RELIABILITY BASED DESIGN OPTIMIZATION OF LONG-SPAN BRIDGES UNDER FLUTTER CONSTRAINT

Doctoral Thesis

by

Ibuki Kusano

Supervised by
Dr. José Ángel Jurado Albarracín-Martinón
Dr. Aitor Baldomir García

La Coruña, June 2015
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この博士号論文を両親、草野次雄、ミチ子に捧げます。
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Abstract

The reliability based design optimization is applied to long-span suspension bridges under probabilistic flutter constraint employing three RBDO methods, namely reliability index approach, performance measure approach and sequential optimization and reliability assessment. Uncertainties in extreme wind speeds at bridge site and flutter derivatives obtained in wind tunnel tests are taken into account. Two cases of RBDO problems are studied; in the first case, the bridge girder volume is to be minimized by varying the thickness of box girder plates while in the second case, the sum of the girder and main cable volume is sought to be minimized by considering both box girder thicknesses as well as the main cable area as design variables. For both cases, the optimum designs must satisfy a predetermined structural reliability level against flutter and other deterministic constraints.

In order to solve this problem, the three RBDO methods mentioned above are programmed in Matlab code, which calls Abaqus finite element models to obtain structural responses and FLAS code, developed by our research group, to perform flutter analysis. Prior to the resolution of the RBDO problem, reliability analyses of bridge flutter are performed in order to obtain the safety level of the original bridge design. The proposed RBDO formulations are applied to two bridge example of the Great Belt East Bridge in Denmark and the Messina Bridge project in Italy. The results obtained by different RBDO methods are then compared for their accuracy and computational efficiency.
Resumen

Se ha aplicado optimización probabilista (RBDO) a puentes colgantes de gran vano considerando condiciones de flameo mediante tres métodos diferentes cuyas denominaciones anglosajonas son Reliability Index Approach, Performance Measure Approach y Sequential Optimization and Reliability Assessment. Se considera la existencia de incertidumbre en la velocidad extremal de viento en el emplazamiento del puente, y en las funciones de flameo obtenidas experimentalmente en túnel de viento. Se plantean dos casos de optimización probabilista; en el primero únicamente se optimiza el volumen del cajón del puente variando los espesores de las chapas que forman el cajón, mientras que en el segundo se minimiza tanto el volumen de los cables principales como el tablero del puente. En ambos casos, se desea obtener diseños óptimos que satisfagan un valor prefijado de la seguridad estructural frente a flameo además de otras restricciones de tipo determinista.

Para resolver este problema, se programan en Matlab los métodos de RBDO mencionados anteriormente, código que ejecuta un modelo de elementos finitos realizado en Abaqus para obtener las respuestas estructurales y el código FLAS, desarrollado por nuestro grupo de investigación, para realizar el análisis a flameo. Antes de resolver el problema de RBDO se realizan distintos análisis de fiabilidad para conocer el nivel de seguridad estructural del puente con su diseño original. Las formulaciones de RBDO propuestas se han aplicado a dos ejemplos de puentes colgantes de gran vano: el Great Belt East Bridge en Dinamarca y el proyecto del Puente de Messina en Italia. Los resultados obtenidos por los diferentes métodos de RBDO se comparan en términos de precisión y eficiencia computacional.
Resumo

Aplicouse optimización probabilista (RBDO) a pontes colgantes de gran van considerando condicións de flameo mediante tres métodos diferentes, que teñen por nome en inglés, Reliability Index Approach, Performance Measure Approach e Sequential Optimization and Reliability Assessment. Considérase a existencia de incerteza na velocidade extremal de vento no emprazamento da ponte, e nas funcións de flameo obtidas experimentalmente en túnel de vento. Estúdanse dous casos de optimización probabilista; no primeiro unicamente optimízase o volume do caixón da ponte variando os espesores das chapas que forman o caixón, mentres que no segundo minimízase tanto o volume dos cables principais como o taboleiro da ponte. En ambos os casos, desexáse obter deseños óptimos que satisfagan un valor prefixado da seguridade estrutural fronte a flameo ademais doutras restricións de tipo determinista.

Para resolver este problema, prográmanse en Matlab os métodos de RBDO mencionados anteriormente, código que executa un modelo de elementos finitos realizado en Abaqus para obter as respostas estruturais e o código FLAS, desenvolvido polo noso grupo de investigación, para realizar a análise a flameo. Antes de resolver o problema de RBDO realizanse distintas análises de fiabilidade para coñecer o nivel de seguridade estrutural da ponte co seu deseño orixinal. As formulacións de RBDO propostas aplicáronse a dous exemplos de pontes colgantes de gran van: o Great Belt East Bridge en Dinamarca e o proxecto da Ponte de Messina en Italia. Os resultados obtidos polos diferentes métodos de RBDO compáranse en termos de precisión e eficiencia computacional.
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CHAPTER 1

INTRODUCTION

1.1 Motivation and research objective

The span-lengths of long-span suspension bridges have increased dramatically in the last century. Akashi-Kaikyo Bridge in Japan built in 1995 has the center span length of 1991 meters while the Great Belt East Bridge in Denmark opened in the same year consists of 1624 meter span. The ever-improving construction technologies make bridge spans each time longer, yet at the same time, the bridge structures become more flexible and more prone to flutter. Flutter is an important aeroelastic phenomenon, which may potentially cause the collapse of a structure. Therefore, it is essential to take into account this aeroelastic instability for the design of long-span bridges.

The structural optimization is an important design tool to save cost, especially for large-scale structures. For a structural design, both the load that a structure has to bear (wind actions, earthquake loads, traffic loads, etc.) and the structural resistance as in the case of flutter velocity calculations, contain uncertainties. We cannot simply
neglect uncertainties, especially in parameters with high randomness for structural calculations.

Compared to traditional deterministic optimizations, the Reliability Based Design Optimization (RBDO) performs structural optimization considering system uncertainties to minimize structural weight while satisfying a predetermined structural safety level. The uncertainties are taken into account as a form of random variables, which are included in the limit state functions that in turn constitute probabilistic constraints. In recent years, several reliability methods have demonstrated their capabilities to evaluate these probability constraints.

In RBDO problems, instead of employing safety factors to account for the overall system uncertainties as in the traditional deterministic optimizations, we take into account precise information of the uncertainties of each parameter that affect the structural responses. Consequently, the RBDO can provide more accurate and competitive solutions to an optimization problem than a traditional deterministic optimization.

Although many researchers have worked on the RBDO applied to different types of structures, especially in aerospace field where reducing weight is critical, there has been no research on the RBDO applied to long-span suspension bridges considering probabilistic flutter constraint. Therefore this is the topic that has been studied in this research.

Prior to the application of the RBDO, reliability analyses of the bridge under study were carried out in order to obtain the reliability level of the original bridge design. This reliability index of the original design serves as a reference value when we set the target reliability of the structure for the RBDO problem. The reliability analysis can be a useful tool to identify which random variables are more relevant than others on the structural safety. Some cases of deterministic optimization were also carried out
before the application of the RBDO to see the viability of design optimization applied to long-span bridges.

In order to demonstrate the applicability of the RBDO formulation to bridge structures, some modern long-span bridges with aerodynamic box decks were analyzed. Plate thicknesses of box girders as well as main cable area were chosen as design variables.

### 1.2 Organization of chapters

In this section, the distribution of the entire work is explained, which are divided into seven chapters.

Chapter 2 deals with structural reliability analysis. The stochastic approach of structural analysis using reliability index is introduced by contrasting it with the traditional deterministic approach using safety factor. The advantages of employing probabilistic approach are emphasized. Then different reliability methods are discussed which can be grouped into moment and sampling methods. A modification of a moment method, the First Order Reliability Method (FORM) to solve some convergence problems is described. The transformation of non-normal distribution to normal equivalent distribution is explained. Finally two examples of a cantilever beam and a 10-bar truss illustrate the use of some of the moment and sampling methods.

In Chapter 3, Reliability Based Design Optimization is presented. Two-level methods of Reliability Index Approach (RIA) and Performance Measure Approach (PMA) and a decoupled approach of Sequential Optimization and Reliability Assessment (SORA) are described in detail. The modification of a reliability method of the Hybrid Mean Value (HMV) to improve convergence is proposed by the author. Application examples of a mathematical problem, a multiple limit state problem, a
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Introduction

buckling beam and a 10-bar truss structure illustrate the performance of each RBDO method.

Chapter 4 discusses the RBDO approach of long-span suspension bridges under flutter constraint, which is the most innovative part of this thesis. The importance of considering flutter instability in the design of long-span bridges is discussed, followed by a description of hybrid method to compute flutter velocity. Then the formulation of reliability analysis of long-span suspension bridges under flutter is explained with relevant reliability parameters. Finally the formulation of RBDO applied to long-span suspension bridges considering flutter is presented using the three RBDO methods discussed in Chapter 3. In order to solve the problem, a computer code was created, and the workflows using each method are illustrated by flow diagrams.

In Chapter 5, the approach proposed in Chapter 4 is applied to the Great Belt East Bridge. Deterministic optimizations were performed first to see the feasibility of structural optimization considering flutter constraints. Then reliability analyses were carried out on the original bridge to obtain the structural reliability considering uncertainties in the extreme wind velocity, structural damping and flutter derivatives. Different cases of reliability analyses were performed by varying random variables set and dispersion in flutter derivatives. Finally the RBDO was performed considering two design variable sets by varying the target reliability indices. The results are graphically presented and the computational efficiency of each method is discussed.

The second application example of the Messina Bridge project is presented in Chapter 6. Just as in Chapter 5, deterministic optimizations were carried out followed by reliability analyses of the original bridge design considering different sets of random variables. Ultimately, the RBDO was performed with three sets of design variables for different target reliability indices. The deck of the Messina Bridge consists of three box girders, two lateral boxes for vehicle traffic and the central box for the railways. This is a very interesting design to apply the RBDO methods because we can obtain different material savings whether working with lateral boxes, or both lateral and the central box
girders. The optimization results using the three RBDO methods were discussed and compared for their computational efficiency.

Chapter 7 summarizes the conclusions drawn from this research and the future lines of research are discussed.
CHAPTER 2

STRUCTURAL RELIABILITY ANALYSIS

2.1 Introduction

Most phenomena that a structure can experience in its lifetime are generally not known or understood with certainty beforehand. For example, the occurrence of extreme loads due to natural events such as earthquakes, strong winds and high waves is simply stochastic and the prediction of structural failure due to such events is inherently a probabilistic problem. The properties of a material, for example, are known to vary slightly from one structural element to another and hence they cannot be characterized adequately by deterministic values. If we simply utilize the “worst case” values in order to assure the structural safety, which is a general procedure for a deterministic approach, this assumption often leads to an overly conservative design. For problems in which uncertainties of variables are relatively small, the deterministic approach may be used. However, when the level of randomness is high, the use of probabilistic approach is necessary to achieve a safe and competitive design.
The purpose of employing a probabilistic approach rather than a deterministic approach is to take into account system uncertainties so that a more realistic structural safety assessment can be achieved. These uncertainties may come from different sources such as stochastic nature of events, lack of accurate data, imprecision of measurement, variability of materials, etc. In general sense, they may be classified into two categories: aleatory and epistemic.

Aleatory or objective uncertainty is also called irreducible or inherent uncertainty according to Haldar et al.\(^{[H1]}\). For example, loading due to naturally occurring events such as earthquakes or hurricane are unpredictable and irreducible. Repeated measurements of the same physical quantity do not give the same value due to the fluctuation of the environment, testing procedure, or variation of instruments, etc. On the other hand, epistemic or subjective uncertainty is a reducible uncertainty that originates from lack of knowledge or data. For example, the level of uncertainty varies depending on the data sample size used to determine a certain parameter. Therefore, in this case, the uncertainty can be reduced simply by using a larger sample size. Another type of subjective uncertainty is related to modelling error. Over-simplifying assumptions in analytical models may introduce undesired level of uncertainty, which may certainly be reduced by the use of more precise data for the analysis.

Two Space Shuttle catastrophes in 1986 and 2003 called the attention of general public on structural safety of manned space vehicles (Figure 2-1). These accidents were caused by the combination of technological defects, unpredictable system conditions and incongruent risk management (Keisner\(^{[K1]}\)). Although these system designs all satisfied the structural restrictions, the uncertainty factors of each system were not directly considered in the design constraints. The consideration of uncertainty in a structure is essential for safe systems. Therefore probabilistic-based methods are useful tool to assess reliability of a structural system, which is too complex to be evaluated by any deterministic methods.
This chapter begins with a discussion of some advantages of the probabilistic approach as opposed to the deterministic approach for structural analysis. After basic concept of structural reliability is described along with some reliability classification examples, some commonly used reliability methods, both moment approaches and simulations, are explained. Finally two application examples of a cantilever and a ten-bar truss are used to demonstrate the efficiency and accuracy of each method.

2.2 Deterministic vs. Probabilistic

In structural designs, the limit state conditions are traditionally checked by factors of safety as $SF=R/S$, where $R$ is the resistance of a structure and $S$ is the load applied to a structure. Even though this concept is quite easy to understand, it suffers from various limitations according to Saouma\textsuperscript{S1}: 1) it does not differentiate between resistance and loading uncertainties; 2) it is restricted to service loads; 3) it does not allow comparison of relative reliabilities among different structures for different performance modes.

In reality, both resistance $R$ and load $S$ are random in nature, which can be characterized by their mean values, $\mu_R$ and $\mu_S$, standard deviations, $\sigma_R$ and $\sigma_S$, and their corresponding probability density functions, $p_R$ and $p_S$ as shown in Figure 2-2. The
deterministic values of resistance and load, $R_N$ and $S_N$ are shown in the figure, which are used to calculate a conventional factor of safety as:

$$SF = \frac{R_N}{S_N}$$

in which, $R_N$ should be greater than $S_N$ with specific margin of safety. As can be seen in the figure, both $R_N$ and $S_N$ are conservative values; $R_N$ may be a couple of standard deviations below the mean value, while $S_N$ may be several standard deviations above the mean value. With a presence of uncertainty, the safety factor depends on uncertainty in both load and resistance, and how the nominal load and resistance values are selected. As a result, the safety factor may not represent the actual margin of safety.

The overlapped area (shaded area in the figure) of the density function, $p_R$ and $p_S$, provides the probability of failure qualitatively, which depends on: 1) relative position of the two curves represented by $\mu_R$ and $\mu_S$; 2) dispersion of the two curves characterized by $\sigma_R$ and $\sigma_S$, i.e., curves can be more slender or wider; 3) skewness of the probability distribution function, $p_R$ and $p_S$. The objective of safe design can be achieved by selecting two curves so that the mean value of the resistance curve is greater than that of the load curve, and the overlapping area of both curves is as small as possible.

![Figure 2-2. Probability density function (PDF) of load and resistance of a system](image)
The conventional safety factor approach achieves this objective by shifting the position of the curves using factors of safety. A more logical approach would be achieving the smallest probability of failure from all three overlapping factors mentioned above.

The factor of safety concept, therefore, is often too conservative and leads to an overly-designed structure. Elishakoff\textsuperscript{[E1]} once mentioned regarding safety factor, “This factor allowed continues to enable constructing safe or nearly safe structures that work. The question is: Could such a methodology be improved? Can we do it better even though the American proverb advised that if it ain’t broke, don’t fix it?”

A structural reliability analysis extends this factor of safety concept to explicitly incorporate different types of uncertainties into random variables. These uncertainties can be represented in terms of probability distribution function (PDF) for the purpose of their quantification. Some descriptors such as moments of a particular distribution function can uniquely define a PDF. There are different types of probability distributions such as normal, uniform, log normal, Gamma, Gumbel distributions and so on. For example, parameters with small coefficients of variation such as Elastic Modulus, Poisson’s ratio as well as other material properties can be represented by normal distribution while fatigue failure, material strength and loading variables may be characterized by log-normal distribution functions (Choi et al.\textsuperscript{[C1]}). The extreme values based on historical data of naturally occurring events such as the maximum wind velocity or the maximum water level in a river are usually described by Gumbel distribution.

By performing a reliability analysis of a structure and obtaining its resulting reliability index, which is a universal indicator of a structural adequacy, an engineer can assess the health of a structure and at the same time, compare its reliability with other similar structures. Figure 2-3 shows a comparison between the deterministic and probabilistic approaches.
2.3 Brief history of structural reliability analysis

The mathematical theory of probability is originated in attempts of analyzing games of chance. An Italian Renaissance mathematician Gerolamo Cardano in the sixteenth century analyzed gambling problems systematically and a century later two French mathematicians, Blaise Pascal and Pierre de Fermat dealt with the likelihood of an event for a dice game and quantified uncertain measures of random events (Renyi[1]).

The concept of reliability emerged from the basis of this probability theory. Reliability theory was originally a tool developed by a 19th-century maritime and life insurance companies to compute profitable rates to charge their customers by predicting the probability of death for a given population or an individual (Franklin[1]). As the mathematical theories of materials and structural behavior evolved in the 20th century, they provided more logical basis of structural designs. At the same time, these theories created a necessary foundation to which probabilistic theory can be applied in order to quantify structural safety. One of the first mathematical formulation of structural
reliability can be attributed to Mayer\textsuperscript{[M1]} in 1920s, which was further developed by Streletzki\textsuperscript{[S2]} and Wierbicki\textsuperscript{[W1]}. They recognized that both load and resistance parameters are random variables and therefore, there exist finite probability of failure. This concept was expanded by Freudenthal\textsuperscript{[F2]} in the 1950s although it was too difficult to evaluate involving functions in lack of digital computers. Reliability became a subject of major engineering interest in that era due to the failure of Vanguard rockets in Cape Canaveral and the airplane accidents of the first commercial jet called the British de Havilland comet (Figure 2-4). However, the reliability theory was not considered as a single discipline until 1961 with the publication of \textit{Multi-component systems and their structures and their reliability} by Birnbaum, Esary and Saunders\textsuperscript{[B1]}. Before that time, engineers were simply applying standard techniques such as queuing, statistics and probability theory to engineering reliability problems. In the following two decades, Cornell\textsuperscript{[C2]}, Hasofer and Lind\textsuperscript{[H2]} presented their pioneering works on practical applications of reliability theory. Cornell proposed a second-moment reliability index in 1969, while Hasofer and Lind formulated the First Order Reliability Method, namely FORM in 1974. Rackwitz and Fiessler\textsuperscript{[R3]} proposed an efficient numerical method to calculate reliability index in 1978. By that time, the reliability methods reached a stage of maturity and they are now available for different applications. It is worth mentioning the works in books developed by Thosft-Christensen and Baker\textsuperscript{[T1]}, Madsen, Krenk and Lind\textsuperscript{[M2]}, Melchers\textsuperscript{[M3]}, and Ayyub and McCuen\textsuperscript{[A1]} to name a few.

\textbf{Figure 2-4. Vanguard rocket exploded seconds after launch at Cape Canaveral in 1957 (left) and the crushed British de Havilland comet}
Two of the most commonly used techniques for structural reliability analysis are the FORM as a moment method and the Monte-Carlo simulation as a sampling method. Ever since the FORM method was formulated in 1970s, it has become one of the most important structural reliability methods. It is now used in various commercial computer codes for practical engineering applications. The application of Monte-Carlo simulation to structural engineering is relatively recent in comparison to FORM because of the need of powerful digital computers. Its main advantage is the direct and simple use of experiments to obtain probabilistic information; however, its high computational cost for simulations remains as a major hurdle for its common application to complex systems. There exist other methods for uncertainty quantification such as the Second-Order Reliability Method (SORM) as an moment method and Latin Hypercube and Importance Sampling as simulation methods. Stochastic expansions such as Polynomial Chaos Expansion can be used to quantify uncertainty and represent output responses in the engineering systems where the system response is computed implicitly (Cameron et al. [C4]).

2.4 Structural reliability evaluation

When a structure or a part of a structure exceeds a specific limit state, the structure is no longer able to perform its required functions. This limit is termed a limit-state. The limit state can be classified into two groups: ultimate and serviceability limit. The ultimate limit states are related to the collapse of all or part of a structure, such as rupture, corrosion, fatigue, deterioration, excessive or premature cracking, or permanent plastic deformation (Madsen [M4]). The occurrences of these types of structural behavior should be very low since they may cause loss of lives and significant economic damages. The serviceability limit states are associated with disruption of normal use of a structure, such as excessive deflections, vibration, local damage etc. Even though there will not be catastrophic events with the presence of such events, the structure may not be usable due to large displacements or vibrations.
A limit state function, also called performance function, is defined depending on the failure modes of a structure, and it is a function of random variables that govern the structural behavior as:

\[ G(x) = R(x) - S(x) \]  \hspace{1cm} (2.2)

or in its normalized form,

\[ G(x) = \frac{R(x)}{S(x)} - 1.0 \]  \hspace{1cm} (2.3)

where \( x \) denotes a vector of basic random variables, \( R \) is the resistance of the structure and \( S \), the load on the structure. Figure 2-5 shows an example of a limit state function in load-resistance space. \( G(x)>0 \) indicates safe region of the structure while \( G(x)<0 \) specifies unsafe region with \( G(x)=0 \) being the failure surface.

\[ f_P(G) = P[G(x) < 0] \]  \hspace{1cm} (2.4)

Or in more general form,
\[ P_f = \int \ldots \int_{G<0} f_x(x_1, x_2, \ldots, x_n) \, dx_1, dx_2, \ldots, dx_n \quad (2.5) \]

where \( f_x(x_1, x_2, \ldots, x_n) \) is the joint probability density function for the basic random variables \( x_1, x_2, \ldots, x_n \). The integration is performed over the failure region, \( G<0 \).

Equation (2.5) is called the full distributional approach and the computation of \( P_f \) using this method is considered as the fundamental equation of reliability analysis[^1]. However, the joint density functions for random variables are rarely available for practical engineering problems, so this direct integration method is seldom used. Instead, engineers normally work with the first and the second moments of random variables.

The mean value and the standard deviation of a limit state, \( \mu_G \) and \( \sigma_G \), can be obtained as

\[ \mu_G = \mu_R - \mu_S \quad (2.6) \]
\[ \sigma_G = \sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S} \quad (2.7) \]

where \( \mu_R \) and \( \mu_S \) are the mean values and \( \sigma_R \) and \( \sigma_S \) are the standard deviations of the resistance and load, and \( \rho_{RS} \) is the correlation factor between \( R \) and \( S \). If there is no correlation between \( R \) and \( S \), the term \( 2\rho_{RS}\sigma_R\sigma_S \) becomes null.
Figure 2-6 shows a PDF of a limit state function $G$. Assuming a case that both $R$ and $S$ are normally distributed and uncorrelated, the probability density function of the limit state can be expressed as:

$$f_G(G) = \frac{1}{\sigma_G \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{G - \mu_G}{\sigma_G} \right)^2 \right]$$

(2.8)

The probability of failure, which is the shaded area of the graph, can be obtained by integrating the PDF as:

$$P_f = \int_{-\infty}^{0} f_G(G) dG$$

(2.9)

$$= \int_{-\infty}^{0} \frac{1}{\sigma_G \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{G - \mu_G}{\sigma_G} \right)^2 \right] dG$$

By changing the variable as: $u = \frac{G - \mu_G}{\sigma_G}$
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\[
P_f = \int_{-\infty}^{\frac{-\mu_G}{\sigma_G}} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} u^2 \right] du
\]

\[
= \Phi\left( \frac{-\mu_G}{\sigma_G} \right)
\]

where \( \varphi \) and \( \Phi \) are the PDF and cumulative distribution function (CDF) of a standard normal random variable. The value, \( \frac{\mu_G}{\sigma_G} \), is called reliability index, \( \beta \).

\[
\beta = \frac{\mu_G}{\sigma_G}
\]

The reliability index represents a relative measure of reliability or confidence in the ability of a structure to perform its function in a satisfactory manner (Streletzki\textsuperscript{[S2]}). The probability of failure can be expressed in terms of reliability index as:

\[
P_f = \Phi(-\beta)
\]

The graphical representation of reliability index is shown for a limit state function in Figure 2-6. The reliability index indicates the distance from the mean value of the limit state, \( \mu_G \), to the failure surface, \( G=0 \) expressed in units of standard deviation. A high value of reliability index implies a large distance from the mean value of the limit state function to the failure surface, which makes the probability of failure (shaded area in the graph) small. Figure 2-7 shows a relation between probability of failure and reliability index by Saouma\textsuperscript{[S1]}.

More and more design codes are incorporating the concept of structural reliability such as American Iron and Steel Institute (AISI)\textsuperscript{[A2]}, Ontario highway bridge design code\textsuperscript{[O1]}, Eurocodes\textsuperscript{[E2]} and so on. Table 2-1 summarizes the reliability
classification on civil engineering works in Eurocodes. As can be seen, the target reliability will be dependent upon the type of structures and the implications of loss of human lives caused by their failures. In this manner, engineers can assess the safety of a structure by controlling its probability of failure based on the reliability index computed from the corresponding reliability analysis and comparing the probability of failure among different structural systems.

![Figure 2-7. Probability of failure vs. reliability index](image)

<table>
<thead>
<tr>
<th>Reliability classes</th>
<th>Consequences for loss of human life, economical, social and environmental consequences</th>
<th>Reliability index $\beta$</th>
<th>Examples of buildings and civil engineering works</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 – high</td>
<td>High</td>
<td>5.2</td>
<td>Bridges, public buildings</td>
</tr>
<tr>
<td>2 – normal</td>
<td>Medium</td>
<td>4.7</td>
<td>Residential and office buildings</td>
</tr>
<tr>
<td>1 – low</td>
<td>Low</td>
<td>4.2</td>
<td>Agricultural buildings, greenhouses</td>
</tr>
</tbody>
</table>

Table 2-1. Eurocode on Reliability classification of civil engineering works
2.5 Methods of structural reliability

The main objective of structural reliability is to predict the performance of a particular structure where there is uncertainty in parameters by computing its corresponding probability of failure. There exist different methods to calculate structural reliability, among which moment and simulation methods are most commonly used. The moment methods have advantages over simulation methods for their computational efficiency, especially when the probability of failure is very small. The major drawback of these methods is that the evaluation of gradients of limit state function with respect to each random variable may be complicated and costly. The sampling methods have an advantage over moment methods for their simple implementation; however, high computational cost is still the key for their application to a complex structural system.

In this section, two moment methods of the First Order Second Moment Method (FOSM) and the First Order Reliability Method (FORM), and two sampling methods of Monte Carlo Simulation (MCS) and Latin Hypercube Sampling (LHS) are presented. A simple cantilever and a 10-bar truss problem demonstrate the use of each method and their computational efficiency and accuracy are discussed.

2.5.1 First-Order Second Moment Method (FOSM)

The FOSM, also called the Mean Value First Order Second Moment (MVFOSM) method uses the first order Taylor series expansions to linearize the limit state function. The original formulation was proposed in 1969 by Cornell[25], who developed it originally for two random variables. As the name, the second moment suggests, it utilizes the first (mean) and the second moments (standard deviation) of the parameters disregarding higher moments. The limit state function is represented by the first order Taylor series at the mean value. If all random variables in the vector \( \mathbf{x} \) are statistically independent, the limit state function at the mean value can be written as:
\[
\tilde{G}(\mathbf{x}) \approx G(\mathbf{\mu}_x) + \nabla G(\mathbf{\mu}_x)^T (\mathbf{x} - \mathbf{\mu}_x) \tag{2.13}
\]

where \( \mathbf{\mu}_x = \{\mu_{x_1}, \mu_{x_2} \ldots \mu_{x_n}\}^T \) and \( \nabla G(\mathbf{\mu}_x) = \left\{ \frac{\partial G(\mathbf{\mu}_x)}{\partial x_1}, \frac{\partial G(\mathbf{\mu}_x)}{\partial x_2}, \ldots, \frac{\partial G(\mathbf{\mu}_x)}{\partial x_n} \right\}^T \)

Then the mean value of the approximation of the limit-state function is

\[
\mathbf{\mu}_{\tilde{G}} \approx E\left[ G(\mathbf{\mu}_x) \right] = G(\mathbf{\mu}_x) \tag{2.14}
\]

where \( E \) is an expectation operator. The variance of the approximate limit-state function is:

\[
\sigma_{\tilde{G}}^2 = Var\left[ \tilde{G}(\mathbf{x}) \right] \approx Var\left[ \nabla G(\mathbf{\mu}_x)^T (\mathbf{x} - \mathbf{\mu}_x) \right] \tag{2.15}
\]

Since \( Var\left[ \nabla G(\mathbf{\mu}_x)^T (\mathbf{x} - \mathbf{\mu}_x) \right] = \left[ \nabla G(\mathbf{\mu}_x)^T \right]^2 Var(\mathbf{x}) \), then

\[
\sigma_{\tilde{G}} = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial G(\mathbf{\mu}_x)}{\partial x_i} \right)^2} \sigma_{x_i} \tag{2.16}
\]

The reliability index is calculated as:

\[
\beta = \frac{\mu_{\tilde{G}}}{\sigma_{\tilde{G}}} \tag{2.17}
\]

Equation (2.17) is the same as Equation (2.11) when the limit state function is linear. Although the method is simple and computationally efficient, it has several disadvantages. Because the method linearizes the limit state function by first-order Taylor series and truncates higher terms, large error can result in case of highly nonlinear limit-state function. Also reliability index is not invariant with mathematically equivalent limit state equations. For example, normalized and non-normalized equations may provide different reliability indices.
2.5.2 First Order Reliability Method (FORM)

The FORM was proposed by Hasofer and Lind\cite{H2} in 1974 to overcome these drawbacks of the FOSM. Since then this method has spread out widely and it has become one of the most important method to evaluate structural reliability. Various commercial codes based on the FORM are available such as STRUREL\cite{R2}. The major improvement of this method over the FOSM is to expand the limit state function at the Most Probable Point of Failure (MPP) rather than at the mean value. The method consists of linearly mapping the basic random variables in a set of normalized and independent variables, $u_i$. The basic random variables must be independent of one another and normally distributed. In the case that the variables are non-normally distributed, their normal distribution equivalents should be obtained as will be explained in Section 2.5.7. The method is generally accurate for practical purposes when the limit state function does not have significant curvature (Bjerager\cite{B2}) and it is computationally efficient especially for small probability of failure compared to sampling methods (Dundulis\cite{D1}).

The first step of this method is to transform the basic random variables into normalized form so that the variables have zero means and unit standard deviations. This common transformation in statistics called Rosenblatt Transformation is described as:

$$ u_i = \frac{x_i - \mu_i}{\sigma_i} \quad (2.18) $$

The failure surface of the limit state function can be rewritten in $u$-space as $G(u)=0$. Then the reliability index, $\beta$ can simply be interpreted as the shortest distance from the origin of the $u$-space to the failure surface. This problem is interpreted as an optimization problem.

$$ \beta = \min \left( \sum_{i=1}^{n} u_i^2 \right) \quad i=1...n \quad (2.19a) $$
subject to  \[ G(\mathbf{u}) = 0 \] \hspace{1cm} (2.19b)

The point on the performance function, \( \mathbf{u}^* \), which indicates the minimum distance from the origin to the failure surface shown in the Figure 2-8 is called the Most Probable Point of failure (MPP). It represents the most probable combination of the random variables in case of structural failure.

In order to solve this optimization problem, there are different algorithms available such as feasible directions, penalty methods, dual methods, Lagrange multiplier methods and so on (Hernandez\(^\text{[H3]}\)). In this case, Lagrange multiplier method is used to solve the problem. The problem of Equation (2.19) is transformed to:

\[
\text{Minimize} \quad L(u_i, \lambda) = \sqrt{\sum_{i=1}^{n} u_i^2} + \lambda G(\mathbf{u}) \quad i=1...n
\]  

(2.20)

The conditions for a stationary point are:

\[
\frac{\partial L}{\partial u_i} = \frac{u_i}{\sqrt{\sum_{i=1}^{n} u_i^2}} + \lambda \frac{\partial G}{\partial u_i} = 0
\] 

(2.21)
\[ \frac{\partial L}{\partial \lambda} = G(\mathbf{u}) = 0 \] (2.22)

\( G(\mathbf{u}) \) at \( \mathbf{u}^* \) can be expressed by the first-order Taylor expansion as

\[ G(\mathbf{u}) \approx G(\mathbf{u}^*) + \sum_{i=1}^{n} \frac{\partial G(\mathbf{u}^*)}{\partial u_i} (u_i - u_i^*) \] (2.23)

From Equation (2.21) we get:

\[ \frac{\partial G(\mathbf{u}^*)}{\partial u_i} = -\frac{u_i}{\lambda \sqrt{\sum_{i=1}^{n} u_i^2}} \] (2.24)

and \( \lambda \) can be deduced as:

\[ \lambda = \frac{1}{\sqrt{\sum_{i=1}^{n} \left( \frac{\partial G(\mathbf{u}^*)}{\partial u_i} \right)^2}} \] (2.25)

By combining Equation (2.22) and (2.23),

\[ G(\mathbf{u}^*) + \sum_{i=1}^{n} \frac{\partial G(\mathbf{u}^*)}{\partial u_i} \cdot u_i - \sum_{i=1}^{n} \frac{\partial G(\mathbf{u}^*)}{\partial u_i} \cdot u_i^* = 0 \] (2.26)

By plugging Equation (2.24) into (2.26)

\[ G(\mathbf{u}^*) - \frac{\sqrt{\sum_{i=1}^{n} u_i^2}}{\lambda} - \sum_{i=1}^{n} \frac{\partial G(\mathbf{u}^*)}{\partial u_i} \cdot u_i^* = 0 \] (2.27)

Then \( \beta \) can be expressed as:
The expression of $\beta$ coincides with the distance formula from the origin of the $u$-space to the hyperplane that approximates the performance function.

The direction cosine of the unit normal vector in $u$-space is written as:

$$
\alpha(u) = -\frac{\partial G(u^*)}{\partial u_i} \frac{1}{\sqrt{\sum_{j=1}^{n} \left( \frac{\partial G(u^*)}{\partial u_j} \right)^2}}
$$

The direction cosine in the original $x$-space is:

$$
\alpha(x) = -\frac{\partial G(x^*)}{\partial x_i} \frac{1}{\sqrt{\sum_{j=1}^{n} \left( \frac{\partial G(x^*)}{\partial x_j} \frac{\sigma_{x_j}}{\sigma_{x_i}} \right)^2}}
$$

The most probable point of failure, MPP, in $u$-space can then be written as:

$$
u^* = \alpha(u)\beta
$$

And in $x$-space

$$
x_i^* = \mu_{x_i} + \alpha \sigma_{x_i} \beta \quad (i=1,2,...,n)
$$

Since $\mathbf{u}^*$ is not known beforehand, an iterative process must be employed to obtain $\beta$. The main steps to compute $\beta$ by employing the FORM are described in the following and illustrated in the flowchart of Figure 2-9.
1. Define the limit state function.

2. Transform the random variables in $u$-space and use the mean value point, $\mu_x$, i.e., $u=0$ as initial values of MPP.

3. In order to approximate the limit state function at the design point $x^k$ ($x$ vector in the $k$th iteration), the gradient of the limit state function with respect to each random variable needs to be evaluated at such point.

4. Calculate reliability index, $\beta$ using Equation (2.28) and its direction cosine from Equation (2.29).

5. The new MPP point is computed in $u$-space from Equation (2.31) and in $x$-space from Equation (2.32).

6. Repeat steps 3-5 until the $\beta$ value converges.
2.5.3 The modification of the FORM

There have been reported cases that the FORM algorithm does not converge well or even diverges depending on the performance function or the initial design point as described by Choi et al.\cite{C1}. The modification of the FORM algorithm was proposed by Baldomir et al.\cite{B3} by introducing a reduction factor when a new design point is computed in Equation (2.31) as:

$$\bar{u}^{k+1} = u^k + \frac{u^{k+1} - u^k}{c}$$  \hspace{1cm} (2.33)
where $\bar{U}^{k+1}$ is a new design point and $c$ is a reduction factor. In order to demonstrate the effectiveness of this method, it was applied to the following mathematical example from Choi et al. The performance function is defined as:

$$G(x_1, x_2) = x_1^3 + x_2^3 - 18$$  \hspace{1cm} (2.34)$$

where $x_1$ and $x_2$ are normally distributed random variables with the mean values, $\mu_{x_1} = 10$ and $\mu_{x_2} = 9.9$, and their respective standard deviations, $\sigma_{x_1} = \sigma_{x_2} = 5$.

A reliability analysis was carried out using the FORM. The results with and without a reduction factor are tabulated in Table 2-2 and 2-3 while the evolution of $x_1$ values are shown graphically in Figure 2-10. As can be clearly seen, in the case without a reduction factor, the value of $x_1$ oscillates and the algorithm does not converge. On the other hand, using the reduction factor of 4, $x_1$ value converges to a value of 2.086. The convergence of the reliability analysis can also be checked by the value of the limit state function, which should be nearly zero at MPP. The limit state function value near convergence with $c=4$ is very small in Table 2-3, while it take a large value in the case without reduction factor in Table 2-2.

In conclusion, the introduction of reduction factor in the case of convergence problems is very effective. For example, in reliability analyses of the Messina Bridge presented in Chapter 6, the reduction factor between 2 and 4 has been employed to achieve convergence.
Figure 2-10. Comparison of the evolution of $x_i$ value with $c=0$ and $c=4$

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$G(x)$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.0000</td>
<td>9.9000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1952.2990</td>
<td>0.9295</td>
</tr>
<tr>
<td>1</td>
<td>6.6808</td>
<td>6.6468</td>
<td>-0.6638</td>
<td>-0.6506</td>
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</tr>
<tr>
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<td>4.5323</td>
<td>4.4877</td>
<td>-1.0935</td>
<td>-1.0825</td>
<td>165.4854</td>
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</tr>
<tr>
<td>3</td>
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<td>3.1717</td>
<td>-1.3725</td>
<td>-1.3457</td>
<td>44.7852</td>
<td>2.1339</td>
</tr>
<tr>
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<td>2.2739</td>
<td>-1.4924</td>
<td>-1.5252</td>
<td>10.1088</td>
<td>2.2001</td>
</tr>
<tr>
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<td>3.0139</td>
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<td>-1.7615</td>
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<td>7</td>
<td>5.2864</td>
<td>9.8130</td>
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<td>1.1651</td>
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<tr>
<td>49</td>
<td>4.3490</td>
<td>8.4847</td>
<td>-1.1302</td>
<td>-0.2831</td>
<td>675.0719</td>
<td>1.1656</td>
</tr>
<tr>
<td>50</td>
<td>8.5191</td>
<td>4.2634</td>
<td>-0.2962</td>
<td>-1.1273</td>
<td>677.7743</td>
<td>1.1651</td>
</tr>
</tbody>
</table>

Table 2-2. The results of the example with $c=0$
Chapter 2

Structural Reliability Analysis

2.5.4 Second-order reliability method (SORM)

When the limit state function is highly non-linear, its approximation by a linear surface may not be satisfactory (Mayer\textsuperscript{[M1]}). The more curvature the limit state function has about the design point, the less accurate the linear surface approximation becomes. The SORM approximate the hyperplane failure surface by a second order surface defined by the second order Taylor series expansion of the limit state function or by a curvature-fitted or point-fitted hyper-parabolic surface. See Fiessler et al\textsuperscript{[F3]}, Breitung\textsuperscript{[B4]}, Der Kiureghian et al\textsuperscript{[D2]}. The major drawback of this method is its high computational cost of evaluating the second-order derivatives.

2.5.5 Monte Carlo Simulation (MCS)

The MCS named after the famous Casino in Monaco is a sampling method that consists of simulating a large amount of experiments artificially by randomly generated sampling sets and observing the results. It originates from the research work of Neumann and Ulam in 1949 in Los Alamos Scientific Laboratory who were investigating radiation shielding (Sobol\textsuperscript{[S3]}). Probability theory predicts that the number of mutually independent outcomes must be very large in order to achieve accuracy,

Table 2-3. The results of the example with \(c=4\)

<table>
<thead>
<tr>
<th>Iter.</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(u_1)</th>
<th>(u_2)</th>
<th>(G(x))</th>
<th>(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.0000</td>
<td>9.9000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1952.2990</td>
<td>0.9295</td>
</tr>
<tr>
<td>1</td>
<td>9.1702</td>
<td>9.0867</td>
<td>-0.1660</td>
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<td>1503.4173</td>
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<td>8.3417</td>
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</tr>
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<td>4</td>
<td>7.0869</td>
<td>7.0358</td>
<td>-0.5826</td>
<td>-0.5728</td>
<td>686.2176</td>
<td>1.4658</td>
</tr>
<tr>
<td>5</td>
<td>6.5102</td>
<td>6.4656</td>
<td>-0.6980</td>
<td>-0.6869</td>
<td>528.2169</td>
<td>1.5708</td>
</tr>
<tr>
<td>6</td>
<td>5.9848</td>
<td>5.9454</td>
<td>-0.8030</td>
<td>-0.7909</td>
<td>406.5187</td>
<td>1.6657</td>
</tr>
<tr>
<td>7</td>
<td>5.5066</td>
<td>5.4715</td>
<td>-0.8987</td>
<td>-0.8857</td>
<td>312.7834</td>
<td>1.7511</td>
</tr>
<tr>
<td>8</td>
<td>5.0723</td>
<td>5.0408</td>
<td>-0.9855</td>
<td>-0.9718</td>
<td>240.5874</td>
<td>1.8277</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>2.0860</td>
<td>2.0744</td>
<td>-1.5828</td>
<td>-1.5651</td>
<td>0.0032</td>
<td>2.2260</td>
</tr>
<tr>
<td>49</td>
<td>2.0860</td>
<td>2.0743</td>
<td>-1.5828</td>
<td>-1.5651</td>
<td>0.0024</td>
<td>2.2260</td>
</tr>
<tr>
<td>50</td>
<td>2.0860</td>
<td>2.0743</td>
<td>-1.5828</td>
<td>-1.5651</td>
<td>0.0018</td>
<td>2.2260</td>
</tr>
</tbody>
</table>
which corresponds to random behavior of games of chance in gambling. Therefore the application of this method to structural engineering was not possible until the appearance of high-speed digital computers in recent years. The roulette wheels in casinos were replaced by mathematical algorithms, which generate a series of pseudo random numbers. This method is used in structural engineering to approximate probability of structural failure considering system uncertainty. It consists of sampling each random variable on aleatory base and evaluating the limit state function. The experiment should be repeated many times in order to achieve accuracy. Therefore the major drawback of MCS is its high computational cost for the need of time-consuming large number of simulations. Nevertheless it will be more and more manageable in the future employing applications with time-saving refinements along with the advancement of high-powered computers.

In the examples of the subsequent section, the Monte Carlo simulation was used to validate the results of FORM employing a Matlab code. Random variables are generated using the form \( x_i = \mu + r_i \sigma \), where \( r_i \) is the standard normal inverse of the randomly generated value between 0 and 1. The limit state function is evaluated with the generated random variables and the probability of failure is computed by the ratio of numbers of failure over the total number of simulations. The random variables are generated in vectorial forms since Matlab handles matrix operations much faster than repeated loop operations.

### 2.5.6 Latin Hypercube Sampling (LHS)

The LHS is a sampling method for generating a set of random parameters from multidimensional distribution. It is generally used to reduce the number of simulations necessary for the MCS to achieve reasonably accurate random distributions. The method is valid with any type of distribution functions.

Latin square is a term used in a statistical sampling that a square grid containing sample position has only one sample in each row and each column. The term, hypercube represents the extension of this concept to an arbitrary number of higher
dimensions for multiple design variables. Therefore LHS, also known as the Stratified Sampling Technique, performs multivariate sampling method with no overlapping designs\cite{S4}. This method was first proposed by McKay et al.\cite{M5} in 1979, while independently equivalent technique was proposed by Eglajs\cite{E3} in 1977.

Using this method, the distribution of each random variable is subdivided into $n$ equally-spaced probability intervals. Each of $n$ intervals has one sample point, which has $1/n$ of distribution probability. The basic steps of the method are illustrated in Figure 2-11. One thing that we should have in mind for multiple random variables cases is that we should try to choose a random variable pair in a way that there is no high correlation among these pairs. For example, if we choose a pair (1,1), (2,2), (3,3), (4,4), they are highly correlated.

\begin{enumerate}
  \item Subdivide the interval into $n$ equally-spaced subintervals.
  \item Choose a value randomly with respect to the PDF in each subinterval.
  \item Repeat the operation for the rest of random variables.
  \item Choose a pair of points so that no overlapping occurs.
\end{enumerate}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2-11.png}
\caption{Latin Hypercube Sampling for 2 random variables case}
\end{figure}

2.5.7 Hasofer Lind – Rackwitz Fiessler method

As mentioned previously, moment reliability methods such as the FORM and the SORM require that the distribution of random variables to be independent and normally distributed. However, many practical engineering problems involve in non-Gaussian random variables. In such cases, it is necessary to transform the random variables into normal-equivalent using normal tail approximation. There exist linear
transformations proposed by Rackwitz and Fiessler\cite{R3}, Chen and Lind\cite{C3} or nonlinear transformations by Rosenblatt\cite{R4} and Nataf\cite{N1}. 

In this section, Hasofer Lind – Rackwitz Fiessler linear transformation method is described. This method provides a normal equivalent by imposing that at the design point, \( x^* \), both probability distribution and cumulative distribution of a non-normal distribution and its equivalent normal distribution are equal as shown in Figure 2-12.

![Figure 2-12: Transformation of non-normal to normal equivalent distribution](image)

**Figure 2-12: Transformation of non-normal to normal equivalent distribution**

Denoting that \( F_{x_i}(x_i) \) and \( f_{x_i}(x_i) \) are cumulative and probability distribution functions, \( x_i \) and \( x_i' \) are the original and equivalent normally distributed random variables. We set that the cumulative functions at \( x^* \) for both original and its equivalent cases are equal:

\[
F_{x_i}(x_i^*) = F_{x_i}(x_i^*) = \Phi \left( \frac{x_i^* - \mu_{x_i'}}{\sigma_{x_i'}} \right) 
\]

(2.35)

where \( \mu_{x_i'} \) is the mean value and \( \sigma_{x_i'} \) is the standard deviation of the equivalent normally distributed variables. Then

\[
\mu_{x_i'} = x_i^* - \sigma_{x_i'} \Phi^{-1} \left( F_{x_i}(x_i^*) \right) 
\]

(2.36)
Likewise, the probability density functions of $x_i$ and $x_i'$ at $x^*$ are the same

$$f_{x_i}(x_i^*) = f_{x_i'}(x_i^*) = \frac{1}{\sigma_{x_i'}} \Phi\left(\frac{x_i^* - \mu_{x_i}}{\sigma_{x_i'}}\right)$$  \hspace{1cm} (2.37)

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right)$ and $y = \frac{x_i^* - \mu_{x_i}}{\sigma_{x_i'}}$.

Substituting Eqn.(2.36) into Eqn. (2.37), we get

$$\sigma_{x_i} = \frac{\phi\left(\Phi^{-1}\left[F_{x_i}(x_i^*)\right]\right)}{f_{x_i}(x_i^*)}$$  \hspace{1cm} (2.38)

### 2.6 Examples of reliability analysis

#### 2.6.1 Cantilever example

Let us now consider an example of reliability analysis applied to a common structure. Figure 2-13 shows a cantilever beam of length $L$ with rectangular cross section of base, $b$ and height $h$, to which a vertical tip load, $P$ is applied. We want to limit the maximum vertical displacement to be less than $L/500$. We study three cases of the problem depending on the number of random variables and the performance functions to define structural failures. In Case 1, two random variables of $P$ and $E$ are studied while in Case 2, four random variables of $P$, $E$, $L$ and $h$ are considered, both under the limit state of the vertical tip displacement. In Case 3, the same random variables as Case 2 are considered with an additional limit state function that restricts the maximum bending moment. For all three cases, the problem was first resolved by the FORM then followed by the MCS and the LHS to compare the results.
I. The resolution by FORM

*Case 1: two random variables of $P$ and $E*$

In this case, we consider two random variables of the load $P$ and the elastic modulus $E$. Both random variables are considered as normally distributed with their mean and standard deviations defined as: $N_P$ (40 kN, 4 kN) and $N_E$ (3.0e+7 kPa, 1.5e+6 kPa). The rest of the relevant data are: $L=3.0$ m; $b=0.25$ m; $h=0.5$ m.

The limit state function that restricts the tip displacement and its derivatives with respect to each random variable are:

$$G(P,E) = \frac{L}{500} - \frac{PL^3}{3EI} \geq 0$$  \hspace{1cm} (2.39a)

$$\frac{\partial G}{\partial P} = -\frac{L^3}{3EI}$$  \hspace{1cm} and \hspace{1cm} $$\frac{\partial G}{\partial E} = \frac{PL^3}{3E^2I}$$  \hspace{1cm} (2.39b)

where $I=(bh^3)/12$. The results of reliability analysis for *Case 1* are summarized in Table 2-4. The algorithm converged quickly and the resulting reliability index of the structure is $\beta=2.532$, which corresponds to $P_f=5.68E-03$. The idea of MPP is that if the structure ever fails, it is most likely that the failure would be produced with this combination of random variable values. In this case, at MPP, $P=48.486$ kN and $E=2.793E+7$ kPa. The tip load increased from the initial value while the elastic modulus decreased as expected. The limit state function is null at the converged MPP.
Table 2-4. The results of Case I with $P$ and $E$ as random variables

Case 2: Four random variables of $P$, $E$, $L$ and $h$

In this case, we consider two additional random variables of beam length $L$ and beam height $h$ to those in Case 1. All the random variables are considered to be normally distributed and uncorrelated among one another. The performance function limits the tip displacement as in Case 1. The data for the additional random variables are: $N_L=(3.0 \text{ m}, 0.06 \text{ m}), N_h=(0.5 \text{ m}, 0.01 \text{ m})$.

$$G(P,E,L,h) = \frac{L}{500} - \frac{4PL^3}{Ebh^3} \geq 0$$

(2.40a)

$$\frac{\partial G}{\partial P} = -\frac{4L^3}{Ebh^3}; \quad \frac{\partial G}{\partial E} = \frac{4PL^3}{E^2bh^3}; \quad \frac{\partial G}{\partial L} = \frac{1}{500} - \frac{12PL^2}{Ebh^3}; \quad \frac{\partial G}{\partial h} = \frac{12PL^3}{Ebh^4}$$

(2.40b)

The results of the reliability analysis for Case 2 are summarized in Table 2-5. The resulting probability of failure, $P_f=0.0202 \ (\beta=2.05)$ is 20-fold larger than that of Case 1 for taking into account additional uncertainties in $L$ and $h$. The MPP values are $P=45.728 \text{ kN}, E=2.872E+7 \text{ kN/m}^2, L=3.039 \text{ m}$ and $h=0.490 \text{ m}$. 

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$P$</th>
<th>$E$</th>
<th>$u_P$</th>
<th>$u_E$</th>
<th>$G(x)$</th>
<th>$dG/du_P$</th>
<th>$dG/du_E$</th>
<th>$\alpha_P$</th>
<th>$\alpha_E$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40.000</td>
<td>3.00E+07</td>
<td>0.000</td>
<td>0.000</td>
<td>1.392E-03</td>
<td>-4.608E-04</td>
<td>2.304E-04</td>
<td>8.944E-01</td>
<td>-4.472E-01</td>
<td>2.702</td>
</tr>
<tr>
<td>5</td>
<td>48.486</td>
<td>2.793E+07</td>
<td>2.122</td>
<td>-1.381</td>
<td>0.000E+00</td>
<td>-4.950E-04</td>
<td>3.223E-04</td>
<td>8.380E-01</td>
<td>-5.456E-01</td>
<td>2.532</td>
</tr>
</tbody>
</table>
Table 2-5. The results of Case 2 considering $P$, $E$, $L$ and $h$ as random variables

Case 3: Four random variables with two limit state functions

We now consider the second limit state function to restrict the bending moment of the beam. The maximum bending moment is limited to $M_{LU}=140$ kNm. The two limit state functions in this case are:

\[
G_1(P, E, L, h) = \frac{L}{500} - \frac{4PE^3}{Ebh^3} \geq 0
\]  
\[ (2.41a) \]
\[
G_2(P, L) = M_{LU} - LP \geq 0
\]  
\[ (2.41b) \]

Table 2-6 summarizes the reliability results of Case 3. The reliability index for the second limit state function, $G_2$, is $\beta=1.624$ ($P_f=0.0522$), which is smaller than that of the first limit state function, $G_1$. Therefore $G_2$ is the active limit state function, which restricts the design of the beam. The probability of failure goes up from 0.0202 in Case 2 to 0.0522 in Case 3.
Table 2-6. The results of case 3 with the two limit state functions

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$P(G_1)$</th>
<th>$E(G_1)$</th>
<th>$L(G_1)$</th>
<th>$h(G_1)$</th>
<th>$P(G_2)$</th>
<th>$L(G_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40.000</td>
<td>3.00E+07</td>
<td>3.000</td>
<td>0.500</td>
<td>40.000</td>
<td>3.000</td>
</tr>
<tr>
<td>1</td>
<td>47.004</td>
<td>2.87E+07</td>
<td>3.036</td>
<td>0.489</td>
<td>46.410</td>
<td>3.019</td>
</tr>
<tr>
<td>2</td>
<td>45.635</td>
<td>2.87E+07</td>
<td>3.040</td>
<td>0.490</td>
<td>46.329</td>
<td>3.022</td>
</tr>
<tr>
<td>3</td>
<td>45.733</td>
<td>2.87E+07</td>
<td>3.039</td>
<td>0.490</td>
<td>46.329</td>
<td>3.022</td>
</tr>
<tr>
<td>4</td>
<td>45.728</td>
<td>2.87E+07</td>
<td>3.039</td>
<td>0.490</td>
<td>46.329</td>
<td>3.022</td>
</tr>
<tr>
<td>5</td>
<td>45.728</td>
<td>2.87E+07</td>
<td>3.039</td>
<td>0.490</td>
<td>46.329</td>
<td>3.022</td>
</tr>
</tbody>
</table>

II. The resolution of the problem by MCS and LHS

The above example was also resolved by sampling methods of the MCS and the LHS for different numbers of simulations whose results are tabulated in Table 2-7. The error was calculated using the $P_f$ of one million simulations by MCS as a reference value.

Figure 2-14 shows sampling points of the MCS and the LHS for 1000 simulations. As explained earlier, the MCS sample points only randomly, while LHS pick points taking into account the number of intervals of the distribution function and the reduction in correlation among random variables. In our example, the number of interval of 100 was chosen.

For all cases, as the number of simulations increase, the probability of failure for each method converges to a certain value. The difference of $P_f$ value for $1E+6$ simulations between MCS and LHS is relatively small; the maximum difference is 1.6% for Case 1. By comparing the results of MSC and LHS for ten thousand and one million simulations, it can be concluded that LHS is more advantageous when working with smaller number of sampling points. The differences in results using MCS $1E+6$ simulations and FORM are very small; the maximum difference is 1.3% for Case 2.
Figure 2-14. Sampling points of a) MCS and b) LHS for Case 1 (1000 simulations)
Table 2-7. Comparison of probability of failure by MCS, LHS and FORM

<table>
<thead>
<tr>
<th></th>
<th>No. Simlns.</th>
<th>Case 1</th>
<th>error %</th>
<th>Case 2</th>
<th>error %</th>
<th>Case 3</th>
<th>error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS</td>
<td>1.E+04</td>
<td>6.50E-03</td>
<td>14.78</td>
<td>1.98E-02</td>
<td>0.58</td>
<td>4.90E-02</td>
<td>5.64</td>
</tr>
<tr>
<td></td>
<td>1.E+05</td>
<td>5.36E-03</td>
<td>5.35</td>
<td>1.94E-02</td>
<td>2.39</td>
<td>5.21E-02</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>1.E+06</td>
<td>5.66E-03</td>
<td>1.99E-02</td>
<td>5.19E-02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LHS</td>
<td>1.E+04</td>
<td>5.30E-03</td>
<td>6.41</td>
<td>1.97E-02</td>
<td>1.08</td>
<td>5.23E-02</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>1.E+05</td>
<td>5.75E-03</td>
<td>1.54</td>
<td>1.96E-02</td>
<td>1.68</td>
<td>5.24E-02</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>1.E+06</td>
<td>5.75E-03</td>
<td>1.59</td>
<td>2.01E-02</td>
<td>1.03</td>
<td>5.19E-02</td>
<td>0.11</td>
</tr>
<tr>
<td>FORM</td>
<td>5.68E-03</td>
<td>0.25</td>
<td>2.02E-02</td>
<td>1.34</td>
<td>5.22E-02</td>
<td>0.50</td>
<td></td>
</tr>
</tbody>
</table>

2.6.2 10-bar truss example

Let us now consider a typical 10-bar truss problem as shown in Figure 2-15. Two point loads of $P$ are applied at Node 2 and 4. There are three types of bars of horizontal, vertical and diagonal whose cross sectional areas are $a_1$, $a_2$, $a_3$ respectively. The vertical displacement at node 2 and the stresses in bar 3 and 7 are checked by their corresponding limit state functions. The allowable vertical tip displacement is set to 0.12 m and the stress in bar 3 and 7 is limited to 2.0E+5 kPa. The rest of the relevant data are as follows:

$P=500$ kN; $L=10$ m; $E=7E+7$ kN/m²

Two cases of the problem are studied. In Case 1, three random variables of the cross sectional area, $a_1$, $a_2$, $a_3$ are considered while in Case 2 six random variables of these three cross sectional areas as well as $P$, $E$ and $L$ are taken into account. Three limit state functions restrict the vertical tip displacement ($G_1$), stress in bar 3 ($G_2$) and stress in bar 7 ($G_3$).
I. The resolution of the problem by FORM

Case 1: 3 random variables of the horizontal, vertical and diagonal cross sectional area of the truss bars: \( a_1, a_2, a_3 \)

We first look at a case in which the three cross sectional areas are considered as random variables under the three limit state functions of \( G_1, G_2 \) and \( G_3 \). Analytical expressions of these limit state functions are used from finite element procedure\(^{[C1]}\). The mean and standard deviations of each variable are as follows.

\[
\begin{align*}
N_{a1} & \ (0.01 \text{ m}^2, 0.001 \text{ m}^2), \ N_{a2} \ (0.0015 \text{ m}^2, 0.00015 \text{ m}^2), \ N_{a3} \ (0.007 \text{ m}^2, 0.0007 \text{ m}^2) \\
\end{align*}
\]

\[
G_1(a_1, a_2, a_3) = d_{allow} - \frac{PL}{a_1 a_3 E} \left( \frac{A+B}{D} \right) \geq 0 \tag{2.42a}
\]

\[
G_2(a_1, a_2, a_3) = \sigma_{allow} - \frac{2P}{a_1} - \frac{Pa_2 a_3 \left( 2\sqrt{2} a_1 + a_3 \right)}{D a_1} \geq 0 \tag{2.42b}
\]
\[ G_3(a_1, a_2, a_3) = \sigma_{\text{allow}} - \frac{2P}{a_3} \left( \frac{4Pa^2_1a_2a_3 + \sqrt{2}Pa_1a_2a_3^2}{D a_3} \right) \geq 0 \] (2.42c)

where  
\[ A = 96\sqrt{2}a_1^3a_2^2 + 4\sqrt{2}a_1^3a_3^2 + 7a_1^2a_3^3 + 26a_2^2a_3^3 \]
\[ B = 80a_1^3a_2a_3 + 304a_1^2a_2^2a_3 + 40a_1a_2a_3^3 + 100\sqrt{2}a_1^2a_2a_3^2 + 116\sqrt{2}a_2a_3^2a_3^2 \]
\[ D = 32a_1^2a_2^2 + 4a_2^2a_3^2 + 12\sqrt{2}a_1^2a_2a_3 + 16\sqrt{2}a_1a_2a_3^2 + a_1^2a_3^2 + 6a_2a_3^2 \]
\[ d_{\text{allow}} = 0.12 \text{ m} \]
\[ \sigma_{\text{allow}} = 2.0 \times 10^5 \text{ kPa} \]

The random variable vector, \( x \), and normalized random variables, \( u_i \), are defined as:

\[ x = [a_1, a_2, a_3] \]
\[ u_i = \frac{a_i - \mu_{a_i}}{\sigma_{a_i}} \quad i = 1, 2, 3 \]

The results of the reliability analyses under these three limit state functions are summarized in Table 2-8 through Table 2-10. The resulting beta for \( G_1, G_2, G_3 \) are 2.97, 4.77 and 2.39 respectively. The smallest beta resulted from these three cases is 2.39 (\( P_f = 0.0229 \)) under the third limit state function that restricts the stress in bar 7, which is the most critical constraint. The MPP value for \( G_3 \) was \( (a_1, a_2, a_3) = (0.0100 \text{ m}^2, 0.00150 \text{ m}^2, 0.00533 \text{ m}^2) \). The change in the area of the diagonal bars is the most significant, which is more than 7 fold increase at MPP.
Table 2-8. The results of Case 1 for the 1st limit state function, $G_1$

<table>
<thead>
<tr>
<th>Iter.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00E-02</td>
<td>7.31E-03</td>
<td>7.79E-03</td>
<td>7.82E-03</td>
<td>7.82E-03</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1.50E-03</td>
<td>1.47E-03</td>
<td>1.48E-03</td>
<td>1.48E-03</td>
<td>1.48E-03</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7.00E-03</td>
<td>5.18E-03</td>
<td>5.56E-03</td>
<td>5.59E-03</td>
<td>5.59E-03</td>
</tr>
<tr>
<td>$G(x)$</td>
<td>2.45E-02</td>
<td>-8.70E-03</td>
<td>-5.60E-04</td>
<td>-2.81E-06</td>
<td>-3.98E-08</td>
</tr>
<tr>
<td>$dG/du_1$</td>
<td>4.69E-03</td>
<td>8.77E-03</td>
<td>7.72E-03</td>
<td>7.66E-03</td>
<td>7.67E-03</td>
</tr>
<tr>
<td>$dG/du_2$</td>
<td>3.25E-04</td>
<td>4.10E-04</td>
<td>3.87E-04</td>
<td>3.86E-04</td>
<td>3.86E-04</td>
</tr>
<tr>
<td>$dG/du_3$</td>
<td>4.53E-03</td>
<td>8.19E-03</td>
<td>7.13E-03</td>
<td>7.05E-03</td>
<td>7.04E-03</td>
</tr>
<tr>
<td>$\beta$</td>
<td>3.75</td>
<td>3.02</td>
<td>2.97</td>
<td>2.97</td>
<td>2.97</td>
</tr>
</tbody>
</table>

Table 2-9. The results of Case 1 for the 2nd limit state function, $G_2$

<table>
<thead>
<tr>
<th>Iter.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>....</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00E-02</td>
<td>9.09E-04</td>
<td>1.67E-03</td>
<td>....</td>
<td>5.23E-03</td>
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<tr>
<td>$a_2$</td>
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<td>1.49E-03</td>
<td>1.50E-03</td>
<td>....</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7.00E-03</td>
<td>7.04E-03</td>
<td>7.00E-03</td>
<td>....</td>
<td>7.01E-03</td>
<td>7.01E-03</td>
</tr>
<tr>
<td>$G(x)$</td>
<td>9.52E+04</td>
<td>-9.36E+05</td>
<td>-4.23E+05</td>
<td>....</td>
<td>-8.41E-01</td>
<td>-3.54E-06</td>
</tr>
<tr>
<td>$dG/du_1$</td>
<td>1.05E+04</td>
<td>1.23E+06</td>
<td>3.70E+05</td>
<td>....</td>
<td>3.81E+04</td>
<td>3.81E+04</td>
</tr>
<tr>
<td>$dG/du_2$</td>
<td>7.95E+01</td>
<td>1.99E+03</td>
<td>9.99E+02</td>
<td>....</td>
<td>2.04E+02</td>
<td>2.04E+02</td>
</tr>
<tr>
<td>$dG/du_3$</td>
<td>-6.37E+01</td>
<td>-5.25E+02</td>
<td>-4.01E+02</td>
<td>....</td>
<td>-1.38E+02</td>
<td>-1.38E+02</td>
</tr>
<tr>
<td>$\beta$</td>
<td>9.09</td>
<td>8.33</td>
<td>7.19</td>
<td>....</td>
<td>4.77</td>
<td>4.77</td>
</tr>
</tbody>
</table>

Table 2-10. The results of Case 1 for the 3rd limit state function, $G_3$

<table>
<thead>
<tr>
<th>Iter.</th>
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<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
<td>1.50E-03</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7.00E-03</td>
<td>4.81E-03</td>
<td>5.28E-03</td>
<td>5.33E-03</td>
<td>5.33E-03</td>
<td>5.33E-03</td>
</tr>
<tr>
<td>$G(x)$</td>
<td>4.74E+04</td>
<td>-2.12E+04</td>
<td>-1.86E+03</td>
<td>-1.69E+01</td>
<td>-1.42E-03</td>
<td>-1.82E-11</td>
</tr>
<tr>
<td>$dG/du_1$</td>
<td>-3.19E+01</td>
<td>-5.46E+01</td>
<td>-4.92E+01</td>
<td>-4.86E+01</td>
<td>-4.86E+01</td>
<td>-4.86E+01</td>
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<td>$dG/du_2$</td>
<td>1.61E+02</td>
<td>3.77E+02</td>
<td>3.14E+02</td>
<td>3.08E+02</td>
<td>3.08E+02</td>
<td>3.08E+02</td>
</tr>
<tr>
<td>$dG/du_3$</td>
<td>1.51E+04</td>
<td>3.17E+04</td>
<td>2.64E+04</td>
<td>2.59E+04</td>
<td>2.59E+04</td>
<td>2.59E+04</td>
</tr>
<tr>
<td>$\beta$</td>
<td>3.13</td>
<td>2.46</td>
<td>2.39</td>
<td>2.39</td>
<td>2.39</td>
<td>2.39</td>
</tr>
</tbody>
</table>
Case 2: 6 random variables of the horizontal, vertical and diagonal cross sectional area of the truss, $a_1$, $a_2$, $a_3$ as well as $P$, $E$ and $L$ under the limit state $G_1$. 

In this case, additional random variables of $P$, $E$ and $L$ are studied as well as those considered in Case 1 under the limit state function of the tip displacement, $G_1$. The random variables and their normalized variables as well as the relevant data for the additional random variables are defined as:

$$x = [a_1, a_2, a_3, P, E, L]$$

$$u_i = \frac{a_i - \mu_{a_i}}{\sigma_{a_i}} \quad i=1, 2, 3$$

$$u_P = \frac{P - \mu_P}{\sigma_P}; \quad u_E = \frac{E - \mu_E}{\sigma_E}; \quad u_L = \frac{L - \mu_L}{\sigma_L}$$

$N_P$ (500 kN, 50 kN), $N_E$ (7.0E+7 kPa, 3.5E+6 kPa), $N_L$ (10.0 m, 0.2 m)

The reliability results are summarized in Table 2-11. The MPP values are $(a_1, a_2, a_3) = (0.00927 \text{ m}^2, 0.00149 \text{ m}^2, 0.00651 \text{ m}^2)$, $P=561.13 \text{kN}$, $E=6.75E+7 \text{kPa}$, $L=10.05 \text{ m}$. The change in $P$ value is the most significant, which is 12% of increase at MPP. This can be verified by its large sensitivity of the performance function with respect to this variable. The resulting $\beta$ has been decreased from 2.97 in Case 1 to 1.76 for taking into account additional random variables.
Table 2-11. The results of Case 2

<table>
<thead>
<tr>
<th>Iter.</th>
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<th>1</th>
<th>2</th>
<th>3</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00E-02</td>
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<td>9.27E-03</td>
<td>9.27E-03</td>
<td>9.27E-03</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1.50E-03</td>
<td>1.49E-03</td>
<td>1.49E-03</td>
<td>1.49E-03</td>
<td>1.49E-03</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7.00E-03</td>
<td>6.52E-03</td>
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<td>6.51E-03</td>
<td>6.51E-03</td>
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<tr>
<td>$P$</td>
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<td>572.97</td>
<td>560.45</td>
<td>561.15</td>
<td>561.13</td>
</tr>
<tr>
<td>$E$</td>
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<td>6.74E+07</td>
<td>6.75E+07</td>
<td>6.75E+07</td>
<td>6.75E+07</td>
</tr>
<tr>
<td>$L$</td>
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<td>10.06</td>
<td>10.06</td>
<td>10.05</td>
<td>10.05</td>
</tr>
<tr>
<td>$G(x)$</td>
<td>2.45E-02</td>
<td>-2.63E-03</td>
<td>1.49E-05</td>
<td>4.08E-07</td>
<td>-1.26E-08</td>
</tr>
<tr>
<td>$dG/du_1$</td>
<td>4.69E-03</td>
<td>6.43E-03</td>
<td>6.31E-03</td>
<td>6.31E-03</td>
<td>6.31E-03</td>
</tr>
<tr>
<td>$dG/du_2$</td>
<td>3.25E-04</td>
<td>4.12E-04</td>
<td>4.03E-04</td>
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<td>4.03E-04</td>
</tr>
<tr>
<td>$dG/du_3$</td>
<td>4.53E-03</td>
<td>6.16E-03</td>
<td>6.04E-03</td>
<td>6.04E-03</td>
<td>6.04E-03</td>
</tr>
<tr>
<td>$dG/du_P$</td>
<td>-9.55E-03</td>
<td>-1.06E-02</td>
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<td>-2.39E-03</td>
<td>-2.39E-03</td>
<td>-2.39E-03</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.93</td>
<td>1.76</td>
<td>1.76</td>
<td>1.76</td>
<td>1.76</td>
</tr>
</tbody>
</table>

II. The resolution of the problem by the MCS

The same problems were resolved by sampling methods of the MCS and the LHS whose results are presented in Table 2-12. The errors were computed using the ten million simulations of MCS as reference values.

For Case 1, when $G_1$ of the tip displacement constraint is considered, there is a large difference of 31.3% between MCS and FORM results. This may be due to the highly non-linear formulation of $G_1$. As explained previously, when there is a large curvature of the limit state function around the design point, the FORM does not approximate probability of failure well. Additionally since we are dealing with relatively small probability of failure, which is in order of 1E-3, slight variation in $P_f$ value causes large relative difference. When $G_3$ of stress constraint in bar 7 is taken into account for Case 1, the difference in results by MCS and FORM is less than 1%, while
for Case 2, in which six random variables are studied under the constraint $G_1$, the difference between $P_f$ by the MCS and the FORM is 11.2%.

Comparing the results of 1E+7 simulations between MCS and LHS, There is no significant difference for Case 1 ($G_3$) and Case 2 ($G_1$), while there is 7.3% difference for Case 1 ($G_1$).

Table 2-13 summarizes the results using the reduced allowable tip displacement from $d=0.12$ m to $d=0.10$ m. The probability of failure has obviously increased for allowing a smaller displacement, and the errors of FORM have reduced significantly. For Case 1, the error of FORM has decreased from 31.3% to 9.1%.

<table>
<thead>
<tr>
<th>No. Simlns.</th>
<th>Case 1 ($G_1$) error %</th>
<th>Case 1 ($G_3$) error %</th>
<th>Case 2 ($G_1$) error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS 1.E+04</td>
<td>1.80E-03 16.69</td>
<td>8.00E-03 4.57</td>
<td>4.68E-02 5.25</td>
</tr>
<tr>
<td>1.E+05</td>
<td>2.46E-03 13.85</td>
<td>8.33E-03 0.64</td>
<td>4.45E-02 0.05</td>
</tr>
<tr>
<td>1.E+06</td>
<td>2.11E-03 2.35</td>
<td>8.32E-03 0.71</td>
<td>4.45E-02 0.05</td>
</tr>
<tr>
<td>1.E+07</td>
<td>2.16E-03 8.38E-03</td>
<td>4.45E-02</td>
<td></td>
</tr>
<tr>
<td>LHS 1.E+04</td>
<td>1.90E-03 12.07</td>
<td>8.30E-03 0.99</td>
<td>4.53E-02 1.87</td>
</tr>
<tr>
<td>1.E+05</td>
<td>2.31E-03 6.91</td>
<td>8.40E-03 0.20</td>
<td>4.46E-02 0.37</td>
</tr>
<tr>
<td>1.E+06</td>
<td>2.01E-03 6.79</td>
<td>8.40E-03 0.21</td>
<td>4.46E-02 0.39</td>
</tr>
<tr>
<td>1.E+07</td>
<td>2.00E-03 7.30</td>
<td>8.38E-03 0.03</td>
<td>4.45E-02 0.02</td>
</tr>
<tr>
<td>FORM</td>
<td>1.48E-03 31.33</td>
<td>8.42E-03 0.49</td>
<td>3.95E-02 11.20</td>
</tr>
</tbody>
</table>

Table 2-12. Comparison of probability of failure by MCS, LHS and FORM ($d_{allow}=0.12$ m for $G_1$ and $\sigma_{allow}=2.0E+5$ kPa for $G_3$)
Table 2-13. Probability of failure with reduced allowable tip displacement ($d_{allow} = 0.10$ m)

Among different reliability methods we have seen in this chapter, the FORM is the most computationally efficient method, which will be used in the reliability analysis examples presented in this work. The formulation of reliability analysis of bridge flutter will consider uncertainties in the extreme wind velocity, flutter derivatives and structural damping. This approach will lead to a large number of random variables. In Chapter 4, the reliability formulation of long-span bridges under flutter is presented, which will be applied to two bridge examples in Chapter 5 and 6.
2.7 References


CHAPTER 3

RELIABILITY BASED DESIGN OPTIMIZATION

3.1 INTRODUCTION

Today's competitive business environment demands engineers to design high-performance products, which should be economic and reliable at the same time. In order to achieve this objective within reduced design time, designers use computational models, such as finite element models and computer-aided engineering methods, which are made possible due to the development of high-speed digital computers in the last half century. The use of computational models permits the simple implementation of structural analyses, sensitivity studies as well as design optimization of structures.

Since any traditional intuitive design process all depends on the capacities or experiences of a designer, it does not generally lead to the best possible design. These drawbacks can be overcome by adapting a design optimization procedure, which consists of logical process of minimizing objective cost function while satisfying required constraints. Different types of design optimization theories can be found in Hernandez\textsuperscript{[H1]}, Vanderplaats\textsuperscript{[V1]} and Arora\textsuperscript{[A1]}. The constraints in design optimization
can be deterministic or probabilistic depending on whether or not we take system uncertainties into consideration. The design optimization plays an important role in structural design because it allows an engineer to derive maximum benefit from available resources.

Conventional deterministic optimization utilizes partial safety factors to count for system uncertainties such as assumptions of static and dynamic loading, material properties, simplifications of component geometries for structural analysis and so on. Because of increasing competitions among industries nowadays, the optimum designs leave very little room for simulation errors or manufacturing imperfections for products. As a result, deterministic designs with scarce safety factor may lead to unreliable designs. On the other hand, the use of large safety factor for lacking the knowledge of uncertainties may produce overly conservative and consequently uncompetitive designs. Libertiny\cite{L1} describes that the use of safety factor is not a direct application of safety to a design, but instead, is a crude estimate of the effect of accumulated errors in the analysis.

The reliability based design optimization (RBDO) provides an alternative to this conventional deterministic optimization and searches for the best middle ground between cost and safety by considering system uncertainties. The structural reliability is achieved by satisfying probabilistic constraints that integrate system uncertainties, which can be quantified by some statistical moments of each random variable. In this manner, an engineer can set a predetermined probability of failure by choosing the corresponding system reliability. For example, instead of limiting the maximum stress or the maximum displacement of a structural member to a specific value, the RBDO restricts the probability of failure due to these failure modes below a predetermined value considering uncertainty in material properties, loading conditions, geometry etc. Therefore RBDO allows an engineer to design a structural system with more meaningful safety level. These methods are becoming more and more popular in different industries such as aerospace\cite{A2}, automotive\cite{G1}, civil, defense\cite{R1} and power
industries because they help to achieve safer design at lower cost. According to Nikolaidis et al.\cite{N1}, the RBDO methods have been used in companies such as General Electric, General Motors, Ford, Boeing, Lockheed Martin to improve dramatically their competitive position and save billions of dollars in engineering design.

Figure 3-1 illustrates the differences between deterministic and reliability-based design optimization considering two design variables in which two constraint functions, $G_1$ and $G_2$ define the feasible region. The deterministic optimal in this case is the point where two constraint functions cross as indicated in the red dot in the figure. However, the constraint functions may shift toward unfeasible region when uncertainties are taken into account. Then the deterministic optimum is no longer a safe design. On the other hand, the reliability-based optimum design is obtained by taking into account system uncertainties with a predetermined reliability level as indicated in green dot in the figure, which is a safer design than the deterministic optimum.

![Figure 3-1: Deterministic optimization vs. reliability-based method](image)

The RBDO methods perform a design optimization to get the best design while carrying out reliability analyses to take into account probabilistic constraints. The basic
formulation of the RBDO is to minimize the cost function while satisfying probabilistic and other deterministic constraints. It is formulated as:

Minimize: \[ \text{Cost} \left( \mathbf{d} \right) \]  
Subject to: \[ P \left[ G_i \left( \mathbf{d}, \mathbf{x} \right) \leq 0 \right] \leq P_f \] \( i = 1, 2, \ldots, m \) \[ g_j \left( \mathbf{d} \right) \leq 0 \] \( j = m+1, \ldots, M \)

where \( \mathbf{d} \) is the vector of design variables, \( \mathbf{x} \) is the vector of random variables, \( \text{Cost} \) is the objective function or cost function, \( P[] \) is the probability operator, \( P_f \) is the allowable probability of failure for each failure mode, \( G_i \) is the \( i \)th performance function, \( g_j \) is a deterministic constraint, \( m \) is the number of performance functions, and \( M \) is the total number of reliability and deterministic constraints. The reliability constraints restrict the probability of each performance function violation to be below a predetermined failure probability.

The evaluation of probabilistic constraints is the key element of the RBDO since it requires considerable computational efforts, accuracy and stability. Some of the most commonly used reliability methods are discussed in Chapter 2 such as FORM, SORM, MCS, LHS, among which the FORM is one of the most widely used method due to its simplicity and efficiency. Yet, its computational cost may be high when the evaluation of associated functions is expensive or the existence of multiple probabilistic constraints. It is still a good option in the cases that the cost of carrying out sampling methods is too high.

The main RBDO formulations can be grouped into three main categories based on the arrangement of design optimization and reliability routines. They are double-loop approach, single-loop approach, and decoupled approach. The development of different methods in each category is explained briefly in the subsequent section.
3.2 The RBDO methods

For any RBDO problem, the selection of the method to solve probabilistic constraints is critical for its computational effort. Many researchers have tried to overcome this difficult task by developing different methods. The two-level approach, also known as double-loop method, is probably the most direct way to solve Equation (3.1) among the three categories.

In this approach, design optimization is performed in the original $x$-space in the outer loop and reliability analyses are carried out in the independent and standard normalized $u$-spaces in the inner loop (see Equation (2.18)) resulting in nested optimization problems. The efficiency and robustness of the RBDO will all depend on the nonlinearity of both optimization and reliability formulations. Any complex structural problem requires the use of finite element methods, which greatly increases the computational cost of evaluating reliability indices. The most commonly used approaches in this category are Reliability Index Approach (RIA) and Performance Measure approach (PMA).

RIA method was proposed by Nikolaidis and Burdisso\(^{[N2]}\) in 1988 employing the FORM to calculate reliability indices in the reliability routine. Although the method was rather simple to formulate, it presented drawbacks of high computational cost and slow convergence, especially in the presence of multiple performance functions.

PMA method was introduced by Tu and Choi\(^{[T1]}\) in 1999 to overcome these disadvantages. The reliability measure was converted into performance measure by solving an inverse reliability problem, which seeks the minimum performance point on the reliability surface. The method arises from an idea that optimizing a complex objective function under simple constraint functions is much easier than optimizing a simple objective function under complex constraint functions (Aoues et al.\(^{[A3]}\)). An efficient search method for the Most Probable Point (MPP) of failure named the Hybrid
Mean Value (HMV) method was incorporated by Choi et al.\cite{C1}, which adaptively uses the Advanced Mean Value (AMV) and the Conjugate Mean Value (CMV) method for convex and concave limit state functions respectively. According to researchers as Lee et al.\cite{L2} and Youn et al.\cite{Y1}, PMA was found to be more computationally efficient and numerically stable than RIA due to its small dependency on probabilistic distribution types (Youn et al.\cite{Y2}).

RIA and PMA approaches are basically inverse of one another and would yield the same solution if the reliability constraints are active at the optimum (Tu et al.\cite{T1}). Nevertheless, even with improved reliability methods of PMA, high computational cost is the major obstacle for the application of two-level methods to complex structural systems.

Many researchers sought for alternative more efficient RBDO formulations to avoid computationally intensive nested optimization problems. A group of methods which separate the upper-level optimization routine from the reliability analysis are called decoupled methods. The Sequential Optimization and Reliability Assessment (SORA) introduced by Du and Chen\cite{D1} in 2004 is one of the most promising approaches in this category. The main idea of SORA is to reformulate the RBDO problem into a sequence of deterministic optimization and reliability analysis. In each cycle, optimization and reliability analysis are carried out to updates the design variables based on the most recent probabilistic results. The method is shown to be computationally efficient compared to two-level methods.

Cheng et al.\cite{C2} proposed the concept of Sequential Approximate Programming (SAP) in 2006, in which the RBDO is transformed into a sequence of sub-programming problems. This approach decomposes the original optimization problem into a sequence of sub-optimization problems, in which an approximate objective function subject to a set of approximate constraints is evaluated. The objective of using approximate functions is to reduce the number of iterations during the optimization. Yet, the
computation of derivatives with respect to design variables as well as random variables is a drawback for SAP when it is compared to SORA, which only requires the derivatives with respect to random variables. Yi et al.\cite{Y3} presented in 2008 the SAP employing PMA, in which the optimality conditions are set at the minimum performance point instead of MPP. Recently Ching and Hsu\cite{C3} formulated a method to transform the reliability constraint into deterministic constraint by introducing the limit state factor and the nominal limit state. The main challenge for using the decoupled approach lies in the formulation of the equivalent RBDO problem in order to reach desired precision (Aoues et al.\cite{A3}).

One of the major concerns in evaluating reliability analysis is that it is formulated as an optimization problem. In order to overcome this disadvantage, some researchers developed mono-level approaches also known as single-loop approaches. It tries to solve the RBDO problem in a single loop avoiding the reliability routine. The reliability constraints are replaced by equivalent optimality constraints or by reformulating the RBDO to obtain a single loop problem. One of the earlier works belongs to Madsen and Friis Hansen\cite{M1} who replaced the probabilistic constraints with Karush-Kuhn-Tucker (KKT) optimality conditions of the FORM so that the convergence can be achieved in both design and random variable at the same time. Kuschel and Rackwitz\cite{K1} improved the approach by reformulating the problem to maximize the reliability under cost constraints. Kirjner-Neto et al.\cite{K2} reformulated the RBDO design and developed a semi-infinite optimization algorithm to solve it. However, the existing formulations of mono-level approach does not guarantee mathematical equivalency to the original two-level problem (Kuschel et al.\cite{K3}).

Among all the RBDO methods discussed above, RIA and PMA in the two-level approach as well as SORA in the decoupled approach are explained in detail in the following section. These methods have been employed to perform the RBDO on long-span bridge examples in this research.
3.2.1 Reliability Index Approach (RIA)

The design optimization of RIA is formulated as:

Minimize: \[ \text{Cost} \left( \mathbf{d} \right) \]  \hspace{1cm} (3.2a)

Subject to: \[ \beta_i \left( \mathbf{d}, \mathbf{x} \right) \geq \beta_i^T \] \hspace{1cm} (3.2b)

\[ g_j \left( \mathbf{d} \right) \leq 0 \] \hspace{1cm} (3.2c)

where \( \beta_i \) is a reliability index and \( \beta_i^T \) is the target reliability index for the \( i \)th limit state function, \( g_j \) is a deterministic constraint. The evaluation of reliability indices leads to an inner optimization loop. FORM is used to obtain \( \beta_i \), which solves the following optimization problems after transforming random variables from their original \( x \)-space to the normalized \( u \)-space:

Minimize: \[ \beta_i = \sqrt{\mathbf{u}^T \cdot \mathbf{u}} \] \hspace{1cm} (3.3a)

Subject to: \[ G_i \left( \mathbf{u} \right) = 0 \] \hspace{1cm} (3.3b)

The solution of this problem, \( \mathbf{u}^* \), is called the Most Probable Point (MPP) of failure, and the reliability index, \( \beta \) is simply the minimum distance point from the origin of the \( u \)-space to the MPP.

Figure 3-2 shows a general work flow of RIA. To a typical optimization loop, an inner loop of reliability routine is added, which is solved by FORM. For each design optimization, reliability routine is performed to compute reliability index for a particular failure mode. Then this reliability index is checked against the target reliability in the probability constraint. Figure 3-3 shows a search scheme of MPP for RIA.
algorithm searches for the minimum-distance point from the mean values of the random variables to the failure surface constrained by $G_i(u) = 0$.

RIA method has an advantage of simple formulation, yet its numerical inefficiency and stability problem are the major obstacles of its application to practical engineering problems.

**Reliability routine**

$$\text{Min } \beta_i = \sqrt{\mathbf{u}^T \cdot \mathbf{u}}$$

$i = 1, 2, ..., m$

Subject to $G_i(u) = 0$

---

![Flowchart of RIA](image)

**Figure 3-2: Flowchart of RIA**
3.2.2 Performance Measure Approach (PMA)

The design optimization process of PMA is formulated as:

Minimize: \[ \text{Cost} (\mathbf{d}) \] \hspace{1cm} (3.4a)

Subject to:
\[ G_i(\mathbf{x}) \geq 0 \hspace{1cm} i=1, 2, \ldots, m \] \hspace{1cm} (3.4b)
\[ g_j(\mathbf{d}) \leq 0 \hspace{1cm} j=m+1, \ldots, M \] \hspace{1cm} (3.4c)

where \( G_i \) is the performance function evaluated by an inverse reliability method. Unlike the reliability routine used for RIA, which tries to find the minimum distance point from the mean values of the random variables to the failure surface subject to null limit state functions, the reliability routine of PMA searches for the lowest performance function that satisfies the condition of \( \beta = \beta^T \). The probabilistic performance measure is obtained in \( u \)-space as:

Figure 3-3: Reliability search scheme for RIA\(^{[A3]}\)
Minimize: \[ G_i(u) \] \quad (3.5a)  
subject to: \[ \beta_i = \beta_i^T \] \quad (3.5b)

In order to find the MPP, only the direction vector has to be determined to satisfy the spherical equality constraint of Equation (3.5b). The algorithm tries to find the minimum performance function that satisfies \( \beta = \beta^T \). The reliability process of PMA is schematized in Figure 3-4 while the search space of MPP is shown in Figure 3-5.

![Figure 3-4: Flowchart of PMA](image-url)
There exist three methods to solve the reliability analyses of PMA as mentioned before such as AMV, CMV and HMV. It was found that the AMV is well-suited for convex performance functions although it exhibits slow convergence or even divergence for concave functions\[^2\]. The CMV works well with concave performance functions, but does not always converge with convex functions. Finally the HMV method takes advantages of both methods by first checking the convexity of the performance function and choosing the appropriate method to use. In this research, the HMV was employed for the reliability phase of PMA method. Since HMV combines AMV and CMV, these methods are described below.

The main steps of the AMV are:

1. The mean value is considered as the initial MPP as $u^0=0$.

2. The gradient vector of the limit state function is evaluated at the current value of $u$.

3. A unit vector is calculated in the opposite direction of the gradient,
\[ n = -\frac{\nabla G(u)}{||\nabla G(u)||} \]  \hspace{1cm} (3.6)

4. The next point is calculated as:

\[ u^k = \beta^T n \]  \hspace{1cm} (3.7)

5. Repeat step 2-4 until the convergence of \( u \).

Figure 3-6 shows an example of the convergence of the AMV algorithm.

The steps for the CMV are as follows.

1. The initial value for the design point is the mean value, \( u^0 = 0 \).

2. The gradient of the limit state function is evaluated at the current value, \( u \).

3. The new search direction is chosen by combining the three consecutive steepest descent directions so that the resulting direction is the diagonal of these directions. The first three iterations for \( k = 0, 1, 2 \), are carried out using the AMV method.
\[ u^k = \beta \frac{n^k + n^{k-1} + n^{k-2}}{n^k + n^{k-1} + n^{k-2}}, \quad k \geq 3 \] (3.8)

4. The steps 2-4 are repeated until the convergence of \( u \).

The HMV method performs the first three consecutive iterations by the AMV to check the convexity of the limit state function. Then one of the two methods is chosen adaptively: the AMV method for a convex function and the CMV method for a concave function. Figure 3-7 and Figure 3-8 show the criterion parameter for convex and concave cases.

The procedure of HMV is:

1. The first three iterations are carried out by the AMV.
2. The criterion parameter, \( \zeta \) for \( k+1 \) cycle is calculated as:
\[ \zeta^{k+1} = (n^{k+1} - n^k) \cdot (n^k - n^{k-1}) \] (3.9)
3. If \( \zeta \geq 0 \), then compute \( u \) by AMV.
   If \( \zeta < 0 \), then compute \( u \) by CMV.
4. Evaluate the performance function and \( \beta \) using the new point, \( u^{k+1} \), and check the convergence. If the algorithm converges, stop the process, otherwise go to Step 5.
5. Compute the gradient of the performance function, \( \nabla G(u^{k+1}) \) and the criterion, \( \zeta \). Set \( k = k+1 \) and go back to Step 3.
3.2.2.1 Modification of the HMV method

In this research, we have encountered cases that the HMV method presented convergence problems. A reduction factor was introduced in the formulation of HMV in order to improve the convergence of the algorithm. For the calculation of direction
vector to obtain the next design point in Equation (3.7) and (3.9), reduction factor, $r$, was introduced in the formulation as:

$$\tilde{n} = n_k + r \cdot (n_{k+1} - n_k)$$  \hspace{1cm} (3.10)

The graphical representation of the modification in the case of $r=0.5$ is shown in Figure 3-9. The reduction in angle for the direction vector, in case of convergence problem, may help the algorithm to converge.

Figure 3-9. Modification of HMV method

Figure 3-10 shows an example of convergence problem using HMV method and its improvement with a reduction factor. The evolution of $u$ value is plotted for the cases with and without a reduction factor. In this case, after introducing the reduction factor of $r=0.5$, we were able to achieve the convergence of the algorithm. For the bridge examples of Great Belt and Messina Bridge in Chapter 5 and 6, the reduction factor of $r=0.5$ and 0.75 were used to achieve convergence in some cases.
3.2.3 SORA

The deterministic design optimization phase of SORA is formulated as:

Minimize: \[ \text{Cost} (d^k) \] \hspace{1cm} (3.11a)

Subject to: \[ G_i(d^k, x^{k-1}) \geq 0 \quad i=1,2,...,m \] \hspace{1cm} (3.11b)

\[ g_j(d^k) \leq 0 \quad j=m+1,...,M \] \hspace{1cm} (3.11c)

where \( k \) indicates the current cycle. \( x^{k-1} \) is the random variable vector calculated from the previous cycle of reliability analysis. In the first cycle, the design optimization is performed with the mean values of random variables. Then the optimum design \( d \) obtained from the above deterministic optimization is fed into the following reliability phase.
Minimize: \[ G_i(u) \] \hspace{1cm} (3.12a)

Subject to: \[ \beta_i = \beta_i^* \] \hspace{1cm} \text{i} = 1, 2, \ldots, m \hspace{1cm} (3.12b)

From this reliability analysis, the MPPs, \( u_i^* \) are obtained. The corresponding random variables \( x_i^k \) are then incorporated into the following cycle of the deterministic design optimization, which starts with the optimum design variables from the previous design cycle as initial values. Since each limit state function is evaluated in the form of probabilistic constraint, the newly obtained MPPs update the probabilistic constraints in the design optimization. The process continues until the objective function is converged with feasible constraint functions. The inverse reliability method is used because of its computational efficiency. The SORA method presents the advantage of fully deterministic, and therefore its implementation is rather simple and can be resolved by common optimization algorithms. Figure 3-11 represents the SORA method schematically.

![Figure 3-11: Flowchart of SORA](image)
There may be cases that the design variables are random variables at the same time. In such cases, the mean values of the random variables are to be designed to minimize the objective function. Since SORA carries out the design optimization deterministically, we need to make sure that the design points fall into the feasible region and satisfy the predetermined target reliability $\beta^T$. Then we need to define a shift vector for the $k$th cycle for the $i$th probabilistic constraint as:

$$ s_i^k = \mu_{x_i}^{k-1} - x_{MPP_i}^{k-1} $$

(3.13)

where $\mu_{x_i}^{k-1}$ is the mean values of random variables for the $k$-1 cycle, $x_{MPP_i}^{k-1}$ is the MPP of the $i$th probabilistic constraint for the $k$-1 cycle. We have shift vectors as many as the number of probabilistic constraints because each probabilistic constraint has a unique MPP. We now redefine the Equation 3.11b as:

$$ G_i(d^k, \mu_x^k - s^k) \geq 0 \quad i=1, 2, …, m $$

(3.14)

Before evaluating the probabilistic constraint functions in the deterministic optimization, the mean value of the random variables should be shifted based on the information of the MPPs obtained in the previous reliability phase. The use of shift vector is illustrated in the multiple limit states example in Section 3.3.2.

### 3.3 Application examples

In order to illustrate the numerical performance of the three RBDO methods discussed in the previous section, four examples are presented: 1) mathematical problem with a single probabilistic constraint, 2) mathematical problem with multiple probabilistic constraints in which the mean values of random variables are designed 3) buckling beam and 4) ten-bar truss structure studied as a reliability example in Chapter 2. Each of the RBDO method was implemented in Matlab code®, and the design
optimization was performed using the Active-set algorithm (Gill\textsuperscript{[G2]}) unless otherwise mentioned. As convergence criteria, the tolerance of absolute change in objective functions and constraints were set to be 1.0E-5. During the reliability routine of RIA, the code checks the absolute change in consecutive beta values as a convergence criterion, while for PMA and SORA, it checks for absolute change in consecutive $u$, which are all set to 1.0E-4.

### 3.3.1 Mathematical problem

This analytical problem (Aoues et al.\textsuperscript{[A3]}) is often used to demonstrate the efficiency of different RBDO methods. It consists of two design variables of $d_1$ and $d_2$ and two normally distributed random variables $x_1$ and $x_2$ with their mean values $\mu_x=(5.0, 3.0)$ and the coefficient of variance of 0.3. Our goal is to get the best design that minimizes the objective function while satisfying the probabilistic constraint, which limits the probability of failure event defined by the limit state function to be less than 1%. The problem is formulated as:

Minimize: \[ F(d) = d_1^2 + d_2^2 \] (3.15a)

Subject to: \[ P[G(d, x) \leq 0] \leq 0.01 \] (3.15b)

where \[ G(d, x) = \frac{1}{5}d_1d_2x_2^2 - x_1 \] (3.15c)

\[ 0 \leq d \leq 15 \] (3.15d)

\[ d_0^0=(2.0, 1.0), \mu_x=(5.0, 3.0), \sigma_x=(1.5, 0.9) \]

**Resolution of the example using RIA method**

First of all, the corresponding target reliability index should be computed for the probability of failure of 1%, which is:
\[ \beta^* = \Phi^{-1}(1 - P_f) = 2.326 \] (3.16)

The evolutions of the design variables in the design space and random variables in \( u \)-space using RIA method are shown in Figure 3-12 a) and b) respectively. As can be seen, the design is improved in each optimization cycle until its convergence while the MPP is searched by the FORM on the limit state function surface of \( G_k(u) = 0 \). The distance between \( u^0 = 0 \) and \( u^7 \) is the \( \beta^* \) in \( u \)-space.
The evolution of design variables, random variables, the objective function as well as $\beta$ are tabulated in Table 3-1 and graphed in Figure 3-13. The algorithm converged in 9 design iterations while the reliability iterations for each design cycle varied between 4 and 6. The total number of iterations between the design optimization and reliability was 64.
Table 3-1: Evolution of design and random variables by RIA

<table>
<thead>
<tr>
<th>iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$F$</th>
<th>$\beta$</th>
<th>rel.iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.000</td>
<td>1.000</td>
<td>4.600</td>
<td>3.391</td>
<td>5.000</td>
<td>-0.510</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1.153</td>
<td>2.307</td>
<td>4.956</td>
<td>3.052</td>
<td>6.651</td>
<td>-0.065</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3.046</td>
<td>2.177</td>
<td>5.496</td>
<td>2.036</td>
<td>14.018</td>
<td>1.121</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2.915</td>
<td>4.698</td>
<td>5.561</td>
<td>1.425</td>
<td>30.563</td>
<td>1.790</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>5.196</td>
<td>4.277</td>
<td>5.528</td>
<td>1.115</td>
<td>45.288</td>
<td>2.124</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>5.251</td>
<td>5.649</td>
<td>5.494</td>
<td>0.962</td>
<td>59.481</td>
<td>2.288</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>5.620</td>
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<td>5.487</td>
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<td>7</td>
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<td>6</td>
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<tr>
<td>8</td>
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<td>5.650</td>
<td>5.487</td>
<td>0.927</td>
<td>63.839</td>
<td>2.326</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
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<td>5.650</td>
<td>5.487</td>
<td>0.927</td>
<td>63.839</td>
<td>2.326</td>
<td>6</td>
</tr>
</tbody>
</table>

| total iter. | 64 |

Table 3-1: Evolution of design and random variables by RIA

![Graph](image-url)
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Figure 3-13: Evolution of RBDO parameters: a) design variables b) random variables c) objective function
Resolution of the example using PMA method

The evolution of the design variables and the MPPs are represented in the design space and \( u \)-space in Figure 3-14. Compared to the resolution by RIA, the design improves more rapidly with less number of iterations. In \( u \)-space, we can clearly see that the algorithm searches for the MPP on the spherical surface of \( \beta = \beta^T \) as opposed to the FORM that search the MPP on the failure surface of \( G_k = 0 \). The HMV method is generally more efficient than the FORM to locate MPP.
The results are also tabulated in Table 3-2 and Figure 3-15. The algorithm converged in 6 design optimization cycles; however, the number of reliability iterations for each reliability routine is higher than that of RIA, which varied between 7 and 19 with the total number of iterations of 78.

![Diagram](image)

**Figure 3-14. Evolution of a) the design variables in the design space and b) the MPPs in u-space**

Table 3-2: Mathematical problem results by PMA
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a) Design variables vs. Iteration:

- $d_1$
- $d_2$

b) Random variables vs. Iteration:

- $x_1$
- $x_2$
Figure 3-15: Evolution of RBDO parameters: a) design variables b) random variables c) objective function

- **Resolution of the example using SORA method**

  Figure 3-16 shows the evolution of design variables in design space and the MPP in $u$-space respectively. As can be seen, the design is progressively improved in each design cycle of deterministic design optimization and reliability analysis. The design quickly improves compared to RIA. The reliability method of SORA is the same as that of PMA, which performs a spherical search of the MPP at $\beta=\beta^T$. 

Figure 3-16. Evolution of the design variables in the design space and the MPP in $\beta$-space

\[
G^0 = 0
\]

\[
G^1 = 0
\]

\[
G^2 = 0
\]

\[
G^{3,4} = 0
\]

(\(k\) is iteration number)
Table 3-3 and Figure 3-17 summarize the evolution of design variables, random variables and objective function using SORA method. As can be seen, a reliability analysis was carried out at the end of each deterministic design optimization and the design was progressively improved. The number of iterations for each reliability routine varied between 6 and 18 with the total number of iterations of 64.

<table>
<thead>
<tr>
<th>design cycle</th>
<th>iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$F$</th>
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<td>5.000</td>
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### Chapter 3
Reliability Based Design Optimization

#### Table 3-3: Mathematical problem results by SORA a) design cycle and b) reliability routine

<table>
<thead>
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<th>Reliability analysis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>rel.iter.</th>
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<td>10</td>
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<tr>
<td>3</td>
<td>5.501</td>
<td>0.928</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>5.487</td>
<td>0.927</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>5.486</td>
<td>0.927</td>
<td>6</td>
</tr>
<tr>
<td><strong>total iter.</strong></td>
<td></td>
<td></td>
<td><strong>64</strong></td>
</tr>
</tbody>
</table>

![Diagram showing design cycles and reliability](image)

Table 3-3: Mathematical problem results by SORA a) design cycle and b) reliability routine

![Graph showing iterations and design variables](image)
Figure 3-17: Evolution of RBDO parameters: a) design variables b) random variables c) objective function
3.3.2 Multiple limit states example

We now consider the following classical mathematical model of RBDO with multiple probabilistic constraints formulated by Youn and Choi [Y2]. In this example, the design variables are random variables at the same time, which means that we are going to design the mean values of the random variables, i.e., \( \mathbf{d} = [\mu(x_1), \mu(x_2)]^T \). The initial design is set to \( \mathbf{d}_0 = [5.0, 5.0]^T \) with a fixed value of standard deviation of \( \sigma_x = [0.3, 0.3]^T \). The target reliability indices are set to \( \beta_i^T = 2.0 \) \( (i=1, 2, 3) \) which correspond to \( P_f = 0.02275 \). The problem is formulated as:

Minimize: \( F(\mathbf{d}) = d_1 + d_2 \) \hspace{1cm} (3.17a)

Subject to: \( P\left[ G_i(\mathbf{d}, \mathbf{x}) \leq 0 \right] \leq \Phi(-\beta_i^T) \) \hspace{1cm} \( i=1, 2, 3 \) \hspace{1cm} (3.17b)

where

\[ G_1(\mathbf{x}) = \frac{x_1^2x_2}{20} - 1 \] \hspace{1cm} (3.17c)

\[ G_2(\mathbf{x}) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1 \] \hspace{1cm} (3.17d)

\[ G_3(\mathbf{x}) = \frac{80}{(x_1^2 + 8x_2 + 5)} - 1 \] \hspace{1cm} (3.17e)

\[ 0 \leq d_i \leq 10 \] \hspace{1cm} (3.17f)

\( \mu_x = [5.0, 5.0]^T, \sigma_x = [0.3, 0.3]^T \)

The results using each RBDO method are shown next. The design variables using all three methods converged to the same values. The MPP values at the optimum for \( G_1 \) and \( G_2 \) are practically the same for all methods because they are active constraints. On the other hand, since \( G_3 \) is not an active constraint, the MPPs resulting
from PMA and SORA are not real MPPs. The algorithms impose so that the MPP lies on the surface of $\beta = \beta^T$; however, the limit state function is not null at such surface.

- Resolution of the problem by RIA method

The evolution of the design variables in the design and $x$-space is graphed in Figure 3-18 while the evolution of design variables, random variables and $\beta$ for the three limit state functions as well as the objective function are shown in Table 3-4 and Figure 3-19. The final design is distanced from both active probabilistic constraints of $G_1$ and $G_2$ by $\beta \cdot \sigma$ in $x$-space. We achieved the optimal design in 5 iterations while the number of reliability iterations for each design cycle varied between 4 and 7. The total number of iterations between the design optimization and reliability was 110.

Figure 3-18. Graphical representation of the resolution by RIA
### Table 3-4. The results of the multiple limit states problem by RIA

<table>
<thead>
<tr>
<th>Iter.</th>
<th>0</th>
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<th>4</th>
<th>5</th>
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<td>4.000</td>
<td>3.005</td>
<td>2.897</td>
<td>2.897</td>
<td>2.897</td>
</tr>
<tr>
<td>obj. func.</td>
<td>10.000</td>
<td>8.000</td>
<td>6.204</td>
<td>6.190</td>
<td>6.192</td>
<td>6.192</td>
</tr>
<tr>
<td>$x_1$ ($G_1$)</td>
<td>2.161</td>
<td>2.410</td>
<td>2.693</td>
<td>2.765</td>
<td>2.765</td>
<td>2.765</td>
</tr>
<tr>
<td>$x_2$ ($G_1$)</td>
<td>4.282</td>
<td>3.442</td>
<td>2.757</td>
<td>2.617</td>
<td>2.617</td>
<td>2.617</td>
</tr>
<tr>
<td>$x_1$ ($G_2$)</td>
<td>4.937</td>
<td>4.244</td>
<td>3.534</td>
<td>3.559</td>
<td>3.560</td>
<td>3.560</td>
</tr>
<tr>
<td>$x_2$ ($G_2$)</td>
<td>2.622</td>
<td>2.576</td>
<td>2.346</td>
<td>2.359</td>
<td>2.359</td>
<td>2.359</td>
</tr>
<tr>
<td>$x_1$ ($G_3$)</td>
<td>5.613</td>
<td>5.763</td>
<td>6.002</td>
<td>6.093</td>
<td>6.093</td>
<td>6.093</td>
</tr>
<tr>
<td>$x_2$ ($G_3$)</td>
<td>5.437</td>
<td>5.224</td>
<td>4.873</td>
<td>4.735</td>
<td>4.735</td>
<td>4.735</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>9.761</td>
<td>5.615</td>
<td>1.877</td>
<td>1.993</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>7.929</td>
<td>4.816</td>
<td>2.462</td>
<td>2.001</td>
<td>2.000</td>
<td>2.000</td>
</tr>
</tbody>
</table>

**rel.iter ($G_1$)**

| rel.iter ($G_1$) | 7 | 6 | 4 | 4 | 4 | 4 |
| rel.iter ($G_2$) | 8 | 8 | 6 | 5 | 5 | 5 |
| rel.iter ($G_3$) | 5 | 6 | 7 | 7 | 7 | 7 |

<table>
<thead>
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<th>Iteration</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design variables</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_1$</td>
<td>5</td>
<td>4.5</td>
<td>4</td>
<td>3.5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$d_2$</td>
<td>5</td>
<td>4</td>
<td>3.5</td>
<td>3</td>
<td>2.5</td>
<td>2</td>
</tr>
</tbody>
</table>

**Total iter.** 110
Figure 3-19. The evolution of a) design variables b) random variables and c) objective function using RIA

- Resolution of the problem by PMA method

The resolution of the problem using PMA is represented graphically in Figure 3-20 while the results of design variables, random variables as well as the objective function are summarized in Table 3-5 and Figure 3-21. In this case, the optimization
algorithm of sequential quadratic programming was used because it converged more quickly than the active-set algorithm. The design converged in 6 iterations and the number of reliability analyses varied between 9 and 11. The total number of iteration was 180.

Figure 3-20. Graphical representation of the resolution by PMA
### Table 3-5. The results of the multiple limit states problem by PMA

<table>
<thead>
<tr>
<th>Iter.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>5.000</td>
<td>4.000</td>
<td>3.423</td>
<td>3.300</td>
<td>3.295</td>
<td>3.295</td>
<td>3.295</td>
</tr>
<tr>
<td>$d_2$</td>
<td>5.000</td>
<td>4.000</td>
<td>3.179</td>
<td>2.917</td>
<td>2.898</td>
<td>2.897</td>
<td>2.897</td>
</tr>
<tr>
<td>obj. func.</td>
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<td>8.000</td>
<td>6.602</td>
<td>6.217</td>
<td>6.192</td>
<td>6.192</td>
<td>6.192</td>
</tr>
<tr>
<td>$x_1$ ($G_1$)</td>
<td>4.457</td>
<td>3.455</td>
<td>2.885</td>
<td>2.769</td>
<td>2.764</td>
<td>2.764</td>
<td>2.764</td>
</tr>
<tr>
<td>$x_2$ ($G_1$)</td>
<td>4.745</td>
<td>3.749</td>
<td>2.912</td>
<td>2.638</td>
<td>2.617</td>
<td>2.617</td>
<td>2.617</td>
</tr>
<tr>
<td>$x_1$ ($G_2$)</td>
<td>4.884</td>
<td>4.045</td>
<td>3.633</td>
<td>3.562</td>
<td>3.560</td>
<td>3.560</td>
<td>3.560</td>
</tr>
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<td>$x_2$ ($G_2$)</td>
<td>4.411</td>
<td>3.402</td>
<td>2.617</td>
<td>2.377</td>
<td>2.359</td>
<td>2.359</td>
<td>2.359</td>
</tr>
<tr>
<td>$x_1$ ($G_3$)</td>
<td>5.485</td>
<td>4.446</td>
<td>3.839</td>
<td>3.708</td>
<td>3.703</td>
<td>3.703</td>
<td>3.703</td>
</tr>
<tr>
<td>$x_2$ ($G_3$)</td>
<td>5.354</td>
<td>4.401</td>
<td>3.612</td>
<td>3.357</td>
<td>3.338</td>
<td>3.338</td>
<td>3.338</td>
</tr>
<tr>
<td>rel.iter ($G_1$)</td>
<td>6</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>rel.iter ($G_2$)</td>
<td>9</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>rel.iter ($G_3$)</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Total iter. 180

![Graph showing design variables over iterations](image)
Figure 3-21. The evolution of a) design variables b) random variables and c) objective function using PMA

- Resolution of the problem by SORA method

The evolution of the design variables using SORA is plotted graphically in Figure 3-22. As explained in Section 3.5.1, when the design variables are random...
variables at the same time, we need to use the shift vector. Since the shift vector is null in the first design cycle, the design goes to the minimum point where the constraint $G_1$ and $G_2$ crosses. Then from the second design cycles on, based on the reliability information of the previous cycle, the design is shifted toward the feasible region in order to guarantee the predetermined reliability value of $\beta^T$. The MPP for $G_3$ resulting from this method is not a real MPP since the constraint $G_3$ is inactive at the optimum.

The evolutions of design variables, random variables as well as objective function by SORA are shown in Table 3-6 and Figure 3-23. The number of reliability iterations for each design cycle varied between 9 and 11 and the total number of iterations was 116.

Figure 3-22. Graphical representation of the resolution by SORA ($i$ is the design cycle, $k$ is the iteration number for $d_i^k$)
Table 3-6. The results of the multiple limit states problem by SORA

<table>
<thead>
<tr>
<th>design cycle</th>
<th>Iter.</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>obj. Func.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>5.000</td>
<td>5.000</td>
<td>10.000</td>
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<td>4.000</td>
<td>8.000</td>
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<tr>
<td></td>
<td>2</td>
<td>3.250</td>
<td>2.750</td>
<td>6.000</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.087</td>
<td>2.169</td>
<td>5.256</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3.104</td>
<td>2.075</td>
<td>5.179</td>
</tr>
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<td></td>
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<td>0</td>
<td>3.114</td>
<td>2.063</td>
<td>5.177</td>
</tr>
<tr>
<td></td>
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<td>3.266</td>
<td>3.042</td>
<td>6.308</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.298</td>
<td>2.872</td>
<td>6.170</td>
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<td></td>
<td>3</td>
<td>3.302</td>
<td>2.869</td>
<td>6.171</td>
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<td>4</td>
<td>3.302</td>
<td>2.869</td>
<td>6.171</td>
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<td>0</td>
<td>3.302</td>
<td>2.869</td>
<td>6.171</td>
</tr>
<tr>
<td></td>
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<td>3.295</td>
<td>2.897</td>
<td>6.192</td>
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<tr>
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<td>3.295</td>
<td>2.897</td>
<td>6.192</td>
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<td>3.295</td>
<td>2.897</td>
<td>6.192</td>
</tr>
<tr>
<td>4</td>
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<tr>
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<td>3.295</td>
<td>2.897</td>
<td>6.192</td>
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<table>
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<tr>
<th>Cycle</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 \ (G_1) )</td>
<td>5.000</td>
<td>2.641</td>
<td>2.773</td>
<td>2.764</td>
<td>2.764</td>
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<tr>
<td>( x_2 \ (G_1) )</td>
<td>5.000</td>
<td>1.694</td>
<td>2.586</td>
<td>2.617</td>
<td>2.617</td>
</tr>
<tr>
<td>( x_1 \ (G_2) )</td>
<td>5.000</td>
<td>3.515</td>
<td>3.569</td>
<td>3.560</td>
<td>3.560</td>
</tr>
<tr>
<td>( x_2 \ (G_2) )</td>
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<td>1.617</td>
<td>2.332</td>
<td>2.359</td>
<td>2.359</td>
</tr>
<tr>
<td>( x_1 \ (G_3) )</td>
<td>5.000</td>
<td>3.510</td>
<td>3.710</td>
<td>3.703</td>
<td>3.703</td>
</tr>
<tr>
<td>( x_2 \ (G_3) )</td>
<td>5.000</td>
<td>2.514</td>
<td>3.309</td>
<td>3.338</td>
<td>3.338</td>
</tr>
<tr>
<td>rel.iter ( (G_1) )</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>rel.iter ( (G_2) )</td>
<td>9</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>rel.iter ( (G_3) )</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Total iter.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>116</td>
</tr>
</tbody>
</table>

Table 3-6. The results of the multiple limit states problem by SORA

a) design cycle and b) reliability routine
Chapter 3

Reliability Based Design Optimization

a) Reliability

b) Random variables

94
3.3.3 Buckling beam example

In this example, we will apply the RBDO methods to a hollow square beam of length $L$ under an axial compressive load $P$ as shown in Figure 3-24. The design variables are the width and thickness of the cross section of the beam, while the axial load and the modulus of elasticity are considered as random variables. The limit state function defines the probabilistic failure mode of the beam by buckling. We set the target reliability index to 5.0, which means that the probability of failure of the structure by buckling should be smaller than 2.87E-7. Deterministic constraints limit the beam stress under $\sigma_e=2.6$ t/cm$^2$ and slenderness ratio $\lambda$ under 200 and the width to thickness ratio smaller than 30. The minimum wall thickness is set to be 0.3 cm. The objective of this problem is to search for the best design to minimize the beam volume while satisfying the reliability constraint as well as other deterministic constraints. The initial
design point of $d^0=(12.0, 0.4)$ was used. The relevant data for the problem are as follows.

$L=300$ cm; $\sigma_e=2.6$ t/cm$^2$

$N_p (80.0 \text{ t}, 12.0 \text{ t}) ; N_E (2100.0 \text{ t/cm}^2, 105.0 \text{ t/cm}^2)$

Buckling limit state: 

$$\frac{P \omega}{4d_1d_2} \leq \sigma_e$$

where 

$$\omega = \frac{2\sigma_e}{1.3\sigma_e + \sigma_E - \sqrt{(1.3\sigma_e + \sigma_E)^2 - 4\sigma_e\sigma_E}}$$

and 

$$\sigma_e = \frac{\pi^2 E}{\lambda^2}$$

The formulation of the problem is the following.

Minimize 

$$F(d) = 4L(d_1d_2 - d_2^2)$$

Subject to: 

$$P[G_i(d, x) \leq 0] \leq 2.87E - 7$$

Figure 3-24: Buckling beam
where
\[ G_1 = \sigma_e - \frac{P_0 \omega}{4d_1 d_2} \]  \hspace{1cm} (3.18c)

\[ g_2: \frac{P}{4d_1 d_2} \leq \sigma_e \]  \hspace{1cm} (3.18d)

\[ g_3: \lambda \leq 200 \]  \hspace{1cm} (3.18e)

\[ g_4: \frac{d_1}{d_2} \leq 30 \]  \hspace{1cm} (3.18f)

\[ g_5: d_2 \geq 0.3 \]  \hspace{1cm} (3.18g)

The resolution of the problem by applying the RBDO methods of RIA, PMA and SORA are summarized in Table 3-7, Table 3-8 and Table 3-9 respectively while the evolutions of RBDO parameters for each method are graphed in Figure 3-25, Figure 3-27 and Figure 3-28. For representation purposes, the design variable \( d_2 \) is multiplied by 20 in Figure 3-25 a), Figure 3-27 a) and Figure 3-28 a) and the random variable \( x_p \) is multiplied by 10 in Figure 3-25 b), Figure 3-27 b) and Figure 3-28 b).

For all three methods, the active constraint at the optimum was the probabilistic buckling constraint and the deterministic constraint \( g_4 \) as can be seen in Figure 3-26. The objective functions from all three methods converged to very similar values with practically same design variables and MPP values.

For both RIA and PMA, the design converged in 7 iterations. For reliability routines, for RIA in general it took 3 iterations to converge, while for PMA it took 4. For SORA, the design converged in 7 design cycles and the number of reliability analyses for each design cycle was 4. The total number of iterations between the design optimization and reliability for RIA, PMA and SORA was 32, 39 and 60 respectively.
Resolution of the buckling example by RIA

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$x_P$</th>
<th>$x_E$</th>
<th>obj. func.</th>
<th>$\beta$</th>
<th>rel.iter</th>
</tr>
</thead>
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<td>0</td>
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<td>40.705</td>
<td>2115.504</td>
<td>5760.000</td>
<td>-3.269</td>
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</tr>
<tr>
<td>1</td>
<td>17.129</td>
<td>1.115</td>
<td>181.838</td>
<td>2026.530</td>
<td>22910.762</td>
<td>8.529</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>19.918</td>
<td>0.765</td>
<td>149.245</td>
<td>2073.701</td>
<td>18288.140</td>
<td>5.776</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>20.338</td>
<td>0.723</td>
<td>144.400</td>
<td>2077.644</td>
<td>17642.724</td>
<td>5.372</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>20.508</td>
<td>0.705</td>
<td>142.145</td>
<td>2079.238</td>
<td>17347.756</td>
<td>5.183</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>20.660</td>
<td>0.688</td>
<td>139.930</td>
<td>2080.683</td>
<td>17060.685</td>
<td>4.998</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>20.655</td>
<td>0.688</td>
<td>139.959</td>
<td>2080.656</td>
<td>17064.815</td>
<td>5.000</td>
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<tr>
<td>7</td>
<td>20.655</td>
<td>0.688</td>
<td>139.959</td>
<td>2080.656</td>
<td>17064.817</td>
<td>5.000</td>
<td>3</td>
</tr>
</tbody>
</table>

Total iter. 32

Table 3-7: Evolution of RBDO parameters by RIA ($d_i$ in cm, $P$ in t and $E$ in t/cm$^2$)
Figure 3-25: Evolution of a) design variables b) random variables c) objective function by RIA
Figure 3-26. Design space at the optimum

- **Resolution of the buckling example by PMA**

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$x_p$</th>
<th>$x_E$</th>
<th>obj. func.</th>
<th>rel.iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0.400</td>
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<td>2009.970</td>
<td>5760.000</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>13.379</td>
<td>0.613</td>
<td>139.537</td>
<td>2034.914</td>
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</tr>
<tr>
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<td>0.726</td>
<td>139.848</td>
<td>2062.684</td>
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</tr>
<tr>
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<td>139.954</td>
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</tr>
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<td>20.625</td>
<td>0.687</td>
<td>139.959</td>
<td>2080.581</td>
<td>16994.482</td>
<td>4</td>
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<tr>
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<td>20.655</td>
<td>0.688</td>
<td>139.959</td>
<td>2080.652</td>
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</tr>
<tr>
<td>6</td>
<td>20.655</td>
<td>0.688</td>
<td>139.959</td>
<td>2080.652</td>
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<td>4</td>
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<tr>
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<td>0.688</td>
<td>139.959</td>
<td>2080.652</td>
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<td>Total iter.</td>
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</table>

Table 3-8: Results of buckling example by PMA ($d_i$ in cm, $P$ in t and $E$ in t/cm$^2$)
Figure 3-27: Evolution of a) design variables b) random variables c) objective function by PMA

- **Resolution of the buckling example by SORA**

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$x_P$</th>
<th>$x_E$</th>
<th>obj. func.</th>
<th>rel.iter</th>
</tr>
</thead>
<tbody>
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<td>17651.273</td>
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<td>17029.515</td>
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<td>17067.047</td>
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<td>20.655</td>
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<td>17064.677</td>
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<td>139.959</td>
<td>2080.652</td>
<td>17064.818</td>
<td>4</td>
</tr>
</tbody>
</table>

Total iter. 60

Table 3-9. The results of buckling problem by SORA ($d_i$ in cm, $x_P$ in t and $x_E$ in t/cm²)
Chapter 3

Reliability Based Design Optimization

a) Design variables (cm)

b) $x_P(t), x_E (\mu cm^2)$
Figure 3-28: Evolution of a) design variables b) random variables c) objective function by SORA

3.3.4 10-bar truss structure

The 10-bar truss problem in Chapter 2 is employed again to carry out the RBDO of the structure. Two point loads, \( P \) are applied as shown in Figure 3-29. Design variables are cross sectional areas of horizontal, vertical and diagonal bars \( d_1, d_2, d_3 \) respectively and the objective function is the volume of the truss. The load \( P \) and elastic modulus \( E \) are set as random variables with the target beta \( \beta_T = 5.0 \), which corresponds to \( P_T = 2.87 \times 10^{-7} \). Three probabilistic constraints are considered: \( G_1 \) limits the vertical displacement at node 2, while \( G_2 \) and \( G_3 \) restrict the stress in member 3 and 7 respectively. \( v_{\text{allow}} \) is the allowable vertical displacement and \( \sigma_{\text{allow}} \) is the allowable stress while \( d^0 \) is the initial design.

\[
N_P \ (500 \text{ kN, 50 kN}), \quad N_E \ (7 \times 10^7 \text{ kN/m}^2, \ 3.5 \times 10^6 \text{ kN/m}^2)
\]
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\[ d^0 = (0.01 \text{ m}^2, 0.0015 \text{ m}^2, 0.007 \text{ m}^2) \]

\[ v_{\text{allow}} = 0.12 \text{ m}; \sigma_{\text{allow}} = 1.75E+5 \text{ kPa}; L = 10.0 \text{ m} \]

![10-bar truss](image)

**Figure 3-29. 10-bar truss**

The formulation of the problem is the following:

Minimize \[ F(d) = 4d_1L + 2d_2L + 4\sqrt{2}d_3L \quad (3.19a) \]

Subject to: \[ P[G_i(d, P, E) \leq 0] \leq P_f^T \quad i = 1, 2, 3 \quad (3.19b) \]

where \[ G_1(d, P, E) = \left| v_{\text{allow}} \right| - \frac{PL}{d_1d_3E} \left( \frac{A+B}{D} \right) \geq 0 \quad (3.19c) \]

\[ G_2(d, P) = \sigma_{\text{allow}} - \left[ \frac{2P}{d_1} + \frac{Pd_2d_3\left(2\sqrt{2}d_1 + d_3\right)}{Dd_1} \right] \geq 0 \quad (3.19d) \]

\[ G_3(d, P) = \sigma_{\text{allow}} - \left[ \frac{2P}{d_3} + \frac{(4Pd_1^2d_3 + \sqrt{2}Pd_1d_2d_3^2)}{Dd_3} \right] \geq 0 \quad (3.19e) \]
where  
\[ A = 96\sqrt{2}d_1^3d_2^2 + 4\sqrt{2}d_1^3d_3^2 + 7d_1^2d_2^3 + 26d_2^2d_3^2 \]
\[ B = 80d_1^3d_2d_3 + 304d_1^2d_2^2d_3 + 40d_1d_2d_3^3 + 100\sqrt{2}d_1^2d_2d_3^2 + 116\sqrt{2}d_2d_3^2d_3^2 \]
\[ D = 32d_1^3d_2^2 + 4d_2^2d_3^2 + 12\sqrt{2}d_1^2d_2d_3 + 16\sqrt{2}d_1d_2^2d_3 + d_2^2d_3^2 + 6d_2d_3^2d_3^2 \]

• Resolution of the problem by RIA

The MPPs in u-space at the optimum design are shown graphically in Figure 3-30. As can be seen, the active constraints at the optimum are the displacement constraint of \( G_1 \) and the stress constraint in member 7 of \( G_3 \), and their corresponding MPP values are \((P, E) = (692.94 \text{ kN, } 5.89\text{E}+7 \text{ kpa})\) and \((P, E) = (750.00 \text{ kN, } 7.00\text{E}+7 \text{ kpa})\). The optimum design is \( \mathbf{d} = (0.01319 \text{ m}^2, 0.00187 \text{ m}^2, 0.00916 \text{ m}^2) \) and the corresponding objective function is 1.083 m³.

\[ \begin{align*}
\mathbf{u}_E &= 0 \\
\mathbf{u}_0 &= \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \\
\mathbf{\beta} &= \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}
\end{align*} \]

\[ \begin{align*}
\mathbf{MPP} (G_1) &= \mathbf{MPP} (G_2) \\
\mathbf{MPP} (G_3) &= \mathbf{MPP} (G_2)
\end{align*} \]

Figure 3-30. The MPPs in u-space at the optimum
Table 3-10 and Figure 3-31 summarize the evolution of the RBDO parameters using RIA. The design converged in 8 iterations and the total number of iterations including the reliability iterations was 84.

<table>
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<tr>
<th>Iter.</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>obj. func.</th>
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<td>1.500E-03</td>
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<th>$x_E (G_2)$</th>
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Table 3-10. Evolution of design variables, random variables, $\beta$ and the number of reliability iterations of the 10-bar truss example using RIA

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<th>$\beta$ ($G_1$)</th>
<th>$\beta$ ($G_2$)</th>
<th>$\beta$ ($G_3$)</th>
<th>rel. iter.($G_1$)</th>
<th>rel. iter.($G_2$)</th>
<th>rel. iter.($G_3$)</th>
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Table 3-10. Evolution of design variables, random variables, $\beta$ and the number of reliability iterations of the 10-bar truss example using RIA
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b)

\[ P(\text{kN}) \]

\[ x_p(G_1), x_p(G_2), x_p(G_3) \]

Iteration

0 2 4 6 8

7.5 \times 10^7

\[ x_e(G_1), x_e(G_2), x_e(G_3) \]

c)
Resolution of the problem by PMA

Figure 3-33 shows the MPPs in $u$-space at the initial design and the optimum design. At the initial design, the constraint, $G_1$ and $G_3$, are violated, however, at the optimum, these two constraints are active. Since both $G_2$ and $G_3$ are independent of $E$, and $G_2$ is not active at the optimum, the MPPs for both constraints take the same value.
Figure 3-32. The MPPs and the probabilistic constraints in $u$-space at the a) initial and b) the optimum design
The evolution of the RBDO parameters using PMA method is summarized in Table 3-11 and Figure 3-33. The optimum design is very similar to that by RIA, which is $d=(0.01321 \text{ m}^2, 0.001873 \text{ m}^2, 0.009142 \text{ m}^2)$ with the objective function of 1.0831 m$^3$. The total number of iterations in this case is 189, which is greater than the case of RIA. The evolution of random variables is not shown in Table 3-11 because the MPPs took the same values throughout the design optimization.

<table>
<thead>
<tr>
<th>Iter.</th>
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<th>$d_2$</th>
<th>$d_3$</th>
<th>obj. func.</th>
</tr>
</thead>
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</table>

$x_F(G_1)$ $x_E(G_1)$ $x_F(G_2)$ $x_E(G_2)$ $x_F(G_3)$ $x_E(G_3)$ rel. iter.(G_1) rel. iter.(G_2) rel. iter.(G_3)

693.02 5.89E+07 750.00 7.00E+07 750.00 7.00E+07 88 44 44

| total iter. | 189 |

Table 3-11. Evolution of design variables and the MPPs at the optimum for the 10-bar truss using PMA
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a) Cross sectional area (m²)

b) P (kN)

Iteration

Iteration
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- Resolution of the problem by SORA

The evolution of the parameters using SORA is presented in Table 3-12 and Figure 3-34 below. The optimum design as well as the objective function is practically
the same as the case using PMA. The optimum design is \( \mathbf{d} = (0.01321 \, \text{m}^2, 0.001873 \, \text{m}^2, 0.009142 \, \text{m}^2) \) and the objective function \( F = 1.0831 \, \text{m}^3 \). The total number of iterations was 103.

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<th>obj. func.</th>
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Table 3-12. Evolution of design and random variables of 10-bar truss using SORA

![Graph showing the evolution of design variables](image)
Chapter 3  
Reliability Based Design Optimization

b) 

\[ P(\text{kN}) \]

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c) 

\[ E(\text{kN/m}^2) \]

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The three RBDO methods of RIA, PMA and SORA discussed in this chapter will be applied to long-span bridges considering aeroelastic constraint. In the formulation, plate thicknesses of box girders as well as the main cable area are considered as design variables while uncertainties in the extreme wind velocity at bridge site and experimentally obtained flutter derivatives are taken into account. In Chapter 4, the formulation of the RBDO applied to long-span bridges under aeroelastic constraint is presented, which will be applied to bridge examples in later chapters.
3.4 References


[L1] Libertiny, G. Z., “Safe design without safety factors”, In American Society of Mechanical Engineers, Dallas, TX, USA, 1997


CHAPTER 4

RBDO FORMULATIONS OF LONG-SPAN SUSPENSION BRIDGES UNDER FLUTTER CONSTRAINT

4.1 Introduction

In the last century we have witnessed spectacular advances in span length of long-span suspension bridges. The Akashi-Kaikyo Bridge in Japan inaugurated to traffic in 1998 has the world-record span length of 1991 meters, and the Gran Belt East Bridge in Denmark opened to public a couple months later is a suspension bridge with the third-longest span length of 1624 meters. The proposed Messina Bridge in Italy that will connect the Sicily Island with the Italian Peninsula consists of the main span length of 3300 meters. The advances in construction technologies make the span length of cable-supported bridges each time longer, yet at the same time, the bridge structures become more flexible and more prone to wind instabilities.
Flutter is one of the most important aeroelastic phenomena to be considered for any long-span bridge project. When this wind instability occurs, the wind forces acting on the bridge deck combined with the deck movement itself cause negative structural damping and increase the deck movement exponentially until the collapse of structures. Since the critical flutter velocity decreases with reduced structural stiffness and damping, suspension bridges with long spans are more likely to suffer this aeroelastic phenomenon. Therefore it is essential to take into account this important failure mode in any long-span bridge design in addition to those design constraints of static and dynamic loadings. Furthermore, for large structures such as long-span bridges, the structural optimization may be an important tool for a design engineer in order to save cost and succeed in the modern competitive market. Consequently, an engineer working with a long-span bridge should seek safety against flutter while minimizing the cost by structural optimization.

As opposed to traditional deterministic optimizations, the Reliability Based Design Optimization (RBDO) performs design optimization considering system uncertainties, which makes this method more reliable. Instead of assigning a fixed value of safety factor on overall structural system to account for the system uncertainties, commonly used reliability methods quantify these uncertainties by some statistical moments. These quantified uncertainties are then integrated into probabilistic constraints, in which a desired reliability level can be specified. These probabilistic constraints as well as deterministic constraints are evaluated in each iteration of the design optimization process. In recent years, there have been applications of the RBDO to different structural systems in aerospace, defense, automobile and civil engineering fields as discussed in Chapter 3. However, there has not been any research on the RBDO applied to long-span suspension bridges under flutter constraint.

In this research, the RBDO of long-span suspension bridges under aeroelastic flutter constraint is studied and these formulations are applied to two long-span suspension bridge examples. The flutter constraint requires a large number of random
variables because it uses experimentally obtained functions that contain uncertainties by nature. The number of random variables can reach as many as one hundred. Three different methods were used to illustrate the performance of the RBDO, namely Reliability Index Approach (RIA), Performance Measure Approach (PMA) and Sequential Optimization and Reliability Assessment (SORA) explained in detail in Chapter 3.

4.2 Aeroelastic instability: Flutter

Aeroelasticity can be defined as a science that studies the fluid-structure interactions of flexible structures caused by wind. Aeroelastic phenomena arise when structural deformations induce additional aerodynamic forces, which in turn may induce additional structural deformations. Such interactions may reach a stable equilibrium or diverge until system failure (Bisplinghoff et al.\textsuperscript{B1}).

Among several aeroelastic phenomena, flutter is one of the most important for the design of long-span bridges because it may cause the destruction of structures. It is an aeroelastic instability in which the wind forces change because of structural deformations while wind modifies the stiffness and damping of the system. Finally when the structural damping becomes null, any small oscillatory movement will be amplified exponentially until the failure of the system (Scanlan\textsuperscript{S1}).

The study of aeroelasticity emerged initially in the aeronautical field and eventually extended to civil engineering. It drew a great deal of attention in bridge engineering field after the well-known collapse of the Tacoma narrow bridge in 1940 (Figure 4-1). Bridge engineers worked on the project at that era were unable to predict this aeroelastic problem nor prevent the bridge from collapsing. The bridge failed after oscillating in a torsional mode at the mild wind speed of 64 km/h according to Washington State Desparment of Transportation\textsuperscript{W1}. Scanlan and Tomko\textsuperscript{S2} explained
that the bridge failed because of flutter phenomenon. This event created ripple effects across the world for the necessity of better understanding of bridge aeroelasticity, which changed the design of suspension bridges ever after; it led to a safer suspension bridge that we are using today.

Figure 4-1. The collapse of the Tacoma Bridge in 1940

There are currently two main approaches for the determination of flutter speed. One is based on a full bridge model tests (Zasso et al.\cite{Z1}) and the other is a hybrid method consisting of an experimental phase of testing a sectional model in an aerodynamic wind tunnel and a subsequent computational phase (Jurado et al.\cite{J1}).

Although there is a fully computational approach to obtain flutter speed, its accuracy is not well-established. When the results from computational fluid dynamics (CFD) methods are more precise, it will be possible to substitute the experimental phase of the hybrid method by a CFD analysis. Then the procedure will be fully computational. Many studies using CFD have recently been carried out trying to obtain aerodynamic coefficients, flutter derivative functions or vortex shedding simulations, etc. See Nieto\cite{N2} and Hernandez\cite{H6, H7}.
The fully-experimental method is based on testing a reduced model of a entire bridge to obtain structural responses under wind loads in a boundary layer wind tunnel. Prior to testing, this type of wind tunnel requires information of statistical characteristics that affect the bridge by modeling a terrain in a small scale (1:2000 to 1:5000) (Meseguer et al.[M1]). Then a larger scale test should be carried out (1:100 to 1:300) with a boundary layer generated using ridges of varying steepness and roughness for the characteristics obtained previously. There are, however, some drawbacks associated with this method. First of all, it requires a large and expensive testing chamber to carry out tests. Secondly, the complete bridge model should maintain the frequency scale and stiffness scale for taking into account aerodynamical and aeroelastic effects simultaneously, which is difficult to achieve for a large complex bridge model. Figure 4-2 shows a complete model of the Akashi Kaikyo Bridge and the wind tunnel of the Ministry of Public Works in Tsukuba where it was tested.

Figure 4-2. The complete model of the Akashi Kaikyo Bridge and the boundary layer wind tunnel (Public Works Research Institute)

4.2.1 Hybrid methods

The second approach of hybrid methods is based on two phases of experimental and computational in determining flutter velocity. In the first phase, a bridge deck sectional model is tested in an aerodynamic wind tunnel to obtain aerodynamic
coefficients and flutter derivatives, which are then used to determine static and aeroelastic forces. An example of such sectional model and a wind tunnel are shown in Figure 4-3.

![Figure 4-3. A sectional bridge deck model of the Messina Bridge and the aerodynamic wind tunnel at the University of La Coruña](image)

The hybrid methods have clear advantages over the first approach of full bridge model testing for not requiring large expensive wind tunnel facilities. Additionally, these methods allow the modification of simple structural parameters in the computational phase. For example, we can modify the cables in the finite element model (Jurado et al.\[J2\], Hernandez et al.\[H1\]), or optimize the mechanical properties of bridge structure (Nieto et al.\[N1\], Hernandez et al.\[H2\]). Nonetheless, these approaches do not provide as much information as the full-bridge testing approach such as the visualization of aeroelastic response of the entire bridge. The group of structural mechanics at the University of La Coruña (Hernandez et al.\[H3\]) has overcome this disadvantage by employing advanced computational visualization to represent bridge movements with high quality images as shown in Figure 4-4.
In this research, the hybrid method was used to study the aeroelastic flutter phenomenon of long-span suspension bridges.

### 4.2.2 Experimental phase

In the first phase of the hybrid method, we have to determine the aeroelastic forces, $f_a$, per unit length of the bridge deck experimentally in an aerodynamic wind tunnel. Scanlan\cite{Scanlan}, known as the father of modern aeroelasticity, describes aeroelastic forces in three components of drag ($D_a$), lift ($L_a$), and moment ($M_a$) in terms of displacement and velocity of the bridge deck as follows.

\[
D_a = \frac{1}{2} \rho V^2 B \left[ K P_1 \left( \frac{v}{V} \right) + K P_2 \left( \frac{\phi}{V} \right) - K^2 P_3 (\phi) + K^2 P_4 \left( \frac{v}{B} \right) - K^2 P_5 \left( \frac{w}{B} \right) \right]
\]

\[
L_a = \frac{1}{2} \rho V^2 B \left[ K H_1 \left( \frac{w}{V} \right) + K H_2 \left( \frac{\phi}{V} \right) + K^2 H_3 \phi + K^2 H_4 \left( \frac{w}{B} \right) - K^2 H_5 \left( \frac{v}{B} \right) \right]
\]

\[
M_a = \frac{1}{2} \rho V^2 B \left[ K A_1 \left( \frac{\dot{w}}{V} \right) + K A_2 B \left( \frac{\phi}{V} \right) + K^2 A_3 \phi + K^2 A_4 \left( \frac{w}{B} \right) - K^2 A_5 \left( \frac{\dot{v}}{B} \right) \right]
\]

where $\rho$ is air density, $V$ is the average wind velocity, $B$ is the deck width, $K$ is the reduced frequency as $K = \omega B / V$, $\omega$ is frequency, $v$, $w$, $\phi$ are horizontal, vertical and
rotational degrees of freedom as shown in Figure 4-5. $P_i^*, H_i^*, A_i^*$ with $i=1…6$ are flutter derivatives associated with each component of aeroelastic forces.

Figure 4-5. Sign convention of the University of La Coruña for flutter analysis

These 18 flutter derivatives, 6 for each force component, must be obtained experimentally, and its procedure is described in detail by León$^{[L1]}$ by testing a bridge sectional model under free vibration. The main steps of the procedure are the followings.

1. Suspend the deck sectional model by various vertical and horizontal springs of known stiffness and attach them to load cells.
2. Record the initial load in the load cells under equilibrium position (Figure 4-6 a)
3. Move the model from its equilibrium position by sustaining it by pneumatic actuators (Figure 4-6 b)
4. Start the data acquisition and let the model vibrate freely by turning off solenoid valve to retract pneumatic actuator. The control program registers the deck movements from load cell data.
5. The control program then calculates aeroelastic stiffness and damping matrices using identification methods as modified Ibrahim time domain method$^{[L1]}$ or iterative least-square method$^{[S3]}$.
This process is repeated several times without wind for improving the precision of the aeroelastic matrices. Then it is performed as many times as necessary with increasing wind velocity for obtaining data points of flutter derivatives. By varying the spring constants of suspension springs, a larger range of reduced velocity can be obtained since the system natural frequencies depend on these spring constants.

Figure 4-6. Aeroelastic test of the Messina Bridge sectional model

Figure 4-7 shows one of the flutter derivatives from the Messina Bridge sectional model test for three angles of attack obtained by León\textsuperscript{[1-1]}. Each of the three lines describes the most representative value set from dispersed data. As can be seen, there is a larger data dispersion associated with increasing reduced velocity. This is because higher wind velocities generate larger vibrations in the test chamber, which causes greater noise in the load cell readings.
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4.2.3 Computational phase

The calculation of structural responses against flutter requires a finite element model of the entire bridge under study as shown in Figure 4-8. This type of structural model usually consists of 3D bar elements with six degrees of freedom. The model is first used to calculate the initial main cable length as well as the initial stress in the main cables and the hanger cables, and then perform a modal analysis to obtain natural frequencies and mode shapes of the entire bridge.

Figure 4-7. Flutter derivative, $A_3^*$ obtained from sectional model test
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RBDO formulations of long-span suspension bridges under flutter constraint

The matrix formulation for the calculation of critical flutter velocity is based on Scanlan’s definition of aeroelastic forces described in an earlier section. We assume that such forces are applied only on the bridge deck because it is the part of structure that interacts with wind when flutter instability occurs (Hernandez et al.\(^{[H4]}\)).

The aeroelastic forces \( f_a \) that exert on the unit length of the bridge deck of the Equation (4.1) can be rewritten as:

\[
\begin{bmatrix}
D_u \\
L_u \\
M_u
\end{bmatrix} = \frac{1}{2} \rho v K B \begin{bmatrix}
P_1^* \\
H_5^* \\
BA_6^*
\end{bmatrix} \begin{bmatrix}
\dot{v} \\
\dot{\phi} \\
\phi
\end{bmatrix} + \frac{1}{2} \rho v^2 K B \begin{bmatrix}
P_6^* \\
H_4^* \\
BA_4^*
\end{bmatrix} \begin{bmatrix}
\dot{v} \\
\dot{\phi} \\
\phi
\end{bmatrix}
\]

which can be expressed in a matrix form as:

\[
f_a = C_a \dot{y} + K_a y
\]  
(4.3)

where \( K_a \) and \( C_a \) are aeroelastic stiffness and damping matrices correspondingly, while \( y \) represents a displacement vector of any node along the deck. The notation of \( y \) is used
here to avoid the confusion with $x$ and $u$ employed as the vector of random variables in the original and the normalized space in the later sections. The global matrix, $K_a$ and $C_a$ for the entire bridge can be obtained by assembling the matrix of each bar element of the deck. The dimension of the matrices coincides with the total number of degree of freedom of the bridge deck.

The system of equations that governs the dynamic behavior of the deck under aeroelastic forces is expressed as (Jurado et al.\[J3\]):

$$M\ddot{y} + C\dot{y} + Ky = f_a$$ \hspace{1cm} (4.4)

where $M$, $C$, $K$ are mass, damping and stiffness matrices. By combining Equation (4.3) and (4.4), we get

$$M\ddot{y} + C\dot{y} + Ky = f_a = C_a\dot{y} + K_a y$$ \hspace{1cm} (4.5)

By rearranging

$$M\ddot{y} + (C - C_a)\dot{y} + (K - K_a)y = 0$$ \hspace{1cm} (4.6)

which represents a movement of a body in free vibration whose damping and stiffness matrices are modified by aeroelastic forces.

In order to solve the problem of Equation (4.6), modal analysis should be performed. The displacement vector can be written as a function of the most relevant mode shapes grouped by the modal matrix, $\Phi$.

$$y = \Phi w e^{\mu t}$$ \hspace{1cm} (4.7)

where $\mu$ and $w$ are complex values. The system can be transformed to
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RBDO formulations of long-span suspension bridges under flutter constraint

\[
\left( \mu^2 \mathbf{I} \mathbf{w} + \mu \mathbf{C}_R \mathbf{w} + \mathbf{K}_R \mathbf{w} \right) \mathbf{e}^{j \omega t} = 0
\]  

(4.8)

where \( \Phi^T \mathbf{M} \Phi = \mathbf{I} \), \( \Phi^T \left( \mathbf{C} - \mathbf{C}_a \right) \Phi = \mathbf{C}_R \), \( \Phi^T \left( \mathbf{K} - \mathbf{K}_a \right) \Phi = \mathbf{K}_R \)

Equation (4.8) becomes a nonlinear eigenvalue problem as:

\[
(\mathbf{A} - \mu \mathbf{I}) \mathbf{w}_\mu \mathbf{e}^{j \omega t} = 0
\]  

(4.9)

where \( \mathbf{A} = \begin{pmatrix} -\mathbf{C}_R & -\mathbf{K}_R \\ \mathbf{I} & 0 \end{pmatrix} \) and \( \mathbf{w}_\mu = \begin{pmatrix} \mu \mathbf{w} \\ \mathbf{w} \end{pmatrix} \)

The solution to this eigenvalue problem is a set of \( m \) couples of conjugate complex eigenvalues, where \( m \) is the number of vibration modes considered in the problem.

\[
\mu_j = \alpha_j + i \beta_j \\
\bar{\mu}_j = \alpha_j - i \beta_j \\
\]  

(4.10)

The real part of the eigenvalue is related to the structural damping while the imaginary part is the frequency of the structural response for a particular combination of modes as:

\[
\omega_j = \beta_j \text{ and } \zeta_j = \frac{-\alpha_j}{\sqrt{\alpha_j^2 + \beta_j^2}} \quad j = 1, \ldots, m
\]  

(4.11)

where \( \zeta_j \) is a structural damping. In order to solve the eigenvalue problem of (4.9), we should realize that the assembly of the matrix \( \mathbf{A} \) requires flutter derivatives, which depends on the reduced frequency of the system, \( K = \omega B/V \). However, the frequency is unknown until the eigenvalue problem is resolved. Therefore the problem must be solved by an iterative process. The procedure of how to obtain aeroelastic response of the bridge deck by solving this eigenvalue problem is shown in Figure 4-9.
1. For a vibration mode, \( j \), the initial frequency is the sum of the last converged frequency plus the difference between the last and the second to the last. For the 1st iteration, use natural frequencies of the system.

2. Resolve the eigenvalue problem, \( (A(\beta_j) - \mu I)w = 0 \) where \( A \) depends on \( \beta_j \).

3. Among \( 2m \) eigenvalues obtained, choose \( \beta_l \), whose value is the closest to \( \beta_j \).

4. If the difference between \( \beta_l \) and \( \beta_j \) is larger than a predefined tolerance value, repeat Step 2 to 4, otherwise, \( \beta \) is considered to be converged.

5. Repeat Step 1-4 for each vibration mode until we complete \( m \) sets of frequencies.

6. Check for repeated eigenvalues, i.e., see if there are any two eigenvalues whose \( \alpha \) and \( \beta \) values are smaller than predetermined tolerance values.

7. If this is the case, then look for another eigenvalue whose \( \beta \) is within a tolerance value, \( \varepsilon_1 \) and \( \alpha \) is greater than a tolerance value, \( \varepsilon_2 \). Replace one of the repeated one with the newly found eigenvalue.
8. Complete the set of eigenvalues.

According to Equation (4.11), \( \alpha \) has to take negative values in order to achieve positive structural damping for diminishing structural oscillations. The critical situation for flutter occurs when the structural damping becomes null, which means that \( \alpha \) in the eigenvalue for any vibration mode becomes from negative to positive. The procedure to obtain the critical flutter velocity is shown schematically in Figure 4-10.

1. Choose an initial wind velocity value, which is small enough to be free from flutter instability.
2. Solve the eigenvalue problem explained previously to get \( m \) set of eigenvalues.
3. Check to see if all \( \alpha \) take negative values. If this is the case, repeat Step 1-3 for increased wind velocity, \( V = V + \Delta V \) until \( \zeta_{\text{min}} \) becomes positive. The critical flutter velocity, \( V_{\text{cr}} \) in which \( \zeta_{\text{min}} = 0 \) can be computed by interpolation. The eigenvector, \( w_{\mu} \) describes the participation of each aeroelastic mode for flutter.
4.2.4 FLAS program

The multimodal flutter analysis program, FLAS was developed by Jurado\textsuperscript{34} at the University of La Coruña to calculate the critical wind speed of long span bridges. It requires natural frequencies of the entire bridge, mode shapes of the deck, 18 flutter derivatives obtained experimentally as well as structural damping, deck width and vibration modes considered in the aeroelastic calculation. The program plots the real and imaginary part of the eigenvalues with respect to wind speed, $\alpha$ and $\beta$, and prints out the critical flutter velocity and its corresponding reduced frequency as an output.
The original FLAS program occasionally did not function properly because of problems associated with repeated eigenvalues. When two eigenvalues are repeated in Step 5 in the procedure to solve eigenvalue problem, one of the aeroelastic modes disappeared and FLAS may give inaccurate results. Figure 4-11 a) shows an example of FLAS problem in which the aeroelastic mode 6 disappeared at the wind velocity of 28 m/s. The problem arises when there are two very similar frequencies after solving the eigenvalue problem. When $\beta_k$ that makes the minimum difference of $|\beta_k - \omega_j|$ is chosen, there may be cases that the program chooses the same eigenvalue for two different aeroelastic modes. In the worst case when the disappeared mode is the flutter causing mode, FLAS simply presents no flutter.

In order to solve this problem, a piece of new code was added by the author, which corresponds to Step 6 and 7 in the procedure to solve eigenvalue problem and “added new code” in the flowchart of Figure 4-9. First of all, when all $m$ sets of eigenvalues are chosen (Step 5), the program checks to see if there are any two repeated eigenvalues; in such case, they have identical real and imaginary parts, $\alpha$ and $\beta$. The code goes back to Step 4 to look among $\alpha_k \pm i\beta_k$ for another eigenvalue with the same imaginary part $\beta_j$, and a different real part $\alpha_k \neq \alpha_j$. Finally one of the repeated eigenvalues is replaced by the new eigenvalue with a different real part. Figure 4-9 b) shows the plots of modified FLAS. The problem presented previously has been solved and flutter instability occurs at 106.49m/s.

This modification significantly improved the performance of FLAS, which was employed for the calculation of flutter velocity throughout the study. For further reading on the topic, refer to Jurado\textsuperscript{[5]}.
Figure 4-11. Example of the FLAS outputs with a) repeated eigenvalue problem and b) added code to filter repeated eigenvalues
4.3 Reliability analysis of long-span suspension bridges under flutter constraint

4.3.1 Introduction

The structural reliability can be defined as the probability that a structural system will perform its function under required service conditions during a specific time period. This probability refers to a particular limit state that a structure must satisfy in order to perform a certain task. In the case of long-span bridges, their structural performance under wind load, especially flutter phenomenon, is one of the most important design considerations.

As explained in earlier sections, there are uncertainties associated with the calculations of flutter velocity values. The primary reason is that the determination of flutter velocity in general is based on a hybrid method, which requires an experimental acquisition of flutter derivatives. These uncertainties may be associated with measurement and instrumental errors in wind tunnel tests as well as those related to the determination of representative flutter derivative curves from limited sample size. Figure 4-12 shows two examples of flutter derivatives, in which we can see three curves that define flutter derivatives for different angles of attack drawn from data points obtained experimentally. We can observe that higher uncertainties are associated with data points at larger wind velocities due to greater noises in the output signal.
There exist some empirical formulas to provide estimate values of flutter wind speed expressed in terms of deck geometry, lowest torsional and bending frequencies, and radius of gyration of deck cross-section, among others. These formulas are used in design codes of different countries, such as the Chinese Wind-resistant Design Specification for Highway Bridges (2004)\(^{[C1]}\); nonetheless, since these methods do not provide accurate results, specific wind tunnel tests should be carried out for each case.

Some researchers have carried out probabilistic bridge analyses against flutter. Ostendfel-Rosenthal et al.\(^{[O1]}\) considered the flutter speed obtained in wind tunnel as a Gaussian random variable affected by other two random variables. In this research extreme wind speeds follow a Gumbel distribution and the FORM is used with a total number of four random variables. Ge et al.\(^{[G1]}\) presented a method to obtain probability of failure due to flutter by the FORM method. In that research four random variables were considered and an empirical flutter speed formula was used to define the limit state function. Pourzeynail and Datta\(^{[P1]}\) considered uncertainty in flutter derivatives for a model with two degrees of freedom of lift and moment. Analytical formulas of six flutter derivatives were employed in their research. Cheng et al.\(^{[C2]}\) presented a reliability flutter study with fourteen random variables using the FORM method. Before applying the reliability method, the authors set an approximation of the limit state
function by the response surface method. Recently, reliability methods have been employed in other bridge analyses such as buffeting by Caracoglia\cite{C3} and Pourzeynali\cite{P2}, buckling of arch bridges by Cheng, J. et al.\cite{C4} and aerostatic stability by Cheng, S. et al.\cite{C5}.

In our research, the modified FORM method (Baldomir et al.\cite{B2}) was applied to compute the probability of failure of long-span suspension bridges due to flutter. In this study, the uncertainties in wind load and each point that defines flutter derivatives were considered, which summed up to a total of approximately 40 random variables for the Great Belt example and one hundred for the Messina Bridge example. Hasofer Lind – Rackwitz Fissler reliability algorithm was programmed in MATLAB code. The limit state function and its first derivative with respect to each random variable were evaluated in each iteration. The computation of flutter velocity was essential at each design point as well as the sensitivity of flutter velocity with respect to each random variable by finite difference method. Flutter velocity is obtained by establishing the dynamic equilibrium equation of the bridge under aeroelastic forces that requires eighteen flutter derivatives obtained experimentally as explained in the previous section. The dynamic equilibrium equation is then resolved by an iterative process using FLAS code (See Section 4.2.4).

4.3.2 Random variables

As explained in Chapter 2, parameters with high uncertainty should be considered as random variables, which may be defined by their first and second statistical moments. In this study, the wind velocity, the structural damping as well as the points that define the eighteen flutter derivatives were considered as random variables.

- Wind velocity
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RBDO formulations of long-span suspension bridges under flutter constraint

Extreme wind velocity at the bridge location is one of the most obvious sources of uncertainty in the calculation of flutter because the occurrence of a particular wind velocity value in a certain return period is simply stochastic based on the past statistical data. This type of uncertainty is referred as inherent uncertainty and is irreducible. Generally, meteorological stations near bridge location can provide historical wind data for the definition of statistical functions.

It is known that a cumulative distribution function of statistical extreme wind values well adjust to an asymptotic extreme-value distribution like Gumbel, Weibull or Frechet. In order to apply the FORM method, any extreme-value distribution function should be converted into a normal-equivalent function. This procedure is detailed in Section 2.6.6.

- Structural damping

According to Davenport et al.\[^{D2}\], structural damping for long-span bridges may vary up to 40% of its mean value. This parameter is used in the reliability analysis when flutter velocity is calculated by solving an eigenvalue problem of Equation (4.9). In this study, the structural damping, $\varsigma$ is defined as a log-normal distributed random variable whose standard deviation is 20% of its mean value. The assumption of log-normal distribution function is based on the fact that the structural damping is always a positive value according to Kareem\[^{K1}\] and Gupta et al.\[^{G2}\].

- Flutter derivatives

As explained in detail in Section 4.2.1, the computation of flutter speed using hybrid method consists of an experimental phase to obtain eighteen flutter derivatives and its subsequent computational phase. Aeroelastic forces are related to bridge deck displacements and their first derivatives as well as flutter derivatives as in Equation (4.1). In the wind tunnel, these flutter derivatives are obtained for different values of reduced velocity as shown in Figure 4-12. Then a curve that best fits the obtained data
set is defined for each flutter derivative, and several representative points are chosen from this curve. This is because the flutter software FLAS uses only some representative flutter derivative points and interpolates intermediate values for the corresponding reduced velocity. These data points are considered to be normally distributed random variables with their defined values as their mean values. Different values of standard deviation were considered to see the sensitivities of the probability of failure on the experimental data dispersion. Figure 4-13 shows an example of flutter derivative, $H_2^*$, in which standard deviation of each data point is represented schematically. Two types of dispersion, constant (Figure 4-13 a) and linearly variable standard deviation (Figure 4-13 b), were considered. The second type is more realistic because it takes into consideration the greater dispersions associated with higher reduced velocities just as observed in actual wind tunnel tests. Since there are four to seven points for each flutter derivative, the total number of random variables associated with all 18 flutter derivatives can be as many as approximately one hundred.

![Figure 4-13. Flutter derivatives $H_2^*$ with a) constant and b) variable standard deviation](image-url)
4.3.3 Limit state function

After the random variables are specified, a limit state function must be defined in order to carry out reliability analyses. For the structural safety of a long-span bridge, the extreme wind velocity must be smaller than the expected flutter speed, which is a function of flutter derivatives and structural damping as:

$$G(x) = V_f(x_i, x_\varsigma) - x_w \quad i = 1, 2, ... n$$ (4.12)

where $x = [x_1, x_2, ..., x_n, x_\varsigma, x_w]$ is a vector of $n+2$ random variables, $V_f$ is flutter velocity, $x_i$ is each point that defines flutter derivatives, $x_\varsigma$ is the structural damping, $x_w$ is the extreme wind velocity and $n$ is the total number of flutter derivatives points. A positive value of the limit state function indicates that the design is in a safe region while the negative value suggests the contrary. The normalized form of the limit state function is commonly used for better convergence of reliability algorithm as:

$$G(x) = \frac{V_f(x_i, x_\varsigma)}{x_w} - 1 \quad i = 1, 2, ... n$$ (4.13)

4.3.4 Reliability method

In this research, the modified FORM method was applied to compute the probability of failure due to flutter, which was explained in detail in Section 2.6.2.1. In order to calculate reliability index of Equation (2.25), the first derivatives of the limit state function with respect to each random variable must be computed. The derivative of the limit state function with respect to $x_w$ takes a simple form of:

$$\frac{\partial G}{\partial x_w} = -\frac{V_f}{x_w^2}$$ (4.14)

The derivatives of the limit state function with respect to $x_i$ and $x_\varsigma$ are expressed as:
The partial derivatives of $V_f$ with respect to $x_i$ and $x_\varsigma$ are obtained by the finite difference method.

### 4.3.5 General work flow of the reliability analysis

The general scheme of the reliability analysis is shown in Figure 4-14. The primary code to perform the reliability analysis in this study is written in Matlab 2010r. The algorithm uses the mean values of random variables, $\mathbf{u}_0=0$, as the initial MPP point. The initial value of $\varsigma$ is written in FLAS input file while the original flutter derivatives are written in each flutter derivative file. Then FLAS is executed to compute flutter velocity using the information of a particular bridge such as natural frequencies, mode shapes, span lengths, deck width, and vibration modes to consider.

In order to calculate the reliability index $\beta$ in Equation (2.28), we need to compute the limit state function and its derivatives with respect to each random variable as in Equation (4.15). The partial derivatives of flutter velocity with respect to structural damping and each flutter derivative point are obtained using the finite difference method. Since FLAS is executed as many times as the number of random variables for each reliability iteration, the number of FLAS execution is quite high. Then the algorithm checks the convergence of $\beta$ as well as the limit state function evaluated at the MPP, which should be smaller than a predetermined value, $\varepsilon$. While the algorithm does not converge, the next point of $\mathbf{u}$ is computed using Equation (2.31) and the new iteration is carried out with the updated values of structural damping and flutter derivatives. The value of $\mathbf{u}$ at the convergence is the MPP.
Figure 4-14: Flow diagram of the reliability analysis
4.4 Reliability based design optimization of long-span suspension bridges under flutter

4.4.1 Introduction

In the previous section, the methodology to obtain reliability index for long-span suspension bridges was discussed considering various random variables. We have seen that the calculation of reliability index itself is an optimization problem to minimize the distance between the mean values of random variables to the failure surface of the limit state function in a normalized space.

This section will present some methods to optimize one of the design characteristics of long-span suspension bridges, specifically the structural weight, under probabilistic flutter constraint. Large structures such as long span bridges require huge material cost, which is one of the major expenses of the construction. Within this material cost, the bridge girder and the main cable constitute the largest part, and consequently, the reduction in material quantity of these elements would be of great importance. There exist different bridge deck designs for long-span bridges, among which truss girders and box girders are the most commonly used.

The aerodynamic box girders are considered as the most advantageous designs for the long-span bridges for the performance against flutter instabilities. They are also advantageous in terms of low maintenance and corrosion protection costs. Since the box deck sections provide larger torsional inertia than open sections, the bridge deck can be dimensioned shorter in height. Consequently, the bridge with box sections exhibit more slender structure.

Thus, bridges with a single box and more recent design of multi-box section are the most interesting designs to be studied using the RBDO analyses. In this research, the Great Belt Bridge in Denmark is chosen as a single-box deck example, and the Messina
Bridge project in Italy, which consists of three box girders, is an example to see the efficiency of the RBDO applied to a multi-box deck section.

For the definition of an optimization problem, the selection of design variables is essential. For example, the design variables of a truss-girder suspension bridge such as the Akashi Kaikyo Bridge (Figure 4-15), may be the cross sectional area of each truss members that form the girder.

In the case of bridge examples studied in this research such as the Gran Belt East Bridge and the Messina Bridge project, the bridge decks consist of one or various aerodynamic box girders as shown in Figure 4-16. Because a large portion of this deck type is made of steel plates, it seems logical to define steel plate thicknesses as design variables. The economic impact of reduction in material cost as a result of structural optimization may be very important. For example, in the case of the Gran Belt East Bridge, the reduction of each millimeter of plate thickness around the girder perimeter results in approximately 1300 metric tons of saving in steel, while for the Messina Bridge project, the same reduction in plate thicknesses can save as much as 2200 tons. Additionally, when we consider the main cable area as a design variable, the economic impact is even more significant for the large volume.

Figure 4-15. Cross section of the Akashi Bridge
The objective of this section is to explain the methodology developed for this research to minimize the bridge weight while satisfying probabilistic constraint against flutter as well as other deterministic constraints. A Matlab code has been developed as the primary code to carry out the entire RBDO process. The active-set algorithm\textsuperscript{[M2]} in the Matlab optimization toolbox was employed as the main optimization algorithm. Abaqus was used first to determine the initial length and stress of the main cables by running a catenary cable model, and to perform both static and modal analysis of the entire bridge. Three RBDO methods of RIA, PMA as well as SORA were written in Matlab code, while flutter velocity was calculated by the FLAS program.

### 4.4.2 Definition of the RBDO problem

In RBDO, the statistical aspects of the random parameters are defined in terms of probabilistic constraints, in which an engineer can specify a desired reliability level. Additionally optimization parameters such as objective function, design variables, deterministic constraints, as well as reliability parameters such as limit state functions and random variables must be declared.
Design variables

Two main cases of design variable sets are studied in this research. The first case (Case I) is to consider only steel plate thicknesses of the box girder under study as shown in Figure 4-17. The second case (Case II) is to include the main cable cross sectional area as an additional design variable to the first case.

![Diagram of box girder and main cable area](image)

**Figure 4-17. Design variables of a box girder and main cable area**

The variation in deck sectional area during the optimization implies a change in total deck weight. This affects the tensile stress in the main cable, which should be limited below a reasonable stress value. Section 4.4.3.2 explains how to determine the cable cross sectional area for a particular value of total deck weight while maintaining the cable tensile stress under a predetermined value. After the cable area is fixed, the initial length and stress of the main cables must be determined. This process explained in detail in Section 4.4.3.1 is iterative since we only know their final geometry, but not the initial cable length.

Objective function

The objective function in this case is the cost function to be minimized, which is the girder volume when plate thicknesses are considered as design variables. In the case that the main cable area is included as a design variable, the sum of the volume of the main cables and the entire deck is considered as the objective function.

Random variables
Among all possible random variables, the most relevant ones in this study are already mentioned in Section 4.3.2, which are extreme wind speeds at the bridge location and flutter derivatives. The structural damping is not considered in the RBDO because of its minor influence over structural reliability as demonstrated in the examples.

The extreme values of wind velocity are the clearest source of uncertainty since the occurrence of a particular value of wind velocity in a certain return period is simply aleatory. The probability functions that represent extreme-values are Gumbel type functions, which should be converted into normal-equivalent function in order to apply the FORM method as explained in Section 2.2.

The flutter derivatives inherently contain uncertainty mainly because their acquisition is experimental. The uncertainty may come from precision of measuring tools, accuracy of tool reading by users, and the preciseness of the curve fitting among all the data points, etc. Each data point of the flutter derivatives is assumed to be normally distributed with linearly increasing standard deviation (0 at $V^*=0$ to 15% of mean values at $V^*=30$).

- Limit state function

The aeroelastic limit state function for the reliability routine is defined in a normalized form as:

$$ G(x) = \frac{V_f(x_i)}{x_w} - 1 \quad i = 1, 2, ..., n $$

(4.16)

where $x = [x_1, x_2, ..., x_n, x_w]$ is a vector of random variables, $V_f$ is flutter velocity, $x_i$ is each point that defines the flutter functions, $x_w$ is extreme wind velocity, $n$ is the total number of data points that define flutter derivatives. Since there are $n$ numbers of data
points as random variables besides the maximum wind velocity, the vector \( \mathbf{x} \) represents a total of \( n+1 \) random variables in the reliability routine.

- Problem formulation

After all the RBDO parameters are defined, we now need to specify design constraints. We considered probabilistic and deterministic constraints in this problem. The probabilistic constraint specifies the required reliability level of the structure on flutter while the deterministic constraints define the design limits and structural performance requirements. There are two cases of RBDO formulations depending on the design variable set as follows.

**Case I**: Steel plate thicknesses as design variables

Min: Girder volume \((d_1, d_2, \ldots, d_m)\) \hspace{1cm} (4.17a)

Subject to \( g_1 : P[G(\mathbf{x}) \leq 0] \leq P_f \)

where \( G(\mathbf{x}) = \frac{V_f(x_i)}{x_w} - 1 \quad i = 1, 2, \ldots n \) \hspace{1cm} (4.17b)

\( g_2 : t' \leq d_j \leq t^* \quad j = 1, 2, \ldots m \) \hspace{1cm} (4.17c)

\( g_3 : \sigma_c = \sigma_{\text{limit}} \) \hspace{1cm} (4.17d)

\( g_4 : \frac{z_d - 1}{z_{\text{max}}} \leq 0 \) \hspace{1cm} (4.17e)

**Case II**: Steel plate thicknesses and main cable area as design variables

Min: Sum of the girder and the main cables volumes \((d_1, d_2, \ldots, d_m, d_c)\) \hspace{1cm} (4.18a)

Subject to \( g_1 : P[G(\mathbf{x}) \leq 0] \leq P_f \)
where $G(x) = \frac{V_i(x_i)}{x_w} - 1 \quad i=1, 2, \ldots n$  

(4.18b)

$g_{2a} : t^i \leq d_j \leq t^v \quad j=1, 2, \ldots m$  

(4.18c)

$g_{2b} : A^i \leq d_c \leq A^v$  

(4.18d)

$g_3 : \sigma_c \leq \sigma_{\text{limit}}$  

(4.18e)

$g_4 : \frac{z_d}{z_{\text{max}}} - 1 \leq 0$  

(4.18f)

where $d_i$ and $d_c$ are the design variables of plate thicknesses and cable area respectively, $n$ is the number of random variables of flutter derivatives, $m$ is the number of design variables of the plate thicknesses, $t^i$ and $t^v$ are side constraints of the plate thickness design variables, $A^i$ and $A^v$ are side constraints of the main cable area design variable, $\sigma_c$ is the maximum tensile stress in the main cable, $\sigma_{\text{limit}}$ is the limiting stress, $z_d$ is the maximum vertical deck displacement due to overload cases, and $z_{\text{max}}$ is the limiting vertical displacement value.

The probabilistic constraint of $g_1$ indicates that the probability of structural failure due to flutter must be smaller than a predetermined value, $P_f$. The constraints from $g_2$ to $g_4$ are deterministic. The constraint, $g_{2a}$ specifies the lower and upper design limits of the girder plate thicknesses for manufacturability and handling limitation, while $g_{2b}$ defines the design limits of cable area for Case II. The constraint of $g_3$ in Case I assigns directly the main cable area so that the cable is at the predetermined value of $\sigma_{\text{limit}}$ while in Case II in which the main cable area is included as a design variable, $g_3$ limits the cable stress under $\sigma_{\text{limit}}$. The constraint $g_4$ requires that the maximum vertical deck displacement under the worst overload case should be under a predetermined value according to the design specification of the particular bridge.
4.4.3 General procedure to solve the RBDO problem

After the general formulation with all the participating parameters is defined, the procedure for the resolution of the RBDO problem is described in this section. Since there are three different RBDO methods employed in this research, only the general work flow is stated here while the detailed procedure for each individual method is explained in the following subsection.

For the entire RBDO process, the main program code is written in Matlab 2010r. Abaqus is employed for the static analysis of the cable model as well as the static and modal analyses of the complete bridge model. The FLAS program calculates the critical flutter velocity using the natural frequencies and the mode shape data obtained from a finite element model in Abaqus as well as flutter derivatives obtained in wind tunnel tests.

There are two major cases of the RBDO procedure: Case I in which only steel plate thicknesses are considered as design variables and Case II in which the main cable area is included as an additional design variable to those in Case I.

Figure 4-18 shows the scheme of the general RBDO procedure. It consists of two main blocks of design optimization and reliability routines. These two optimization problems are either nested or sequential depending on the RBDO method employed.
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Figure 4-18. The general RBDO procedure

The procedures for both Case I and Case II are outlined as follows.

 Optimization routine

1. Calculate girder mechanical properties of bending and torsional stiffness, $I_y(d_j)$, $I_z(d_j)$, $J(d_j)$ for the 3D bar FE model with new design variables (See Appendix). The initial design variables are the values from the original bridge design.

2. For Case I only, compute the corresponding cable area due to the change in deck weight as will be described in Section 4.4.3.2.
3. Calculate the initial cable length and the initial stress of the main cables and hanger cables using the Abaqus cable model as in 4.4.3.1.

4. Run a finite element model of the complete bridge under the static overload case to obtain the maximum vertical deck displacement. For Case II, obtain the maximum main cable axial stress.

5. Run a modal analysis of the finite element model under the self-weight to get natural frequencies and mode shapes of the bridge.

6. Write FLAS input file from the results of step 5 and run FLAS using experimentally obtained flutter derivatives to obtain the critical flutter velocity.

7. The Matlab optimization algorithm checks both probabilistic and deterministic constraints. The parameters in the probabilistic constraint are computed in the reliability analysis routine.

8. The optimization algorithm modifies the design variables.

9. Repeat step 1-8 until the convergence of the algorithm.

- Reliability analysis routine

The Matlab code calculates the MPP using FORM for RIA, and HMV for PMA and SORA. The flutter velocity and its sensitivity with respect to each random variable are computed by launching the FLAS program. The algorithm stops when the convergence criteria are met, which is the convergence of reliability index $\beta$ for FORM and convergence of MPP for HMV.

The following section describes in detail the method developed in this study to determine the initial main cable length and the cable stress under its self-load whenever the design variables change in each iteration, which corresponds to Step 3 in the optimization routine. Then Section 4.4.3.2 explains how to assign the main cable cross sectional area for a particular value of total deck weight so that the main cables are at
the predetermined stress value under static overload case, which corresponds to Step 2 in the optimization routine used only for Case I.

### 4.4.3.1 The initial cable stress and cable position for the analysis

Whenever the design variables are changed, we have to recalculate the total deck weight that the main cables have to support. Additionally, the initial tensile stress and the initial length of the main cables must be determined. A cable under its self-weight has a catenary shape given by an equation found in books as in Hernandez\(^{[H5]}\):

\[
    z = \frac{H}{p_c} \cosh \left( \frac{p_c}{H} x + c_1 \right) + c_2
\]

where \( H \) is the horizontal component of the axial force in the cable, \( p_c \) is the cable weight, \( x \) and \( z \) are the coordinates of the cable, and \( c_1 \) and \( c_2 \) are constants to be determined. Figure 4-19 shows a scheme of the main cable. The dotted line indicates the position of the cable under its self-weight, while the solid line is the final position when loaded with girder. The final position of the mid-span point, \( C_f \) when the cable is loaded with the bridge deck is known; however, the point \( C' \), the mid-span point when the cable is under only its self-weight must be computed.

![Figure 4-19. The scheme of the main cable](image-url)
The determination of the point $C'$ is an iterative process first by guessing an initial counter displacement value, a distance between $C'$ and $C_f$, and solving the above equation that passes through point B and $C'$ knowing that the slope at $C'$ is null. Then the cable is loaded with the deck and the lowest point of the main cable is defined as $C'f$. While $C'f$ does not coincide with $C_f$, the initial assumption of the distance between $C'$ and $C'f$ is updated and the process continues. Assuming that the horizontal component of the axial force in the cable is constant, the catenary equation that passes through the points A and B is solved. Then the section D-E is obtained by symmetry. Once the catenary equations are known, the axial forces in the cable under its self-weight can be calculated as:

$$N = H \cosh \left( \frac{p \cdot x + c_1}{H} \right)$$  \quad (4.20)

The Matlab code carries out this iterative process of obtaining the initial tensile stress and the position of the main cable. With an initial guess of counter displacement, the code solves three implicit catenary equations of Equation (4.19) for the center span with three unknowns of $c_1$, $c_2$ and $H$. Using the results, the code solves the catenary equation of the lateral spans. Then it writes the coordinates of the catenary nodes as well as the initial axial force in the cable under its self-weight in Abaqus format. With known total deck weight, the load applied at each hanger location due to the corresponding girder segment is determined, which is written as a load case. A finite element model of the main cable shown in Figure 4-20 a) is launched considering geometric nonlinearity including internal stresses to check the vertical mid-span displacement under its self-weight, which should be practically null. After the main cables are loaded with the deck, the mid-span displacement is recorded. The value of counter displacement is updated and the process continues until the mid-span position coincides with the final location of the cable. Finally the recorded positions of cable nodes and tensile stress of the main cable as well as the initial tensile stress in the hanger cables are written in Abaqus input format. This process is carried out every time
design variables are modified by optimization algorithms. The updated information is then used to perform static and modal analyses using a finite element model of the entire bridge as shown in Figure 4-20 b).

![Figure 4-20](image1.png)

**Figure 4-20.** a) FEM of the main cable b) FEM of the entire bridge

### 4.4.3.2 Determination of cable area for the change in deck weight

As explained in Section 4.4.2, we have two cases of optimization problems depending on whether the main cable cross sectional area is included as design variable (*Case II*) or not (*Case I*). For *Case I*, the main cable area in the finite element model must be modified in each optimization iteration so that the increased tensile stress in the cables caused by possible increment in deck weight will be at a reasonable value.
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At first glance, the relationship between the main cable area and the cable tensile stress was not obvious. Naturally the cable area should increase with larger deck weight; however, the greater cable area also increases the tensile stress in the cable due to its self-weight.

One way to solve the problem may be an iterative process: starting with an initial cable area, a finite element model of the entire bridge is launched under the worst static overload case providing the maximum cable stress. The cable area is updated in order to adjust to the desired stress in the cable, and the process is repeated until the convergence of the cable area. However, knowing that the RBDO is a computationally intensive process, it is desirable to establish a relationship between the deck weight and cable area without getting into any iterative procedure.

In order to establish this relationship, a series of finite element analyses were carried out under the static overload case by varying cable areas and deck weight. The resulting cable stress was plotted against cable area. Figure 4-21 a) shows an example of such plot for the Messina Bridge example. Since the maximum cable stress for the original design under the overload case was approximately 800 Mpa, which is approximately 50% of the ultimate material strength and considered reasonable, this stress value was used to determine the cable area. Then the cable area was plotted against the deck weight at the cable stress of 800 Mpa as shown in Figure 4-21 b). As can be seen in the graph, the relation between the deck weight and cable area is described simply by a linear equation. The same process was carried out for the Great Belt Bridge to establish such relationship. In each design optimization iteration for Case I, this equation of deck weight and cable area was used to determine the adequate cable area for a given deck weight.
Figure 4-21. a) Main cable stress vs. cable area for different deck weight b) Cable area vs. deck weight at the cable stress at 800 Mpa for Messina example

Case I can be seen as an optimization problem with the constraint of the maximum main cable stress to be always active since we assign the cable area which corresponds to the maximum cable stress value of the original design under the traffic overload case. In Case II, the constraint of the main cable stress $g_3$ explicitly restricts...
the main cable stress since the cable area is considered as an independent design variable.

4.4.3.3 Implementation of the RBDO methods

The general formulations of the three methods employed in this study of RBDO, namely Reliability Index Approach (RIA), Performance Measure Approach (PMA), and Sequential Optimization and Reliability Assessment (SORA) were explained in detail in Chapter 3. In the subsequent sections, each RBDO formulation applied to long-span suspension bridges under a probabilistic flutter constraint is defined.

4.4.3.3.1 RIA

This classical RBDO formulation is a double-loop approach and it is characterized by the use of FORM method for its reliability routine. The formulations of the design optimization and the reliability routines for Case I are defined as follows.

\[
\begin{align*}
\text{Design optimization routine:} & \\
\min & \quad \text{girder volume (d)} \\
\text{s.t.} & \\
& \beta(d, x) \geq \beta^R \\
& l' \leq d_j \leq t' \\
& \sigma_c = \sigma_{\text{max}} \\
& \frac{z_d}{z_{\text{max}}} - 1 \leq 0
\end{align*}
\]

\[
\begin{align*}
\text{Reliability routine:} & \\
\min & \quad \beta = \sqrt{u^T \cdot u} \\
\text{s.t.} & \\
& G(u) = \frac{V_f(u)}{\mu_w + u \sigma_w} = 0 \\
& i = 1, \ldots, n
\end{align*}
\]

where \( u_i \) and \( u_w \) are normalized random variables of flutter derivatives and extreme wind velocity respectively, \( \mu_w \) and \( \sigma_w \) are the mean and standard deviation of extreme wind velocity.

Figure 4-22 shows the process of RIA schematically. For each design iteration, after calculating the girder mechanical properties and carrying out modal and static analyses, reliability analysis is performed using FORM. The natural frequencies and
mode shapes of this particular design are used to compute flutter speed while the random variables are updated each time for the search of the MPP. The converged $\beta$ from the reliability routine is then compared to the target reliability, $\beta^T$, in the probabilistic constraint while the deterministic constraints are evaluated as well. While the design is not optimum, the optimization algorithm modifies the design and the process continues until the convergence of the objective function is achieved with feasible constraints functions.
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Figure 4-22. Flow diagram of RIA
4.4.3.3.2 PMA

The characteristic of this popular two-level method is based on the use of the inverse reliability analysis in the probabilistic routine. It is based on a principle that minimizing a complex function under simple constraints is more efficient than minimizing a simple function under complex constraints (Aoues et al.\cite{A1}). The formulation of PMA for Case I is as follows.

Design optimization routine:

\[
\begin{align*}
\text{min:} & \quad \text{girder volume } (\mathbf{d}) \\
\text{s.t.:} & \quad G_1: \frac{V_f(x_1^{\text{MPP}}, \ldots, x_g^{\text{MPP}})}{x_w^{\text{MPP}}} - 1 \geq 0 \\
& \quad g_2: t_l \leq d_j \leq t_u \\
& \quad g_3: \sigma_c = \sigma_{\text{max}} \\
& \quad g_4: \frac{z_d}{z_{\text{max}}} - 1 \leq 0
\end{align*}
\]

Reliability routine:

\[
\begin{align*}
\text{min:} & \quad G(\mathbf{u}) = \frac{V_f(u_i)}{\mu_w + u_i\sigma_w} - 1 \quad i=1,\ldots, n \\
\text{s.t.:} & \quad \beta = \beta^T
\end{align*}
\]

The flow diagram of PMA is shown in Figure 4-23. The process is similar to RIA except for the algorithm of the reliability routine and the probabilistic constraint. The reliability algorithm performs the spherical search of the MPP at $\beta = \beta^T$ and checks for the convergence of $u$. Then the converged $x_w$ is checked against the flutter velocity $V_f$ in the probabilistic constraint in the design optimization. The process continues until the convergence of the optimization algorithm.
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Figure 4-23. Flow chart of PMA
4.4.3.3.3 **SORA**

The main idea of the decoupled method, SORA, lies in transforming the RBDO into two sequential phases of deterministic optimization and reliability analysis, which makes the reliability phase independent from the design optimization. This approach is proven to be efficient because of its deterministic nature of the optimization and can be solved by any classical optimization algorithm. The formulation of this method applied to our study is written as:

\[
\text{Design optimization routine:} \quad \min: \text{girder volume } (d^k) \quad \min: \ G(u) = \frac{V_f(u)}{u_{w,k} \sigma_{w,k} + \mu_w} - 1 \quad (4.23)
\]

\[
\text{s.t. } G_i : \frac{V_f(x_{MPP}^{MPP}, \ldots, x_n^{MPP})}{x_{w,k-1}^{MPP}} - 1 \geq 0
\]

\[
g_2 : t_f \leq d^k \leq t_u
\]

\[
g_3 : \sigma_c = \sigma_{\text{max}}
\]

\[
g_4 : \frac{z_d}{z_{\text{max}}} - 1 \leq 0
\]

The flowchart of the RBDO process using SORA is shown in Figure 4-24. The design optimization is carried out deterministically using the mean values of random variables as an initial value of the MPP. FLAS is executed to compute flutter speed and the probabilistic and deterministic constraints are evaluated. Then the optimized design as well as the natural frequencies and mode shapes of the optimum design are fed into the reliability routine, in which reliability analysis is performed using HMV method. The algorithm checks for the convergence of the objective function and feasibility of constraint functions. While the design is not converged, the updated random variables are fed back into the design optimization and the process continued until the convergence of the algorithm.
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Figure 4-24. Flow chart of the SORA method
In the following chapters, the RBDO methods will be applied to two of the most relevant long-span suspension bridges in the world: The Great Belt East Bridge and the Messina strait Bridge. The Great Belt East Bridge is the third longest suspension bridge in the world with a slender aerodynamic box girder while the proposed Messina Bridge will be the suspension bridge with the longest center span with a triple-box girder. By employing the RBDO methods, we can achieve competitive designs taking into system uncertainties.
4.5 References


[H5] Hernandez, S., “Análisis lineal y no lineal de estructuras de barras”, School of civil engineering, University of La Coruña, 2011

Chapter 4  
RBDO formulations of long-span suspension bridges under flutter constraint


Chapter 4  
RBDO formulations of long-span suspension bridges under flutter constraint


CHAPTER 5

THE GREAT BELT BRIDGE EXAMPLE

5.1 Introduction

The RBDO formulations developed for long-span suspension bridges under flutter constraint in Chapter 4 is now applied to a bridge project of the Great Belt East Bridge. This bridge was built to connect Denmark to Mainland Europe for improving transportation and trade. It is not only one of the longest-span bridges in the world, but also a great engineering as well as cultural icon of the country.

Denmark consists of the Jutland peninsula and more than 400 islands, among which two largest are Zealand where the capital, Copenhagen, is located and Funen (Figure 5-1). These two islands are connected by the Great Belt link across the Great Belt strait. Two bridges forming a part of the Great Belt crossing is the 6.6 km-long West Bridge between Funen and Sprogø and the 6.8 km-long East Bridge between Sprogø and Zealand. This chapter deals with this East suspension Bridge.
Chapter 5

The Great Belt Bridge Example

Figure 5-1. Geographical location of the Gran Belt Bridge

The plans to build a fixed link across the Great Belt strait existed ever since 1930s, although the project was not initiated until the Danish parliament approved its construction in 1987. The project dealt with a construction of a suspension bridge to carry motorways and a bored tunnel for railways. The design of the bridge was initiated by the design consultants COWI and Ramboll in 1991[11]. Since the East Bridge crosses the international navigation route with a large volume of marine traffic, about 20,000 ships a year from the Baltic Sea to the North Sea (Storebælt publication[12]), the bridge had to be designed in a way that large ships could pass through the main span of the bridge safely even under severe weather conditions. A series of studies was undertaken on theoretical ship collision to the bridge and safe navigation, which determined the main span length (Scott[12]).
Open to traffic in 1999, the Great Belt East Bridge shown in Figure 5-2 is a suspension bridge with 1624 meter span, making it the world's third-longest suspension bridge. The steel bridge deck is a slender aerodynamic box girder of 31 meter wide and 4 meter deep (Figure 5-3). The box girder was chosen over plate or truss girders for its lower construction and maintenance cost. The external shape of the girder suits for its structural and aerodynamic performances (Larsen\textsuperscript{[L1]}). One of the characteristics of this bridge is that the girder is continuous over the full cable supported length of 2694 meter without any expansion joint at the pylons. The omission of a cross beam at the pylon clearly exhibits the transfer of vertical loads to the main cables. Two reinforced concrete pylons reach a height of 254 meters because of the main cable sag ratio of 1/9. The resulting bridge structure is highly flexible.
5.2 Structural model

A 3D-beam finite element model in Abaqus 6.12 (Figure 5-5) was created to carry out modal analyses as well as static analyses of the bridge. The model consists of 1257 elements and 747 nodes with 4,452 degrees of freedom. The bridge girder is modeled as a beam element, which is supported by hanger cables every 24 meters to transfer their vertical loads to the main cables. The deck is continuous throughout three spans without any vertical support at towers although it counts with lateral supports at the pylons. Each of the two main cables has a fixed connection to the bridge girder at the mid span by locking devices, which consist of cable clamps held by trussed triangles.
attached to the girder. These devices help to minimize deflections under asymmetric traffic load. The boundary conditions are imposed at the anchorages and the tower foundations. There are total of 225 nodes along the bridge girder, approximately 12 meters apart.

![Figure 5-5. Structural model of the Great Belt Bridge](image)

The geometrical and mechanical properties of the model used in the calculations are summarized in Table 5-1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center span length (m)</td>
<td>1624</td>
</tr>
<tr>
<td>Lateral span length (m)</td>
<td>535</td>
</tr>
<tr>
<td>Main cable sag (m)</td>
<td>180</td>
</tr>
<tr>
<td>Distance between main cables (m)</td>
<td>27</td>
</tr>
<tr>
<td>Total deck width (m)</td>
<td>31</td>
</tr>
<tr>
<td>Deck height (m)</td>
<td>4.4</td>
</tr>
<tr>
<td>Girder moment of inertia, Iy (m^4)</td>
<td>3.59</td>
</tr>
<tr>
<td>Girder moment of inertia, Iz (m^4)</td>
<td>75.22</td>
</tr>
<tr>
<td>Girder polar moment of inertia, J (m^4)</td>
<td>7.96</td>
</tr>
<tr>
<td>Mass per unit length of deck (t/m)</td>
<td>14.75</td>
</tr>
<tr>
<td>Main cable cross sectional area (m^2)</td>
<td>0.453</td>
</tr>
</tbody>
</table>

**Table 5-1. Geometrical and mechanical properties of the structural model**
5.3 **Static analysis of the bridge under traffic load**

Static analyses using the Abaqus finite element model in Figure 5-5 were performed to study the structural performance of the bridge under the traffic overload case defined in BS 5400, the British standard code of practice for the design and construction of steel, concrete and composite bridges\textsuperscript{[W1]}. There are six motorways on the bridge deck measuring 3.8 meters wide for each lane and 22.6 meters in total. According to the code, the traffic load is given by \( 9 \text{kN/m} / 3.8 \text{ m} = 2.4 \text{kN/m}^2 \). Two lanes are assumed to be loaded fully while the other four lanes are loaded with \( 1/3 \) of the load.

Because the bridge structure is highly flexible, the static calculations were carried out in two steps as described in Chapter 4. In the first step, the initial cable length as well as the initial stress of the two main cables and hanger cables was determined using an Abaqus cable model. In the subsequent step, based on that initial condition, a static analysis was performed using the Abaqus model of the complete bridge. The maximum vertical deck displacement of the original Great Belt Bridge under the traffic overload case was 2.505 meters, which is \( 1/648 \) of the span length. The limiting vertical displacement value for the subsequent optimization problems was chosen to be \( 1/500 \) of the span length, which is 3.248 meters. The maximum main cable stress under the same load case was 565 MPa. Since the ultimate tensile strength of the cable is 1570 MPa\textsuperscript{[S1]}, this stress value is considered reasonable.

5.4 **Flutter analysis**

Prior to computing flutter speed of the Great Belt Bridge, the Abaqus finite element model was used to calculate the natural frequencies and the mode shapes of the bridge. Because of the large flexibility of the structure, the modal analysis was performed in two steps. In the first step, the initial stresses of the main cables and hanger cables are calculated under the self-load of the bridge, and in the subsequent
step, modal analysis was performed with overall stiffness of the structure. Table 5-2 lists the natural frequencies and the vibration modes of the finite element model of the Great Belt Bridge obtained by the author, which are compared to the data by Larsen[1].

<table>
<thead>
<tr>
<th>Mode</th>
<th>Type</th>
<th>Kusano</th>
<th>Larsen</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LS</td>
<td>0.051</td>
<td>0.052</td>
</tr>
<tr>
<td>2</td>
<td>VS</td>
<td>0.098</td>
<td>0.100</td>
</tr>
<tr>
<td>3</td>
<td>VA</td>
<td>0.112</td>
<td>0.115</td>
</tr>
<tr>
<td>4</td>
<td>LA</td>
<td>0.119</td>
<td>0.123</td>
</tr>
<tr>
<td>5</td>
<td>VS</td>
<td>0.131</td>
<td>0.135</td>
</tr>
<tr>
<td>8</td>
<td>VA</td>
<td>0.177</td>
<td>0.184</td>
</tr>
<tr>
<td>9</td>
<td>LA</td>
<td>0.184</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>LS</td>
<td>0.187</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>VA</td>
<td>0.189</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>LS</td>
<td>0.208</td>
<td>0.187</td>
</tr>
<tr>
<td>15</td>
<td>VS</td>
<td>0.216</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>VS</td>
<td>0.250</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>LA</td>
<td>0.277</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>VA</td>
<td>0.283</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>L/TS</td>
<td>0.283</td>
<td>0.278</td>
</tr>
<tr>
<td>22</td>
<td>VS</td>
<td>0.285</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>L/TS</td>
<td>0.287</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>LA</td>
<td>0.307</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>VS</td>
<td>0.332</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>VA</td>
<td>0.334</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>LS</td>
<td>0.340</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>VS</td>
<td>0.356</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>V/TA</td>
<td>0.385</td>
<td>0.383</td>
</tr>
<tr>
<td>35</td>
<td>VS</td>
<td>0.398</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5-2. Frequencies and vibration modes of the Great Belt Bridge**

Note: L (lateral), V (vertical), T (torsional), S (symmetric), A (asymmetric)

Figure 5-6 shows some of the most relevant mode shapes of the original Great Belt Bridge. It should be noted that for flutter calculations of this type of bridges, it is important to include the first torsional mode. In our case, the first torsional symmetric
mode occurs at the mode 21 and the second torsional mode at the mode 24 for the original bridge design. However, the vibration modes change as the girder design changes during the optimization process. For example, for a girder with a uniform plate thickness of 5 mm, the first torsional mode is the mode 19 while for a girder with 10 mm-thick plate, the torsional mode changes to mode 20. In both cases, the first vertical and lateral symmetric modes such as mode 1, 2, 5 and 10 remain unchanged.

Note: lateral, vertical, and torsional displacement
In order to perform flutter analysis, experimentally obtained 18 flutter derivatives are required as explained in Chapter 4. In this study, flutter derivatives obtained by Cobo\textsuperscript{[C1]} as shown in Figure 5-7 were employed to calculate the critical flutter velocity. Flutter derivatives with null values are not shown in the figure.
A series of flutter analyses were performed on the original bridge design with different combination of vibration modes of the deck. Since the first torsional mode changes depending on the girder design, all the symmetric torsional modes for different design, mode 19 to 22, 24 and 30, are included in the study as well as some of the first
vertical and lateral symmetric modes as mode 1, 2, 4, 5, 11, 12, 13. Consequently, the total number of modes taken into account in this study is 13. Figure 5-8 shows the FLAS output, in which we can observe that the first torsional aeroelastic mode 20 is causing flutter. The critical flutter velocity of 80.32 m/s was obtained at the reduced frequency of 0.489.

![FLAS output of the original Great Belt Bridge](image)

**Figure 5-8. FLAS output of the original Great Belt Bridge**

### 5.5 Deterministic optimization

In order to check the feasibility of structural optimization of the Great Belt East Bridge considering steel plate thicknesses of the girder as well as the main cable area as design variables under aeroelastic constraint, several deterministic optimizations were performed first.
We considered two cases of deterministic optimization problems for the Great Belt Bridge. In Case I, the bridge girder volume is to be minimized by varying the plate thicknesses of the girder from $d_1$ to $d_4$ as shown in Figure 5-9. In Case II, the sum of the volumes of the box girder and the two main cables is to be minimized by considering the girder plate thicknesses as well as the main cable area as design variables ($d_1$ to $d_5$ in Figure 5-9).

For both cases, the bridge design must satisfy a series of constraint functions such as flutter, the maximum stress in the main cables and the maximum vertical deck displacement under the traffic overload case as well as side limits of the design variables. Section 5.3 discussed about the maximum stress in the main cables of the original design under the traffic overload case, which was 565 MPa. In the deterministic optimization, this value was used to limit the main cable stress. The formulations of these two cases are described as follows.

**Figure 5-9: Design variables for the deterministic optimization**

### 5.5.1 Case I

In this case, we considered the four steel plate thicknesses of the box girder as design variables. The objective function to be minimized is the volume of the girder. The formulation of Case I is shown next.

\[
\text{Min: girder volume (d)} \quad (5.1a)
\]

\[
g_{i, i}': 5.0 \text{ mm} \leq d_i \leq 20.0 \text{ mm} \quad i = 1, 2, \ldots, 4 \quad (5.1b)
\]
where \( d_i \) is a design variable, \( V_f \) is critical flutter velocity of the bridge, \( V_{limit} \) is an arbitrary limiting flutter velocity, \( z_d \) is the maximum vertical displacement due to the traffic overload case, and \( z_{max} \) is the limiting vertical displacement, \( \sigma_c \) is the maximum main cable stress.

\[
g_1^I \text{ are the side constraints of the design variables, } g_2^I \text{ is the deterministic aeroelastic constraint, } g_3^I \text{ is the stress constraint of the main cables, which is forced to be active. The main cable area is modified based on this constraint, and the relationship between the deck weight and the main cable area is described in Equation (5.2). } g_4^I \text{ restricts the maximum vertical deck displacement under the traffic overload case.}
\]

As explained in Section 4.4.3.2, every time the girder design changes, which affects the total girder weight, the main cable area should be adjusted so that the main cables are at a reasonable stress level. A study was conducted in order to establish a relationship between the girder weight and the appropriate main cable area, which is explained in detail next.

- **Updating the main cable cross sectional area during optimization**

In order to establish the relationship between the deck weight and the appropriate main cable area, the Abaqus FE model was run by varying both the deck
weight and the main cable area. The maximum main cable tensile stress under the traffic overload case was recorded each time, and the results are shown in Figure 5-10.

![Figure 5-10. Maximum main cable stress vs. main cable area for different deck weight](image)

For the original bridge design, the main cable area and the deck weight are 0.453 m² and 14.75 t/m respectively while the maximum tensile stress of the main cable under the static overload case is 565 MPa. For design purposes, whenever the deck weight is updated, we want to assign a main cable area in the finite element model so that the maximum cable stress is at 565 MPa under the traffic overload case. The cable area was then plotted against the total deck weight at this tensile stress as shown in Figure 5-11. As can be seen, the main cable sectional area to be assigned for a particular deck weight can be expressed simply by a linear equation:

\[
A_c = 0.0241 \cdot DW + 0.1046
\]

(5.2)

where \(A_c\) is a cable area in m² and \(DW\) is deck weight in t/m. Equation (5.2) was employed for all deterministic and probabilistic optimizations in this chapter except in Case II in which the main cable area is explicitly considered as a design variable.
5.5.2 *Case II*

In this case, we considered a total of five design variables, the four design variables from *Case I* as well as the main cable area. The objective function is the sum of the box girder and the two main cables volumes. Side constraints of the cable area is added to the formulation.

The main purpose of this case is instead of assuming that the maximum cable stress constraint is active at the optimum as in *Case I*, the constraint of $g_i^\mu$ restricts the maximum cable stress by an inequality constraint for having the main cable area as a design variable. The formulation of *Case II* is shown below.

Min: sum of the main cables and girder volumes ($d$) \hspace{1cm} (5.3a)

$$g_{1,i}^\mu : 5.0 \text{ mm} \leq d_i \leq 20.0 \text{ mm} \hspace{1cm} i=1, 2, ..., 4 \hspace{1cm} (5.3b_1)$$

$$g_{15}^\mu : 0.1 \text{ m}^2 \leq d_5 \leq 2.0 \text{ m}^2 \hspace{1cm} (5.3b_2)$$
The Matlab “fmincon” optimizer was used with active-set algorithm to solve the optimization problems. The termination tolerance values of the objective function and the constraint functions were set to 1E-5.

Knowing that the flutter velocity of the original bridge design is 80.32 m/s, arbitrary limiting flutter velocities of 75 and 85 m/s were chosen to see how the bridge design would change with different limiting flutter speeds.

### 5.5.3 Deterministic optimization procedure

The main procedures of the deterministic optimization for both cases are described as follows.

1. Calculate the mechanical properties of the girder cross section such as area, moments of inertia, \( I_y(d_i), I_z(d_i), J(d_i), A(d_i) \), with the initial design variables. See Appendix for details.
2. For Case I, determine the main cable cross sectional area using Equation (5.2).
3. Write the mechanical properties of the deck and the main cable area in Abaqus input files.
4. Calculate the initial main cable position and the initial tensile stresses in the main cables and the hanger cables and update the data in Abaqus input files.
5. Launch two Abaqus FE models: for static analysis under the traffic overload case to get the maximum vertical deck displacement and the maximum stress in
the main cable, and modal analysis to obtain natural frequencies and vibration mode shapes.

6. Launch FLAS to obtain critical flutter speed.
7. The constraint functions are evaluated and the optimization algorithm modifies the design.
8. Repeat the procedure 1-7 with a new design until the algorithm converges.

5.5.4 Deterministic optimization results

The results of the deterministic optimization are shown next. We want to see how the design variables actually change by varying the deterministic limiting flutter velocity constraint. This step is important before proceeding to more complex probabilistic optimization problems of the bridge structure.

- **Case I: four design variables of the box girder plate thicknesses**

  In this case, four design variables of \(d_1\) to \(d_4\) in Figure 5-9 were studied while the girder volume was defined as the objective function. Two limiting flutter speeds of 75 m/s and 85 m/s were chosen arbitrarily. The initial girder design was \(d=[12.0, 10.0, 10.0, 10.0]\) (in mm) with the corresponding objective function value of 2834.10 m³.

  Figure 5-12 and Figure 5-13 show the evolutions of design variables and objective function for Case I while Table 5-3 and Table 5-4 summarize the optimum designs and the number of FLAS and Abaqus executions. For the case with the limiting flutter velocity of 75 m/s, in which the required critical flutter speed value was reduced from the original design, the girder became lighter as expected. The optimum design of the top plate thickness, \(d_1\) has decreased the most, followed by the upper side plate thicknesses of \(d_2\). The objective function has decreased by about 12% from the original design. On the other hand, when we increased the limiting critical flutter speed to 85 m/s, the girder became heavier by approximately 13%. For both cases, the aeroelastic
constraint was active at the optimum. The computational time for both cases is considered reasonable as can be seen in the results.

![Figure 5-12](image1.png)

Figure 5-12. Evolutions of the design variables (left) and the objective function (right) for Case I: limiting $V_f=75\text{m/s}$

![Figure 5-13](image2.png)

Figure 5-13. Evolutions of the design variables (left) and the objective function (right) for Case I: limiting $V_f=85\text{m/s}$

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>obj. func.</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.0</td>
<td>9.378</td>
<td>8.178</td>
<td>9.569</td>
<td>8.706</td>
<td>2498.04</td>
<td>-11.86</td>
<td></td>
</tr>
</tbody>
</table>

Table 5-3. Optimum designs for Case I
Table 5-4. Number of FLAS and Abaqus executions for Case I

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>No. of FLAS executions</th>
<th>No. of Abaqus executions</th>
<th>Computational time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.0</td>
<td>51</td>
<td>51</td>
<td>1.2</td>
</tr>
<tr>
<td>85.0</td>
<td>77</td>
<td>77</td>
<td>1.4</td>
</tr>
</tbody>
</table>

- **Case II: five design variables of the box girder plate thicknesses and the main cable area**

  Case II deals with an additional design variable of the main cable area to those in Case I. The objective function is defined as the sum of the volumes of the box girder and the two main cables. Two deterministic flutter constraints of 75 and 85 m/s were considered as in Case I. The original design is $\mathbf{d}=[12.0, 10.0, 10.0, 10.0, 0.453]$ (d1 to d4 in mm, d5 in m²) and its corresponding objective function of the sum of the girder and two cable volumes is 5375.54 m³.

  Figure 5-14 and Figure 5-15 show the evolutions of the optimization parameters while Table 5-5 and Table 5-6 present the optimum designs and the number of FLAS and Abaqus executions for Case II. For the case with the limiting flutter velocity of $V_f=75$ m/s, most of the design variables have been reduced in values, especially the top plate thickness of $d_1$ while the objective function has been reduced by 8.1%. Both aeroelastic and the maximum cable stress constraints were active at the optimum. When the limiting flutter velocity was increased to $V_f=85$ m/s, the design variables of $d_2$ and $d_3$ have increased by about 30% while $d_1$ did not practically change. The objective function has increased by 3.4%, and the maximum cable stress constraint was active at the optimum in this case. The number of FLAS and Abaqus executions have increased greatly compared to the four design variables case. For example, in the case of the limiting $V_f=75$ m/s, the number of Abaqus and FLAS executions has increased by more than 4 times, which is also reflected in the five to six-fold computational time.
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Figure 5-14. Evolutions of the design variables (left) and the objective function (right) for Case II: limiting $V_f=75\text{m/s}$

Figure 5-15. Evolutions of the design variables (left) and the objective function (right) for Case II: limiting $V_f=85\text{m/s}$

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>obj. func.</th>
<th>obj. func.</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.0</td>
<td>8.737</td>
<td>9.300</td>
<td>11.046</td>
<td>9.026</td>
<td>0.428</td>
<td>4939.72</td>
<td>-8.11</td>
</tr>
<tr>
<td>85.0</td>
<td>11.974</td>
<td>13.177</td>
<td>6.356</td>
<td>10.727</td>
<td>0.460</td>
<td>5556.76</td>
<td>3.37</td>
</tr>
</tbody>
</table>

Table 5-5. Optimum designs for Case II
In these deterministic optimization examples, we have verified that the design optimization was in fact feasible by setting a deterministic limiting flutter speed as a design constraint. This step was necessary before getting into more complex RBDO problems under probabilistic flutter constraint. Prior to performing RBDO, reliability analysis of the original bridge design under flutter limit state was performed to obtain the reference reliability index, which is presented in the following section.

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>No. of FLAS executions</th>
<th>No. of Abaqus executions</th>
<th>Computational time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.0</td>
<td>224</td>
<td>224</td>
<td>7.0</td>
</tr>
<tr>
<td>85.0</td>
<td>290</td>
<td>290</td>
<td>8.1</td>
</tr>
</tbody>
</table>

Table 5-6. Number of FLAS and Abaqus executions for Case II
5.6 Reliability analysis

5.6.1 Introduction

The procedure of performing reliability analyses of long-span bridges under flutter constraint was discussed in Section 4.3. In this section, these formulations were applied to the Great Belt East Bridge. The FORM method was employed to calculate the probability of failure of the bridge due to flutter. This method was programmed in MATLAB code, which computes the reliability index of the structure under flutter in order to obtain the corresponding probability of failure for different combinations of random variables. Flutter speed is obtained by first establishing a dynamic equilibrium equation of the bridge under aeroelastic force, which requires experimentally-obtained flutter derivatives, mode shapes and natural frequencies of the bridge under study. Then the dynamic equilibrium equation is solved by an iterative process using FLAS code to compute flutter speed. The Abaqus finite element model in Figure 5-5 calculates the mode shapes and natural frequencies of the bridge.

The limit state function that specifies the probabilistic failure mode is defined as a difference between the flutter velocity and the extreme wind velocity at the bridge site. The random variables considered in this study are the points that define flutter derivatives (Figure 5-7), the extreme wind velocity at the bridge location as well as the structural damping of the bridge. The uncertainties in flutter derivatives and structural damping affect the computation of flutter velocity while the statistical data of extreme wind influence the load that the bridge has to bear. Reliability analyses were performed taking into account different combinations of random variables to see the effects of some variables on overall structural reliability. The random variables taken into account in the study as well as the limit state function are explained in the subsequent section.
5.6.2 Random variables

- **Wind velocity**

  The assessment of wind data is one of the most important tasks for the design of a long-span bridge since it affects directly the overall lateral stability of the structure. In the case of the Great Belt East Bridge, a meteorological mast was erected in 1977, which was 70 meter-high equipped with sonic anemometers, wind vanes, pressure sensors and thermometers for collecting wind data. Continuous recordings of mean and gust wind speeds, wind direction and static pressure were carried out for more than 10 years\[S1\]. In this research, the omnidirectional extreme wind speed of 43.0 m/s is used according to the Danish Wind code of DS 410\[D1\], which corresponds to the mean wind at 70 meter height in 100-year return period. This value is more conservative than the estimated values from the wind study, which is approximately 10% smaller. The standard deviation of the wind is taken from the wind study documented in Storæbelt publication, whose value is 3.89 m/s. In general, statistical extreme wind values are expressed as an asymptotic extreme-value distribution like Gumbel function. Then the values of statistical moments need to be transformed into their corresponding normal-equivalent prior to the application of FORM.

- **Structural damping**

  The structural damping, $\zeta$, is defined as a log-normal distributed random variable: the mean value is 0.00318 and the standard deviation is 20% of the mean value.

- **Flutter derivatives**

  As explained in Section 4.2, a hybrid method was used for the computation of flutter speed in this research, which involves in an experimental phase to obtain flutter derivatives and its subsequent computational phase to calculate flutter velocity. We considered that this experimental phase is subject to uncertainties. Each flutter
derivative function is represented by several points as shown in Figure 5-16, and intermediate values are interpolated by cubic spline functions during the flutter calculation. In order to account for the larger data dispersion with increasing wind speed in the wind tunnel, we studied two cases of linearly increasing dispersions: 0 at $V^* = 0$ and $\sigma_{\text{max}} = 0.15 \mu_x$ for the first case and $\sigma_{\text{max}} = 0.3 \mu_x$ for the second case at $V^* = 30$.

Among 18 flutter derivatives, only those listed in Figure 5-7, $A_1$ through $A_4$ and $H_1$ through $H_4$, were considered as random variables in reliability analyses because the rest of the flutter derivatives have nearly zero values. Since each flutter function is defined by 4 to 6 data points, the total number of random variables from flutter derivatives reaches as many as 42 for the reliability calculations.

![Figure 5-16. Representation of a flutter derivative $A_3^*$: each point on the graph is considered as a normally-distributed random variable](image)

5.6.3 Limit state function

The limit state function for the reliability analysis is defined as:

$$G(x) = V_f(x_i, x_j) - x_w \quad i = 1, 2, \ldots n \quad (5.4)$$

or in a normalized form as:
where \( \mathbf{x} \) is a vector of random variables, \( V_f \) is flutter velocity, \( x_i \) is each point that defines flutter derivatives, \( x_s \) is structural damping, \( x_w \) is extreme wind speed, \( n \) is a total number of points that define flutter derivatives. Since we take into account the extreme wind velocity and structural damping besides flutter derivatives as random variables, the vector, \( \mathbf{x} \) represents a total of \( n+2 \) variables in the reliability analyses.

As stated in the previous section, the statistical function of extreme wind speeds is a Gumbel-type function; however, a normal-equivalent distribution function must be obtained in order to use the FORM method. The following section describes the procedure to obtain the normal-equivalent of extreme wind velocity probability density function of the Great Belt Bridge.

### 5.6.4 Normal-equivalent density function of extreme wind speed

Using the relations, \( \mu_w = \mu + \gamma \beta \) and \( \sigma_w = \beta \pi / \sqrt{6} \) where \( \gamma = 0.57721 \) is Euler-Mascheroni constant, \( \mu_w \) and \( \sigma_w \) are the mean value and dispersion of extreme wind speed, the coefficients of the Gumbel equation can be obtained. In our case, \( \mu = 41.2493 \) and \( \beta = 3.0330 \). The Gumbel probability density and cumulative density function can be expressed as:

\[
\begin{align*}
  f_g(x) &= \frac{1}{\beta} \exp\left(-\frac{x - \mu}{\beta}\right) \cdot \exp\left[-\exp\left(-\frac{x - \mu}{\beta}\right)\right] \\
  F_g(x) &= \exp\left[-\exp\left(-\frac{x - \mu}{\beta}\right)\right]
\end{align*}
\]

The normal-equivalent standard deviation and mean value, \( \sigma_n \) and \( \mu_n \) are obtained using Hasofer Lind\textsuperscript{[H1]} – Rackwitz Fiessler\textsuperscript{[R1]} (HL-RF) method as follows.
\( \sigma_N = \frac{\Phi^{-1}[F_G(x^*)]}{f_G(x^*)} = 3.720 \text{ m/s} \)

\( \mu_N = x^* - \sigma_N \Phi \left( F_G(x^*) \right) = 42.340 \text{ m/s} \)

The probability density function and cumulative density function of the Gumbel and its normal-equivalent are shown in Figure 5-17 and Figure 5-18.

Figure 5-17. Probability density function of Gumbel and its normal equivalent for the extreme wind speeds
Theoretically this concept of transformation from Gumbel function to its normal-equivalent should be applied in each iteration of reliability analysis when wind velocity moves from its mean value to the MPP (Most probable Point of failure). However, knowing that the MPP occurs at far more than 3 standard deviations away from the mean value, carrying out the transformation at such point makes little sense since the probability density function at such point is extremely small. Therefore, the mean value and the standard deviation of the normal equivalent at the mean design point of the Gumbel function were used throughout the studies.

5.6.5 Different cases of the reliability analysis

After all the reliability parameters as well as the limit state function are defined, we are now ready to carry out reliability analyses of the Great Belt Bridge under the flutter limit state function. The flow chart of the detailed procedure of reliability analysis can be found in Section 4.3.5. Various combinations of random variables were considered in this study to see their influences on overall structural safety, which are grouped into four cases shown below.
Case A: extreme wind velocity, $x_w$ as a single random variable

Case B: $x_w$, structural damping, $x_\zeta$, and the points from one of the 8 flutter derivatives

Case C: $x_w$, $x_\zeta$ and the points from 4 flutter derivatives from the same type, $A^*$ or $H^*$

Case D: $x_w$, $x_\zeta$ and all the points from 8 flutter derivatives

These cases of reliability analyses were performed utilizing a Linux-based cluster with 5094.4 GFLOP’s peak power and total memory of 1792 GB. The termination criterion for all the reliability analyses is set so that the difference of any two consecutive beta values should be smaller than 1E-4.

5.6.6 Reliability results

- Case A: wind velocity as a single random variable

In this case, only the extreme wind velocity, $x_w$ was considered as a random variable. Since none of the flutter derivatives or structural damping varies, $V_f$ takes a deterministic value. Then the reliability index can be calculated directly from Equation (2.25) and (2.28) presented in Chapter 2 since the gradient of the limit state function is constant. Using the limit state function of non-normalized form of Equation (5.4), its gradient is:

$$\frac{\partial G(x_w)}{\partial x_w} = -1 \quad \text{or} \quad \frac{\partial G}{\partial u_w} = -\sigma_w$$

$$\lambda = \frac{1}{\sqrt{\sum_{i=1}^{\mu} \left( \frac{\partial G}{\partial x} u^*_i \right)^2}} = \frac{1}{\sigma_w}$$
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therefore, \[ \beta = \frac{G(u^*) - \sum_{i=1}^{n} \frac{\partial G(u^*)}{\partial u_i} u_i^*}{\sqrt{\sum_{i=1}^{n} \left( \frac{\partial G(u^*)}{\partial u_i} \right)^2}} = \frac{V_f - \mu_{w*}}{\sigma_w} = 10.3817 \]

The corresponding probability of failure is \( P_f = 1.502 \times 10^{-25} \).

- **Case B: \( x_w, x_\zeta \) and the points from one of the eight flutter derivatives**

In this case, one of the 8 flutter derivatives, the structural damping and the extreme wind velocity were considered as random variables. Although it is not very realistic to perform reliability analysis considering only one flutter derivative as random variables since all flutter derivatives are subject to uncertainties, this case allows us to identify which flutter derivatives are more influential than others on overall structural reliability. Reliability analyses were also performed without structural damping to see the impact of including this random variable on reliability. The number of random variables for each subcase is between 5 and 7. The resulting reliability indices with different random variables sets are presented in Table 5-7.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables (N)</th>
<th>( \sigma_{\text{max}=0.15\mu_x} ) linear variation</th>
<th>( \sigma_{\text{max}=0.3\mu_x} ) linear variation</th>
<th>( \beta )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_w, x_\zeta, A_{1,1}, \ldots, A_{1,3} )</td>
<td>5</td>
<td></td>
<td></td>
<td>9.864</td>
<td>9.210</td>
</tr>
<tr>
<td>( x_w, x_\zeta, A_{2,1}, \ldots, A_{2,4} )</td>
<td>6</td>
<td></td>
<td></td>
<td>10.013</td>
<td>7.309</td>
</tr>
<tr>
<td>( x_w, x_\zeta, A_{3,1}, \ldots, A_{3,5} )</td>
<td>7</td>
<td></td>
<td></td>
<td>10.037</td>
<td>9.193</td>
</tr>
<tr>
<td>( x_w, x_\zeta, A_{4,1}, \ldots, A_{4,5} )</td>
<td>7</td>
<td></td>
<td></td>
<td>10.355</td>
<td>10.324</td>
</tr>
<tr>
<td>( x_w, x_\zeta, H_{1,1}, \ldots, H_{1,4} )</td>
<td>6</td>
<td></td>
<td></td>
<td>10.366</td>
<td>10.357</td>
</tr>
<tr>
<td>( x_w, x_\zeta, H_{2,1}, \ldots, H_{2,5} )</td>
<td>7</td>
<td></td>
<td></td>
<td>10.363</td>
<td>10.363</td>
</tr>
<tr>
<td>( x_w, x_\zeta, H_{3,1}, \ldots, H_{3,4} )</td>
<td>6</td>
<td></td>
<td></td>
<td>9.890</td>
<td>8.975</td>
</tr>
<tr>
<td>( x_w, x_\zeta, H_{4,1}, \ldots, H_{4,5} )</td>
<td>7</td>
<td></td>
<td></td>
<td>10.352</td>
<td>10.309</td>
</tr>
</tbody>
</table>
Table 5-7. Reliability results of Case B, one flutter derivative, $x_w$ and $x_\omega$ as random variables

For the case with 15% dispersion on the mean value at $V^*=30$, two of the most influential flutter derivatives were $A_{1^*}$ and $H_{3^*}$ while for the 30% dispersion case, $A_{2^*}$ is far more relevant than other flutter derivatives. The difference in $\beta$ values of $A_{2^*}$ between 15% and 30% dispersion cases were significant: for the cases in which structural damping is included as random variable, $\beta$ for the case with 15% dispersion is $10.013$ ($P_f=6.68e-24$) while $\beta$ for the case with 30% dispersion is $7.309$ ($P_f=1.35e-13$).

Table 5-8 and Figure 5-19 present the initial values of the random variable of $A_{2^*}$ and the MPP values with 15% and 30% dispersion at $V^*=30$ while Figure 5-20 shows the FLAS output of both cases. As can be seen, for the case with 30% dispersion, the second and the third random variables at the MPP were varied by 48.2% and -20.5% respectively. On the other hand, the maximum variation of random variables for the case with 15% dispersion is the third random variable with 9.6% difference. Since the flutter speed is sensitive to the second and the third points of the flutter derivative $A_{2^*}$, the flutter velocity at MPP with 30% standard deviation has decreased to 61.76 m/s. On the other hand, for the case with 15% dispersion case, the change in flutter velocity from the original design is about 1 m/s.
In general, the flutter derivatives of $A_1^*$ through $A_3^*$ are important because they are related to aeroelastic moment of the bridge deck, and so is $H_3^*$, which involves in the rotational movement of the deck as can be seen in Equation (4.2).

Regarding computational time, the reliability analysis of $A_2^*$, for example, performed 24 iterations before convergence using the reduction factor of $c=3$. FLAS was executed 96 times and the computational time was approximately 0.4 hours. The computational time of other flutter derivatives were very similar.

<table>
<thead>
<tr>
<th>$V^*$</th>
<th>initial values</th>
<th>MPP values</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>15% $\sigma$</td>
<td>30% $\sigma$</td>
</tr>
<tr>
<td>6.0</td>
<td>-0.25</td>
<td>-0.250</td>
<td>-0.240</td>
</tr>
<tr>
<td>10.0</td>
<td>-0.55</td>
<td>-0.544</td>
<td>-0.285</td>
</tr>
<tr>
<td>12.0</td>
<td>-0.7</td>
<td>-0.632</td>
<td>-0.844</td>
</tr>
<tr>
<td>20.0</td>
<td>-1.52</td>
<td>-1.528</td>
<td>-1.488</td>
</tr>
</tbody>
</table>

Table 5-8. The MPP values of $A_2^*$ for 15% and 30% dispersions at $V^*$=30

![Graph showing initial and MPP values for 15% and 30% $\sigma$ of random variables $A_2^*$](image)

Figure 5-19. Initial and MPP values for 15% and 30% $\sigma$ of random variables $A_2^*$
Figure 5-20. FLAS outputs for $A_2^*$ flutter derivative at MPP with a) 15% dispersion and b) 30% dispersion over the mean value at $V^*=30$
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- **Case C:** \( x_w, x_\zeta \) and the data points from four flutter derivatives of the same type, \( A^* \) and \( H^* \)

  In *Case C*, the flutter derivatives of the same type, \( A^* \) and \( H^* \) are studied as random variables as well as \( x_w \) and \( x_\zeta \). The maximum number of random variables has been increased to 23. The reliability results are tabulated in Table 5-9. As can be seen, the random variable set including \( A^* \) is more influential than that of \( H^* \). The smallest \( \beta \) resulted from the random variables set of \( A^*, x_w \) and \( x_\zeta \) is 7.103, which corresponds to \( P_f \) of 4.42E-12. For the case with \( A^* \) without \( x_\zeta \), 25 reliability iterations were carried out using the reduction factor of \( c=3 \). The number of FLAS executions was 550 and the computational time was about 1.3 hours.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables ((N))</th>
<th>variable ( \sigma ) (0.15 \mu_x)</th>
<th>variable ( \sigma ) (0.3 \mu_x)</th>
<th>( \beta )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_w, x_\zeta, A_1^<em>, A_2^</em>, A_3^<em>, A_4^</em> )</td>
<td>21</td>
<td>9.396</td>
<td>7.103</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_w, A_2^<em>, A_3^</em>, A_4^* )</td>
<td>22</td>
<td>9.402</td>
<td>7.136</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_w, x_\zeta, H_1^<em>, H_2^</em>, H_3^<em>, H_4^</em> )</td>
<td>22</td>
<td>9.879</td>
<td>8.956</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_w, H_1^<em>, H_2^</em>, H_3^<em>, H_4^</em> )</td>
<td>23</td>
<td>9.894</td>
<td>8.968</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5-9. Reliability results: 8 flutter derivatives of the same type, \( x_w \) and \( x_\zeta \) as random variables

- **Case D:** \( x_w, x_\zeta \) and all the data points from eight flutter derivatives

  Finally all the flutter derivatives as well as the extreme wind and structural damping were considered as random variables in the reliability analysis, which is the most realistic case of all. Table 5-10 summarizes the reliability indices of two random variables set with and without structural damping. The number of random variables has increased to as many as 44. The values of \( \beta \) have decreased from *Case C* for taking into account more random variables in each case. For the case with 43 random variables and
σ = 0.15μx, 25 reliability iterations were carried out before convergence of the algorithm. The number of FLAS executions was 1032 and the computational time was approximately 2.3 hours.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables (N)</th>
<th>variable σ 0.15μx</th>
<th>variable σ 0.3μx</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_w, x_c, A_1, A_2, A_3, H_1, H_2, H_3, H_4)</td>
<td>44</td>
<td>8.833</td>
<td>6.897</td>
</tr>
<tr>
<td>(x_w, A_1, A_2, A_3, A_4, H_1, H_2, H_3, H_4)</td>
<td>43</td>
<td>8.845</td>
<td>6.914</td>
</tr>
</tbody>
</table>

Table 5-10. Reliability results: eight flutter derivatives, \(x_w\) and \(x_c\) as random variables

To achieve convergence, the use of reduction factor of \(c=3\) was employed for all cases. Figure 5-21 shows the evolutions of \(\beta\) and the limit state function without reduction factor for the case with 43 random variables with \(\sigma = 0.15\mu_x\). As can be seen, the limit state function oscillates about \(G=0\) and the algorithm does not quite converge. On the other hand, by employing a reduction factor of \(c=3\), the algorithm converges nicely as shown in Figure 5-22.
Figure 5-21. Evolutions of: a) $\beta$ and b) $G(x)$ without reduction factor
There is a large difference in $\beta$ between the cases with 15 and 30% dispersions at $V^*=30$; for the case without the structural damping, $\beta$ for 15% dispersion is 8.845 ($P_f=4.57E-19$) while $\beta$ for 30% dispersion is 6.914 ($P_f=2.36E-12$). Table 5-11 and Figure 5-23 present the initial and the MPP values of the flutter derivative, $A^*_2$ for 15% and 30% dispersions without structural damping while Figure 5-24 shows the FLAS outputs for both cases. The large difference in reliability indices results from the significant difference in flutter velocities between these two cases, which mainly comes from the variations in the random variables of the flutter derivative, $A^*_2$ at the MPP just as in Case B.

**Table 5-11. The MPP values of $A^*_2$ for 15% and 30% dispersions at $V^*=30$**

<table>
<thead>
<tr>
<th>$V^*$</th>
<th>initial values</th>
<th>MPP values</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>15% $\sigma$</td>
<td>30% $\sigma$</td>
</tr>
<tr>
<td>6.0</td>
<td>-0.25</td>
<td>-0.250</td>
<td>-0.238</td>
</tr>
<tr>
<td>10.0</td>
<td>-0.55</td>
<td>-0.524</td>
<td>-0.319</td>
</tr>
<tr>
<td>12.0</td>
<td>-0.7</td>
<td>-0.658</td>
<td>-0.830</td>
</tr>
<tr>
<td>20.0</td>
<td>-1.52</td>
<td>-1.526</td>
<td>-1.490</td>
</tr>
</tbody>
</table>
Figure 5-23. Initial and MPP values for 15% and 30% \( \sigma \) of random variables of \( A_2^* \)
We have seen so far that the structural damping has little influence on the structural reliability of the Great Belt Bridge. Therefore, it is not included as a random variable to perform the RBDO of the Great Belt in the subsequent section.
5.7 Reliability Based Design Optimization

The formulation of the RBDO problem explained in detail in Section 4.4 is now applied to the Great Belt East Bridge. The bridge structure is to be optimized by varying the thicknesses of the steel plates that form the aerodynamic box girder and the area of the two main cables. The probabilistic flutter constraint as well as other deterministic constraints is taken into account. The three RBDO methods of RIA, PMA and SORA discussed in Chapter 3 are employed to perform the RBDO of the bridge structure and the results from these methods are discussed.

5.7.1 Definition of the RBDO problem

The RBDO parameters taken into consideration in this study as well as the RBDO formulation are described as follows.

- Design variables
  Two design variable sets are considered in this example. Case I involves in a set of four design variables of the steel plate thicknesses that form the aerodynamic box girder while Case II includes the main cable area as an additional design variable to those in Case I. Figure 5-25 illustrates the location of each design variable.

- Objective function
The objective function for Case I is the girder volume while that of Case II is the sum of the volumes of the box girder and two main cable.

- Random variables

Based on the reliability analysis of Section 5.6, flutter derivatives and the extreme wind velocity are considered as random variables while the structural damping is disregarded for its little influence over the structural reliability. The points that define eight flutter derivatives of $A_1^*$ to $A_4^*$ and $H_1^*$ to $H_4^*$ shown in Figure 5-7 are assumed to be normally distributed random variables with linearly variable standard deviation of 0 at $V^*=0$ and 15% of the mean value at $V^*=30$. The normal-equivalent mean and standard deviation values of the extreme wind velocity can be found in Section 5.6.4. The total number of random variables is 43 and the reliability index of the original design with the corresponding random variables is $\beta=8.845$.

- Limit state function

The aeroelastic limit state function is the same as in the reliability analyses, which is

$$G(x) = \frac{V_f(x_i)}{x_{sw}} - 1 \quad i = 1, 2, \ldots, n$$

where $n$ is the number of points to define flutter derivatives. The vector $x$ represents $n+1$ random variables.

- Problem formulation

The RBDO formulation of Case I is defined as follows.

**Case I:** four girder plate thicknesses as design variables

Min: Girder volume ($d$)  

$$g_i^I : P[G(x) \leq 0] \leq P_f$$

\[ (5.6a) \]

\[ (5.6b) \]
where $G(x) = \frac{V_i(x_i)}{x_w} - 1$  

$$g_{2,j}^l : 5 \text{ mm} \leq d_j \leq 20 \text{ mm} \quad j = 1, 2, ..., 4$$  

$$g_3^l : \sigma_c = 565 \text{ MPa}$$  

$$g_4^l : \frac{z_d}{z_{\text{max}}} - 1 \leq 0 \quad \text{where } z_{\text{max}} = \frac{L}{500}, L = 1624 \text{ m}$$  

where $\sigma_c$ is the maximum main cable stress and $z_d$ is the maximum vertical deck displacement, both under the static overload case.

The constraint $g_1$ is the probabilistic flutter constraint while $g_2$ defines the side constraints of the design variables. The constraint of $g_3$ assigns a main cable area whenever the girder design is changed (Equation (5.2)) so that the maximum main cable stress under the static overload case is 565 MPa as explained in Section 5.5.1. The constraint function $g_4$ limits the maximum vertical deck displacement below $z_{\text{max}}$ under the traffic overload case. Two target $P_f$ were defined by choosing the corresponding target reliability, $\beta^T$, to see how the optimum designs change by varying the target reliability indices. The RBDO was first performed with the reference value of $\beta^T = 8.845$ (Table 5-10), and then two cases of $\beta^T = 8.0$ and $\beta^T = 10.0$ were studied.

The formulation of Case II is defined as follows.

**Case II**: five design variables of the steel plate thicknesses and the main cable area

Min: sum of the girder and the main cable volumes ($d$)  

$$g_1^u : P[G(x) \leq 0] \leq P_f$$
where \( G(x) = \frac{V_i(x_i)}{x_w} - 1 \) \( i = 1, 2, \ldots, n \) \( (5.7b) \)

\[ \begin{align*}
g^{II}_{2a,j} : & \quad 5.0 \text{ mm} \leq d_j \leq 20.0 \text{ mm} \quad j = 1, 2, \ldots, 4 \quad (5.7c) \\
g^{II}_{2b} : & \quad 0.1 \text{ m}^2 \leq d_5 \leq 2.0 \text{ m}^2 \quad (5.7d) \\
g^{II}_3 : & \quad \sigma_c \leq 565 \text{ MPa} \quad (5.7e) \\
g^{II}_4 : & \quad \frac{z_d}{z_{max}} - 1 \leq 0 \quad \text{where} \quad z_{max} = \frac{L}{500}; L=1624 \text{ m} \quad (5.7f)
\]

The side constraints of the cable area are added to the formulation of Case I, while the constraint \( g_3 \) in this case limits the maximum main cable stress by an inequality constraint. The main purpose of Case II is instead of assuming that the maximum cable stress constraint of \( g_3 \) is active at the optimum, the constraint of \( g^{II}_3 \) restricts the maximum cable stress by an inequality constraint for having the main cable area as an independent design variable. The target reliability was set to \( \beta^T=8.0 \) in order to compare the results with those in Case I with the same target reliability.

The Matlab “fmincon” optimizer was employed with active-set algorithm to carry out the optimization routine. The termination tolerance of the objective function and the constraint functions was set to 1E-4. The stopping criterion of reliability routines for FORM was the difference between any two consecutive beta values to be smaller than 5E-3, and for HMV, the tolerance of any \( u \) values to be smaller than 5E-3.

### 5.7.2 RBDO cases

The RBDO cases with two sets of design variables are considered with different target reliability indices as follows.

**Case I:** four design variables of the girder plate thicknesses for the target reliability indices of \( \beta^T=8.845, \beta^T=10.0 \) and \( \beta^T=8.0 \).
**Case II**: five design variables of the girder plate thicknesses and the main cable area for $\beta_T^r=8.0$

### 5.7.3 RBDO results

The RBDO results are presented next. For Case I, the RBDO was first performed on the original bridge design with the reliability index of $\beta_T^r=8.845$. The results of this subcase were then used to compare with the optimum designs from the subsequent cases with increased and reduced target reliability.

- **Case I-1: four design variables of the girder plate thicknesses for $\beta_T^r=8.845$**

  In this subcase, four design variables of $d_1$ through $d_4$ in Figure 5-25 are considered as design variables while the objective function is the volume of the box girder. We want to optimize the structure while maintaining the same reliability level of $\beta_T^r=8.845$ as the original design. The design of the original bridge is $d=[12.0, 10.0, 10.0, 10.0]$ (in mm) and its corresponding objective function of the girder volume is 2834.09 m$^3$.

  Table 5-12 shows the optimum designs and the objective functions using the three RBDO methods. In general, $d_1$ and $d_2$ have increased slightly while $d_3$ has decreased from the original design. Nonetheless the objective functions from the three methods converged to very similar values. The box girder volume has decreased by an average of 0.6% while the total volume of the box girder and the main cables has decreased by 0.3%. The optimum design by PMA is used to compare with those in the following subcases with increased and reduced target reliability indices.
Table 5-12. Baseline design for the four design variables case

- **Case I-2: four design variables of the girder plate thicknesses for $\beta^T=10.0$**

  In this subcase, the target reliability was increased to $\beta^T=10.0$ from the reliability value of the original bridge design. This target reliability was chosen arbitrarily by the author to see how the girder design changes by increasing the required structural reliability level. Figure 5-26 and Figure 5-27 present the evolution of the design variables and the objective functions using the three RBDO methods while Table 5-13 and Table 5-14 summarize the optimum designs and the numbers of FLAS and Abaqus executions.

  The objective functions using the three methods converged to very similar values; however, the thicknesses of the girder are different depending on the method. The optimum designs of RIA and PMA are similar while that of SORA is different. By increasing the target reliability to $\beta^T=10.0$, the objective function of the girder volume has increased by an average of 15% and the main cable volume has increased by 5% compared to the design of Case I-1. This increment in the cable volume is due to the maximum cable stress constraint of $g_i^L$. It is important to note that the cable area in Case I is an implicit design variable assigned by Equation (5.2). The volume of the two main cables constitutes 47% of the total volume of the box girder and the two main cables. In result, the total volume of the box girder and the main cables has increase by around 10.8%.

  The numbers of FLAS and Abaqus executions are linked to the formulations of each RBDO method: the number of FLAS executions is primarily related to the number of reliability iterations while that of Abaqus executions is associated with the number of
design iterations. The nested formulation of the two-level method involves in high number of reliability iterations while the decoupled formulation is associated with a small number of reliability iterations and a high number of design iterations. The time duration of FLAS execution is based on the number of aeroelastic modes, the number of nodes along the bridge deck from the FE model to consider in the flutter calculation, and the flutter velocity of the bridge for a particular design. It takes approximately 10 seconds to run the program for the original bridge design ($V_f=80.32$ m/s). Since the flutter velocity obviously changes depending on the design of the bridge and the flutter derivatives during the reliability analysis to obtain the MPPs, the time duration of FLAS execution also changes. The modal and static analyses, which are launched simultaneously, take about 15 seconds to complete. Nevertheless, its computational time also depends on factors such as the types of nodes in which the computation is carried out in the cluster as well as the time required for the communications between these nodes. In this subcase, the computational time of SORA is about one third of RIA. The elevated numbers of FLAS executions for RIA and PMA are related to difficulties in convergence in the reliability routines. To achieve convergence, the reduction factor of $c=3$ for RIA and $r=0.5$ for PMA and SORA were employed. The reliability iterations for RIA varied between 11 and 27 while that of PMA and SORA oscillated between 15 and 33.
Figure 5-26. Evolutions of four design variables using a) RIA b) PMA c) SORA ($\beta^T = 10.0$)
Figure 5-27. Evolution of the objective functions for the four design variables case ($\beta_T = 10.0$)

<table>
<thead>
<tr>
<th>$\beta_T = 10.0$</th>
<th>$d_1$ (in mm)</th>
<th>$d_2$ (in mm)</th>
<th>$d_3$ (in mm)</th>
<th>$d_4$ (in mm)</th>
<th>obj. func. ($m^3$)</th>
<th>obj. func. ($m^3$)</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA</td>
<td>15.063</td>
<td>12.739</td>
<td>9.925</td>
<td>11.426</td>
<td>3247.89</td>
<td>15.13</td>
<td>11.00</td>
</tr>
<tr>
<td>PMA</td>
<td>14.795</td>
<td>12.921</td>
<td>10.261</td>
<td>11.512</td>
<td>3244.65</td>
<td>15.01</td>
<td>10.92</td>
</tr>
</tbody>
</table>

Table 5-13. Optimum design variables and objective functions for $\beta_T = 10.0$

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>39528</td>
<td>38736</td>
<td>7801</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>91</td>
<td>58</td>
<td>1092</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>93.3</td>
<td>80.0</td>
<td>31.3</td>
</tr>
</tbody>
</table>

Table 5-14. The number of FLAS and Abaqus executions
Case I-3: four design variables of the girder plate thicknesses for \( \beta^T = 8.0 \)

In this subcase, the target reliability was reduced to \( \beta^T = 8.0 \) from the structural reliability level of the original bridge design. Figure 5-28 and Figure 5-29 present the evolutions of the design variables and the objective functions while Table 5-15 and Table 5-16 summarize the optimum designs and the numbers of FLAS and Abaqus executions.

The objective functions using the three RBDO methods all converged to similar values; however, the distributions of the girder material are different depending on the method, especially \( d_2 \) and \( d_3 \). For reducing the target reliability to \( \beta^T = 8.0 \), we have achieved to save an average of 9.6% of the girder volume from the design of Case I-1. Since the main cable volume has decreased by 5.3% due to the maximum cable stress constraint, the reduction in total volume was approximately 7.0%.

In order to achieve convergence, the reduction factor of \( c = 3 \) was employed for RIA while \( r = 0.5 \) was used for PMA. The number of reliability iterations for RIA was 8 or 9 while that for PMA oscillates between 11 and 22. The reliability iterations for SORA varied between 17 and 20. SORA was the most computationally efficient, which took approximately one third and one fourth of the computational time of RIA and PMA respectively.

![Graph](image-url)  

a) RIA
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Figure 5-28. Evolutions of four design variables using a) RIA b) PMA c) SORA for $\beta^T=8.0$
Figure 5-29. Evolution of the objective functions for the four design variables case ($\beta^T=8.0$)

<table>
<thead>
<tr>
<th>$\beta^T=8.0$</th>
<th>$d_1$ (in mm, obj. func. in m³)</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
</table>

Table 5-15. Optimum designs for the four design variables case ($\beta^T=8.0$)

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>23472</td>
<td>36324</td>
<td>7433</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>96</td>
<td>52</td>
<td>162</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>53.8</td>
<td>71.9</td>
<td>16.0</td>
</tr>
</tbody>
</table>

Table 5-16. Number of FLAS and Abaqus executions for the four design variables case ($\beta^T=8.0$)
Case II: five design variables of the plate thicknesses of the box girder as well as the main cable area for $\beta^T=8.0$

In Case II, we have considered a total of five design variables including the main cable area, which are $d_1$ to $d_5$ in Figure 5-25 while the objective function is the sum of the volumes of the box girder and two main cables. The initial design is $d=[12.0, 10.0, 10.0, 10.0, 0.453]$ ($d_1$ to $d_4$ in mm, $d_5$ in m$^2$) and its corresponding objective function is 5375.54 m$^3$.

The target reliability was set to $\beta^T=8.0$ as in Case I-3. The evolutions of design variables and the objective functions are presented in Figure 5-30 and Figure 5-31 while the optimum designs and the computational time are summarized Table 5-17 and Table 5-18. For representation purposes, the design variable of the main cable area, $d_5$, is multiplied by 20.

The objective functions of Case II using the three methods converged to similar values while the design variables, especially $d_3$ and $d_4$ are different depending on the methods. For reducing the target reliability to $\beta^T=8.0$, the girder volume has decreased by an average of 9.0% and the main cable volume has decreased by 4.7%. As a result, the total volume of the box girder and the two main cables has decreased by approximately 7.0%. To achieve convergence, the reduction factor of $c=3$ was employed for RIA while no reduction factor was used for PMA or SORA. The reliability iterations for RIA varied between 10 and 27, while those of PMA and SORA oscillate primarily between 8 and 23. For some cases for both PMA and SORA, the number of reliability iterations reached as many as 90 for difficulties in convergence. The computational time for all three methods is similar in this subcase.

Compared to Case I-3, the objective functions of the total volume of the box girder and the two main cables are very similar, whose differences are less than 0.5% although the optimum designs are different. For all three methods, both the aeroelastic
and the maximum main cable stress constraints are active at the optimum. The computational time of *Case I* is approximately 67% of *Case II* for RIA and 20% for SORA.

![Graph](image1.png)  

**a)** RIA

![Graph](image2.png)  

**b)** PMA
Figure 5-30. Evolutions of five design variables using a) RIA b) PMA c) SORA for $\beta^T = 8.0$

Figure 5-31. Evolution of the objective functions for the four design variables case ($\beta^T = 8.0$)
Table 5-17. Optimum designs for the five design variables case ($\beta^T=8.0$)

<table>
<thead>
<tr>
<th>$\beta^T$=8.0</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>girder vol.</th>
<th>obj. func.</th>
<th>girder vol.</th>
<th>obj. func.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA</td>
<td>9.113</td>
<td>9.400</td>
<td>10.903</td>
<td>9.069</td>
<td>0.432</td>
<td>2570.43</td>
<td>4992.16</td>
<td>-8.89</td>
<td>-6.73</td>
</tr>
<tr>
<td>PMA</td>
<td>9.349</td>
<td>9.206</td>
<td>7.324</td>
<td>10.066</td>
<td>0.431</td>
<td>2570.66</td>
<td>4986.17</td>
<td>-8.88</td>
<td>-6.84</td>
</tr>
<tr>
<td>SORA</td>
<td>9.514</td>
<td>9.750</td>
<td>7.841</td>
<td>8.500</td>
<td>0.428</td>
<td>2560.10</td>
<td>4960.96</td>
<td>-9.26</td>
<td>-7.31</td>
</tr>
</tbody>
</table>

Table 5-18. Number of FLAS and Abaqus executions for the five design variables case ($\beta^T=8.0$)

<table>
<thead>
<tr>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>37656</td>
<td>25956</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>91</td>
<td>94</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>80.0</td>
<td>72.9</td>
</tr>
</tbody>
</table>

In Case I, only the girder plate thicknesses were considered as design variables while the main cable area was restricted by an equality constraint in order to assure that the main cables were at the reasonable stress level during the optimization process. Consequently, we imposed that the maximum main cables stress constraint to be active at the optimum. On the other hand, in Case II the main cable area is considered as an independent design variable and the maximum main cable constraint is not forced to be active.

The resulting objective functions from both cases are similar because the maximum main cable stress constraint is active in Case II. Therefore we can confirm the validity of Case I for assuming the optimality criteria of the maximum cable stress, $\sigma=565$ MPa. Since the computational time for Case I is as low as 20% of Case II for SORA, we can conclude that the formulation of Case I is more computationally efficient.
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5.8 References


[I1] Indian Institution of Bridge Engineering, “Cable stayed, supported and suspension bridges”, Hyderabad, India, 1999


[S1] The Storebælt publication, “East Bridge”, Denmark, 1998


6.1 Bridge description

The Messina Strait Bridge is a long-planned bridge construction project that will link the island of Sicily to the southern Italian mainland. If completed, it will be the suspension bridge with the longest center-span in the world of 3300 meters surpassing the current record holder, Akashi strait bridge in Japan with 1991 meter-span.

The geographical location and the virtual view of the proposed bridge are shown in Figure 6-1 and Figure 6-2. The Messina Strait is a part of the Mediterranean Sea separating the island of Sicily to the west from the mainland Italy to the east connecting the Ionian Sea with the Tyrrhenian Sea. The minimum width of the strait is approximately 3 kilometers between Capo Peloro in Sicily and Torre Cavallo in Calabria where the link is planned. The southern Italy today is economically unstable compared to its northern counterpart. Since the ferry traffic is congested during peak hours, the construction of a permanent link is believed to encourage its economic
growth and social regeneration. According to Stretto di Messina\textsuperscript{[S1]}, the economic gain due to the construction of a bridge is estimated to be as much as the construction cost.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{messina_bridge_location.png}
\caption{Geographical location of the projected Messina Bridge\textsuperscript{[M1]} (the dotted yellow line represents the bridge site)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{messina_bridge_view.png}
\caption{Virtual view of the Messina Bridge}
\end{figure}
The idea of joining Calabria and Sicily has been existed ever since Roman time. Yet, engineering challenges caused by the deep water, fast sea current, high wind and seismic activities have been the impediments for the bridge construction. The Messina Strait Company, (Stretto di Messina S.p.A) was established by the Italian government in 1981 to study the bridge design, construction, operation and management of the maritime traffic. They proposed numerous solutions including both bored and floating tunnels as well as single and multi-span bridges, among which a single-span suspension bridge was chosen because of the seismic and high sea current conditions of the bridge site by minimizing the subsea and geotechnical work as well as its construction and maintenance cost. In 2004, Stretto Di Messina issued a tender notice to select a general contractor to carry out the final design and construction, and in the following year a general contractor was chosen after a design competition. However, in 2006 the Italian Parliament voted against the plan questioning the bridge's viability and the ability of Italian treasury to bear its share of the construction cost. In 2009 the Italian government reopened the shelved project claiming that the engineering, political, and financial obstacles have been cleared. Yet, it is still unknown when the project can be initiated because of the heavy debt of the Italian government and the global recession.

There exist various versions of the bridge design because of the nature of the project; this research is based on the bridge design of the preliminary design of November 30, 2004\cite{C1, C2}. The proposed bridge is a suspension bridge with 3300 meter main span and two 183 meter lateral spans (Figure 6-3). The 61-meter wide triple box girder connected by transversal girders every 30 meters longitudinally provides 6 traffic lanes and two railways. The deck is suspended by hanger cables, which are looped over the two main cables of approximately two square-meters each hanging from 382 meter-high towers (Figure 6-4 to Figure 6-6).
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Figure 6-3. Side view of the Messina Bridge

Figure 6-4. Cross section of the triple-box girder

Figure 6-5. Tower details
6.2 Structural model

In order to carry out the aeroelastic and structural analyses of the Messina Bridge, a 3D-beam finite element model in Abaqus was created (Figure 6-7). The model consists of 2913 elements with 12,528 degrees of freedom. Each of the triple boxes is represented by a beam element, which are connected by transversal beam elements. The deck is suspended by main cables through vertical hanger cables attached to the cross girders. The boundary conditions are imposed at the location of the anchorages, the foundation of the two towers, and the tower-cable connections. There are a total of 245 nodes longitudinally along the deck, spaced approximately 15 meters apart. The geometrical and mechanical properties of the model used in the calculations are summarized in Table 6-1.
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Figure 6-7. Structural model of the Messina Bridge

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total deck length (m)</td>
<td>3666</td>
</tr>
<tr>
<td>Center span length (m)</td>
<td>3300</td>
</tr>
<tr>
<td>Lateral span length (m)</td>
<td>183</td>
</tr>
<tr>
<td>Distance between tower-anchorage, Sicily side (m)</td>
<td>960</td>
</tr>
<tr>
<td>Distance between tower-anchorage, Calabria side (m)</td>
<td>810</td>
</tr>
<tr>
<td>Total deck width (m)</td>
<td>61.13</td>
</tr>
<tr>
<td>Center box area (m$^2$)</td>
<td>0.341</td>
</tr>
<tr>
<td>Center box moment of inertia, Iy (m$^4$)</td>
<td>0.286</td>
</tr>
<tr>
<td>Center box moment of inertia, Iz (m$^4$)</td>
<td>1.847</td>
</tr>
<tr>
<td>Center box polar moment of inertia, J (m$^4$)</td>
<td>0.653</td>
</tr>
<tr>
<td>Lateral box area (m$^2$)</td>
<td>0.645</td>
</tr>
<tr>
<td>Lateral box moment of inertia, Iy (m$^4$)</td>
<td>0.623</td>
</tr>
<tr>
<td>Lateral box moment of inertia, Iz (m$^4$)</td>
<td>11.02</td>
</tr>
<tr>
<td>Lateral box polar moment of inertia, J (m$^4$)</td>
<td>1.375</td>
</tr>
<tr>
<td>Girder density (T/m$^3$)</td>
<td>7.85</td>
</tr>
</tbody>
</table>

Table 6-1. Geometrical and mechanical properties of the structural model of Messina
6.3 Nonlinear structural analysis

Because of the flexibility of the structure, the structural calculations were carried out in two steps: the first step to adjust the initial stress of the main cables and hanger cables iteratively due to the self-weight of the bridge deck, and the subsequent step of running static analyses under traffic loads.

First of all, the finite element model was launched under its self-weight to check for its equilibrium as explained in Chapter 4. The maximum deck displacement in the center span was 0.45 m, which is reasonable considering the 3.3 km-long center span length and the cable mesh size of approximately 30 m.

Then a series of static analyses with the original mechanical properties were performed under the static loading cases specified in the design specification for the Messina Bridge. Both road and railway loads refer to the Italian code, D.M. 04.05.90 and I/SC/PS-OM/2298. According to the specification, there are two types of loads to be considered for the global structural design: variable dense load to evaluate the bearing capacity of the retaining and supporting structural system, main cable and saddles, and sparse variable load to evaluate the structural response in terms of runnability, deformation and comfort. In order to evaluate the overall performance of the bridge and check the maximum displacement of the deck, the sparse variable load was taken into consideration.

This load case involves in the global deformation of the bridge due to the road-rail traffic under quasi steady state. For roadway, 5 kN/m of distributed load was applied for each lane, while for the railways, two loads of 88 kN/m along 750 m were applied. Among different load combinations, we have applied the distributed traffic loads on one of the lateral spans as well as half of the center span, and two train loads as indicated in Figure 6-8. The maximum vertical displacement of the original deck was 5.27 m, which is 1/620 of the span length. The maximum tensile stress of 800 MPa in
the main cables was about 43% of the ultimate material strength of 1860 MPa. Both the vertical displacement of the deck and the tensile stress in the main cables of the original bridge design are considered reasonable.

![Diagram of traffic and railway load case](image)

**Figure 6-8. The worst traffic and railway load case**

### 6.4 FLUTTER ANALYSIS

The structural model presented in the previous section was employed to calculate the natural frequencies and the vibration modes of the Messina Bridge. The modal analyses were carried out in two steps just as in the previous section because of the large flexibility of the structure. In the first step, the initial main cable length as well as the initial stresses in the main and hanger cables was calculated followed by non-linear modal analyses with the corresponding overall stiffness of the bridge structure. Table 6-2 and Figure 6-9 summarize some of the natural frequencies and vibration mode shapes of the Messina Bridge, which are compared to the data by Diana et al[^D1].
### Table 6-2. Natural frequencies of the Messina Bridge

<table>
<thead>
<tr>
<th>Mode</th>
<th>Type</th>
<th>Freq (Hz) Kusano</th>
<th>Freq (Hz) Diana</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LS</td>
<td>0.0309</td>
<td>0.033</td>
</tr>
<tr>
<td>2</td>
<td>LA</td>
<td>0.0573</td>
<td>0.059</td>
</tr>
<tr>
<td>3</td>
<td>VA</td>
<td>0.0606</td>
<td>0.061</td>
</tr>
<tr>
<td>4</td>
<td>VS</td>
<td>0.0811</td>
<td>0.080</td>
</tr>
<tr>
<td>5</td>
<td>LS</td>
<td>0.0860</td>
<td>0.084</td>
</tr>
<tr>
<td>6</td>
<td>TA</td>
<td>0.0868</td>
<td>0.081</td>
</tr>
<tr>
<td>7</td>
<td>VA</td>
<td>0.0925</td>
<td>0.093</td>
</tr>
<tr>
<td>8</td>
<td>VS</td>
<td>0.0980</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>TS/LA</td>
<td>0.1032</td>
<td>0.097</td>
</tr>
<tr>
<td>10</td>
<td>VA</td>
<td>0.1033</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>LA</td>
<td>0.1042</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>VS</td>
<td>0.1078</td>
<td>0.107</td>
</tr>
<tr>
<td>13</td>
<td>LS</td>
<td>0.1142</td>
<td></td>
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<tr>
<td>14</td>
<td>VA</td>
<td>0.1279</td>
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<td>15</td>
<td>LA</td>
<td>0.1347</td>
<td></td>
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<tr>
<td>16</td>
<td>TS/LA</td>
<td>0.1356</td>
<td>0.129</td>
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<tr>
<td>17</td>
<td>VS/LA</td>
<td>0.1448</td>
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<td>18</td>
<td>LA</td>
<td>0.1461</td>
<td></td>
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<tr>
<td>19</td>
<td>VS</td>
<td>0.1565</td>
<td></td>
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<td>20</td>
<td>VS</td>
<td>0.1586</td>
<td></td>
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<tr>
<td>21</td>
<td>VS</td>
<td>0.1603</td>
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<td>22</td>
<td>VS</td>
<td>0.1620</td>
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<tr>
<td>23</td>
<td>LA/TA</td>
<td>0.1658</td>
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<tr>
<td>24</td>
<td>VS</td>
<td>0.1678</td>
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</tr>
<tr>
<td>25</td>
<td>VA</td>
<td>0.1734</td>
<td></td>
</tr>
</tbody>
</table>

*Note: V(vertical), L(lateral), T(torsional), S(symmetric), A(asymmetric)*
Note: lateral, dashed line vertical, and dotted line torsional displacement
Figure 6-9. Vibration modes of the Messina Bridge
As explained in the previous chapter, 18 flutter derivatives obtained in wind tunnel test were necessary in order to calculate flutter velocity. A sectional model of the Messina Bridge scaled to 1/100 was utilized to obtain 18 flutter derivatives by Leon\textsuperscript{[L1]} at the wind tunnel of the University of La Coruña as shown in Figure 6-10. The flutter derivatives obtained in the test were represented graphically in Figure 6-11, in which the horizontal axis represents reduced velocity, $V^*$. 

![Figure 6-10. Wind tunnel test of the Messina bridge sectional model](image)

![Graphs of $A_1^*$ and $A_2^*$](image)
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- **A3***: Graph showing a linear increase.
- **A4***: Graph showing a linear decrease.
- **A5***: Graph showing a linear decrease.
- **A6***: Graph showing a linear decrease.
- **H1***: Graph showing a decrease with a local maximum.
- **H2***: Graph showing a linear decrease.
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Figure 6-11. Flutter derivatives of Messina obtained in the wind tunnel of the University of La Coruña by León

The FLAS code developed by the University of La Coruña was employed to compute flutter velocity using the natural frequencies and mode shapes of the bridge obtained in the modal analysis as well as other basic information of the bridge such as span length, number of elements in each span, aeroelastic modes to consider as well as structural damping.

First of all, FLAS was run with the first 25 vibration modes, whose results are shown in Figure 6-12. The top half of the graph shows the evolution of negative alpha values vs. wind velocity, which are related to the damping of the structure. Flutter occurs when the alpha value of the aeroelastic mode 6 goes from negative to positive, which is a point of null structural damping at the wind speed of 102.69 m/s. Any increment of wind velocity from this point will increase the vibration exponentially. The
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The bottom half is the evolution of beta values, which are related to frequencies of the structure (see Equation (4.11)). The frequency of the mode 6 gradually decreases as flutter occurs at the reduced frequency of 0.246.

In order to reduce computational time in the phase of optimization, the seven most relevant modes of 1, 2, 3, 4, 5, 6 and 11 were chosen for flutter analyses by combining some of the first vertical, lateral and torsional modes as shown in Figure 6-13. It should be noted that it is important to include the first torsional mode in the flutter computation. Since the flutter velocity using these seven modes varies from the case with 25 modes only by 0.03%, these seven modes were used for flutter calculations in the subsequent sections of reliability analysis and reliability based design optimization.

Figure 6-12. Flutter analysis results: FLAS output with 25 modes
6.5 DETERMINISTIC OPTIMIZATION

Before getting into the reliability based design optimization of the Messina Bridge, several deterministic optimization cases were studied to check the viability of the optimization problem on the bridge structure. Two main cases of design variables were studied. In Case I, we considered the steel plate thicknesses of different girder edges as design variables ($d_1$ to $d_6$ in Figure 6-14). The objective function to be minimized is the volume of the box girders. In Case II, the main cables area is included as an additional design variable to those in Case I ($d_1$ to $d_7$ in Figure 6-14). The objective function in this case is the sum of the three box girders and the two main cables volumes. For both cases, the bridge design must satisfy a series of constraint functions as described in each formulation.
6.5.1 Case I

The formulation of Case I is presented in Equation (6.1). The objective of this optimization problem is to minimize the bridge girder volume by varying the plate thicknesses of the box girders, $d_i$ to $d_6$ in Figure 6-14.

\begin{align}
\text{Min: Girder volume (d)} & \quad (6.1a) \\
& \quad (6.1b) \\
& \quad (6.1c) \\
& \quad (6.1d) \\
& \quad (6.1e)
\end{align}

where $V_f$ is the flutter velocity of the current design, $V_{\text{limit}}$ is the limiting flutter speed, $d_i$ is the design variable of plate thicknesses, $\sigma_c$ is the maximum tensile stress in the main cable, $z_d$ is the maximum vertical deck displacement due to the static overload case, and $z_{\text{max}}$ is the limiting displacement value.

The constraint $g_1^f$ limits the minimum flutter velocity of the bridge while $g_{2,i}^f$ restricts the side limits of the design variables. $g_3^f$ limits the maximum main cable stress.
by an equality constraint, and \( g_4 \) restricts the maximum vertical deck displacement under the traffic overload case.

As explained in Section 4.4.3.2, the main cable area needs to be adjusted every time the girder design is changed so that the main cables are at reasonable stress level. The maximum cable stress of the original design under traffic overload case was used as a reference value. A study was carried out in order to establish a relationship between the deck weight and the main cable area, which is described next.

- Updating the main cable cross sectional area

In each iteration of the optimization problem with new plate thicknesses, the deck weight changes, and this in turn changes the tensile stress in the main cables. Knowing that the main cable weight for the original design is greater than the deck weight, it is important to dimension the cable cross sectional area adequately. The finite element model was run varying both deck weight and the cable sectional area, and the result of the cable tensile stress with respect to the cable area for different deck weight are shown in Figure 6-15. The dimension of the main cable area is determined so that the maximum cable tensile stress under the static overload case is 800 MPa, which is approximately 40% of the ultimate material strength and is considered reasonable. The cable area was then plotted against the deck weight at the cable stress of 800 MPa in Figure 6-16. As can be observed, the appropriate cable sectional area for a particular deck weight is expressed simply by a linear equation as:

\[
Ac = 0.0583 \cdot DW + 0.2292
\]

(6.2)
Figure 6-15. Cable stress vs. Cable area for different deck weight

Figure 6-16. Cable area vs. Total deck weight for cable stress of 800 MPa
6.5.2 Case II

The formulation of Case II is presented in Equation (6.3). The sum of the volumes of the box girders and the two main cables is to be minimized by considering the girder plate thicknesses and the main cable area as design variables. The side constraint of the main cable area is added to the formulation of Case I. The maximum main cable stress in Case II is limited by an inequality constraint of $g^{II}_3$ for having the main cable area as an independent design variable.

As explained in Chapter 5, the main purpose of Case II is instead of assuming that the maximum cable stress is active at the optimum as in Case I, the constraint $g^{II}_3$ restricts the maximum cable stress by an inequality constraint. The formulation of Case II is as follows.

\[
\text{Min: Sum of the girder and main cables volumes} \ (d) \quad (6.3a)
\]

\[
g^{II}_{1} : \frac{V_f}{V_{\text{limit}}} - 1 \geq 0 \quad (6.3b)
\]

\[
g^{II}_{2} : 5.0 \ \text{mm} \leq d_i \leq 30.0 \ \text{mm} \quad (6.3c)
\]

\[
g^{II}_{2,7} : 1.0 \ \text{m}^2 \leq d_i \leq 3.0 \ \text{m}^2 \quad (6.3d)
\]

\[
g^{II}_{3} : \sigma_c \leq 800 \ \text{Mpa} \quad (6.3e)
\]

\[
g^{II}_{4} : \frac{z_d}{z_{\text{max}}} - 1 \leq 0 \ (z_{\text{max}} = \frac{L}{500}; \ L = 3300 \ \text{m}) \quad (6.3f)
\]

The Matlab “fmincon” optimizer was employed with active-set algorithm to solve the deterministic optimization problems. The termination tolerance values of the objective function and the constraint functions were set to 1E-5.
6.5.3 Deterministic optimization procedure

The optimization steps of both cases are detailed below. The procedures for the two cases are slightly different depending on the design variables set. An iterative scheme is used to determine the initial main cable length and initial tensile stress of the main and hanger cables.

1. Calculate new mechanical properties, \( I_y(d_i), I_z(d_i), J(d_i), A(d_i) \) with the new design variables. See Appendix for the detail.
2. For Case I, compute the appropriate main cable cross sectional area due to changes in deck weight using Equation (6.2).
3. Run the iterative process explained in Section 4.4.3.2 in order to compute the initial main cable length and the initial tensile stresses in the main cables and hanger cables.
4. Launch the Abaqus finite element model of the complete bridge to perform static analysis to obtain the maximum deck displacement and the maximum cable stress under the static overload case.
5. Launch the same FE model for modal analysis using the stiffness matrix of the complete bridge under its self-weight in order to get natural frequencies and mode shapes.
6. Write FLAS input file from the modal analysis results of 5 and launch FLAS for flutter analysis with new frequencies and vibration modes.
7. The Matlab optimization algorithm evaluates all constraints and objective function and modifies the design variables
8. Repeat Step 1-7 until the convergence of the algorithm.

6.5.4 Deterministic optimization results

The results of the deterministic optimization are shown next. The purpose of this section is to demonstrate the feasibility of design optimization of the Messina Bridge under aeroelastic constraint. We want to see if the design variables actually change
under different values of limiting flutter velocity constraint. It is important to know if this deterministic optimization works so that we can proceed later on to the reliability based design optimization of the bridge structure under flutter constraint.

**Case Ia: Three design variables of the lateral box girder**

In *Case Ia*, three design variables of the plate thicknesses of the two lateral box girders, \(d_1\) to \(d_3\) in Figure 6-14 were considered while the objective function is the volume of the lateral girders. Two limiting flutter constraint of \(V_f=98\) m/s and 105 m/s were chosen arbitrarily knowing that the flutter velocity of the original design is 102.72 m/s. The original design is \(d=[14.0, 14.0, 14.0]\) in mm, and the corresponding objective function is 4732.07 m³. Figure 6-17 and Figure 6-18 present the evolutions of the design variables and the objective functions for the limiting flutter velocity of 98 m/s to 105 m/s respectively while Table 6-3 and Table 6-4 summarizes the optimum designs and the number of FLAS and Abaqus executions for each case.

When the imposed minimum flutter speed is reduced to 98 m/s, the girder became lighter from the original design while the limiting speed is increased to 105 m/s, the girder became heavier as expected. It has been verified that the girder design in fact changed as the limiting flutter velocity is altered. The computation was carried out in approximately 5 hours for \(V_f=98\) m/s, which is considered reasonable. For both cases, the aeroelastic constraint and the maximum stress constraint in the main cables were active at the optimum while the vertical displacement constraint remained inactive.
Figures 6-17 and 6-18 illustrate the evolution of design variables (left) and the objective function (right) for Case Ia:

- For Case Ia, limiting $V_f = 98$ m/s.
- For Case Ia, limiting $V_f = 105$ m/s.

Table 6-3 presents the optimum designs for Case Ia:

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>$d_1$ (mm)</th>
<th>$d_2$ (mm)</th>
<th>$d_3$ (mm)</th>
<th>obj. func.</th>
<th>obj. func. % variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.0</td>
<td>9.140</td>
<td>8.156</td>
<td>13.279</td>
<td>3610.57</td>
<td>-23.70</td>
</tr>
<tr>
<td>105.0</td>
<td>16.579</td>
<td>17.565</td>
<td>14.751</td>
<td>5384.40</td>
<td>13.79</td>
</tr>
</tbody>
</table>

Table 6-3. Optimum designs for Case Ia
Table 6-4. Number of FLAS and Abaqus executions for Case Ia

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>No. of FLAS executions</th>
<th>No. of Abaqus executions</th>
<th>Computational time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.0</td>
<td>97</td>
<td>97</td>
<td>5.3</td>
</tr>
<tr>
<td>105.0</td>
<td>56</td>
<td>56</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Case Ib: Six design variables of the plate thicknesses for both lateral and central boxes girders

In Case Ib, a total of six design variables, three plate thicknesses of the lateral girder ($d_1$ throught $d_3$) and three of the central girder ($d_4$ throught $d_6$), were studied while the sum of the three box girder volumes was considered as the objective function. Figure 6-19 and Figure 6-20 show the evolutions of the design variables and the objective functions for the limiting flutter speed of 98 m/s and 105 m/s respectively while Table 6-5 and Table 6-6 present the optimum designs and the number of FLAS and Abaqus executions for both cases. Just as in Case Ia, the girder design became lighter from the original design as we require smaller limiting flutter velocity of 98 m/s, while the girder became heavier in the contrary case. Compared to Case Ia, the computational time has tripled for the case with the limiting velocity of 105 m/s, while for the limiting $V_f$=98 m/s, it only increased by 20%. The active constraint at the optimum for both cases was the aeroelastic constraint.
Figure 6-19. Evolution of the design variables (left) and the objective function (right) for Case Ib: limiting $V_f = 98$ m/s

Figure 6-20. Evolution of the design variables (left) and the objective function (right) for Case Ib: limiting $V_f = 105$ m/s

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>105.0</td>
<td>16.035</td>
<td>16.302</td>
<td>13.983</td>
<td>17.211</td>
<td>12.001</td>
<td>15.206</td>
<td>6500.92</td>
<td>8.64</td>
</tr>
</tbody>
</table>

Table 6-5. Optimum designs for Case Ib
Table 6-6. Number of FLAS and Abaqus executions for Case Ib

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>No. of FLAS executions</th>
<th>No. of Abaqus executions</th>
<th>Computational time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.0</td>
<td>81</td>
<td>81</td>
<td>6.3</td>
</tr>
<tr>
<td>105.0</td>
<td>129</td>
<td>129</td>
<td>7.1</td>
</tr>
</tbody>
</table>

- **Case II: Seven design variables of the six plate thicknesses as well as the main cable area**

This case deals with an additional design variable of the main cable area to all the design variables in Case Ib while the objective function is the sum of the volumes of the three box girders and the two main cables. Figure 6-21 and Figure 6-22 demonstrate the evolutions of the seven design variables and the objective functions for the limiting flutter velocity of 98 m/s and 105 m/s respectively while Table 6-7 and Table 6-8 summarize the optimum designs and the number of FLAS and Abaqus executions for this case. For representation purposes, the design variable of $d_7$ is multiplied by 10 in the graphs. For the reduced limiting flutter velocity, the girder became lighter from the original design as the previous cases, but for the increased limiting velocity, the objective function did not increase from the original design. This indicates that the volume of the original bridge design can be reduced while achieving a higher flutter velocity than the original design. The computational time has increased compared to Case Ib, especially for the increased limiting velocity, which has been doubled. Both aeroelastic and maximum cable stress constraints were active at the optimum.
Figure 6-21. Evolution of the design variables (left) and the objective function (right) for *Case II*: limiting $V_f = 98$ m/s

Figure 6-22. Evolution of the design variables (left) and the objective function (right) for *Case II*: limiting $V_f = 105$ m/s

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>105.0</td>
<td>19.930</td>
<td>7.687</td>
<td>6.892</td>
<td>29.906</td>
<td>14.743</td>
<td>25.164</td>
<td>1.850</td>
<td>25811.80</td>
<td>-4.06</td>
</tr>
</tbody>
</table>

Table 6-7. Optimum designs for *Case II*
In these deterministic optimization examples, we have demonstrated the feasibility of the optimization problem of the Messina Bridge considering girder plate thicknesses and the main cable area as design variables under aeroelastic flutter constraint. Therefore, in Section 6.7, the reliability based design optimization of the Messina Bridge will be performed considering uncertainties in flutter constraint. Prior to carrying out RBDO, reliability analyses of the bridge flutter will be performed in the next section in order to obtain the probability of failure of the original design due to flutter.

<table>
<thead>
<tr>
<th>$V_f$ (m/s)</th>
<th>No. of FLAS executions</th>
<th>No. of Abaqus executions</th>
<th>Computational time (hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.0</td>
<td>122</td>
<td>122</td>
<td>8.7</td>
</tr>
<tr>
<td>105.0</td>
<td>254</td>
<td>254</td>
<td>14.5</td>
</tr>
</tbody>
</table>

Table 6-8. Number of FLAS and Abaqus executions for Case II
6.6 RELIABILITY ANALYSIS OF THE MESSINA BRIDGE UNDER FLUTTER CONSTRAINT

The formulation of reliability analysis explained in Section 4.3 is now applied to the Messina Bridge example. The objective of this reliability analysis is to obtain the probability of failure of the initial bridge design due to the flutter limit state considering system uncertainties. The relevant reliability parameters such as random variables and the limit state function are defined in the subsequent sections. In this study, the wind velocity, the structural damping as well as the points that define all eighteen flutter derivatives (see Figure 6-11) are considered as random variables. This reliability analysis of the bridge flutter is based on the study carried out by Baldomir [B1] in our research group. For further reading on reliability analyses of the Messina Bridge, refer to a published paper by Baldomir et al. [B2].

6.6.1 Random variables

- Wind velocity

The extreme wind velocity at the bridge location is one of the clearest sources of uncertainty, which affects the wind load that the bridge structure has to support. In the case of Messina Bridge project, the wind data at the bridge site was extrapolated based on the statistical data collected from four different meteorological stations. Additionally, new anemometers were installed near the project site for the acquisition of more accurate data. The structural engineering group at the University of Genoa [D2] processed this huge amount of metrological data and determined that the Gumbel type probability function adjusted the measured data very well. The statistical functions of extreme wind speeds for the Messina Bridge project are described in Section 6.6.3. This extreme wind velocity was used to evaluate the probabilistic flutter constraint in each iteration of the reliability analysis.
• **Structural damping**

In this study, the structural damping, \( x_\varsigma \), is defined as a log-normal distributed random variable: the mean value is 0.00318 and its dispersion is 20% over the mean value. This parameter affects the flutter speed of the bridge structure along with flutter derivatives.

• **Flutter derivatives**

The experimentally obtained 18 flutter derivatives of the Messina Bridge, which takes part in the calculation of flutter speed, can be found in Figure 6-11. In the wind tunnel tests, flutter derivatives are obtained at different reduced velocity. After a line that best fit each flutter derivative is defined, several points are chosen to describe the line. In the calculation of flutter speed, the flutter values between any two points are interpolated by a cubic spline function.

These points that define each flutter derivative are considered to be normally distributed random variable with their defined values as mean values as can be seen in the example of flutter derivative, \( H_2^* \) in Figure 4-13. Different values of standard deviation were considered for this study to observe the sensitivities of the probability of failure over the experimental data dispersion. We considered both constant standard deviation values of 5 and 15% over the mean values as well as linearly variable standard deviations (0 at \( V^* = 0 \) and 15% at \( V^* = 30 \)). Since there are four to seven data points for each flutter derivative (See Figure 6-11), there are as many as 87 random variables associated with flutter derivatives.

### 6.6.2 Limit state function

The limit state function to define the structural failure due to flutter is the difference between the flutter velocity and the extreme wind velocity as:

\[
G(x) = V_f(x_i, x_\varsigma, x_v) - x_v, \quad i = 1, 2, \ldots, n
\]  

(6.4)
Or in its normalized form:

\[ G(x) = \frac{V_f(x_i, x_{\zeta})}{x_w} - 1 \quad i = 1, 2, ..., n \] (6.5)

where \( x \) is a vector of random variables, \( V_f \) is flutter velocity, \( x_i \) is each point that defines the flutter functions, \( x_{\zeta} \) is structural damping, \( x_w \) is extreme wind velocity, \( n \) is a total number of data points. Since there are \( n \) numbers of data points as random variables besides the extreme wind velocity and the structural damping, the vector, \( x \) represents a total of \( n+2 \) random variables in the reliability analysis.

The following section describes the procedure to obtain a normal-equivalent of the extreme wind velocity probability function of the Messina Bridge.

### 6.6.3 Definition of normal-equivalent density function

According to the studies carried out by the University of Genoa, the Gumbel probability density function best describes the extreme wind velocity data collected near the project site. The probability density function and the cumulative distribution functions are defined as:

\[ f_{x_w}(x_w) = \frac{1}{\beta} \exp\left(-\frac{x_w - \mu}{\beta}\right) \cdot \exp\left[-\exp\left(-\frac{x_w - \mu}{\beta}\right)\right] \] (6.6)

\[ F_{x_w}(x_w) = \exp\left[-\exp\left(-\frac{x_w - \mu}{\beta}\right)\right] \] (6.7)

where \( \beta \) is 3.69 and \( \mu \) is 26.393 m/s. The mean value and the standard deviation of the random variable are:

\[ \mu_{x_w} = \mu + \gamma \beta = 28.5229 \text{ m/s} \]
\[ \sigma_{x_i} = \frac{\beta \pi}{\sqrt{6}} = 4.7327 \text{ m/s} \]

where \( \gamma \) is Euler-Mascheroni constant, which is 0.5772156649.

As explained in Section 2.5.6, Rackwitz-Fiessler method is applied to estimate the normal equivalent mean and standard deviation values. This method imposes that at the design point, \( x^* \), the cumulative distribution function (CDF) as well as the probability distribution function (PDF) for both the non-normal and the normal functions to be equivalent. Equation (2.35) and (2.37) are applied at the design point to obtain the normal equivalent standard deviation and mean value as:

\[ \sigma_{x_i} = \frac{\phi\left(\Phi^{-1}\left[F_{x_i}(x_i^*)\right]\right)}{f_{x_i}(x_i^*)} = 4.525 \text{ m/s} \]

\[ \mu_{x_i} = x_i^* - \sigma_{x_i} \Phi\left(F_{x_i}(x_i^*)\right) = 27.720 \text{ m/s} \]

Figure 6-23 and Figure 6-24 show the Gumbel function and its normal equivalent for PDF and CFD.
As explained in Section 5.6.4, the transformation from Gumbel to its normal-equivalent was not performed in each iteration of reliability analysis because the MPP is
located more than 3 standard deviations away and the pdf of such point is very small. Thus, the mean value and the standard deviation of the normal equivalent at the Gumbel mean value were used throughout the reliability analysis.

### 6.6.4 Calculation of reliability

Before performing the reliability analysis, the following steps are necessary:

- To obtain the 18 flutter derivatives experimentally in a wind tunnel
- To run a non-linear FE model of the entire bridge under the self-weight to obtain a stiffness matrix and to perform a modal analysis.

The detailed explanation of the procedure of reliability analysis is in Chapter 2. Once the flutter derivatives and the modal information of the bridge are obtained, the reliability analysis is performed as follows. The flow chart of Figure 6-25 shows the outline of the process.

1. Set the mean values of the random variables as an initial design point, i.e., 
   \[ x_0 = \mu_X. \]
2. Run the FLAS code to obtain flutter velocity with the corresponding random variable values. The natural frequencies and modal information of the bridge are used in the calculation.
3. Evaluate the limit state function, \( G(x_k). \)
4. Compute the gradients of the limit state function, \( \nabla G(x_k). \) The partial derivatives of \( V_f \) with respect to the flutter derivatives points and structural damping are evaluated by the finite difference method. Thus this step requires launching the FLAS as many times as the number of the finite difference evaluations.
5. Calculate the reliability index, \( \beta. \)
6. Calculate the new point of the flutter derivatives, \( x_{k+1} \) using Eqn. (4.19).
7. Repeat the step 1-6 until \( \beta \) converges.
8. Obtain the MPP and $x^*$, and compute the probability of failure, $P_f$.

![Flow chart of the reliability analysis of the Messina Bridge](image)

**Figure 6-25. Flow chart of the reliability analysis of the Messina Bridge**

### 6.6.5 Cases of the reliability analysis

Various cases of random variables were considered in this study to investigate the influences of each random variable on overall structural safety as follows.

*Case A*: the wind velocity, $x_w$ as a single random variable

*Case B*: $x_w$ and the structural damping, $x_\varsigma$

*Case C*: $x_w$, $x_\varsigma$ and the points from one of the 18 flutter derivatives
Case D: $x_w$, $x_\varsigma$ and and the points from 6 flutter derivatives from the same type, $A^*$, $H^*$, or $P^*$

Case E: $x_w$, $x_\varsigma$ and all the data points from 18 flutter derivatives

The termination criterion for all the reliability analyses is set so that the difference of any two consecutive beta values should be smaller than 1E-4.

6.6.6 Reliability results

- **Case A: wind velocity as a random variable**

In this case, only the wind velocity, $x_w$ was considered as a single random variable. Since none of the flutter derivatives values vary, $V_f$ takes a deterministic value. The reliability index can be calculated directly from the equation presented in the previous chapter since the gradient of the limit state function is constant.

Using the limit state function of non-normalized form (Equation (6.4)), its gradient is constant.

$$\frac{\partial g(x_w)}{\partial x_w} = -1 \quad \text{or} \quad \frac{\partial g}{\partial u_w} = -\sigma_w$$

$$\lambda = \frac{1}{\sqrt{\sum_{i=1}^{n} \left( \frac{\partial g}{\partial x} (u^*) \right)^2}} = \frac{1}{\sigma_w}$$

Therefore,

$$\beta = \frac{g(u^*) - \sum_{i=1}^{n} \frac{\partial g(u^*)}{\partial u_i} u_i^*}{\sqrt{\sum_{i=1}^{n} \left( \frac{\partial g(u^*)}{\partial u_i} \right)^2}} = \frac{0 - \left( -\sigma_w \cdot \frac{V_f - \mu_w}{\sigma_w} \right)}{\sigma_w} = 16.538$$
This reliability index value corresponds to a probability of failure, $P_f = 9.45 \times 10^{-62}$.

- **Case B: $x_w$ and the structural damping, $x_\varsigma$ as random variables**

  In the second case, the random variable of structural damping $x_\varsigma$ was added to the reliability analysis. The resulting reliability index was slightly reduced to $\beta = 16.47$ ($P_f = 2.91 \times 10^{-61}$) from Case A. The reliability index and the performance function converged quickly as shown in Figure 6-26.

![Figure 6-26. The evolutions of beta and the performance function for Case B](image)

- **Case C: $x_w$, $x_\varsigma$ and the points from one of the 18 flutter derivatives**

  Case C is a situation in which the points that define one of the 18 flutter derivatives are added as random variables to those in Case B. This is not a very realistic case because all 18 flutter derivatives contain uncertainties for being obtained from experimental tests. However, this case provides us very useful information of which flutter derivatives have more influence on structural safety of the bridge. The reliability analyses were performed for each case of the 18 flutter derivatives considering four different data dispersions: constant dispersions of 5% and 15% over the mean values and linearly variable dispersions of 0 at $V^* = 0$ and 15% and 30% over the mean values.
at $V^*=30$. The total number of reliability analyses was 72. The obtained results are shown in Table 6-9.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables (N)</th>
<th>$\sigma_{max}=0.15\mu_x$</th>
<th>$\sigma_{max}=0.3\mu_x$</th>
<th>const. $\sigma=5%$</th>
<th>const. $\sigma=15%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_w, x, A_{1,1}$</td>
<td>8</td>
<td>16.436</td>
<td>16.380</td>
<td>16.461</td>
<td>16.423</td>
</tr>
<tr>
<td>$x_w, x, A_{2,1}$</td>
<td>8</td>
<td>14.795</td>
<td>12.418</td>
<td>16.303</td>
<td>13.517</td>
</tr>
<tr>
<td>$x_w, x, A_{3,1}$</td>
<td>8</td>
<td>12.573</td>
<td>10.875</td>
<td>15.267</td>
<td>12.177</td>
</tr>
<tr>
<td>$x_w, x, A_{4,1}$</td>
<td>8</td>
<td>16.469</td>
<td>16.465</td>
<td>16.465</td>
<td>16.462</td>
</tr>
<tr>
<td>$x_w, x, A_{5,1}$</td>
<td>5</td>
<td>16.460</td>
<td>16.431</td>
<td>16.465</td>
<td>16.460</td>
</tr>
<tr>
<td>$x_w, x, A_{6,1}$</td>
<td>7</td>
<td>16.445</td>
<td>16.386</td>
<td>16.465</td>
<td>16.437</td>
</tr>
<tr>
<td>$x_w, x, H_{1}^{+}$</td>
<td>7</td>
<td>16.405</td>
<td>16.268</td>
<td>16.456</td>
<td>16.371</td>
</tr>
<tr>
<td>$x_w, x, H_{2}^{+}$</td>
<td>8</td>
<td>15.624</td>
<td>14.321</td>
<td>16.254</td>
<td>14.879</td>
</tr>
<tr>
<td>$x_w, x, H_{3}^{+}$</td>
<td>7</td>
<td>16.044</td>
<td>15.546</td>
<td>16.379</td>
<td>15.974</td>
</tr>
<tr>
<td>$x_w, x, H_{4}^{+}$</td>
<td>7</td>
<td>16.470</td>
<td>16.467</td>
<td>16.466</td>
<td>16.465</td>
</tr>
<tr>
<td>$x_w, x, H_{5}^{+}$</td>
<td>6</td>
<td>16.224</td>
<td>15.813</td>
<td>16.430</td>
<td>16.198</td>
</tr>
<tr>
<td>$x_w, x, H_{6}^{+}$</td>
<td>6</td>
<td>16.470</td>
<td>16.470</td>
<td>16.466</td>
<td>16.472</td>
</tr>
<tr>
<td>$x_w, x, P_{1}^{+}$</td>
<td>5</td>
<td>16.470</td>
<td>16.466</td>
<td>16.467</td>
<td>16.467</td>
</tr>
<tr>
<td>$x_w, x, P_{2}^{+}$</td>
<td>6</td>
<td>16.470</td>
<td>16.467</td>
<td>16.466</td>
<td>16.466</td>
</tr>
<tr>
<td>$x_w, x, P_{3}^{+}$</td>
<td>7</td>
<td>15.506</td>
<td>14.537</td>
<td>16.275</td>
<td>15.478</td>
</tr>
<tr>
<td>$x_w, x, P_{4}^{+}$</td>
<td>7</td>
<td>16.466</td>
<td>16.453</td>
<td>16.465</td>
<td>16.463</td>
</tr>
<tr>
<td>$x_w, x, P_{5}^{+}$</td>
<td>7</td>
<td>16.432</td>
<td>16.311</td>
<td>16.464</td>
<td>16.419</td>
</tr>
<tr>
<td>$x_w, x, P_{6}^{+}$</td>
<td>6</td>
<td>16.439</td>
<td>16.353</td>
<td>16.461</td>
<td>16.432</td>
</tr>
</tbody>
</table>

Table 6-9. Reliability results of Case B, the data points that define only one flutter derivative are considered as random variables besides $x_w$ and $x$.

It can be clearly seen that among all 18 flutter derivatives, $A_{3}^{*}$ is the most significant on the structural safety of the Messina bridge. For the data dispersion of $0.3\mu_x$ at $V^*=30$, the reliability index of $A_{3}^{*}$ is 10.875, which corresponds to the
probability of failure of 7.58E-28. $A_2^*$ is the second most influential while $H_2^*$ and $P_3^*$ are important among the flutter derivatives of the same type. It is interesting to note that these most relevant flutter derivatives have sub index of 2 or 3, which means that they are associated with the rotation of the bridge deck as can be seen in Eqn. (4.1). The rest of the flutter derivatives have minimum influence on the structural safety. For the computation of reliability analysis of $A_3^*$, 30 iterations of the reliability routine were performed until the convergence using the reduction factor of $c=3$ while FLAS was executed a total of 240 times. The computational time in this case was approximately 4 hours. Figure 6-27 shows the evolutions of beta and the limit state function without the reduction factor while Figure 6-28 shows the case with the reduction factor of $c=3$. As can be seen, we had difficulties achieving convergence without the reduction factor as $\beta$ value oscillated and the limit state function did not reduce beyond 0.01. On the other hand, using the reduction factor of $c=3$, $\beta$ converged well with null limit state function value.
Figure 6-27. Evolutions of: a) $\beta$ and b) $G(x)$ for $c=1$
**Case D:** $x_w$, $x_\omega$ and the data points from six flutter derivatives of the same type, $A^*$, $H^*$, or $P^*$

In this case, the six flutter derivatives of the same type were considered as random variables at the same time in addition to the extreme wind velocity and the structural damping. The $A^*$ type flutter derivatives define aeroelastic moments about the longitudinal axis of the bridge deck, which produces torsional movements of the deck. The $H^*$ type derivatives involve in aeroelastic lift force, which causes vertical displacement of the deck. The $P^*$ type derivatives describe an aeroelastic drag force, which produces lateral movements of the deck (Equation 4.1, Figure 4-5). The obtained results are tabulated in Table 6-10.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables ($N$)</th>
<th>$\beta_{\sigma_{max}=0.15\mu_x}$</th>
<th>$\beta_{\sigma_{max}=0.3\mu_x}$</th>
<th>$\beta_{\sigma=5%}$</th>
<th>$\beta_{\sigma=15%}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_w, x_\omega, A^<em>_1, A^</em>_2, A^<em>_3, A^</em>_4, A^*_5$</td>
<td>34</td>
<td>12.472</td>
<td>10.779</td>
<td>15.528</td>
<td>12.004</td>
</tr>
<tr>
<td>$x_w, x_\omega, H^<em>_1, H^</em>_2, H^<em>_3, H^</em>_4, H^<em>_5, H^</em>_6$</td>
<td>30</td>
<td>15.453</td>
<td>14.301</td>
<td>16.154</td>
<td>14.651</td>
</tr>
</tbody>
</table>

![Figure 6-28. Evolutions of: a) $\beta$ and b) $G(x)$ for $c=3$](image)
Table 6-10. Reliability results: 6 flutter derivatives of the same type are considered as random variables as well as $x_w$ and $x_\varsigma$.

The most significant flutter derivative type on the structural safety is clearly $A^*$ with $\beta=10.779$ ($P_f=2.16E-27$) for the case with $\sigma=0.3\mu_x$ due mainly to the reduction in reliability caused by $A_3^*$ and $A_2^*$. For the case of type $A^*$, 31 iterations of reliability analysis were necessary for the convergence of the algorithm using the step size reduction factor of $c=4$ while the number of execution of FLAS was 1054. The computational time for the reliability analysis was approximately 22 hours.

It is worth mentioning that for the results of Table 6-10, the aeroelastic mode that causes flutter for all three cases were different. In the case of flutter derivative type $A^*$, the aeroelastic mode 2 causes flutter, while for the other cases of type $H^*$ and $P^*$, the mode 6 causes flutter. Figure 6-29 through Figure 6-31 shows the real part of eigenvalues at MPP for each case.

| $x_w$, $x_\varsigma$, $P_1^*$, $P_2^*$, $P_3^*$, $P_4^*$, $P_5^*$ | 27 | 15.488 | 14.507 | 16.267 | 15.445 |

Figure 6-29. Flutter mode at MPP considering all $A^*$ type flutter derivatives.
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Figure 6-30. Flutter mode at MPP considering all $H^*$ type flutter derivatives

Figure 6-31. Flutter mode at MPP considering all $P^*$ type flutter derivatives
• **Case E: \(x_w, x_\zeta\) and all the data points from 18 flutter derivatives**

The last case is the most realistic situation since all the points that define flutter derivatives are considered as random variables. The total number of random variables is 90 and the reliability index was computed for four different standard deviations of the experimental data and the results are presented in Table 6-11.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables ((N))</th>
<th>(\sigma_{\text{max}}=0.15\mu_x) linear variation</th>
<th>(\sigma_{\text{max}}=0.3\mu_x) linear variation</th>
<th>const. (\sigma=5%)</th>
<th>const. (\sigma=15%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_w, x_\zeta, A_1, \ldots, A_L, H_1, \ldots, H_6, P^I_1, \ldots, P^I_6)</td>
<td>90</td>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12.300</td>
<td>10.539</td>
<td>15.217</td>
<td>12.163</td>
</tr>
</tbody>
</table>

**Table 6-11. Reliability results: all the flutter derivatives as random variables**

The reliability index in the case with variable \(\sigma_{0.3\mu_x}\) is reduced to 10.539 \((P_f = 2.86\times10^{-26})\). The number of FLAS executions in this case was 2970, and the convergence was achieved after 34 iterations using a reduction step factor of \(c=4\). The computational time for the reliability analysis was approximately 36 hours.

It should be noted that for the constant 15% dispersion case, the resulting \(\beta\) in **Case E** is larger than the case in which only \(A^*\) set as random variables in **Case D**. If we compare these two cases in Figure 6-29 and Figure 6-32, we can see that the aeroelastic mode that cause flutter in **Case D** at MPP is the mode 2 while in **Case E** the mode 6 is causing flutter. Therefore even though the level of uncertainties is higher in **Case E**, \(\beta\) in this case is larger due to the change in the combination of modes that produces flutter instability.
Table 6-12 shows two random variable sets of all flutter derivative and $A_3^*$ with and without structural damping for the case of 15% variable dispersion. It should be noted that for random variable set of $A_3^*$, $\beta$ with $x_\varsigma$ is larger than $\beta$ without $x_\varsigma$. This occurs because the aeroelastic mode that causes flutter for both cases is different: mode 2 with $x_\varsigma$ and mode 6 without $x_\varsigma$.

As can be observed, the structural damping has the minimum effect on the overall structural reliability: the differences between the cases with and without $x_\varsigma$ for both random variable sets are less 1%. Moreover, in the cases without $x_\varsigma$, since the difference between the random variables set of all flutter derivatives and only $A_3^*$ is 1.2%, in the subsequent section of RBDO, the seven random variables of $A_3^*$ and $x_w$ are considered as random variables.
The reliability index values that we are dealing in this section are very large due to the significant difference between the mean wind speed at the bridge location \( \mu_w = 28.52 \text{ m/s} \) and the deterministic flutter wind speed \( V_f = 102.72 \text{ m/s} \). Moreover, the flutter speed of the original bridge design is set to very high due to the design requirement of the project.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>no. of variables ((N))</th>
<th>variable ( \sigma 0.15\mu_x )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_w, x_c, A_1, ..., A_6, H_1, ..., H_6, P_1', ..., P_6' )</td>
<td>90</td>
<td>12.299</td>
<td></td>
</tr>
<tr>
<td>( x_w, A_1', ..., A_6', H_1', ..., H_6', P_1', ..., P_6' )</td>
<td>89</td>
<td>12.300</td>
<td></td>
</tr>
<tr>
<td>( x_w, x_c, A_{3,1}, ..., A_{3,n} )</td>
<td>8</td>
<td>12.573</td>
<td></td>
</tr>
<tr>
<td>( x_w, A_{3,1}', ..., A_{3,n}' )</td>
<td>7</td>
<td>12.450</td>
<td></td>
</tr>
</tbody>
</table>

Table 6-12. Comparison of reliability results between with and without \( x_c \)
6.7 RELIABILITY BASED DESIGN OPTIMIZATION OF
THE MESSINA BRIDGE UNDER FLUTTER CONSTRAINT

The reliability based design optimization formulations under flutter constraint explained in detail in Section 4.4 are now applied to the proposed Messina Bridge example in this section. The RBDO is carried out to minimize the cost function, which is either the volume of the box girders or the sum of the volumes of the girders and the main cables depending on the design variables set while satisfying a predetermined structural reliability level under flutter. Three RBDO methods of RIA, PMA and SORA are employed to study this example. The RBDO of the Messina Bridge can be found in a paper published by Kusano et al. [K1]. The RBDO parameters considered in this study are described in the following section.

6.7.1 Definition of RBDO parameters

- **Design variables**

  Two cases of design variable sets are studied: *Case I* in which only plate thicknesses of the girders are considered as design variables and *Case II* in which the main cable cross sectional area is added to the variable set in *Case I*.

  Two subcases for *Case I* were considered. *Case Ia* involves in three design variables of the plate thicknesses for each edge of the lateral box girders, and *Case Ib* with six design variables of the plate thicknesses for each edge of both lateral and central box girders. For the three design variables case, the thicknesses of the central box were maintained as the original design. Figure 6-33 shows the location of the three design variables \( d_1 \) through \( d_3 \) as well as the six design variables \( d_1 \) through \( d_6 \).

  In *Case II*, the main cable area was included as an additional design variable to the six design variable case in *Case Ib*, resulting in a total of seven design variables.
Objective function

The bridge deck of the Messina Bridge consists of two lateral and one central box girders. The objective function for Case Ia is the volume of the two lateral girders while for Case Ib, it is the volume of the two lateral and the central girders. For Case II, the objective function is the sum of the volumes of the girders and the main cables.

Random variables

Among all the random variables studied in the reliability analyses of the Messina Bridge in the previous section, the extreme wind velocity and the flutter derivative $A_3^*$ were chosen to perform the RBDO on this bridge. The structural damping was discarded for its small influence over the structural reliability. According to the reliability study in the previous section, $A_3^*$ was found to be the most influential parameter on the reliability index of the bridge. Since the difference in $\beta$ between the random variable set with all flutter derivatives and another set with only $A_3^*$ was only 1.2% for the linearly variable dispersion of 0.15$\sigma$, the RBDO was performed with these seven random variables of $A_3^*$ as well as the extreme wind velocity. Refer to Section 6.6.6 Case E for details.

The normal-equivalent of the Gumbel-type extreme wind velocity function is defined in Section 6.6.3 along with its equivalent mean value and standard deviation. For the flutter derivative points, their defined values are considered as mean values with linearly variable standard deviations with respect to the reduced velocity (0% at $V^*=0$ to 15% of mean value at $V^*=30$).

Limit state function
The limit state function is the aeroelastic constraint, which is defined as:

\[
G(x) = \frac{V_f(x)}{x_w} - 1 \quad i = 1, 2, \ldots, n
\]  

(6.8)

Since there are \( n \) number of random variables associated with flutter derivatives, the random variable vector, \( x \) represents a total of \( n+1 \) random variables.

### 6.7.2 Problem formulation

One of the main characteristics of the RBDO formulations is the use of probabilistic constraints, in which system uncertainties are taken into account and an engineer can set a desired level of structural reliability. In our case, the probabilistic aeroelasticity constraint limits the system failure due to flutter below the predetermined values of probability of failure. The reference value of reliability index of the original design is \( \beta = 12.45 \) considering as random variables, wind velocity and flutter derivative data of \( A_3 \) with linearly variable dispersion of 0.15\( \mu \) at \( V^* = 30 \). This reliability index represents the probability of failure of 7.0E-36. In the calculation of the RBDO, a bridge designer can choose a desired target reliability index value. If we reduce the target \( \beta \) from the original design, we expect the design variables to decrease. On the other hand, if we increase the target \( \beta \), the design variable should increase in order to achieve greater structural safety.

Two main cases of the RBDO formulations of the Messina Bridge example were studied depending on the design variable set. In Case I, the design variables are plate thicknesses of the box girders while the objective function is the volume of the girders as follows.

**Case I:** Girder plate thicknesses as design variables

Min: Area (\( d \)) \hspace{1cm} (6.9a)
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\[ g_1 : P\left[ G(x) \leq 0 \right] \leq P_f \quad \text{where} \quad G(x) = \frac{V_f(x)}{x_w} - 1 \quad (6.9b) \]

\[ g_{2,i}^I : 5.0 \text{ mm} \leq d_i \leq 30.0 \text{ mm} \quad i = 1, 2, \ldots, 6 \quad (6.9c) \]

\[ g_3 : \sigma_c = 800 \text{ Mpa} \quad (6.9d) \]

\[ g_4 : \frac{z_d}{z_{\max}} - 1 \leq 0 \quad \text{with} \quad z_{\max} = \frac{L}{500} ; L=3300 \text{ m} \quad (6.9e) \]

The constraint \( g_1 \) is the probabilistic flutter constraint, in which the target reliability is defined in terms of \( P_f \). The constraint \( g_{2,i}^I \) are the side constraints of the steel plate, while \( g_4 \) limits the maximum vertical deck displacement under the static overload case. \( g_3 \) is an equality constraint of the maximum main cable stress as explained in Section 4.4.3.2, which assigns the main cable area whenever girder weight varies so that the maximum tensile stress of the cable is 800 MPa under the static overload case.

In Case II, we take into account the main cable area as an additional independent design variable to those in Case I. The objective function in this case is the sum of volumes of the box girders and the two main cables. The new constraint \( g_{2,7}^{II} \) is added to define the side constraints of the main cable area. The constraint \( g_4 \) in this case restricts the maximum main cable tensile stress by an inequality constraint. The main purpose of this case is instead of assuming that the maximum cable stress constraint is active at the optimum as in Case I, we restrict the maximum cable stress by an inequality constraint for having the cable area as an independent design variable.

**Case II**: Steel plate thicknesses and cable area as design variables

\[ \text{Min: Volume (d)} \quad (6.10a) \]
\[ g_1 : P[G(x) \leq 0] \leq P_f \quad \text{where} \quad G(x) = \frac{V(x)}{x_w} - 1 \quad (6.10b) \]

\[ g_{2,i}^H : 5.0 \text{ mm} \leq d_i \leq 30.0 \text{ mm} \quad i = 1, 2, \ldots, 6 \quad (6.10c) \]

\[ g_{2,7}^H : 1.0 \text{ m}^2 \leq d_7 \leq 2.0 \text{ m}^2 \quad (6.10d) \]

\[ g_3^F : \sigma_c \leq 800 \text{ Mpa} \quad (6.11e) \]

\[ g_4 : \frac{z_d}{z_{\text{max}}} - 1 \leq 0 ; \quad z_{\text{max}} = \frac{L}{500} ; \quad L = 3300 \text{ m} \quad (6.11f) \]

### 6.7.3 RBDO cases

There are three cases of the RBDO problems in this research depending on the number of design variables. In each case, the RBDO was first performed on the bridge structure with the reliability index of the original bridge design of \( \beta^T = 12.45 \) so that its optimum design can serve to compare with the results with increased and reduced target reliability of \( \beta^T = 13.0 \) and \( \beta^T = 12.0 \).

**Case Ia:** three design variables of the lateral box girder plate thicknesses for \( \beta^T = 12.45, \beta^T = 13.0 \) and \( \beta^T = 12 \).

**Case Ib:** six design variables consisting of three plate thicknesses of the lateral box and three plate thicknesses of the central box girder for \( \beta^T = 12.45, \beta^T = 13.0 \) and \( \beta^T = 12 \).

**Case II:** seven design variables of the six design variables of Case Ib as well as the main cable area for \( \beta^T = 12.45, \beta^T = 13.0 \) and \( \beta^T = 12 \).
6.7.4 RBDO results

- **Case Ia-1: three design variables of the lateral box girder plate thicknesses for $\beta^T=12.45$**

In *Case Ia*, three design variables of the lateral box girder, $d_1$ through $d_3$ in Figure 6-33, were taken into account while the plate thicknesses of the central box girder were maintained as the original design. The objective function is the volume of the two lateral box girders. The original lateral box girder design is $d=[14.0, 14.0, 14.0]$ (in mm) and its corresponding volume of the two box girders is $4732.07 \text{ m}^3$.

First of all, the RBDO was performed on the original bridge design with the target reliability of $\beta^T=12.45$ in order to use this subcase to compare with the other target reliability cases. Table 6-13 shows the optimum designs and the objective functions using the three RBDO methods. The design variables of RIA and PMA did not vary very much from the initial design; however, the short edge of $d_3$ using SORA has decreased by 17%. All the objective functions have converged to a very similar value, which has increased approximately by 0.7% from the original design. On the other hand, the two main cable volumes have decreased by about 11% from the original design due to the maximum cable stress constraint. Thereby, the total volumes of the box girders and the main cables using the three methods have decreased by approximately 9%. The optimum design using PMA is used to compare with increased and reduced target reliability cases subsequently.

<table>
<thead>
<tr>
<th>$\beta^T=12.45$</th>
<th>$(d_i \text{ in mm}, \text{ obj. func. in m}^3)$</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$d_1$</td>
<td>$d_2$</td>
</tr>
<tr>
<td>RIA</td>
<td>14.084</td>
<td>14.257</td>
</tr>
<tr>
<td>PMA</td>
<td>14.152</td>
<td>14.184</td>
</tr>
<tr>
<td>SORA</td>
<td>14.110</td>
<td>14.510</td>
</tr>
</tbody>
</table>

**Table 6-13. Optimum designs for the three design variables case ($\beta^T=12.45$)**
• **Case Ia-2: three design variables of the lateral box girder plate thicknesses for $\beta^T=13.0$**

In this subcase, the target reliability index was set to $\beta^T=13.0$, which was increased from the reference value of the original bridge design. This target reliability value was chosen arbitrarily by the author. We would like to see how the girder design changes by varying the target reliability index.

Figure 6-34 and Figure 6-35 present the evolution of the design variables and the objective functions using the three RBDO methods. Table 6-14 summarizes the optimum designs as well as the variations in the objective functions compared to the original design while Table 6-15 shows the number of FLAS and Abaqus executions for each method.

As can be seen, the objective functions using the three methods converged to very similar values; however, the optimum design variables are different among the three methods. RIA and PMA present similar tendency; $d_1$ and $d_2$ are much larger than $d_3$ while for SORA, the optimum values of the three design variables are in the closer range. Although the values of $d_3$ vary among the three methods, it has little effect on the objective function because of its short length. Moreover, the variation in thickness of this segment does not influence the aeroelastic behavior of the bridge notably. By increasing the reliability index from 12.45 to 13.0, the objective functions of the lateral box girders volume have increased by 13.5 to 14% compared to Case Ia-1. The volume of the two main cables has increased by 4.5% due to the maximum stress constraint in the main cables. Since the cable volume constitutes about 78% of the total sum of the main cables and girder volumes, the variation in the volume of the complete bridge is dominated by the change in the main cable volume, which has increased by approximately 6%. The probabilistic flutter constraint is active at the optimum for all methods.
The numbers of FLAS executions for RIA and PMA are similar while that of SORA is about 30% of RIA. However, the number of Abaqus executions of SORA is much larger than those by RIA or PMA. This is because the number of FLAS and Abaqus executions is linked to the formulations of each RBDO method as explained in Chapter 5. Time duration of a FLAS execution is based on various factors such as the number of aeroelastic modes, the number of nodes along the bridge deck from the FE model to consider in flutter calculation as well as the flutter velocity of the bridge for a particular design. It takes approximately one minute for the original Messina Bridge ($V_f=102.72$ m/s). For the Abaqus FE model, it takes about 40 seconds to run either static or modal analyses, which are launched simultaneously. The computational time of PMA and RIA are similar while that of SORA is about 40% lower. The number of reliability iterations for RIA varies between 15 and 16 while that of PMA varies between 15 and 19. The number of reliability iterations of SORA fluctuates between 9 and 26. In order to achieve convergence, the reduction factor of $c=3$ was used for RIA (Section 2.5.2.1) while the reduction factors of $r=0.5$ and $r=0.75$ (Section 3.2.2.1) were used for PMA and SORA respectively.
Figure 6-34. Evolution of the three design variables ($\beta^T=13$)
Figure 6-35. Evolution of the objective functions for the three design variables case ($\beta^T=13$)

<table>
<thead>
<tr>
<th>$\beta^T=13$</th>
<th>$d_1$ in mm, obj. func. in m$^3$</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>obj. func.</td>
<td>total volume</td>
</tr>
<tr>
<td>RIA</td>
<td>16.948</td>
<td>5431.33</td>
</tr>
<tr>
<td>PMA</td>
<td>16.815</td>
<td>5412.78</td>
</tr>
<tr>
<td>SORA</td>
<td>16.312</td>
<td>5408.16</td>
</tr>
</tbody>
</table>

Table 6-14. Optimum designs for the three design variables case ($\beta^T=13$)

<table>
<thead>
<tr>
<th>$\beta^T=13$</th>
<th>$d_1$ in mm, obj. func. in m$^3$</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>obj. func.</td>
<td>total volume</td>
</tr>
<tr>
<td>RIA</td>
<td>16.948</td>
<td>5431.33</td>
</tr>
<tr>
<td>PMA</td>
<td>16.815</td>
<td>5412.78</td>
</tr>
<tr>
<td>SORA</td>
<td>16.312</td>
<td>5408.16</td>
</tr>
</tbody>
</table>

Table 6-15. Number of FLAS and Abaqus executions for the three design variables case ($\beta^T=13$)

<table>
<thead>
<tr>
<th>$\beta^T=13$</th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>2786</td>
<td>2870</td>
<td>883</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>22</td>
<td>23</td>
<td>260</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>44.4</td>
<td>47.0</td>
<td>28.5</td>
</tr>
</tbody>
</table>
Case Ia-3: three design variables of the lateral box girder plate thicknesses for $\beta^T=12.0$

In this subcase, the target reliability index was reduced to $\beta^T=12$ from the reference value of $\beta=12.45$. Figure 6-36 and Figure 6-37 show the evolution of design variables and the objective functions while Table 6-16 and Table 6-17 summarize the optimum designs and the number of FLAS and Abaqus executions.

The objective function has decreased as expected since we have reduced the target reliability. The optimum objective functions among the three methods are very similar; however the distributions of the girder material among the three methods are distinct. The PMA and RIA have similar optimum design variables while those of SORA are different, especially the design variable of $d_3$. By reducing the target reliability to $\beta^T =12$, we have achieved the reduction in the objective function of the lateral girder volume by as much as 11.8% compared to the design for $\beta^T =12.45$ while the reduction of the main cable volume was approximately 4% due to the main cable constraint. In result, the total reduction in the girders and main cables volumes was around 5.3%.

The computational time of SORA is the lowest, which is about half of RIA. The number of reliability iterations of RIA varied between 15 and 16 while that of PMA fluctuated between 11 and 22. The number of reliability iterations for SORA varied between 8 and 12. During the reliability routine of RIA and SORA, the reduction factor of $c=3$ and $r=0.5$ were employed to achieve convergence. The aeroelastic constraint as well as the maximum cable stress constraint was active at the optimum for all three methods.
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a) RIA

b) PMA
Figure 6-36. Evolution of the three design variables ($\beta^T = 12$)

Figure 6-37. Evolution of the objective functions for the three design variables case ($\beta^T = 12$)
Table 6-16. Optimal designs for the three design variables case ($\beta^T = 12$)

<table>
<thead>
<tr>
<th>$\beta^T$=12</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA</td>
<td>11.728</td>
<td>11.294</td>
<td>13.672</td>
<td>4214.43</td>
<td>-11.59</td>
</tr>
<tr>
<td>PMA</td>
<td>11.701</td>
<td>11.229</td>
<td>13.694</td>
<td>4201.24</td>
<td>-11.87</td>
</tr>
<tr>
<td>SORA</td>
<td>11.351</td>
<td>12.156</td>
<td>8.569</td>
<td>4206.00</td>
<td>-11.77</td>
</tr>
</tbody>
</table>

Table 6-17. Number of FLAS and Abaqus executions for the three design variables case ($\beta^T = 12$)

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>2772</td>
<td>2562</td>
<td>752</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>28</td>
<td>34</td>
<td>486</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>40.9</td>
<td>35.3</td>
<td>22.0</td>
</tr>
</tbody>
</table>

- Case 1b-1: six design variables of the lateral and central box girders plate thicknesses for $\beta^T = 12.45$

In Case 1b, the number of design variables was increased to six including all edges of the lateral and central box girders, $d_i$ through $d_6$ as shown in Figure 6-33. The objective function is the volume of the three box girders, and the initial design of the box girders is $d=[14.0, 14.0, 14.0, 16.0, 12.0, 14.0]$ (in mm) and its corresponding volume of the three box girders is 5983.75 m$^3$.

The target reliability was set to $\beta^T =12.45$ in order to compare this results with those in the following subcases with increased or reduced target reliability indices. Table 6-18 shows the optimum designs using the three RBDO methods. In general, the design variables of the central girder have reduced greatly, for some cases to the minimum values. The objective functions using PMA and SORA have decreased by about 3% while that of RIA has been reduced by only 0.2%. The reductions in the main cable volume due to the maximum cable stress constraint were about 12%, and the reduction in the total volume taking into account the three box girders and the two main
cables were approximately 10%. As in the three-design variable case, the results in the following subcases will be compared to this result by PMA.

<table>
<thead>
<tr>
<th>$\beta^T$=12.45</th>
<th>$d_1$ in mm, obj. func. in m$^3$</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA 14.194</td>
<td>18.264</td>
<td>6.469</td>
</tr>
<tr>
<td>PMA 14.434</td>
<td>16.818</td>
<td>15.630</td>
</tr>
<tr>
<td>SORA 15.609</td>
<td>16.374</td>
<td>12.780</td>
</tr>
</tbody>
</table>

Table 6-18. The baseline design for the six design variables case ($\beta^T$=12.45)

- **Case Ib-2: six design variables of the lateral and central box girders plate thicknesses for $\beta^T$=13**

In this subcase, we have increased the target reliability to $\beta^T$=13. Figure 6-40 and Figure 6-41 show the evolutions of design variables and objective functions for the three methods while Table 6-19 and Table 6-20 summarize the optimum design values and the number of FLAS and Abaqus executions.

In general, the plate thicknesses of the lateral girders, $d_1$ through $d_3$ have increased while those of the central box girder, $d_4$ through $d_6$ have decreased. Since the lateral box girders have greater impact on the torsional inertia of the girder, the increment of these design variables contributes to greater flutter speeds. By increasing the reliability index to $\beta^T$=13, the objective function of the girder volume by RIA, for example, has increased approximately by 9% compared to the design of Case Ib-1. After taking into account the reduction in the main cables volume according to the cable constraint, the overall increase of the girder and the main cables volume was 5.6%. The probabilistic flutter constraint was active for all three methods.

For the reliability routine of RIA, the reduction factor of $c$=3 was used while for PMA and SORA, $r$=0.5 and $r$=0.75 were employed. The number of reliability iterations for RIA was 15 or 16, while that for PMA varied between 15 and 27. For SORA it
varied between 12 and 27. The high number of FLAS executions for PMA is due to the slow convergence of the algorithm during the reliability analysis. The computational time of SORA is five to seven times smaller than that of RIA and PMA respectively.
Figure 6-38. Evolution of the six design variables ($\beta^T = 13$)

Figure 6-39. Evolution of the objective functions for the six variables case ($\beta^T = 13$)
Table 6-19. Optimum designs for the six design variables case ($\beta^T = 13$)

<table>
<thead>
<tr>
<th>$\beta^T = 13$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>obj. func.</th>
<th>obj. func.</th>
<th>total volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMA</td>
<td>14.522</td>
<td>21.131</td>
<td>18.556</td>
<td>9.579</td>
<td>9.916</td>
<td>5.001</td>
<td>6443.00</td>
<td>10.74</td>
<td>5.95</td>
</tr>
<tr>
<td>SORA</td>
<td>16.439</td>
<td>19.923</td>
<td>15.697</td>
<td>5.004</td>
<td>9.708</td>
<td>5.004</td>
<td>6323.85</td>
<td>8.69</td>
<td>4.81</td>
</tr>
</tbody>
</table>

Table 6-20. Number of FLAS and Abaqus executions for the six design variables case ($\beta^T = 13$)

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>12320</td>
<td>14609</td>
<td>963</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>115</td>
<td>114</td>
<td>340</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>174.5</td>
<td>255.1</td>
<td>35.2</td>
</tr>
</tbody>
</table>

- **Case Ib-3: six design variables of the lateral and central box girder thicknesses for $\beta^T = 12$**

We now reduce the target reliability to $\beta^T = 12$ in this subcase. Figure 6-38 and Figure 6-39 present the evolutions of the design variables as well as the objective functions for each method, while Table 6-21 and Table 6-22 summarize the optimal designs and the number of FLAS and Abaqus executions for each method.

As we decreased the target reliability, all design variables at the optimum were reduced from the original design, especially those of the lateral girder. The distribution of girder material by PMA and RIA are very similar while that for SORA, especially $d_3$ is different, yet the objective functions among the three methods converged to similar values. The objective functions of the girder volume have decreased by about 6% compared to the design in *Case Ib-1*, and the total volume of the box girders and the main cables using the three methods have been reduced by approximately 3.5%. The aeroelastic and the maximum main cable stress constraints were active at the optimum for all three methods.
In order to achieve convergence, the use of SQP algorithm as well as the reduction factor of $r=0.75$ was necessary for PMA while for RIA the reduction factor $c=3$ was employed. The number of reliability iterations for RIA varied between 15 and 16 while those for PMA oscillate between 11 and 24. The reliability iterations of SORA varied between 15 and 24. The decoupled formulation of SORA is the most computationally efficient method, which is less than half of the computational time of PMA and RIA.
Figure 6-40. Evolution of the six design variables ($\beta T = 12$)

Figure 6-41. Evolution of the objective functions for the six design variables case ($\beta T = 12$)
Table 6-21. Optimum designs for the six design variables case ($\beta_T = 12$)

<table>
<thead>
<tr>
<th>$\beta_T = 12$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>obj. func.</th>
<th>% variation</th>
<th>obj. func.</th>
<th>total volume</th>
</tr>
</thead>
</table>

Table 6-22. Number of FLAS and Abaqus executions for the six design variables case ($\beta_T = 12$)

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>6538</td>
<td>5285</td>
<td>640</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>66</td>
<td>43</td>
<td>240</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>84.0</td>
<td>71.0</td>
<td>30.8</td>
</tr>
</tbody>
</table>

- **Case II-1: seven design variables of the plate thicknesses of the lateral and central box girders as well as the main cable area for $\beta_T = 12.45$**

Finally, we added the main cable area to the design variable set of Case Ib to carry out the RBDO with a total of seven design variables, which are $d_1$ to $d_7$ in Figure 6-33. The objective function is the sum of the volumes of the three box girders and the two main cables. The initial design is $d = [14.0, 14.0, 14.0, 16.0, 12.0, 14.0, 2.0]$ (in mm and $d_7$ in m$^2$) and its corresponding objective function is 26903.75 m$^3$.

Just as the previous cases, we first performed the RBDO on the original bridge design with the target reliability of $\beta_T = 12.45$, and the resulting optimum designs are shown in Table 6-23. The design variables of the central girder have decreased greatly, in some cases to the minimum design values while some of the design variables of the lateral girders have increased. The volumes of the three girders have decreased by about 17.6% while the sum of the box girders and the main cable volumes has decreased by approximately 11% from the original bridge. The design of PMA is used for the comparison with the results in the following subcases.
Table 6-23. Optimum designs for the seven design variables case ($\beta_T=12.45$)

<table>
<thead>
<tr>
<th>$\beta_T$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>obj. func.</th>
<th>girder obj. func.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA</td>
<td>14.078</td>
<td>16.647</td>
<td>19.476</td>
<td>5.000</td>
<td>7.688</td>
<td>5.000</td>
<td>1.75</td>
<td>24104.35</td>
<td>-17.53</td>
</tr>
<tr>
<td>PMA</td>
<td>14.976</td>
<td>16.967</td>
<td>10.412</td>
<td>5.000</td>
<td>5.000</td>
<td>5.000</td>
<td>1.75</td>
<td>24085.95</td>
<td>-17.85</td>
</tr>
<tr>
<td>SORA</td>
<td>14.088</td>
<td>17.484</td>
<td>13.595</td>
<td>5.001</td>
<td>5.273</td>
<td>5.000</td>
<td>1.76</td>
<td>24147.17</td>
<td>-17.54</td>
</tr>
</tbody>
</table>

- **Case II-2: seven design variables of the plate thicknesses of the lateral and central box girders as well as the main cable area for $\beta_T=13$**

In this subcase, we carry out the RBDO on the bridge structure considering a total of seven design variables with the target reliability of $\beta_T=13$. The evolutions of the design variables and the objective functions are presented in Figure 6-42 and Figure 6-43 while the optimum solutions and the number of FLAS and Abaqus executions are summarized in Table 6-24 and Table 6-25. For representation purposes, the design variable of $d_7$ is multiplied by 10 in Figure 6-42. Since the RBDO employing RIA did not converge, only the results of PMA and SORA are presented.

Both aeroelastic and maximum main cable stress constraints were active at the optimum for both methods. In general, the design variables of the lateral girders have increased while those of the central girder have decreased. For increasing the target reliability to $\beta_T=13$, the girder volume has gained about 10% compared to the design in **Case II-1** while the cable volume has decreased by about 4%. In result, the objective function has increased by approximately 5%.

Compared to **Case Ib-2**, in which six design variables are considered with the maximum cable stress limited by the equality constraint, the variations of the total volume of the girders and main cable are practically the same with less than 1% difference between these two cases. However, the computational time of SORA considering seven design variables is more than 5 times larger than that of six design variables case. As a result, we can conclude that the solution of **Case Ib-2** is more
efficient than Case II-2. The computational time of PMA did not vary much from Case Ib-2.

For both SORA and PMA, the reduction factor of $r=0.5$ was employed during the reliability analysis. The number of reliability iterations for PMA and SORA varied between 17 and 23.

Figure 6-42. Evolution of the seven design variables ($\beta^T = 13$)
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Figure 6-43. Evolution of the objective functions for the seven design variable case ($\beta^T=13$)

Table 6-24. Optimum designs for the seven design variables case ($\beta^T=13$)

<table>
<thead>
<tr>
<th>$\beta^T = 13$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>obj. func.</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMA</td>
<td>17.347</td>
<td>19.943</td>
<td>11.119</td>
<td>5.000</td>
<td>5.000</td>
<td>5.000</td>
<td>1.825</td>
<td>25407.5</td>
<td>9.84</td>
</tr>
<tr>
<td>SORA</td>
<td>17.297</td>
<td>18.667</td>
<td>19.866</td>
<td>5.083</td>
<td>7.015</td>
<td>5.004</td>
<td>1.823</td>
<td>25366.8</td>
<td>9.56</td>
</tr>
</tbody>
</table>

Table 6-25. Number of FLAS and Abaqus executions

<table>
<thead>
<tr>
<th></th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>13300</td>
<td>3755</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>100</td>
<td>2878</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>278.0</td>
<td>182.2</td>
</tr>
</tbody>
</table>

- **Case II-3: seven design variables of the plate thicknesses of the lateral and central box girders as well as the main cable area for $\beta^T = 12$**

  In this subcase, the target reliability was reduced to $\beta^T = 12$ considering seven design variables. Figure 6-44 and Figure 6-45 show the evolutions of the design
variables as well as the objective functions while Table 6-26 and Table 6-27 summarize the optimal designs and the number of FLAS and Abaqus executions for each method.

For the three methods, both maximum cable stress and aeroelastic constraints were active at the optimum. Regarding the distribution of the girder material, the design variables of the lateral girder did not decrease as much as those in the central girder. The main cable area tends to go to its minimum value because of its large volume in the objective function. For reducing the target reliability to $\beta^T = 12$, we have achieved the reduction of girder volume by around 7% and the total volume of the box girders and the main cable volumes by as much as 4% compared to the design in Case II-1.

Compared to Case Ib-3 in which 6 design variables are considered with the maximum cable stress limited by the equality constraint for the same target reliability of $\beta^T = 12$, the difference in the girder volume is less than 1.1%. Although the distributions of the girder material are different; for example, the optimum design variables of the central girder, $d_4$ and $d_6$, for RIA and SORA in this case led to their minimum design values while none of the design variables researched their minimum values in Case Ib-3. The computational time has increased by 2 to 5 times for having the main cable area as a design variable.

The number of reliability iterations for RIA varied between 15 and 16, while that for PMA and SORA oscillated between 11 and 19. The computational time of SORA is slightly larger than that of PMA because of its high number of Abaqus executions.
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a) RIA

b) PMA
Figure 6-44. Evolution of the seven design variables ($\beta^T = 12$)

Figure 6-45. Evolution of the objective functions for the seven design variables case ($\beta^T = 12$)
In Case I, we assumed that the maximum main cable constraint was active at the optimum, and the cable area was adjusted according to the Equation (6.1). On the other hand in Case II, the main cable area is a design variable and the maximum cable stress is not forced to be active at the optimum. For comparing the results in Case I and Case II, the optimum results are very similar because the maximum cable stress constraint is in fact active. For $\beta^T=12$, The computational time for SORA in Case II-2 is about five times larger for SORA, twice for PMA and three times for RIA compared to those in Case I b-2. Therefore we can conclude that the assumption of optimality criteria of the cable stress is valid and Case I is more computationally efficient than Case II.

### Table 6-26. Optimum designs for the seven design variables case ($\beta^T=12$)

<table>
<thead>
<tr>
<th>$\beta^T=12$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>obj. func.</th>
<th>obj. func.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIA</td>
<td>13.137</td>
<td>14.511</td>
<td>11.862</td>
<td>5.000</td>
<td>5.000</td>
<td>5.000</td>
<td>1.700</td>
<td>23109.7</td>
<td>-7.47</td>
</tr>
<tr>
<td>SORA</td>
<td>12.723</td>
<td>15.006</td>
<td>10.045</td>
<td>5.009</td>
<td>6.588</td>
<td>5.000</td>
<td>1.701</td>
<td>23122.8</td>
<td>-7.37</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d_1$ to $d_6$ in mm, $d_7$ in m$^2$, obj. func. in m$^3$</th>
<th>% variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>obj. func. girder</td>
<td>$%$ variation</td>
</tr>
</tbody>
</table>

### Table 6-27. Number of FLAS and Abaqus executions for the seven design variables case ($\beta^T=12$)

<table>
<thead>
<tr>
<th></th>
<th>RIA</th>
<th>PMA</th>
<th>SORA</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of FLAS execution</td>
<td>12824</td>
<td>9345</td>
<td>3029</td>
</tr>
<tr>
<td>No. of Abaqus execution</td>
<td>132</td>
<td>90</td>
<td>2132</td>
</tr>
<tr>
<td>Computational time (hour)</td>
<td>223.0</td>
<td>125.2</td>
<td>150.0</td>
</tr>
</tbody>
</table>

In Case I, we assumed that the maximum main cable constraint was active at the optimum, and the cable area was adjusted according to the Equation (6.1). On the other hand in Case II, the main cable area is a design variable and the maximum cable stress is not forced to be active at the optimum. For comparing the results in Case I and Case II, the optimum results are very similar because the maximum cable stress constraint is in fact active. For $\beta^T=12$, The computational time for SORA in Case II-2 is about five times larger for SORA, twice for PMA and three times for RIA compared to those in Case II-2. Therefore we can conclude that the assumption of optimality criteria of the cable stress is valid and Case I is more computationally efficient than Case II.


### 6.8 References


[D2] DISEG-Dipartimento di Ingegneria Structurale e Geotecnica, Universita degli Studi di Genova, “Calutazione del vento di proggetto”, Ponte sulle Stretto Di Messina, 2004


CHAPTER 7

CONCLUSIONS

7.1 General conclusions from the reliability analysis and the Reliability Based Design Optimization (RBDO) methods

- For performing structural analyses, the probabilistic approach can provide more precise results, which are adjusted to real uncertainty that affects structural performances, compared to the traditional deterministic approach employing partial safety factors.
- The First Order Reliability Method (FORM) is a computationally efficient method, which works well even with a large number of random variables.
- Sampling methods such as Monte Carlo (MCS) and Latin Hypercube (LHS) methods have the advantage of simple application; however, the need of a large number of simulations is still an obstacle for their application to large complex systems. In general, LHS is more precise than MCS for fewer numbers of simulations.
Chapter 7

Conclusions

- The use of reduction factor proposed by our research group in the case of convergence problem in FORM is very effective as demonstrated in the bridge examples.
- The RBDO can provide more competitive optimum designs than the conventional deterministic optimization for taking into account specific uncertainty in each parameter that affects structural responses. Moreover, the RBDO guarantees a predetermined structural safety level that the deterministic optimization does not provide.
- The decoupled method of Sequential Optimization and Reliability Assessment (SORA) is more computationally efficient than the two-level methods of Reliability Index Approach (RIA) and Performance Measure Approach (PMA) due to its formulation. Within the two-level methods, PMA is generally more efficient than RIA for employing the inverse reliability method.
- The two-level methods are generally computationally intensive because they require a large number of reliability analyses. Although the decoupled method of SORA requires a greater number of deterministic design optimization iterations, its computational cost is lower because it involves in a smaller number of complete cycles (deterministic optimization followed by reliability analysis).
- The modification of Hybrid Mean Value (HMV) method employing a reduction factor proposed by the author is effective in the case of convergence problems of the algorithm as demonstrated in the two bridge examples.

7.2 Conclusions from the reliability analysis and the RBDO applied to long-span suspension bridges under aeroelastic constraint

- It is feasible to apply the RBDO methods to long-span suspension bridges considering probabilistic flutter constraint along with other deterministic constraints.
• The proposed approach of employing Matlab code as a main code, Abaqus for finite element analyses and FLAS code developed by the University of Coruña for flutter computations was effective for the resolution of the RBDO problem.

• By performing reliability analyses considering only one flutter derivative as well as the extreme wind velocity and the structural damping as random variables, we were able to identify which flutter derivatives were more relevant than others on the structural reliability. This information is very useful in order to obtain more precise results for certain flutter derivatives in wind tunnel tests. It may also be used to identify those flutter derivatives with little influence on the structural reliability so that we can reduce the number of random variables in the RBDO problem.

• By performing reliability analyses with the same type of flutter derivative, the flutter derivatives of type $A^*$ were identified to be the most influential flutter derivative type, which are related to the moment of the deck. In general, flutter derivatives with sub-indices of 2 and 3 are also important, which are related to the rotational displacement and velocity due to torsion of the deck.

• By carrying out reliability analyses with different standard deviations for experimental data, we were able to see the influence of data dispersion on the overall structural reliability. It has been observed that the data dispersions increased with greater wind velocity in the wind tunnel tests. Therefore we considered a linearly increasing standard deviation with wind velocity for flutter derivatives.

• For the Great Belt Bridge example, $A_1^*$, $A_2^*$, $A_3^*$ and $H_3^*$ are the most relevant flutter derivatives on structural reliability, which are employed in the RBDO problem. Using a smaller dispersion of flutter derivatives ($\sigma_{\text{max}}=0.15\mu$), the differences in reliability indices among all flutter derivatives are within 5%; however, when the maximum dispersion value is increased to ($\sigma_{\text{max}}=0.3\mu$), $A_2^*$ becomes the most important flutter derivative on the structural reliability, and the differences in reliability indices increase to 30%. This is because flutter velocity is very sensitive to the dispersion of the random variables of $A_2^*$. 

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• For the Messina Bridge example, the flutter derivatives of $A_2^*$, $A_3^*$ and $H_2^*$, $P_3^*$ are some of the most important functions on the reliability against flutter, among which $A_3^*$ reduces the structural reliability the most. Therefore, the points that define this flutter derivative were considered as random variables in the RBDO problem. Comparing the constant standard deviation cases of 5% and 15% on flutter derivatives, reliability indices of $A_2^*$ and $A_3^*$ were reduced by 17% and 20% respectively while other flutter derivatives did not vary significantly.

• It has been verified that the structural damping had little influence on the structural reliability for both Messina and Great Belt. Therefore it was disregarded in the RBDO formulation.

• For the RBDO problems, when $\beta^T$ was reduced from that of the original design, the optimum objective function decreased from the original design, while $\beta^T$ was increased, the opposite occurred as expected.

• For the Messina example, even though the optimum objective functions from different methods were very similar, the optimum design of the short edge, $d_3$ of the lateral box girder varied noticeably. This is because it had a small contribution to the objective function and little influence on the aeroelastic behavior of the bridge.

• When we consider six design variables of both lateral and central box girders for the Messina Bridge, the design variables of the lateral girders in general increased while those of the central girder decreased. This is because the design variables of the lateral girders contributed more on the torsional inertia of the bridge deck, which in turn increased the flutter velocity of the structure.

• For the Messina example with 6 design variables and $\beta^T=12$, reduced from the reference $\beta=12.45$, the reduction in total volume from the original design was approximately 15% while for the Great Belt example with 4 design variables and $\beta^T=8$, decreased from the reference value of $\beta=8.85$, the reduction in volume was about 7%.

• For both bridge examples, we verified the optimality criterion of the maximum cable stress imposed in Case I. The objective functions of both Case I and Case II were
very similar because the maximum cable stress constraint was active in Case II. Therefore we can conclude that Case I is a more computationally efficient approach.

- The computational time to solve a RBDO problem depends on the method employed. Among the three RBDO methods studied in this research, the decoupled method of SORA is generally the most computationally efficient, which took for some cases only a third of time using the two-level methods of RIA and PMA. For example, in Case I-2 of the Great Belt Bridge, the computational time of SORA was 31 hours while those of RIA and PMA were 93 and 80 hours respectively. In this case, four design variables and 43 random variables were taken into account in the problem.

- The process that takes the most computational time to resolve the RBDO problem is the computation of flutter velocity using FLAS. The execution time of FLAS depends on the number of aeroelastic modes, the number of deck nodes included in the flutter analysis as well as the flutter velocity value because a larger number of iterations are necessary as flutter velocity increases. For the case of the Great Belt with 77 nodes along the deck and 13 aeroelastic modes, the execution time was 10 seconds for the flutter speed of 80.32 m/s. In the case of the Messina Bridge with 225 nodes and 7 aeroelastic modes, the computational time was approximately 60 seconds for the flutter speed of 102.72 m/s.

- Since the two-level methods of RIA and PMA require significantly greater numbers of flutter evaluations compared to SORA, their computational costs are higher.

### 7.3 Future lines of research

In this research, the design optimization was carried out on long-span bridges considering uncertainties in the flutter constraint. There are some aspects that can be included in the future studies.

Instead of considering a single probabilistic constraint of flutter, we can include multiple probabilistic constraints in the RBDO formulations. For example, the deterministic constraints that we considered in the study such as the maximum vertical
displacement and the maximum cable stress constraint under the traffic overload case can be probabilistic. In this case, we can consider uncertainties in material properties, loads and dimensions etc. We can also consider other aeroelastic phenomena such as buffeting or vortex shedding excitation as additional probabilistic constraints.

Another interesting study may be to consider a shape optimization of bridge deck. When we change the shape of a box girder, flutter derivatives obviously change. Then we need to define flutter derivatives using a computational method. There are several possible computational approaches to obtain flutter derivatives. For example, one approach is to get flutter derivatives directly using computational fluid dynamic (CFD) method. Another simpler approach is to obtain aerodynamic coefficients using CFD, and subsequently to define flutter derivatives employing the quasi-static approach. However, neither of the approaches presents precision for the case with complex-shaped deck sections.

In this study, the moment method of FORM was used for the reliability analysis. It would be interesting to use different reliability methods such as stochastic polynomial expansion, two-point adaptive nonlinear approximation, or the second-order method of SORM.

In the reliability analyses of the bridge examples, we considered that all the random variables are independent random variables and therefore, there was no correlation among them. We may consider some correlations among the random variables of the same flutter derivative because they are not totally independent of one another.
Appendix A

Mechanical properties of the deck

1. Great Belt Bridge mechanical properties

Design variables:

\( d_1 \): top plate thickness (segment 1)

\( d_2 \): bottom plate thickness (segment 2)

\( d_3 \): upper side plate thickness (segment 3)

\( d_4 \): lower side plate thickness (segment 4)

The coordinate of the midpoint of each line segment with respect to the global reference point

Segment 1: \(( y'_1, z'_1 )\)

Segment 2: \(( y'_2, z'_2 )\)

Segment 3: \(( y'_3, z'_3 )\)

Segment 4: \(( y'_4, z'_4 )\)

Line length

\( L_i \): length of each line segment
Angles

\( \alpha_i \) (i=3, 4): inclination of each line segment with respect to horizontal line

I. Compute the center of gravity of the deck

1. **Top plate** (2.496\% inclination)

\[ \alpha_i = \tan^{-1}(0.02496) \]

\[ z_1 = z_1 + \frac{d_1}{2} \cdot \cos(\alpha_i) \]

\[ y_1 = y_1 - \frac{d_1}{2} \cdot \sin(\alpha_i) \]

\[ A_1 = L_1 \cdot d_1 \]

2. **Bottom plate**

\[ z_2 = z_2 - \frac{d_2}{2} \]

\[ y_2 = y_2 \]

\[ A_2 = L_2 \cdot d_2 \]

3. **Upper side plate**

\[ \alpha_3 = 26.565 \cdot \frac{\pi}{180} \]

\[ z_3 = z_3 + \frac{d_3}{2} \cdot \cos(\alpha_3) \]

\[ y_3 = y_3 - \frac{d_3}{2} \cdot \sin(\alpha_3) \]

\[ A_3 = L_3 \cdot d_3 \]
Appendix A

Mechanical properties of the bridge deck

4. Lower side plate

\[ \alpha_4 = 26.565 \cdot \frac{\pi}{180} \]

\[ z_4 = z' + \frac{d_4}{2} \cdot \cos(\alpha_4) \]

\[ y_4 = y' + \frac{d_4}{2} \cdot \sin(\alpha_4) \]

\[ A_4 = L_4 \cdot d_4 \]

\[ A_{\text{deck}} = 2 \cdot \sum_{i=1}^{4} A_i \]

The center of gravity of the deck is:

\[ z_{\text{deck}} = 2 \cdot \frac{1}{A_{\text{deck}}} \sum_{i=1}^{4} A_i \cdot z_i \]

\[ y_{\text{deck}} = 0 \]

II. Compute the moment of inertia of the deck

1. Top plate

\[ I'_{y}(1) = \frac{1}{12} \cdot L_1 \cdot (d_1)^3 \]

\[ I'_{z}(1) = \frac{1}{12} \cdot d_1 \cdot L^3 \]

Rotate and translate with respect to the c.g. of the plates

\[ I'_{y}(1) = 2 \cdot \left[ I'_{y}(1) \cdot \cos^2(\alpha_i) + I'_{z}(1) \cdot \sin^2(\alpha_i) + A_i \cdot (z_i - z_{\text{deck}})^2 \right] \]

\[ I'_{z}(1) = 2 \cdot \left[ I'_{y}(1) \cdot \sin^2(\alpha_i) + I'_{z}(1) \cdot \cos^2(\alpha_i) + A_i \cdot (y_i)^2 \right] \]
2. **Bottom plate**

\[ I_y(2) = \frac{1}{12} \cdot 2y_s \cdot (d_2)^3 \]
\[ I_z(2) = \frac{1}{12} \cdot (2y_s)^3 \cdot d_2 \]

Translate with respect to the c.g. of the plates
\[ I_y(2) = I_y(2) + A_2 \cdot (z_s - z_{\text{deck}})^2 \]

3. **Upper side plate**

\[ I_y(3) = \frac{1}{12} \cdot L_3 \cdot (d_3)^3 \]
\[ I_z(3) = \frac{1}{12} \cdot d_3 \cdot (L_3)^3 \]

Rotate and translate with respect to the c.g. of the plates
\[ I_y(3) = 2 \left[ I_y(3) \cdot \cos^2(\alpha_3) + I_z(3) \cdot \sin^2(\alpha_3) + A_3 \cdot (z_3 - z_{\text{deck}})^2 \right] \]
\[ I_z(3) = 2 \left[ I_y(3) \cdot \sin^2(\alpha_3) + I_z(3) \cdot \cos^2(\alpha_3) + A_3 \cdot (y_3)^2 \right] \]

4. **Lower side plate**

\[ I_y(4) = \frac{1}{12} \cdot L_4 \cdot (d_4)^3 \]
\[ I_z(4) = \frac{1}{12} \cdot d_4 \cdot (L_4)^3 \]

Rotate and translate with respect to the c.g. of the plates
\[ I_y(4) = 2 \left[ I_y(4) \cdot \cos^2(\alpha_4) + I_z(4) \cdot \sin^2(\alpha_4) + A_4 \cdot (z_4 - z_{\text{deck}})^2 \right] \]
\[ I_z(4) = 2 \left[ I_y(4) \cdot \sin^2(\alpha_4) + I_z(4) \cdot \cos^2(\alpha_4) + A_4 \cdot (y_4)^2 \right] \]
Total moment of inertia of the plates

\[ I_{y_{\text{deck}}} = \sum_{i=1}^{4} I_{y}(i) \]

\[ I_{z_{\text{deck}}} = \sum_{i=1}^{4} I_{z}(i) \]

The center of gravity of Stiffeners

\[ A_{\text{stiff}} = 358991.3835 \]

\[ z_{\text{stiff}} = 2032.6516 \quad \text{with respect to the global reference point} \]

\[ I'_{y_{\text{stiff}}} = 2.6796e + 12 \]

\[ I_{y_{\text{stiff}}} = I'_{y_{\text{stiff}}} - A_{\text{stiff}} \cdot z_{\text{stiff}}^2 \]

\[ I_{z_{\text{stiff}}} = 1.9036e + 13 \]

Find the overall c.g. including stiffeners

\[ A_{\text{total}} = A_{\text{deck}} + A_{\text{stiff}} \]

\[ z = \frac{1}{A_{\text{total}}} \left[ A_{\text{stiff}} \cdot z_{\text{stiff}} + A_{\text{deck}} \cdot z_{\text{deck}} \right] \]

\[ I_{y} = I_{y_{\text{deck}}} + A_{\text{deck}} \cdot (z_{\text{deck}} - z)^2 + I'_{y_{\text{stiff}}} + A_{\text{stiff}} \cdot (z_{\text{stiff}} - z)^2 \]

\[ I_{z} = I_{z_{\text{deck}}} + I_{z_{\text{stiff}}} \]

The polar moment of inertia of the deck

\[ \omega = 108549500 + 0.5 \cdot A_{\text{deck}} \]

\[ \delta = 2 \cdot \sum_{i=1}^{4} \frac{L_i}{(d_i)^3} \]

\[ J = 4 \cdot \frac{\omega^2}{\delta} \]
2. Messina Bridge mechanical properties

a) Lateral box girder

Design variables:

\( d_1 \): top plate thickness in mm (segment 1)

\( d_2 \): bottom plate thickness in mm (segment 2, 3 and 4)

\( d_3 \): side plate thickness in mm (segment 5 and 6)

The coordinate of the midpoint of each line segment with respect to the global reference point

Segment 1 (top plate): \((y'_1, z'_1)\)

Segment 2 (bottom circular plate): \((y'_2, z'_2)\)

Segment 3 (bottom arc plate): \((y'_3, z'_3)\)
Segment 4 (bottom straight plate): \((y_4', z_4')\)

Segment 5 (side straight plate): \((y_5', z_5')\)

Segment 6 (side circular plate): \((y_6', z_6')\)

**Line length**

\(L_i\): length of each line segment (mm)

**Angles**

\(\beta_i\) \((i=2, 3, 6)\): half angle of each circular/arc section

\(\alpha_i\) \((i=1, 4, 5)\): inclination of each line segment with respect to horizontal line

\(\alpha_i\) \((i=2, 3, 6)\): angle between the line that connects the mid-point and the center of circle/arc and horizontal line

**Radius of circle/arc segment**

\(r_2\): radius of the circular segment 2 (mm)

\(r_3\): radius of the arc segment 3 (mm)

\(r_6\): radius of the circular segment 6 (mm)

---

**I. To compute the center of gravity of the lateral box girder with respect to the global reference point**

**1. Top plate**

\[\alpha_i = \tan^{-1}(0.02)\]

\[y_i = y_i' + \frac{d_i}{2} \sin(\alpha_i)\]

\[z_i = z_i' + \frac{d_i}{2} \cos(\alpha_i)\]

\[A_i = L_i \cdot d_i\]

**2. Bottom circular plate**
\[
\beta_2 = \frac{49.605 \cdot \pi}{180}
\]

\[
y_{c2} = r_2 \cdot \left(1 - \frac{\sin(\beta_2)}{\beta_2}\right)
\]

\[
\alpha_2 = 40.93 \cdot \frac{\pi}{180}
\]

\[
y_2 = y_{ref}(2) - [y_{c2} - 0.5 \cdot d_2 \cdot \cos(\alpha_2)]
\]

\[
z_2 = z_2' + [y_{c2} - 0.5 \cdot d_2 \cdot \sin(\alpha_2)]
\]

\[
A_2 = d_2 \cdot \left(r_2 + \frac{d_2}{2}\right) \cdot \beta_2
\]

3. **Bottom arc plate**

\[
\beta_3 = \frac{36.631 \cdot \pi}{180}
\]

\[
y_{c3} = r_3 \cdot \left(1 - \frac{\sin(\beta_3)}{\beta_3}\right)
\]

\[
\alpha_3 = 84.052 \cdot \frac{\pi}{180}
\]

\[
y_3 = y_3' - [y_{c3} - 0.5 \cdot d_2 \cdot \cos(\alpha_3)]
\]

\[
z_3 = z_3' + [y_{c3} - 0.5 \cdot d_2 \cdot \sin(\alpha_3)]
\]

\[
A_3 = d_3 \cdot \left(r_3 + \frac{d_3}{2}\right) \cdot \alpha_3
\]

4. **Bottom straight plate**

\[
\alpha_4 = 12.368 \cdot \frac{\pi}{180}
\]

\[
y_4 = y_4' + \frac{d_4}{2} \cdot \sin(\alpha_4)
\]
Appendix A

Mechanical properties of the bridge deck

\[ z_4 = z_4' + \frac{d_5}{2} \cos(\alpha_4) \]

\[ A_4 = L_4 \cdot d_2 \]

5. Side straight plate

\[ \alpha_5 = 38.660 \cdot \frac{\pi}{180} \]

\[ y_5 = y_5' + \frac{d_5}{2} \sin(\alpha_5) \]

\[ z_5 = z_5' + \frac{d_5}{2} \cos(\alpha_5) \]

\[ A_5 = L_5 \cdot d_3 \]

6. Side circular plate

\[ \beta_6 = \frac{39.806 \cdot \pi}{2} \cdot \frac{\pi}{180} \]

\[ y_{c6} = \left( r_6 + \frac{d_3}{2} \right) \left( 1 - \frac{\sin(\beta_6)}{\beta_6} \right) \]

\[ \alpha_6 = 71.214 \cdot \frac{\pi}{180} \]

\[ y_6 = y_6' - \left[ y_{c6} - 0.5 \cdot d_3 \cdot \cos(\alpha_6) \right] \]

\[ z_6 = z_6' + \left[ y_{c6} - 0.5 \cdot d_3 \cdot \sin(\alpha_6) \right] \]

\[ A_6 = d_3 \left( r_6 + \frac{d_3}{2} \right) \cdot \beta_6 \]

The total area of the deck is:

\[ A_{\text{deck}} = \sum_{i=1}^{6} A_i \]

The center of gravity of the deck is:
Appendix A

Mechanical properties of the bridge deck

\[ y_{deck} = \frac{1}{A_{deck}} \sum_{i=1}^{6} A_i \cdot y_i \]

\[ z_{deck} = \frac{1}{A_{deck}} \sum_{i=1}^{6} A_i \cdot z_i \]

II. To compute the moment of inertia

1. Top plate

\[ I_{y'}(1) = \frac{1}{12} L \cdot (d_{y})^3 \]
\[ I_{z'}(1) = \frac{1}{12} \cdot d_{z}L^3 \]

Rotate and translate with respect to the global c.g.

\[ I_{y'}(1) = I_{y'}(1) \cdot \cos^2(\alpha_i) + I_{z'}(1) \cdot \sin^2(\alpha_i) + A_i \cdot (z_i - z_{deck})^2 \]
\[ I_{z'}(1) = I_{y'}(1) \cdot \sin^2(\alpha_i) + I_{z'}(1) \cdot \cos^2(\alpha_i) + A_i \cdot (y_i - y_{deck})^2 \]

2. Bottom circular plate

\[ I_{y'}(2) = \left( r_1 + \frac{d_{2}}{2}\right)^3 \cdot d_2 \left[ \beta_2 + \sin(\beta_2) \cdot \cos(\beta_2) - \frac{2 \cdot \sin^2(\beta_2)}{\beta_2} \right] \]
\[ I_{z'}(2) = \left( r_1 + \frac{d_{2}}{2}\right)^3 \cdot d_2 \left[ \beta_2 - \sin(\beta_2) \cdot \cos(\beta_2) \right] \]

Rotate and translate to the global c.g.

\[ I_{y'}(2) = I_{y'}(2) \cdot \cos^2(\alpha_2) + I_{z'}(2) \cdot \sin^2(\alpha_2) + A_i \cdot (z_2 - z_{deck})^2 \]
\[ I_{z'}(2) = I_{y'}(2) \cdot \sin^2(\alpha_2) + I_{z'}(2) \cdot \cos^2(\alpha_2) + A_i \cdot (y_2 - y_{deck})^2 \]

3. Bottom arc plate

\[ I_{y'}(3) = \left( r_2 + \frac{d_{2}}{2}\right)^3 \cdot d_2 \left[ \alpha_3 + \sin(\alpha_3) \cdot \cos(\alpha_3) - \frac{2 \cdot \sin^2(\alpha_3)}{\alpha_3} \right] \]
Appendix A  
Mechanical properties of the bridge deck

\[ I'_x(3) = \left( r_2 + \frac{d_2}{2} \right)^3 \cdot d_2 \cdot \left[ \alpha_3 - \sin(\alpha_3) \cdot \cos(\alpha_3) \right] \]

Rotate and translate to the global c.g.

\[ I_y(3) = I'_y(3) \cdot \cos^2(\alpha_3) + I'_x(1) \cdot \sin^2(\alpha_3) + A_3 \cdot (z_3 - z_{\text{deck}})^2 \]

\[ I_z(3) = I'_y(3) \cdot \sin^2(\alpha_3) + I'_x(3) \cdot \cos^2(\alpha_3) + A_3 \cdot (y_3 - y_{\text{deck}})^2 \]

4. Bottom straight plate

\[ I'_y(4) = \frac{1}{12} \cdot L_4 \cdot (d_2)^3 \]

\[ I'_z(4) = \frac{1}{12} \cdot d_2 \cdot (L_4)^3 \]

Rotate and translate to the global c.g.

\[ I_y(4) = I'_y(4) \cdot \cos^2(\alpha_4) + I'_x(4) \cdot \sin^2(\alpha_4) + A_4 \cdot (z_4 - z_{\text{deck}})^2 \]

\[ I_z(4) = I'_y(4) \cdot \sin^2(\alpha_4) + I'_x(4) \cdot \cos^2(\alpha_4) + A_4 \cdot (y_4 - y_{\text{deck}})^2 \]

5. Side straight plate

\[ I'_y(5) = \frac{1}{12} \cdot L_5 \cdot (d_3)^3 \]

\[ I'_z(5) = \frac{1}{12} \cdot d_3 \cdot (L_5)^3 \]

Rotate and translate to the global c.g.

\[ I_y(5) = I'_y(5) \cdot \cos^2(\alpha_5) + I'_x(5) \cdot \sin^2(\alpha_5) + A_5 \cdot (z_5 - z_{\text{deck}})^2 \]

\[ I_z(5) = I'_y(5) \cdot \sin^2(\alpha_5) + I'_x(5) \cdot \cos^2(\alpha_5) + A_5 \cdot (y_5 - y_{\text{deck}})^2 \]

6. Side circular plate

\[ I'_y(6) = \left( r_3 + \frac{d_3}{2} \right)^3 \cdot d_3 \cdot \left[ \alpha_6 + \sin(\alpha_6) \cdot \cos(\alpha_6) - \frac{2 \cdot \sin^2(\alpha_6)}{\alpha_6} \right] \]
Mechanical properties of the bridge deck

\[ I_z(6) = \left( r_z + \frac{d_z}{2} \right)^3 \cdot d_z \cdot \left[ \alpha_6 - \sin(\alpha_6) \cdot \cos(\alpha_6) \right] \]

Rotate and translate to the global c.g.

\[
I_y(6) = I_y'(6) \cdot \cos^2(\alpha_6) + I_z'(6) \cdot \sin^2(\alpha_6) + A_6 \cdot (z_6 - z_{deck})^2
\]

\[
I_z(6) = I_y'(6) \cdot \sin^2(\alpha_6) + I_z'(6) \cdot \cos^2(\alpha_6) + A_6 \cdot (y_6 - y_{deck})^2
\]

The moment of inertia of the deck is:

\[
I_{y,\text{deck}} = \sum_{i=1}^{6} I_y(i)
\]

\[
I_{z,\text{deck}} = \sum_{i=1}^{6} I_z(i)
\]

The area, center of gravity and moment of inertia with respect to the global axis

\[
A_{\text{stiff}} = 227207.9934
\]

\[
y_{\text{stiff}} = 7343.0053
\]

\[
z_{\text{stiff}} = -72.6118
\]

\[
I_{y,\text{stiff}} = 1.9331E+11
\]

\[
I_{z,\text{stiff}} = 1.5783E+13
\]

The moment of inertia with respect to the c.g. of the stiffeners

\[
I_{y,\text{stiff}} = I_{y,\text{stiff}} - A_{\text{stiff}} \cdot z_{\text{stiff}}^2
\]

\[
I_{z,\text{stiff}} = I_{z,\text{stiff}} - A_{\text{stiff}} \cdot y_{\text{stiff}}^2
\]

\[
A_{\text{total}} = A_{\text{deck}} + A_{\text{stiff}}
\]

The global c.g. and the moment of inertia with respect to the global axis are:
Appendix A  Mechanical properties of the bridge deck

\[ y_{cg} = \frac{1}{A_{\text{total}}} \left[ A_{\text{stiff}} \cdot y_{\text{stiff}} + A_{\text{deck}} \cdot y_{\text{deck}} \right] \]

\[ z_{cg} = \frac{1}{A} \left[ A_{\text{stiff}} \cdot z_{\text{stiff}} + A_{\text{deck}} \cdot z_{\text{deck}} \right] \]

\[ I_{y_{\text{total}}} = I_{y_{\text{deck}}} + A_{\text{deck}} \cdot (z_{\text{deck}} - z_{cg})^2 + I_{y_{\text{stiff}}} + A_{\text{stiff}} \cdot (z_{\text{stiff}} - z_{cg})^2 \]

\[ I_{z_{\text{total}}} = I_{z_{\text{deck}}} + A_{\text{deck}} \cdot (y_{\text{deck}} - y_{cg})^2 + I_{z_{\text{stiff}}} + A_{\text{stiff}} \cdot (y_{\text{stiff}} - y_{cg})^2 \]

Polar moment of inertia is calculated without considering that of stiffeners because of their small influences on overall inertia (2-3%)

\[ \omega = 26554600.9529 + 0.5 \cdot A_{\text{deck}} \]

\[ \delta = \frac{A_1}{(d_1)^2} + \frac{A_2}{(d_2)^2} + \frac{A_3}{(d_2)^2} + \frac{A_4}{(d_3)^2} + \frac{A_5}{(d_3)^2} \]

\[ J = 4 \cdot \frac{\omega^2}{\delta} \]
b) Central box girder

Design variables:

\( d_4 \): top plate thickness in mm (segment 7)
\( d_5 \): bottom plate thickness in mm (segment 8)
\( d_6 \): side plate thickness in mm (segment 9)

The coordinate of the midpoint of each line segment with respect to the global reference point

- Segment 7 (top plate): \((y'_7, z'_7)\)
- Segment 8 (bottom plate): \((y'_8, z'_8)\)
- Segment 9 (side plate): \((y'_9, z'_9)\)

Line length
Appendix A

Mechanical properties of the bridge deck

$L_i$: length of each line segment (mm)

**Angles**

$\beta_9$: angle of arc section

**Radius of arc segment**

$r_9$: radius of the arc segment 9 (mm)

1. To compute the center of gravity of the central box girder

7. **Top plate (2% inclination)**

$$\alpha_i = \tan^{-1}(0.02)$$

$$z_7 = z_7 - \frac{d_4}{2} \cdot \cos(\alpha_7)$$

$$A_7 = L_7 \cdot d_4$$

8. **Side plate**

$$z_8 = z_8'$$

$$A_8 = L_8 \cdot d_5$$

9. **Bottom plate**

$$r_9 = 6500.0$$

$$\beta_9 = \frac{69.607768}{2} \cdot \frac{\pi}{180}$$

$$y_{z9} = \left(r_9 + \frac{d_6}{2}\right) \cdot \left(1 - \frac{\sin(\beta_9)}{\beta_9}\right)$$

$$z_9 = z_9' - \left(y_{z9} - \frac{d_6}{2}\right)$$
Appendix A  

Mechanical properties of the bridge deck

\[ A_y = d_y \left( r_0 + \frac{d_0}{2} \right) \cdot 2\beta \]  

average radius times thickness

The center of gravity is therefore,

\[ z_{\text{deck}} = \frac{2 \cdot A_\gamma \cdot z_\gamma + 2 \cdot A_\delta \cdot z_\delta + A_9 \cdot z_9}{2 \cdot (A_\gamma + A_\delta) + A_9} \]

\[ y_{\text{deck}} = 0 \]

II. To compute moment of inertia

7. Top plates

\[ I_{\text{y}}(7) = \frac{1}{12} L_\gamma \cdot (d_4)^3 \]

\[ I_{\text{z}}(7) = \frac{1}{12} d_4 (L_\gamma)^3 \]

\[ \alpha_\gamma = \tan^{-1}(0.02) \]

Rotate and translate to the overall c.g.

\[ I_{\text{y}}(7) = 2 \left[ I_{\text{y}}(7) \cdot \cos^2(\alpha_\gamma) + I_{\text{z}}(7) \cdot \sin^2(\alpha_\gamma) + A_7 \cdot \left( z_\gamma - z_{\text{deck}} \right)^2 \right] \]

\[ I_{\text{z}}(7) = 2 \left[ I_{\text{y}}(7) \cdot \sin^2(\alpha_\gamma) + I_{\text{z}}(7) \cdot \cos^2(\alpha_\gamma) + A(7) \cdot \left( y_\gamma \right)^2 \right] \]

8. Side plates

\[ I_{\text{y}}(8) = \frac{1}{12} d_5 \cdot (L_8)^3 \]

\[ I_{\text{z}}(8) = \frac{1}{12} (d_5)^3 \cdot L_8 \]

Translate to the overall c.g.
Appendix A

Mechanical properties of the bridge deck

\[ I_y(8) = 2 \left[ I_y'(8) + A_k \cdot \left( z_k - z_{deck} \right)^2 \right] \]

\[ I_z(8) = 2 \left[ I_z'(8) + A_k \cdot \left( y_k + \frac{d_z}{2} \right)^2 \right] \]

9. **Bottom plates**

\[ \beta_9 = \frac{69.607768 \cdot \pi}{2 \cdot 180} \]

\[ I_y'(9) = \left( r_9 + \frac{d_9}{2} \right)^3 \cdot \frac{d_9}{2} \cdot \left( \beta_9 \sin(\beta_9) \cdot \cos(\beta_9) - \frac{2\sin^2(\beta_9)}{\beta_9} \right) \]

Translate to the overall c.g.

\[ I_y(9) = I_y'(9) + A(9) \cdot \left( z_9 - z_{deck} \right)^2 \]

\[ I_z(9) = \left( r_9 + \frac{d_9}{2} \right)^3 \cdot \frac{d_9}{2} \cdot \left[ \beta_9 - \sin(\beta_9) \cdot \cos(\beta_9) \right] \]

Total moment of inertia of the plates is therefore,

\[ I_{y,\text{deck}} = \sum_{i=1}^{3} I_y(i) \]

\[ I_{z,\text{deck}} = \sum_{i=1}^{3} I_z(i) \]

Compute the area and inertia of the stiffeners

\[ A_{\text{stiff}} = 87063.1159 \]

\[ z_{\text{stiff}} = 1014.6681 \]

\[ I_{y,\text{stiff}} = 1.5671e+11 \quad \text{with respect to the global reference point} \]

\[ I_z = 4.2634e+11 \]

\[ I_{y,\text{stiff}} = I_{y,\text{stiff}} - A_{\text{stiff}} \cdot \left( z_{\text{stiff}} \right)^2 \quad \text{with respect to the c.g. of the stiffeners} \]
The center of gravity of the total section (plates + stiffeners)

\[ A_{total} = A_{deck} + A_{stiff} \]

\[ z_{cg} = \frac{1}{A_{total}} \left( z_{deck} \cdot A_{deck} + z_{stiff} \cdot A_{stiff} \right) \]

Moment of inertia of the total section

\[ I_y = I_{y_{deck}} + A_{deck} \cdot (z_{deck} - z_{cg})^2 + I_{y_{stiff}} + A_{stiff} \cdot (z_{stiff} - z_{cg})^2 \]

\[ I_z = I_{z_{deck}} + I_{z_{stiff}} \]

Polar moment of inertia of the deck

\[ \omega_{deck} = 13882286.9713 + 0.5 \cdot A_{deck} \]

\[ \delta_{deck} = 2 \cdot \frac{A_1}{d_4^2} + 2 \cdot \frac{A_2}{d_5^2} + \frac{A_3}{d_6^2} \]

\[ J = \frac{4 \cdot \omega_{deck}^2}{\delta_{deck}} \]
Resumen

1. Introducción

A medida que aumenta la longitud de vano en puentes soportados por cables junto con los avances tecnológicos, la estructura del puente se vuelve más flexible y más propensa a flamear. El flameo es una inestabilidad aeroelástica importante que se produce cuando las fuerzas de viento que inciden sobre el tablero combinadas con los movimientos que éste experimenta se produce una situación de amortiguamiento negativo dando lugar a un crecimiento exponencial de los movimientos del tablero hasta llegar al colapso de la estructura. Para grandes estructuras, como puentes de gran vano, es importante mantener la seguridad frente a flameo minimizando el coste.

En comparación con la optimización determinista, que emplea coeficientes de seguridad parciales para considerar el conjunto de incertidumbres en un sistema, la optimización en régimen probabilista (RBDO) permite optimizar estructuras teniendo en cuenta información precisa sobre las incertidumbres de parámetros que afectan a la respuesta estructural manteniendo un nivel de seguridad especificado. La incertidumbre se considera mediante la introducción de variables aleatorias en las funciones de estado límite que constituyen las condiciones de diseño probabilistas. Estas condiciones de diseño se evalúan iterativamente durante el proceso de optimización. De este modo, la optimización probabilista puede proporcionar soluciones más precisas y competitivas que la optimización determinista.
Aunque muchos investigadores han trabajado en optimización probabilista aplicada a diferentes tipos de estructuras, especialmente en el campo aeroespacial donde reducir peso es crítico, no ha habido investigación sobre RBDO aplicado a puentes de gran vano considerando como condición probabilista el fenómeno del flameo. Por tanto, éste será el tema tratado en esta investigación.

Antes de la aplicación de optimización probabilista se realizan análisis probabilistas de los puentes que se estudian para conocer el nivel de seguridad del puente con su diseño original. El índice de fiabilidad obtenido servirá como valor referencia cuando se especifiquen distintos valores exigidos durante el problema de optimización probabilista. Este análisis de fiabilidad resulta útil además para identificar qué variables aleatorias influyen más en la seguridad estructural.

Para demostrar la aplicación de la optimización probabilista en estructuras de puentes se han analizado puentes de gran vano con tableros formados por cajones aerodinámicos. Se han elegido como variables de diseño tanto el espesor de las chapas que forman los cajones metálicos como el área de los cables principales.

2. **Análisis a flameo en puentes de gran vano**

En esta investigación la velocidad de flameo en puentes de gran vano se ha calculado utilizando un método híbrido constituido por una fase experimental, realizando ensayos de un modelo seccional del tablero del puente en un túnel de viento, y segunda fase computacional. En la primera fase se realizan ensayos en un túnel de viento aerodinámico de un modelo seccional del tablero para obtener los coeficientes aerodinámicos y las funciones de flameo. En la fase computacional se calcula la velocidad de flameo mediante la resolución de la ecuación de equilibrio dinámico del tablero de forma iterativa bajo las cargas aerolásticas a partir de las funciones de flameo obtenidas experimentalmente, y las frecuencias y modos de vibración del puente completo calculado mediante un modelo de elementos finitos. Se utiliza el código FLAS,
desarrollado por nuestro grupo de investigación, para el cálculo de la velocidad de
flameo en puentes. El código ha sido modificado por el autor para solucionar un
problema relacionado con la obtención de autovalores duplicados que impedía la
resolución correcta del problema, y por tanto, se mejora su funcionamiento.

3. Análisis de fiabilidad de puentes frente a la inestabilidad de
flameo

Los análisis de fiabilidad proporcionan información sobre la probabilidad de
fallo de una estructura respecto a un estado límite teniendo en cuenta las incertidumbres
que afectan al sistema. En este estudio de análisis de fiabilidad de puentes de gran vano
frente a flameo se ha considerado la existencia de incertidumbre en los valores extremos
de la velocidad de viento, el amortiguamientostructural y las funciones de flameo
obtenidas experimentalmente.

El régimen extremal de viento en el emplazamiento del puente constituye una
fuente clara de incertidumbre, que generalmente se define como una función de
probabilidad tipo Gumbel basada en las medidas proporcionadas por las estaciones
meteorológicas ubicadas cerca del puente. El amortiguamiento estructural puede variar
hasta un 40% respecto a su valor medio de según los trabajos de varios investigadores.
Para calcular la velocidad de flameo utilizando el método híbrido se necesitan 18
funciones de flameo que se obtienen en un túnel de viento para distintas velocidades
reducidas. En la etapa computacional se utilizarán los valores obtenidos
experimentalmente y que definen las funciones de flameo para calcular la velocidad de
flameo. Se asume que esta fase experimental está sujeta a incertidumbre, y por tanto, los
valores que definen las funciones de flameo son variables aleatorias normalmente
distribuidas. Se consideran los valores experimentales como los valores medios
mientras que se supone una desviación típica linealmente variable con la velocidad de
viento aplicada en los ensayos. De esta forma se pretende recoger el aumento de
incertidumbre existente en los ensayos a medida que se incrementa la velocidad de viento. La función de estado límite que define la situación de fallo se formula como la diferencia entre la velocidad de flameo del puente y el valor extremo de la velocidad de viento en el emplazamiento del puente. La probabilidad de fallo debida al estado límite a flameo se calcula mediante el método de fiabilidad de primer orden (FORM). Este método obtiene el índice de fiabilidad, que representa la distancia más corta desde el valor medio de las variables aleatorias a la superficie de fallo, definida por la función de estado límite. Luego se obtiene la probabilidad de fallo a partir del correspondiente índice de fiabilidad. El algoritmo utilizado en el método FORM a veces tiene problemas de convergencia en función de la forma de la función de estado límite y del punto inicial considerado. Este problema ha sido resuelto en otros trabajos de nuestro grupo de investigación mediante la utilización de un factor de reducción en la formulación.

4. Optimización en régimen probabilista de puentes de gran vano considerando la condición de flameo

Las estructuras de gran envergadura como puentes de gran vano conllevan un enorme coste material, siendo uno de los mayores costes para llevarlas a cabo. Dentro de este coste material, el cajón y los cables principales constituyen una gran parte del conjunto de la estructura y por tanto una reducción de la cantidad de material en estos elementos debería ser de relevancia. De acuerdo a lo comentado anteriormente, el planteamiento de optimización en régimen probabilista puede proporcionar un diseño más preciso y competitivo que el obtenido mediante optimización determinista.

En un problema común de optimización probabilista el proceso de diseño se realiza en el espacio original de las variables aleatorias mientras que el análisis de fiabilidad se lleva a cabo en el espacio normal estandarizado de las variables aleatorias. En función de la disposición en la que se realicen estos procesos se definen métodos de resolución en doble bucle y métodos de resolución desacoplados. En esta investigación
se han utilizado los métodos de doble bucle Reliability Index Approach (RIA) y Performance Measure Approach (PMA). Como método desacoplado se ha utilizado el método Sequential Optimization and Reliability Assessment (SORA).

En los métodos de doble bucle se realiza un bucle de optimización externa mientras que el análisis de fiabilidad se realiza en un bucle interno. Este último también es un proceso de optimización que permite obtener los índices de fiabilidad. En el caso del método RIA se utiliza el método FORM mientras que en el caso del PMA se utiliza el método híbrido (HMV). El método PMA se basa en la idea de que, en general, es más sencillo minimizar una función objetivo compleja sujeta a restricciones sencillas que el problema inverso. El algoritmo de fiabilidad en el método PMA se formula de forma inversa que en el método PMA. El método HMV utiliza de forma adaptativa tanto el método “Advanced Mean-Value (AMV)” como el método “Conjugate Mean-Value (CMV)” para funciones convexas y cóncavas de la función de estado límite, respectivamente.

El método desacoplado SORA resuelve el problema de RBDO en una secuencia de optimizaciones deterministas seguidas de sus respectivos análisis de fiabilidad. Este método es efectivo debido a la naturaleza determinista de la formulación. En este caso el método HMV es el utilizado en el análisis de fiabilidad.

En esta investigación se aplica RBDO a puentes de gran vano considerando la condición de flameo. Los parámetros considerados en la formulación del problema de optimización se describen a continuación:

- Variables de diseño

Se consideran dos conjuntos de variables de diseño en este estudio. El primero (Caso I) es el formado por los espesores de las chapas del cajón y el segundo (Caso II) incluye el área de los cables principales como una variable de diseño adicional a las del Caso I.
• Función objetivo

La función objetivo es la función de coste que va a ser minimizada y que en este caso es el volumen de los cajones para el *Caso I* mientras que es la suma del volumen de los cajones y de los cables principales para el *Caso II*.

• Variables aleatorias

Se consideran variables aleatorias la velocidad de viento extremal y las funciones de flameo utilizadas en el problema de optimización. El amortiguamiento estructural se ha considerado en los análisis de fiabilidad estructural pero no se considera en el problema de optimización debido a la pequeña influencia sobre la seguridad estructural.

• Función de estado límite

La función de estado límite es la diferencia entre la velocidad de flameo del puente y el viento extremal en el emplazamiento de la estructura.

En la formulación del problema se considera la situación de flameo como una condición de diseño probabilista mientras que otras restricciones tales como límites laterales de las variables de diseño, tensión en los cables principales y la flecha máxima del tablero del puente bajo la sobrecarga de uso debida al tráfico se han considerado como restricciones de tipo determinista.

Los tres métodos de RBDO arriba mencionados se han programado en un código de Matlab para la resolución del problema de optimización. Se utiliza un método de gradiente implementado en Matlab, denominado ‘active-set’ dentro la función denominada ‘fmincon’. Durante el proceso de optimización se calculan los gradientes de la función objetivo y de las condiciones de diseño respecto a las variables de diseño.

Además, para llevar a cabo el análisis de fiabilidad se calculan las derivadas parciales de las funciones de flameo respecto a las variables aleatorias mediante el
método de las diferencias finitas. Como consecuencia de ello es necesario ejecutar un número elevado de veces los programas FLAS y Abaqus durante el proceso de optimización.

A continuación se muestran los pasos necesarios para resolver el problema de RBDO:

- **Subrutina de optimización:**
  1. Calcular las propiedades mecánicas del tablero \( A(d_j), I_y(d_j), I_z(d_j), J(d_j) \) para modificar el modelo de elementos finitos a partir de los valores de las nuevas variables de diseño. El valor inicial de las variables de diseño se corresponden con los valores del diseño original del puente.
  2. Se calcula el área del cable correspondiente al nuevo peso del tablero, solo para el **Caso I**.
  3. Calcular la longitud inicial de los cables principales así como la tensión inicial de los mismos y de las péndolas utilizando un modelo del cable realizado en Abaqus.
  4. Realizar un análisis estático no lineal del Puente complete bajo el caso de carga pésimo a la hora de analizar la flecha máxima del tablero. Para el **Caso II** se obtiene la máxima tensión normal del cable.
  5. Realizar un análisis modal del modelo de elementos finitos bajo peso propio para obtener los modos y frecuencias naturales de vibración del puente.
  6. Escribir el archivo de entrada del programa FLAS a partir de los resultados del paso 5 y ejecutar FLAS para obtener la velocidad crítica de flameo utilizando las funciones de flameo obtenidas experimentalmente.
  7. El algoritmo de optimización de Matlab chequea el cumplimiento tanto de las condiciones de diseño probabilista como las deterministas. Las condiciones de diseño probabilistas se calculan en la subrutina de fiabilidad estructural.
Appendix B

Summary in Spanish

8. El algoritmo de optimización modifica las variables de diseño.
9. Repetir pasos 1-8 hasta convergencia del algoritmo.

♦ Subrutina de análisis de fiabilidad estructural

El código de Matlab calcula el MPP utilizando FORM en el caso de utilizar el método RIA, y HMV para los métodos PMA y SORA. Se obtiene la velocidad de flameo y su sensibilidad respecto a cada variable aleatoria mediante la ejecución del programa FLAS. El algoritmo termina cuando se cumplen los criterios de convergencia, que son la convergencia del índice de fiabilidad $\beta$ para el método FORM y convergencia del MPP para el método HMV.

Los métodos de RBDO mencionados se han aplicado a dos ejemplos de puentes colgantes que son el puente del Great Belt y el puente proyectado para el estrecho de Messina.

- El puente del Great Belt

El puente del Great Belt en Dinamarca es un Puente colgante con 1624 metros abierto al tráfico en 1999. Está formado por un cajón aerodinámico continuo en toda su longitud sin juntas en las torres. Las dos torres de hormigón armado del puente alcanzan 254 metros de altura tratándose por tanto de una estructura altamente flexible. Se ha realizado un modelo de elementos finitos del puente completo en Abaqus para obtener tanto los modos y frecuencias de vibración como el análisis estático. Se obtuvo una velocidad de flameo de 80.32 m/s mediante el programa FLAS utilizando las funciones de flameo obtenidas experimentalmente y una combinación de modos proporcionados por la etapa computacional. Tras un análisis de fiabilidad se obtiene un índice de fiabilidad para el diseño original del puente de $\beta=8.83$ teniendo en cuenta las 44 variables aleatorias definidas, lo que se corresponde con una probabilidad de fallo de $P_f=5.23E-19$. Se considera una desviación típica linealmente variable desde $\sigma=0$ para una velocidad reducida de $V^*=0$ hasta $\sigma_{\text{max}}=0.15\mu$ para $V^*=30$. Posteriormente se lleva a
cabo la optimización probabilista de la estructura del puente fijando los índices de fiabilidad en $\beta^T=8.845$, $\beta^T=10.0$ y $\beta^T=8.0$ para el Caso I, y $\beta^T=8.0$ para el Caso II utilizando los tres métodos de RBDO mencionados previamente.

- El puente sobre el estrecho de Messina

El puente proyectado en el estrecho de Messina conectando Sicilia con la península de Italia tendrá el vano más largo del mundo con 3300 metros. El tablero de 61 metros de ancho está formado por tres cajones conectados por vigas transversales cada 30 metros y dará servicio al tráfico rodado mediante 6 carriles y al tráfico ferroviario mediante dos líneas. Los cables principales de aproximadamente 2 metros cuadrados de sección transversal cuelgan de dos torres que alcanzan los 382 metros de altura. Para llevar a cabo el análisis aeroelástico y estructural del puente se realiza un modelo de barras 3D en el programa Abaqus. Este modelo se ha utilizado para obtener las frecuencias y modos naturales de vibración que junto con las funciones de flameo obtenidas experimentalmente permiten calcular la velocidad de flameo, que este caso resulta ser de 102.72 m/s. También se llevan a cabo análisis de fiabilidad de acuerdo con el método FORM. El valor obtenido del índice de fiabilidad es de 12.30 considerando un total de 90 variables aleatorias, lo que se corresponde con una probabilidad de fallo de $P_f=4.52E-35$. De nuevo se considera una desviación típica linealmente variable con la velocidad de viento en el ensayo experimental hasta un máximo de $\sigma_{max}=0.15\mu$ para una velocidad reducida de $V^*=30$.

Finalmente, la optimización probabilista se realiza considerando dos situaciones en función de las variables de diseño incluidas en el problema. En el Caso Ia, se consideran 3 variables de diseño, correspondientes a los espesores de chapa de los cajones laterales y se fijan los siguientes valores del índice de fiabilidad: $\beta^T=12.45$, $\beta^T=13.0$, $\beta^T=12.0$. En el Caso Ib, se consideran 6 variables de diseño como consecuencia de añadir los 3 espesores de chapa que definen el cajón central de este puente y se establecen los siguientes índices de fiabilidad a alcanzar: $\beta^T=12.45$, $\beta^T=13.0$, $\beta^T=12.0$. Finalmente en el Caso II, se añade el área de los cables principales como una
variable de diseño adicional al conjunto considerado para el Caso Ib, y se resuelve el problema de optimización probabilista para $\beta^T=12.45$, $\beta^T=13.0$, $\beta^T=12.0$.

5. **Conclusiones generales de los métodos de fiabilidad estructural y de optimización en régimen probabilista (RBDO)**

- El análisis probabilista puede proporcionar resultados más adecuados y ajustados a la incertidumbre real que existe en los parámetros que afectan a una estructura que los proporcionados por el empleo de coeficientes parciales de seguridad.
- Se verifica que el método de fiabilidad de primer orden (FORM) es computacionalmente eficiente incluso para un número elevado de variables aleatorias.
- Los métodos de muestreo tales como Montecarlo y Latin Hypercube (LHS) tienen la ventaja de ser fáciles de implementar, sin embargo, necesitan un gran número de simulaciones y todavía es un obstáculo para su aplicación en sistemas complejos. En general, LHS es más preciso que MCS para un número reducido de simulaciones.
- El uso del factor de reducción comentado en esta investigación para la aplicación del método FORM ha resultado muy eficiente, tal y como se demuestra en los ejemplos analizados.
- La optimización probabilista proporciona diseños más competitivos que la optimización convencional determinista pues tiene en cuenta la incertidumbre específica que tiene cada parámetro que afecta a la respuesta estructural. Además asegura que se cumple un nivel de seguridad especificado, cosa que no sucede cuando se realiza la optimización en régimen determinista.
- El método desacoplado SORA es más eficiente computacionalmente que los métodos de dos niveles RIA y PMA debido a su formulación. Entre los métodos de doble bucle el PMA es en general más eficiente debido a su formulación inversa respecto al método RIA.
Los métodos de doble bucle son en general más costosos en términos computacionales puesto que requieren un mayor número de análisis de fiabilidad. Aunque el método desacoplado SORA requiere un mayor número de iteraciones en el proceso de cada optimización determinista, su coste computacional es menor puesto que el número de ciclos completos (optimización determinista seguido del análisis de fiabilidad) realizados es pequeño.

La modificación del método HMV mediante la introducción de un factor de reducción propuesto en esta investigación ha resultado efectiva para solucionar problemas de convergencia de este algoritmo, tal y como se demuestra en los ejemplos.

6. **Conclusiones de la aplicación del análisis de fiabilidad y de la optimización probabilista en puentes colgantes de gran vano bajo restricciones aeroelásticas**

- Se demuestra la posibilidad de aplicar de forma satisfactoria optimización en régimen probabilista a puentes de gran vano considerando condiciones de flameo y otras restricciones deterministas.
- El planteamiento propuesto de utilizar un código propio de Matlab junto con los programas Abaqus, para realizar el modelo de elementos finitos, y el programa FLAS desarrollado en el grupo de Mecánica de Estructuras de la Universidad de A Coruña para el cálculo de la velocidad de flameo, ha resultado efectivo para la resolución del problema de optimización probabilista.
- Mediante la realización de diversos análisis de fiabilidad considerando como variables aleatorias solamente una función de flameo, el amortiguamiento estructural y la velocidad de viento extremal es posible identificar qué funciones de flameo son más relevantes que otras sobre la fiabilidad estructural. Esta información es muy útil, pues indica que sería deseable incrementar la precisión a la hora de realizar los
ensayos en túnel de viento para determinadas funciones de flameo. Este estudio también resulta útil para descartar aquellas funciones de flameo que tengan poca influencia en la seguridad estructural y de este modo poder reducir el problema de optimización probabilista (RBDO).

- Tras realizar los análisis de fiabilidad considerando como variables aleatorias los valores que definen uno de los tipos de funciones de flameo, se ha comprobado que las funciones de flameo tipo $A^*$ son las más influyentes en la seguridad del puente. Las funciones de flameo con subíndice 2 y 3 son las más influyentes, siendo las que están relacionadas con el giro y la velocidad de giro a torsión del tablero.

- Mediante la realización de análisis de fiabilidad con diferentes valores de la desviación típica en los datos experimentales se ha podido conocer su influencia en la seguridad estructural para esta tipología de puentes. De la realización de los ensayos se deduce que la dispersión de los resultados aumenta con la velocidad de viento en el túnel. Por ello, se ha considerado una desviación típica linealmente variable con la velocidad de viento en los ensayos.

- En el ejemplo del Great Belt las funciones de flameo más relevantes son $A_1^*$, $A_2^*$, $A_3^*$ y $H_3^*$ y son las que se consideran como variables aleatorias en el problema de optimización. Cuando se realiza el análisis de fiabilidad considerando cada función de flameo como variable aleatoria de forma independiente y para una dispersión baja de los datos experimentales ($\sigma_{\text{max}}=0.15\mu$), se observa que la variación del índice de fiabilidad es menor al 5%; sin embargo, cuando se aumenta el valor máximo de la dispersión a $\sigma_{\text{max}}=0.3\mu$, $A_2^*$ se convierte en la más influyente sobre la seguridad estructural, aumentando la diferencia en el índice de fiabilidad al 30%. Esto se debe a que la velocidad de flameo es altamente sensible a la dispersión de los valores que definen la función $A_2^*$.

- En el ejemplo de Messina las funciones de flameo $A_2^*$, $A_3^*$, $H_2^*$ y $P_3^*$ son las más importantes en la seguridad frente a flameo, siendo $A_3^*$ la función que más reduce la seguridad estructural cuando se considera variable aleatoria. Por tanto, los valores que definen esta función de flameo serán los considerados en la optimización.
probabilista. Para una desviación típica de valor constante del 5% y 15% en las funciones de flameo, los índices de fiabilidad para $A_2^*$ y $A_3^*$ se obtiene una reducción del índice de fiabilidad del 17% y 20% respectivamente, mientras que la variación es muy pequeña para el resto de funciones.

- Se ha verificado que el amortiguamiento estructural tiene poca influencia sobre el análisis de fiabilidad planteado tanto para el ejemplo de Gran Belt como el de Messina. Por tanto, no se ha considerado en el problema de RBDO.

- Como se esperaba, en los resultados de la optimización probabilista se observa que cuando el índice de fiabilidad fijado es menor que el del modelo original la función objetivo disminuye mientras que sucede lo contrario al aumentar el índice de fiabilidad.

- En el caso del ejemplo de Messina, aunque se obtienen valores muy similares de la función objetivo para los diferentes métodos de RBDO utilizados, el espesor del lado corto del cajón lateral $d_3$ varía sustancialmente. Esto es debido a la pequeña contribución que esta variable de diseño tiene en la función objetivo así como en la respuesta del puente frente a flameo.

- Cuando se consideran 6 variables de diseño correspondientes a los espesores de chapa en cajones laterales y central, se produce un incremento de su valor en las asignadas a los cajones laterales y una reducción de espesores en el cajón central. Esto se debe a que las variables de diseño asociadas a los cajones laterales contribuyen a aumentar la rigidez torsional del tablero y por tanto a incrementar la velocidad de flameo del puente.

- En el ejemplo de Messina con 6 variables de diseño cuando se reduce la fiabilidad de $\beta=12.45$ en el diseño original a $\beta^T=12$, la reducción en volumen que se obtiene tras la optimización es del 15%. En el caso del Great Belt con 4 variables de diseño y reduciendo el índice de fiabilidad exigido de $\beta=8.85$ en el diseño original a $\beta^T=8$ la reducción de volumen es aproximadamente del 7%.

- En ambos ejemplos se cumple el criterio optimizante de máxima tensión en el cable establecido para el Caso I. El valor óptimo de la función objetivo en ambos casos
Caso I y Caso II son muy similares pues se demuestra en el Caso II que la condición de tensión en el cable es una condición activa. Por tanto, se puede concluir que el planteamiento del Caso I es computacionalmente más efectivo.

- El tiempo de computación para resolver el problema de optimización probabilista depende del método de RBDO utilizado. Entre los 3 métodos analizados el método desacoplado SORA es en general más eficiente computacionalmente, llegando a utilizar sólo un tercio del tiempo empleado por los métodos de doble bucle RIA y PMA. Así por ejemplo, en el Caso I-2 con $\beta^T=10.0$ del ejemplo del Gran Belt el tiempo de computación con el método SORA fue de 31 horas mientras que en el RIA y PMA fue de 93 y 80 horas, respectivamente. Este problema contenía 4 variables de diseño y 43 variables aleatorias.

- El proceso que mayor tiempo de computación requiere al resolver el problema de optimización es el cálculo de la velocidad de flameo con FLAS. Se observa que el tiempo de ejecución depende del número de modos aeroelásticos considerados, del número de nudos del tablero que se incluyen en el análisis a flameo, así como del valor de la velocidad de flameo, pues cuanto más alta sea más iteraciones serán necesarias. En el caso del Gran Belt con 77 nudos en el tablero y 13 modos aeroelásticos el tiempo de cálculo de la velocidad de flameo ($V_f=80.32$ m/s) es aproximadamente de 10 segundos. En el caso de Messina con 225 nudos en el tablero y 7 modos aeroelásticos el tiempo de computación es aproximadamente de 60 segundos, con una velocidad de flameo de $V_f=102.72$ m/s.

- De acuerdo con la conclusión anterior los métodos de doble bucle (RIA, PMA) necesitan evaluar muchas más veces la velocidad de flameo que el método SORA, de ahí su mayor coste computacional.
Appendix C

In accordance with the University regulations approved in 2012, the summary of this thesis is presented in Galician.

Resumo

1. Introdución

A medida que aumenta a lonxitude de van en pontes soportadas por cables xunto cos avances tecnolóxicos, a estrutura da ponte vólvese máis flexible e máis propensa a flamear. O flameo é unha inestabilidade aeroelástica importante que se produce cando as forzas de vento que inciden sobre o taboleiro combinadas cos movementos que este experimenta prodúcese unha situación de amortiguamento negativo dando lugar a un crecemento exponencial dos movementos do taboleiro ata chegar ao colapso da estrutura. Para grandes estruturas, como pontes de gran van, é importante manter a seguridade fronte o flameo e poder minimizar o custo.

En comparación coa optimización determinista, que emprega coeficientes de seguridade parciais para considerar o conxunto de incertezas nun sistema, a optimización en réxime probabilista (RBDO) permite optimizar estruturas tendo en conta información precisa sobre as incertezas de parámetros que afectan á resposta estrutural mantendo un nivel de seguridade especificado. A incerteza considérase mediante a utilización de variables aleatorias incluídas nas funcións de estado límite que constitúen as condicións de deseño probabilistas. Estas condicións de deseño avalíanse iterativamente durante o proceso de optimización. Deste xeito, a optimización probabilista pode proporcionar soluções máis precisas e competitivas que a optimización determinista.

Aínda que moitos investigadores traballaron en optimización probabilista aplicada a diferentes tipos de estruturas, especialmente no campo aeroespacial onde
reducir peso é crítico, non houbo investigación sobre RBDO aplicado a pontes de gran van considerando como condición probabilista o fenómeno do flameo. Por tanto, este será o tema tratado nesta investigación.

Antes da aplicación de optimización probabilista realizanse análises probabilistas das pontes que se estudan para coñecer o nivel de seguridade da ponte co seu deseño orixinal. O índice de fiabilidade obtido servirá como valor referencia cando se especifiquen distintos valores esixidos durante o problema de optimización probabilista. Esta análise de fiabilidade resulta útil ademais para identificar que variables aleatorias inflúen máis na seguridade estrutural.

Para demostrar a aplicación da optimización probabilista en estruturas de pontes analizáronse pontes de gran van con taboleiros formados por caixóns aerodinámicos. Elixíronse como variables de deseño tanto o espesor das chapas que forman os caixóns metálicos como a área dos cables principais.

2. **Análise a flameo en pontes de gran van**

Nesta investigación a velocidade de flameo en pontes de gran van calculouse utilizando un método híbrido constituído por unha fase experimental, realizando ensaios dun modelo seccional do taboleiro da ponte nun túnel de vento, e unha segunda fase computacional. Na primeira fase realizanse ensaios nun túnel de vento aerodinámico dun modelo seccional do taboleiro para obter os coeficientes aerodinámicos e as funcións de flameo. Na fase computacional calcúlase a velocidade de flameo mediante a resolución da ecuación de equilibrio dinámico do taboleiro de forma iterativa baixo as cargas aeroelásticas a partir das funcións de flameo obtidas experimentalmente, e das frecuencias e modos de vibración da ponte completa calculado mediante un modelo de elementos finitos. Utilízase o código FLAS, desenvolvido polo noso grupo de investigación, para o cálculo da velocidade de flameo en pontes. O código foi modificado polo autor para solucionar un problema relacionado coa obtención de...
autovalores duplicados que impedía a resolución correcta do problema, e por tanto, mellórase o seu funcionamento.

3. **Análise de fiabilidade de pontes frente á inestabilidade de flameo**

As análises de fiabilidade proporcionan información sobre a probabilidade de fallo dunha estrutura respecto dun estado límite tendo en conta as incertezas que afectan ó sistema. Neste estudo de análise de fiabilidade de pontes de gran van fronte a flameo considerouse a existencia de incerteza nos valores extremos da velocidade de vento, o amortiguamento estrutural e as funcións de flameo obtidas experimentalmente.

O réxime extremal de vento no emprazamento da ponte constitúe unha fonte clara de incerteza, que xeralmente se define como unha función de probabilidade tipo Gumbel baseada nas medidas proporcionadas polas estacións meteorolóxicas situadas preto da ponte. O amortiguamento estrutural pode variar ata un 40% respecto do seu valor medio de segundo os traballos de varios investigadores. Para calcular a velocidade de flameo utilizando o método híbrido necesitanse 18 funcións de flameo que se obtenen nun túnel de vento para distintas velocidades reducidas. Na etapa computacional utilizaranse os valores obtidos experimentalmente e que definan as funcións de flameo para calcular a velocidade de flameo. Asúmese que esta fase experimental está suxeita a incerteza, e por tanto, os valores que definan as funcións de flameo son variables aleatorias normalmente distribuídas. Considéranse os valores experimentais como os valores medios mentres que se supón unha desviación típica linealmente variable coa velocidade de vento aplicada nos ensaios. Desta forma preténdese recoller o aumento de incerteza existente nos ensaios a medida que se incrementa a velocidade de vento. A función de estado límite que define a situación de fallo formúlase como a diferenza entre a velocidade de flameo da ponte e o valor extremo da velocidade de vento no emprazamento da ponte. A probabilidade de fallo debida ao estado límite a flameo calcúlase mediante o método de fiabilidade de primeira
orde (FORM). Este método obtén o índice de fiabilidade, que representa a distancia máis curta desde o valor medio das variables aleatorias á superficie de fallo, definida pola función de estado límite. Logo obtense a probabilidade de fallo a partir do correspondente índice de fiabilidade. O algoritmo utilitzado no método FORM ás veces ten problemas de converxencia en función da forma da función de estado límite e do punto inicial considerado. Este problema foi resolto noutros traballos do noso grupo de investigación mediante a utilización dun factor de redución na formulación.

4. **Optimización en réxime probabilista de pontes de gran van considerando a condición de flameo**

As estruturas de gran envergadura como pontes de gran van levan un enorme custo material, sendo un dos maiores custos para levalas a cabo. Dentro deste custo material, o caixón e os cables principais constitúen unha gran parte do conxunto da estrutura e por tanto unha redución da cantidade de material nestes elementos debería ser de relevancia. De acordo ao comentado anteriormente, a formulación de optimización en réxime probabilista pode proporcionar un deseño máis preciso e competitivo que o obtido mediante optimización determinista.

Nun problema común de optimización probabilista o proceso de deseño realizase no espazo orixinal das variables aleatorias mentres que a análise de fiabilidade levase a cabo no espazo normal estandarizado das variables aleatorias. En función da disposición na que se realicen estes procesos definense métodos de resolución en dobre bucle e métodos de resolución desacoplados. Nesta investigación utilizáronse os métodos de dobre bucle Reliability Index Approach (RIA) e Performance Measure Approach (PMA). Como método desacoplado utilizouse o método Sequential Optimization and Reliability Assessment (SORA).

Nos métodos de dobre bucle realizase un bucle de optimización externa mentres que a análise de fiabilidade realizase nun bucle interno. Este último tamén é un proceso
de optimización que permite obter os índices de fiabilidade. No caso do método RIA utilizase o método FORM mentres que no caso do PMA utilizase o método híbrido (HMV). O método PMA baséase na idea de que, en xeral, é máis sinxelo minimizar unha función obxectivo complexa suxeita a restricións sinxelas que o problema inverso. O algoritmo de fiabilidade no método PMA formúlase de forma inversa que no método PMA. O método HMV utiliza de forma adaptativa tanto o método Advanced Mean-Value (AMV) como o método Conjugate Mean-Value (CMV) para funcións convexas e cóncavas da función de estado límite, respectivamente.

O método desacoplado SORA resolve o problema de RBDO nunha secuencia de optimizacions deterministas seguidas dos seus respectivos análises de fiabilidade. Este método é efectivo debido á natureza determinista da formulación. Neste caso o método HMV é o utilizado na análise de fiabilidade.

Nesta investigación aplícase RBDO a pontes de gran van considerando a condición de flameo. Os parámetros considerados na formulación do problema de optimización describense a continuación:

- **Variables de deseño**

  Considéranse dous conxuntos de variables de deseño neste estudo. O primeiro (*Caso I*) é o formado polos espesores das chapas do caixón e o segundo (*Caso II*) inclúe a área dos cables principais como unha variable de deseño adicional ás do *Caso I*.

- **Función obxectivo**

  A función obxectivo é a función de custo que vai ser minimizada e que neste caso é o volume dos caixóns para o *Caso I* mentres que é a suma do volume dos caixóns e dos cables principais para o *Caso II*.

- **Variables aleatorias**
Considéranse variables aleatorias a velocidade de vento extremal e as funcións de flameo utilizadas no problema de optimización. O amortiguamento estrutural considerouse nas análises de fiabilidade estrutural pero non se considera no problema de optimización debido á pequena influencia sobre a seguridade estrutural.

- Función de estado límite

A función de estado límite é a diferenza entre a velocidade de flameo da ponte e o vento extremal no emprazamento da estrutura.

Na formulación do problema considérase a situación de flameo como unha condición de deseño probabilista mentres que outras restricións tales como límites laterais das variables de deseño, tensión nos cables principais e a frecha máxima do taboleiro da ponte baixa a sobrecarga de uso debida ao tráfico consideráronse como restricións de tipo determinista.

O tres métodos de RBDO arriba mencionados programáronse nun código de Matlab para a resolución do problema de optimización. Utilízase un método de gradiente implementado en Matlab, denominado ‘active-set’ dentro a función denominada ‘fmincon’. Durante o proceso de optimización calcúlanse os gradientes da función obxectivo e das condicións de deseño respecto das variables de deseño.

Ademais, para levar a cabo a análise de fiabilidade calcúlanse as derivadas parciais das funcións de flameo respecto das variables aleatorias mediante o método das diferenzas finitas. Como consecuencia diso é necesario executar un número elevado de veces os programas FLAS e Abaqus durante o proceso de optimización.

A continuación móstranse os pasos necesarios para resolver o problema de RBDO:

- Subrutina de optimización:
1. Calcular as propiedades mecánicas do taboleiro $A(d_j)$, $I_y(d_j)$, $I_z(d_j)$, $J(d_j)$ para modificar o modelo de elementos finitos a partir dos valores das novas variables de deseño. O valor inicial das variables de deseño correspondéndose cos valores do deseño orixinal da ponte.

2. Calcúlase a área do cable correspondente ao novo peso do taboleiro, só para o Caso I.

3. Calcular a lonxitude inicial dos cables principais así como a tensión inicial dos mesmos e das péndolas utilizando un modelo do cable realizado en Abaqus.

4. Realizar unha análise estática non lineal da ponte completa baixo o caso de carga pésimo á hora de analizar a frecha máxima do taboleiro. Para o Caso II obtense a máxima tensión normal do cable.

5. Realizar unha análise modal do modelo de elementos finitos baixo peso propio para obter os modos e frecuencias naturais de vibración da ponte.

6. Escribir o arquivo de entrada do programa FLAS a partir dos resultados do paso 5 e executar FLAS para obter a velocidade crítica de flameo utilizando as funcións de flameo obtidas experimentalmente.

7. O algoritmo de optimización de Matlab chequea o cumprimento tanto das condicións de deseño probabilista como as deterministas. As condicións de deseño probabilistas calcúlanse na subrutina de fiabilidade estrutural.

8. O algoritmo de optimización modifica as variables de deseño.


- Subrutina de análise de fiabilidade estrutural

O código de Matlab calcula o MPP utilizando FORM no caso de utilizar o método RIA, e HMV para os métodos PMA e SORA. Obtense a velocidade de flameo e as súas sensibilidades respecto de cada variable aleatoria mediante a execución do programa FLAS. O algoritmo termina cando se cumpren os criterios de converxencia,
que son a converxencia do índice de fiabilidade $\beta$ para o método FORM e a converxencia do MPP para o método HMV.

Os métodos de RBDO mencionados aplicáronse a dous exemplos de pontes colgantes que son a ponte do Great Belt e a ponte proyectada para o estreito de Messina.

- A ponte do Great Belt

A ponte do Great Belt en Dinamarca é unha Ponte colgante con 1624 metros aberto ao tráfico en 1999. Está formado por un caixón aerodinámico continuo en toda a súa lonxitude sen xuntas nas torres. As dúas torres de formigón armado da ponte alcanzan 254 metros de altura tratándose por tanto dunha estrutura altamente flexible. Realizouse un modelo de elementos finitos da ponte completa en Abaqus para obter tanto os modos e frecuencias de vibración como a análise estática. Obtívose unha velocidade de flameo de 80.32 m/s mediante o programa FLAS utilizando as funcións de flameo obtidas experimentalmente e unha combinación de modos proporcionados pola etapa computacional. Tras unha análise de fiabilidade obtense un índice de fiabilidade para o deseño orixinal da ponte de $\beta=8.83$ tendo en conta as 44 variables aleatorias definidas, o que se corresponde cunha probabilidade de fallo de $P_f=5.23E-19$. Considérase unha desviación típica linealmente variable desde $\sigma=0$ para unha velocidade reducida de $V^*=0$ ata $\sigma_{\text{max}}=0.15\mu$ para $V^*=30$. Posteriormente lévase a cabo a optimización probabilista da estrutura da ponte fixando os índices de fiabilidade en $\beta^T=8.845$, $\beta^T=10.0$ e $\beta^T=8.0$ para o Caso I, e $\beta^T=8.0$ para o Caso II utilizando os tres métodos de RBDO mencionados previamente.

- A ponte sobre o estreito de Messina

A ponte proyectada no estreito de Messina conectando Sicilia coa península de Italia terá o van máis longo do mundo con 3300 metros. O taboleiro de 61 metros de ancho está formado por tres caixóns conectados por vigas transversais cada 30 metros e dará servizo ao tráfico rodado mediante 6 carrís e ao tráfico ferroviario mediante dúas liñás. Os cables principais de aproximadamente 2 metros cadrados de sección
transversal colgan de dúas torres que alcanzan os 382 metros de altura. Para levar a cabo a análise aeroelástica e estrutural da ponte realizase un modelo de barras 3D no programa Abaqus. Este modelo utilizouse para obter as frecuencias e modos naturais de vibración que xunto coas funcións de flameo obtidas experimentalmente permiten calcular a velocidade de flameo, que neste caso resulta ser de 102.72 m/s. Tamén se leva a cabo unha análise de fiabilidade de acordo co método FORM. O valor obtido do índice de fiabilidade é de 12.30 considerando un total de 90 variables aleatorias, o que se corresponde cunha probabilidade de fallo de \( P_f = 4.52 \times 10^{-35} \). De novo considérase unha desviación típica linealmente variable coa velocidade de vento no ensaio experimental ata un máximo de \( \sigma_{max} = 0.15 \mu \) para una velocidade reducida de \( V^* = 30 \).

Finalmente, a optimización probabilista realizase considerando dúas situacións en función das variables de deseño incluídas no problema. No **Caso Ia**, considéranse 3 variables de deseño, correspondentes aos espesores de chapa dos caixóns laterais e fixanse os seguintes valores do índice de fiabilidade: \( \beta_T = 12.45, \beta_T = 13.0, \beta_T = 12.0 \). No **Caso Ib**, considéranse 6 variables de deseño como consecuencia de engadir os 3 espesores de chapa que definen o caixón central desta ponte e establécense os seguintes índices de fiabilidade a alcanzar: \( \beta_T = 12.45, \beta_T = 13.0, \beta_T = 12.0 \). Finalmente no **Caso II**, engádese a área dos cables principais como unha variable de deseño adicional ao conxunto considerado para o **Caso Ib**, e resólvese o problema de optimización probabilista para \( \beta_T = 12.45, \beta_T = 13.0, \beta_T = 12.0 \).

5. **Conclusións xerais dos métodos de fiabilidade estrutural e de optimización en réxime probabilista (RBDO)**

- A análise probabilista pode proporcionar resultados máis adecuados e axustados á incerteza real que existe nos parámetros que afectan a unha estrutura que os proporcionados polo emprego de coeficientes parciais de seguridade.
Verifícase que o método de fiabilidade de primeira orde (FORM) é computacionalmente eficiente mesmo para un número elevado de variables aleatorias.

Os métodos de simulación tales como Montecarlo e Latin Hypercube (LHS) teñen a vantaxe de ser fáciles de implementar, con todo, necesitan un gran número de simulacións e aínda é un obstáculo para a súa aplicación en sistemas complexos. En xeral, o LHS é máis preciso que o MCS para un número reducido de simulacións.

O uso do factor de redución comentado nesta investigación para a aplicación do método FORM resultou moi eficiente, tal e como se demostra nos exemplos analizados.

A optimización probabilista proporciona deseños máis competitivos que a optimización convencional determinista pois ten en conta a incerteza específica que ten cada parámetro que afecta á resposta estrutural. Ademais asegura que se cumpre un nivel de seguridade especificado, cousa que non sucede cando se realiza a optimización en réxime determinista.

O método desacoplado SORA é máis eficiente computacionalmente que os métodos de dous niveis RIA e PMA debido á súa formulación. Entre os métodos de dobre bucle o PMA é en xeral máis eficiente debido á súa formulación inversa respecto ao método RIA.

Os métodos de dobre bucle son en xeral máis custosos en termos computacionais posto que requiren un maior número de análise de fiabilidade. Aínda que o método desacoplado SORA require un maior número de iteracións no proceso de cada optimización determinista, o seu custo computacional é menor posto que o número de ciclos completos (optimización determinista seguido da análise de fiabilidade) realizados é pequeno.

A modificación do método HMV mediante a introdución dun factor de redución proposto nesta investigación resultou efectiva para solucionar problemas de converxencia deste algoritmo, tal e como se demostra nos exemplos.
6. **Conclusións da aplicación da análise de fiabilidade e da optimización probabilista en pontes colgantes de gran van baixo restricións aeroelásticas**

- Demóstrase a posibilidade de aplicar de forma satisfactoria optimización en réxime probabilista a pontes de gran van considerando condicións de flameo e outras restricións deterministas.
- A formulación proposta de utilizar un código proprio de Matlab xunto cos programas Abaqus, para realizar o modelo de elementos finitos, e o programa FLAS desenvolvido no grupo de Mecánica de Estruturas da Universidade da Coruña para o cálculo da velocidade de flameo, resultou efectivo para a resolución do problema de optimización probabilista.
- Mediante a realización de diversas análises de fiabilidade considerando como variables aleatorias soamente unha función de flameo, o amortiguamento estrutural e a velocidade de vento extremal é posible identificar qué funcións de flameo son máis relevantes que outras sobre a fiabilidade estrutural. Esta información é moi útil, pois indica que sería desexable incrementar a precisión á hora de realizar os ensaios en túnel de vento para determinadas funcións de flameo. Este estudo tamén resulta útil para descartar aquelas funcións de flameo que teñan pouca influencia na seguridade estrutural e deste xeito poder reducir o problema de optimización probabilista (RBDO).
- Tras realizar as análises de fiabilidade considerando como variables aleatorias os valores que definen un dos tipos de funcións de flameo, comprobouse que as funcións de flameo tipo \( A^* \) son as máis influentes na seguridade da ponte. As funcións de flameo con subíndice 2 e 3 son as máis influentes, sendo as que están relacionadas co xiro e a velocidade de xiro a torsión do taboleiro.
- Mediante a realización da análise de fiabilidade con diferentes valores da desviación típica nos datos experimentais púidose coñecer a súa influencia na seguridade estrutural para esta tipoloxía de pontes. Da realización dos ensaios dedúcese que a
dispersión dos resultados aumenta coa velocidade de vento no túnel. Por iso, considerouse unha desviación típica linealmente variable coa velocidade de vento nos ensaios.

- No exemplo do Great Belt as funcións de flameo máis relevantes son \( A_1^*, A_2^*, A_3^* \) e \( H_3^* \) e son as que se consideran como variables aleatorias no problema de optimización. Cando se fai a análise de fiabilidade considerando cada función de flameo como variable aleatoria de forma independente e para unha dispersión baixa dos datos experimentais (\( \sigma_{max}=0.15 \mu \)), obsérvase que a variación do índice de fiabilidade e menor ó 5%; sen embargo, cando se aumenta o valor máximo da dispersión a \( \sigma_{max}=0.3 \mu \), \( A_2^* \) convírtese na máis influínte sobre a seguridade estrutural, aumentando a diferenza no índice de fiabilidade ata o 30%. Isto é debido a que a velocidade de flameo é altamente sensible á dispersión dos valores que definen a función \( A_2^* \).

- No exemplo de Messina as funcións de flameo \( A_2^*, A_3^*, H_2^* \), \( H_2^* \) e \( P_3^* \) son as máis importantes na seguridade fronte a flameo, sendo \( A_3^* \) a función que máis reduce a seguridade estrutural cando se considera como variable aleatoria. Por tanto, os valores que definen esta función de flameo serán os considerados na optimización probabilista. Para unha desviación típica de valor constante do 5% e 15% nas funcións de flameo, os índices de fiabilidade para \( A_2^* \) e \( A_3^* \) obtense unha redución do índice de fiabilidade do 17% e 20% respectivamente, mentres que a variación é moi pequena para o resto de funcións.

- Verificouse que o amortiguamento estrutural ten pouca influencia sobre a análise de fiabilidade exposto tanto para o exemplo de Gran Belt como o de Messina. Por tanto, non se considerou no problema de RBDO.

- Como se esperaba, nos resultados da optimización probabilista obsérvase que cando o índice de fiabilidade fixado é menor que o do modelo orixinal a función obxectivo diminúe mentres que sucede o contrario ao aumentar o índice de fiabilidade.

- No caso do exemplo de Messina, aínda que se obteñen valores moi similares da función obxectivo para os diferentes métodos de RBDO utilizados, o espesor do lado
curto do caixón lateral $d_3$ varía substancialmente. Isto é debido á pequena contribución que esta variable de deseño ten na función obxectivo así como na resposta da ponte fronte a flameo.

- Cando se consideran 6 variables de deseño correspondentes aos espesores de chapa en caixóns laterais e central, producéuse un incremento do seu valor nas asignadas aos caixóns laterais e unha redución de espesores no caixón central. Isto débese a que as variables de deseño asociadas aos caixóns laterais contribúen a aumentar a rixidez torsional do taboleiro e por tanto a incrementar a velocidade de flameo da ponte.

- No exemplo de Messina con 6 variables de deseño cando se reduce a fiabilidade de $\beta=12.45$ no deseño orixinal a $\beta^T=12$, a redución en volume que se obtén tras a optimización é do 15%. No caso do Great Belt con 4 variables de deseño e reducindo o índice de fiabilidade esixido de $\beta=8.85$ no deseño orixinal a $\beta^T=8$ a redución de volume é aproximadamente do 7%.

- En ambos exemplos constátase o criterio optimizante de máxima tensión no cable establecido no *Caso I*. O valor óptimo da función obxectivo en ambos os casos *Caso I* e *Caso II* son moi similares pois se demostra no *Caso II* que a condición de tensión no cable é unha condición activa. Por tanto, pódese concluír que a formulación presentada no *Caso I* é computacionalmente máis efectiva.

- O tempo de computación para resolver o problema de optimización probabilista depende do método de RBDO utilizado. Entre os 3 métodos analizados o método desacoplado SORA é en xeral máis eficiente computacionalmente, chegando a utilizar só un terzo do tempo empregado polos métodos de dobre bucle RIA e PMA. Así por exemplo, no *Caso I-2* con $\beta^T=10.0$ do exemplo do Gran Belt o tempo de computación co método SORA foi de 31 horas mentres que no RIA e no PMA foi de 93 e 80 horas, respectivamente. Este problema contiña 4 variables de deseño e 43 variables aleatorias.

- O proceso que maior tempo de computación require ao resolver o problema de optimización é o cálculo da velocidade de flameo con FLAS. Obsérvase que o tempo de execución depende do número de modos aeroelásticos considerados, do número
de nós do taboleiro que se inclúen na análise a flameo, así como do valor da velocidade de flameo, pois canto máis alta sexa máis iteracións serán necesarias. No caso do Gran Belt con 77 nós no taboleiro e 13 modos aeroelásticos o tempo de cálculo da velocidade de flameo ($V_f=80.32$ m/s) é aproximadamente de 10 segundos. No caso de Messina con 225 nós no taboleiro e 7 modos aeroelásticos o tempo de computación é aproximadamente de 60 segundos, con unha velocidade de flameo de $V_f=102.72$ m/s.

- De acordo coa conclusión anterior os métodos de dobre bucle (RIA, PMA) necesitan avaliar moitas máis veces a velocidade de flameo que o método SORA, de aí o seu maior custo computacional.