ON A FEM–BEM FORMULATION FOR AN EXTERIOR QUASILINEAR PROBLEM IN THE PLANE*

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Abstract. We use a version of the FEM–BEM method introduced by Costabel [*Boundary Elements IX*, Vol. 1, C. A. Brebbia et al., eds., Springer-Verlag, 1987] and Han [*J. Comput. Math.*, 8 (1990), pp. 223–232] to discretize an exterior quasilinear problem. We provide error estimates for the Galerkin method and propose a fully discrete scheme based on simple quadrature formulas. Furthermore, we show that these numerical integration schemes preserve the optimal rates of convergence. Finally, we present results of numerical experiments involving our discretization method.

Key words. boundary element, finite element, nonlinear problems

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1. Introduction. In this paper we consider a discretization procedure for an exterior quasilinear problem which consists in a combination of finite elements (FEM) with boundary elements (BEM). There are two principal classes of FEM–BEM formulations. The first one relies on the Johnson–Nedelec method introduced in [14] for the Laplace equation. This formulation has also been used for FEM–BEM discretizations of exterior Stokes problems; cf. [22, 18]. The second one is based on the so-called *symmetric FEM–BEM approach* introduced independently by Costabel [5] and Han [12]. This approach turned out to be more suitable for the elasticity system; cf. [11, 6]. It has also been successfully generalized to nonlinear boundary value problems that become homogeneous and linear with constant coefficients outside a bounded region. In these extensions, the error analysis is always given when the coefficients satisfy conditions that make the nonlinear operator strongly monotone and Lipschitz continuous; cf. [11, 10, 6, 2]. The advantage in this case is that Céa's lemma is satisfied. What to do when these conditions do not hold is discussed here.

In [24] (cf. also [20, 21, 8, 15]), Xu provides a very powerful tool to deal with the numerical analysis of nonlinear problems on bounded domains when no version of Céa's lemma is available. The technique consists in linearizing the nonlinear partial differential equation at a given isolated solution (see hypothesis (2.6)) and considering its finite element discretization. We show here that this approach can be straight-forwardly extended to exterior nonlinear problems without using discrete Green's functions but at the expense of some restrictions on the type of nonlinearity. We could not handle the general case since no bounds are known for discrete Green's functions associated with FEM–BEM formulations.

The main result of this paper concerns contributions to the analysis of a fully discrete nonlinear FEM–BEM formulation. We had to answer two principal difficulties. On the one hand, unless the nonlinear operator is Lipschitz continuous and strongly monotone, Strang's lemma is not satisfied and there is no general framework to control

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the effect of numerical quadratures on convergence. We show here that the method described in [24] can be completed, at least for our quasilinear problem, in order to deal with the effect of numerical integration.

On the other hand, one has to take some care with the pseudosingular behavior of the kernels associated with the integral operators when using numerical quadratures. We present here a modified version of the symmetric FEM–BEM approach that allows us to avoid these singularities. We take advantage of the techniques given in [13, 7] to compute in the global matrix the coefficients corresponding to boundary integrals by quadrature formulas.

The rest of the paper is organized as follows. In section 2, we consider an exterior quasilinear equation as a model problem and make some regularity hypotheses on the coefficients and the solution of the continuous problem. Then, we present a new version of the symmetric FEM–BEM formulation. This new version is equivalent to the usual one at the continuous level, but it leads to a different discrete scheme that offers some additional advantages. In section 3, we describe the triangulation of the domain and prove a technical tool related to approximation of curved finite elements on Sobolev spaces with a noninteger index. The error analysis of the Galerkin scheme is given in section 4 and a family of full discretizations of the complete system of equations is presented in section 5. Estimates of the quadrature error are also reported in this section. Finally, section 6 is devoted to numerical experiments.

Sobolev spaces. Given an open set $\mathcal{O}$ in the plane, we consider the Hilbertian Sobolev spaces $H^m(\mathcal{O})$ endowed with their usual norms $\| \cdot \|_{m,\mathcal{O}}$. The corresponding seminorms are denoted $| \cdot |_{m,\mathcal{O}}$. The spaces $W^{m,\infty}(\mathcal{O})$ are those Sobolev spaces derived from $L^\infty(\mathcal{O})$ (cf.[1]). Their norms and seminorms are denoted by $\| \cdot \|_{m,\infty,\mathcal{O}}$ and $| \cdot |_{m,\infty,\mathcal{O}}$, respectively.

We will also consider periodic Sobolev spaces. Let $\mathcal{C}^\infty$ be the space of 1-periodic infinitely often differentiable real valued functions of a single variable. Given $g \in \mathcal{C}^\infty$, we define its Fourier coefficients

$$\widehat{g}(k) := \int_0^1 g(s)e^{-2k\pi\imath s}ds.$$

Then for $r \in \mathbf{R}$ we define the Sobolev space H^r to be the completion of $\mathcal{C}^\infty$ with the norm

$$\|g\|_r := \left(\sum_{k\in\mathbf{Z}}(1+|k|^2)^r|\widehat{g}(k)|^2 \right)^{1/2}.$$

It is well known (see [16]) that H^r are Hilbert spaces and that $H^{r_1} \subset H^{r_2}$ if $r_1 > r_2$, the inclusion being dense and compact. Moreover, the H^0-inner product

$$\langle \lambda, \mu \rangle := \int_0^1 \lambda(s)\mu(s)ds$$

can be extended to represent the duality of H^{-r} and H^r $\forall r$. We will keep the same notation for this duality bracket.

Throughout this paper, C, with or without subscripts, denotes a generic constant independent of the discretization parameter h.

2. The model problem. Let Ω_0 be a bounded and simply connected domain in $\mathbf{R}^2$ with sufficiently smooth boundary Γ_0. We consider a closed curve Γ_1 contained

in $\mathbf{R}^2 \setminus \overline{\Omega}_0$ and denote by Ω^{nl} the annular region bounded by Γ_0 and Γ_1. We also denote by Ω^l the complement of $\overline{\Omega}_0 \cup \overline{\Omega}^{nl}$ in $\mathbf{R}^2$.

We consider three continuous nonlinear functions $\beta_i : \overline{\Omega}^{nl} \times \mathbf{R} \to \mathbf{R}$ $(i = 0, 1, 2)$ such that the derivatives $(\partial \beta_i / \partial s)$, $(\partial^2 \beta_i / \partial s^2)$ and $(\partial \beta_i / \partial x_j)$, $(i = 0, 1, 2)$, $(j = 1, 2)$, are continuous in $\overline{\Omega}^{nl} \times \mathbf{R}$. We need to approximate a function u that satisfies

$$
\begin{aligned}
-\mathrm{div}\,(\nabla u + 1_{\Omega^{nl}}(x)\boldsymbol{\beta}(x,u)) + 1_{\Omega^{nl}}(x)\beta_0(x,u) &= 0 && \text{in} \quad \mathbf{R}^2 \setminus \overline{\Omega}_0, \\
u &= 0 && \text{on} \quad \Gamma_0, \\
u(x) &= O(1) && \text{as} \quad |x| \to \infty,
\end{aligned}
$$
(2.1)

where $\boldsymbol{\beta}(x,u) = (\beta_1(x,u), \beta_2(x,u))^T$ and $1_{\Omega^{nl}}(\cdot)$ is the indicator function of the set Ω^{nl}.

Some existence and uniqueness results for this type of problem are given in [11] under some conditions on the coefficients β_i. We will not dwell on such issues, but instead, we assume that (2.1) has at least one solution and provide error estimates for an approximate solution obtained from a FEM–BEM discretization scheme.

We introduce an artificial boundary Γ that contains in its interior the set $\overline{\Omega}_0 \cup \overline{\Omega}^{nl}$. Thus, the closed curve Γ divides Ω^l in two regions, a bounded domain denoted Ω_1^l and Ω_2^l, which is the unbounded region exterior to Γ. We denote $\Omega := \Omega^{nl} \cup \Gamma_1 \cup \Omega_1^l$ and introduce the space

$$
X := \{v \in H^1(\Omega); \quad v|_{\Gamma_0} = 0\}.
$$

We assume that u, the solution of problem (2.1), satisfies

$$
u|_\Omega \in X \cap H^{1+\sigma}(\Omega) \quad \text{with} \quad 0 < \sigma < 1.
$$

Notice that, even in the linear case, we cannot expect a greater amount of regularity ($\sigma = 1$) since, in general, the normal derivative of the solution has a jump across Γ_1. However, a Sobolev embedding theorem implies that such a solution is continuous in $\overline{\Omega}$ and thus, the semilinear form

$$
a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Omega^{nl}} \boldsymbol{\beta}(x,u) \cdot \nabla v \, dx + \int_{\Omega^{nl}} \beta_0(x,u) v \, dx
$$

is well defined $\forall v \in X$. It follows that u satisfies in Ω the following variational formulation:

$$
a(u,v) - \int_\Gamma \frac{\partial u}{\partial \nu} v \, d\sigma = 0 \quad \forall v \in X,
$$
(2.2)

where ν is the unit normal vector to Γ oriented from Ω to Ω_2^l.

The symmetric FEM–BEM method introduced in [5] consists in coupling (2.2) with two boundary integral identities relating the trace of u and the normal derivative $\frac{\partial u}{\partial \nu}$ to each other. These boundary integral equations arise from Green's representation formula of u in Ω_2^l and the jump of the layer potentials.

In contrast to the Johnson–Nédélec FEM–BEM coupling procedure [14], the symmetric FEM–BEM approach allows one to take the artificial boundary Γ polygonal. This is the choice made by all authors (cf. [5, 11, 10, 2, 19]) since, at first glance, this seems to be more convenient for the treatment of the discrete problem. Instead, we

assume here that Γ is a $\mathcal{C}^\infty$ boundary and parameterize this curve in order to change all functions defined on Γ by periodic functions.

Then let $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ be a smooth regular 1-periodic parametric representation of the curve Γ:

$$|\mathbf{x}'(s)| > 0 \qquad \forall s \in \mathbf{R}, \qquad \text{and} \qquad \mathbf{x}(t) \neq \mathbf{x}(s), \qquad 0 < |t - s| < 1.$$

We define by means of $\mathbf{x}(\,\cdot\,)$ the parameterized trace on Γ as the extension of

$$\gamma : \mathcal{C}^\infty(\overline{\Omega}) \to H^0,$$
$$u \mapsto \gamma u(\,\cdot\,) := u|_\Gamma(\mathbf{x}(\,\cdot\,))$$

to $H^1(\Omega)$. The resulting linear application $\gamma : H^1(\Omega) \to H^{1/2}$ is bounded and onto; cf. Theorem 8.15 of [16].

We now consider the following integral operators:

$$\mathcal{V}g(\,\cdot\,) := \int_0^1 V(\,\cdot\,, t)g(t)\, dt, \qquad\qquad \mathcal{K}g(\,\cdot\,) := \int_0^1 K(\,\cdot\,, t)g(t)\, dt,$$

where

$$V(s, t) := -\frac{1}{2\pi} \log|\mathbf{x}(s) - \mathbf{x}(t)| \quad \text{and} \quad K(s, t) := \frac{1}{2\pi} \frac{(\mathbf{x}(s) - \mathbf{x}(t)) \cdot \nu(\mathbf{x}(t))}{|\mathbf{x}(s) - \mathbf{x}(t)|^2} |\mathbf{x}'(t)|.$$

These operators are parameterized versions of the simple and double layer potentials, respectively. We recall some well-known properties of the integral operators $\mathcal{V}$ and $\mathcal{K}$.

LEMMA 2.1. *The operators* $\mathcal{K} : H^\theta \to H^\theta$ *and* $\mathcal{V} : H^\theta \to H^{\theta+1}$ *are bounded* $\forall \theta$. *Moreover there exists* $\alpha > 0$ *such that*

$$\langle \mu, \mathcal{V}\mu \rangle \geq \alpha \|\mu\|_{-1/2}^2 \qquad \forall \mu \in H_0^{-\frac{1}{2}},$$

where $H_0^{-\frac{1}{2}} := \{\mu \in H^{-\frac{1}{2}}; \quad \langle \mu, 1 \rangle = 0\}$.

Proof. Notice that $K : \mathbf{R}^2 \to \mathbf{R}$ is $\mathcal{C}^\infty$ and 1-periodic in both its variables. Then the first result follows from standard arguments of integral operators in Sobolev spaces. The proof of boundedness and $H_0^{-\frac{1}{2}}$-ellipticity of $\mathcal{V}$ may be found, for instance, in [16]. □

It is straightforward to show that by parameterizing the integrals on Γ in the traditional symmetric FEM–BEM method (cf. [11, 5, 12]) we arrive at the following global weak formulation of (2.1) (a similar strategy is used in [17] for the Johnson–Nédélec FEM–BEM method):

find $(u, \lambda) \in X \times H_0^{-\frac{1}{2}};$

$$(2.3) \qquad a(u, v) + \langle \tfrac{d}{dt}(\gamma v), \mathcal{V}\tfrac{d}{dt}(\gamma u) \rangle - \langle \lambda, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})\gamma v \rangle = 0 \qquad \forall v \in X,$$

$$\langle \mu, \mathcal{V}\lambda \rangle + \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})\gamma u \rangle = 0 \qquad \forall \mu \in H_0^{-\frac{1}{2}},$$

where $\mathcal{I}$ is the identity operator. The auxiliary unknown λ is related to the normal derivative of u on Γ by

$$\lambda(\,\cdot\,) := \frac{\partial u}{\partial \nu}(\mathbf{x}(\,\cdot\,))|\mathbf{x}'(\,\cdot\,)|.$$

The zero mean value condition imposed on λ is a consequence of the asymptotic behavior of u. Without this condition u would behave like $c \log|x| + O(1)$, with $c \neq 0$, when $|x| \to \infty$.

Let $\mathbf{W} := X \times H_0^{-\frac{1}{2}}$. We simplify the notation and denote $\mathbf{u} := (u, \lambda)$, $\mathbf{v} := (v, \mu)$ and $\mathbf{z} := (z, \xi)$ the elements (u, λ), (v, μ), and (z, ξ) of $\mathbf{W}$. The space $\mathbf{W}$ is provided with its natural Hilbertian norm:

$$\|\mathbf{v}\|_W^2 := \|v\|_{1,\Omega}^2 + \|\mu\|_{-1/2}^2.$$

We introduce the semilinear form

$$A(\mathbf{u}, \mathbf{v}) := a(u, v) + d(\mathbf{u}, \mathbf{v}),$$

where

$$d(\mathbf{u}, \mathbf{v}) := b(\lambda, \mu) + b\left(\frac{d}{dt}(\gamma u), \frac{d}{dt}(\gamma v)\right) - c(\lambda, \gamma v) + c(\mu, \gamma u)$$

with the following definitions of the bilinear forms $b(\,\cdot\,,\,\cdot\,)$ and $c(\,\cdot\,,\,\cdot\,)$:

$$b(\lambda, \mu) := \langle \mu, \mathcal{V}\lambda \rangle, \qquad c(\mu, \gamma v) := \left\langle \mu, \left(\frac{1}{2}\mathcal{I} - \mathcal{K}\right)\gamma v \right\rangle.$$

We easily deduce from Lemma 2.1 and the continuity of γ and $\frac{d}{dt} : H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$ that $d(\,\cdot\,,\,\cdot\,) : \mathbf{W} \times \mathbf{W} \to \mathbf{R}$ is bounded. In terms of these notations, problem (2.3) is reduced to

(2.4)
$$\begin{aligned} &\text{find } \mathbf{u} \in \mathbf{W}; \\ &A(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{W}. \end{aligned}$$

Let us consider the bilinear form $A'(u;\,\cdot\,,\,\cdot\,) : \mathbf{W} \times \mathbf{W} \to \mathbf{R}$ defined by

$$\begin{aligned} A'(u; \mathbf{v}, \mathbf{z}) := \int_\Omega \nabla v \cdot \nabla z \, dx &+ \int_{\Omega^{nl}} (\partial \boldsymbol{\beta}/\partial s)(x, u) \cdot \nabla z v \, dx \\ &+ \int_{\Omega^{nl}} (\partial \beta_0/\partial s)(x, u) v z \, dx + d(\mathbf{v}, \mathbf{z}), \end{aligned}$$

where we denoted $(\partial \boldsymbol{\beta}/\partial s)(x, u) := ((\partial \beta_1/\partial s)(x, u), (\partial \beta_2/\partial s)(x, u))^T$.

Let $\mathbf{W}'$ be the dual of $\mathbf{W}$ pivotal to $L^2(\Omega) \times H^0$ and denote by $(\,\cdot\,,\,\cdot\,)$ the duality bracket between $\mathbf{W}'$ and $\mathbf{W}$. Notice that $A'(u;\,\cdot\,,\,\cdot\,)$ is bounded on $\mathbf{W} \times \mathbf{W}$ since the functions $(\partial \beta_i/\partial s)(\,\cdot\,, u(\,\cdot\,))$, $(i = 0, 1, 2)$, are continuous in $\overline{\Omega}^{nl}$. Then there exists an operator $\mathcal{A} : \mathbf{W} \to \mathbf{W}'$ such that

$$(\mathcal{A}\mathbf{v}, \mathbf{z}) = A'(u; \mathbf{v}, \mathbf{z}) \quad \forall \mathbf{v}, \mathbf{z} \in \mathbf{W}.$$

By virtue of Lemma 2.1, it is straightforward that for a sufficiently large constant $k \geq 0$ there exists $\alpha_1 > 0$ such that

(2.5)
$$(\mathcal{A}\mathbf{v}, \mathbf{v}) + k\|v\|_{0,\Omega}^2 \geq \alpha_1 \|\mathbf{v}\|_W^2 \quad \forall \mathbf{v} \in \mathbf{W}.$$

Let $i : X \to X'$ be the canonical injection. As X is compactly embedded in $L^2(\Omega)$, we deduce that operator $\mathcal{J} : \mathbf{W} \to \mathbf{W}'$ defined by $\mathcal{J}(\mathbf{v}) := (i(v), 0)$ is also compact. Thus, the Fredholm alternative applies for $\mathcal{A}$. We assume here that

(2.6)
$$A'(u; \mathbf{v}, \mathbf{z}) = 0 \quad \forall \mathbf{z} \in \mathbf{W} \quad \Longrightarrow \quad \mathbf{v} = 0.$$

This implies that $\mathcal{A} : \mathbf{W} \to \mathbf{W}'$ is an isomorphism.

3. Finite elements with curved triangles. For simplicity of exposition, in the rest of the paper we restrict ourselves to polygonal boundaries Γ_0 and Γ_1. Given $h := 1/N$, with N a positive integer, let $\{t_i := (i-1)\,h; \quad i = 1, \cdots, N+1\}$ be the induced uniform partition of $[0,1]$. We denote by Ω_h the polygonal domain whose vertices lying on Γ are $\Delta_h = \{\mathbf{x}(t_i) : i = 1, \ldots, N+1\}$. Let τ_h be a triangulation of $\overline{\Omega}_h$ by triangles T of diameter h_T not greater than $\max|\mathbf{x}'(s)|h$. We assume that any vertex of a triangle lying on the exterior boundary of $\partial\Omega_h$ belongs to Δ_h. We also suppose that Γ_1 does not cut through any element of τ_h and that the family of triangulations $\{\tau_h\}_h$ is quasi-uniform, i.e., it is regular and there exists $\tau > 0$ such that

$$\min_{T \in \tau_h} h_T \geq \tau h \quad \forall h > 0.$$

We recall that the regularity of $\{\tau_h\}_h$ can be expressed by demanding that the angles of the triangulations are bounded below by a positive constant, independently of h.

We obtain from τ_h a triangulation $\widetilde{\tau}_h$ of $\overline{\Omega}$ by replacing each triangle of τ_h with one side along the exterior part of $\partial\Omega_h$ by the corresponding curved triangle.

Let $\widetilde{T}$ be a curved triangle of $\widetilde{\tau}_h$. We denote by P_1, P_2, and P_3 its vertices, numbered in such a way that P_2 and P_3 are endpoints of the curved side of $\widetilde{T}$. Let $t_i, t_{i+1} \in [0,1]$ be such that $\mathbf{x}(t_i) = P_2$ and $\mathbf{x}(t_{i+1}) = P_3$. Then, the vectorial function $\varphi : [0,1] \to \mathbf{R}^2$ defined by

$$\varphi(t) := \mathbf{x}(t_i + t\,h)$$

is a parameterization of the curved side of $\widetilde{T}$.

Let $\widehat{T}$ be the reference triangle with vertices $\widehat{P}_1 := (0,0)$, $\widehat{P}_2 := (1,0)$ and $\widehat{P}_3 := (0,1)$. Consider the affine map F_T defined by $F_T(\widehat{P}_i) = P_i$ for $i \in \{1,2,3\}$. Also, consider the function $\Theta_T : \widehat{T} \to \mathbf{R}^2$ given by

$$\Theta_T(\hat{x}_1, \hat{x}_2) := \frac{\hat{x}_1}{1 - \hat{x}_2} \left(\varphi(\hat{x}_2) - (1 - \hat{x}_2)P_2 - \hat{x}_2 P_3 \right),$$

where the limiting value has to be taken as $\hat{x}_2 \to 1$. We then introduce the $\mathcal{C}^\infty$ mapping $\widetilde{F}_T : \widehat{T} \to \mathbf{R}^2$ given by

$$\widetilde{F}_T := F_T + \Theta_T.$$

It is proved in Theorem 22.4 of [25] that this transformation is a homeomorphism that maps one-to-one $\widehat{T}$ onto the curved triangle $\widetilde{T}$ in such a way that $\widetilde{F}_T(\hat{P}_i) = P_i$ for $i = 1, 2, 3$. Furthermore, the image of edge $\hat{P}_2\hat{P}_3$ is the curved side of $\widetilde{T}$ and the two other edges of $\widehat{T}$ are transformed linearly under $\widetilde{F}_T$ to the straight sides of $\widetilde{T}$.

A finite element is defined on $\widetilde{T}$ by a triplet $(\widetilde{T}, P_1(\widetilde{T}), \Sigma_T)$, where $P_1(\widetilde{T})$ is the image under $\widetilde{F}_T$ of the space $P_1(\widehat{T})$ of polynomials of degree no greater than 1 on $\widehat{T}$:

$$P_1(\widetilde{T}) = \{p : \ \widetilde{T} \to \mathbf{R}; \ p = \hat{p} \circ \widetilde{F}_T^{-1}, \ \hat{p} \in P_1(\widehat{T})\},$$

and $\Sigma_T = \{N_i, \ i = 1, 2, 3\}$ is a set of linear functionals defined by $N_i(\phi) = \phi(P_i) \quad \forall \phi \in P_1(\widetilde{T})$ and $\forall i = 1, 2, 3$. It is easy to show that Σ_T is $P_1(\widetilde{T})$-unisolvent (cf. [4]), i.e., if $\phi \in P_1(\widetilde{T})$ and $N_i(\phi) = 0 \ \forall i = 1, 2, 3$ then $\phi = 0$. It is also important to note that a function $\phi \in P_1(\widetilde{T})$ that satisfies $N_i(\phi) = N_j(\phi) = 0$ for $1 \leq i \neq j \leq 3$ vanishes on side P_iP_j of $\widetilde{T}$.

Under the assumption of regularity of τ_h, Theorem 22.4 in [25] proves that, if h is sufficiently small, the Jacobian $J(\widetilde{F}_T)(\cdot)$ of $\widetilde{F}_T(\cdot)$ does not vanish on $\widehat{T}$ and the following estimates are satisfied:

(3.1)
$$\|J(\widetilde{F}_T)\|_{0,\infty,\hat{T}} = O(h_T^2), \qquad |\widetilde{F}_T|_{1,\infty,\hat{T}} = O(h_T),$$
$$|\widetilde{F}_T^{-1}|_{1,\infty,\tilde{T}} = O(h_T^{-1}).$$

We denote by K an arbitrary triangle of $\widetilde{\tau}_h$. Then, $P_1(K)$ is the usual space of polynomials of degree no greater than one if K is a straight triangle and the space $P_1(\widetilde{T})$ defined above if K is a curved triangle $\widetilde{T}$. We define the finite dimensional subspace $X_h \subset X$ by

$$X_h = \{v \in X; \quad v|_K \in P_1(K) \quad \forall K \in \widetilde{\tau}_h\}.$$

We will need a finite element approximation property for functions in Sobolev spaces with noninteger index. Such a result may be found in Theorem 3.3 of [23] in the case of straight triangles. The generalization to curved elements is given below.

LEMMA 3.1. *There exists a constant $C > 0$ independent of h such that*

(3.2)
$$\inf_{v_h \in X_h} \|v - v_h\|_{1,\Omega} \le Ch^\sigma \|v\|_{1+\sigma,\Omega} \quad \forall v \in V \cap H^{1+\sigma}(\Omega).$$

Proof. Let $\widehat{v}$ be a continuous function on the reference triangle $\widehat{T}$ and v defined on $K \in \widetilde{\tau}_h$ by $v(F_K(\widehat{x})) = \widehat{v}(\widehat{x})$, where $F_K(\cdot)$ is the usual affine mapping when K is a straight triangle and is given by the application $\widetilde{F}_T(\cdot)$ if K is a curved triangle $\widetilde{T}$. Let π_K be the nodal interpolation operator. The associated nodal interpolation operator $\widehat{\pi}$ on the reference triangle $\widehat{T}$ satisfies $(\pi_K v)(F_K(\widehat{x})) = \widehat{\pi}\widehat{v}(\widehat{x}) \ \forall \widehat{x}$ in $\widehat{T}$. Hence, by the Sobolev embedding $H^{1+\sigma}(\widehat{T}) \hookrightarrow \mathcal{C}^0(\widehat{T})$, a change of variable and properties (3.1) we obtain

$$|v - \pi_K v|_{1,K} \le c|\widehat{v} - \widehat{\pi}\widehat{v}|_{1,\hat{T}}.$$

Now, since $\widehat{\pi}$ leaves invariant linear functions we deduce from a generalized Bramble–Hilbert lemma (cf. Theorem 2.3.1 of [23]) that

$$|\widehat{v} - \widehat{\pi}\widehat{v}|_{1,\hat{T}} \le C|\widehat{\nabla}\widehat{v}|_{\sigma,\hat{T}},$$

where $\widehat{\nabla}$ is the gradient with respect to the $\widehat{x}$ variable. Let us define the matrix valued function $B(\widehat{x}) = (\partial_i(F_K)_j(\widehat{x}))_{i,j}$. Applying the chain rule formula we compute easily that

$$B^{-T}\widehat{\nabla}\widehat{v}(\widehat{x}) = \nabla v \circ F_K(\widehat{x}),$$

where B^{-T} is the inverse matrix of B^T, the transpose of B. By definition

$$|\widehat{\nabla}\widehat{v}|_{\sigma,\hat{T}}^2 = \int_{\hat{T}}\int_{\hat{T}} \frac{|\widehat{\nabla}\widehat{v}(\widehat{x}) - \widehat{\nabla}\widehat{v}(\widehat{y})|^2}{|\widehat{x} - \widehat{y}|^{2+2\sigma}} \, d\widehat{x}d\widehat{y}$$
$$= \int_K\int_K \frac{|B^T(\widehat{x})\nabla v(x) - B^T(\widehat{y})\nabla v(y)|^2}{|\widehat{x} - \widehat{y}|^{2+2\sigma}} J(F_K^{-1})(x)J(F_K^{-1})(y) \, dxdy,$$

where $J(F_K^{-1})(\cdot) = J(F_K)^{-1}(\cdot)$ is the Jacobian of F_K^{-1}. When K is a straight triangle, B is a constant matrix and it is straightforward to deduce that

$$|\widehat{\nabla}\widehat{v}|_{\sigma,\hat{T}}^2 \le Ch^{2\sigma}|\nabla v|_{\sigma,K}^2.$$

It remains to show the same inequality when K is a curved triangle $\widetilde{T}$. From (3.1) we deduce that

$$|\widehat{\nabla}\widehat{v}|^2_{\sigma,\hat{T}} \le Ch^{-4} \int_{\tilde{T}} \int_{\tilde{T}} \frac{|B^T(\widehat{x})\nabla v(x) - B^T(\widehat{y})\nabla v(y)|^2}{|\widehat{x}-\widehat{y}|^{2+2\sigma}} \, dxdy$$

and by the triangle inequality

$$|\widehat{\nabla}\widehat{v}|^2_{\sigma,\hat{T}} \le 2Ch^{-4}\{I_1 + I_2\},$$

where

$$I_1 = \int_{\tilde{T}} \int_{\tilde{T}} |B^T(\widehat{x})|^2 \frac{|\nabla v(x) - \nabla v(y)|^2}{|\widehat{x}-\widehat{y}|^{2+2\sigma}} \, dxdy$$

and

$$I_2 = \int_{\tilde{T}} \int_{\tilde{T}} |\nabla v(y)|^2 \frac{|B^T(\widehat{x}) - B^T(\widehat{y})|^2}{|\widehat{x}-\widehat{y}|^{2+2\sigma}} \, dxdy.$$

Again using (3.1), we may bound the first term I_1 as follows:

$$I_1 \le Ch^2 \int_{\tilde{T}} \int_{\tilde{T}} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x-y|^{2+2\sigma}} \frac{|x-y|^{2+2\sigma}}{|\widehat{x}-\widehat{y}|^{2+2\sigma}} \, dxdy \le Ch^{4+2\sigma}|\nabla v|^2_{\sigma,\tilde{T}},$$

where the last inequality is a consequence of (3.1) and the mean value theorem since

$$|x-y| = |\widetilde{F}_T(\widehat{x}) - \widetilde{F}_T(\widehat{y})| \le Ch|\widehat{x}-\widehat{y}|.$$

Now we write the second term I_2 as follows:

$$I_2 = \int_{\tilde{T}} \int_{\tilde{T}} \frac{|B^T(\widehat{x}) - B^T(\widehat{y})|^2}{|\widehat{x}-\widehat{y}|^2} \frac{|x-y|^{2\sigma}}{|\widehat{x}-\widehat{y}|^{2\sigma}} \frac{|\nabla v(y)|^2}{|x-y|^{2\sigma}} \, dxdy,$$

and use the same arguments given above together with (3.1) to obtain

$$I_2 \le Ch^{4+2\sigma} \int_{\tilde{T}} \left(\int_{\tilde{T}} \frac{dx}{|x-y|^{2\sigma}} \right) |\nabla v(y)|^2 \, dy.$$

We easily deduce from the hypotheses on $\{\tau_h\}_h$ that there exists a constant c such that $K \subset B(y,ch)$ for any $K \in \widetilde{\tau}_h$ and $\forall y \in K$, where $B(y,ch)$ is the ball centered at y of radius ch. Hence,

$$I_2 \le Ch^{4+2\sigma} \int_{\tilde{T}} \left(\int_{B(y,ch)} \frac{dx}{|x-y|^{2\sigma}} \right) |\nabla v(y)|^2 \, dy$$

and changing to polar coordinates to evaluate exactly the internal integral yields

$$I_2 \le Ch^6 |v|^2_{1,\tilde{T}}.$$

From the estimates on I_1 and I_2 we deduce that

$$|\widehat{\nabla}\widehat{v}|^2_{\sigma,\hat{T}} \le C\{h^{2\sigma}|\nabla v|^2_{\sigma,\tilde{T}} + h^2|v|^2_{1,\tilde{T}}\},$$

and the result follows. $\quad\square$

We now introduce a finite dimensional subspace $S_h \subset H_0^{-\frac{1}{2}}$ which consists of 1-periodic and zero meanvalue piecewise constant functions on the uniform partition: $0 = t_1 < t_2 < \cdots < t_{N+1} = 1$. We refer to [3] for the following approximation property:

(3.3) $$\inf_{\mu \in S_h} \|\lambda - \mu\|_{-1/2} \le Ch^\sigma \|\lambda\|_{\sigma-1/2} \quad \forall\lambda \in H^{\sigma-1/2} \cap H_0^{-\frac{1}{2}},$$

where C is a constant independent of h.

4. The discrete problem. We denote $\mathbf{W}_h := X_h \times S_h$ and introduce the discrete problem associated with (2.4):

$$
\begin{aligned}
&\text{find } \mathbf{u}_h := (u_h, \lambda_h) \in \mathbf{W}_h; \\
&A(\mathbf{u}_h, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{W}_h.
\end{aligned}
\tag{4.1}
$$

From (3.2), (3.3), and the density of regular functions in X and $H^{-\frac{1}{2}}$, we deduce the following approximation property:

$$
\lim_h \inf_{\mathbf{v}_h \in \mathbf{W}_h} \|\mathbf{v} - \mathbf{v}_h\|_W = 0 \quad \forall \mathbf{v} \in \mathbf{W}.
\tag{4.2}
$$

Theorem 10.1.2 of [3] assures that under conditions (2.5), (2.6), and (2.4) there exists $h_0 \in (0,1]$ such that the following inf-sup condition is satisfied:

$$
\sup_{\mathbf{z} \in \mathbf{W}_h} \frac{A'(u; \mathbf{v}, \mathbf{z})}{\|\mathbf{z}\|_W} \geq \alpha_2 \|\mathbf{v}\|_W \quad \forall \mathbf{v} \in \mathbf{W}_h,
\tag{4.3}
$$

for some constant $\alpha_2 > 0$ independent of h $\forall h < h_0$.

It is easy to deduce from (4.3) that the Galerkin projection $\mathbf{P}_h : \mathbf{W} \to \mathbf{W}_h$ given by

$$
\mathbf{P}_h \mathbf{v} := (P_h v, \rho_h \mu) \in \mathbf{W}_h;
$$

$$
A'(u; \mathbf{P}_h \mathbf{v}, \mathbf{z}) = A'(u; \mathbf{v}, \mathbf{z}) \qquad \forall \mathbf{z} \in \mathbf{W}_h
$$

is well defined $\forall h < h_0$. Furthermore, this operator satisfies

$$
\|\mathbf{v} - \mathbf{P}_h \mathbf{v}\|_W \leq C \inf_{\mathbf{v}_h \in \mathbf{W}_h} \|\mathbf{v} - \mathbf{v}_h\|_W,
\tag{4.4}
$$

for some constant C independent of $\mathbf{v}$ and h $\forall h < h_0$. Hence, we deduce from Lemma 3.1 and (3.3) that, if h is sufficiently small,

$$
\|\mathbf{v} - \mathbf{P}_h \mathbf{v}\|_W \leq C h^\sigma (\|v\|_{1+\sigma,\Omega}^2 + \|\mu\|_{-1/2+\sigma}^2)^{1/2},
\tag{4.5}
$$

$\forall \mathbf{v} \in \mathbf{W} \cap (H^{1+\sigma}(\Omega) \times H^{-1/2+\sigma})$.

LEMMA 4.1. *A function* $\mathbf{u}_h \in \mathbf{W}_h$ *is a solution of* (4.1) *if and only if the following equation is satisfied:*

$$
A'(u; \mathbf{u} - \mathbf{u}_h, \mathbf{v}) = R(u; u_h, v) \quad \forall \mathbf{v} \in \mathbf{W}_h,
$$

where

$$
\begin{aligned}
R(u; u_h, v) := \int_{\Omega^{nl}} \Big(\int_0^1 \big[(\partial^2 \boldsymbol{\beta}/\partial s^2)(x, u + t(u_h - u)) \cdot \nabla v \\
+ (\partial^2 \beta_0/\partial s^2)(x, u + t(u_h - u)) v \big] (1 - t) \, dt \Big) (u - u_h)^2 \, dx.
\end{aligned}
$$

Proof. Let $\eta(t) := A(\mathbf{u} + t(\mathbf{u}_h - \mathbf{u}), \mathbf{v})$. The result follows from identity

$$
\eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1 - t) \, dt
$$

and the fact that $A(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}_h, \mathbf{v}) = 0$ $\forall \mathbf{v} \in \mathbf{W}_h$. □

LEMMA 4.2. *Let* $\mathbf{M}_h := \{\mathbf{v} \in \mathbf{W}_h; \quad \|v\|_{0,\infty,\Omega} \leq 1 + \|u\|_{0,\infty,\Omega}\}$. *There exists a constant $C > 0$ independent of h such that*

$$|R(u;v,z)| \leq C\|u-v\|_{1,\Omega}^2 \|\mathbf{z}\|_W \quad \forall \mathbf{v} \in \mathbf{M}_h \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

Proof. By virtue of

$$\|u + t(v-u)\|_{0,\infty,\Omega} \leq 1 + 2\|u\|_{0,\infty,\Omega} \quad \forall v \in \mathbf{M}_h$$

and the Cauchy–Schwarz inequality we obtain that

$$|R(u;v,z)| \leq C_0 \|u-v\|_{L^4(\Omega)}^2 \|z\|_{1,\Omega},$$

where

$$C_0 \geq \max_i \left\{ \sup_{\substack{x \in \overline{\Omega}^{nl} \\ |s| \leq 1 + 2\|u\|_{0,\infty,\Omega}}} |(\partial^2 \beta_i / \partial s^2)(x,s)| \right\}.$$

The result follows from the fact that X is embedded continuously in $L^4(\Omega)$. □

We define the nonlinear mapping $\mathbf{\Upsilon} : \mathbf{W}_h \to \mathbf{W}_h$ as follows: given $\mathbf{v} \in \mathbf{W}_h$, $\mathbf{\Upsilon v}$ is the unique solution of

$$(4.6) \qquad A'(u; \mathbf{\Upsilon v}, \mathbf{z}) = A'(u; \mathbf{u}, \mathbf{z}) - R(u;v,z) \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

To prove the continuity of this operator, we consider a sequence $(\mathbf{v}_n) \in \mathbf{W}_h$ that converges to an element $\mathbf{v}$ of $\mathbf{W}_h$. The following identity

$$A'(u; \mathbf{\Upsilon v} - \mathbf{\Upsilon v}_n, \mathbf{z}) = R(u;v_n,z) - R(u;v,z),$$

together with (4.3) give the estimate

$$\|\mathbf{\Upsilon v} - \mathbf{\Upsilon v}_n\|_W \leq C \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{|R(u;v_n,z) - R(u;v,z)|}{\|\mathbf{z}\|_W}.$$

Hence, $\lim_n \mathbf{\Upsilon v}_n = \mathbf{\Upsilon v}$ since the limit of the right-hand side is zero. We are now ready to prove the main result of this section.

THEOREM 4.3. *Let $u \in X \cap H^{1+\sigma}(\Omega)$ be a solution of problem* (2.1), *with $0 < \sigma < 1$ and assume that* (2.6) *is satisfied. Then, there exists $h_0 \in (0,1]$ such that the discrete problem* (4.1) *has a solution $\mathbf{u}_h \in \mathbf{W}_h$ satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_W \leq Ch^\sigma,$$

for some constant C independent of h $\forall h < h_0$.

Proof. We define the set

$$\mathbf{B}_h := \{\mathbf{v} \in \mathbf{W}_h; \quad \|\mathbf{v} - \mathbf{P}_h\mathbf{u}\|_W \leq h^\sigma\}.$$

We will prove that, if h is sufficiently small, $\mathbf{B}_h \subset \mathbf{M}_h$. Let $\mathbf{v} \in \mathbf{B}_h$, the triangle inequality gives

$$\|v\|_{0,\infty,\Omega} \leq \|u-v\|_{0,\infty,\Omega} + \|u\|_{0,\infty,\Omega},$$

$$(4.7) \qquad \|u - v\|_{0,\infty,\Omega} \leq \|u - P_h u\|_{0,\infty,\Omega} + \|P_h u - v\|_{0,\infty,\Omega},$$

and

$$\|u - P_h u\|_{0,\infty,\Omega} \leq \|u - \pi_h u\|_{0,\infty,\Omega} + \|\pi_h u - P_h u\|_{0,\infty,\Omega},$$

where $\pi_h : \mathcal{C}^0(\overline{\Omega}) \to X_h$ is the pointwise linear interpolation operator.

Now we use the fact that $\{\tau_h\}_h$ is quasi-uniform to obtain the following inverse inequality (cf. Theorem 3.4 in [23]):

$$(4.8) \qquad \|w\|_{0,\infty,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} \|w\|_{1,\Omega} \quad \forall w \in X_h.$$

Thus,

$$\|P_h u - v\|_{0,\infty,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} \|P_h u - v\|_{1,\Omega} \leq C h^\sigma \left(\log \frac{1}{h} \right)^{1/2}$$

and

$$\|\pi_h u - P_h u\|_{0,\infty,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} \|\pi_h u - P_h u\|_{1,\Omega}$$

$$\leq C \left(\log \frac{1}{h} \right)^{1/2} (\|u - P_h u\|_{1,\Omega} + \|u - \pi_h u\|_{1,\Omega}).$$

It follows from Lemma 3.1 and (4.5) that

$$\|\pi_h u - P_h u\|_{0,\infty,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} h^\sigma (\|u\|_{1+\sigma,\Omega}^2 + \|\lambda\|_{-1/2+\sigma}^2)^{1/2}.$$

Finally, proceeding as in Lemma 3.1, we obtain the following interpolation error estimate:

$$\|u - \pi_h u\|_{0,\infty,\Omega} \leq C h^\sigma \|u\|_{1+\sigma,\Omega}.$$

Consequently, when h is sufficiently small,

$$\|u - v\|_{0,\infty,\Omega} \leq 1,$$

and $\mathbf{v} \in \mathbf{M}_h$.

Now, by definition of $\mathbf{P}_h$, we may write (4.6) as follows:

$$A'(u; \mathbf{\Upsilon v} - \mathbf{P}_h \mathbf{u}, \mathbf{z}) = -R(u; v, z) \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

We deduce from (4.3) and Lemma 4.2 that

$$\|\mathbf{\Upsilon v} - \mathbf{P}_h \mathbf{u}\|_W \leq C \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{A'(u; \mathbf{\Upsilon v} - \mathbf{P}_h \mathbf{u}, \mathbf{z})}{\|\mathbf{z}\|_W} \leq C \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{|R(u; v, z)|}{\|\mathbf{z}\|_W}$$

$$\leq C \|u - v\|_{1,\Omega}^2.$$

Hence,

$$\|\mathbf{\Upsilon v} - \mathbf{P}_h \mathbf{u}\|_W \leq 2C \{ \|u - P_h u\|_{1,\Omega}^2 + \|P_h u - v\|_{1,\Omega}^2 \}$$

$$\leq 2C \{ h^{2\sigma} (\|u\|_{1+\sigma,\Omega}^2 + \|\lambda\|_{-1/2+\sigma}^2) + h^{2\sigma} \} \leq h^\sigma,$$

if h is sufficiently small.

An application of Brouwer's fixed point theorem shows that there exists $\mathbf{u}_h \in \mathbf{W}_h$ such that $\mathbf{\Upsilon u}_h = \mathbf{u}_h$ and we deduce from Lemma 4.1 that $\mathbf{u}_h$ is a solution of (4.1). Furthermore,

$$\|\mathbf{u} - \mathbf{u}_h\|_W \leq \|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_W + \|\mathbf{P}_h\mathbf{u} - \mathbf{u}_h\|_W \leq Ch^\sigma,$$

where we used (4.5) and the fact that $\mathbf{u}_h \in \mathbf{B}_h$.   □

Finally, we point out that the same technique given in [24] can be easily reproduced here to prove a local uniqueness result for (4.1).

5. Analysis of the fully discrete method. The implementation of our method requires numerical quadratures for all the integrals appearing in (4.1). In practice one solves

(5.1)
$$\text{find } \mathbf{u}_h^* \in \mathbf{W}_h;$$
$$A_h(\mathbf{u}_h^*, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{W}_h,$$

where $A_h(\cdot, \cdot)$ is an approximation of $A(\cdot, \cdot)$ that we will define below.

Consider first a quadrature formula on the reference triangle

$$\widehat{Q}(\widehat{\phi}) := \sum_{l=1}^L \widehat{\omega}_l \widehat{\phi}(\widehat{z}_l) \simeq \int_{\hat{T}} \widehat{\phi}.$$

On each $K \in \widetilde{\tau}_h$ we define

$$Q_K(\phi) := \widehat{Q}(|J(F_K)|\phi \circ F_K) = \sum_{l=1}^L \widehat{\omega}_l |J(F_K)|(\widehat{z}_l)\phi(F_K(\widehat{z}_l)) \simeq \int_K \phi(x)\, dx.$$

We assume that the weights $\widehat{\omega}_l$ are positive and satisfy

(5.2)
$$\sum_{l=1}^L \widehat{\omega}_l = \frac{1}{2}.$$

This induces us to define $a_h(\cdot, \cdot)$ by

$$a_h(u, v) = a_h^{nl}(u, v) + a_h^l(u, v) \quad \forall u, v \in X_h,$$

where

$$a_h^{nl}(u, v) = \sum_{K \subset \Omega^{nl}} Q_K(\boldsymbol{\beta}(x, u) \cdot \nabla v\, dx + \beta_0(x, u)v)$$

and

$$a_h^l(u, v) = \sum_{K \subset \Omega} Q_K(\nabla u \cdot \nabla v).$$

Now we turn to define an approximation $b_h(\cdot, \cdot)$ of $b(\cdot, \cdot)$ on $S_h \times S_h$. To this end, we need a basic two-dimensional quadrature formula on the unit square

$$\mathcal{Z}(g) := \sum_{k=1}^d \eta_k g(z_k) \simeq \int_0^1 \int_0^1 g(z)\, dz.$$

We will assume that $\mathcal{Z}$ is exact for all polynomials of degree no greater that 1 in each variable. Let us consider the following decomposition of V:

$$V(s,t) = -\frac{1}{4\pi} \log{(s-t)^2} + F(s,t).$$

Notice that F is of class $\mathcal{C}^\infty$ in the domain $\{(s,t); \quad |s-t| < 1\}$. We follow [13, 7] and approximate

$$V_{i,j} := -\frac{1}{4\pi} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \log{(s-t)^2} + \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} F(s,t)$$

by computing the first integral exactly and using $\mathcal{Z}$ for the second one. In order to avoid the neighborhood of $\{(s,t); \quad |s-t| = 1\}$, one may compute approximations $\widetilde{V}_{i,j}$ of $V_{i,j}$ for indices i, j that satisfy $|i-j| \le N/2$ and recover $\widetilde{V}_{i,j}$ for $i,j = 1,\dots,N$ by periodicity, i.e.,

$$V_{i,j} \simeq \widetilde{V}_{i,j} := -\frac{1}{4\pi} \int_{t_{\underline{i}}}^{t_{\underline{i}+1}} \int_{t_{\underline{j}}}^{t_{\underline{j}+1}} \log{(s-t)^2} + h^2 \mathcal{Z}(F(t_{\underline{i}} + h\cdot, t_{\underline{j}} + h\cdot)),$$

where

$$(\underline{i},\underline{j}) = \begin{cases} (i,j) & \text{if } |i-j| \le N/2, \\ (i,j-N) & \text{if } i-j > N/2, \\ (i-N,j) & \text{if } j-i > N/2. \end{cases}$$

Now, for any λ and μ in S_h, we define

$$b(\lambda,\mu) \simeq b_h(\lambda,\mu) := \sum_{i,j=1}^{N} \lambda_i \mu_j \widetilde{V}_{i,j},$$

where λ_i and μ_i are the constant values of λ and μ on (t_i, t_{i+1}). It is important to choose $\mathcal{Z}$ with nodes z_k symmetric with respect to $\{(s,t) \in [0,1] \times [0,1]; \quad s = t\}$ in order to inherit the symmetry of V and obtain coefficients that satisfy $\widetilde{V}_{i,j} = \widetilde{V}_{j,i}$. The following result is proved in [13, 7].

LEMMA 5.1. *There exists a constant C independent of h such that*

$$|b(\lambda,\mu) - b_h(\lambda,\mu)| \le Ch\|\lambda\|_{-1/2}\|\mu\|_{-1/2} \quad \forall \lambda, \mu \in S_h.$$

By construction, for any $v \in X_h$, the one variable function $t \mapsto v(\mathbf{x}(t))$ belongs to the space T_h of 1-periodic piecewise linear and continuous functions on the uniform partition $0 = t_1 < t_2 < \cdots < t_{N+1} = 1$. Hence $\frac{d}{dt}v(\mathbf{x}(t)) \in S_h$ and therefore, we may approximate $b(\frac{d}{dt}\gamma u, \frac{d}{dt}\gamma v)$ by $b_h(\frac{d}{dt}\gamma u, \frac{d}{dt}\gamma v)$ $\forall u, v \in X_h$. Furthermore, by virtue of Lemma 5.1 and the continuity of $\frac{d}{dt} : H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$ and γ,

$$(5.3) \quad \left| b\left(\frac{d}{dt}\gamma u, \frac{d}{dt}\gamma v\right) - b_h\left(\frac{d}{dt}\gamma u, \frac{d}{dt}\gamma v\right) \right| \le Ch\|u\|_{1,\Omega}\|v\|_{1,\Omega} \qquad \forall u, v \in X_h.$$

To conclude with the approximation of the boundary terms, we need to substitute $c(\cdot,\cdot)$ by a sufficiently close bilinear form $c_h(\cdot,\cdot)$ on $S_h \times T_h$. Let $\{l_i, \quad i =$

$1, \ldots, N\}$ be the nodal basis of T_h, i.e., the functions of T_h that satisfy $l_i(t_j) = \delta_{i,j}$. For any $v \in X_h$ and $\mu \in S_h$ we define

$$c(\mu, \gamma v) \simeq c_h(\mu, \gamma v) := \sum_{i=1}^{N} h\mu_i \frac{\gamma v(t_{i+1}) + \gamma v(t_i)}{4} - \sum_{i,j=1}^{N} h^2 \mu_i \gamma v(t_j) \mathcal{Z}(K_{i,j}),$$

where

$$K_{i,j}(s,t) := K(t_i + sh, t_{j-1} + th)l_j(t_{j-1} + th) + K(t_i + sh, t_j + th)l_j(t_j + th).$$

LEMMA 5.2. *There exists a constant C independent of h such that*

$$|c(\mu, \gamma v) - c_h(\mu, \gamma v)| \leq Ch\|v\|_{1,\Omega}\|\mu\|_{-1/2} \quad \forall v \in X_h \ \forall \mu \in S_h.$$

Proof. Let $\mathcal{E} := \mathcal{Z} - \int_0^1 \int_0^1$ be the error functional. By construction, we have the estimate

$$|c(\mu, \gamma v) - c_h(\mu, \gamma v)| \leq Ch^2 \sum_{i,j=1}^{N} |\mu_i||\gamma v(t_j)||\mathcal{E}(K_{i,j})|,$$

$\forall v \in X_h$ and $\forall \mu \in S_h$. Using the fact that $\mathcal{Z}$ is of degree 1, it follows readily from the Bramble–Hilbert lemma that

$$|\mathcal{E}(K_{i,j})| \leq C|K_{i,j}|_{2,\infty,(0,1)\times(0,1)} \leq C_1 h^2.$$

Thus,

$$|c(\mu, \gamma v) - c_h(\mu, \gamma v)| \leq Ch^4 \sum_{i=1}^{N} |\mu_i| \sum_{j=1}^{N} |\gamma v(t_j)|$$

$$= Ch^3 \int_0^1 |\mu| \sum_{j=1}^{N} |\gamma v(t_j)| \leq Ch^2 \|\mu\|_0 \left(h \sum_{j=1}^{N} \gamma v(t_j)^2 \right)^{1/2},$$

where in the last step we used the Cauchy–Schwarz inequality for both the integral and the sum.

We conclude by the equivalence of the norms $g \mapsto \|g\|_0$ and $g \mapsto (h \sum_{i=1}^{N} g(t_i)^2)^{1/2}$ on T_h and the inverse inequality

$$\|\mu\|_0 \leq Ch^{-\frac{1}{2}}\|\mu\|_{-1/2} \quad \forall \mu \in S_h. \qquad \square$$

Let us now show that the approximate semilinear form

$$A_h(\mathbf{u}, \mathbf{v}) := a_h(u,v) + b_h(\lambda, \mu) + b_h\left(\frac{d}{dt}(\gamma u), \frac{d}{dt}(\gamma v)\right) - c_h(\lambda, \gamma v) + c_h(\mu, \gamma u)$$

is sufficiently close to $A(\,\cdot\,, \,\cdot\,)$. We begin by recalling the following classical results; cf. [25].

LEMMA 5.3. *Let K be a triangle of $\widetilde{\tau}_h$. There exists a constant C independent of K such that*

(5.4) $$\left| \int_K fp \, dx - Q_K(fp) \right| \leq Ch_K \sqrt{mes(K)}\|f\|_{1,\infty,K}\|p\|_{1,K}$$

and

$$(5.5) \qquad \left| \int_K \nabla p \cdot \nabla q \, dx - Q_K(\nabla p \cdot \nabla q) \right| \leq C h_K \|p\|_{1,K} \|q\|_{1,K},$$

$\forall f \in W^{1,\infty}(K)$ *and* $\forall p, q \in P_1(K)$.

We easily deduce from (5.5) that

$$(5.6) \qquad \left| \int_\Omega \nabla u \cdot \nabla v \, dx - a_h^l(u,v) \right| \leq C h \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \forall u, v \in X_h.$$

LEMMA 5.4. *Let* $\mathbf{B}_h^* = \{\mathbf{v} \in \mathbf{W}_h; \quad \|\mathbf{v} - \mathbf{u}_h\|_W \leq h^\sigma\}$, *where* $\mathbf{u}_h \in \mathbf{B}_h$ *is the solution of problem* (4.1). *There exists a constant* $C > 0$ *such that*

$$(5.7) \qquad |A(\mathbf{v}, \mathbf{z}) - A_h(\mathbf{v}, \mathbf{z})| \leq Ch(1 + \|\mathbf{v}\|_W)\|\mathbf{z}\|_W \quad \forall \mathbf{v} \in \mathbf{B}_h^* \, \forall \mathbf{z} \in \mathbf{W}_h.$$

Proof. From Lemma 5.1, Lemma 5.2, (5.3), and (5.6), we deduce that we have to prove only the following estimate:

$$I_1 + I_2 \leq Ch(1 + \|v\|_{1,\Omega})\|z\|_{1,\Omega},$$

where

$$I_1 = \left| \int_{\Omega^{nl}} \boldsymbol{\beta}(x,v) \cdot \nabla z \, dx - \sum_{K \subset \Omega^{nl}} Q_K(\boldsymbol{\beta}(x,v) \cdot \nabla z) \right|$$

and

$$I_2 = \left| \int_{\Omega^{nl}} \beta_0(x,v) z \, dx - Q_K(\beta_0(x,v)z) \right|.$$

Let us show that $\|v\|_{0,\infty,\Omega}$ is bounded independently of h $\forall \mathbf{v} \in \mathbf{B}_h^*$. We use the triangular inequality and the fact that $\mathbf{B}_h \subset \mathbf{M}_h$ to obtain the estimate

$$\|v\|_{0,\infty,\Omega} \leq \|u_h\|_{0,\infty,\Omega} + \|u_h - v\|_{0,\infty,\Omega} \leq 1 + \|u\|_{0,\infty,\Omega} + \|u_h - v\|_{0,\infty,\Omega}.$$

On the other hand, by (4.8),

$$\|u_h - v\|_{0,\infty,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} \|u_h - v\|_{1,\Omega} \leq C \left(\log \frac{1}{h} \right)^{1/2} h^\sigma.$$

It follows that for h sufficiently small

$$\|v\|_{0,\infty,\Omega} \leq 2 + \|u\|_{0,\infty,\Omega} \quad \forall v \in \mathbf{B}_h^*.$$

The rest of the proof follows from an idea of Feistauer; cf. [9] (see also Theorem 36.2 of [25]). We give here a sketch of the method for the sake of completeness. We take in (5.4) $f(x) = \beta_i(x, v(x))|_K$ $(i = 1, 2)$ and $p(x) = 1$ and notice that

$$\|\beta_i(\cdot, v(\cdot))\|_{1,\infty,K} \leq \|\beta_i(\cdot, v(\cdot))\|_{0,\infty,K} + \sum_{j=1}^2 \|(\partial \beta_i / \partial x_j)(\cdot, v(\cdot))\|_{0,\infty,K}$$
$$+ \|(\partial \beta_i / \partial s)(\cdot, v(\cdot))\|_{0,\infty,K} |\nabla(v|_K)|.$$

Summing over the triangles contained in $\overline{\Omega}^{nl}$, we obtain the following estimate:

$$I_1 \le Ch \sum_{K \subset \overline{\Omega}^{nl}} \mathrm{mes}(K)|\nabla(z|_K)|\,(1 + |\nabla(v|_K)|)$$

and deduce that

$$I_1 \le Ch(1 + \|v\|_{1,\Omega})\|z\|_{1,\Omega}$$

by the Cauchy–Schwarz inequality and the fact that $\mathrm{mes}(K)^{1/2}|\nabla(p|_K)| = |p|_{1,K}$ for any linear function p.

Similarly, using (5.4) with $f(x) = \beta_0(x,v(x))|_K$ and $p(x) = z|_K(x)$ leads to the same estimate for I_2. $\quad\square$

In the rest of this section we show that problem (5.1) has at least one solution $\mathbf{u}_h^*$ which is, asymptotically, as close to $\mathbf{u}$ as $\mathbf{u}_h$.

LEMMA 5.5. *A function* $\mathbf{u}_h^* \in \mathbf{W}_h$ *is a solution of* (5.1) *if and only if*

$$A'(u_h; \mathbf{u}_h^* - \mathbf{u}_h, \mathbf{z}) = A(\mathbf{u}_h^*, \mathbf{z}) - A_h(\mathbf{u}_h^*, \mathbf{z}) - R(u_h; u_h^*, z) \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

Proof. Let $\eta_h(t) = A(\mathbf{u}_h + t(\mathbf{u}_h^* - \mathbf{u}_h), \mathbf{z})$. The result follows from identity

$$\eta_h(1) = \eta_h(0) + \eta_h'(0) + \int_0^1 \eta_h''(t)(1 - t)\, dt$$

and the fact that $A(\mathbf{u}_h, \mathbf{z}) = A_h(\mathbf{u}_h^*, \mathbf{z}) = 0 \ \forall z \in X_h$. $\quad\square$

THEOREM 5.6. *Under hypotheses of Theorem* 4.3 *there exists* $h_1 \in (0,1]$ *such that,* $\forall h < h_1$, *problem* (5.1) *has a solution* $\mathbf{u}_h^* \in \mathbf{W}_h$ *that satisfies*

$$\|\mathbf{u} - \mathbf{u}_h^*\|_W \le Ch^\sigma.$$

Proof. First of all, we notice that

$$\|u_h + t(v - u_h)\|_{0,\infty,\Omega} \le 2 + \|u\|_{0,\infty,\Omega} \quad \forall v \in \mathbf{B}_h^* \quad \forall t \in [0,1].$$

Thus, proceeding as in Lemma 4.2, we may show that there exists a constant C independent of h such that

$$|R(u_h; v, z)| \le C\|u_h - v\|_{1,\Omega}^2 \|\mathbf{z}\|_W \quad \forall \mathbf{v} \in \mathbf{B}_h^* \, \forall \mathbf{z} \in \mathbf{W}_h.$$

Let us define the nonlinear mapping $\mathbf{\Upsilon}^* : \mathbf{W}_h \to \mathbf{W}_h$ by

$$A'(u_h; \mathbf{\Upsilon}^* \mathbf{v}, \mathbf{z}) = A'(u_h; \mathbf{u}_h, \mathbf{z}) + A(\mathbf{v}, \mathbf{z}) - A_h(\mathbf{v}, \mathbf{z}) - R(u_h; v, z).$$

This application is well defined and continuous since the bilinear form $A'(u_h; \cdot, \cdot) : \mathbf{W}_h \times \mathbf{W}_h \to \mathbf{R}$ satisfies the inf-sup condition

$$(5.8) \qquad \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{A'(u_h; \mathbf{v}, \mathbf{z})}{\|\mathbf{z}\|_W} \ge C\|\mathbf{v}\|_W \quad \forall \mathbf{v} \in \mathbf{W}_h,$$

for some constant C independent of h, if h is sufficiently small. Indeed,

$$
\begin{aligned}
|A'(u; \mathbf{v}, \mathbf{z}) - A'(u_h; \mathbf{v}, \mathbf{z})| = \Big| & \int_{\Omega^{nl}} \int_0^1 (\partial^2 \boldsymbol{\beta}/\partial s^2)(x, u + t(u_h - u))\nabla z(u - u_h)v\, dt\, dx \\
& + \int_{\Omega^{nl}} \int_0^1 (\partial^2 \beta_0/\partial s^2)(x, u + t(u_h - u))z(u - u_h)v\, dt\, dx \Big| \\
\le\ & C \int_{\Omega^{nl}} \{|\nabla z||(u - u_h)v| + |(u - u_h)vz|\}\, dx,
\end{aligned}
$$

where

$$C \geq \left(\max_i \sup_{\substack{x \in \overline{\Omega}^{nl} \\ |s| \leq 2 + \|u\|_{0,\infty,\Omega}}} |(\partial^2 \beta_i / \partial s^2)(x,s)| \right).$$

By the Cauchy–Schwarz inequality and the continuous embedding of $H^1(\Omega)$ in $L^4(\Omega)$ we deduce that

$$|A'(u; \mathbf{v}, \mathbf{z}) - A'(u_h; \mathbf{v}, \mathbf{z})| \leq C\|u - u_h\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \|z\|_{1,\Omega}$$
$$\leq C_1 \|u - u_h\|_{1,\Omega} \|v\|_{1,\Omega} \|z\|_{1,\Omega}.$$

Finally, Theorem 4.3 implies that for h sufficiently small

$$|A'(u; \mathbf{v}, \mathbf{z}) - A'(u_h; \mathbf{v}, \mathbf{z})| \leq Ch^\sigma \|z\|_{1,\Omega} \|v\|_{1,\Omega}$$

and inequality (5.8) follows.

We are now ready to prove that $\mathbf{\Upsilon}^*(\mathbf{B}_h^*) \subset \mathbf{B}_h^*$, let $\mathbf{v} \in \mathbf{B}_h^*$,

$$\|\mathbf{\Upsilon}^*\mathbf{v} - \mathbf{u}_h\|_W \leq C \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{A'(u_h; \mathbf{\Upsilon}^*\mathbf{v} - \mathbf{u}_h, \mathbf{z})}{\|\mathbf{z}\|_W}$$
$$\leq C \sup_{\mathbf{z} \in \mathbf{W}_h} \frac{|A(\mathbf{v}, \mathbf{z}) - A_h(\mathbf{v}, \mathbf{z})|}{\|\mathbf{z}\|_W} + C_1 \|v - u_h\|_{1,\Omega}^2.$$

Consequently, we deduce from Lemma 5.4 that

$$\|\mathbf{\Upsilon}^*\mathbf{v} - \mathbf{u}_h\|_{1,\Omega} \leq C_3\{(1 + \|\mathbf{v}\|_W)h^{1-\sigma} + h^\sigma\}h^\sigma < h^\sigma \quad (0 < \sigma < 1)$$

for h sufficiently small, since $\|v\|_{1,\Omega}$ is bounded. We conclude by Brouwer's fixed point theorem as in Theorem 4.3. □

6. Numerical results. In this section, we present results of numerical experiments. We take $\Omega_0 = (-0.5, 0.5) \times (-0.5, 0.5)$, $\Omega^{nl} = (-1.5, 1.5) \times (-1.5, 1.5) \setminus \overline{\Omega}_0$ and the artificial boundary Γ is the circle centered at the origin of radius 3. We present results of numerical experiments for problem (5.1) when β_i vanishes identically for $i = 1, 2$ and

$$\beta_0(x, s) = f(x) - \frac{s}{\sqrt{1+s^2}} 1_{\Omega^{nl}}(x).$$

The Dirichlet boundary condition on Γ_0 and the function f are chosen in such a way that the exact solution is $u(x, y) = x/(x^2 + y^2)$.

We use Newton's method to solve the nonlinear discrete equations. As an initial guess for Newton's algorithm we take the discrete solution of the linear problem obtained by dropping the nonlinear term $\frac{u}{\sqrt{1+u^2}} 1_{\Omega^{nl}}(x)$ from the equation. The iterations of Newton's method are performed until the stopping criterion $|U^{n+1} - U^n| \leq 10^{-6}|U^n|$ is satisfied. Here, U^n is the vectorial representation of the solution at step n of Newton's method. Table 5.1 shows the number of iterations and the error in the discrete L^∞-norm as we vary the discretization parameter h.

TABLE 5.1
Convergence history of Newton's method versus the mesh parameter.

h	Iterations	$\max_i |(u - u_h)(x_i)|$
1/18	4	1.11×10^{-1}
1/36	4	$3,53 \times 10^{-2}$
1/72	4	$1,82 \times 10^{-2}$
1/144	4	$7,14 \times 10^{-3}$

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