A low-order mixed finite element method for a class of quasi-Newtonian Stokes flows. Part I: a priori error analysis

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Abstract

We present a mixed finite element method for a class of non-linear Stokes models arising in quasi-Newtonian fluids. Our results include, as a by-product, a new mixed scheme for the linear Stokes equation. The approach is based on the introduction of both the flux and the tensor gradient of the velocity as further unknowns, which yields a twofold saddle point operator equation as the resulting variational formulation. We prove that the continuous and discrete formulations are well posed, and derive the associated a priori error analysis. The corresponding Galerkin scheme is defined by using piecewise constant functions and Raviart–Thomas spaces of lowest order.

Keywords: Mixed finite element method; Twofold saddle point formulation; Stokes equation

1. Introduction

In the recent papers [3,16] we analyzed dual-mixed formulations for non-linear boundary value problems in plane elasticity. In the case of incompressible materials, we considered the non-Newtonian model from [5,7], and applied the dual-mixed approach from [11] to study its solvability and finite element approximations. Since the non-linear constitutive law depends on the strain tensor, we introduced this variable and the rotation as further unknowns, which yielded a twofold saddle point operator equation as the resulting variational formulation. Then, we extended the well known PEERS space and defined a stable Galerkin scheme, for which a Bank–Weiser type a posteriori error analysis was also developed.

The purpose of the present paper is to extend those results to the case of quasi-Newtonian flows whose kinematic viscosities are a non-linear monotone function of the gradient of the velocity of the fluid. Actually, since this constitutive equation does not depend on the strain tensor but just on the above

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mentioned gradient, the present analysis is much simpler than in [3,16], and leads to a stable Galerkin scheme with low-order finite element subspaces. Indeed, the extended PEERS space is not needed any more, and it suffices to consider piecewise constant functions and Raviart–Thomas spaces of order zero. In addition, the monotonicity certainly includes the linear case, and hence we obtain as a by-product a new mixed finite element method for the usual Stokes equations.

In order to describe the boundary value problem of interest, we now let Ω be a bounded and simply connected domain in $\mathbf{R}^2$ with Lipschitz-continuous boundary Γ. Our purpose is to determine the velocity $\mathbf{u} := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and the pressure p of a non-linear Stokes fluid occupying the region Ω under the action of an external force. More precisely, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for $(\mathbf{u}, p)$ in appropriate spaces such that

$$\begin{aligned} & -\mathbf{div}(\psi(|\nabla \mathbf{u}|)\nabla \mathbf{u} - p\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega, \\ & \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\mathbf{div}$ and div are the usual vector and scalar divergence operators, $\nabla \mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $\mathbf{R}^2$, $\mathbf{I}$ is the identity matrix of $\mathbf{R}^{2\times 2}$, and $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is the non-linear kinematic viscosity function of the fluid. We remark that $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$ must satisfy the compatibility condition $\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu} \, \mathrm{d}s = 0$, where $\boldsymbol{\nu}$ is the unit outward normal to Γ. Hereafter, given any Hilbert space S, we denote by S^2 and $S^{2\times 2}$ the spaces of vectors and tensors of order 2, respectively, with entries in S, provided with the product norms induced by the norm of S. In addition, for any $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbf{R}^{2\times 2}$, we adopt the notations $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^2 \tau_{ii}$, $\boldsymbol{\zeta} : \boldsymbol{\tau} := \sum_{i,j=1}^2 \zeta_{ij}\tau_{ij}$, and $\boldsymbol{\tau}^t := (\tau_{ji})$.

The kind of non-linear Stokes problem given by (1.1) appears in the modeling of a large class of non-Newtonian fluids (see, e.g. [2,18,19,23]). In particular, the Ladyzhenskaya law for fluids with large stresses (see [18]), also known as power law, is given by $\psi(t) := \kappa_0 + \kappa_1 t^{\beta-2} \; \forall t \in \mathbf{R}^+$, with $\kappa_0 \geqslant 0$, $\kappa_1 > 0$, and $\beta > 1$, and the Carreau law for viscoplastic flows (see, e.g. [19,23]) reads $\psi(t) := \kappa_0 + \kappa_1(1 + t^2)^{(\beta-2)/2} \; \forall t \in \mathbf{R}^+$, with $\kappa_0 \geqslant 0$, $\kappa_1 > 0$, and $\beta \geqslant 1$.

We now let $\psi_{ij} : \mathbf{R}^{2\times 2} \to \mathbf{R}$ be the mapping given by $\psi_{ij}(\mathbf{r}) := \psi(|\mathbf{r}|) r_{ij}$ for all $\mathbf{r} := (r_{ij}) \in \mathbf{R}^{2\times 2}$, for all $i, j \in \{1, 2\}$, and define the tensor $\boldsymbol{\psi} : \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ by $\boldsymbol{\psi}(\mathbf{r}) := (\psi_{ij}(\mathbf{r}))$ for all $\mathbf{r} \in \mathbf{R}^{2\times 2}$. Then, throughout this paper we assume that ψ is of class C^1 and that there exist $C_1, C_2 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbf{R}^{2\times 2}$, there holds

$$|\psi_{ij}(\mathbf{r})| \leqslant C_1 \|\mathbf{r}\|_{\mathbf{R}^{2\times 2}}, \qquad \left| \frac{\partial}{\partial r_{kl}} \psi_{ij}(\mathbf{r}) \right| \leqslant C_1 \quad \forall i, j, k, l \in \{1, 2\} \tag{1.2}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} \psi_{ij}(\mathbf{r}) s_{ij} s_{kl} \geqslant C_2 \|\mathbf{s}\|_{\mathbf{R}^{2\times 2}}^2. \tag{1.3}$$

It is easy to check that the Carreau law satisfies (1.2) and (1.3) for all $\kappa_0 > 0$, and for all $\beta \in [1, 2]$. In particular, with $\beta = 2$ we recover the usual linear Stokes model.

We recall here that the non-linear model satisfying the power law (with $\kappa_0 = 0$) was studied in [20] by using a dual-mixed variational formulation based on inverting the relation $\tilde{\boldsymbol{\sigma}} = \psi(|\nabla \mathbf{u}|)\nabla \mathbf{u}$ to obtain $\nabla \mathbf{u}$ as an explicit function of $\tilde{\boldsymbol{\sigma}}$. It is important to emphasize that this relation is required to perform the corresponding integration by parts, after multiplying by a suitable test function (see also Eq. (1.5) below), which is the starting point in the derivation of variational formulations of dual-mixed type. However, we remark that this procedure cannot be applied to the Carreau law since such explicit inversion formula is not available in this case.

Certainly, one could also deal with (1.1) without requiring the inversion of that relation. In fact, multiplying the partial differential equations by test functions $\mathbf{v} \in [H^1(\Omega)]^2$ and $q \in L^2(\Omega)$, and integrating by parts, we get

$$
\begin{aligned}
&\int_\Omega \psi(|\nabla \mathbf{u}|)\nabla \mathbf{u} : \nabla \mathbf{v}\,\mathrm{d}x - \int_\Omega p \operatorname{div} \mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x \quad \forall \mathbf{v} \in [H^1(\Omega)]^2, \\
&\int_\Omega q \operatorname{div} \mathbf{u}\,\mathrm{d}x = 0 \quad \forall q \in L^2(\Omega),
\end{aligned} \tag{1.4}
$$

which constitutes the usual primal-mixed variational formulation of (1.1) (see, e.g. [17] for the well known linear case). In this setting, the velocity $\mathbf{u}$ lives in the space $[H^1(\Omega)]^2$, and hence the corresponding finite element subspace needs to be a subset of the continuous functions. In addition, the Dirichlet boundary condition, being essential and non-homogeneous, cannot be incorporated neither in the continuous and discrete formulations nor in the definitions of the spaces involved, and therefore one is necessarily lead to a non-conforming Galerkin scheme.

Instead of primal-mixed methods, in the present work we are interested in a dual-mixed variational formulation for the boundary value problem (1.1). In this case the velocity $\mathbf{u}$ becomes an unknown in $[L^2(\Omega)]^2$, which gives more flexibility to choose the associated finite element subspace (in particular, piecewise constant functions becomes a feasible choice). In addition, the Dirichlet boundary condition, being now natural, is incorporated directly into the right hand sides (linear functionals) of the continuous and discrete formulations. Indeed, multiplying $\nabla \mathbf{u}$ by a test function $\boldsymbol{\tau}$ and integrating by parts, we obtain

$$
\int_\Omega \nabla \mathbf{u} : \boldsymbol{\tau}\,\mathrm{d}x = -\int_\Omega \mathbf{u} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \int_\Gamma \boldsymbol{\tau}\boldsymbol{\nu} \cdot \mathbf{u}\,\mathrm{d}s, \tag{1.5}
$$

which, after replacing $\mathbf{u}|_\Gamma$ by $\mathbf{g}$, yields

$$
\int_\Omega \mathbf{u} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \int_\Omega \nabla \mathbf{u} : \boldsymbol{\tau}\,\mathrm{d}x = \int_\Gamma \boldsymbol{\tau}\boldsymbol{\nu} \cdot \mathbf{g}\,\mathrm{d}s. \tag{1.6}
$$

Another important advantage of using a dual-mixed method lies on the possibility of introducing further unknowns with a clear physical meaning. In this way, they are approximated directly, which avoids any numerical postprocessing yielding additional sources of error. Then, the conservativity properties are transferred to some of these unknowns (for instance, continuity of the normal components of the stresses), which, as we will show below, can also be approximated with finite elements of very low order.

On the other hand, it is important to mention that additional variables such as $\nabla \mathbf{u}$ and other tensors are also used in least-squares finite element methods (see, e.g. [6]). In this approach, the saddle point optimization arising from a primal-mixed formulation like (1.4) is replaced by an unconstrained minimization leading to symmetric and positive definite systems that are much easier to solve than the primal-mixed Galerkin scheme.

The mixed finite element method proposed in the present paper simply relies on the introduction of the stress and gradient of the velocity tensors as auxiliary unknowns, and it does not require any inversion process, whence the resulting variational formulation shows, as in [3,16], a twofold saddle point structure. Therefore, the abstract theory for this kind of operator equation (see, e.g. [8,9,11,14,15]), which constitutes a generalization of the well known Babuška–Brezzi theory, can also be applied to the present situation. In particular, efficient iterative methods to solve the associated linear systems are available (see, e.g. [12,13]). The extension of this approach to kinematic viscosity functions not satisfying (1.2) or (1.3), which includes the Carreau law with $\kappa_0 = 0$ or $\beta > 2$, will be reported in a separate work.

The rest of this paper is organized as follows. In Section 2 we derive the continuous variational formulation of (1.1) and prove that it is well posed. We include here a subsection containing the main abstract results for the solvability and Galerkin approximations of twofold saddle point operator equations. Finally, the associated mixed finite element scheme is studied in Section 3. We introduce there the finite element subspaces of low order, prove that the discrete scheme is uniquely solvable, and derive the corresponding quasi-optimal error estimate and rate of convergence.

2. The continuous variational formulation

2.1. The twofold saddle point equation

We introduce first $\boldsymbol{\sigma} := \psi(|\nabla \mathbf{u}|)\nabla \mathbf{u} - p\mathbf{I}$ and $\mathbf{t} := \nabla \mathbf{u}$ in Ω as additional unknowns. In this way, according to the definition of the tensor $\boldsymbol{\psi}$, the non-linear constitutive law and the equilibrium equation become, respectively,

$$\boldsymbol{\sigma} = \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in } \Omega. \tag{2.1}$$

In addition, since $\operatorname{div}(\mathbf{u}) = \operatorname{tr}(\nabla \mathbf{u})$, the incompressibility condition can be rewritten as $\operatorname{tr}(\mathbf{t}) = 0$ in Ω. Consequently, multiplying the relation $\mathbf{t} = \nabla \mathbf{u}$ by a tensor $\boldsymbol{\tau}$, integrating by parts, using that $\mathbf{u} = \mathbf{g}$ on Γ (see (1.6)), and then testing appropriately the equations of (2.1) and the incompressibility of the fluid, we arrive at the following mixed variational formulation of (1.1): Find $(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s}\,\mathrm{d}x - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\,\mathrm{d}x - \int_\Omega p\operatorname{tr}(\mathbf{s})\,\mathrm{d}x = 0,\\
&-\int_\Omega \boldsymbol{\tau} : \mathbf{t}\,\mathrm{d}x - \int_\Omega q\operatorname{tr}(\mathbf{t})\,\mathrm{d}x - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x = -\langle \boldsymbol{\tau}\boldsymbol{v}, \mathbf{g}\rangle_\Gamma,\\
&-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x
\end{aligned} \tag{2.2}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q, \mathbf{v}) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$.

Hereafter, $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing of $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product, and $H(\mathbf{div};\Omega)$ is the space of tensors $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ satisfying $\mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2$. It is well known that $H(\mathbf{div};\Omega)$, provided with the inner product $\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\boldsymbol{\zeta}, \mathbf{div}\boldsymbol{\tau}\rangle_{[L^2(\Omega)]^2}$, is a Hilbert space, where $\langle \cdot, \cdot \rangle_{[L^2(\Omega)]^{2\times 2}}$ and $\langle \cdot, \cdot \rangle_{[L^2(\Omega)]^2}$ stand for the usual inner products of $[L^2(\Omega)]^{2\times 2}$ and $[L^2(\Omega)]^2$, respectively.

Before continuing the analysis, we remark that (2.2) is not uniquely solvable since adding $(0, c\mathbf{I}, -c, 0)$ to $(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u})$, for any $c \in \mathbf{R}$, yields further solutions of this problem. Therefore, in order to guarantee uniqueness, we proceed as in [1] (see also [4,16]) and require additionally that $\int_\Omega \operatorname{tr}\boldsymbol{\sigma}\,\mathrm{d}x = 0$, which leads to the introduction of a Lagrange multiplier $\xi \in \mathbf{R}$ as a further unknown.

Consequently, the mixed variational formulation of (1.1) is re-stated as follows: Find $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \xi) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbf{R}$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s}\,\mathrm{d}x - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\,\mathrm{d}x - \int_\Omega p\operatorname{tr}(\mathbf{s})\,\mathrm{d}x = 0,\\
&-\int_\Omega \boldsymbol{\tau} : \mathbf{t}\,\mathrm{d}x - \int_\Omega q\operatorname{tr}(\mathbf{t})\,\mathrm{d}x - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \xi\int_\Omega \operatorname{tr}\boldsymbol{\tau}\,\mathrm{d}x = -\langle \boldsymbol{\tau}\boldsymbol{v}, \mathbf{g}\rangle_\Gamma,\\
&-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,\mathrm{d}x + \eta\int_\Omega \operatorname{tr}\boldsymbol{\sigma}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x
\end{aligned} \tag{2.3}$$

for all $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, q, \mathbf{v}, \eta) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div}; \Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbf{R}$. We note here that one knows in advance that $\xi = 0$. In fact, it suffices to take $\boldsymbol{\tau} = \mathbf{I}$ and $q = -1$ in the second equation of (2.3), and use the compatibility condition for the Dirichlet data $\mathbf{g}$. However, we do keep this artificial unknown since it is needed to insure the symmetry of the whole formulation.

Next, we notice that (2.3) has a twofold saddle point structure. Indeed, let us introduce the spaces $X_1 := [L^2(\Omega)]^{2\times 2}$, $M_1 := H(\mathbf{div}; \Omega) \times L^2(\Omega)$, $M := [L^2(\Omega)]^2 \times \mathbf{R}$, and define the operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, and $\mathbf{B} : M_1 \to M'$, and the functionals $(\mathbf{G}, \mathbf{F}) \in M_1' \times M'$, as follows:

$$[\mathbf{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \boldsymbol{\psi}(\mathbf{r}) : \mathbf{s}\, dx, \quad [\mathbf{B}_1(\mathbf{r}), (\boldsymbol{\tau}, q)] := -\int_\Omega \boldsymbol{\tau} : \mathbf{r}\, dx - \int_\Omega q\, \mathrm{tr}(\mathbf{r})\, dx, \tag{2.4}$$

$$[\mathbf{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \eta)] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx + \eta \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\, dx, \tag{2.5}$$

$$[\mathbf{G}, (\boldsymbol{\tau}, q)] := -\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g} \rangle_\Gamma \quad \text{and} \quad [\mathbf{F}, (\mathbf{v}, \eta)] := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \tag{2.6}$$

for all $\mathbf{r}, \mathbf{s} \in X_1$, $(\boldsymbol{\tau}, q) \in M_1$, and $(\mathbf{v}, \eta) \in M$, where $[\cdot, \cdot]$ stands for the duality pairing induced by the corresponding operators and functionals.

Then, it is easy to see that (2.3) can also be stated as: Find $\vec{\mathbf{t}} := (\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \xi)) \in X_1 \times M_1 \times M$ such that

$$\begin{aligned} &[\mathbf{A}_1(\mathbf{t}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{s}), (\boldsymbol{\sigma}, p)] = 0, \\ &[\mathbf{B}_1(\mathbf{t}), (\boldsymbol{\tau}, q)] + [\mathbf{B}(\boldsymbol{\tau}, q), (\mathbf{u}, \xi)] = [\mathbf{G}, (\boldsymbol{\tau}, q)], \\ &[\mathbf{B}(\boldsymbol{\sigma}, p), (\mathbf{v}, \eta)] = [\mathbf{F}, (\mathbf{v}, \eta)] \end{aligned} \tag{2.7}$$

for all $\vec{\mathbf{s}} := (\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \eta)) \in X_1 \times M_1 \times M$.

The abstract theory for this kind of twofold saddle point operator equation is already available (see, e.g. [8,9]), and their main results are collected in the following subsection.

2.2. Abstract theory for twofold saddle point equations

Let X_1, M_1, and M be Hilbert spaces, and consider a non-linear operator $\mathbf{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : M_1 \to M'$, with adjoints $\mathbf{B}_1' : M_1 \to X_1'$ and $\mathbf{B}' : M \to M_1'$, respectively. Then, we are interested in the following non-linear variational problem: *Given* $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$, *find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *such that*

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O} \\ \mathbf{B}_1 & \mathbf{O} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{H} \\ \mathbf{G} \\ \mathbf{F} \end{pmatrix}. \tag{2.8}$$

We have the following theorem.

Theorem 2.1. *Let* $\tilde{M}_1 := \ker(\mathbf{B})$, *define* $V_1 := \{\mathbf{s} \in X_1 : [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0\ \forall \boldsymbol{\tau} \in \tilde{M}_1\}$, *and let* $\Pi_1 : X_1' \to V_1'$ *be the canonical imbedding defined by* $\Pi_1(\mathbf{H}) = \mathbf{H}|_{V_1}$ *for all* $\mathbf{H} \in X_1'$. *Assume that*

(i) *the non-linear operator* $\mathbf{A}_1 : X_1 \to X_1$ *is Lipschitz-continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\tilde{\mathbf{t}} \in X_1$, *the non-linear operator* $\Pi_1 \mathbf{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone with a monotonicity constant* $\alpha > 0$ *independent of* $\tilde{\mathbf{t}}$.

(ii) *there exists* $\beta > 0$ *such that for all* $v \in M$

$$\sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{||\boldsymbol{\tau}||_{M_1}} \geqslant \beta ||v||_M; \tag{2.9}$$

(iii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in \tilde{M}_1$

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \geqslant \beta_1 ||\boldsymbol{\tau}||_{M_1}. \tag{2.10}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *solution of* (2.8). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$||(\mathbf{t}, \boldsymbol{\sigma}, u)||_{X_1\times M_1\times M} \leqslant C\{||\mathbf{H}|| + ||\mathbf{G}|| + ||\mathbf{F}|| + ||\mathbf{A}_1(0)||\}.$$

Proof. See Theorem 2.4 in [8] (see also Theorem 2.1 in [14], Theorem 1 in [9], or Theorem 4.1 in [15]). □

Now, let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively. Then the Galerkin scheme associated with (2.8) reads as follows: Given $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$, find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned}
&[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] = [\mathbf{H}, \mathbf{s}_h],\\
&[\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\tau}_h), u_h] = [\mathbf{G}, \boldsymbol{\tau}_h],\\
&[\mathbf{B}(\boldsymbol{\sigma}_h), v_h] = [\mathbf{F}, v_h]
\end{aligned} \tag{2.11}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \in X_{1,h} \times M_{1,h} \times M_h$.

The discrete analogue of Theorem 2.1 is established next.

Theorem 2.2. *Let* $\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : [\mathbf{B}(\boldsymbol{\tau}_h), v_h] = 0\ \forall v_h \in M_h\}$, *define* $V_{1,h} := \{\mathbf{s}_h \in X_{1,h} : [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0\ \forall \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$ *and let* $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ *be the canonical imbedding. Further, let* $\mathbf{A}_{1,h} := p_h' \mathbf{A}_1 : X_1 \to X_{1,h}'$ *where* $p_h : X_{1,h} \to X_1$ *is the canonical injection with adjoint* $p_h' : X_1' \to X_{1,h}'$. *Assume that*

(i) *the non-linear operator* $\mathbf{A}_{1,h} : X_1 \to X_{1,h}'$ *is Lipschitz-continuous with a Lipschitz constant* $\gamma_h > 0$, *and for any* $\tilde{\mathbf{t}} \in X_{1,h}$, *the non-linear operator* $\Pi_{1,h}\mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V_{1,h}'$ *is strongly monotone with a monotonicity constant* $\alpha_h > 0$ *independent of* $\tilde{\mathbf{t}}$.

(ii) *there exists* $\beta_h > 0$ *such that for all* $v_h \in M_h$

$$\sup_{\substack{\boldsymbol{\tau}_h\in M_{1,h}\\ \boldsymbol{\tau}_h\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), v_h]}{||\boldsymbol{\tau}_h||_{M_1}} \geqslant \beta_h ||v_h||_M; \tag{2.12}$$

(iii) *there exists* $\beta_{1,h} > 0$ *such that for all* $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h\in X_{1,h}\\ \mathbf{s}_h\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \geqslant \beta_{1,h} ||\boldsymbol{\tau}_h||_{M_1}. \tag{2.13}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *solution of* (2.11). *Moreover, there exists* $C_h > 0$, *independent of the solution, such that*

$$||(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||_{X_1 \times M_1 \times M} \leqslant C_h\{||\mathbf{H}_h|| + ||\mathbf{G}_h|| + ||\mathbf{F}_h|| + ||\mathbf{A}_{1,h}(0)||\},$$

where $\mathbf{H}_h := \mathbf{H}|_{X_{1,h}}$, $\mathbf{G}_h := \mathbf{G}|_{M_{1,h}}$, *and* $\mathbf{F}_h := \mathbf{F}|_{M_h}$.

Proof. See Theorem 3.2 in [8] (see also Theorem 3.1 in [14], Theorem 3 in [9], or Theorem 4.2 in [15]). □

Finally, concerning the error analysis, we have the following result.

Theorem 2.3. *Assume that all the hypotheses of both Theorems* 2.1 *and* 2.2 *are satisfied, and let* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *be the unique solutions of* (2.8) *and* (2.11), *respectively. In addition, suppose that there exist positive constants* $\tilde{\gamma}$, $\tilde{\alpha}$, $\tilde{\beta}$, *and* $\tilde{\beta}_1$ *such that* $\gamma_h \leqslant \tilde{\gamma}$, $\alpha_h \geqslant \tilde{\alpha}$, $\beta_h \geqslant \tilde{\beta}$, *and* $\beta_{1,h} \geqslant \tilde{\beta}_1$ *for all h. Then, there exists* $C > 0$, *independent of h, such that the following Céa error estimate holds*:

$$||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)|| \leqslant C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in X_{1,h} \times M_{1,h} \times M_h}} ||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)||. \tag{2.14}$$

Proof. See Section 4 in [8] (see also Theorem 3.3 in [14] or Theorem 5 in [9]). □

2.3. Solvability of the continuous formulation

We need the following technical result.

Lemma 2.1. *Let* $H_0(\mathbf{div}; \Omega) := \{\boldsymbol{\tau} \in H(\mathbf{div}; \Omega) : \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau})\,\mathrm{d}x = 0\}$. *Then, for any* $\mathbf{v} \in [L^2(\Omega)]^2$ *there holds*

$$\sup_{\substack{\hat{\boldsymbol{\tau}} \in H_0(\mathbf{div};\Omega) \\ \hat{\boldsymbol{\tau}} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\hat{\boldsymbol{\tau}}\,\mathrm{d}x}{||\hat{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}}.$$

Proof. Since $H_0(\mathbf{div}; \Omega) \subseteq H(\mathbf{div}; \Omega)$, we easily see that

$$\sup_{\substack{\hat{\boldsymbol{\tau}} \in H_0(\mathbf{div};\Omega) \\ \hat{\boldsymbol{\tau}} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\hat{\boldsymbol{\tau}}\,\mathrm{d}x}{||\hat{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}} \leqslant \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}}. \tag{2.15}$$

Now, it is not difficult to realize that $H(\mathbf{div}; \Omega) = H_0(\mathbf{div}; \Omega) \oplus \mathbf{R}\mathbf{I}$. Then, given a tensor $\boldsymbol{\tau} \in H(\mathbf{div}; \Omega)$ with $\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x > 0$, we let $\hat{\boldsymbol{\tau}} \in H_0(\mathbf{div}; \Omega)$ and $c \in \mathbf{R}$ such that $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}} + c\mathbf{I}$. It follows that $\mathbf{div}\boldsymbol{\tau} = \mathbf{div}\hat{\boldsymbol{\tau}}$ and $||\boldsymbol{\tau}||^2_{[L^2(\Omega)]^{2\times 2}} = ||\hat{\boldsymbol{\tau}}||^2_{[L^2(\Omega)]^{2\times 2}} + 2c^2|\Omega|$, which yields $||\boldsymbol{\tau}||^2_{H(\mathbf{div};\Omega)} = ||\hat{\boldsymbol{\tau}}||^2_{H(\mathbf{div};\Omega)} + 2c^2|\Omega|$, and hence $||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)} \geqslant ||\hat{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}$. According to the above we deduce that

$$\frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} = \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\hat{\boldsymbol{\tau}}\,\mathrm{d}x}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} \leqslant \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\hat{\boldsymbol{\tau}}\,\mathrm{d}x}{||\hat{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}},$$

which implies that

$$\sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} \leqslant \sup_{\substack{\hat{\boldsymbol{\tau}} \in H_0(\mathbf{div};\Omega) \\ \hat{\boldsymbol{\tau}} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\hat{\boldsymbol{\tau}}\,\mathrm{d}x}{||\hat{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}}. \tag{2.16}$$

In this way, (2.15) and (2.16) provide the required result. □

The main theorem concerning the solvability of (2.3) can be established now.

Theorem 2.4. *There exists a unique* $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \xi) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbf{R}$ *solution of problem* (2.3). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|\vec{\mathbf{t}}\| \leqslant C\{\|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2}\}.$$

Proof. The proof reduces to show that the hypotheses of Theorem 2.1 are satisfied by the formulation (2.7). In fact, we first observe that for each $\tilde{\mathbf{r}} \in X_1$ the Gâteaux derivative $\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ is a bilinear form on $X_1 \times X_1$, which is uniformly bounded and uniformly X_1-elliptic. In fact, using the definitions of $\mathbf{A}_1$ and $\boldsymbol{\psi}$, we find that

$$\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) = \int_\Omega \left\{ \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \psi_{ij}(\tilde{\mathbf{r}}) r_{kl} s_{ij} \right\} \mathrm{d}x \quad \forall \mathbf{r}, \mathbf{s} \in X_1,$$

which, according to (1.2) and (1.3), implies the existence of positive constants $\tilde{C}_1$ and $\tilde{C}_2$, such that

$$|\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})| \leqslant \tilde{C}_1 \|\mathbf{r}\|_{X_1} \|\mathbf{s}\|_{X_1} \quad \text{and} \quad \mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \geqslant \tilde{C}_2 \|\mathbf{s}\|_{X_1}^2 \tag{2.17}$$

for all $\tilde{\mathbf{r}}, \mathbf{r}, \mathbf{s} \in X_1$. Then, it is well known that the above properties yield the strong monotonicity and Lipschitz continuity of the non-linear operator $\mathbf{A}_1$.

We now check that the linear operator $\mathbf{B}$ verifies the inf–sup condition on $M_1 \times M$. Given $(\mathbf{v}, \eta) \in M$, we have

$$\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{v},\eta)]}{\|(\boldsymbol{\tau},q)\|_{M_1}} \geqslant \frac{[\mathbf{B}(\eta\mathbf{I},0),(\mathbf{v},\eta)]}{\|\eta\mathbf{I}\|_{H(\mathbf{div};\Omega)}} = (2|\Omega|)^{1/2}|\eta|. \tag{2.18}$$

Next, using Lemma 2.1 in the first equality below, we deduce that

$$\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{v},\eta)]}{\|(\boldsymbol{\tau},q)\|_{M_1}} \geqslant \sup_{\substack{\hat{\boldsymbol{\tau}}\in H_0(\mathbf{div};\Omega)\\ \hat{\boldsymbol{\tau}}\neq 0}} \frac{[\mathbf{B}(\hat{\boldsymbol{\tau}},0),(\mathbf{v},\eta)]}{\|\hat{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau},0),(\mathbf{v},0)]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)\\ \boldsymbol{\tau}\neq 0}} \frac{-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}},$$

which, similarly as shown in Theorem 4.3 of [11], yields the existence of $\hat{\beta} > 0$ such that

$$\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{v},\eta)]}{\|(\boldsymbol{\tau},q)\|_{M_1}} \geqslant \hat{\beta}\|\mathbf{v}\|_{[L^2(\Omega)]^2}. \tag{2.19}$$

Therefore, (2.18) and (2.19) provide the continuous inf–sup condition for $\mathbf{B}$.

We now introduce the null space of the operator $\mathbf{B}$, that is, $\tilde{M}_1 := \{(\boldsymbol{\tau},q) \in M_1 : [\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{v},\eta)] = 0 \ \forall (\mathbf{v},\eta) \in M\}$, which gives $\tilde{M}_1 := \{(\boldsymbol{\tau},q) \in M_1 : \mathbf{div}\,\boldsymbol{\tau} = 0 \text{ in } \Omega \text{ and } \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\,\mathrm{d}x = 0\}$. It follows that there exists $\beta_1 > 0$ such that for all $(\boldsymbol{\tau},q) \in \tilde{M}_1$ there holds

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),(\boldsymbol{\tau},q)]}{\|\mathbf{s}\|_{X_1}} \geqslant \beta_1\|(\boldsymbol{\tau},q)\|_{M_1}. \tag{2.20}$$

In fact, we prove (2.20), the continuous inf–sup condition for $\mathbf{B}_1$, by bounding below the expression

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),(\boldsymbol{\tau},q)]}{\|\mathbf{s}\|_{X_1}}$$

with suitable choices of $\mathbf{s} \in X_1$. If $\|q\|_{L^2(\Omega)} \leqslant \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$ we take $\mathbf{s} := -(\boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I})$ and then apply the equivalence result given by Lemma 3.1 in [1] for tensors $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ satisfying $\int_\Omega \mathrm{tr}(\boldsymbol{\tau})\,\mathrm{d}x = 0$. Similarly, if $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \leqslant \|q\|_{L^2(\Omega)}$ we just consider $\mathbf{s} := -q\mathbf{I} + \boldsymbol{\tau}$. We omit further details and refer the reader to Theorem 3.1 below for a similar procedure.

Consequently, noting that $\mathbf{A}_1(0)$ is the null functional, a straightforward application of Theorem 2.1 completes the proof. □

3. The mixed finite element scheme

We assume for simplicity that Γ is a polygonal curve, and let $\{\mathscr{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ by triangles T of diameter h_T such that $h := \max\{h_T : T \in \mathscr{T}_h\}$ and $\bar{\Omega} = \cup\{T : T \in \mathscr{T}_h\}$. For each $T \in \mathscr{T}_h$ we let $\mathrm{RT}_0(T)$ be the local Raviart–Thomas space of order zero, that is,

$$\mathrm{RT}_0(T) := \mathrm{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right\},$$

where $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ is a generic vector of $\mathbf{R}^2$. In addition, given a non-negative integer k and a subset $\mathscr{S}$ of $\mathbf{R}^2$, we let $\mathbf{P}_k(\mathscr{S})$ be the space of polynomials defined on $\mathscr{S}$ of degree $\leqslant k$.

Then we define the following finite element subspaces:

$$X_{1,h} := \{\mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \mathbf{s}|_T \in [\mathbf{P}_0(T)]^{2\times 2}\ \forall T \in \mathscr{T}_h\},$$

$$M^{\boldsymbol{\sigma}}_{1,h} := \left\{\boldsymbol{\tau} := (\tau_{ij}) \in H(\mathbf{div};\Omega) : (\tau_{i1}\tau_{i2})^t|_T \in \mathrm{RT}_0(T)\ \forall i \in \{1,2\},\ \forall T \in \mathscr{T}_h\right\},$$

$$M^p_{1,h} := \{q \in L^2(\Omega) : q|_T \in \mathbf{P}_0(T)\ \forall T \in \mathscr{T}_h\},$$

$$M_{1,h} := M^{\boldsymbol{\sigma}}_{1,h} \times M^p_{1,h},$$

$$M^{\mathbf{u}}_h := \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_T \in [\mathbf{P}_0(T)]^2\ \forall T \in \mathscr{T}_h\}$$

and

$$M_h := M^{\mathbf{u}}_h \times \mathbf{R}.$$

Hence, the Galerkin scheme associated with (2.7) is: Find $\vec{\mathbf{t}}_h := (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \xi_h)) \in X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned}&[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), (\boldsymbol{\sigma}_h, p_h)] = 0,\\&[\mathbf{B}_1(\mathbf{t}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h, \xi_h)] = [\mathbf{G}, (\boldsymbol{\tau}_h, q_h)],\\&[\mathbf{B}(\boldsymbol{\sigma}_h, p_h), (\mathbf{v}_h, \eta_h)] = [\mathbf{F}, (\mathbf{v}_h, \eta_h)]\end{aligned} \tag{3.1}$$

for all $\vec{\mathbf{s}}_h := (\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)) \in X_{1,h} \times M_{1,h} \times M_h$.

The following theorem establishes that (3.1) is well posed and provides the corresponding quasi-optimal error estimate.

Theorem 3.1. *There exists a unique* $\vec{\mathbf{t}}_h \in X_{1,h} \times M_{1,h} \times M_h$ *solution of the Galerkin scheme* (3.1). *Moreover, there exist* c, $C > 0$, *independent of* h, *such that*

$$\|\vec{\mathbf{t}}_h\| \leqslant c\{\|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2}\}$$

and

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\| \leqslant C \inf_{\vec{\mathbf{s}}_h \in X_{1,h} \times M_{1,h} \times M_h} \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|.$$

Proof. We now apply the abstract Theorems 2.2 and 2.3. To this end, and since we already established that $\mathbf{A}_1$ is strongly monotone and Lipschitz-continuous, it only remains to show that $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the corresponding discrete inf–sup conditions with constants independent of h.

Given $(\mathbf{v}_h, \eta_h) \in M_h$, we have the discrete analogue of (2.18), that is,

$$\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h} \\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} \geqslant \frac{[\mathbf{B}(\eta_h \mathbf{I}, 0), (\mathbf{v}_h, \eta_h)]}{\|\eta_h \mathbf{I}\|_{H(\mathbf{div};\Omega)}} = (2|\Omega|)^{1/2} |\eta_h|. \tag{3.2}$$

Now, since $M^{\boldsymbol{\sigma}}_{1,h} = \hat{M}^{\boldsymbol{\sigma}}_{1,h} \oplus \mathbf{R}\mathbf{I}$, with $\hat{M}^{\boldsymbol{\sigma}}_{1,h} = M^{\boldsymbol{\sigma}}_{1,h} \cap H_0(\mathbf{div};\Omega)$, we also deduce, using the discrete analogue of Lemma 2.1 in the first equality below, that

$$\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h} \\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} \geqslant \sup_{\substack{\hat{\boldsymbol{\tau}}_h \in \hat{M}^{\boldsymbol{\sigma}}_{1,h} \\ \hat{\boldsymbol{\tau}}_h \neq 0}} \frac{[\mathbf{B}(\hat{\boldsymbol{\tau}}_h, 0), (\mathbf{v}_h, \eta_h)]}{\|\hat{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h, 0), (\mathbf{v}_h, 0)]}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}$$

$$= \sup_{\substack{\boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{-\int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h \, \mathrm{d}x}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}$$

and hence

$$\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h} \\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} \geqslant \sup_{\boldsymbol{\tau}_{h,i} \neq 0} \frac{-\int_\Omega v_{h,i} \mathrm{div}\, \boldsymbol{\tau}_{h,i} \, \mathrm{d}x}{\|\boldsymbol{\tau}_{h,i}\|_{H(\mathrm{div};\Omega)}} \quad \forall i \in \{1,2\}, \tag{3.3}$$

where $v_{h,i}$ and $\boldsymbol{\tau}_{h,i}$ are the ith component and ith row of the vector $\mathbf{v}_h$ and tensor $\boldsymbol{\tau}_h$, respectively. Then, using the properties of the equilibrium interpolation operator (see, e.g. [4,22]), as we did in Lemma 5.6 of [15] (see also Lemma 4.3 of [21]), one can show that there exists $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{h,i} \neq 0} \frac{-\int_\Omega v_{h,i} \mathrm{div}\, \boldsymbol{\tau}_{h,i} \, \mathrm{d}x}{\|\boldsymbol{\tau}_{h,i}\|_{H(\mathrm{div};\Omega)}} \geqslant \tilde{\beta} \|v_{h,i}\|_{L^2(\Omega)} \quad \forall i \in \{1,2\}. \tag{3.4}$$

In this way, (3.2)–(3.4) imply the discrete inf–sup condition for $\mathbf{B}$.

On the other hand, the discrete kernel of $\mathbf{B}$ is defined by

$$\tilde{M}_{1,h} := \{(\boldsymbol{\tau}_h, q_h) \in M_{1,h} : [\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)] = 0 \ \forall (\mathbf{v}_h, \eta_h) \in M_h\},$$

which, according to the definition of $\mathbf{B}$ and the properties of the subspaces $M^{\boldsymbol{\sigma}}_{1,h}$ and M_h, yields $\tilde{M}_{1,h} = \tilde{M}^{\boldsymbol{\sigma}}_{1,h} \times M^p_{1,h}$, with $\tilde{M}^{\boldsymbol{\sigma}}_{1,h} := \{\boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h} : \mathbf{div}\, \boldsymbol{\tau}_h = 0 \text{ in } \Omega \text{ and } \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h)\, \mathrm{d}x = 0\}$.

We prove now the discrete inf–sup condition for $\mathbf{B}_1$. Given $(\boldsymbol{\tau}_h, q_h) \in \tilde{M}_{1,h}$, we assume first that $\|q_h\|_{L^2(\Omega)} \leqslant \|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}$ and define $\tilde{\mathbf{s}}_h := -(\boldsymbol{\tau}_h - \frac{1}{2} \mathrm{tr}\,(\boldsymbol{\tau}_h)\mathbf{I})$, which verifies $\mathrm{tr}\,(\tilde{\mathbf{s}}_h) = 0$. Since $\mathbf{div}\, \boldsymbol{\tau}_h = 0$ in Ω, we deduce that $\tilde{\mathbf{s}}_h$ belongs to $X_{1,h}$, and hence

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\mathbf{s}_h\|_{X_1}} \geqslant \frac{[\mathbf{B}_1(\tilde{\mathbf{s}}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\tilde{\mathbf{s}}_h\|_{X_1}} = \left\| \boldsymbol{\tau}_h - \frac{1}{2} \mathrm{tr}\,(\boldsymbol{\tau}_h) \mathbf{I} \right\|_{[L^2(\Omega)]^{2\times 2}}.$$

Thus, applying the equivalence result given by Lemma 3.1 in [1], we conclude that there exists $\tilde{\beta}_1 > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\mathbf{s}_h\|_{X_1}} \geqslant \tilde{\beta}_1 \|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \geqslant \frac{\tilde{\beta}_1}{2} \|(\boldsymbol{\tau}_h, q_h)\|_{M_1}. \tag{3.5}$$

Next, we assume that $\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \leqslant \|q_h\|_{L^2(\Omega)}$ and define $\hat{\mathbf{s}}_h := -q_h\mathbf{I} + \boldsymbol{\tau}_h$. It follows easily that $\hat{\mathbf{s}}_h \in X_{1,h}$, and therefore

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\mathbf{s}_h\|_{X_1}} \geqslant \frac{[\mathbf{B}_1(\hat{\mathbf{s}}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\hat{\mathbf{s}}_h\|_{X_1}} = \frac{2\|q_h\|^2_{L^2(\Omega)} - \|\boldsymbol{\tau}_h\|^2_{H(\mathbf{div};\Omega)}}{\|-q_h\mathbf{I} + \boldsymbol{\tau}_h\|_{[L^2(\Omega)]^{2\times 2}}} \geqslant \hat{\beta}_1 \|q_h\|_{L^2(\Omega)} \geqslant \frac{\hat{\beta}_1}{2} \|(\boldsymbol{\tau}_h, q_h)\|_{M_1},$$

with $\hat{\beta}_1 = \frac{1}{2(1+\sqrt{2})}$. This inequality and (3.5) provide the discrete inf–sup condition for $\mathbf{B}_1$.

Consequently, a straightforward application of Theorems 2.2 and 2.3 completes the proof. □

We now recall the following approximation properties of the subspaces $X_{1,h}$, $M^{\boldsymbol{\sigma}}_{1,h}$, $M^{p}_{1,h}$, and $M^{\mathbf{u}}_{h}$, respectively, which follow from classical error estimates for projection and equilibrium interpolation operators (see, e.g. [22])

($\mathrm{AP}_{1,h}$) For all $\mathbf{s} \in [H^1(\Omega)]^{2\times 2}$ there exists $\mathbf{s}_h \in X_{1,h}$ such that

$\|\mathbf{s} - \mathbf{s}_h\|_{[L^2(\Omega)]^{2\times 2}} \leqslant Ch\|\mathbf{s}\|_{[H^1(\Omega)]^{2\times 2}}$.

($\mathrm{AP}^{\boldsymbol{\sigma}}_{1,h}$) For all $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$ with $\mathbf{div}\,\boldsymbol{\tau} \in [H^1(\Omega)]^2$, there exists $\boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h}$ such that

$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \leqslant Ch\{\|\boldsymbol{\tau}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\boldsymbol{\tau}\|_{[H^1(\Omega)]^2}\}$.

($\mathrm{AP}^{p}_{1,h}$) For all $q \in H^1(\Omega)$ there exists $q_h \in M^{p}_{1,h}$ such that

$\|q - q_h\|_{L^2(\Omega)} \leqslant Ch\|q\|_{H^1(\Omega)}$.

($\mathrm{AP}^{\mathbf{u}}_{h}$) For all $\mathbf{v} \in [H^1(\Omega)]^2$ there exists $\mathbf{v}_h \in M^{\mathbf{u}}_{h}$ such that

$\|\mathbf{v} - \mathbf{v}_h\|_{[L^2(\Omega)]^2} \leqslant Ch\|\mathbf{v}\|_{[H^1(\Omega)]^2}$.

Then we have the following result on the rate of convergence of the mixed finite element scheme (3.1).

Theorem 3.2. *Let* $\vec{\mathbf{t}} := (\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \xi))$ *and* $\vec{\mathbf{t}}_h := (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \xi_h))$ *be the unique solutions of the continuous and discrete formulations* (2.7) *and* (3.1), *respectively. Assume that* $\mathbf{t} \in [H^1(\Omega)]^{2\times 2}$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^{2\times 2}$, $\mathbf{div}\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $p \in H^1(\Omega)$, *and* $\mathbf{u} \in [H^1(\Omega)]^2$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\| \leqslant Ch\Big\{\|\mathbf{t}\|_{[H^1(\Omega)]^{2\times 2}} + \|\boldsymbol{\sigma}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\boldsymbol{\sigma}\|_{[H^1(\Omega)]^2} + \|p\|_{H^1(\Omega)} + \|\mathbf{u}\|_{[H^1(\Omega)]^2}\Big\}.$$

Proof. It is a consequence of the Céa estimate from Theorem 3.1 and the approximation properties stated above. □

Finally, we remark that an a posteriori error analysis yielding a reliable and quasi-efficient estimate for our mixed finite element method, together with several numerical results, are provided in the second part of this work (see [10]).

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