally distribution prices, $\log(S)$ has a normal distribution with μ given in eq. 9.3 and standard deviation $\sigma\sqrt{T}$. Expressed in terms of the cumulative probability, the digital call option price is

$$V_0^D = e^{-rT}\mathbb{P}\big[S_T \geq K\big] = e^{-rT}\mathbb{P}\big[\log(S_T) \geq \log(K)\big] = e^{-rT}\mathcal{N}(d_2),$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative function and d_2 is given by

$$d_2 = \frac{\ln\Big(\frac{S_0}{K}\Big) + \Big(r - \frac{\sigma^2}{2}\Big)T}{\sigma\sqrt{T}}.$$

To value this discontinuous payoff, one possible market consensus is to use a piecewise linearisation, by valuing a combination of vanilla calls, denoted by C, on the underlying asset. The linearised payoff is equivalent to a call spread: long $1/\epsilon$ calls with strike $(K - \epsilon)$ and short $1/\epsilon$ calls with strike K. [7] The calls have the same expiration date as the digital with final payoff

$$1/\epsilon \cdot \max\big(S_T - (K - \epsilon), 0\big) - 1/\epsilon \cdot \max\big(S_T - K, 0\big),$$

where the factor ϵ is chosen by the trader. The payoff of the digital call can be thought of as a limit of a call spread

$$V_T^D = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(C_T(K - \epsilon) - C_T(K)\Big) = -\frac{\partial C_T(K)}{\partial K}.$$

We examine the BS model when applied for the P&L explanation. The P&L analysis provides users with a coherent breakdown of the drivers of P&L movements between two points in time with reference to a selected number of pricing factors. Most of the methodologies for P&L analysis essentially involve performing an approximation of the portfolio using a Taylor expansion of the change in the value metric in terms of the risk factors which

[7] A call spread can be selected by choosing the smallest spread which results in a position size considered small enough to be liquid, either by representing a real possibility for purchase in the market or by being representable in the firm's risk reports by delta positions that can be achieved with reasonable liquidity.

are defined as instantaneous stresses or parallel shocks and thus rely on the first and higher–order sensitivities of the assets in the portfolios.

The P&L explanation comes with challenges owing to the complexities of derivative instruments. This approach works very well for simple derivatives (or linear) products that are explained by the main greeks of the model. Yet, this becomes more complicated with larger market moves and more exotic products, besides the inherent model uncertainty of the model used.[8] Even though mathematically European digital options are easy to price, computing accurate greeks is quite difficult as the model assumptions and in particular the model distribution can dramatically change their values through the life of an option.

In general, there are two fundamental differences between digital options and standard vanilla options: mixed convexity and discontinuous payoffs. The convexity of the digital option changes sign depending on whether it is in–the–money or out–of–the–money. Besides, with time approaching the expiry date, the sensitivities to certain risk factors change their behavior making the option very difficult to hedge. Another source of model risk arises from the usage of the call spread which becomes apparent in the case of unexpected and sudden shifts. The main risk however arises when the underlying price oscillates around the strike price near expiration. This may lead to unexpected, strong variations of the risk exposure of the banks, especially for second order–exposures like Gamma or Volga.

The behaviour of some of the option sensitivities with respect to the position of the current price of the underlying or the time to expiration suggests to link this effect to the curvature of the underlying space. Higher–order sensitivities compare information in two different states (points on the underlying model space) and so distance indicates the difference between them. If this difference is characterized only based on the Euclidean geometry, some of the background information may be lost and the result may not fit the practical significance of the financial analysis but rather cause deviations. Geometrically, the space would not be flat but curved, and the curvature would change depending on the different factors influencing the dynamic behavior of the option price during its life.

To illustrate this point we apply our proposal to a portfolio V consisting of a single digital call option priced under the BS model.[9] In order to explain the potential factors that may influence the changes in the portfolio over a given period we decompose the daily P&L of the portfolio using a second order Taylor expansion. The daily P&L for the digital option with the underlying local geometry of the model space with coordinate system (μ, σ), given market data, can be calculated as

$$
\begin{aligned}
dV_t = V_t - V_{t-1} \quad &\approx \quad \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma \partial S}d\sigma dS \\
&+ \quad \frac{1}{2}\left(\frac{\partial^2 V}{\partial \sigma^2} - \Gamma^r_{\sigma\sigma}\frac{\partial V}{\partial r} - \Gamma^\sigma_{\sigma\sigma}\frac{\partial V}{\partial \sigma}\right)d\sigma^2 + \frac{1}{2}\left(\frac{\partial^2 V}{\partial r^2} - \Gamma^r_{rr}\frac{\partial V}{\partial r} - \Gamma^\sigma_{rr}\frac{\partial V}{\partial \sigma}\right)dr^2 \\
&+ \quad \underbrace{\left(\frac{\partial^2 \mathbf{V}}{\partial \mathbf{r} \partial \sigma} - \boldsymbol{\Gamma}^{\mathbf{r}}_{\mathbf{r}\sigma}\frac{\partial \mathbf{V}}{\partial \mathbf{r}} - \boldsymbol{\Gamma}^{\sigma}_{\mathbf{r}\sigma}\frac{\partial \mathbf{V}}{\partial \sigma}\right)\mathbf{dr}d\sigma + \frac{\partial^2 V}{\partial t^2}dt^2}_{\text{Not included since it is negligible}} + \textbf{residuals} \\
&\approx \quad \text{Euclidean Taylor expansion} + \text{Curvature correction} + \textbf{residuals},
\end{aligned}
$$

[8]For example, since mispricing of the BS model for out–of–money options is large, the performance is strongly affected by the extent to which theoretical model characteristics or market data (skew/smile) are taken into account in their implementation.

[9]Digital options that might be part of a widely diversified portfolio are in some cases approached and examined in an isolated manner, especially in those cases where the underlying is threatening to finish close to the strike and the difficulty in its management increases.

where correction terms are with respect to the curvature of the underlying space, S_t is the underlying, t is the time to expiration, σ is the volatility and r is interest rate respectively. [10]

First we calculate the P&L under the assumption of Euclidean geometry and then compare its daily approximation errors with the one rising from the P&L under certain curved geometry. Using this approach, we can analyse and evaluate the impact of the model risk on the performance of the Taylor approximation. We consider a simple digital call written on the USD/CAD exchange rate in which the institution pays on 24th June 2017, the strike $K = 1.325$ being fixed on 24th June 2016 ($T = 1$) for the initial price of the underlying at $S_0 = 1.282$. The time evolution of the underlying with the realized volatility and interest rate are depicted in Fig. 9.2. To illustrate the proposed approach we further consider other underlying assets whose results are included in the next Section 9.5.2.

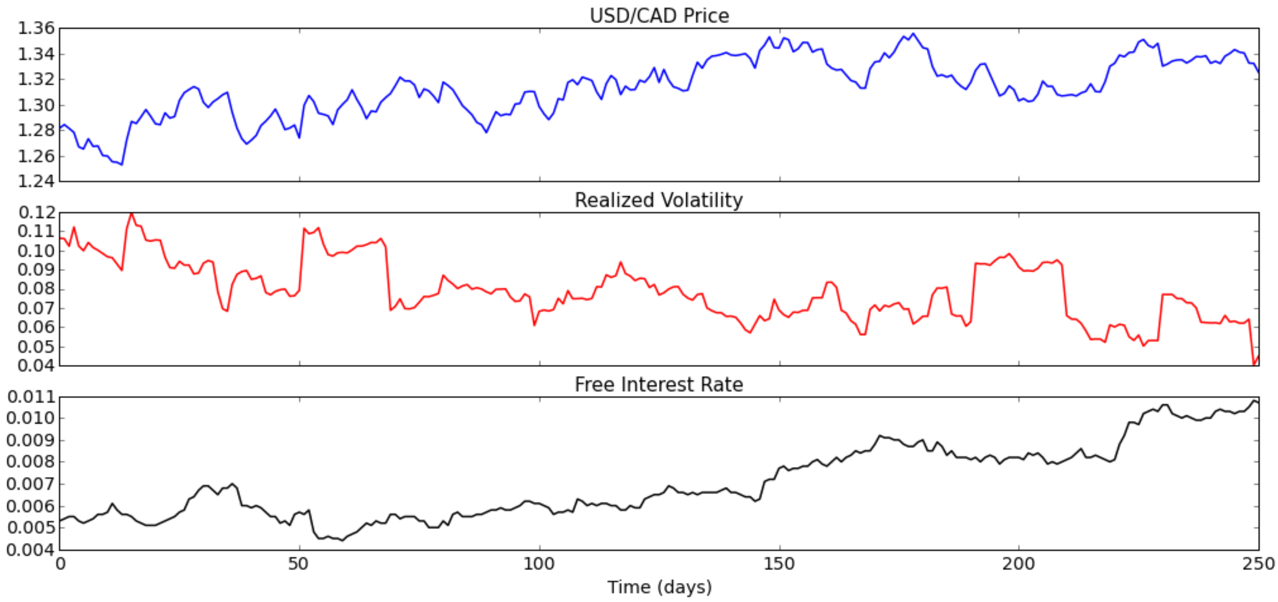

Figure 9.2: Evolution of the underlying USD/CAD exchange rate with the realized volatility and interest rate spread ($r = r_d - r_f$).

We start by considering the P&L explanation as done usually in practice, under the Euclidean geometry, i.e. all Christoffel symbols are equal to zero, $\Gamma^k_{ij} = 0$. In general the P&L attribution will, as with most analysis of surplus processes, leave some unexplained amounts. The error (unexplained part) under the Euclidean geometry is depicted in Fig. 9.3. We present the total error which is just the difference between the market at time t and the theoretical P&L calculated using the Taylor expansion one day before t, i.e. $(P\&L - Theoretical\ P\&L)$.

Generally speaking, the main goal of the theoretical P&L is to perfectly explain the dynamics of the option price. The key thing here is that the P&L always exhibit variance so the fact that the errors are not identically zero is

[10] Usually the effect of the interest rate sensitivity is limited and included in the residual term as well as other higher–order derivatives. The effect of interest rate shifts are usually less important than those of other factors, most notably the price and volatility of the underlying instrument. For these derivatives a single sensitivity to a parallel shift in yield curve usually suffices — this is ρ. To illustrate our approach we include also the second differential with respect to the movement of the interest rate since this may become relevant for longer lived options. Refer to Fig. 9.5.

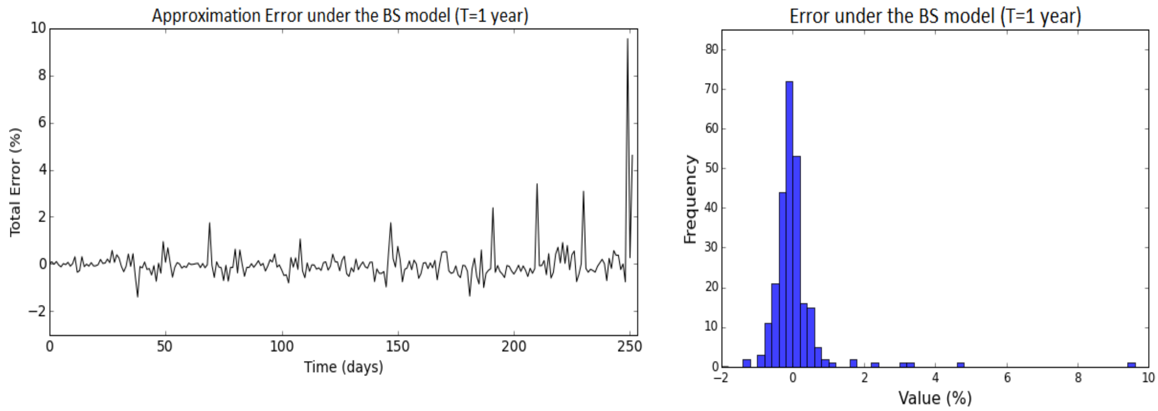

Figure 9.3: Total errors under the BS model with the assumption of Euclidean geometry and total P&L of 0.043. The initial price of the underlying asset is given by $S_0 = 1.28166$ and $S_T = 1.33263$ at expiry with fixed $K = 1.325$. Descriptive statistics for errors: mean = 0.000017, std. dev. = 0.0084, skew = 6.93.

not of particular importance in itself when it comes to P&L explanation.[11] Rather one has to consider other characteristics of the errors and define an acceptable deviation representing the difference between the actual movements of the option price and the movement implied by a given linear combination of the risk factors. Deviations, among others, arise due to the following issues: model limitations (changes in data not been mapped to any of the risk factors or actual shape of movements), higher–order terms in Taylor expansion not quantified, or inconsistent usage of the model with the underlying assumptions.

According to Figs. 9.2 and 9.3, the closer to expiry the larger the approximation errors. Besides it can be seen that the errors were sensitive to changes in the stock price, the realized volatility and also the interest rates. The explanation of these results is in line with the theory and the variance in daily errors can be linked to the deviations of the BS limitations, such as not taking into account jumps in volatility or in price, from the real market behavior. As the BS model is a good approximation to market reality in *static* situations, the second–order approximation assuming an underlying Euclidean geometry stays meaningfully close to the option price for small variations in price and volatility.

Most errors do not exceed 1% in absolute value, but can be of up to 10%. This means that in this particular case the largest fraction of daily errors which correspond to no jumps is small in absolute values, whereas for jump events the error in the BS model is largely affected.

9.5 Choice of a Riemannian metric

One of the basic questions is to determine the geometrical structure, in particular to define a Riemannian metric and an affine connection on the corresponding statistical manifold which would appropriately reflect the dynamical behaviour of the model, the underlying S_t, the payoff and the market data. Any smooth manifold will

[11] For example, frequency of the rebalancing: In theory, the model assumes continuous dynamics and so requires continuous trading which is not feasible in practice (e.g. transaction costs or market incompleteness).

admit a large family of Riemannian metrics which may display diverse geometrical properties. Goodness of the metric depends on its ability to encode second–order information about the problem. Different descriptions of the space give rise to different families of metrics, and hence to different natural geometries.

For the sensitivity analysis that explores the underlying model space locally, it may be advantageous to consider different geometries of the underlying space each having different properties that may be of benefit in different forms of analysis and specific applications. For example, the motivating requirement for asymmetry in statistical inference is captured in the preferred point metric and associated geometry (see [43]), whereas an assessment of the conditional variance of a maximum likelihood estimator is performed by the observed Fisher–Rao information matrix (see [53]). Another possibility is to extract a proper metric from the observed data to examine the dynamic behaviour of the model with respect to the behavior of the financial market.

9.5.1 Fisher–Rao information metric

As mentioned in Sec. 9.3, the underlying space for a digital option is a statistical manifold possessing a Riemannian structure which can be endorsed with the Fisher–Rao information metric, see [96]. For the parametric density $p(\theta)$, the Fisher–Rao information matrix is given by $g = \mathbb{E}_\theta[\nabla_\theta l(\theta)\cdot\nabla_\theta l(\theta)^T]$ where $l(\theta) = \log p(\theta)$ is the log–likelihood function, ∇ is the gradient of the log–likelihood at θ (implicitly depending on $\mathcal{X}$) and $\mathbb{E}_\theta$ denotes expectation taken with respect to $p(\theta)$.

Amari [7] showed that a wide range of metrics for comparing probabilities reduce to a simple function of the Fisher–Rao metric when the density functions are close to each other. Examples of such metrics include Kullback–Leiber (KL), Bhattarcharrya, Matusita–Hellinger or Jensen–Shannon divergence. KL divergence, also known as information gain or relative entropy, is a non–symmetric measure of the difference between two probability distributions p and q, but it is also a measure of the expected number of extra bits required to code samples from p when using a code based on q.[12] Moreover, it is also known that the KL divergence is the Riemannian distance under the Levi–Civita connection, refer to [7] again, with the approximation being valid up to second order terms. The minimization of the KL divergence is equivalent to maximum–likelihood estimation, i.e. maximizing the likelihood of data under our estimate is equal to minimizing the difference between our estimate and the real data distribution. The explicit form of the Fisher–Rao metric for the family of log–normal distributions is:

$$g\big(r(t),\sigma(t)\big) = \begin{bmatrix} g_{rr} & g_{r\sigma} \\ g_{\sigma r} & g_{\sigma\sigma} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sigma(t)^2} & 0 \\ 0 & \dfrac{2}{\sigma(t)^2} \end{bmatrix}$$

and the Christoffel symbols of the second kind associated with the Levi–Civita connection are:

$$\Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\sigma} \\ \Gamma^r_{r\sigma} & \Gamma^r_{\sigma\sigma} \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{\sigma(t)} \\ -\dfrac{1}{\sigma(t)} & 0 \end{bmatrix}, \quad \Gamma^\sigma_{ij} = \begin{bmatrix} \Gamma^\sigma_{rr} & \Gamma^\sigma_{r\sigma} \\ \Gamma^\sigma_{r\sigma} & \Gamma^\sigma_{\sigma\sigma} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\sigma(t)} & 0 \\ 0 & -\dfrac{1}{\sigma(t)} \end{bmatrix}.$$

It is well known [100] that, after a change of coordinates, the length element ds^2 coincide with the length element of the Poincaré half–space model of hyperbolic geometry. This means that the underlying space when equipped

[12] See [7] for a proof that the KL divergence between two infinitesimally close distributions is half of the square of their Riemannian distance:

$$D_{KL}\big(p(x|\theta)\,\|\,p(x|\theta+d\theta)\big) \approx \frac{1}{2} g_{ij} d\theta^i d\theta^j.$$

with its Fisher–Rao information metric becomes a space of constant negative curvature. The covariant version of the second–order derivatives is

$$\nabla_g^2 V_{\sigma\sigma} = \frac{\partial^2 V}{\partial \sigma^2} - \Gamma^{\sigma}_{\sigma\sigma}\frac{\partial V}{\partial \sigma} - \Gamma^{r}_{\sigma\sigma}\frac{\partial V}{\partial r} = \frac{\partial^2 V}{\partial \sigma^2} + \frac{\mathbf{1}}{\sigma}\frac{\partial \mathbf{V}}{\partial \sigma},$$

$$\nabla_g^2 V_{rr} = \frac{\partial^2 V}{\partial r^2} - \Gamma^{\sigma}_{rr}\frac{\partial V}{\partial \sigma} - \Gamma^{r}_{rr}\frac{\partial V}{\partial r} = \frac{\partial^2 V}{\partial r^2} - \frac{\mathbf{1}}{\mathbf{2}\sigma}\frac{\partial \mathbf{V}}{\partial \sigma}.$$

Figures 9.4 and 9.5 show that the correction under the negatively curved geometry induced by the Fisher–Rao information metric mainly affects the second order derivative with respect to volatility.

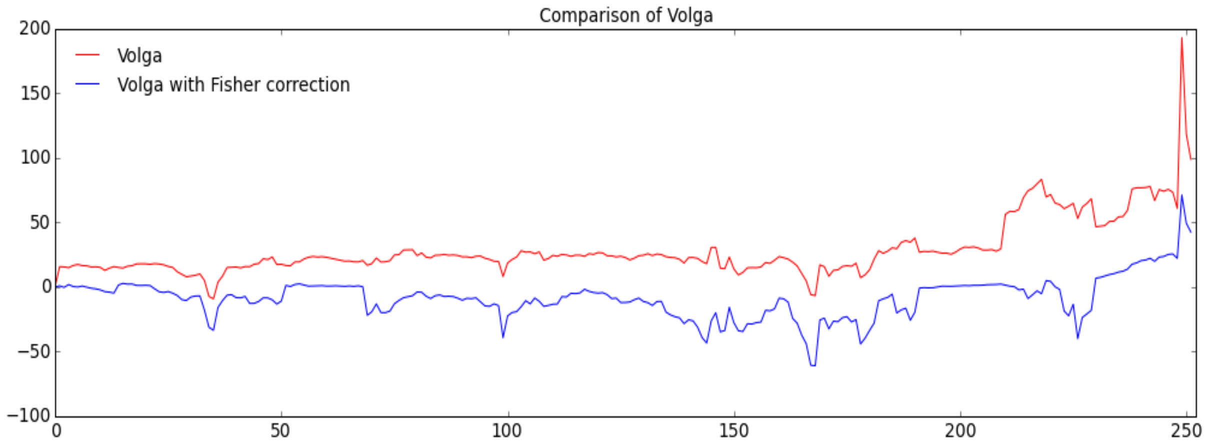

Figure 9.4: Change in the second order derivatives with respect to σ, $\frac{\partial^2 V}{\partial \sigma^2}$.

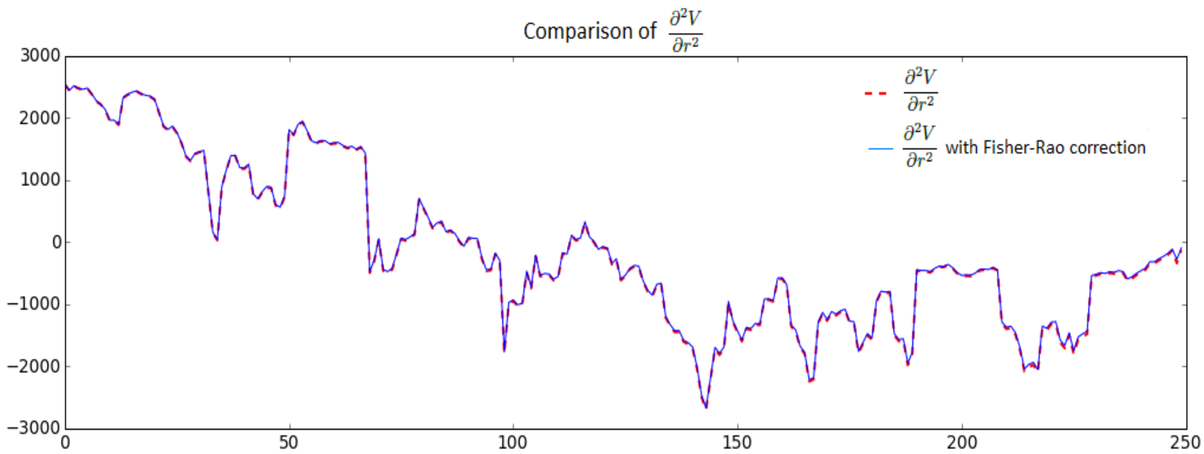

Figure 9.5: Change in the second–order derivatives with respect to r, $\frac{\partial^2 V}{\partial r^2}$.

The resulting daily approximation errors under the Fisher–Rao geometry are shown in Fig. 9.6. In comparison with Fig. 9.3, Fig. 9.6 illustrates the decrease in approximation errors, especially around the maturity, with mean -0.054%. Correcting for the nonlinearities implied by the changing volatility decreases the extreme errors and increases the number of errors around zero, i.e. the overall performance of the P&L approximation is improved.

Figure 9.7 compares the absolute approximation errors under the Euclidean geometry and the geometry induced by the Fisher–Rao information metric. Large positive values represent large improvements of the Fisher–

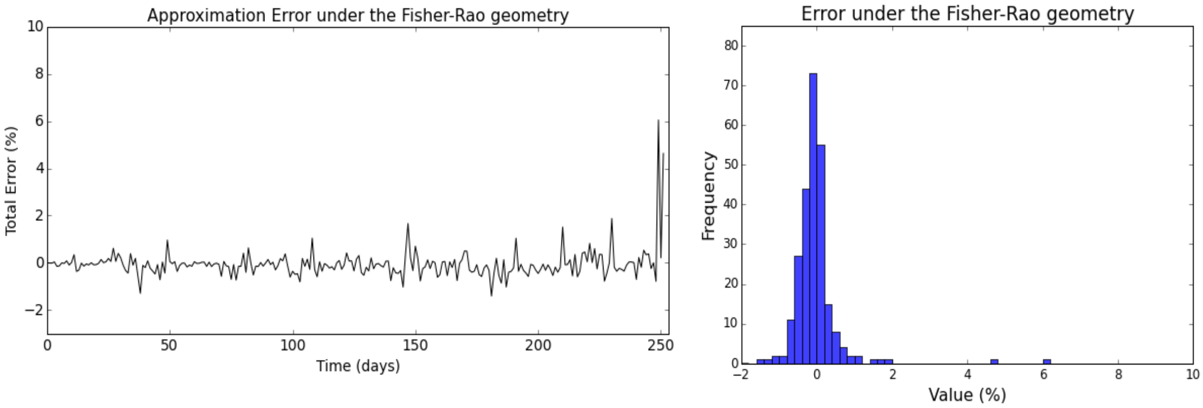

Figure 9.6: Total errors under the Fisher–Rao geometry. The initial price of the underlying asset is given by $S_0 = 1.28166$ and $S_T = 1.33263$ at expiry with fixed $K = 1.325$. Descriptive statistics: mean = -0.054%, std. dev. = 0.0063, skew = 5.42.

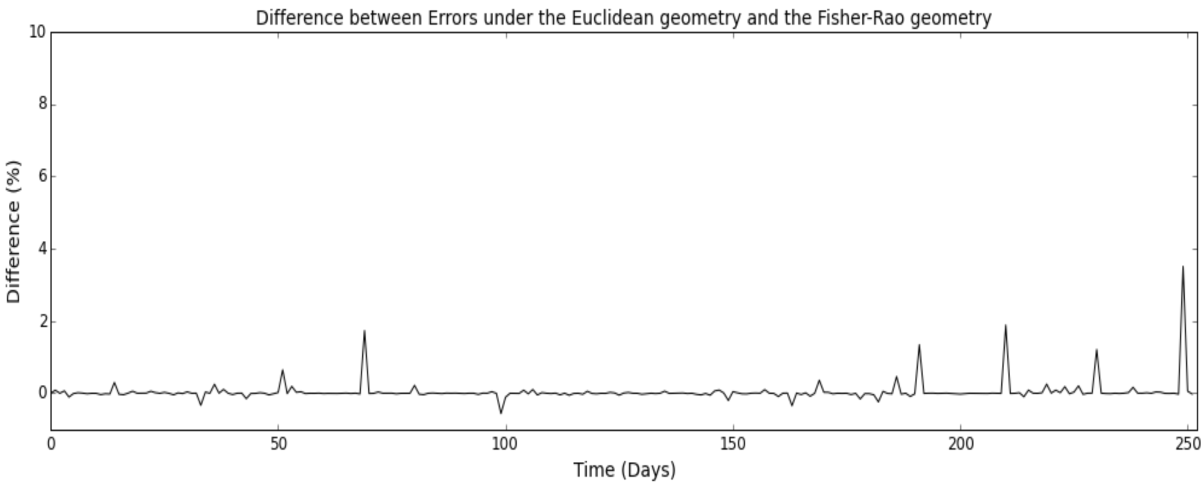

Figure 9.7: Difference between the absolute approximation errors under the Euclidean geometry and the Riemannian geometry associated with the Fisher–Rao information metric, i.e. $Total_t = (|Error_{Euclidean}| - |Error_{Fisher}|)_t$.

Rao metric. The almost absence of negative values means that Fisher–Rao rarely worsens the calculations. Values close to zero mean either that the error is small in both cases or that there is almost no improvement. Thus, we can notice an overall improvement in the approximation. The correction for the model risk changes with respect to the dynamics implied by the model, particularly variations in the volatility of the underlying, with considerable enhancement during larger shifts in data.

9.5.2 Christoffel symbols based on critical points of digital options

The Fisher geometry seems a natural choice if all we know is the underlying statistical family of the given model. The risks involved in trading digital options are, however, not directly related only to the underlying asset. The main risk arises when the price of the underlying oscillates around the strike (barrier) near expiration. In general, the discontinuity in the payoff will make Gamma or Vega flip sign at the barrier. Delta and Gamma magnitudes attain maximum around the barrier and the closer to expiry the higher the magnitude of these risks. Moreover these sensitivities become large as $t \to T$ for $S \approx e^{-(r-q)t}K$. Additionally, given the mixed convexity of the second derivatives, the impacts of large up or down movements are not symmetric, so direction matters.

Considering general characteristics specific to digital contracts or the particular usage of the model, it may be possible to choose a geometry that is better suited for the given problem.[13] Tuning the geometry of the variety with respect to some of the influencing factors of the given contract such as moneyness, distance to barrier (current value of the state variable) or time to maturity of the option, may improve the overall P&L explanation power when compared to the Euclidean or Fisher–Rao metrics.

Defining a metric is not an intuitive task. Metrics are defined in terms of a local inner product and it may be difficult to understand the implications of a specific choice on the resulting geometry. Alternatively, we may consider to determine directly the curvature of the manifold through the Christoffel symbols for a given connection.[14] Considering the time to maturity (different behaviour closer to expiry), distance to K and the direction of the changes in volatility, interest rates and stock price, we propose to use the Christoffel symbols given in Table 9.1, where the constants, a_{it} and b_{it} for $i = 1, 2, 3, 4$ are fitted for this particular example.[15] In this particular example, the constants take the values shown in Table 9.1.

The Christoffel symbols, and so the curvature adjustments, depend both on the parameters that define the underlying probability distribution, such as volatility and interest rate, but also factors that characterize the given digital contract, e.g. changing behaviour closer to expiration, the importance of the direction of the movement. These are local, thus capturing local variations.

As Figs. 9.8 and 9.9 indicate, tuning the curvature towards more specific characteristics of the digital option shows some improvement in the approximation. Figure 9.9 compares the Estimated geometry with the Euclidean and the Fisher–Rao geometries. As positive values represent the decrease in errors induced by the Estimated ge-

[13]Note that the choice of a geometry is problem dependent, i.e. following the same idea other geometries can be derived for various families of contracts such as barrier or Bermuda options, or that are better suited for other applications such as hedging.

[14]The curvature of the manifold is determined by the connection that is defined by the Christoffel symbols. Thus we can directly work with the Christoffel symbols.

[15]Note that this choice is only a first attempt, the existence and the construction of a covariant derivative that would be theoretically consistent with the P&L explanation for certain underlying assets and contracts and that will outperform the standard techniques is left for further research. See Sec. 7.

Christoffel symbols	for $S_t < Ke^{-r_t t}$ and $t < \frac{1}{2}T$	otherwise	a_{it}
Γ^r_{rr}	$a_{1t}\dfrac{1}{r_t\sigma_t}$	$b_{1t}\dfrac{1}{r_t\sigma_t}$	0.7
$\Gamma^\sigma_{\sigma\sigma}$	$a_{2t}\dfrac{1}{\sigma_t}$	$b_{2t}\dfrac{1}{\sigma_t}$	$0.6 \cdot sign(\sigma_t - \sigma_{t-1})$
$\Gamma^r_{\sigma\sigma}$	$a_{3t}\dfrac{r_t}{\sigma_t^2}$	$b_{3t}\dfrac{r_t}{\sigma_t^2}$	$-5 \cdot sign(\sigma_t - \sigma_{t-1})$
Γ^σ_{rr}	$a_{4t}\dfrac{1}{r_t\sigma_t^2}$	$b_{4t}\dfrac{1}{r_t\sigma_t}$	-1.4

Table 9.1: Christoffel symbols fitted to the data: Estimated geometry.

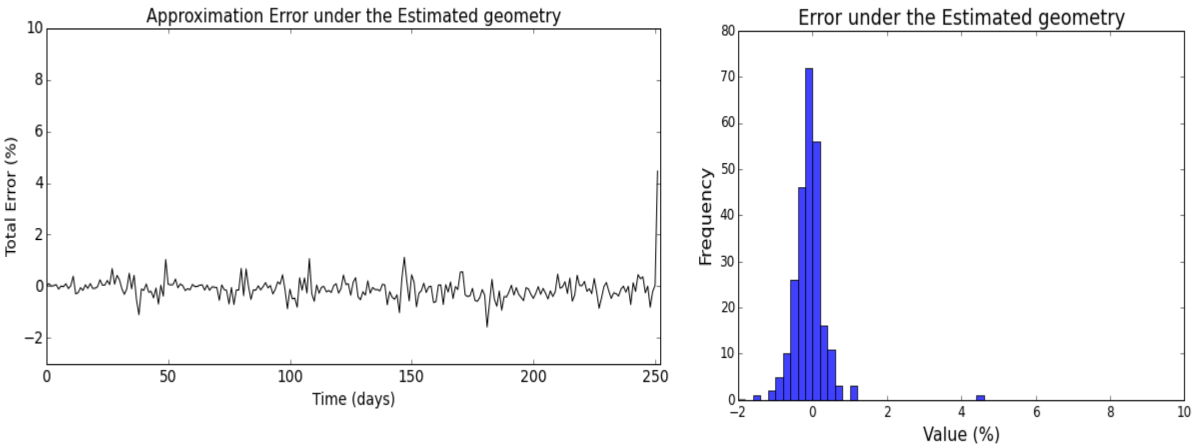

Figure 9.8: Total approximation errors under the Estimated geometry. The initial price of the underlying asset is given by $S_0 = 1.28166$ and $S_T = 1.33263$ at expiry with fixed $K = 1.325$. Descriptive statistics: mean = -0.09%, std. dev. = 0.0084, skew = 4.015.

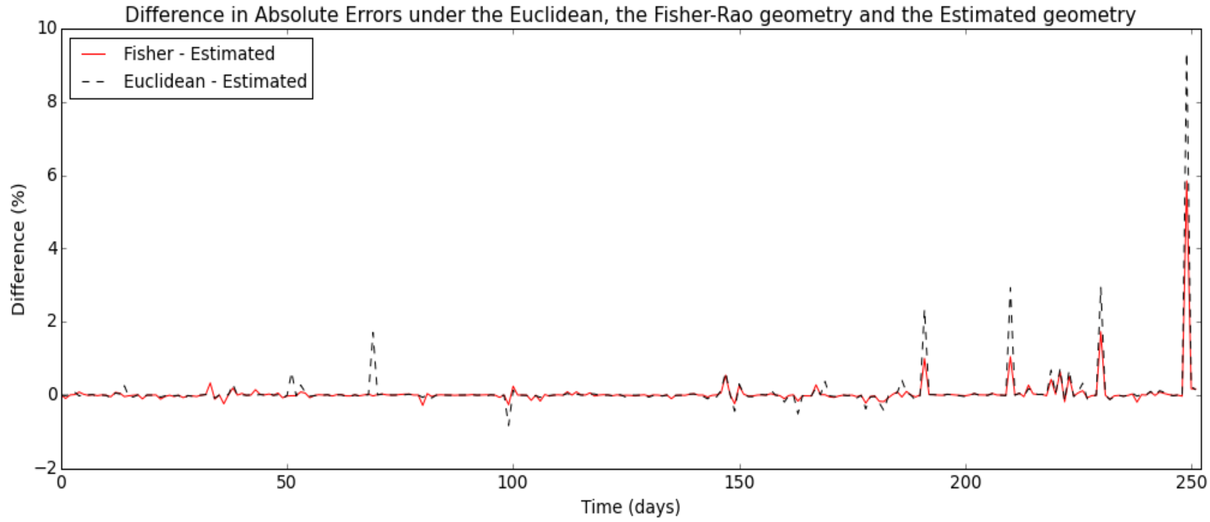

Figure 9.9: Difference between the absolute approximation errors under the Euclidean geometry, the Riemannian geometry associated with the Fisher–Rao information metric and the Estimated geometry, i.e. $(Fisher - Estimated)_t = (|Error_{Fisher}| - |Error_{Estimated}|)_t$ and $(Euclidean - Estimated)_t = (|Error_{Euclidean}| - |Error_{Estimated}|)_t$.

ometry and negative values mean worsening, we can conclude that the overall performance is enhanced also with respect to the geometry induced by the Fisher–Rao information metric. The curvature expression reveals the following behaviour of the model: with increasing volatility the model risk inherent in the P&L explanation under the BS model increases, exactly when the curvature of the underlying model space plays a crucial role.

9.6 Other Applications

We present two other applications for the daily P&L explanation of digital options written on a synthetic underlying asset following a Heston model and on an actual equity index (volatility index VIX). In both examples we examine the impact of the curvature of the underlying variety (model space) on the P&L explanation under the BS model. Based on the results in subsequent subsections, we can conclude that introduction of curvature improves the usage of a given model and equivalently reduces the inherent model risk.

9.6.1 Digital Option on synthetic underlying asset following Heston model

The aim of this example is to examine the effect of the curvature when the dynamics of the underlying asset resemble the properties that are assumed by the BS model. Under the BS model, the volatility and interest rate are assumed to be at most time dependent, which is not enough in many situations. We consider an underlying asset simulated by the Heston model plus an interest rate following the Vasicek mode, see eqs. 9.6 and 9.7 below. The parameters of these models are set so that the dynamics of the underlying asset moves towards the assumptions of the BS model, i.e. small volatility of volatility and volatility of interest rate (see Fig. 9.10).

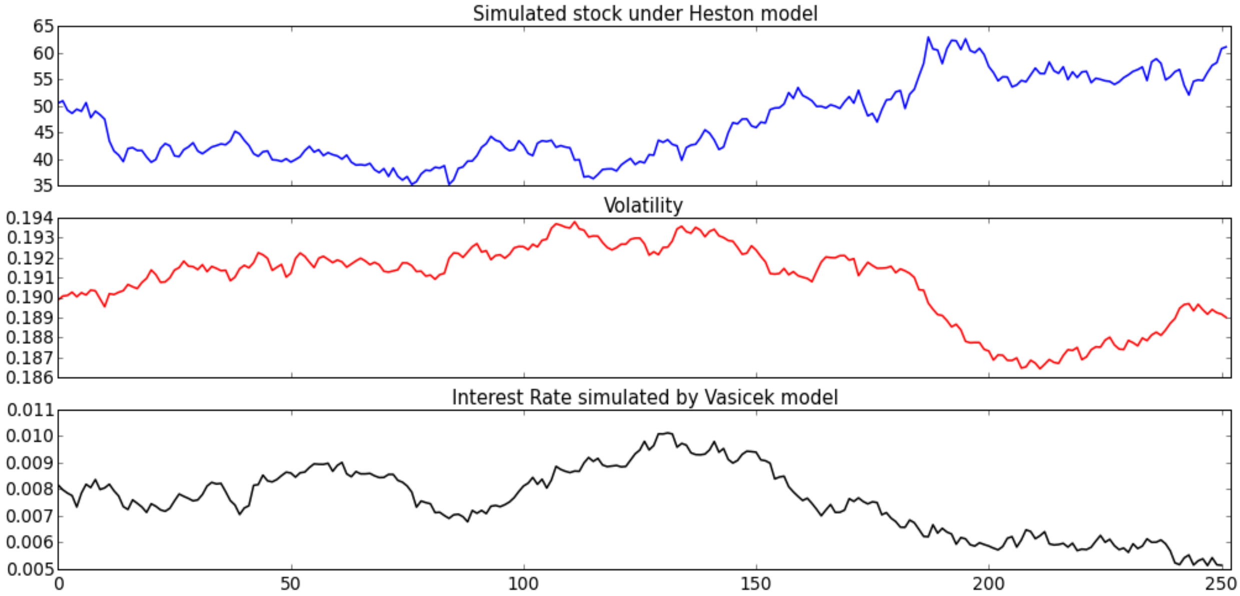

Figure 9.10: Evolution of the simulated underlying asset with its volatility by the Heston model and the interest rate by the Vasicek model ($S_0 = 50.54, S_T = 61.11, K = 65$; parameters for the Vasicek model: $r_0 = 0.01, \alpha = 0.13, \beta = 0.02, \sigma = 0.005$ and parameters of Heston model: $v_t = 0.188, \bar{v}_t = 0.01, \lambda = 0.01, \rho = -0.2174, \eta = 0.21$).

The Heston model assumes that the asset price and its volatility follow some random processes, refer to [66]. The stochastic process of the asset price S_t is log–normally distributed and the stochastic volatility is assumed to follow a positive increasing function of a mean–reversion process. The asset dynamics are given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dW_{2,t} \end{aligned} \tag{9.6}$$

with $\langle dW_{1,t}, dW_{2,t} \rangle = \rho dt$, where μ is the drift and the instantaneous variance of the stock price v_t itself is a stochastic process. λ is the speed of reversion of v_t to its long–term mean $\bar{v}$. We can think of λ as the rate at which the variance reverts back to its long term average value. η is the volatility of the variance process v_t (often called the volatility of volatility). $W_{1,t}$ and $W_{2,t}$ are two dependent Wiener processes with correlation coefficient ρ.

We consider that the interest rate follows the Vasicek model, an Ornstein–Uhlenbeck process whose dynamics are given by the following SDE:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t, \quad r_{t_0} = r_0 \tag{9.7}$$

where $\alpha, \beta > 0$, and σ are constants, and W_t is a Wiener process under the risk neutral measure modelling the random market risk factor and is assumed to be independent from $W_{1,t}$ and $W_{2,t}$. The long term mean is given by $-\alpha/\beta$.

The evolution of the underlying price, its volatility and interest rate are shown in Fig. 9.10. Compared to the evolution of the foreign exchange rate given in Fig.9.2, the simulated asset dynamics, as expected, are not subject to any large price, volatility or interest rate movements (jumps), and so resemble assumptions under the BS model.

The daily approximation errors of the second order Taylor expansion under the Euclidean geometry are depicted in Fig. 9.11. As Fig. 9.11 illustrates, the second–order approximation under the Euclidean structure stays meaningfully close to the option price for small variations in price, volatility and interest rate. As expected, there is a higher variation closer to expiration and when the asset price oscillates around strike. These factors are not taken into account in the sensitivities based on the BS model and so in this situations the model incorporates high model risk since it is used in areas for which it was not developed.

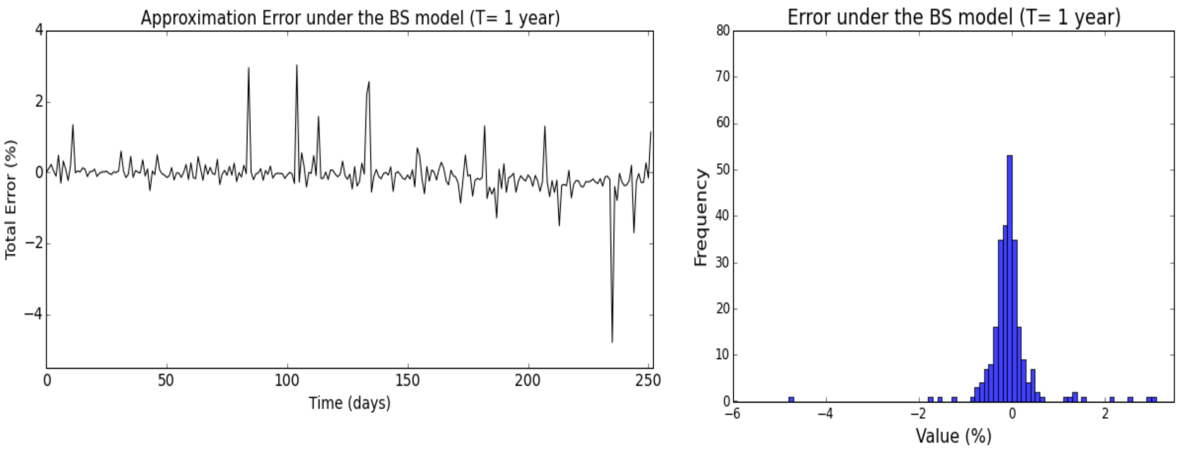

Figure 9.11: Total approximation errors under the Euclidean geometry. The initial price of the underlying asset is given by $S_0 = 50.54$ and at expiry $S_T = 61.11$ with fixed $K = 65$. Descriptive statistics: mean = -0.0058%, std. dev. = 0.006, skew = -0.367.

Figures 9.12, 9.13, 9.14 and 9.15 illustrate the average improvement of the covariant Taylor expansion for the daily P&L explanation of a digital option under the Fisher–Rao geometry and the Estimated geometry in comparison with errors under the Euclidean geometry. In comparison with the evolution of the approximation errors for the exchange rate given in Figures 9.6 and 9.7 under the Fisher–Rao geometry and Figures 9.8 and 9.9 under the Estimated geometry we can see that the curvature corrections are rather small. This is in line with the theoretical assumptions, since the BS model is assumed to be a good approximation of the market reality in static situations or in a equilibrium state on the financial market.

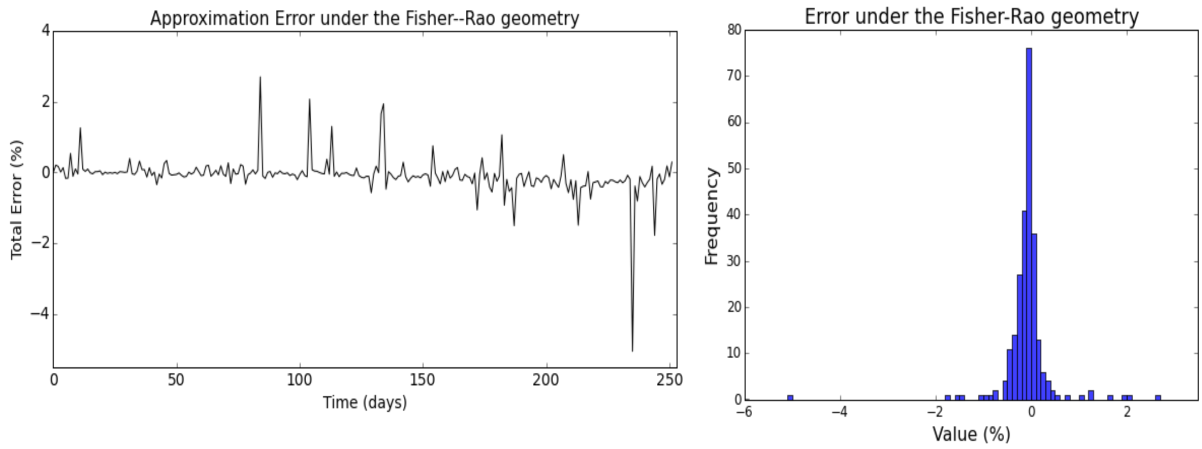

Figure 9.12: Total approximation errors under the Fisher–Rao geometry. The initial price of the underlying asset is given by $S_0 = 50.54$ and $S_T = 61.11$ at expiry with fixed $K = 65$. Descriptive statistics: mean = -0.086%, std. dev. = 0.0052, skew = -2.37.

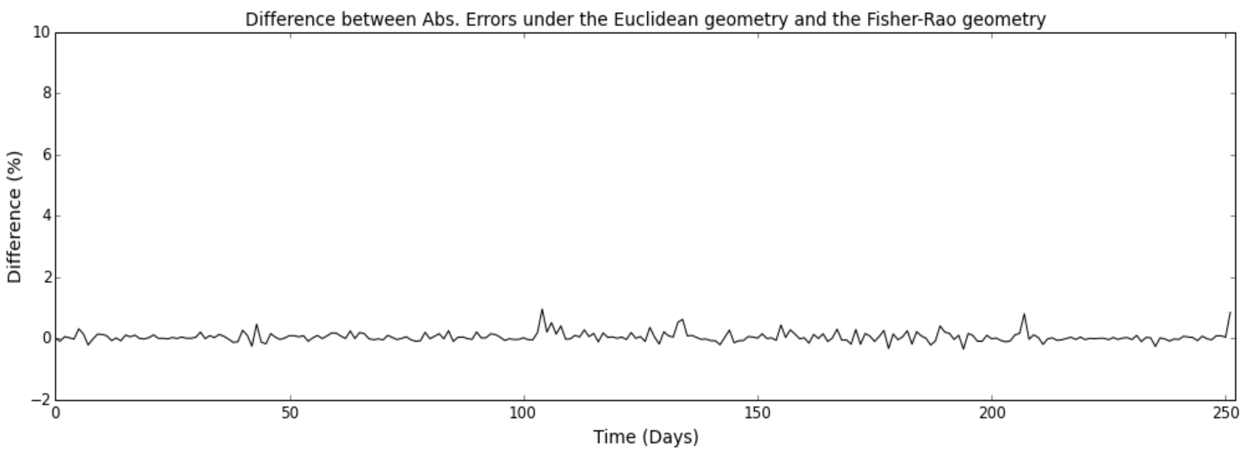

Figure 9.13: Difference between the absolute approximation errors under the Euclidean geometry and the Riemannian geometry associated with the Fisher–Rao information metric. The initial price of the underlying asset is given by $S_0 = 50.54$ and $S_T = 61.11$ at expiry with fixed $K = 65$.

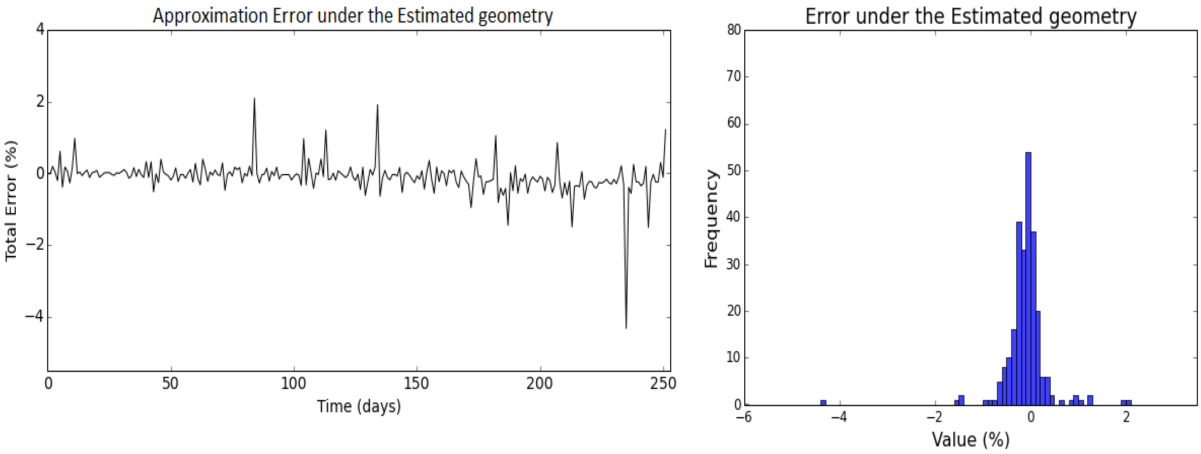

Figure 9.14: Total approximation errors under the Estimated geometry. The initial price of the underlying asset is given by $S_0 = 50.54$ and $S_T = 61.11$ at expiry with fixed $K = 65$. Descriptive statistics: mean = -0.10%, std. dev. = 0.0046, skew = -2.3.

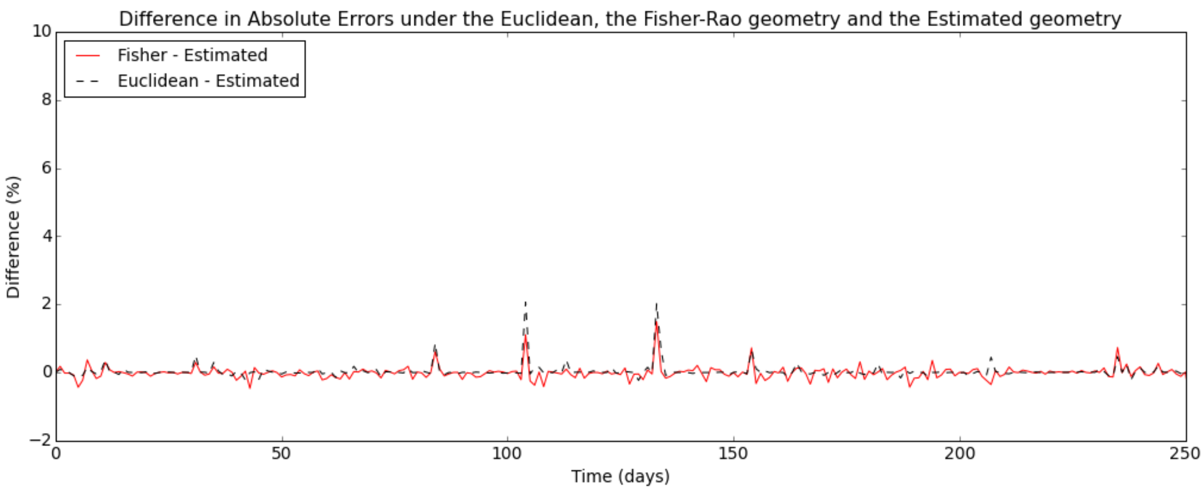

Figure 9.15: Difference between the approximation error under the Euclidean geometry and the Estimated geometry. The initial price of the underlying asset is given by $S_0 = 50.54$ and $S_T = 61.11$ at expiry with fixed $K = 65$.

9.6.2 Digital Option on Volatility Index on Equity

We consider the P&L explanation for a digital option on the volatility index VIX in which the institution pays on $3rd$ January 2018, the strike $K = 13$ being fixed on $4th$ January 2017 $(T = 1)$ for the initial price $S_0 = 11.85$. The time evolution of the underlying with the realized volatility and interest rate are depicted in Fig. 9.16.

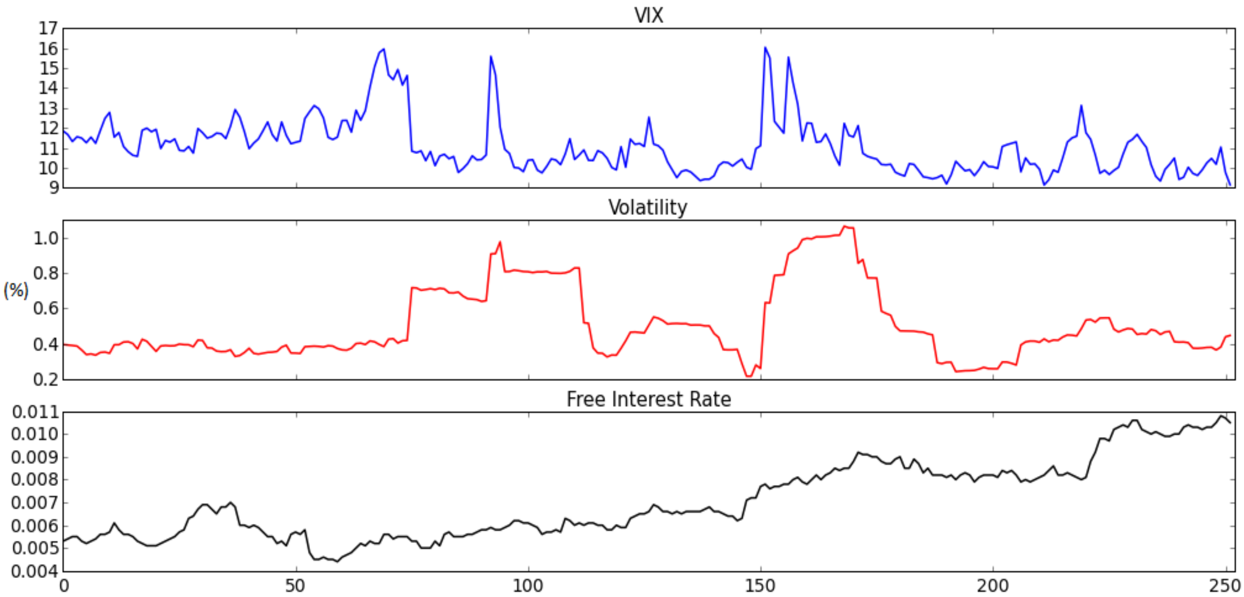

Figure 9.16: Evolution of the underlying VIX volatility index with the realized volatility and interest rate.

The daily approximation errors under the Euclidean geometry are depicted in Fig. 9.17. As can be seen, there is a relation between the errors and the daily volatility and price changes, bigger changes mean bigger approximation errors. The largest fraction of daily errors is small in absolute value as they refer to small variations in price and volatility.

In Figures 9.18, 9.19, 9.20 and 9.21 we illustrate the average improvement of the covariant Taylor expansion for daily P&L explanation of a digital option by comparing the Taylor expansion with the Euclidean geometry, the Fisher–Rao geometry and the Estimated geometry. Whereas negative values in Figs. 9.19 and 9.21 mean deterioration of the approximation, positive values represent the decrease in daily errors and so improvement in the P&L explanation, we can conclude that overall performance is enhanced by the Fisher–Rao information metric and even more by the Estimated geometry. Thus, accounting for a proper geometry of the underlying model variety may significantly improve the usage of the model.

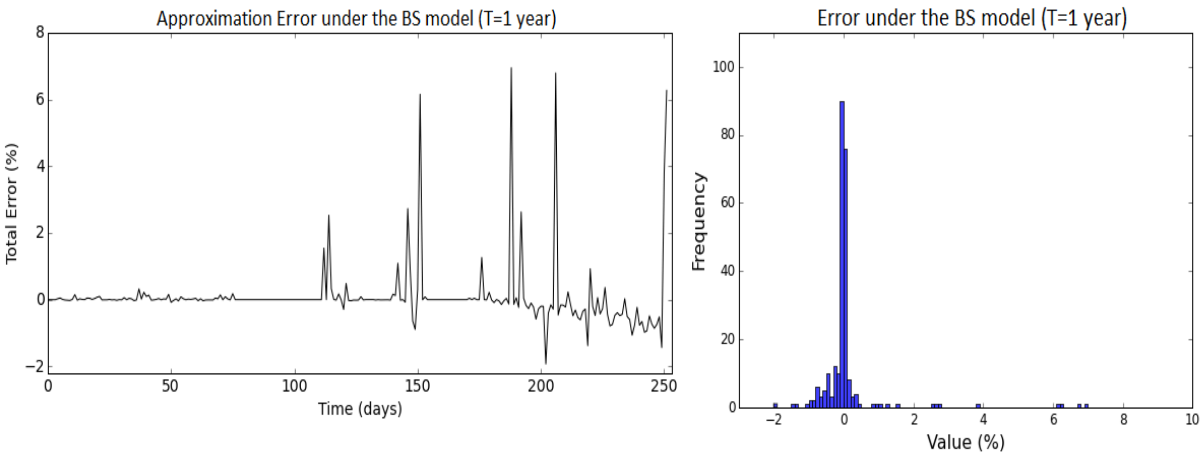

Figure 9.17: Total approximation errors under the Euclidean geometry. The initial price of the underlying asset is given by $S_0 = 11.85$ and $S_T = 9.15$ at expiry with fixed $K = 13$. Descriptive statistics: mean = 0.075%, std. dev. = 0.0097, skew = 5.15.

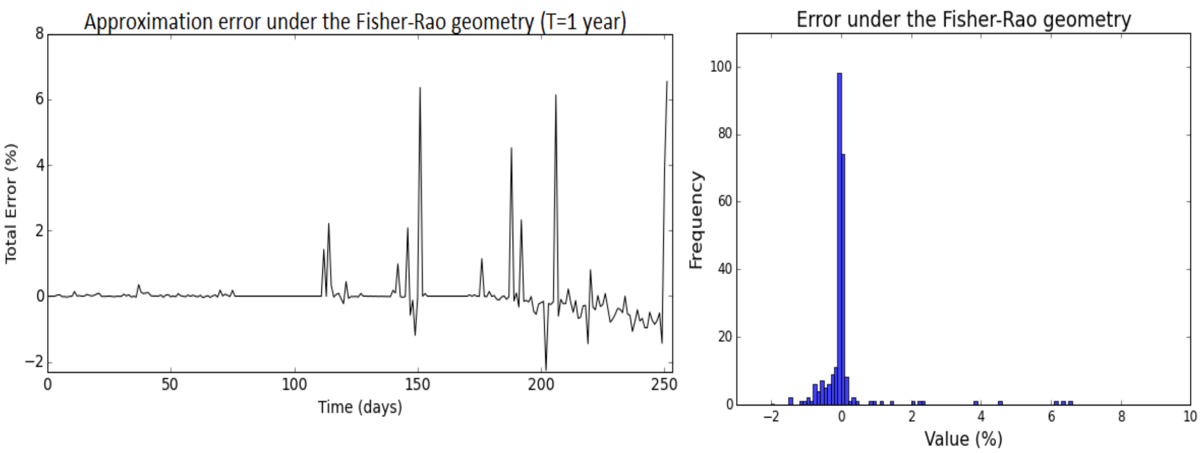

Figure 9.18: Total approximation errors under the Fisher–Rao geometry. The initial price of the underlying asset is given by $S_0 = 11.85$ and $S_T = 9.15$ at expiry with fixed $K = 13$. Descriptive statistics: mean = 0.047%, std. dev. = 0.0089, skew = 5.038.

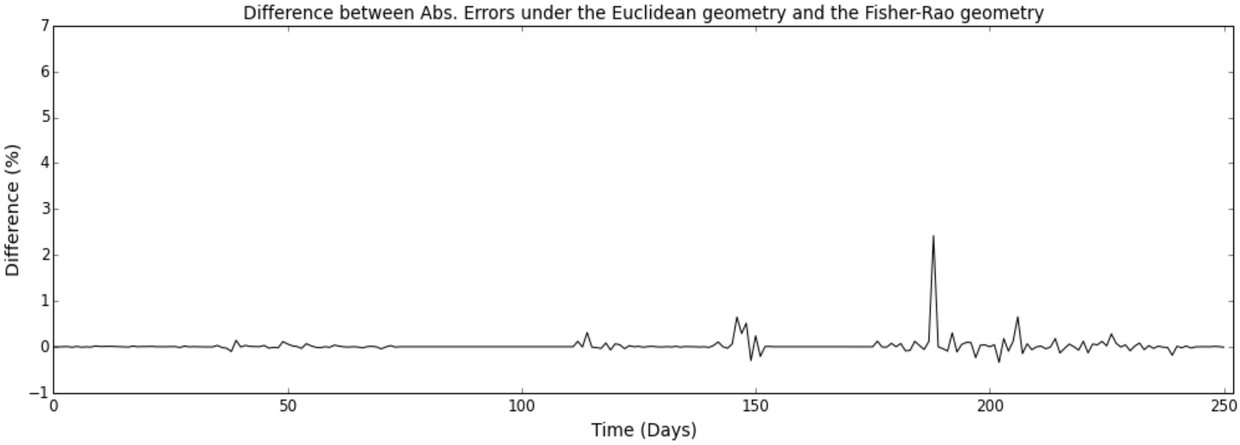

Figure 9.19: Difference between the absolute approximation errors under the Euclidean geometry and the Riemannian geometry associated with the Fisher–Rao information metric. The initial price of the underlying asset is given by $S_0 = 11.85$ and $S_T = 9.15$ at expiry with fixed $K = 13$.

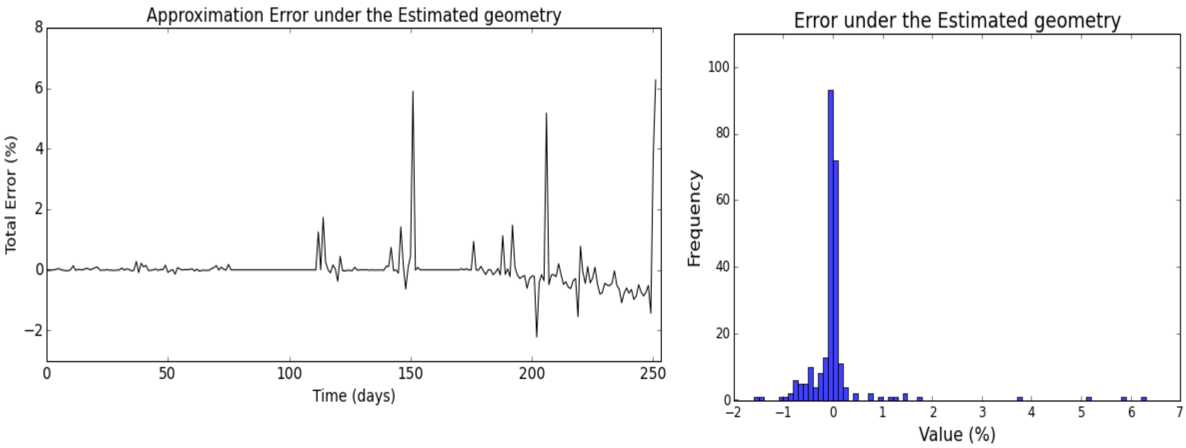

Figure 9.20: Total approximation errors under the Estimated geometry. The initial price of the underlying asset is given by $S_0 = 11.85$ and $S_T = 9.15$ at expiry with fixed $K = 13$. Descriptive statistics: mean = 0.019%, std. dev. = 0.0077, skew = 5.32.

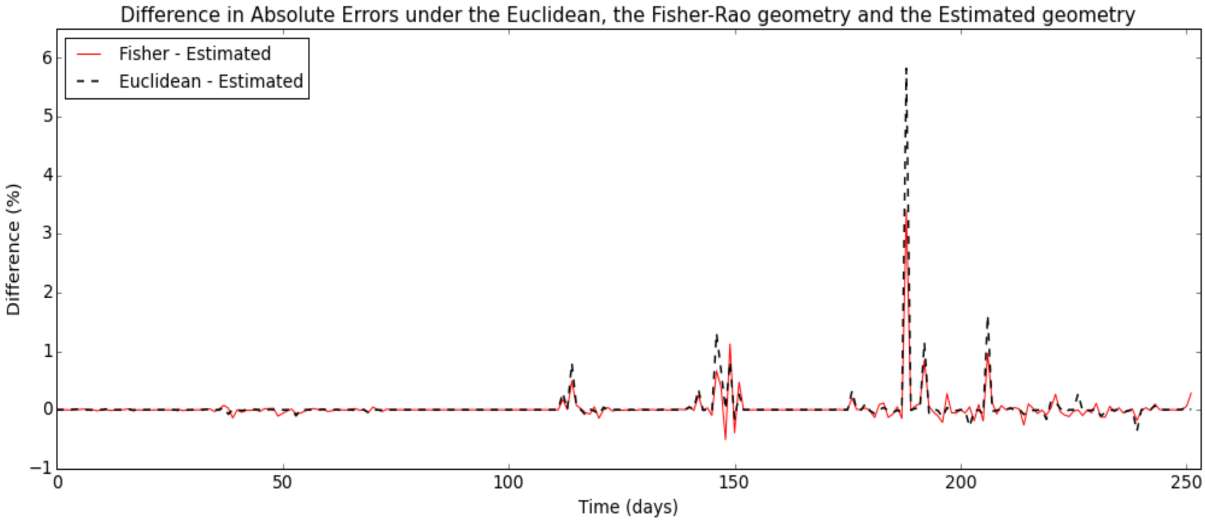

Figure 9.21: Difference between the absolute approximation errors under the Euclidean geometry, the Riemannian geometry associated with the Fisher–Rao information metric and the Estimated geometry.

9.7 Conclusions and Further Research

The broad aim of the present contribution is to encourage the inclusion of differential geometry to improve on the usage of a given model; equivalently, to reduce the inherent model risk which is inextricably related to its use (linking with Chapter 6 and Chapter 8). The authors have shown that considering a proper non–Euclidean geometry for the underlying model variety may improve the overall model performance.

The authors have tried to achieve the aforementioned objectives by selecting a widespread and problematic model, payoff and usage: P&L explanation of digital options via the Black–Scholes model. Not only several underlying and market conditions have been assessed but also curved and flat geometries.

This study is of a starting–point nature, suggesting many directions for further research. We would like to highlight the following:

1. Since curvature is affected by the choice of a particular family of models, their usage as well as data, it might be interesting to explore calibration algorithms for selecting most appropriate Riemannian metrics.
2. Choice of a proper metric for other usages besides the P&L explanation or sensitivity analysis (where the differential calculus is applied) and also to other data, models and their associated usages.
3. Of utmost interest in our opinion is to perform theoretical analysis to asses, among others, when improvements are expected and to what order, to identify areas when model risk is amplified or to improve the detection of model failures. In other words, to establish a deeper link with quantification of model risk and model validation.

Chapter 10

CONCLUSIONS AND FURTHER RESEARCH

"A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding." (Isaac Newton)

In the present contribution, we have addressed the management and the quantification of model risk with the objective to design a general framework, while taking into account both internal policies and regulatory issues, applicable to most of the modelling techniques currently employed by financial institutions. The proposed framework leads to an objective assessment of model risk while covering the most important aspects of the model risk management, such as data, calibration, mathematical foundation, model performance, and, most importantly, usage of the model. In this concluding chapter, we summarize implications of our contribution and suggest possible venues for a future research.

The first part of the dissertation was dedicated to the introduction to the main concepts of the capital and risk management, and to the description of some of the most relevant approaches and methodologies currently employed by financial institutions, covering several approaches and models for capital allocation, managing credit risk (see Section 2.1), market risk (see Section 2.2), operational risk (see Section 3.1) and other relevant non–financial risks (e.g., business risk, macro forecasts, reputational risk, or IT risk; for more details see Section 3.2). In addition, we have provided a brief overview of models used for pricing financial derivatives (see Section 2.3), stress–testing and provisioning (see Sections 4.2 and 4.1). The final section of this Part I was dedicated to the interconnections between different types of risks and various methods used for the aggregation (see Section 4.3). When properly designed and implemented, these models represent a valuable source in the decision making process, but they all need well–conceived processes to improve model utility while identifying, quantifying and mitigating the potential risk emerging from their usage.

All models bear model risk, no matter how complex or how accurate they may be, as they always represent a simplification of reality to certain degree. The use of ever more sophisticated quantitative models allows a financial institution to better understand and control their processes while at the same time exposing it to other kind of risks linked to the potential inadequacy of the model employed. Besides, markets are neither perfect nor efficient, regulations and new laws can change everything, a missing or neglected factor, inadequate data, improper calibration, or risk ignorance may have superior influence. It is an endless list of what might go wrong.

Although, the complete elimination of model risk is not possible, implementing an approach that combines rigorous risk management structures, as prescribed by the supervisory guidances [33], with prudent detailed quantification, may be an effective strategy to mitigate and control it. In the second part of this work we have addressed the issue of an effective model risk management (MRM) by contributing to a movement from efforts focused primarily on qualitative assessment of model risk, to a more advanced stage based on the quantitative model risk measurement. We have introduced a general framework for the quantification of model risk using differential geometry [89] and information theory [7] while providing a sound mathematical definition of model risk using weighted Riemannian manifolds, applicable to most modelling techniques using measure theory as a starting point (for more details we refer the reader to Chapter 6). The approach represents an objective assessment of model risk, given in financial terms, and provides a practical way to solve a theoretically difficult and complex problem meanwhile allowing issues to be properly prioritized, resources efficiently allocated, and adequate cushion against losses established. We also presented an example of model risk quantification in the realm of capital allocation for credit risk (see Chapter 7) and an example on how to further use this framework for the reduction of model risk in the case of P&L explanation for digital options (see Chapter 9), to illustrate how a practitioner may identify the relevant abstract concepts and put them to work.

This dissertation contributes to the literature by developing an approach applicable to most of the models applied by financial institutions in their day–to–day operations while covering the main areas of MRM, including the methodological choice of the model, its implementation, but also the whole process through which the model is deployed within the organization, thereby encompassing data, calibration, usage, and its performance. The approaches in the literature, to our very best knowledge, are limited in these same two ways: They consider very particular mathematical techniques and are usually very focused on selected aspects of model risk management.

Given the growing risk posed by models, MRM is becoming a significant part of the overall management of financial institutions that will require an objective assessment of model risk. Banks can therefore benefit from a structured approach to model risk and a general framework for the quantification that can cope with relevant aspects of model risk management and valid across most of the financial models employed. In addition to the quantification, our proposed framework offers a way to approach data uncertainties by studying perturbation and metrics defined on the samples, which are then transmitted to the weighted Riemannian manifold through the fitting process (for further details see Chapter 8). The framework is, therefore, furthermore suitable, for example, for sensitivity analysis by using the directional and total derivatives,[1] for stress testing, including regulatory and planification exercises, or for checking the validity of the approximations done through the modelling process, testing model validity and stability, or applicable in other tests and analysis used in MRM.

As such, the proposed methodology as well as its practical implementation might be of interest not only to risk managers but also to domestic and international regulators. The value of sophisticated MRM extends well beyond the satisfaction of regulatory requirement. A robust MRM allows firms to make efficient allocation of resources, maximize the benefit they achieve from their investment or prevent issues from occurring, before they arise through reduced likelihood of poor decisions, improved control efficiency, better understanding of model assumptions, limitations and outputs, increased confidence in the business planning processes. Furthermore, it may

[1] The directional derivative would allow us to estimate the individual contribution to the output variance of each perturbation in the input, as well as to assess the sensitivity to missing properties, model assumptions or missing values. With the total derivative, determined by the pushforward map, we could measure the total contribution in a small neighbourhood of a given initial sample. Additionally, they both provide an indication of wrong model assumptions, bad model fitting or of influential observations that could somewhat distort the parameter estimates leading in some cases to inaccurate or wrong outputs. For more details we refer the reader to Section 8.2.

enhance reputation of the overall risk management function as MRM feeds into the overall approach to managing risks within an organisation which can in turn influence credit ratings and the ability to retain and attract new business. Finally, a strong MRM process can lead to improved capital usage and more stable profits. As such, this can act as a competitive advantage for an institution.

Research is a never–ending activity, and natural extensions of what has been presented can be approached in several ways, all of which we find to be both of theoretical and of practical interest. We shall finish by naming just a few of them:

The proposed framework represents a general methodology on how to objectively assess and control model risk. It can be further tailored and made more efficient for specific models, algorithms or applications by selecting a appropriate Riemannian metric for specific family of models and their usages, tuning the choice of the neighbourhood and the kernel for a specific application and implementation, or by suggesting a proper norm depending on the usage of the model or of the framework itself. Examples of the practical implementation of this framework and the specific selection of the concepts mentioned above were proposed in Chapter 7 for a model used for capital allocation in case of credit risk and in Chapter 9 for P&L explanation for digital option. As, however, financial institutions employ a wide range of techniques and highly diverse models for various purposes, a careful and detailed study of this sort needs to be taken for each particular model or a family of models.

Motivated by the extensive theory on Banach spaces in the realms of for example functional analysis, the study of Banach spaces over weighted Riemannian manifolds shall broaden our understanding of the properties of these spaces as as their application to the quantification of model risk.

In regard to Chapter 8, it will be of interest to extent the framework using sub–manifolds in the case of hidden variables and/or incomplete data, as well as to deepen in the influence of a fitting process on the model risk management. Note that depending on the particular estimation or calibration procedure, the resulting model selection may vary. For instance, any of these approaches can be applied in fitting models that have a continuous–valued response: method of moments, minimizing residuals, maximizing the likelihood, homogeneity of modelled classes, and predictive accuracy in partitioned data, each giving rise to different estimated models. In general, there are two main problems: non–uniqueness and error or risk propagation. First, the fitting problem is not unique which causes several models to fit the same data. Second, the selection of the model design and the estimated parameters may be affected by limited and erroneous input data, by the non–uniqueness of parameter estimated, as well as the subjectivity in defining a good fit to observed data. As such, even a successful application of fitting process does not necessarily justify high confidence in the predictive capability of a financial model.

The study presented in Chapter 9 is of starting–point nature, suggesting many directions for further analyses and research with the objective to establish a deeper link between the curvature of model manifolds with quantification of model risk and model validation. We would like to highlight the following:

1. Since curvature is affected by the choice of a particular family of models, their usage as well as data, it might be interesting to explore calibration algorithms for selecting most appropriate Riemannian metrics.

2. Choice of a proper metric for other usages besides the P&L explanation or sensitivity analysis (where the differential calculus is applied) and also to other data, models and their associated usages.

3. Of utmost interest in our opinion is to perform theoretical analysis to asses, among others, when improvements are expected and to what order, to identify areas when model risk is amplified or to improve the detection of model failures. In other words, to establish a deeper link with quantification of model risk and model validation.

As illustrated in Chapter 7 and Chapter 9, the issue of quantifying the uncertainty between two models or data samples, can be determined by a distance of the geodesic connecting the corresponding probability distributions. The geodesic equation described a reversible dynamics whose solution is the trajectory between considered probability distributions. It is well known that the Riemannian curvature of a manifold is closely related to the behaviour of the geodesics. Positive Riemannian curvature implies that the nearby geodesics oscillate about one another while in the case of negative curvature, the geodesics rapidly diverge from each other. The behaviour of this divergence is determined by the geodesic equation.

For instance, imposing different curvatures on $\mathcal{M}$ enables us to study the influence of changes on the behaviour of models or data samples. The concept of curvature may represent characteristic of economic or financial markets, business, additional information about the inputs or eliminating some type of error, among many others.

Resumen extenso

Los modelos financieros son muy relevantes para el éxito de los negocios financieras. Proporcionan métodos cuantitativos o sistemas que aplican teorías y metodologías para transformar datos de entrada en estimaciones cuantitativas para la toma de decisiones.

Por definición, los modelos simplifican la realidad para servir a un propósito específico enfocándose deliberadamente en aspectos particulares de los fenómenos y degradan o ignoran los que consideran despreciables. Por lo tanto, el uso de modelos expone a las instituciones financieras al riesgo del modelo, entendido como el potencial de pérdida financiera o de reputación debido a errores en el desarrollo, implementación o uso del modelo (uso de datos erróneos, cambios o usos del modelo no autorizados o uso de un modelo fuera del propósito de para el que ha sido desarrollado), solo por nombrar algunos. El riesgo del modelo puede ser particularmente alto, especialmente bajo condiciones de estrés o combinado con otros eventos desencadenantes interrelacionados. Además del riesgo del modelo de un modelo individual, muchos modelos utilizan como entradas las salidas de otros modelos, de modo que incluso un pequeño error en un modelo puede combinarse o amplificarse cuando su salida incorrecta se emplea en otros modelos. Por lo tanto, la comprensión de las capacidades y limitaciones de los supuestos subyacentes es clave cuando se maneja un modelo y sus resultados.

De acuerdo con la Junta de Gobierno del Sistema de la Reserva Federal (Fed) y la Oficina del Control de la Moneda (OCC), el riesgo del modelo se define como

> *"[. . .] el potencial de consecuencias adversas de decisiones basadas en resultados e informes de modelos incorrectos o mal utilizados. El riesgo del modelo puede generar pérdidas financieras, malas decisiones comerciales y estratégicas, o daños a la reputación del banco."*

Luego, la Fed identifica las dos causas principales del riesgo del modelo (uso inapropiado y errores fundamentales). Además, afirman que el riesgo del modelo debe ser manejado y tratado con la misma severidad que cualquier otro tipo de riesgo y que los bancos deben identificar las fuentes del riesgo del modelo y evaluar su magnitud.

La cuantificación, como parte esencial de la gestión del riesgo de modelo, es necesaria para una gestión coherente y una comunicación efectiva de las debilidades y limitaciones del modelo para los responsables de la toma de decisiones y los usuarios, así como para evaluar objetivamente el riesgo del modelo en el contexto de la posición general de la organización. La cuantificación del riesgo del modelo debe considerar la incertidumbre derivada de la selección de las técnicas matemáticas (por ejemplo, centrarse en ajustar una distribución normal, dejando de lado otras familias de distribución), la metodología de calibración (por ejemplo, diferentes algoritmos de optimización pueden derivar valores de parámetros diferentes) y las limitaciones en los datos de muestra (por ejemplo, base de datos dispersa o incompleta).

A pesar del aumento de la consciencia sobre el riesgo del modelo y la comprensión de su impacto significativo, no existen normas establecidas globalmente en el sector financiero ni de mercado sobre su definición y cuantificación exactas, a pesar de que los reguladores requieren una gestión del riesgo modelo adecuada. Esto se debe principalmente a la complejidad del problema de la evaluación del riesgo de modelo. La cuantificación plantea varios desafíos que provienen de la alta diversidad de modelos, la amplia gama de técnicas y los diferentes usos de los modelos, entre otros factores. Algunos resultados de modelos determinan las decisiones, otros resultados de los mismos proporcionan una fuente de información de gestión, algunas salidas se utilizan además como entradas en otros modelos, o los resultados del modelo pueden ser completamente anulados por el juicio del experto. Además, como toda metodología de cuantificación integral, se debería al menos considerar los datos utilizados para construir el modelo, sus fundamentos matemáticos, la infraestructura de TI, el rendimiento general y (lo que es más importante) el uso. Sin mencionar que para cuantificar el riesgo del modelo se necesita otro modelo, que nuevamente es propenso al riesgo de modelo.

El objetivo de esta tesis fue abordar la cuantificación del riesgo modelo. Para ello, hemos introducido un marco general para la cuantificación del riesgo modelo, utilizando la geometría diferencial y la teoría de la información, que es potencialmente aplicable a la mayoría de las técnicas de modelado actualmente en uso en las instituciones financieras. Este marco cubre aspectos relevantes de la gestión de riesgos modelo: uso, rendimiento, fundamentos matemáticos, calibración o datos. Las diferencias entre los modelos están determinadas por la distancia bajo una métrica riemanniana.

En este marco, los modelos están representados por distribuciones de probabilidad particulares que pertenecen a un conjunto de medidas de probabilidad $\mathcal{M}$ (la variedad de los modelos), disponibles para el modelado. Suponemos que el modelo examinado p_0 está representado por una distribución de probabilidad que puede ser parametrizada de manera única usando un vector de parámetros n–dimensional $\theta_0 = (\theta_0^1, \ldots, \theta_0^n)$, es decir $p_0 = p(x, \theta_0) \in \mathcal{M}$.

Sobre la variedad $\mathcal{M}$ definimos un núcleo no lineal K, que atribuye una relevancia relativa a todo modelo alternativo y asigna la credibilidad de las hipótesis subyacentes que harían otros modelos parcial o relativamente preferibles al modelo nominal seleccionado p_0. La particular elección de K depende de la sensibilidad del modelos, el análisis de escenarios, la relevancia de los resultados en la toma de las decisiones, el negocio, los objetivos propuestos, o la incertidumbre de los fundamentos del modelo. El requerimiento de que K sea una función de densidad suave en $\mathcal{M}$ induce una nueva medida de probabilidad absolutamente contínua ζ con respecto a volumen de Rieman dv definido por

$$\zeta(U) = \int_U d\zeta = \int_U K dv(p),$$

donde $U \subseteq \mathcal{M}$ es un entorno abierto de p_0 que contiene modelos alternativos que no están muy lejos en un sentido cuantificado por la relevancia de las propiedades que faltan y las limitaciones del modelo (i.e. la incertidumbre en la selección del modelo).

La medida de riesgo de modelo considera el uso del modelo representado por una aplicación $f\colon \mathcal{M} \to \mathbb{R}$, con $p \mapsto f(p)$, que a cada modelo p asigna el resultado $f(p)$. Entonces, el riesgo de modelo se mide como una norma de las diferencias de los resultados del modelo sobre la variedad riemaniana con peso K, dotada de la métrica de Fisher–Rao y la conexión de Levi–Civita.

Esta aproximación puede usarse para cuantificar el riesgo de modelo mediante el embebimiento $\mathcal{M} \hookrightarrow \mathcal{M}'$ donde $\mathcal{M}' = \{p(x,\theta) : \theta \in \Theta; dim(\theta) \geq dim(\theta_0)\}$ es una variedad riemanniana que contiene una colección de distribuciones de probabilidad creadas variando θ_0, añadiendo propiedades o considerando la incertumbre en los datos o en la calibración. Además, proporcionamos una aplicación de este marco para cuantificar el riesgo de modelos en problemas de asignación de capital por riesgo de crédito.

A continuación, profundizamos aún más en la influencia de la incertidumbre de los datos en el riesgo del modelo al tiempo que relacionamos las incertidumbres que rodean a los datos con la estructura y el desarrollo del modelo. Para evaluar y controlar mejor la incertidumbre en los datos, movemos la configuración para la cuantificación del riesgo del modelo a un espacio muestral a través de un proceso de muestreo que está vinculado al proceso de ajuste utilizado para el desarrollo.

Basándonos en el proceso de ajuste de datos $\bar{\pi} : \bar{S} \to \mathcal{M}$, que asocia a cada muestra $x \in \bar{S}$ una distribución de probabilidad $p(x,\theta) = \bar{\pi} \in \mathcal{M}$, definimos el espacio cociente a partir $\bar{S}$ donde las clases de equivalencia contienen muestras que son igualmente válidas con respecto a la variedad de modelos $\mathcal{M}$. Denotamos $S = \bar{S}/\sim = \{[x] : x \in \bar{S}\}$ el correspondiente espacio cociente y la proyección correspondiente por $\delta : \bar{S} \to S$.

Para garantizar la consistencia entre los datos y los modelos, y para asegurar la unicidad y compatibilidad del proceso de muestreo, requerimos que $\bar{\pi}$ y $\bar{\sigma}$, con $Dom(\bar{\sigma}) = \mathcal{M}$, verifiquen

$$\bar{\pi}\bar{\sigma}\bar{\pi} = \bar{\pi}.$$

Estos requerimientos no son muy restrictivos, ya que la mayoría de los algoritmos de estimación, como el de máxima verosimilitud, el método de los momentos, los mínimos cuadrados o la estimación bayesiana, así como muchos procesos de calibración caen dentro de este marco cuando se aplican las restricciones apropiadas (si es necesario).

En particular, esto implica que no todos los datos son aceptables. Por ejemplo, datos con muy pocos elementos no aseguran que el proceso de ajuste funcione como inverso de $\bar{\sigma}$. Bajo esta hipótesis, la estuctura geométrica de la variedad de modelos $\mathcal{M}$ puede ser transferida (*pulled back*) desde $\mathcal{M}$ hasta S induciendo una estructura riemaniana adecuanda en S. De este modo, se pueden tener en cuenta directamente las no linealidades entre las diferentes muestras y evaluar the manera objetiva las diferencias.

Haciendo el *pull back* de la estructura de la variedad de modelos, introducimos una estructura riemaniana consistente en el espacio de datos que nos permite investigar y cuantificar el riesgo de modelo trabajando simplemente con los datos. Así, a pesar de las diferencias aparentes entre S y $\mathcal{M}$, siendo el primero el espacio de observaciones y la segunda la variedad estadística de los modelos, ambos pueden ser dotados de la misma estructura matemática y ser localmente equivalentes desde un punto de vista geométrico. Así, se puede elegir entre trabajar con modelos en $\mathcal{M}$ o con datos o muestras en S.

La estructura riemanniana en el espacio de los datos proporciona un nuevo modo de investigar los datos a través de la distancia inducida entre los datos por la métrica *pull back*, y evaluar la propagación de las perturbaciones. La posibilidad de trabajar con datos en lugar de con los modelos ofrece una alternativa computacional donde la información sobre el modelos se resume de una manera tratable, más fácil de aplicar por parte de la institución y más facil de evaluar la incertidumbre en los datos.

El marco es adecuado para el análisis de sensibilidad utilizando derivadas direccionales y totales, pruebas de estrés, es decir, para ejercicios de regulación y planificación, o para verificar la validez de las aproximaciones realizadas a través del proceso de modelado, probar la validez y estabilidad del modelo o aplicar en otros pruebas y análisis utilizados en el modelo de gestión de riesgos.

El siguiente objetivo fue resaltar la importancia de la geometría del espacio subyacente en los modelos financieros, centrándose en cálculos diferenciales en el contexto más general de una variedad riemanniana. La medida de curvatura que da una idea de lo que un objeto geométrico se desvía de ser plano y proponemos usarla en dos contextos. En primer lugar, puede verse como un medio para controlar y reducir el riesgo del modelo. Por ejemplo, al surgir de la incertidumbre de medición de los datos, debido a estrategias de calibración inexactas, la especificación errónea del modelo o incluso el uso incorrecto, así como el resultado de un desajuste entre la geometría subyacente del espacio y la geometría euclidiana canónica que a menudo, implícita o explícitamente, se supone para el modelo. En segundo lugar, incluso en el caso de una variedad de modelo plano subyacente, las diferentes geometrías pueden ajustarse mejor al uso particular de un modelo y así aumentar el rendimiento general.

Ejemplificamos estas ideas considerando la aplicación a la explicación P&L basada en el desarrollo de Taylor de las opciones digitales con el modelo Black–Scholes (BS) y demostramos la mejora al comparar los resultados bajo geometrías euclidianas y no euclidianas. Elegimos el modelo BS debido a sus limitaciones y desventajas conocidas para establecer el precio de las opciones digitales, en consecuencia cuando se utiliza para la explicación P&L, a fin de resaltar el riesgo del modelo. Las motivaciones para esto son dobles. En primer lugar, al usar el modelo de BS podemos centrarnos en el problema de la especificación errónea del modelo. En segundo lugar, con la elección de este modelo, podemos abordar el vínculo entre la curvatura y el riesgo del modelo inherente.

El desarrollo de Taylor de segundo orden del precio de la opción, V, esta dado por

$$\begin{aligned} dV \approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial^2 V}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 V}{\partial r^2}dr^2 + \frac{\partial^2 V}{\partial t^2}dt^2\right) \\ + \frac{\partial^2 V}{\partial S \partial \sigma}dSd\sigma + \frac{\partial^2 V}{\partial S \partial r}dSdr + \frac{\partial^2 V}{\partial r \partial \sigma}drd\sigma + \textbf{error}. \end{aligned}$$

Desde una perspectiva geométrica, el desarrollo de Taylor se calcula bajo la geometría euclidiana de los datos y del modelo. La clave de nuestro argumento es darse cuenta de que, al igual que muchos otros modelos financieros, la hipótesis subyacente de la fijación de precios de las opciones está codificada como un modelo estadístico que captura todas las suposiciones con respecto al probable comportamiento no lineal del sistema y las dependencias del modelo con respecto a los datos.

Por ejemplo, en el caso de Black–Scholes, el modelo estadístico representa una familia de distribuciones de probabilidad log-normales que forma las variedad estadística curvada $\mathcal{M}$ con coordenadas μ y σ. Un punto en la variedad $\mathcal{M}$, determinado por (μ_i, σ_i), representa una distribución de probabilidad específica $p(x \mid \mu_i, \sigma_i)$ definida en el espacio euclidiano de los datos $\mathcal{X}$. El precio de la opción V se corresponde con un funcional sobre la variedad $\mathcal{M}$, de modelo que la estructura subyacente en $\mathcal{M}$ induce una estructura riemaniana en el correspondiente espacio de modelos de opciones.

Por lo tanto, para reducir los posibles riesgos de modelo, sugerimos modificar el desarrollo de segundo orden y las griegas de orden superior [2] con respecto a los parámetros del modelo, considerando la corrección debida a la curvatura del espacio subyacente. El desarrollo covariante de Taylor de segundo orden del precio de opción que tiene en cuenta la estructura del modelo subyacente puede escribirse

$$\begin{aligned} dV &\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{\partial^2 V}{\partial S \partial \sigma}dSd\sigma + \frac{\partial^2 V}{\partial S \partial r}dSdr \\ &+ \frac{\partial^2 V}{\partial r \partial \sigma}drd\sigma + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \sigma^2}d\sigma^2 - \mathbf{\Gamma}^{\sigma}_{\sigma\sigma}\frac{\partial \mathbf{V}}{\partial \sigma} - \mathbf{\Gamma}^{\mathbf{r}}_{\sigma\sigma}\frac{\partial \mathbf{V}}{\partial \mathbf{r}} + \frac{\partial^2 V}{\partial r^2}dr^2 - \mathbf{\Gamma}^{\sigma}_{\mathbf{rr}}\frac{\partial \mathbf{V}}{\partial \sigma} - \mathbf{\Gamma}^{\mathbf{r}}_{\mathbf{rr}}\frac{\partial \mathbf{V}}{\partial \mathbf{r}}\right) + \text{error} \\ &\approx \text{Desarrollo de Taylor euclidiano} + \text{Corrección de la curvatura} + \text{residuos.} \end{aligned}$$

Dado que una variedad de Rieman es localmente euclidiana, dentro de un entorno pequeño y para direcciones con una pequeña curvatura, los términos de curvatura a menudo pueden ser despreciados. Sin embargo, para curvaturas o entornos más grandes, es necesario calcular explícitamente los términos de curvatura, aunque solo sea para mostrar que son realmente insignificantes.

Aplicando estas ideas a varios ejemplos que cubren varias condiciones subyacentes y de mercado, pero también geometrías curvas y planas, hemos demostrado que considerar una geometría no euclidiana adecuada para la variedad del modelo subyacente puede mejorar el rendimiento general del modelo y reducir el riesgo del modelo. Teniendo en cuenta lo anterior y una consideración de las limitaciones de la geometría plana cuando se aplica a los mercados financieros, se sugiere la introducción de una geometría riemanniana en la variedad del modelo $\mathcal{M}$. La consideración de la geometría del problema abre nuevas posibilidades de modelado debido a la introducción de un nuevo grado de libertad de modelado.

En resumen, la presente contribución ofrece una amplia gama de aplicaciones que cubren la cuantificación del riesgo del modelo, el análisis de sensibilidad y las pruebas de estrés. Además, proporciona una forma de reducir el riesgo del modelo que está intrínsecamente relacionada con el uso de un modelo y la mejora del rendimiento general del mismo. Además, abre una nueva perspectiva sobre el modelado financiero desde los datos y el modelo. Como tal, nuestro marco ofrece beneficios directos para la administración y el uso de modelos financieros al tiempo que permite a la organización una asignación eficiente de recursos, reduce la probabilidad de malas decisiones, mejora la eficiencia del control, aumenta la confianza en los procesos de planificación comercial, mejora el uso de capital y maximiza el beneficio que obtienen de su inversión, entre otros.

La estructura de esta tesis doctoral

Este documento doctoral se divide esencialmente en dos partes principales: una introducción general al tema y el desarrollo de un marco para la cuantificación del riesgo de modelo y su aplicación a algunos de los modelos actualmente en uso — enmarcado por una breve introducción y las conclusiones finales.

La intención principal de esta tesis es la evaluación objetiva del riesgo de modelo. Con el fin de comprender la importancia de esta cuestión, hay que comprender la estructura de gobierno y de organización de las instituciones financieras, la alta diversidad de modelos empleado, la amplia gama de técnicas, los diferentes usos de los modelos dentro de las instituciones financieras, las limitaciones y debilidades de la modelos, entre otros. Por ello, la Parte I se dedica a una introducción al riesgo y la gestión del capital de las instituciones financieras con el objetivo de presentar y explicar cómo los principales bancos utilizan modelos cuantitativos en sus operaciones diarias para la

[2] Las sensibilidades del precio de la opción o las sensibilidades de los factores de la opción son aproximaciones usadas para determinar el cambio en el precio de la opción debido a un cambio en el valor de una de las entradas o parámetros del modelo, comúnmente llamadas griegas.

planificación de capital y presupuesto, medición de riesgos, derivados de precios o préstamos y hacer hincapié en su naturaleza subyacente basada en la teoría de medida. Por otra parte, señalamos y discutimos algunos tipos de riesgos de modelo asociados con el uso de estos modelos y miramos las muchas cosas que pueden salir mal en la gestión de estos modelos (desde su concepción hasta su uso final) y en qué medida los resultados están mal con respecto a algún punto de referencia (datos históricos, modelo externo o interno alternativos).

La Parte I consta de cinco capítulos. El Capítulo 1 describe los modelos utilizados para el cálculo de capital económico y regulatorio. En el Capítulo 2 establecemos los modelos de gestión y de control utilizados para los riesgos de crédito y de mercado, tales como el riesgo de crédito (véase la Sección 2.1), riesgo de mercado (véase la Sección 2.2), los riesgos asociados a los derivados financieros (véase la Sección 2.3) y los riesgos estructurales y de balance (véase la Sección 2.4). El Capítulo 3 se dedica a los riesgos no financieros y se divide en dos secciones, la Sección 3.1 cubre el riesgo operacional y la Sección 3.2 representa una breve introducción a la otra parte de riesgos no financieros, como los relacionados con la informática y la cibernética, el riesgo legal, los riesgos de incumplimiento, el riesgo que surge de las previsiones macro y el riesgo de terceras partes. El Capítulo 4 cubre los modelos utilizados para las pruebas de estrés (véase la Sección 4.1) y la evaluación de las reservas (véase la Sección 4.2). Finalmente, en la Sección 4.3 identificamos las posibles fuentes de riesgo de modelo dentro de cada sector y señalamos las inter–relaciones entre los diferentes tipos de riesgos.

La Parte II está orientada al riesgo de modelo, con el objetivo primero de presentar la complejidad del problema y los desafíos con su gestión, y diseñar un marco general para la cuantificación del riesgo del modelo que tenga en cuenta las diferentes fuentes del mismo durante todo el ciclo de vida de un modelo, siendo aplicable a la mayoría de los modelos financieros actualmente en uso. Esta parte comprende 6 capítulos, acompañada de una introducción al riesgo del modelo, al tiempo que describe los principales desafíos en su cuantificación y gestión.

El Capítulo 5 resalta las fuentes potenciales de riesgo de modelo que pueden ocurrir durante todo el ciclo de vida de un modelo y analiza las formas posibles de mitigar, controlar o reducir algunas de las fuentes. Como la eliminación completa del riesgo del modelo no es posible, la implementación de un enfoque que combine una estructura de gestión de riesgos rigurosa con una cuantificación detallada prudente es de gran importancia. A continuación, en el Capítulo 6 se propone un enfoque general para la cuantificación del riesgo del modelo en el marco de la geometría diferencial y la teoría de la información. Introducimos una medida de riesgo modelo en una variedad estadística donde los modelos están representados por una función de distribución de probabilidad, que es capaz de hacer frente a aspectos relevantes de la gestión de riesgo modelo y tiene el potencial de evaluar muchos de los enfoques matemáticos utilizados actualmente en las instituciones financieras, como riesgo de crédito, riesgo de mercado, fijación de precios y cobertura de derivados, riesgo operativo, macroprevisiones o XVA (ajustes de valoración). El capítulo 7 está dedicado a un ejemplo empírico del cálculo del riesgo modelo de un modelo de riesgo de crédito utilizado para la evaluación de capital, con una ilustración detallada de cómo un profesional puede identificar los conceptos abstractos relevantes y ponerlos a trabajar.

El objetivo principal del Capítulo 8 es profundizar en la influencia de la incertidumbre de los datos en el riesgo del modelo al relacionar las incertidumbres de los datos con la estructura del modelo. Haciendo el *pull back* de la métrica de la variedad de modelos se introduce una estructura riemanniana consistente en el espacio muestral que nos permite cuantificar el riesgo del modelo, trabajando con datos. En la práctica, ofrece una alternativa computacional, facilita la aplicación de la intuición empresarial y la asignación de la incertidumbre de los datos, así como la comprensión del riesgo del modelo a partir de los datos y del modelo. El Capítulo 9 analiza la posible implementación de los principios discutidos anteriormente para mejorar el uso de un modelo y para reducir el riesgo del modelo inherente con la aplicación a la explicación P&L de las opciones digitales.

En última instancia, las conclusiones financieras en el Capítulo 10 resumen los resultados principales y resaltan las ventajas y beneficios del marco propuesto, incluyendo posibles direcciones para el trabajo futuro.

RESUMO EXTENSO

Os modelos financeiros son moi relevantes para o éxito dos negocios financeiros. Proporcionan métodos cuantitativos ou sistemas que aplican teorías e metodoloxías para transformar datos de entrada en estimacións cuantitativas para a toma de decisións.

Por definición, os modelos simplifican a realidade para servir a un propósito específico enfocándose deliberadamente en aspectos particulares dos fenómenos e degradan ou ignoran os que consideran despreciables. Por tanto, o uso de modelos expón ás institucións financeiras ao risco do modelo, entendido como o potencial de perda financeira ou de reputación debido a erros no desenvolvemento, implementación ou uso do modelo (uso de datos erróneos, cambios ou usos do modelo non autorizados ou uso dun modelo fora do propósito para o que foi desenvolvido), só por nomear algúns. O risco do modelo pode ser particularmente alto, especialmente baixo condicións de tensións ou combinado con outros eventos interrelacionados. Ademais do risco do modelo dun modelo individual, moitos modelos utilizan como entradas as saídas doutros modelos, de modo que mesmo un pequeno erro nun modelo pode combinarse ou amplificarse cando a súa saída incorrecta emprégase noutros modelos. Por tanto, a comprensión das capacidades e limitacións dos supostos subxacentes é clave cando se manexa un modelo e os seus resultados.

De acordo coa Xunta de Goberno do Sistema da Reserva Federal (Fed) e a Oficina do Control da Moeda (OCC), o risco do modelo defínese como

> *"[...] o potencial de consecuencias adversas de decisións baseadas en resultados e informes de modelos incorrectos ou mal utilizados. O risco do modelo pode xerar perdas financeiras, malas decisións comerciais e estratéxicas, ou danos á reputación do banco."*

Logo, a Fed identifica as dúas causas principais do risco do modelo (uso inapropiado e erros fundamentais). Ademais, afirman que o risco do modelo debe ser manexado e tratado coa mesma severidade que calquera outro tipo de risco e que os bancos deben identificar as fontes do risco do modelo e avaliar a súa magnitude.

A cuantificación, como parte esencial da xestión do risco de modelo, é necesaria para unha xestión coherente e unha comunicación efectiva das debilidades e limitacións do modelo para os responsables da toma de decisións e os usuarios, así como para avaliar obxectivamente o risco do modelo no contexto da posición xeral da organización. A cuantificación do risco do modelo debe considerar a incerteza derivada da selección das técnicas matemáticas (por exemplo, centrarse en axustar unha distribución normal, deixando de lado outras familias de distribucións), a metodoloxía de calibración (por exemplo, diferentes algoritmos de optimización poden derivar valores de parámetros diferentes) e as limitacións nos datos da mostra (por exemplo, base de datos dispersa ou incompleta).

A pesar do aumento da consciencia sobre o risco do modelo e a comprensión do seu impacto significativo, non existen normas establecidas globalmente no sector financeiro nin de mercado sobre a súa definición e cuantificación exactas, a pesar de que os reguladores requiren unha xestión do risco de modelo axeitada. Isto débese principalmente á complexidade do problema da avaliación do risco de modelo. A cuantificación expón varios desafíos que proveñen da alta diversidade de modelos, a ampla gama de técnicas e os diferentes usos dos modelos, entre outros factores. Algúns resultados de modelos determinan as decisións, outros resultados dos mesmos proporcionan unha fonte de información de xestión, algunhas saídas utilízanse ademais como entradas noutros modelos, ou os resultados do modelo poden ser completamente anulados polo xuízo do experto. Ademais, como toda metodoloxía de cuantificación integral, deberíase polo menos considerar os datos utilizados para construír o modelo, os seus fundamentos matemáticos, a infraestrutura de TI, o rendemento xeral e (o que é máis importante) o uso. Sen mencionar que para cuantificar o risco do modelo necesítase outro modelo, que novamente é propenso ao risco de modelo.

O obxectivo desta tese foi abordar a cuantificación do risco modelo. Para iso, introducimos un marco xeral para a cuantificación do risco modelo, utilizando a xeometría diferencial e a teoría da información, que é potencialmente aplicable á maioría das técnicas de modelado actualmente en uso nas institucións financeiras. Este marco cobre aspectos relevantes da xestión de riscos modelo: uso, rendemento, fundamentos matemáticos, calibración ou datos. As diferenzas entre os modelos están determinadas pola distancia baixo unha métrica riemanniana.

Neste marco, os modelos están representados por distribucións de probabilidade particulares que pertencen a un conxunto de medidas de probabilidade $\mathcal{M}$ (a variedade dos modelos), dispoñibles para o modelado. Supoñemos que o modelo examinado p_0 está representado por unha distribución de probabilidade que pode ser parametrizada de maneira única usando un vector de parámetros n– dimensional $\theta_0 = (\theta_0^1, \ldots, \theta_0^n)$, é dicir $p_0 = p(x, \theta_0) \in \mathcal{M}$. Sobre a variedade $\mathcal{M}$ definimos un núcleo non lineal K, que atribúe unha relevancia relativa a todo modelo alternativo e asigna a credibilidade das hipóteses subxacentes que farían outros modelos parcial ou relativamente preferibles ao modelo nominal seleccionado p_0. A particular elección de K depende da sensibilidade dos modelos, a análise de escenarios, a relevancia dos resultados na toma das decisións, o negocio, os obxectivos propostos, ou a incerteza dos fundamentos do modelo. O requirimento de que K sexa unha función de densidade suave en $\mathcal{M}$ induce unha nova medida de probabilidade absolutamente contínua ζ con respecto ao volume de Rieman dv definido por

$$\zeta(U) = \int_U d\zeta = \int_U K dv(p),$$

donde $U \subseteq \mathcal{M}$ é unha contorna aberta de p_0 que contén modelos alternativos que non están moi lonxe nun sentido cuantificado pola relevancia das propiedades que faltan e as limitacións do modelo (i.e. a incerteza na selección do modelo).

A medida de risco de modelo considera o uso do modelo representado por unha aplicación $f\colon \mathcal{M} \to \mathbb{R}$, con $p \mapsto f(p)$, que a cada modelo p asigna o resultado $f(p)$. Entón, o risco de modelo mídese como unha norma das diferenzas dos resultados do modelo sobre a variedade riemaniana con peso K, dotada da métrica de Fisher–Rao e a conexión de Levi– Civita.

Esta aproximación pode usarse para cuantificar o risco de modelo mediante o embebemento $\mathcal{M} \hookrightarrow \mathcal{M}'$ onde $\mathcal{M}' = \{p(x, \theta) : \theta \in \Theta;\, dim(\theta) \geq dim(\theta_0)\}$ é unha variedade riemanniana que contén unha colección de distribucións de probabilidade creadas variando θ_0, engadindo propiedades ou considerando a incerteza nos datos ou na calibración. Ademais, proporcionamos unha aplicación deste marco para cuantificar o risco de modelos en problemas de asignación de capital por risco de crédito.

A continuación, afondamos aínda máis na influencia da incerteza dos datos no risco do modelo á vez que relacionamos as incertezas que rodean aos datos coa estrutura e o desenvolvemento do modelo. Para avaliar e controlar mellor a incerteza nos datos, movemos a configuración para a cuantificación do risco do modelo a un espazo muestral a través dun proceso de mostraxe que está vinculado ao proceso de axuste utilizado para o desenvolvemento.

Baseándonos no proceso de axuste de datos $\bar{\pi} : \bar{S} \to \mathcal{M}$, que asocia a cada mostra $x \in \bar{S}$ unha distribución de probabilidade $p(x, \theta) = \bar{\pi} \in \mathcal{M}$, definimos o espazo cociente a partires de $\bar{S}$, onde as clases de equivalencia conteñen mostras que son igualmente válidas con respecto á variedade de modelos $\mathcal{M}$. Denotamos $S = \bar{S}/\sim = \{[x] : x \in \bar{S}\}$ o correspondente espazo cociente e a proxección correspondente por $\delta : \bar{S} \to S$.

Para garantir a consistencia entre os datos e os modelos, e para asegurar a unicidade e compatibilidade do proceso de mostraxe, requirimos que $\bar{\pi}$ e $\bar{\sigma}$, con $Dom(\bar{\sigma}) = \mathcal{M}$, verifiquen

$$\bar{\pi}\bar{\sigma}\bar{\pi} = \bar{\pi}.$$

Estes requirimentos non son moi restritivos, xa que a maioría dos algoritmos de estimación, como o de máxima verosimilitude, o método dos momentos, os mínimos cadrados ou a estimación bayesiana, así como moitos procesos de calibración caen dentro deste marco cando se aplican as restricións apropiadas (se é necesario). En particular, isto implica que non todos os datos son aceptables. Por exemplo, datos con moi poucos elementos non aseguran que o proceso de axuste funcione como inverso de $\bar{\sigma}$. Baixo esta hipótese, a estuctura xeométrica da variedade de modelos $\mathcal{M}$ pode ser transferida (*pulled back*) desde $\mathcal{M}$ ata S inducindo unha estrutura riemaniana adecuada en S. Deste xeito, pódense ter en conta directamente as non linealidades entre as diferentes mostras e avaliar de maneira obxectiva as diferenzas.

Facendo o *pull back* da estrutura da variedade de modelos, introducimos unha estrutura riemaniana consistente no espazo de datos que nos permite investigar e cuantificar o risco de modelo traballando simplemente cos datos. Así, a pesar das diferenzas aparentes entre S e $\mathcal{M}$, sendo o primeiro o espazo de observacións e a segunda a variedade estatística dos modelos, ambos poden ser dotados da mesma estrutura matemática e ser localmente equivalentes desde un punto de vista xeométrico. Así, pódese elixir entre traballar con modelos en $\mathcal{M}$ ou con datos ou mostras en S.

A estrutura riemanniana no espazo dos datos proporciona un novo modo de investigar os datos a través da distancia inducida entre os datos pola métrica *pull back*, e avaliar a propagación das perturbacións. A posibilidade de traballar con datos en lugar de cos modelos ofrece unha alternativa computacional onde a información sobre os modelos resúmese dunha maneira tratable, máis sinxela de aplicar por parte da institución e máis fácil de avaliar a incerteza nos datos.

O marco é adecuado para a análise de sensibilidade utilizando derivadas direccionais e totais, probas de tensións, é dicir, para exercicios de regulación e planificación, ou para verificar a validez das aproximacións realizadas a través do proceso de modelado, probar a validez e estabilidade do modelo ou aplicar noutros probas e análises utilizados no modelo de xestión de riscos.

O seguinte obxectivo foi resaltar a importancia da xeometría do espazo subxacente nos modelos financeiros, centrándose en cálculos diferenciais no contexto máis xeral dunha variedade riemanniana. A medida de curvatura que dá unha idea do que un obxecto xeométrico desvíase de ser plano e propoñemos usala en dous contextos. En primeiro lugar, pode verse como un medio para controlar e reducir o risco do modelo. Por exemplo, ao xurdir da incerteza de medición dos datos, debido a estratexias de calibración inexactas, a especificación errónea do modelo ou mesmo o uso incorrecto, así como o resultado dun desaxuste entre a xeometría subxacente do espazo e a xeometría euclidiana canónica que a miúdo, implícita ou explícitamente, suponse para o modelo. En segundo lugar, mesmo no caso dunha variedade de modelo plano subxacente, as diferentes xeometrías poden axustarse mellor ao uso particular dun modelo e así aumentar o rendemento xeral.

Exemplificamos estas ideas considerando a aplicación á explicación P&L baseada no desenvolvemento de Taylor das opcións dixitais co modelo Black–Scholes (BS) e demostramos a mellora ao comparar os resultados baixo xeometrías euclidianas e non euclidianas. Eliximos o modelo BS debido ás súas limitacións e desvantaxes coñecidas para establecer o prezo das opcións dixitais, en consecuencia cando se utiliza para a explicación P&L, a fin de resaltar o risco do modelo. As motivacións para isto son dobres. En primeiro lugar, ao usar o modelo de BS podemos centrarnos no problema da especificación errónea do modelo. En segundo lugar, coa elección deste modelo, podemos abordar o vínculo entre a curvatura e o risco do modelo inherente. O desenvolvemento de Taylor de segunda orde do prezo da opción, V, esta dado por

$$
\begin{aligned}
dV \approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial^2 V}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 V}{\partial r^2}dr^2 + \frac{\partial^2 V}{\partial t^2}dt^2\right) \\
+ \frac{\partial^2 V}{\partial S \partial \sigma}dS d\sigma + \frac{\partial^2 V}{\partial S \partial r}dS dr + \frac{\partial^2 V}{\partial r \partial \sigma}dr d\sigma + \textbf{erro}.
\end{aligned}
$$

Desde unha perspectiva xeométrica, o desenvolvemento de Taylor calcúlase baixo a xeometría euclidiana dos datos e do modelo. A clave do noso argumento é darse conta de que, do mesmo xeito que moitos outros modelos financeiros, a hipótese subxacente da fixación de prezos das opcións está codificada como un modelo estatístico que captura todas as suposicións con respecto ao probable comportamento non lineal do sistema e as dependencias do modelo con respecto aos datos.

Por exemplo, no caso de Black–Scholes, o modelo estatístico representa unha familia de distribucións de probabilidade log-normais que forma a variedade estatística curvada $\mathcal{M}$ con coordenadas μ e σ. Un punto na variedade $\mathcal{M}$, determinado por (μ_i, σ_i), representa unha distribución de probabilidade específica $p(x \mid \mu_i, \sigma_i)$ definida no espazo euclidiano dos datos $\mathcal{X}$. O prezo da opción V correspóndese cun funcional sobre a variedade $\mathcal{M}$, de modelo que a estrutura subxacente en $\mathcal{M}$ induce unha estrutura riemaniana no correspondente espazo de modelos de opcións.

Por tanto, para reducir os posibles riscos de modelo, suxerimos modificar o desenvolvemento de segunda orde e as gregas de orde superior[3] con respecto aos parámetros do modelo, considerando a corrección debida á curvatura do espazo subxacente. O desenvolvemento covariante de Taylor de segunda orde do prezo de opción que ten en conta a estrutura do modelo subxacente pode escribirse

[3] As sensibilidades do prezo da opción ou as sensibilidades dos factores da opción son aproximacións usadas para determinar o cambio no prezo da opción debido a un cambio no valor dunha das entradas ou parámetros do modelo, comunmente chamadas gregas.

$$
\begin{aligned}
dV &\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{\partial^2 V}{\partial S \partial \sigma}dSd\sigma + \frac{\partial^2 V}{\partial S \partial r}dSdr \\
&+ \frac{\partial^2 V}{\partial r \partial \sigma}drd\sigma + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \sigma^2}d\sigma^2 - \mathbf{\Gamma}^{\sigma}_{\sigma\sigma}\frac{\partial \mathbf{V}}{\partial \sigma} - \mathbf{\Gamma}^{\mathbf{r}}_{\sigma\sigma}\frac{\partial \mathbf{V}}{\partial \mathbf{r}} + \frac{\partial^2 V}{\partial r^2}dr^2 - \mathbf{\Gamma}^{\sigma}_{\mathbf{rr}}\frac{\partial \mathbf{V}}{\partial \sigma} - \mathbf{\Gamma}^{\mathbf{r}}_{\mathbf{rr}}\frac{\partial \mathbf{V}}{\partial \mathbf{r}}\right) + \text{error} \\
&\approx \text{Desenvolvemento de Taylor euclidiano} + \text{Corrección de la curvatura} + \text{residuos.}
\end{aligned}
$$

Dado que unha variedade de Riemann é localmente euclidiana, dentro dunha contorna pequena e para direccións cunha pequeña curvatura, os termos de curvatura a miúdo poden ser desprezados. Con todo, para curvaturas ou contornas máis grandes, é necesario calcular explícitamente os termos de curvatura, aínda que só sexa para mostrar que son realmente insignificantes.

Aplicando estas ideas a varios exemplos que cobren varias condicións subxacentes e de mercado, pero tamén xeometrías curvas e planas, demostramos que considerar unha xeometría non euclidiana adecuada para a variedade do modelo subxacente pode mellorar o rendemento xeral do modelo e reducir o risco do modelo. Tendo en conta o anterior e unha consideración das limitacións da xeometría plana cando se aplica aos mercados financeiros, suxírese a introdución dunha xeometría riemanniana na variedade do modelo $\mathcal{M}$. A consideración da xeometría do problema abre novas posibilidades de modelado debido á introdución dun novo grao de liberdade de modelado.

En resumo, a presente contribución ofrece unha ampla gama de aplicacións que cobren a cuantificación do risco do modelo, a análise de sensibilidade e as probas de tensións. Ademais, proporciona unha forma de reducir o risco do modelo que está intrínsecamente relacionada co uso dun modelo e a mellora do rendemento xeral do mesmo. Ademais, abre unha nova perspectiva sobre o modelado financeiro desde os datos e o modelo. Como tal, o noso marco ofrece beneficios directos para a administración e o uso de modelos financeiros á vez que permite á organización unha asignación eficiente dos recursos, reduce a probabilidade de malas decisións, mellora a eficiencia do control, aumenta a confianza nos procesos de planificación comercial, mellora o uso de capital e maximiza o beneficio que obteñen do seu investimento, entre outros.

A estrutura desta tese doutoral

Este documento doutoral divídese esencialmente en dous partes principais: unha introdución xeral ao tema e o desenvolvemento dun marco para a cuantificación do risco de modelo e a súa aplicación a algúns dos modelos actualmente en uso — enmarcado por unha breve introdución e as conclusións finais.

A intención principal desta tese é a avaliación obxectiva do risco de modelo. Co fin de comprender a importancia desta cuestión, hai que comprender a estrutura de goberno e de organización das institucións financeiras, a alta diversidade de modelos empregado, a ampla gama de técnicas, os diferentes usos dos modelos dentro das institucións financeiras, as limitacións e debilidades da modelos, entre outros. Por iso, a Parte I dedícase a unha introdución ao risco e a xestión do capital das institucións financeiras co obxectivo de presentar e explicar como os principais bancos utilizan modelos cuantitativos nas súas operacións diarias para a planificación de capital e orzamento, medición de riscos, derivados de prezos ou préstamos e facer fincapé na súa natureza subxacente baseada na teoría de medida. Por outra banda, sinalamos e discutimos algúns tipos de riscos de modelo asociados co uso destes modelos e miramos as moitas cousas que poden saír mal na xestión destes modelos (desde a súa concepción ata o seu uso final) e en que medida os resultados están mal con respecto a algún punto de referencia (datos históricos, modelo externo ou interno alternativos).

A Parte I consta de cinco capítulos. O Capítulo 1 describe os modelos utilizados para o cálculo de capital económico e regulatorio. No Capítulo 2 establecemos os modelos de xestión e de control empregados para os riscos de crédito e de mercado, tales como o risco de crédito (véxase a Sección 2.1), risco de mercado (véxase a Sección 2.2), os riscos asociados aos derivados financeiros (véxase a Sección 2.3) e os riscos estruturais e de balance (véxase a Sección 2.4). O Capítulo 3 dedícase aos riscos non financeiros e divídese en dos seccións, a Sección 3.1 cobre o risco operacional y a Sección 3.2 representa unha breve introdución á outra parte de riscos non financeiros, como os relacionados coa informática e a cibernética, o risco legal, os riscos de incumprimento, o risco que xorde das previsións macro e o risco de terceiras partes. O Capítulo 4 cobre os modelos utilizados para as probas de tensións (véxase a Sección 4.1) e a avaliación das reservas (véxase a Sección 4.2). Finalmente, na Sección 4.3 identificamos as posibles fontes de risco de modelo dentro de cada sector e sinalamos as inter–relacións entre os diferentes tipos de riscos.

A Parte II está orientada ao risco de modelo, co obxectivo primeiro de presentar a complexidade do problema e os desafíos coa súa xestión, e deseñar un marco xeral para a cuantificación do risco do modelo que teña en conta as diferentes fontes do mesmo durante todo o ciclo de vida dun modelo, sendo aplicable á maioría dos modelos financeiros actualmente en uso. Esta parte comprende 6 capítulos, acompañada dunha introdución ao risco do modelo, á vez que describe os principais desafíos na súa cuantificación e xestión.

O Capítulo 5 resalta as fontes potenciais de risco de modelo que poden ocorrer durante todo o ciclo de vida dun modelo e analiza as formas posibles de mitigar, controlar ou reducir algunhas das fontes. Como a eliminación completa do risco do modelo non é posible, a implementación dun enfoque que combine unha estrutura de xestión de riscos rigorosa cunha cuantificación detallada prudente é de gran importancia. A continuación, no Capítulo 6 proponse un enfoque xeral para a cuantificación do risco do modelo no marco da xeometría diferencial e a teoría da información. Introducimos unha medida de risco modelo nunha variedade estatística onde os modelos están representados por unha función de distribución de probabilidade, que é capaz de facer fronte a aspectos relevantes da xestión de risco modelo e ten o potencial de avaliar moitos dos enfoques matemáticos utilizados actualmente nas institucións financeiras, como risco de crédito, risco de mercado, fixación de prezos e cobertura de derivados, risco operativo, macroprevisiones ou XVA (axustes de valoración). O Capítulo 7 está adicado a un exemplo empírico do cálculo do risco modelo dun modelo de risco de crédito utilizado para a avaliación de capital, cunha ilustración detallada de como un profesional pode identificar os conceptos abstractos relevantes e poñelos a traballar.

O obxectivo principal do Capítulo 8 é afondar na influencia da incerteza dos datos no risco do modelo ao relacionar as incertezas dos datos coa estrutura do modelo. Facendo o *pull back* da métrica da variedade de modelos introdúcese unha estrutura riemanniana consistente no espazo muestral que nos permite cuantificar o risco do modelo, traballando con datos. Na práctica, ofrece unha alternativa computacional, facilita a aplicación da intuición empresarial e a asignación da incerteza dos datos, así como a comprensión do risco do modelo a partir dos datos e do modelo. O Capítulo 9 analiza a posible implementación dos principios discutidos anteriormente para mellorar o uso dun modelo e para reducir o risco do modelo inherente coa aplicación á explicación P&L das opcións dixitais.

En última instancia, as conclusións financeiras no Capítulo 10 resumo os resultados principais e resaltan as vantaxes e beneficios do marco proposto, incluíndo posibles direccións para o traballo futuro.

Bibliography

[1] ABASTO, D. F., AND KUST, M. P. Model01: Quantifying the risk of incremental model changes. `http://ssrn.com/abstract=2492256.` (2014).

[2] ABRAMOV, V., LOWDERMILK, M., AND ZHOU, X. A practical guide to market risk model validations. `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2916853` (2017).

[3] ACERBI, C. Spectral measures of risk: A coherent representation of subjective risk aversion. J*ournal of Banking and Finance 26(7)* (2002), 1505–1518.

[4] ADAMS, R. A. Sobolev spaces. Academic Press (1975).

[5] ALLEN, B. Internal affairs. R*isk 19* (June, 2006), 45–49.

[6] AMARI, S., BARNDORFF-NIELSEN, O., KASS, R., LAURITZEN, S., AND RAO, C. Differential geometry in statistical inference. I*MS Lecture Notes–Monograph Series 10, Hayward, California, USA* (1987).

[7] AMARI, S. I. Differential–geometrical methods in statistics. Lecture Notes on Statistics *1* (1985).

[8] BANCO SANTANDER. Pillar III disclosures. `http://www.santander.com/csgs` (2015).

[9] BANK FOR INTERNATIONAL AND SETTLEMENT (BIS). International convergence of capital measurement and capital standards. a revised framework, comprehensive version. `http://www.bis.org/publ/bcbs128.htm` (2006).

[10] BANK FOR INTERNATIONAL SETTLEMENT (BIS). Review of the Pillar 3 disclosure requirements. `http://www.bis.org/publ/bcbs286.pdf` (2014).

[11] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Principles for the management of credit risk. h*ttps://www.bis.org/publ/bcbsc125.pdf*.

[12] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Principles for effective risk data aggregation and risk reporting. h*ttps://www.bis.org/bcbs/publ* (2013).

[13] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). The four lines of defence model for financial institutions. F*SI Occasional Paper No 11., ISSN 1020–9999 (online)* (2015).

[14] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Revisions to the standardised approach for credit risk. h*ttps://www.bis.org/bcbs/publ/d347.pdf* (2015).

[15] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Interest rate risk in the banking book. h*ttps://www.bis.org/bcbs/publ/d368.pdf* (2016).

[16] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Minimum capital requirements for market risk. h*ttps://www.bis.org/bcbs/publ/d352.pdf* (2016).

[17] BANK FOR INTERNATIONAL SETTLEMENTS (BIS). Reducing variation in credit risk-weighted assets - constraints on the use of internal model approaches. h*ttps://www.bis.org/bcbs/publ/d362.pdf* (2016).

[18] BANK OF ENGLAND. Stress testing the uk banking system: 2018 guidance for participating banks and building societies. *https://www.bankofengland.co.uk/-/media/boe/files/stress-testing/2018/stress-testing-the-uk-banking-system-2018-guidance.pdf* (2018).

[19] BASEL COMMITTEE ON BANKING SUPERVISION. Basel III counterparty credit risk - frequently asked questions. *http://www.bis.org/publ/bcbs228.pdf* (2012).

[20] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). International convergence of capital measurement and capital standards. a revised framework. *http://www.bis.org/publ/bcbsca.htm* (2004).

[21] BASEL COMMITTEE ON BANKING SUPERVISION BCBS. Principles for sound stress testing practices and supervision. *http://www.bis.org/publ/bcbs155.pdf* (2009).

[22] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Revision to the basel ii market risk framework. Basel (2009).

[23] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Guidance for national authorities operating the countercyclical capital buffer. B*ank for International Settlements, Basel* (2010).

[24] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Basel iii: A global regulatory framework for more resilient banks and banking systems. *http://www.bis.org/publ/bcbs189.pdf* (2011).

[25] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Basel III: A global regulatory framework for more resilient banks and banking systems. *http://www.bis.org/publ/bcbs189.pdf* (2011).

[26] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Basel III: International framework for liquidity risk measurement, standards, and monitoring. *http://www.bis.org/publ/bcbs188.pdf* (2011).

[27] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Revisions to the basel ii market risk framework. *http://www.bis.org/publ/bcbs193.pdf* (2011).

[28] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Basel III: leverage ratio framework and disclosure requirements. *http://www.bis.org/publ/bcbs270.pdf* (2014).

[29] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Part2: The first Pillar - minimum capital requirements. *http://www.bis.org/publ/bcbs128b.htm* (2016).

[30] BASEL COMMITTEE ON BANKING SUPERVISION (BCBS). Range of practices and issues in economic capital frameworks. B*ank for International Settlement (BIS)* (March, 2009).

[31] BLUHM, C., OVERBECK, L., AND WAGNER, C. Introduction to credit risk modeling. Florida, USA: Chapman & Hall/CRC Press (2016).

[32] BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM (FRB) AND OFFICE OF THE COMPTROLLER OF THE CURRENCY (OCC). Guidance on model risk management. *www.federalreserve.gov/boarddocs/srletters/2011/sr1107.htm* (2011).

[33] BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM (FRS) AND OFFICE OF THE COMPTROLLER OF THE CURRENCY, OCC (2011–12). The four lines of defence model for financial institutions. S*upervisory Guidance on Model Risk Management Sec. III* (2011-12), 3.

[34] Boucher, C. M., Daníelsson, J., Kouontchou, P., and Maillet, B. Risk models–at–risk. Journal of Banking & Finance *44(1)* (2014), 72–92.

[35] Branger, N., and Schlag, C. Model risk: A conceptual framework for risk measurement and hedging. EFMA 2004 Basel Meetings Paper (2004).

[36] Breuer, T., and Csiszár, I. Measuring model risk. P*apers 1301.4832,* arXiv.org (2013).

[37] Brigo, D., and Liinev, J. On the distributional distance between the libor and the swap market models. preprint (2002).

[38] Chen, W.W.S. Finding gaussian curvature of lifespan distribution. Applied Mathematics *5* (2014), 3392–3400.

[39] Christodoulakis, G., and Satchell, S. The validity of credit risk model validation methods. The Analytics of Risk Model Validation, MA: Academic Pres *27–44* (2008).

[40] Committee, B., et al. *An Explanatory Note on the Basel II IRB Risk Weight Functions. B*asel Committee on Banking Supervision, 2004.

[41] Cont, R. Model uncertainty and its impact on the pricing of derivative instruments. Mathematical Finance *16* (2006), 519–547.

[42] Courtault, J. M., Kabanov, Y., Bru, B., Crépel, P., Lebon, I., and Le Marchand, A. Louis bachelier on the centenary of théorie de la spéculation. M*athematical Finance 10(3)* (2000), 339–353.

[43] Critchley, F., Marriott, P., and Salmon, M. Preferred point geometry and statistical manifolds. The Annals of Statistics (1993), 1197–1224.

[44] Csiszár, I., and Breuer, T. An information geometry problem in mathematical finance. International Conference on Networked Geometric Science of Information. Springer International Publishing (2015).

[45] Cummins, J. D., Lewis, C. M., and Wei, R. The market value impact of operational loss events for us banks and insurers. J*ournal of Banking and Finance 30(10)* (2006), 2605–2634.

[46] Dahlin, F., and Storkitt, S. Estimation of loss given default for low default portfolios.

[47] De Weert, F. Bank and insurance capital management. J*ohn Wiley & Sons 516* (2011).

[48] Deloitte. Basel III framework: The butterfly effect. h*ttp://www2.deloitte.com* (2015).

[49] Directive of the European Parliament and of the Council. Article 22 of the european capital requirement directives (CRD 2006/48/EC). `http://eur-lex.europa.eu` (2006).

[50] Do Carmo, M. P. Riemannian geometrys. Birkhauser *1* (1992).

[51] Doff, R. Defining and measuring business risk in an economic–capital framework. T*he Journal of Risk Finance 9(4)* (2008), 317–333.

[52] Downd, K., B. D., and Cairns, A. Long–term value at risk. *The Journal of Risk Finance 5, N. 2* (2004), 52–57.

[53] EFRON, B. Defining the curvature of a statistical problem (with applications to second order efficiency). The Annals of Statistics (1975), 1189–1242.

[54] ELLES, J., AND LEMAIRE, L. Selected topics in harmonic maps. Conference Board of the Mathematical Sciences by the American Mathematical Society Providence, Rhode Island (1983).

[55] EUROPEAN BANKING AUTHORITY (EBA). Supervisory review and evaluation process (SREP) and pillar 2. `https://www.eba.europa.eu/regulation-and-policy/supervisory-review-and-evaluation-srep-and-pillar-2/-/activity-list/NRWsAs5hcSDu/more`.

[56] EUROPEAN BANKING AUTHORITY (EBA). Eba guidelines on stressed value at risk (stressed var), eba/gl/2012/2. `https://www.eba.europa.eu/documents/10180/104547/EBA-BS-2012-78--GL-on-Stressed-VaR-.pdf` (2012).

[57] EUROPEAN BANKING AUTHORITY (EBA). The guidelines for common procedures and methodologies for the supervisory review and evaluation process (srep). `https://www.eba.europa.eu/documents/10180/935249/EBA-GL-2014-13+(Guidelines+on+SREP+methodologies+and+processes).pdf` (2014).

[58] EUROPEAN BANKING AUTHORITY (EBA). Guidelines on the management of interest rate risk arising from non–trading activities. *EBA/GL/2015/08* (2015).

[59] FEDERAL RESERVE SYSTEM. Comprehensive capital analysis and review (CCAR). h*ttps://www.federalreserve.gov/supervisionreg/ccar-by-year.htm* (2011).

[60] FEDERAL RESERVE SYSTEM (FED). Supervisory capital assessment program. h*ttps://www.federalreserve.gov/newsevents/files/bcreg20090507a1.pdf* (2009).

[61] FL: THE INSTITUTE OF INTERNAL AUDITORS INC. The three lines of defense in effective risk management and controls. A*ltamonte Springs,* 3LinesofDefenseinEffectiveRiskManagementandControl (January 2013).

[62] FOR INTERNATIONAL SETTLEMENT (BIS), B. Interest rate risk in the banking book. consultative document. `http://www.bis.org/bcbs/publ/d319.htm` (2015).

[63] GIBSON, R. Model risk: concepts, calibration and pricing. London, UK: Risk Books (2000).

[64] GLASSERMAN, P., AND XU, X. Robust risk measurement and model risk. Quantitative Finance (September 2013).

[65] GOVERNORS OF THE FEDERAL RESERVE SYSTEM (FRS). SR 11-7: Guidance on model risk management. `http://www.federalreserve.gov/bankinforeg/srletters/sr1101a1.pfd` (2011).

[66] HESTON, S. L. A closed–form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies *6(2)* (1993), 327–343.

[67] HIRSA, A., AND NEFTCI, S. N. An introduction to the mathematics of financial derivatives. Academic Press (2013).

[68] HULL, J., AND SUO, W. A methodology for assessing model risk and its application to the implied volatility function model. Journal of Financial and Quantitative Analysis *37* (2002), 297–318.

[69] HULL, J. C., AND BASU, S. Options, futures, and other derivatives. Pearson Education India (2016).

[70] IFRS AND THE US FINANCIAL ACCOUNTING STANDARDS BOARD (FASB). International financial reporting standard 3 business combinations. IFRS3 (2008).

[71] INTERNATIONAL FINANCIAL REPORTING STANDARD (IFRS). Ifrs 9 financial instruments. h*ttps://www.ifrs.org/issued-standards/list-of-standards/ifrs-9-financial-instruments/* (2014).

[72] JACKWERTH, J. Option implied risk-neutral distributions and implied binomial trees: A literature review. `https://papers.ssrn.com/sol3/papers.cfm/abstract_id=183705` (1999).

[73] JACOBS, M. J. The quantification and aggregation of model risk: perspective on potential approaches. I*nt. J. Financial Engineering and Risk Management 2, No. 2* (2015).

[74] JONES, D. Emerging problems with the basel capital accord: regulatory capital arbitrage and related issues. J*ournal od Banking* & *Finance 24* (2000), 35–58.

[75] KAROUI, N., JEANBLANC-PICQUÉ, M., AND SHREVE, S. Robustness of the black and scholes formula. Mathematical Finance *8, No. 2* (1998), 93–126.

[76] KAUFMANN, R. Long–term risk management. *PhD Thesis, ETH Zurich* (2004).

[77] KERKHOF, J., MELENBERG, B., AND SCHUMACHER, H. Model risk and capital reserves. Journal of Banking and Finance *34* (2010), 267–279.

[78] KORN, R., AND KORN, E. Option pricing and portfolio optimization: modern methods of financial mathematics. American Mathematical Soc. *31* (2001).

[79] KRAJČOVIČOVÁ, Z., AND PÉREZ VELASCO, P. Model risk and differential geometry applied to sensitivity analysis. *Available at SSRN: https://ssrn.com/abstract=3204132* (2018).

[80] KRAJČOVIČOVÁ, Z., PÉREZ VELASCO, P., AND VÁZQUEZ, C. A new approach to quantification of model risk for practitioners. Accepted for publication in Journal of Computational Finance (2018).

[81] KRAJČOVIČOVÁ, Z., PÉREZ VELASCO, P., AND VÁZQUEZ, C. Quantification of model risk: Data uncertainty. In International Conference on Geometric Science of Information. Lecture notes. Springer, Cham (November 2017), 523–531.

[82] KRUSE, T., SCHNEIDER, J. C., AND SCHWEIZER, N. What's in a ball? constructing and characterizing uncertainty sets. Chichester, West Sussex, UK: John Wiley & Sons (2011).

[83] KYDLAND, F. E., AND PRESCOTT, E. C. Time to build and aggregate fluctuations. Econometrica *50* (November 1982), 1345–70.

[84] LAURENCE, P. Quantitative modeling of derivative securities: From theory to practice. R*outledge* (2017).

[85] MCGUIRE, W. Best practices for reducing model risk exposure. B*ank Accounting & Finance 9–14, April–May* (2007).

[86] MORINI, M. Understanding and managing model risk: A practical guide for quants, traders and validators. C*hichester, West Sussex, UK: John Wiley* & *Sons* (2011).

[87] MORINI, M. Understanding and managing model risk: A practical guide for quants, traders and validators. John Wiley & Sons (2011).

[88] MUKHERJEE, S., WU, Q., AND ZHOU, D. X. Learning gradients on manifolds. B*ernoulli* (2010), 181–207.

[89] MURRAY, M. K., AND RICE, J. Differential geometry and statistics. M*onographs on Statistics & Applied Probability, 48. Florida, USA: CRC Press* (1993).

[90] NAVAB, N., PLUIM, J.P.W., AND VIERGEVER, M.A. Medical image computing and computer-assisted intervention (miccai 2010). *Berlin, Germany:* Springer–Verlag (2010).

[91] NICOLAESCU, L. I. Lectures on the geometry of manifolds.

[92] ON BANKING SUPERVISION (BCBS), B. C. International convergence of capital measurement and capital standards. `http://www.bis.org/publ/bcbs107.htm` (2006).

[93] ON BANKING SUPERVISION (BCBS), B. C. Revisions to the basel ii market risk framework. `http://www.bis.org/publ/bcbs193.pdf` (2011).

[94] ON THE GLOBAL FINANCIAL SYSTEM, C. Stress testing by large financial institutions: Current practice and aggregation issues. *Bank for International Settlement* (April, 2000).

[95] RAO, C. R. Information and accuracy attainable in the estimation of statistical parameters. B*ull. Calcutta. Math. Soc. 37(3)* (1945), 81–91.

[96] RAO, C. R. Differential metrics in probability spaces. D*ifferential geometry in statistical inference 10* (1987), 217–240.

[97] REBONATO, R. Theory and practice of model risk management. modern risk management: A history. RiskWaters Group, London (2002), 223–248.

[98] RISK NET. Data processing model of bank credit evaluation system. `http://www.risk.net/data/basel/pdf/basel_risk_0404_tech1.pdf` (2004).

[99] ROBBIN, J. Extrinsic differential geometry. Lecture notes `http://www.math.wisc.edu/~robbin/` (2008).

[100] SAID, S., BOMBRUN, L., AND BERTHOUMIEU, Y. Warped riemannian metrics for location–scale models. a*rXiv preprint arXiv:1707.07163* (2017).

[101] SAID, S., BOMBRUN, L., BERTHOUMIEU, Y., AND MANTON, J. H., . Riemannian gaussian distributions on the space of symmetric positive definite matrices. IEEE Transactions on Information Theory *63(4)* (2017), 2153–2170.

[102] SALTELLI, A., TARANTOLA, S., CAMPOLONGO, F., AND RATTO, M. Sensitivity analysis in practice: A guide to assessing scientific models. Chichester, UK: John Wiley & Sons (2013).

[103] SCANDIZZO, S. The validation of risk models: A handbook for practitioners. S*pringer* (2016).

[104] SHIMA, H., AND YAGI, K. Geometry of hessian manifolds. D*ifferential Geometry and its Applications 7(3)* (1997), 277–290.

[105] SIBBERTSEN, P., STAHL, G., AND LUEDTKE, C. Measuring model risk. Hannover Economic Papers (HEP) *dp–409, Leibniz Universität Hannover, Wirtschaftswissenschaftliche Fakultät* (2008).

[106] SIMS, C. A. Macroeconomics and reality. Econometrica *48* (January 1980), 1–48.

[107] STANTADER, G. Annual report banco santander. *Available at* `www.santander.com/csgs` (2014).

[108] STERN, A. l^p change of variables inequalities on manifolds. arXiv preprint arXiv:1004.0401 (2010).

[109] SWANY, V. Basel III: Implications for indian banking. I*BS Hyderabad* (2015).

[110] THE EUROPEAN COMMISSION. The Directive on Markets in Financial Instruments repealing Directive 2044/39/EC and the Regulation on Markets in Financial Instruments MiFID II/MiFIR. The EU Official Journal (2014).

[111] THE EUROPEAN PARLIAMENT AND COUNCIL. Solvency II Directive (Directive 2009/138/EC). The EU Parliament and Council (2014).

[112] THE EUROPEAN PARLIAMENT AND COUNCIL. The Capital Requirements Regulation and Directive (CRR/CRD). The EU Parliament and Council (2016).

[113] THE EUROPEAN PARLIAMENT AND THE COUNCIL OF THE EUROPEAN UNION. The capital adequacy of investment firms and credit institutions (recast), directive 2006/49/ec. `http://eur-lex.europa.eu/legal-content` (14 June 2006).

[114] THE INSTITUTE OF RISK MANAGEMENT. Operational risk modelling: common practices and future development. I*nternational Model Industry Forum, Executive Summary* (2015).

[115] TWEEDIE, M. C. K. Statistical properties of inverse gaussian distributions . The Annals of Mathematical Statistics *28* (1957), 362–377.

[116] UNITED STATES FEDERAL LAW . Dodd–frank wall street reform and consumer protection act. h*ttps://www.gpo.gov/fdsys/pkg/PLAW-111publ203/html/PLAW-111publ203.htm* (2010).

[117] WANG, J. N., Y. J. H., AND CHENG, N. Y. P. How accurate is the square–root–of–time rule in scaling tail risk: A global study. *Journal of Banking and Finance 35(5)* (2011), 1158–1169.

[118] WILLMORE, T. Riemannian geometry. O*xford, UK: Oxford University Press* (1993).