An eddy current problem related to electromagnetic forming

Alfredo Bermúdez, Rafael Muñoz, Pilar Salgado
Dept. de Matemática Aplicada, Universidade de Santiago de Compostela, Spain
Carlos Reales
Dept. de Matemáticas y Estadística, Universidad de Córdoba, Colombia
Rodolfo Rodríguez
Dept. de Ingeniería Matemática, Universidad de Concepción, Chile

Abstract

Electromagnetic forming (EMF) is a high velocity cold forming process for electrically conductive metals. The aim of this paper is to analyze a numerical method to solve a transient axisymmetric eddy current problem arising from mathematical modelling of this process. We deal with a degenerate parabolic partial differential equation. Well-posedness can be proved by using regularization arguments. For numerical solution, a finite element method in space combined with an implicit Euler time discretization is proposed. Error estimates are obtained and numerical results are shown.

1 Introduction

In the Electromagnetic Forming (EMF) process the electromagnetic forces are used to deform metallic workpieces. A transient electric current is introduced in a coil using a capacitor bank and high-speed switches. It produces a magnetic field that from Faraday's law induces eddy currents in the workpiece. The magnetic field, together with the eddy currents, originate the Lorentz forces that drive the deformation of the workpiece [5, 8, 10].

From the mathematical point of view, the motion of the workpiece introduces two difficulties to the problem. First, the domain changes along the time. Second, the velocity in the workpiece produces currents that in principle should be added in the Ohm's law. While the difficulties arising from this additional term have been studied in [3] with a fixed domain, in EMF typically these currents are not significant, so they are neglected in the present paper.

The axisymmetry allows us to write the problem in terms of the azimuthal component of a magnetic vector potential defined in a meridional section of the domain (see, for instance, [1]). This leads to consider a transient problem where the term involving the time derivative only appears in a part of the domain, which changes with time. The eddy current model must be coupled with an adequate mechanical model for the deformation of the workpiece but in this paper we take the motion of the workpiece as a data.

The outline of this paper is as follows: in Section 2, we describe the transient eddy current model and introduce a magnetic vector potential formulation under axisymmetric assumptions. In Section 3, a well-posed weak formulation is stated. In Section 4, we introduce the finite element space discretization and obtain error estimates. In Section 5, we propose a backward Euler scheme for time discretization and prove error estimates for the fully discretized problem. In Section 6, some numerical results for an EMF device are shown.

2 Statement of the problem

Two different geometries can be seen in Figure 1. In order to have a domain with cylindrical symmetry, we replace the coil by several concentric rings having toroidal geometry and carrying the same current intensity. For numerical purposes it is convenient to cut the whole space. More specifically, we introduce a three dimensional cylinder $\widetilde{\Omega}$ of radius R and height L containing the coil and the workpiece. Then, by the cylindrical symmetry, we can work in a meridional section of $\widetilde{\Omega}$ denoted by Ω. Let $\Omega_{\mathrm{S}} := \Omega_1 \cup \cdots \cup \Omega_m$ where Ω_k $k = 1, \cdots, m$ are the meridional sections of the coil. Let Ω_t be the meridional section of the workpiece at time t. We assume that $\Omega_t \cap \Omega_{\mathrm{S}} = \emptyset$ for all t. Let $\Omega_t^{\mathrm{A}} := \Omega \setminus (\Omega_{\mathrm{S}} \cup \Omega_t)$ be the section of the domain occupied by air. Finally, let Γ_0 be the intersection between $\partial\Omega$ and the symmetry axis ($r = 0$), and $\Gamma_D := \partial\Omega \setminus \Gamma_0$ (see Figure 2).

We will use standard notation in electromagnetism:

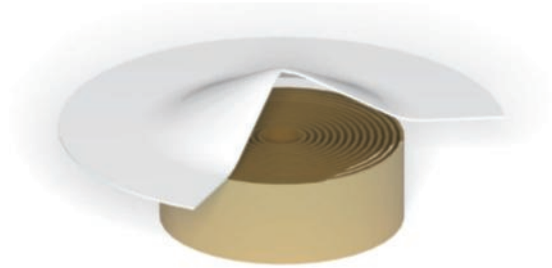

Figure 1: Sketch of the 3D-domain of the EMF System.

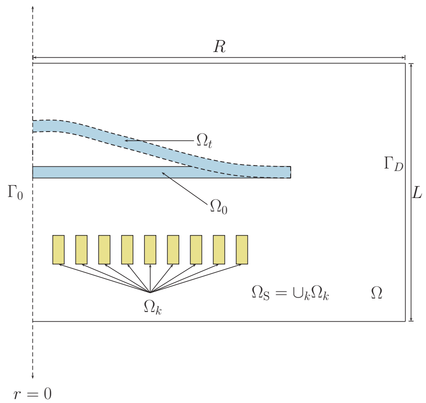

Figure 2: Sketch of the meridional section of the EMF system.

- $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic induction,
- $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{J}$ is the current density,
- μ is the magnetic permeability, σ is the electric conductivity.

The magnetic permeability μ is taken as a positive constant in the whole domain. The conductivity σ vanishes outside the workpiece. This piece can be made of different materials, each with a different conductivity. We will make this assumption more precise below;

by the moment we just assume

$$0 < \underline{\sigma} \leq \sigma \leq \overline{\sigma}, \qquad \text{in the workpiece,}$$
$$\sigma = 0, \qquad \text{outside of the workpiece.}$$

In this kind of problem, the electric displacement can be neglected in Ampère's law, leading to the so called eddy current model:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J}, \tag{2.1}$$
$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0}, \tag{2.2}$$
$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.3}$$

This system must be completed with the following relations:

$$\boldsymbol{B} = \mu \boldsymbol{H},$$

and

$$\boldsymbol{J} = \begin{cases} \sigma \boldsymbol{E}, & \text{in the workpiece,} \\ \boldsymbol{J}_{\mathrm{S}}, & \text{in the coil, (data),} \\ \mathbf{0}, & \text{in the air.} \end{cases} \tag{2.4}$$

Thus, the current density $\boldsymbol{J}$ is taken as data in the coil and unknown in the workpiece Ω_t. Since σ vanishes outside Ω_t, the relation above can be written in a single equation as follows:

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}}.$$

We assume that all the physical quantities are independent of the angular coordinate θ and that the source current density field has only azimuthal non-zero component, i.e.,

$$\boldsymbol{J}(t, r, \theta, z) = J(t, r, z)\boldsymbol{e}_\theta.$$

Proceeding as in [1] and [3] it can be proved that

$$\boldsymbol{H}(t, r, \theta, z) = H_r(t, r, z)\boldsymbol{e}_r + H_z(t, r, z)\boldsymbol{e}_z,$$
$$\boldsymbol{B}(t, r, \theta, z) = B_r(t, r, z)\boldsymbol{e}_r + B_z(t, r, z)\boldsymbol{e}_z,$$
$$\boldsymbol{E}(t, r, \theta, z) = E(t, r, z)\boldsymbol{e}_\theta,$$

Moreover, because of (2.3), we can introduce a magnetic vector potential $\boldsymbol{A}$ for $\boldsymbol{B}$,

$$\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}, \tag{2.5}$$

of the form

$$\boldsymbol{A}(t, r, \theta, z) = A(t, r, z)\boldsymbol{e}_\theta \tag{2.6}$$

and such that (2.2) leads to

$$-\boldsymbol{E} = \frac{\partial \boldsymbol{A}}{\partial t}.$$

Therefore, the Maxwell equation system (2.1)-(2.3) can be rewritten in terms of the vector potential $\boldsymbol{A}$ as follows:

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) = \boldsymbol{J} = J\boldsymbol{e}_\theta,$$

where

$$J = \begin{cases} 0 & \text{in } \Omega_t^{\mathrm{A}}, \\ -\sigma(t)\dfrac{\partial A}{\partial t} & \text{in } \Omega_t, \\ J_{\mathrm{S}} & \text{in } \Omega_{\mathrm{S}} \text{ (data)}. \end{cases} \tag{2.7}$$

Thus, we are lead to the following parabolic-elliptic problem:

$$\begin{cases} \sigma(t)\dfrac{\partial A}{\partial t}\boldsymbol{e}_\theta + \mathbf{curl}\left(\dfrac{1}{\mu}\,\mathbf{curl}\,(A\boldsymbol{e}_\theta)\right) & = 0 \quad \text{in } \Omega_t, \\ \mathbf{curl}\left(\dfrac{1}{\mu}\,\mathbf{curl}\,(A\boldsymbol{e}_\theta)\right) & = J_{\mathrm{S}}\boldsymbol{e}_\theta \quad \text{in } \Omega_{\mathrm{S}}, \\ \mathbf{curl}\left(\dfrac{1}{\mu}\,\mathbf{curl}\,(A\boldsymbol{e}_\theta)\right) & = 0 \quad \text{in } \Omega_t^{\mathrm{A}}. \end{cases} \tag{2.8}$$

3 Weak formulation

Let $L^2_r(\Omega)$ be the weighted Lebesgue space of all measurable functions A defined in Ω such that

$$\|A\|^2_{L^2_r(\Omega)} := \int_\Omega |A|^2 r\,dr\,dz < \infty.$$

The weighted Sobolev space $H^k_r(\Omega)$ consists of all functions in $L^2_r(\Omega)$ whose derivatives up to the order k are also in $L^2_r(\Omega)$. We define the norms and semi-norms in the standard way; in particular

$$|A|^2_{H^1_r(\Omega)} := \int_\Omega \left(|\partial_r A|^2 + |\partial_z A|^2\right) r\,dr\,dz.$$

Let $L^2_{1/r}(\Omega)$ be the weighted Lebesgue space of all measurable functions A defined in Ω such that

$$\|A\|^2_{L^2_{1/r}(\Omega)} := \int_\Omega \frac{|A|^2}{r}\, dr\, dz < \infty.$$

Let us define the Hilbert space $\widetilde{H}^1_r(\Omega)$ by

$$\widetilde{H}^1_r(\Omega) := \left\{A \in H^1_r(\Omega) : A \in L^2_{1/r}(\Omega)\right\}$$

with the norm

$$\|A\|_{\widetilde{H}^1_r(\Omega)} := \left(\|A\|^2_{H^1_r(\Omega)} + \|A\|^2_{L^2_{1/r}(\Omega)}\right)^{1/2}.$$

Finally, let

$$\mathcal{V} := \{A \in \widetilde{H}^1_r(\Omega) : \ A = 0 \,\mathrm{on}\, \Gamma_D\}.$$

Since part of our domain Ω changes with time, we need to define a reference domain $\widehat{\Omega}$ and an application

$$\begin{aligned} X_t : \widehat{\Omega} &\longrightarrow \Omega_t, \\ \widehat{x} &\longmapsto X_t(\widehat{x}), \end{aligned}$$

transforming $\widehat{\Omega}$ into Ω_t (see Figure 3). We assume X_t is a sufficiently smooth diffeomorphism with respect to space and differentiable with respect to time. A usual way to define

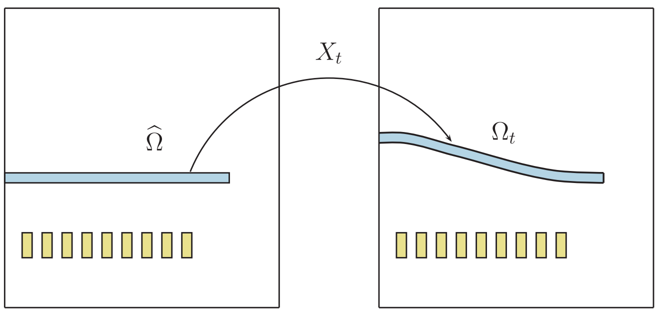

Figure 3: Reference domain.

X_t is through a vector field $\boldsymbol{v}$ which represents the velocity of the workpiece and which is taken as a data in our analysis. More precisely, from now on we assume $\boldsymbol{v}$ is continuous with respect to time and continuously differentiable and globally Lipschitz with respect to space. We take $\widehat{\Omega} := \Omega_{t=0}$, and define $t \mapsto X_t$ as the solution of the following problem:

$$\begin{aligned} \frac{\partial X_t}{\partial t}(\widehat{x}) &= \boldsymbol{v}(t, X_t(\widehat{x})), \\ X_0(\widehat{x}) &= \widehat{x}. \end{aligned}$$

In the appendix section a detailed description of this process can be found. On the other hand, the conductivity σ is taken such that

$$\sigma(t,x) = \widehat{\sigma}(\widehat{x}), \tag{3.1}$$

where $x = X_t(\widehat{x})$ and $\widehat{\sigma}$ is the conductivity in the reference domain $\widehat{\Omega}$, which is a measurable function and satisfies

$$0 < \underline{\sigma} \leq \widehat{\sigma}(\widehat{x}) \leq \overline{\sigma}, \qquad \hat{x} \in \hat{\Omega}.$$

This means that σ only depends on t through x and, from a physical point of view, that the conductivity of each material point remains constant along the process.

Let us introduce the following non-cylindrical open subset of $\Omega \times (0,T)$,

$$Q := \{(x,t) : \ x \in \Omega_t, \ t \in (0,T)\}.$$

Let us consider the following Banach spaces of functions defined in Q :

$$L^p_r(Q) := \{\varphi : Q \to \mathbb{R} \text{ measurable with } \int_0^T \int_{\Omega_t} |\varphi|^p r \, dr \, dz \, dt < \infty\},$$

endowed with the norm

$$\|\varphi\|_{L^p_r(Q)} := \left(\int_0^T \int_{\Omega_t} |\varphi|^p r \, dr \, dz \, dt \right)^{1/p},$$

and

$$W^{1,p}_r(Q) := \{\varphi \in L^p_r(Q) : \ \frac{\partial \varphi}{\partial t} \in L^p_r(Q), \ \frac{\partial \varphi}{\partial r} \in L^p_r(Q), \ \frac{\partial \varphi}{\partial z} \in L^p_r(Q)\},$$

endowed with the norm

$$\|\varphi\|_{W^{1,p}_r(Q)} := \left(\|\varphi\|^p_{L^p_r(Q)} + \|\frac{\partial \varphi}{\partial t}\|^p_{L^p_r(Q)} + \|\frac{\partial \varphi}{\partial r}\|^p_{L^p_r(Q)} + \|\frac{\partial \varphi}{\partial z}\|^p_{L^p_r(Q)} \right)^{1/p}.$$

Moreover we denote $H^1_r(Q) := W^{1,2}_r(Q)$.

Next we deduce a variational formulation of (2.8) and prove it is well-posed. For this purpose, let us multiply (2.8) by a test vector field $Z\boldsymbol{e}_\theta$ with $Z \in \mathcal{V}$, integrate over Ω and use a Green's formula to obtain

$$\int_{\Omega_t} \sigma \frac{\partial A}{\partial t} Z r\, dr\, dz + a(A, Z) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z r\, dr\, dz, \tag{3.2}$$

where

$$a(A, Z) := \int_\Omega \frac{1}{\mu} \,\mathbf{curl}\, A\boldsymbol{e}_\theta \cdot \mathbf{curl}\, Z\boldsymbol{e}_\theta r\, dr\, dz.$$

It is shown in [7, Propositions 2.1 and 3.1] that a is $\mathcal{V}$-elliptic; namely, there exists $\alpha > 0$ such that

$$a(Z, Z) \geq \alpha \|Z\|^2_{\widetilde{H}^1_r(\Omega)} \quad \forall Z \in \mathcal{V}.$$

Notice that (3.2) corresponds to a degenerate parabolic problem because the term including the partial derivative of A with respect to time is only defined in Ω_t. The proof of the following theorem can be found in [2].

Theorem 1. *Let $J_{\mathrm{S}} \in H^1(0, T; L^2_r(\Omega_{\mathrm{S}}))$ and $A^0 \in \widetilde{H}^1_r(\Omega_0)$. Then, there exists a unique solution $A \in L^2(0, T; \mathcal{V})$, with $\frac{\partial A}{\partial t} \in L^2_r(Q)$, to the weak problem,*

$$\begin{aligned} \int_{\Omega_t} \sigma \partial_t A Z r\, dr\, dz + a(A, Z) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z r\, dr\, dz \quad \forall Z \in \mathcal{V}, a.e.\ t \in [0, T] \\ A(0) = A^0 \ \text{in}\ \Omega_0. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} &\|\partial_t A\|_{L^2_r(Q)} + \|A\|_{L^\infty(0,T;\mathcal{V})} \\ &\quad \leq C \left\{ \|A^0\|^2_{\widetilde{H}^1_r(\Omega_0)} + \int_0^T \|J_{\mathrm{S}}(t)\|^2_{L^2_r(\Omega_{\mathrm{S}})}\, dt + \int_0^T \|\partial_t J_{\mathrm{S}}(t)\|^2_{L^2_r(\Omega_{\mathrm{S}})} dt \right\}. \end{aligned} \tag{3.4}$$

Remark 1. *Since $A \in L^2(0, T; \mathcal{V})$ and $\frac{\partial A}{\partial t} \in L^2_r(Q)$ then $A \in H^1_r(Q)$. From the trace theorem this implies that $A|_{\Omega_0 \times \{0\}} \in L^2_r(\Omega_0 \times \{0\}) \simeq L^2_r(\Omega_0)$. Thus the initial condition in (3.3) makes sense.*

Theorem 2. *We make the same assumptions as in Theorem 1. Then the solution A satisfies $\sqrt{t}\, \partial_t A \in L^2(0, T; \mathcal{V})$.*

Proof: See [2].

□

4 Semi-discrete problem. Finite element approximation

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω where h is the mesh-size. Let

$$\mathcal{V}_h := \{A_h \in \mathcal{V} : A_h|_T \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}.$$

Let us emphasize that in principle we do not assume that the meshes are fitted to Ω_0.

We introduce the following semi-discrete problem: find $A_h \in L^2(0,T;\mathcal{V}_h)$ with $\partial_t A_h \in L^2(Q)$ such that

$$\begin{aligned} \int_{\Omega_t} \sigma \partial_t A_h Z_h r\, dr\, dz + a(A_h, Z_h) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z_h r\, dr\, dz \quad \forall Z_h \in \mathcal{V}_h, \\ A_h(0)|_{\Omega_0} = A_h^0|_{\Omega_0} \end{aligned} \tag{4.1}$$

where A_h^0 has to satisfy the following conditions (see the proof of Theorem 1):

$$A_h^0|_{\Omega_0} \to A^0 \quad \text{in } L_r^2(\Omega_0), \tag{4.2}$$

$$\|A_h^0\|_{\widetilde{H}_r^1(\Omega)} \leq C\|A^0\|_{\widetilde{H}_r^1(\Omega_0)}. \tag{4.3}$$

To obtain A_h^0 we proceed as in the proof of Theorem 1. Let $\widehat{A}^0 \in \mathcal{V}$ be as in that proof and A_h^0 the Clément interpolant in $\mathcal{V}_h$ of $\widehat{A}^0$ as defined in [9, Section 7]. From the properties proved in this reference we have that

$$\|A_h^0\|_{\widetilde{H}_r^1(\Omega)} \leq C\|\widehat{A}^0\|_{\widetilde{H}_r^1(\Omega)}$$

and

$$\|A_h^0 - \widehat{A}^0\|_{L_r^2(\Omega)} \longrightarrow 0, \text{ as } h \to 0.$$

Therefore, straightforward computations allow us to conclude (4.2) and (4.3).

Similar to the proof of Theorem 1 one can show that problem (4.1) has a unique solution for $A^0 \in \widetilde{H}_r^1(\Omega_0)$ and $J_{\mathrm{S}} \in H^1(0,T;L_r^2(\Omega_{\mathrm{S}}))$. Moreover, the following estimate holds:

$$\begin{aligned} \operatorname*{sup\,ess}_{t\in[0,T]} \int_{\Omega_t} \sigma(A_h(t))^2 r\, dr\, dz + \int_0^T \|A_h(s)\|^2_{\widetilde{H}_r^1(\Omega)}\, ds \leq C\Big\{\int_{\Omega_0} \sigma(A^0)^2 r\, dr\, dz \\ + \int_0^T \|J_{\mathrm{S}}(s)\|^2_{L_r^2(\Omega_{\mathrm{S}})}\, ds\Big\}. \end{aligned}$$

Now we are in a position to prove error estimates for the computed vector potential A_h as well as for the physical quantities of interest that can be derived from it, namely, the approximations $\boldsymbol{B}_h$ and $\boldsymbol{J}_h$ of the magnetic induction $\boldsymbol{B}$ and the current density $\boldsymbol{J}$. According to (2.5) and (2.6), let us define

$$\boldsymbol{B}_h := \mathbf{curl}(A_h \boldsymbol{e}_\theta).$$

The current density $\boldsymbol{J}$ in the workpiece is given by $\boldsymbol{J} = \sigma(-\frac{\partial A}{\partial t})\boldsymbol{e}_\theta$. Hence we define the computed current density as follows:

$$\boldsymbol{J}_h := -\sigma \left(\frac{\partial A_h}{\partial t} \right) \boldsymbol{e}_\theta \quad \text{in } \Omega_t.$$

The following error estimates is proved in [2]

Theorem 3. *Let A and A_h be the solutions of problems (3.2) and (4.1), respectively. Let $\boldsymbol{B}$ be defined by (2.5) and (2.6) and $\boldsymbol{J}$ by (2.4) and (2.7). Let $\boldsymbol{B}_h$ and $\boldsymbol{J}_h$ be defined as above. If $A \in H^1(0,T;H_r^2(\Omega))$, then, there exists a positive constant C, independent of h, such that*

$$\|A - A_h\|_{L^\infty(0,T;L_r^2(\Omega_t))} \leq C\left\{\|A(0) - A_h(0)\|_{L_r^2(\Omega_0)} + h^2\|A\|_{H^1(0,T;H_r^2(\Omega))}\right\}, \quad (4.4)$$

$$\|\boldsymbol{B} - \boldsymbol{B}_h\|_{L^2(0,T;L_r^2(\Omega))} \leq C\left\{\|A(0) - A_h(0)\|_{L_r^2(\Omega_0)} + h\|A\|_{H^1(0,T;H_r^2(\Omega))}\right\}, \quad (4.5)$$

$$\|A - A_h\|_{L^\infty(0,T;L_r^2(\Omega))} \leq C\left\{\|A(0) - A_h(0)\|_{\widetilde{H}_r^1(\Omega)} + h\|A\|_{H^1(0,T;H_r^2(\Omega))}\right\}, \quad (4.6)$$

$$\|\boldsymbol{J} - \boldsymbol{J}_h\|_{L^2(0,T;L_r^2(\Omega_t))} \leq C\left\{\|A(0) - A_h(0)\|_{\widetilde{H}_r^1(\Omega)} + h\|A\|_{H^1(0,T;H_r^2(\Omega))}\right\}. \quad (4.7)$$

5 Fully discrete problem

Let us consider a uniform partition of the time interval $[0,T]$: $\{t^k := k\Delta t,\ k = 1,\dots,N\}$ with time step $\Delta t := \frac{T}{N}$. For time discretization we use the backward Euler approximation:

$$\int_{\Omega_{t^k}} \sigma(t^k) \frac{\partial A_h}{\partial t} Z_h r\, dr\, dz \approx \frac{1}{\Delta t} \int_{\Omega_{t^k}} \sigma(t^k) \left(A_h^k - A_h^{k-1}\right) Z_h r\, dr\, dz.$$

Thus, the fully discrete approximation of our problem is defined as follows:

Given $A_h^0 \in \mathcal{V}_h$, for $k = 1,\dots,N$, find $A_h^k \in \mathcal{V}_h$ such that

$$\frac{1}{\Delta t} \int_{\Omega_{t^k}} \sigma(t^k) \left(A_h^k - A_h^{k-1}\right) Z_h r\, dr\, dz + a(A_h^k, Z_h) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}(t^k) Z_h r\, dr\, dz\,, \forall Z_h \in \mathcal{V}_h. \quad (5.1)$$

The previous scheme needs an initial data in a neighborhood of Ω_0 containing Ω_{t^1}. Let us assume that $A(0)$ is known in all Ω and take A_h^0 as an approximation of $A(0)$. The following theorem has been proved in [2].

Theorem 4. *If $J_{\mathrm{S}} \in H^1(0,T;L_r^2(\Omega))$, then problem (5.1) has a unique solution and there exists a positive constant C such that*

$$\max_{1\leq k\leq N} \|A_h^k\|_{\widetilde{H}_r^1(\Omega)} \leq C\left\{\|A_h^0\|_{\widetilde{H}_r^1(\Omega)} + \|J_{\mathrm{S}}\|_{H^1(0,T;L_r^2(\Omega))}\right\}.$$

Remark 2. *Since the domain where the derivative of A is approximated changes with time, terms like $\int_{\Omega_{t^k}} \sigma(t^k)A_h^{k-1}$ appear in the numerical scheme. This is the reason why we cannot follow a more standard approach as that used for the semidiscrete problem. Anyway we have succeeded in proving the stability of the fully discrete scheme by assuming further regularity for J_{S}.*

Our next goal is to get error estimates for the solution of the discrete problem (5.1). To do this we introduce some notation. Given $(\phi^0,\ldots,\phi^N) \in \mathbb{R}^{N+1}$, we define the backward difference quotient

$$\bar{\partial}\phi^k := \frac{\phi^k - \phi^{k-1}}{\Delta t}, \qquad k = 1,\ldots,N.$$

For A being the solution of (3.3) and A_h^k that of (5.1), we write

$$A(t^k) - A_h^k = \delta_h^k + \rho_h^k,$$

with

$$\delta_h^k := \mathbf{P}_h A(t^k) - A_h^k, \quad k = 1,\ldots,N,$$

and

$$\rho_h^k := A(t^k) - \mathbf{P}_h A(t^k), \quad k = 0,\ldots,N.$$

To define δ_h^0 we use the approximation A_h^0 of $A(0)$:

$$\delta_h^0 := \mathbf{P}_h A(0) - A_h^0.$$

Finally, we define the truncation errors

$$\tau^k := \bar{\partial}A(t^k) - \partial_t A(t^k), \quad k = 1,\ldots,N.$$

The first step is to estimate δ_h^k in terms of ρ_h^k and τ^k. The proofs of the next lemmas can be found in [2]

Lemma 5. *Let $A \in \mathcal{C}^0([0,T];\mathcal{V}) \cap \mathcal{C}^1([0,T];L_r^2(\Omega))$ be the solution of problem (3.3), and let $\tau^k := \bar{\partial}A(t^k) - \partial_t A(t^k)$. Then*

$$\Delta t \sum_{k=1}^{N} \int_{\Omega_{t^k}} \sigma(t^k)(\bar{\partial}\delta_h^k)^2 r\, dr\, dz + \max_{1\le k\le N} \|\delta_h^k\|^2_{\widetilde{H}_r^1(\Omega)} \le C\|\delta_h^0\|^2_{\widetilde{H}_r^1(\Omega)} + C\Delta t \sum_{k=1}^{N} \left\{ \|\bar{\partial}\rho_h^k\|^2_{L_r^2(\Omega_{t^k})} + \|\tau^k\|^2_{L_r^2(\Omega_{t^k})} \right\}$$

Lemma 6. *Let A be the solution of problem (3.3). There exists C independent of h and Δt such that, if $A \in H^1(0,T;H_r^2(\Omega))$, then*

$$\left(\Delta t \sum_{k=1}^{N} \|\bar{\partial}\rho_h^k\|^2_{L_r^2(\Omega_{t^k})} \right)^{1/2} \le Ch^2 \|A\|_{H^1(0,T;H_r^2(\Omega))}$$

and, if $A \in H^2(0,T;L_r^2(\Omega))$, then

$$\left(\Delta t \sum_{k=1}^{N} \|\tau^k\|^2_{L_r^2(\Omega_{t^k})} \right)^{1/2} \le C\Delta t \|A\|_{H^2(0,T;L_r^2(\Omega))}.$$

Now we give error estimates for the computed vector potential A_h^k as well as for the physical quantities of interest that can be derived from it: approximations $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$ of the magnetic induction $\boldsymbol{B}$ and the current density $\boldsymbol{J}$, which are defined as follows:

$$\boldsymbol{B}_h^k := \mathbf{curl}(A_h^k \boldsymbol{e}_\theta),$$

$$\boldsymbol{J}_h^k := -\sigma(t^k)\bar{\partial}A_h^k \boldsymbol{e}_\theta \quad \text{in } \Omega_{t^k}.$$

The error estimates for this quantities will be a consequence of Lemmas 5 and 6. The former depends on the particular approximation A_h^0 of $A(0)$ used in problem (5.1). In fact, recall that A_h^0 appears in the definition of δ_h^0. If the solution to problem (3.3) is sufficiently smooth at time $t = 0$, namely $A(0) \in H_r^2(\Omega) \cap \mathcal{V}$, then we can take

$$A_h^0 := \mathcal{I}_h A(0),$$

where we denote by $\mathcal{I}_h$ the Lagrange interpolant operator. In such a case we have the following result.

Theorem 7. *Let A and A_h^k be the solutions of problems (3.3) and (5.1), respectively. Let $\boldsymbol{B} = \mathbf{curl}(A\boldsymbol{e}_\theta)$, $\boldsymbol{J} = -\sigma\,(\partial_t A)|_{\Omega_t}\boldsymbol{e}_\theta$ and $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$ be as defined above. If $A \in H^1(0,T;H_r^2(\Omega)\cap\mathcal{V})\cap H^2(0,T;L_r^2(\Omega))$ and $A_h^0 := \mathcal{I}_h A(0)$, then there exists C independent of h, and Δt such that*

$$\max_{1\le k\le N}\|\boldsymbol{B}(t^k)-\boldsymbol{B}_h^k\|_{L_r^2(\Omega)} \le C\left\{h\|A\|_{H^1(0,T;H_r^2(\Omega))}+\Delta t\|A\|_{H^2(0,T;L_r^2(\Omega))}\right\},$$

$$\left\{\Delta t\sum_{k=1}^{N}\|\boldsymbol{J}(t^k)-\boldsymbol{J}_h^k\|^2_{L_r^2(\Omega_{t^k})}\right\}^{1/2} \le C\left\{h\|A\|_{H^1(0,T;H_r^2(\Omega))}+\Delta t\|A\|_{H^2(0,T;L_r^2(\Omega))}\right\}.$$

6 Numerical tests

We have used the numerical method to compute the current density and the Lorentz force in an example taken from an electromagnetic forming process. We consider the geometry and physical data of the axisymmetric electromagnetic forming test described in [8] (see Figure 4) which is a classical benchmark (see [8, 10] for more details). The geometrical and physical data are given in Table 1.

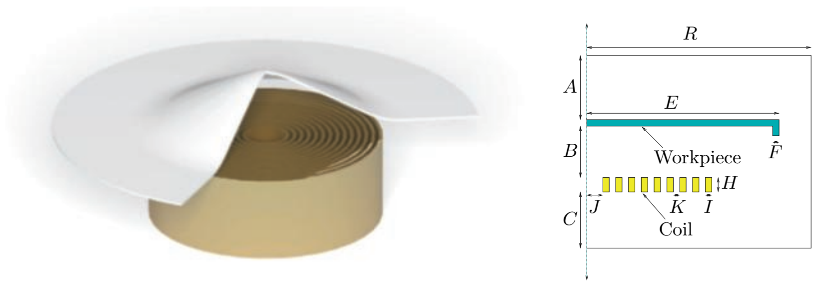

Figure 4: EMF. Geometry of the benchmark problem.

The current density $J_{\rm S}$ was assumed to be constant in each turn of the coil. It was obtained from [8] where the corresponding intensity was reported. Figure 5 shows this intensity during the whole process. To determine Ω_t, we assumed that the workpiece is a rigid body moving under the action of a Lorentz force $\mathbf{f} = \boldsymbol{J}\times\boldsymbol{B}$. To compute $\mathbf{f}$, we made a preliminary estimate of $\boldsymbol{J}$ and $\boldsymbol{B}$ by solving the electromagnetic model with the workpiece at a fixed domain Ω_0, as described in Section 6.3 of [3].

Table 1: Test 2. Geometrical data and physical parameters:

Thickness of the workpiece (F):	0.0012 m
Height of the tool coil (H):	0.0115 m
Width of each turn coil (I):	0.0025 m
Distance between coil turns (K):	0.0003 m
Distance coil-workpiece (B):	0.002 m
Vertical distance from coil to bottom (C):	0.05 m
Vertical distance from workpiece to the top (A):	0.05 m
Width of the workpiece (E):	0.115 m
Width of the rectangular box (R):	0.2 m
Number of coil turns:	9
Electrical conductivity of metal (σ):	25900 (Ohm m)$^{-1}$
Magnetic permeability of all materials (μ):	$4\pi 10^{-7}$ Hm^{-1}
Final time (T):	90μs

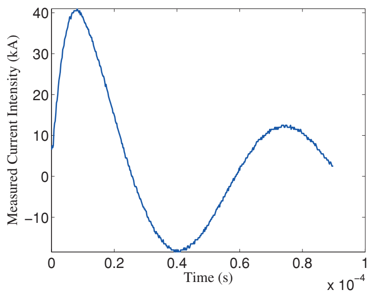

Figure 5: Test 2. Current intensity (kA) vs. time (s).

We have used the mesh shown in Figure 11 which is more refined in the zone occupied by Ω_t for $t \in [0, T]$. We have used a low order integration rule as in the previous test to compute the integrals of piecewise smooth functions. Figure 7 shows the resulting velocity of the rigid workpiece. Figures 8-10 show the computed current densities at 10ms, 35ms, and 90ms, respectively.

Acknowledgements

The research of these authors was partially supported by Xunta de Galicia under research project INCITE09 207 047 PR, by Ministerio de Ciencia e Innovación (Spain) under re-

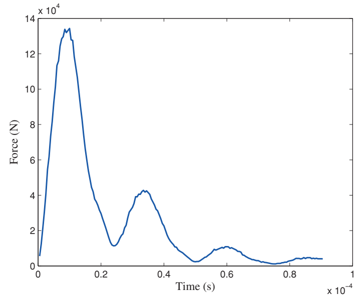

Figure 6: Test 2. Lorentz force in N vs. time (s).

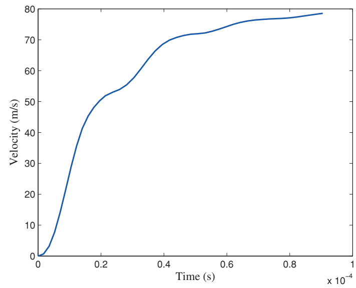

Figure 7: Test 2. Velocity (m/s) vs. time (s).

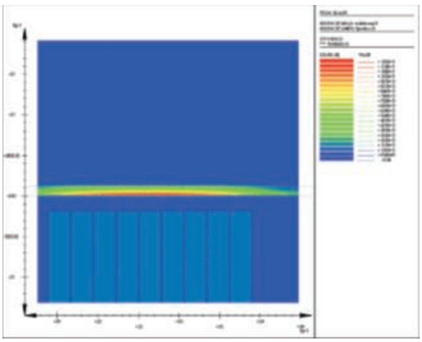

Figure 8: Test 2. Current density in A/m^2 at $10\mu s$.

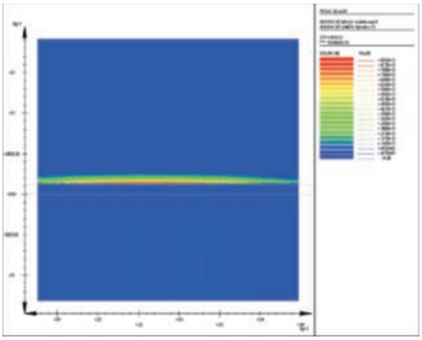

Figure 9: Test 2. Current density in A/m^2 at $35\mu s$.

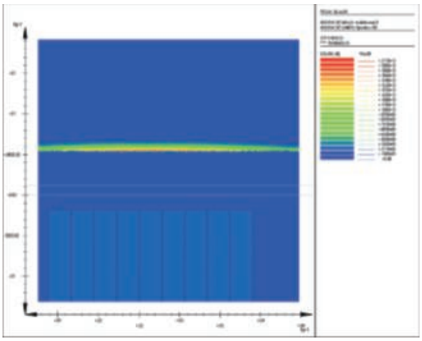

Figure 10: Test 2. Current density in A/m^2 at $90\mu s$.

search projects Consolider MATHEMATICA CSD2006-00032 and MTM2008-02483 and by FONDAP in Applied Mathematics (Chile).

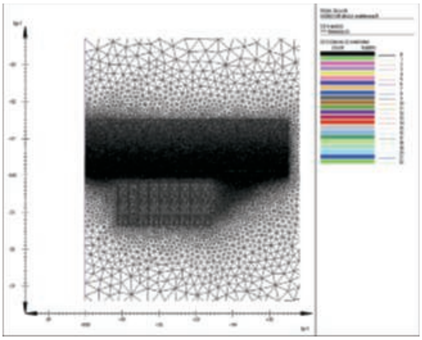

Figure 11: Test 2. Meshes zoom.

References

[1] A. Bermúdez, C. Reales, R. Rodríguez, and P. Salgado, *Numerical analysis of a finite element method to solve the axisymmetric eddy current model of an induction furnace.* IMA Journal of Numerical Analysis (doi:10.1093/imanum/drn063).

[2] A. Bermúdez, R. Muñoz, C. Reales, R. Rodríguez, and P. Salgado, *Numerical analysis of a transient eddy current problem arising from electromagnetic forming.* In preparation.

[3] A. Bermúdez, C. Reales, R. Rodríguez, and P. Salgado, *Mathematical and numerical analysis of a transient eddy current axisymmetric problem involving velocity terms.* Preprint del Departamento de Ingeniería Matemática de la Universidad de Concepción.

[4] K. E. Brenan, S.L. Campbell, and L.R. Petzold *Numerical solution of initial-value problems in differential-algebraic equations*, Philadelphia, SIAM, 1996.

[5] A. El-Azab, M. Garnich, and A. Kapoor, *Modeling of the electromagnetic forming of sheet metals: state-of-art an future needs*, J. Mat. Process. Tech., 142 (2003), pp. 744–754.

[6] A. Ern and J. Guermond. *Theory and Practice of Finite Elements,* New York, Springer, 2004.

[7] J. Gopalakrishnan and J. Pasciak. *The convergence of V-cycle multigrid algorithms for axisymmetric Laplace and Maxwell equations,* Math. Comp., 75 (2006), pp. 1697–1719.

[8] M. Kleiner, A. Brosius, H. Blum, F. Suttmeier, M. Stiemer, Svendsen, J. Unger, and S. Reese., *Benchmark simulation for coupled electromagnetic-mechanical metal forming processes*, University of Dortmund, Chair of Forming Technology.

[9] B. Mercier and G. Raugel. *Resolution d'un problème aux limites dans un ouvert axisymétrique par éléments finis en r, z et séries de Fourier en θ,* RAIRO, Anal. Numér., 16 (1982), pp. 405–461.

[10] M. Stiemer, O. J. Unger, B. Svendsen and H. Blum, *Algorithmic formulation and numerical implementation of coupled electromagnetic-inelastic continuum models for electromagnetic metal forming*, Internat. J. Numer. Methods Engrg., 68 (2006), pp. 1697–1719.