Stabilized dual-mixed method for the problem of linear elasticity with mixed boundary conditions

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Abstract

We extend the applicability of the augmented dual-mixed method introduced recently in [4, 5] to the problem of linear elasticity with mixed boundary conditions. The method is based on the Hellinger–Reissner principle and the symmetry of the stress tensor is imposed in a weak sense. The Neuman boundary condition is prescribed in the finite element space. Then, suitable Galerkin least-squares type terms are added in order to obtain an augmented variational formulation which is coercive in the whole space. This allows to use any finite element subspaces to approximate the displacement, the Cauchy stress tensor and the rotation.

Key words: Mixed finite element, stabilization, linear elasticity, mixed boundary condition.
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1. Introduction

Mixed finite element methods are typically used in linear elasticity to avoid the effects of locking. They also allow to approximate directly unknowns of physical interest, such as the stresses. We consider here the mixed method of Hellinger and Reissner, that provides simultaneous approximations of the displacement $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$. The symmetry of the stress tensor prevents the extension of the standard dual-mixed formulation of the Poisson equation to this case. In general, the symmetry of $\boldsymbol{\sigma}$ is imposed weakly, through the introduction of the rotation as an additional unknown, and stable mixed finite elements for the linear elasticity problem involve many degrees of freedom (see, for instance, [2]).

The application of stabilization techniques allows to use simpler finite element subspaces, including convenient equal-order interpolations that are generally unstable within the mixed approach. Recently, a new stabilized mixed finite element method was presented in [3] for the problem of linear elasticity in the plane. This method leads to a well-posed, locking-free Galerkin scheme for any choice of finite element subspaces when homogeneous Dirichlet boundary conditions are prescribed. Moreover, in the simplest case it requires less degrees of freedom than the classical PEERS or BDM. The method was successfully extended in [4] to the case of non-homogeneous Dirichlet boundary conditions; the three-dimensional version can be found in [5].

The case of mixed boundary conditions, which is the most usual in practice, was studied only in [3]. There the non-homogeneous Neumann boundary condition is imposed in a weak sense, which entails some difficulties. Namely, it is necessary to introduce the trace of the displacement on the Neumann boundary as an extra unknown and the resulting variational formulation has a saddle-point structure. As a consequence, the well-posedness and convergence of the Galerkin scheme has to be studied for each particular choice of discrete spaces. In particular, the method proposed in [3] requires the use of an independent mesh of the Neumann boundary that has to satisfy a compatibility condition with the mesh induced by the triangulation of the domain. This turns the implementation of the method difficult, specially if one wants to apply adaptive refinement algorithms. On the other hand, due to some technical difficulties, it is not possible to extend the method analyzed in [4,5] to the case of mixed boundary conditions if the Neumann boundary condition is imposed weakly.

In this work, we consider the mixed method of Hellinger and Reissner and impose the Neumann boundary condition in a strong sense. Then, we follow [4, 5] and add appropriate Galerkin least-squares type terms. In this way, we avoid the difficulties from [3] and extend the results from [4, 5] to the linear elasticity problem with mixed boundary conditions. Specifically, we obtain a Galerkin scheme that is well-posed and free of locking for any choice of finite element subspaces. As an example, we define a family of finite element subspaces and provide the corresponding error estimates.

The rest of the paper is organized as follows. In Section 2, we recall the usual dual-mixed variational formulation of the linear elasticity problem with mixed boundary conditions. In Section 3, we introduce and analyze the augmented dual-mixed variational formulation. Finally, in Section 4, we analyze the corresponding discrete scheme and show that it is well-posed and free of locking for any choice of finite element subspaces. We also provide optimal error estimates for a family of finite element subspaces. Throughout the paper C denotes a generic constant.

2. Dual-mixed variational formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded and simply connected domain with a Lipschitz-continuous boundary Γ, and let Γ_D and Γ_N be two disjoint subsets of Γ such that Γ_D and Γ_N have positive measure and $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. We consider the problem of linear elasticity with non-homogeneous mixed boundary conditions, that is, given a volume force $\mathbf{f} \in [L^2(\Omega)]^d$, a prescribed displacement $\mathbf{u}_D \in [H^{1/2}(\Gamma_D)]^d$ and a traction $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^d$, we look for the displacement vector field $\mathbf{u}$ and the stress tensor field $\boldsymbol{\sigma}$ of an isotropic linear elastic material occupying the region Ω :

$$\left\{ \begin{array}{ll} -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f}, & \boldsymbol{\sigma} = \mathcal{C}\, \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma_D, & \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N, \end{array} \right. \tag{2.1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, $\mathbf{n}$ is the unit outward normal to Γ, and $\mathcal{C}$ is the elasticity operator determined by Hooke's law, that is,

$$\mathcal{C}\, \boldsymbol{\zeta} \;:=\; \lambda \operatorname{tr}(\boldsymbol{\zeta})\, \mathbf{I} \;+\; 2\, \mu\, \boldsymbol{\zeta}\,, \quad \forall\, \boldsymbol{\zeta} \,\in\, [L^2(\Omega)]^{d\times d}\,,$$

where $\mathbf{I}$ denotes the identity matrix of $\mathbb{R}^{d\times d}$ and λ, $\mu > 0$ are the Lamé parameters. We recall that the inverse operator $\mathcal{C}^{-1}$ is known explicitly:

$$\mathcal{C}^{-1}\, \boldsymbol{\zeta} \;:=\; \frac{1}{2\,\mu}\, \boldsymbol{\zeta} \;-\; \frac{\lambda}{2\,\mu\,(d\,\lambda + 2\,\mu)} \operatorname{tr}(\boldsymbol{\zeta})\, \mathbf{I}\,, \qquad \forall\, \boldsymbol{\zeta} \,\in\, [L^2(\Omega)]^{d\times d}\,.$$

Our aim is to extend the applicability of the augmented mixed finite element method introduced in [4, 5] to problem (2.1). With that purpose, we consider the mixed method of Hellinger and

Reissner, that provides simultaneous approximations of the displacement $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$, and impose the symmetry of $\boldsymbol{\sigma}$ in a weak sense, through the introduction of the rotation $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})$ as an additional unknown. Then, the inverted constitutive law can be written in the form

$$\nabla\mathbf{u} - \boldsymbol{\gamma} = \mathcal{C}^{-1}\boldsymbol{\sigma}, \quad \text{in } \Omega. \tag{2.2}$$

We multiply the equilibrium equation and equation (2.2) by tests functions, integrate by parts and use the Dirichlet boundary condition to obtain the following dual-mixed variational formulation of problem (2.1): Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H_{\mathbf{g}} \times Q$ such that

$$\left\{ \begin{array}{lcll} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},(\mathbf{u},\boldsymbol{\gamma})) & = & \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_{\Gamma_D}, & \forall\, \boldsymbol{\tau} \in H_{\mathbf{0}}, \\ b(\boldsymbol{\sigma},(\mathbf{v},\boldsymbol{\eta})) & = & -\displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}, & \forall\,(\mathbf{v},\boldsymbol{\eta}) \in Q, \end{array} \right. \tag{2.3}$$

where $H_{\mathbf{g}} := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \boldsymbol{\tau}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N\}$, $H_{\mathbf{0}} := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \boldsymbol{\tau}\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N\}$, $Q := [L^2(\Omega)]^d \times [L^2(\Omega)]^{d\times d}_{\mathtt{skew}}$, and the bilinear forms $a : H(\mathbf{div};\Omega) \times H(\mathbf{div};\Omega) \to \mathbb{R}$ and $b : H(\mathbf{div};\Omega) \times Q \to \mathbb{R}$ are defined by

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\zeta} : \boldsymbol{\tau}, \qquad b(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}, \tag{2.4}$$

for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ and for all $(\mathbf{v},\boldsymbol{\eta}) \in Q$.

We remark that, in contrast to [3], we impose the Neumann boundary condition in a strong sense. This will allow us to derive an augmented variational formulation that will be coercive in the whole space, so that the corresponding Galerkin scheme will be well-posed and free of locking for any choice of finite element subspaces.

In what follows we assume, without lost of generality, that $\mathbf{g} = \mathbf{0}$ and consider $a : H_{\mathbf{0}} \times H_{\mathbf{0}} \to \mathbb{R}$ and $b : H_{\mathbf{0}} \times Q \to \mathbb{R}$. Next we state an auxiliary result that will be applied to prove that problem (2.3) is well-posed.

Lemma 2.1. *There exists $C > 0$, depending only on Γ_N and Ω, such that*

$$C\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} \le \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{d\times d}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^d}, \quad \forall\,\boldsymbol{\tau} \in H_{\mathbf{0}},$$

where $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{d}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$ denotes the deviator of tensor $\boldsymbol{\tau}$.

Proof. For $d = 2$, apply Lemmas 2.1 and 2.2 in [3]. The proof for $d = 3$ is analogous. □

Theorem 2.1. *Problem* (2.3) *is well-posed.*

Proof. It is clear that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined in (2.4) are both bounded. On the other hand, applying Lemma 2.1 it is easy to see that $a(\cdot,\cdot)$ is coercive in the kernel of $b(\cdot,\cdot)$, $V = \{\boldsymbol{\tau} \in H_{\mathbf{0}} : \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{in } \Omega\}$, with a coercivity constant that only depends on Γ_N and Ω. Finally, the inf-sup condition for $b(\cdot,\cdot)$ in $H_{\mathbf{0}} \times Q$ is proved in Lemma 4.3 in [1] for $d = 2$; the proof for $d = 3$ is analogous. □

3. The augmented dual-mixed variational formulation

Now, we follow the ideas in [4, 5] and enrich the dual-mixed variational formulation (2.3) with residuals arising from the constitutive and equilibrium equations, and from the relation that defines the rotation as a function of the displacement. We also add a term dealing with the Dirichlet boundary condition. The problem consists in finding $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H} := H_{\mathbf{0}} \times [H^1(\Omega)]^d \times [L^2(\Omega)]^{d\times d}_{\mathtt{skew}}$ such that

$$A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \,=\, L(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}\,, \tag{3.1}$$

where the bilinear form $A : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and the linear functional $L : \mathbf{H} \to \mathbb{R}$ are defined by

$$\begin{aligned} A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \,&:=\, a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) \,-\, b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) \\ &+\, \kappa_1 \int_\Omega \big(\, \boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{C}^{-1}\, \boldsymbol{\sigma} \,\big) : \big(\, \boldsymbol{\varepsilon}(\mathbf{v}) + \mathcal{C}^{-1}\, \boldsymbol{\tau} \,\big) \,+\, \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{div}(\boldsymbol{\tau}) \\ &+\, \kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}) \right) : \left(\boldsymbol{\eta} + \frac{1}{2}(\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathtt{t}}) \right) \,+\, \kappa_4 \int_{\Gamma_D} \mathbf{u} \cdot \mathbf{v}, \end{aligned} \tag{3.2}$$

and

$$L(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \,:=\, \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_{\Gamma_D} \,+\, \int_\Omega \mathbf{f} \cdot (\, \mathbf{v} \,-\, \kappa_2\, \mathbf{div}(\boldsymbol{\tau}) \,) \,+\, \kappa_4 \int_{\Gamma_D} \mathbf{u}_D \cdot \mathbf{v}\,, \tag{3.3}$$

for all $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}$, where the stabilization parameters, κ_1, κ_2, κ_3 and κ_4, are positive numbers.

In what follows, we consider the following inner product in $\mathbf{H}$:

$$\langle (\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \rangle_{\mathbf{H}} \,:=\, (\boldsymbol{\sigma}, \boldsymbol{\tau})_{H(\mathbf{div};\,\Omega)} + (\mathbf{u}, \mathbf{v})_{[H^1(\Omega)]^d} + (\boldsymbol{\gamma}, \boldsymbol{\eta})_{[L^2(\Omega)]^{d\times d}}\,,$$

and denote the corresponding induced norm by $||\cdot||_{\mathbf{H}}$. We remark that $\mathbf{H}$ is a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathbf{H}}$. Our aim now is to derive sufficient conditions on the stabilization parameters, κ_1, κ_2, κ_3 and κ_4, that allow us to ensure that the bilinear form $A(\cdot, \cdot)$ is strongly coercive and bounded in $\mathbf{H}$, with constants independent of λ.

Lemma 3.1. *There exists a constant $C_0 > 0$ such that*

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{d\times d}} + \|\mathbf{v}\|^2_{[L^2(\Gamma_D)]^d} \,\geq\, C_0\, \|\mathbf{v}\|^2_{[H^1(\Omega)]^d}, \quad \forall\, \mathbf{v} \in [H^1(\Omega)]^d\,.$$

Proof. The proof of this inequality is analogous to that of Lemma 3.1 in [4] for $d = 2$ and Lemma A.2 in [5] for $d = 3$. □

Theorem 3.1. *Assume that $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ are positive parameters independent of λ. Then, there exists a positive constant M, independent of λ, such that*

$$|\, A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}))\, | \,\leq\, M\, \|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})\|_{\mathbf{H}}\, \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}}\,, \quad \forall\, (\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}),\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \,\in\, \mathbf{H}\,.$$

If, moreover,

$$\kappa_1 < 2\,\mu \quad \text{ and } \quad \kappa_3 < C_0\, \min(\kappa_1 + \kappa_3, \kappa_4)\,, \tag{3.4}$$

then there exists a positive constant α, independent of λ, such that

$$A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \,\geq\, \alpha\, \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|^2_{\mathbf{H}}\,, \quad \forall (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \,\in\, \mathbf{H}\,,$$

and the augmented formulation (3.1) *is well-posed, with a stability constant independent of λ.*

Proof. It is easy to check that $A(\cdot,\cdot)$ and $L(\cdot)$ are bounded in $\mathbf{H}$, with constants independent of λ. Now, let $(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) \in \mathbf{H}$. Proceeding as in [4, 5], we obtain that

$$A((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) \;\geq\; \big(1-\frac{\kappa_1}{2\mu}\big)\frac{1}{2\mu}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{d\times d}} \,+\, \kappa_2\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^d}$$
$$+\,(\kappa_1+\kappa_3)\,\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{d\times d}} \,-\, \kappa_3|\mathbf{v}|^2_{[H^1(\Omega)]^d} \,+\, \kappa_4\|\mathbf{v}\|^2_{[L^2(\Gamma_D)]^d} \,+\, \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{d\times d}}\,.$$

Then, if $\kappa_1 \in (0, 2\mu)$, applying Lemmas 2.1 and 3.1, we obtain that

$$\begin{aligned} A((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) \quad &\geq \quad C\,\min\big(\big(1-\frac{\kappa_1}{2\mu}\big)\frac{1}{2\mu},\kappa_2\big)\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} \,+\, \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{d\times d}} \\ &+ \quad \big(C_0\,\min(\kappa_1+\kappa_3,\kappa_4) \,-\, \kappa_3\big)\,\|\mathbf{v}\|^2_{[H^1(\Omega)]^d}\,, \end{aligned}$$

where C is the constant of Lemma 2.1. So, under conditions (3.4), the bilinear form $A(\cdot,\cdot)$ is strongly coercive in $\mathbf{H}$ and the Theorem follows applying the Lax-Milgram Lemma. □

We remark that in practice we can choose $\kappa_4 \geq \kappa_1+\kappa_3$. In that case, if $C_0 \geq 1$, condition $(3.4)_2$ is fulfilled automatically.

4. The augmented mixed finite element method

Given a finite dimensional subspace $\mathbf{H}_h \subseteq \mathbf{H}$, the Galerkin scheme associated to (3.1) reads: find $(\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h) \in \mathbf{H}_h$ such that

$$A((\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h)) \,=\, L(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h) \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h) \in \mathbf{H}_h\,. \tag{4.1}$$

We have the following result.

Theorem 4.1. *Assume that the stabilization parameters, κ_1, κ_2, κ_3 and κ_4, satisfy the assumptions of Theorem 3.1 and let $\mathbf{H}_h$ be any finite element subspace of $\mathbf{H}$. Then, the Galerkin scheme* (4.1) *is well-posed and there exists a positive constant C, independent of h and λ, such that*

$$\|(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})-(\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h)\|_{\mathbf{H}} \,\leq\, C \inf_{(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h)\in\mathbf{H}_h} \|(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})-(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h)\|_{\mathbf{H}}\,. \tag{4.2}$$

Proof. The result follows from Theorem 3.1 and the Lax-Milgram Lemma. □

In order to define explicit finite element subspaces $\mathbf{H}_h$ of $\mathbf{H}$, we let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of meshes of $\bar{\Omega}$ made up of triangles if $d=2$ or tetrahedra if $d=3$. We assume that for all $h>0$, $\bar{\Omega} = \cup\{\,T : T\in\mathcal{T}_h\,\}$ and each vertex in $\bar{\Gamma}_D\cap\bar{\Gamma}_N$ is a vertex of $\mathcal{T}_h$. Given an element $T\in\mathcal{T}_h$, we denote by h_T its diameter and define the mesh size $h := \max\{\,h_T : T\in\mathcal{T}_h\,\}$. In addition, given an integer $k\geq 0$ and $T\in\mathcal{T}_h$, we denote by $\mathcal{P}_k(T)$ the space of polynomials in d variables defined in T of total degree less than or equal to k, and $\mathcal{RT}_k(T)$ denotes the local Raviart-Thomas space of order k (see [2]). Then, we define the finite element subspaces

$$H_h^{\boldsymbol{\sigma}} := \big\{\,\boldsymbol{\tau}_h \in H_{\mathbf{0}} \,:\, \boldsymbol{\tau}_h|_T \,\in\, [\mathcal{RT}_k(T)^{\mathtt{t}}]^d, \quad \forall\,T\,\in\,\mathcal{T}_h\,\big\}\,,$$

$$H_h^{\mathbf{u}} := \big\{\,\mathbf{v}_h \,\in\, [C(\bar{\Omega})]^d \,:\, \mathbf{v}_h|_T \,\in\, [\mathcal{P}_{k+1}(T)]^d, \quad \forall\,T\,\in\,\mathcal{T}_h\,\big\}\,,$$

$$H_h^{\boldsymbol{\gamma}} := \big\{\,\boldsymbol{\eta}_h \,\in\, [L^2(\Omega)]^{d\times d}_{\mathtt{skew}} \,:\, \boldsymbol{\eta}_h|_T \,\in\, [\mathcal{P}_k(T)]^{d\times d}, \quad \forall\,T\,\in\,\mathcal{T}_h\,\big\}\,,$$

where $^{\mathtt{t}}$ indicates that the rows of tensors in $H_h^{\boldsymbol{\sigma}}$ are piecewise local Raviart-Thomas elements.

Theorem 4.2. *Let* $\mathbf{H}_h := H_h^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}}$ *and let* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ *be the unique solutions to problems* (3.1) *and* (4.1)*, respectively. Then, if* $\boldsymbol{\sigma} \in [H^s(\Omega)]^{d\times d}$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in [H^s(\Omega)]^d$*,* $\mathbf{u} \in [H^{1+s}(\Omega)]^d$ *and* $\boldsymbol{\gamma} \in [H^s(\Omega)]^{d\times d}$*, for some* $1 \leq s \leq k+1$*, there exists* $C > 0$*, independent of* h *and* λ*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \leq$$

$$\leq C\, h^s \left(\|\boldsymbol{\sigma}\|_{[H^s(\Omega)]^{d\times d}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^s(\Omega)]^d} + \|\mathbf{u}\|_{[H^{1+s}(\Omega)]^d} + \|\boldsymbol{\gamma}\|_{[H^s(\Omega)]^{d\times d}} \right).$$

Proof. It is a consequence of Céa's estimate (4.2) and the approximation properties of the subspaces involved (see [2]). □

5. Conclusions

We extended the applicability of the augmented dual-mixed method introduced in [4, 5] to the linear elasticity problem with mixed boundary conditions. The main idea is to impose the Neumann boundary condition in a strong sense. This allows to obtain a Galerkin scheme that is well-posed and free of locking for any choice of finite element subspaces, improving the results in [3]. We propose a family of approximations and prove the corresponding error estimates. In the simplest case, the method presented here is less expensive and easier to implement than the one presented in [3], since we do not need to approximate the trace of the displacement on Γ_N and we only use one mesh of the domain. Finally, we emphasize that our results are valid in two and three dimensions.

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