Introduction to inverse problems

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Motivation

This introductory course to the field of inverse problems aims at making sensitive to the public the ill-posed character of these types of problems.

Let us give a very intuitive definition of what an inverse problem is.

A direct problem is a problem where one deduces effects from causes, whereas in the inverse problem one wants to find the causes which lead to some determined effects one can notice or measure.

The direct problem is usually well posed according to Hadamard, that is:

i) The solution exists

ii) It is unique

iii) It depends continuously on the data

Unlike this, the inverse problems are usually ill-posed because one (or more) of the conditions i), ii) or iii) is not fulfilled, (usually ii) and iii)). The same effects may be induced by various causes ...

The inverse problems occur very often in everyday life: in image processing, bioengineering (such that the inverse electro-cardiographic problem where one wants to recover the potential on the heart from measurements on the torso) or mechanics where one wants to recover some hidden flaws such as cracks, holes or inhomogeneities from overdetermined boundary data.

Within this introductory course, we will be concerned with inverse problems defined by overspecified boundary data. We will focus on the inverse problem of determining a Robin parameter and that of solving a Cauchy problem.

Let us note that the Robin coefficient may be recovered by solving a Cauchy problem.

We consider the following problem: provided a partial differential operator is known within a domain Ω, recover the value of the boundary data on a part Γ_u of its boundary, overspecified data being available on the remaining part Γ_d of the boundary. This problem will be called data completion problem, it is as a Cauchy problem for the partial differential operator and the manifold Γ_u.

This kind of problem arises in many industrial, engineering or biomedical applications under various forms: identification of boundary conditions, exchange or loss factor on unreachable part of the boundary, expansion of measured surface fields inside a body from partial boundary measurements, but it also can be the first step in general parameters identification problems where only partial boundary data are under control.

The more common problem, borrowed from thermostatics, consists in recovering the temperature in a given domain when its distribution and the heat flux are known over accessible region of the boundary. We shall be presenting the issue in the framework of thermostatics, which is mathematically identical to the electrostatics case encountered in electric impedance tomography. Note that this procedure is extendable to elastostatics or any other symmetric linear elliptic problem.

1 Identifiability, stability, regularization

This section is devoted to basic concepts regarding the inverse problems.
All the results regarding the inverse Robin problem come from the paper by Slim Chaabane and Mohamed Jaoua [17].

1.1 Identifiability

To illustrate the first concept (uniqueness) or identifiability let us consider the following Robin inverse problem [17].

We are interested in determining the Robin coefficient q of some material of which a body occupying the connected domain $\Omega \in \mathbb{R}^2$ or $\mathbb{R}^3$ is composed. To this end we shall use boundary measurements of the temperature on some part K of the boundary $\partial\Omega$, which is assumed to be of $C^{1,1}$ regularity, and, moreover, let γ, Γ_D and Γ_N be three open subsets of the boundary such that,

$$\partial\Omega = \overline{\gamma} \cup \overline{\Gamma_D} \cup \overline{\Gamma_N}.$$

The direct problem associated with the inverse one we deal with is therefore the following:

$$\begin{cases} \triangle u &= 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} &= \varphi \text{ on } \Gamma_N, \\ u &= 0 \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} + qu &= 0 \text{ on } \gamma. \end{cases} \tag{1.1}$$

We use the language of the thermal imaging, φ thus being the prescribed heat flux on Γ_N

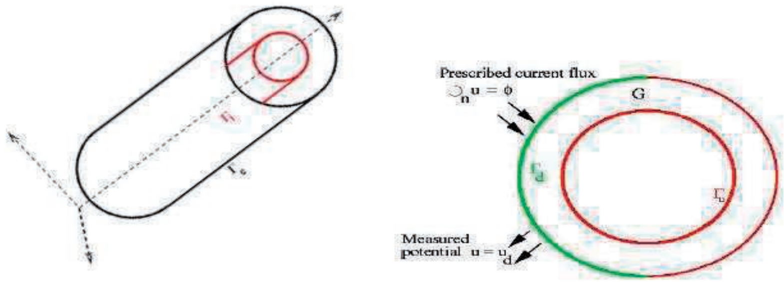


Figure 1: Corrosion detection

($\varphi \neq 0$ on Γ_N), and q the unknown heat-exchange function, which has to be determined by measuring the temperature on some open subset K of Γ_N; $f = u\,|_K$. However, problem (1.1) might also be viewed as a model for a corrosion detection problem by voltage measurements (see, for example, [27, 28]): q is thus the corrosion coefficient, φ the current flux prescribed on Γ_N and u the electrostatic potential.

Given the flux φ and the measured temperature f, the inverse problem is after defining an appropriate set $\mathcal{Q}_{ad}$ of admissible heat-exchange coefficients, the following:

Find $q \in \mathcal{Q}_{ad}$ such that: the u solution of (1.1) also verifies $u\,|_K = f$.

Let V be the following space :

$$V = \left\{ u \in H^1(\Omega);\ u \mid_{\Gamma_D} = 0 \right\} ;$$

V is a Hilbert space with respect to the inner product defined by :

$$< u, v >= \int_\Omega < \nabla u, \nabla v > .$$

Denoting by $\mid . \mid_{1,\Omega}$ the norm derived from this inner product, let α be the norm of the trace operator

$$\tau : V \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$$

$$u \longrightarrow u \mid_{\partial\Omega}$$

while considered as a mapping from V equipped with the energy norm, onto $L^2(\partial\Omega)$:

$$\alpha = \sup_{v \in V; v \neq 0} \frac{\mid v \mid_{0,\partial\Omega}}{\mid v \mid_{1,\Omega}}.$$

The set of admissible Robin coefficients is defined as follows:

$$\mathcal{Q}_{ad} = \left\{ q \in C^0(\overline{\gamma});\ \min_{x \in \overline{\gamma}} q(x) > -\frac{1}{\alpha^2} \right\}.$$

The following identifiability result presented in Theorem 1 then holds.

Theorem 1. ***(uniqueness)*** *Let q_1 and q_2 be two elements of $\mathcal{Q}_{ad}$, and $(u_i)_{i=1,2}$ be the solutions of problem (1.1) with $(q_i)_{i=1,2}$ as a Robin coefficient. Suppose that $u_1 \mid_K = u_2 \mid_K$. Then $q_1 = q_2$.*

Proof: Let q_1 and q_2 be two elements of $\mathcal{Q}_{ad}$ such that: $u_1 \mid_K = u_2 \mid_K$, and let us denote by w their difference ($w = u_1 - u_2$), which is a solution of the following problem:

$$\begin{cases} \triangle w &= 0 \text{ in } \Omega, \\ w &= 0 \text{ on } K, \\ \dfrac{\partial w}{\partial n} &= 0 \text{ on } \Gamma_N. \end{cases}$$

By using Holmgren's unique continuation theorem, we get $w \equiv 0$ in Ω, which means that $u_1 = u_2 \in \Omega$ and therefore :

$$\begin{cases} \dfrac{\partial u_1}{\partial n} + q_1 u_1 &= 0 \text{ on } \gamma. \\ \dfrac{\partial u_1}{\partial n} + q_2 u_1 &= 0 \text{ on } \gamma, \end{cases}$$

Thus

$$u_1(q_1 - q_2) = 0 \text{ on } \gamma. \tag{1.2}$$

Let us assume that $q_1 \neq q_2$. Thanks to the continuity of q_1 and q_2, we can find some open subset ϑ of γ with positive measure such that

$$(q_1 - q_2)(x) \neq 0 \ , \quad \forall x \in \vartheta.$$

Equation (1.2) then yields $u_1 \equiv 0$ on ϑ, and u_1 is therefore a solution of the Cauchy problem

$$\begin{cases} \triangle u_1 &= 0 \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \vartheta, \\ \dfrac{\partial u_1}{\partial n} &= 0 \quad \text{on } \vartheta. \end{cases}$$

If we use Holmgren's theorem again, we get $u_1 \equiv 0 \in \Omega$, which is in contradiction with $\varphi \not\equiv 0$ on Γ_N. □

Remark 1. *Dropping the Dirichlet boundary condition* ($\Gamma_D = \emptyset$), *the same result holds with the space of admissible Robin coefficients below:*

$$\mathcal{Q}_{ad} = \left\{ q \in C^0(\overline{\gamma});\ q(x) \geq 0;\ q(x) \neq 0 \right\} ,$$

the non-negativity of ϕ being necessary to ensure that the direct problem is well posed (see, for example, Garabedian [24]).

Cracks identification:

To illustrate further identifiability issue let us consider now the inverse problem of cracks recovery.

The forward problem:

Given Ω, $\sigma \subset \Omega$ and $\Phi \in L^2(\Omega)$, find the solution u of:

$$(FP) \begin{cases} \triangle u &= 0 \quad \text{in } \Omega \backslash \sigma, \\ \dfrac{\partial u}{\partial n} &= 0 \quad \text{on } \sigma, \\ \dfrac{\partial u}{\partial n} &= \Phi \quad \text{on } \partial\Omega\,. \end{cases}$$

The inverse problem:

Given Ω, Φ, f, find σ such that the solution of the forward problem (FP) in $\Omega \backslash \sigma$ also verifies: $u \mid_{\Gamma_0} = f$.

It was already proven that a single measurement is usually not enough (see [22]); a single crack by overdetermined boundary data is illustrated in Figure 3.

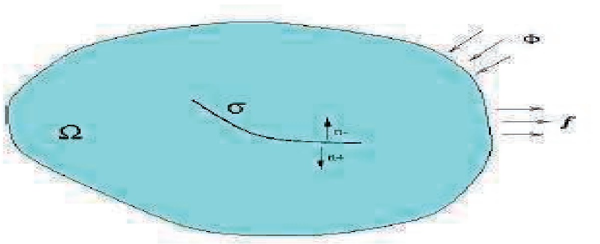


Figure 2: Cracks identification

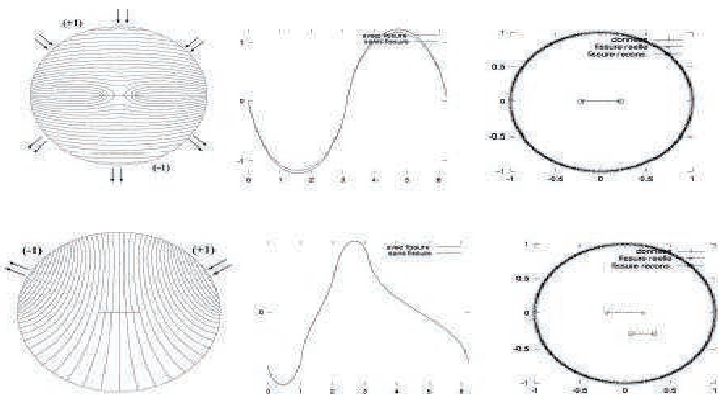


Figure 3: Identification of cracks from a single measurement

1.2 Stability

The measurements are assumed to be performed on a non-empty open subset K of Γ_N. Roughly speaking, stability means that small errors in the measurements would yield small perturbations on the unknown coefficient q. To formalize the idea, let us consider the mapping η defined by

$$\eta : \mathcal{Q}_{ad} \longrightarrow L^2(K)$$

$$q \longrightarrow f = u_q \mid_K$$

The identifiability result proved above means that η is injective, and therefore, that the restriction

$$\eta : \mathcal{Q}_{ad} \longrightarrow \eta(\mathcal{Q}_{ad})$$

is invertible. Limiting our search to any compact subset of $\mathcal{Q}_{ad}$, a weak stability result, which is merely the continuity of the inverse operator η^{-1}, can be obtained as a straightforward consequence of the uniqueness theorem (Andrieux et al [7]).

Local Lipschitz stability

To prove local Lipschitz stability, the Lagrangian differentiation with respect to the domain has been repeatedly used as a basic and somewhat powerful tool for the study of geometric inverse problems ([6, 7, 12], etc). The Robin boundary condition has also been studied in [10], in the framework of 2D cracks recovery a local Lipschitz stability result being proved using the same technique. Actually, the latter also works for differentiation with respect to the field or boundary coefficients, as was stated many years ago by Simon [45].

Now given $q \in \mathcal{Q}_{ad}$ and $\psi \in \mathcal{Q}_{ad}$, there exists some real number $h_0 > 0$, depending on ϕ and ψ, such that

$$h \in] - h_0, h_0[\quad \Longleftrightarrow \quad q^h := (q + hr) \in \mathcal{Q}_{ad}.$$

Let u^h be the solution of problem (1.1) for q^h as a Robin coefficient. We then have the following proposition.

Proposition 1. *There exist u^1 and $\varepsilon(h)$ in $H^1(\Omega)$ such that*

$$u^h = u^0 + hu^1 + h\varepsilon(h) \tag{1.3}$$

where $\lim_{h\to 0} \mid \varepsilon(h) \mid_{1,\Omega} = 0$, *$u^0$ is solution of (1.1) with q as a Robin coefficient, and u^1 is solution of the following problem:*

$$\left\{ \begin{array}{l} \textit{Find } u^1 \textit{ in } V \textit{ such that,} \\ \displaystyle\int_\Omega < \nabla u, \nabla v > + \int_\gamma q u^1 v = - \int_\gamma r u^0 v \quad \textit{for all } v \in V. \end{array} \right. \tag{1.4}$$

Proof: The variational formulation of problem (1.1) is:

$$\int_\Omega < \nabla u, \nabla v > + \int_\gamma quv = -\int_{\gamma_N} rv \quad \text{for all } v \in V. \tag{1.5}$$

Let $V^\star$ be the dual space of V, and define the following mapping:

$$\Lambda :]-h_0, h_0[\times V(\gamma) \longrightarrow V^\star$$

$$(h,u) \longrightarrow \left\{ \int_\Omega < \nabla u, \nabla v > + \int_\gamma q^h uv - \int_{\gamma_N} \varphi v \right\}$$

the solution u^h of problem (1.1) with q^h as a Robin coefficient is therefore the solution of $\Lambda(h, u^h) = 0$. The mapping Λ being linear with respect to u, its partial derivative with respect to u ($\frac{\partial \Lambda}{\partial u}(0,u) = \Lambda(0,.)$) turns out to be an isomorphism between V and $V^\star$. By the implicit function theorem, it turns out that u^h is some C^1 function of h, provided h is small enough. This yields expansion (1.3), with $u^1 \in V$, and $\lim_{h\to 0} \mid \varepsilon(h) \mid_{1,\Omega} = 0$.

Now, plugging this expansion in the variational formulation (1.5) of the direct problem with $q^h = q + hr$ as a Robin coefficient, and identifying in both sides of the equation terms of the same order in h, we derive that:

- u^0 solves problem (1.1) with q as a Robin coefficient
- u^1 solves problem (1.4)

which ends the proof. □

Theorem 2. ***(Local Lipschitz stability).*** *Assume that $\Psi \neq 0$ on γ, and denote by $f^h = u^h \mid_K$. Then*

$$\lim_{h\to 0} \frac{\mid f^h - f^0 \mid_{0,K}}{\mid h \mid} > 0. \tag{1.6}$$

Proof: According to expansion (1.3), (1.6) is equivalent to

$$\mid u \mid_{0,K} > 0.$$

Let us then assume that $u^1 = 0$ on K. In this case, equation (1.4) gives that u^1 is a solution of the Cauchy problem

$$\begin{cases} \triangle u_1 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } K, \\ \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \Gamma_N, \end{cases}$$

which by Holmgren's theorem leads to

$$u^1 \equiv 0 \text{ in } \Omega.$$

Equation (1.4) then yields to

$$\int_\gamma r u^0 v = 0 \ , \quad \forall v \in V$$

from which we derive

$$r u^0 = 0 \quad \text{ a.e. in } \gamma.$$

From the continuity of ψ, and the fact that $r \neq 0$ on γ, we derive the existence of some open subset ϑ of γ where

$$u^0 = 0 \text{ on } \vartheta.$$

The Robin boundary condition yields to:

$$\frac{\partial u^0}{\partial n} = 0 \text{ on } \vartheta.$$

Using Holmgren's theorem again, we derive that $u^0 \equiv 0$ in Ω, which is in contradiction with our assumption that $\varphi \neq 0$ on Γ_N. □

Instability: The Hadamard's classical example

In 1923 Hadamard [25] provided a fundamental example which shows that the solution of a Cauchy problem for Laplace's equation does not depend continuously upon the data [see figure 4]. The example is as follows: consider the solution $u = (u_k)_k$ to the Cauchy problem in the half plane

$$\begin{cases} \Delta u & = 0 & \text{in } \{(x,y) \in \mathbb{R}^2 \backslash y > 0\}, \\ u(x,0) & = 0, & \text{for every } x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial n}(x,0) & = \dfrac{\sin nx}{n} & \text{for every } x \in \mathbb{R}. \end{cases} \tag{1.7}$$

We have $u_k = \dfrac{\sin kx \, \sinh ky}{n^2}$, it turns out that

$$\frac{\partial u_k}{\partial n}(x,0) \to 0 \text{ uniformly as } k \to \infty,$$

whereas, for $y > 0$, we have $u_k(x,y) = \dfrac{\sin kx \sinh ky}{k^2}$ blows up when $k \to \infty$:

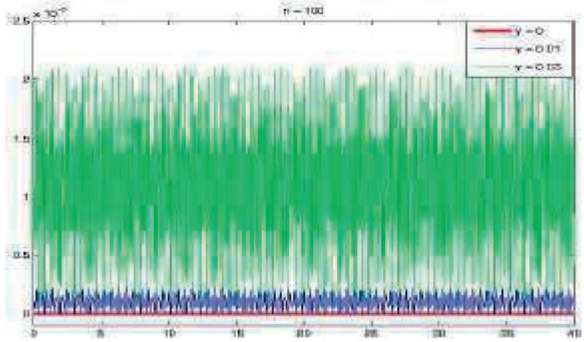


Figure 4: Exponential blow up of the higher frequencies

1.3 Regularization

The content of the present subsection is borrowed to Michel Kern's Course [29]. Let us consider an abstract frame: E and F are two Hilbert spaces and $A \in \mathcal{L}(E, F)$.

Solving a linear inverse problem returns to

$$\begin{cases} \text{Find } \hat{x} \in E \text{ such that} \\ A\hat{x} = \hat{z}, \end{cases}$$

where $\hat{z}$ is the measurement and $\hat{x}$ the unknown.

Usually $\hat{z} \in F$. However, we often have:

1. $R(A) \neq F$ $(R(A) \subset F)$,

2. $R(A)$ is not closed.

Difficulty (1) is not serious, one can limitate to $R(A)$ instead of F, provided $R(A)$ is closed. Should this last assumption fail, we are in trouble since:

$$\boxed{R(A) \text{ closed} \Leftrightarrow A^{-1} \text{ is continuous}}$$

Proof: We suppose A is injective and denote by A^{-1}

$$A^{-1} : R(A) \longrightarrow E,$$

its inverse operator.

- If $R(A)$ is closed in F, it's a Hilbert space. Thus $A : E \longrightarrow R(A)$ is linear continuous and bijective (since it's injective). Thus it's an isomorphism and A^{-1} is continuous.
- Suppose now A^{-1} is continuous. Therfore $R(A) = (A^{-1})^{-1}(E)$. Since E is closed and A^{-1} is continuous, $R(A)$ is closed.

□

Example: The Robin inverse problem [17].

We are recovering q from $u|_{\Gamma_N}$, knowing that $\dfrac{\partial u}{\partial n}$ has also been prescribed on Γ_N.

Now q is such that $\dfrac{\partial u}{\partial n} + qu = 0 \Rightarrow q = -\dfrac{\frac{\partial u}{\partial n}}{u}|_{\gamma}$.

Thus, we try to recover u and $\dfrac{\partial u}{\partial n}$ on γ from the pair (f, ϕ) on Γ_N which is $(u, \dfrac{\partial u}{\partial n})$ on Γ_N. Our operator A is thus

$$A : H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(\gamma) \longrightarrow H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(\Gamma_N)$$

$$\left(u, \frac{\partial u}{\partial n}\right)_{\gamma} \longrightarrow \left(u, \frac{\partial u}{\partial n}\right)_{\Gamma_N} .$$

We have also seen that A^{-1} is not continuous (Hadamard) because of the exponential magnification of the higher frequency components.

We seek now to solve $A\hat{x} = \hat{z}$. However, the measurement $\hat{z}$ has been polluted by noise. Instead of $A\hat{x} = \hat{z}$, we are thus trying to solve $Ax = z$ where $z = \hat{z} + \delta$.

Instead of solving $Ax = z$, we then try to minimize $||A\hat{x} - \hat{z}||_F^2$.

The minimization problem is

$$(LS) \begin{cases} \text{Find } \hat{x} \in E \text{ such that} \\ \min_{x \in E} ||Ax - \hat{z}||_F^2 = ||A\hat{x} - \hat{z}||_F^2 . \end{cases}$$

Two cases show up :

1) $\exists \hat{x} \in E$ such that $A\hat{x} = \hat{z} (\hat{z} \in R(A))$. Therefore $x = \hat{x}$ and the problem is solved ....

But this solution $\hat{x}$ does not depend continuously on the data (Hadamard).

2) $\hat{z} \notin R(A)\ldots$ and in most cases $\hat{z} \in \overline{R(A)}$. Therefore the optimization problem has no solution. Its minimum is zero, but there is no $x \in E$ realizing that minimum.

In case of noise, $\hat{z} \notin R(A)$ but we don't know it: we only know some measurement z^δ such that

$$||\hat{z} - z^\delta||_F = \delta\,.$$

In order to overcome these difficulties, Tikhonov has proposed a regularization procedure: instead of solving (LS), let us solve (LS_ε)

$$(LS_\varepsilon)\left\{\begin{array}{c}\text{Find } x \in E \text{ minimizing}\\ ||Ax - \hat{z}||_F^2 + \varepsilon^2||x - x_0||_E^2\,.\end{array}\right.$$

Proposition 2. *Problem (LS_ε) is equivalent to*

$$(NE_\varepsilon)\ (A^\star A + \varepsilon^2 I)x = A^\star \hat{z} + \varepsilon^2 x_0\,.$$

(NE_ε) has a unique solution that continuously depends on $\hat{z}$.

Proof:

$$\begin{array}{rcl} J(x) & = & ||Ax - \hat{z}||_F^2 + \varepsilon^2||x - x_0||_E^2,\\ & = & < Ax - \hat{z}, Ax - \hat{z} >_F + \varepsilon^2 < x - x_0, x - x_0 >_E,\\ & = & < A^\star Ax, x >_E - 2 < Ax, \hat{z} >_F + \varepsilon^2||x||_E^2 - 2\varepsilon^2 < x, x_0 > + \varepsilon^2||x_0||_E^2 + ||\hat{z}||_F^2,\\ & = & < A^\star Ax, x >_E + \varepsilon^2||x||_E^2 - 2 < x, A^\star\hat{z} + \varepsilon^2 x_0 >_F + \varepsilon^2||x_0||_E^2 + ||\hat{z}||_F^2,\\ & = & < (A^\star A + \varepsilon^2 I)x, x >_E - 2 < x, A^\star\hat{z} + \varepsilon^2 x_0 >_F + cst. \end{array}$$

$J(x)$ is quadratic with respect to x, and its minimum is characterized by $J'(x) = 0$.

$$J'(x) = 2(A^\star A + \varepsilon^2 I)x - 2(A^\star\hat{z} + \varepsilon^2 x_0).$$

Thus

$$J'(x) = 0 \Leftrightarrow (A^\star A + \varepsilon^2 I)x = A^\star\hat{z} + \varepsilon^2 x_0.$$

Does this problem has a unique solution?

$A^\star A + \varepsilon^2 I$ is coercive on E

$$\begin{array}{rcl} < (A^\star A + \varepsilon^2 I)x, x >_E & = & < A^\star Ax, x >_E + \varepsilon^2 < x, x >_E,\\ & \geq & \varepsilon^2||x||_E^2\,. \end{array}$$

It is thus continuous and bijective (automorphism of E).
Therefore, $A^\star A + \varepsilon^2 I$ is bijective and (NE_ε) has a unique solution.

$$(NE_\varepsilon) \quad (A^\star A + \varepsilon^2 I)x = A^\star \hat{z} + \varepsilon x_0$$

Thus

$$\begin{array}{lll} < A^\star Ax, x >_E + \varepsilon^2 ||x||_E^2 & \leq & ||A^\star \hat{z}||_E ||x||_E + \varepsilon^2 ||x||_E ||x_0||_E, \\ ||Ax||_F^2 + \varepsilon^2 ||x||_E^2 & \leq & ||A^\star \hat{z}||_E ||x||_E + \varepsilon^2 ||x||_E ||x_0||_E, \end{array}$$

and

$$||x||_E \leq \frac{1}{\varepsilon^2} ||A^\star \hat{z}||_E + ||x_0||_E \, .$$

We have thus continuity with respect to the data and a blowing up happens when $\varepsilon \to 0$.

□

Let us now look at the problem with noisy data.

Assumptions:

1) There is an ideal observation $\hat{z} \in R(A)$

$$A\hat{x} = \hat{z} \, .$$

2) Moreover, we suppose that (Baumeister's assumption)

$$\hat{x} \in R(A)^\star \Longleftrightarrow \hat{x} = A^\star w, \quad w \in F \, .$$

3) Finaly let z^δ be the noisy data measurements

$$||z^\delta - \hat{z}||_F = \delta \, .$$

$\dfrac{||z^\delta - \hat{z}||_F}{||\hat{z}||_F}$ is called the noise/signal ratio.

We thus look at the solution of the following problem

$$\left\{ \begin{array}{c} x_{\delta\varepsilon} \in E, \\ \min_x \{ ||Ax - z_\delta||_F^2 + \varepsilon^2 ||x - x_0||_E^2 \} \, , \end{array} \right.$$

where $x_{\delta\varepsilon}$ will thus solve:

$$(a)\quad A^\star A x_{\delta\varepsilon} + \varepsilon^2 x_{\delta\varepsilon} = A^\star z_\delta + \varepsilon^2 x_0,$$

Let x_ε solve the Tikhonov problem with the ideal observation:

$$\begin{cases} x_\varepsilon \in E, \\ \min_x ||Ax - \hat{z}||_F^2 + \varepsilon^2 ||x - x_0||_E^2. \end{cases}$$

Therefore x_ε solves

$$(b)\quad A^\star A x_\varepsilon + \varepsilon^2 x_{\delta\varepsilon} = A^\star \hat{z} + \varepsilon^2 x_0.$$

Using (a) and (b), we get

$$A^\star A(x_{\delta\varepsilon} - x_\varepsilon) + \varepsilon^2 (x_{\delta\varepsilon} - x_\varepsilon) = A^\star (z_\delta - \hat{z}),$$

from which we derive

$$\begin{aligned} ||x_{\delta\varepsilon} - x_\varepsilon||_E &\leq \frac{||A^\star||}{\varepsilon^2} ||z_\delta - \hat{z}||_F, \\ &\leq ||A^\star|| \frac{\delta}{\varepsilon^2}. \end{aligned}$$

□

Now, let us estimate the second part of the error, which is $||x_\varepsilon - \hat{x}||_E$.

$$\hat{x} \text{ solves } A\hat{x} = \hat{z} \Rightarrow A^\star A \hat{x} = A^\star \hat{z}.$$

Thus

$$A^\star A(x_\varepsilon - \hat{x}) + \varepsilon^2 x_\varepsilon = \varepsilon^2 x_0.$$

Let us suppose $x_0 = 0$. Thus

$$A^\star A(x_\varepsilon - \hat{x}) = -\varepsilon^2 x_\varepsilon + \varepsilon^2 \hat{x} - \varepsilon^2 \hat{x} = -\varepsilon^2 (x_\varepsilon - \hat{x}) + \varepsilon^2 \hat{x},$$

and

$$\begin{aligned} < A^\star A(x_\varepsilon - \hat{x}), x_\varepsilon - \hat{x} >_E &= -\varepsilon^2 < x_\varepsilon - \hat{x}, x_\varepsilon - \hat{x} >_E + \varepsilon^2 < \hat{x}, x_\varepsilon - \hat{x} >_E, \\ &= -\varepsilon^2 < x_\varepsilon - \hat{x}, x_\varepsilon - \hat{x} >_E + \varepsilon^2 < A^\star w, x_\varepsilon - \hat{x} >_E, \end{aligned}$$

which yields

$$||A(x_\varepsilon - \hat{x})||_F^2 = -\varepsilon^2 ||x_\varepsilon - \hat{x}||_E^2 + \varepsilon^2 < w, A(x_\varepsilon - \hat{x}) >_F,$$

which gives

$$||A(x_\varepsilon - \hat{x})||_F^2 \leq -\varepsilon^2 ||x_\varepsilon - \hat{x}||_E^2 + \varepsilon^2 ||w||_F ||A(x_\varepsilon - \hat{x})||_F \,.$$

Now we use

$$ab \leq \frac{\varepsilon^2}{2} a^2 + \frac{1}{2\varepsilon^2} b^2,$$

$$||w||_F ||A(x_\varepsilon - \hat{x})||_F \leq \frac{\varepsilon^2}{2} ||w||_F^2 + \frac{1}{2\varepsilon^2} ||A(x_\varepsilon - \hat{x})||_F^2.$$

Thus

$$||A(x_\varepsilon - \hat{x})||_F^2 \leq -\varepsilon^2 ||x_\varepsilon - \hat{x}||_E^2 + \frac{\varepsilon^4}{2} ||w||_F^2 + \frac{1}{2} ||A(x_\varepsilon - \hat{x})||_F^2,$$

from which we drive

$$\frac{1}{2} ||A(x_\varepsilon - \hat{x})||_F^2 \leq \frac{\varepsilon^4}{2} ||w||_F^2 - \varepsilon^2 ||x_\varepsilon - \hat{x}||_E^2.$$

Thus

$$||x_\varepsilon - \hat{x}||_E^2 \leq \frac{\varepsilon^2}{2} ||w||_F^2.$$

Gathering these two estimates, we get

$$||x_{\delta\varepsilon} - \hat{x}||_E \leq ||A^\star|| \frac{\delta}{\varepsilon^2} + \frac{\varepsilon}{\sqrt{2}} ||w||_F \,.$$

Problems:

1) We don't know $||w||$ or $||A^\star||$ and can't thus determine the right value of ε.

2) The computed solution is not exact, even though the data are noise free.

This is the reason why we shall be studying alternative regularisation methods.

2 The Robin coefficient recovery

Model problem: The Robin one

$$\begin{cases} \triangle u & = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} & = \Phi \text{ on } \Gamma_N, \text{ prescribed current flux} \\ u & = f \text{ on } \Gamma_N, \text{ measurements} \\ \dfrac{\partial u}{\partial n} + qu & = 0 \text{ on } \gamma. \end{cases}$$

Suppose q is determined. Therefore, we have twice too many prescribed data on Γ_N, which allows to solve two problems: a Dirichlet problem (D) and a Neumann problem (N)

$$(D)\begin{cases}\triangle u^D & = 0 \text{ in } \Omega,\\ u^D & = f \text{ on } \Gamma_N,\\ \dfrac{\partial u^D}{\partial n}+qu^D & = 0 \text{ on } \gamma\end{cases}, \qquad (N)\begin{cases}\triangle u^N & = 0 \text{ in } \Omega,\\ \dfrac{\partial u^N}{\partial n} & = \Phi \text{ on } \Gamma_N,\\ \dfrac{\partial u^N}{\partial n}+qu^N & = 0 \text{ on } \gamma.\end{cases}$$

Provided q is the actual impedance, we have

$$u^D(q) = u^N(q).$$

Therefore, let us minimize the misfit between u^D and u^N in order to get the actual q. Many misfit functions may be proposed to that end.

The energy error functional has been introduced in [37] in the context of *a posteriori* estimators in the finite elements method. Within the inverse problem community this functional has been introduced in [31, 32, 33] in the context of parameter identification. It has been widely exploited in the same context in [15]. It has also been used for Robin type boundary condition recovering [17]. For lacking boundary data recovering (i.e. Cauchy problem resolution) in the context of Laplace operator, the energy error functional has been introduced in [5, 4]. A study of similar techniques can be found in [14] and the analysis found in these papers uses elements taken from the domain decomposition framework [42]. There is a large amount of literature in the field of Cauchy's problems.

- Ordinary least squares:

$$\begin{aligned} J(q) &= \int_{\Gamma_N} |u^D(q) - u^N(q)|^2,\\ &= \int_{\Gamma_N} |f - u^N(q)|^2. \end{aligned}$$

  We have seen above that this is not the best choice, because of the blowing up of the Cauchy solution away from the prescription boundary.

- Energy least squares:

$$\int_\Omega |\nabla(u^D - u^N)|^2 + \int_\gamma |u^D - u^N|^2.$$

The advantages of such an approach is that it has a physical meaning: it may be interpreted as a constitutive law error functional. Furthermore, it is expected to have a stabilyzing feature in so far as it is a distributed least squares (not a boundary integral).

Theorem 3. *There exists a unique solution* $q \in \mathcal{Q}_{ad}$ *such that*

$$(O) \quad J(q) \leq J(q') \ , \quad \forall q' \in \mathcal{Q}_{ad},$$

and q *is the solution of the inverse problem*

$$(IP) \begin{cases} \text{Find } q \in \mathcal{Q}_{ad} \text{ such that the solution } u \text{ of} \\ (FP) \begin{cases} \triangle u & = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} & = \varphi \text{ on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + qu & = 0 \text{ on } \gamma. \end{cases} \\ \text{also verifies } u = f \text{ on } \Gamma_N, \end{cases}$$

provided (IP) *has a unique solution (identifiability)* $q \in \mathcal{Q}_{ad}$.

Proof:

• Let q solve (IP) therefore $J(q) = 0$, since $u(q) = u^D$ and also $u(q) = u^N$ (this is (FP)) and

$$J(q) \leq J(q') \ \forall q' \in \mathcal{Q}_{ad},$$

• Suppose q_1 and q_2 are two solutions of (O).
Therefore $u_1^D = u_1^N$ and $u_2^D = u_2^N$, since $J(q_1) = J(q_2) = 0$. It follows that both u_1^N and u_2^N solve the inverse problem and thus $u_1^N = u_2^N$, which yields $q_1 = q_2$. □

2.1 The constitutive Kohn-Vogelius gradient algorithm

In order to implement the gradient algorithm, we need to calculate the gradient $J'(q).r$. To that end, both expansion of u_D^h and u_N^h are needed (with respect to h). We prove that (see for details [17])

$$u_N^h = u_N^0 + hu_N^1 + h\varepsilon(h).$$

We have already proved that:

$$u_D^h = u_D^0 + hu_D^1 + h\eta(h),$$

where u_D^1 and u_N^1 solve

$$(N_1)\ \begin{cases} u_N^1 \in V = H^1(\Omega) \\ \displaystyle\int_\Omega \nabla u_N^1 \nabla v + \int_\gamma q u_N^1 v = -\int_{\Gamma_N} r u_N^0 v\,, \quad \forall v \in H^1(\Omega)\,, \end{cases}$$

$$(D_1)\ \begin{cases} u_D^1 \in V_0(\Omega) \\ \displaystyle\int_\Omega \nabla u_D^1 \nabla v + \int_\gamma q u_D^1 v = -\int_{\Gamma_N} r u_D^0 v\,, \quad \forall v \in V_0(\Omega)\,. \end{cases}$$

Theorem 4. *Derivative of the error-function.*

$$J'(q).r = \lim_{h\to 0} \frac{J(q_h) - J(q)}{\lambda} = \int_\gamma r(|u_D^0|^2 - |u_N^0|^2),$$

with $q_h = q + hr$.

Proof: We can write

$$J(q) = J_D(q) + J_N(q) + J_{DN}(q)$$

$J_N(q) = \int_\Omega |\nabla u_N|^2 + \int_\gamma q|u_N|^2$
$J_D(q) = \int_\Omega |\nabla u_D|^2 + \int_\gamma q|u_D|^2$
and finally

$$J_{DN}(q) = -2\left\{\int_\Omega \nabla u_D \nabla u_N + \int_\gamma q u_D u_N\right\}.$$

This latter is constant since

$$\int_\Omega \nabla u_D \nabla u_N = -\int_\Omega \triangle u_N u_D + \int_\Gamma \frac{\partial u_N}{\partial n} u_D$$

But $u_D = f$ and $\dfrac{\partial u_N}{\partial n} = \varphi$ on Γ_N and $\triangle u_N = 0$ in Ω.
Therefore

$$\int_\Omega \nabla u_D \nabla u_N = \int_{\Gamma_N} \varphi f + \int_\gamma \frac{\partial u_N}{\partial n} u_D\,.$$

Since $\dfrac{\partial u_N}{\partial n} = -q u_N$, we get

$$\int_\Omega \nabla u_D \nabla u_N = \int_{\Gamma_N} \varphi f - \int_\gamma q u_N u_D,$$

and thus

$$\int_\Omega \nabla u_D \nabla u_N + \int_\gamma q u_N u_D = \int_{\Gamma_N} \varphi f = cst.$$

This term will not interveign in the derivative.

Now, using the asymptotic expansion above, we get that

$$J'_D(q).r = 2\int_\Omega \nabla u_D^0 \nabla u_D^1 + 2\int_\gamma q u_D^0 u_D^1 + \int_\gamma r|u_D^0|^2,$$

and similarly

$$J'_N(q).r = 2\int_\Omega \nabla u_N^0 \nabla u_N^1 + 2\int_\gamma q u_N^0 u_N^1 + \int_\gamma r|u_N^0|^2 .$$

Now, using formulation (N_1) with $v = u_N^0$, we get

$$J'_N(q).r = -2\int_\gamma r|u_N^0|^2 + \int_\gamma r|u_N^0|^2 = -\int_\gamma r|u_N^0|^2.$$

Using also (D_1) with $v = u_D^0$, we get

$$J'_D(q).r = 2[\int_\Omega \nabla u_D^0 \nabla u_D^1 + \int_\gamma q u_D^0 u_D^1] + \int_\gamma r|u_D^0|^2,$$

But

$$\int_\Omega \nabla u_D^0 \nabla u_D^1 + \int_\gamma q u_D^0 u_D^1 = -\int_\Omega \triangle u_D^0 u_D^1 + \int_\Gamma \frac{\partial u_D^0}{\partial n} u_D^1 + \int_\gamma q u_D^0 u_D^1 .$$

u_D^0 verifies $\dfrac{\partial u_D^0}{\partial n} + q u_D^0 = 0$ on γ and $\dfrac{\partial u_D^0}{\partial n} = \varphi$ on Γ_N, whereas u_D^1 vanishes on Supp φ. Therefore

$$\int_\Omega \nabla u_D^0 \nabla u_D^1 + \int_\gamma q u_D^0 u_D^1 = 0$$

and thus

$$J'_D(q).r = \int_\gamma r|u_D^0|^2.$$

The derivative is announced

$$J'(q).r = \int_\gamma r(|u_D^0|^2 - |u_N^0|^2) .$$

□

We will refer to this algorithm as the Kohn-Vogelius one (KV-algorithm), because Kohn and Vogelius introduced this error functional to the inverse problems community.

<u>The K-V algorithm</u>

- Choose $q^0 \in \mathcal{Q}_{ad}$
  - Calculate $u^D(q)$ and $u^N(q)$
  - Compute $J'(q)$
- $q^{n+1} = q^n - \rho J'(q^n)$

It is nothing but the gradient algorithm applied to the KV functional.

1. As all gradient algorithm, it is slow.

2. Understand why it is?

$$J'(q) \cdot r = \int_\gamma r(u_D^2(q) - u_N^2(q))\,.$$

Suppose $r = \sin(n\sigma)$. Therefore, $J'(q)\sin(n\sigma)$ is the n^{th} Fourier coefficient of $(u_D^2 - u_N^2)$. Provided u_D and u_N are smooth, the components of the gradient then decrease fast. It turns out that the gradient search is thus nothing but a search along the lowest frequencies. The algorithm is therefore expected to squeeze the higher frequencies components, and consequently to behave in a stable way (since instabilities show up in the higher frequencies)

2.2 Stability of the Kohn-Vogelius algorithm (or/and robustness)

$$(IP)\left\{\begin{array}{l} \text{Find } q \in \mathcal{Q}_{ad} \text{ such that } u \\ \left\{\begin{array}{lcl} \triangle u & = & 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} & = & \varphi \text{ on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + qu & = & 0 \text{ on } \gamma. \end{array}\right. \\ \text{also solve } u(q) = f. \end{array}\right. \quad , \quad (OP)\left\{\begin{array}{l} \text{Find } q \in \mathcal{Q}_{ad} \text{ such that } u \\ J(q) \le J(q')\ , \quad \forall q' \in \mathcal{Q}_{ad} \end{array}\right.$$

1) If the measured data f are "exact", then $(IP) \Leftrightarrow (OP)$

2) The measurements are noisy. Thus

(a) The inverse problem has no solution

(b) The optimization problem may have.

Theorem 5. *Suppose $f_n \in H^{\frac{1}{2}}(\Gamma_N)$ is a sequence of "noisy measurements" such that $f_n \to f \in H^{\frac{1}{2}}(\Gamma_N)$. Therefore:*

1) $\forall n \in \mathbb{N}$, the optimization problem (OP_n) with data f_n has at least one solution

2) q_n being any solution of (OP_n), we have $\lim_{n\to\infty} ||q_n - q||_{0,\gamma} = 0$.

Proof: It's somewhat technical (see Chaabane et al. [17]). □

Remarks:

1) This result is a stability one for the optimization problem, not for the (IP). Indeed, q_n has no conection with any inverse problem, whereas q has. However, this theorem tells us it makes sense to solve the optimization problem with approximate data (f_n) since we get an approximation q_n of the desired impedance q.

2) BUT usually noisy data f_n are not expected to belong to $H^{\frac{1}{2}}(\Gamma_N)$.
We have $f_\varepsilon = f + \varepsilon,\ \varepsilon \in L^2(\Gamma_N)$ and $||\varepsilon||_{L^2(\Gamma_N)} = \epsilon$.
What is usually run is:

(a) Smooth f_ε by some method, such as cubic splines and get $\tilde{f}_\varepsilon$ which is smooth enough ..., say $\tilde{f}_\varepsilon \in H^{\frac{1}{2}}(\Gamma_N)$ (actually $\tilde{f}_\varepsilon \in C^2(\Gamma_N)$).

But how close to the actual data f is $\tilde{f}_\varepsilon$? We can prove that (Chaabane et al. [18])

$$\begin{array}{rcl} ||\tilde{f}_\varepsilon - f||_{0,\infty,\Gamma_N} & \leq & c(\epsilon + h^2), \\ . & & \qquad\qquad h = [\text{ splining path }] \\ ||\tilde{f}_\varepsilon - f||_{1,\infty,\Gamma_N} & \leq & c(\frac{\epsilon}{h} + h). \end{array}$$

Therefore, choosing $h \sim \sqrt{\epsilon}$, we get

$$||\tilde{f}_\varepsilon - f||_{0,\infty,\Gamma_N} \leq c\epsilon$$

and

$$||\tilde{f}_\varepsilon - f||_{1,\infty,\Gamma_N} \leq c\sqrt{\epsilon}\,.$$

(b) Use the smoothed $\tilde{f}_\varepsilon$ data to solve (OP).

Therefore, get $\tilde{q}_\varepsilon$ and, using the stability theorem

$$\lim_{\varepsilon\to 0} ||\tilde{q}_\varepsilon - q||_{0,\gamma} = 0.$$

This is a robustness result, though not a quantitative one. □

3 Solving Cauchy problems

This section is due to a strong collaboration with Amel Ben Abda.

3.1 Data completion

$$\begin{cases} \nabla \cdot k(x)\nabla u = 0 & \text{in} \quad \Omega \\ k(x)\nabla u \cdot n = \Phi & \text{on} \quad \Gamma_d \\ u = T & \text{on} \quad \Gamma_d\,. \end{cases} \tag{3.1}$$

Let us consider the Cauchy problem (3.1). Assuming the data (Φ, T) are compatible, i.e. that this pair is indeed the trace and normal derivative of a unique function u, extending the data means finding (φ, t) such that:

$$\begin{cases} \nabla \cdot k(x)\nabla u = 0 & \text{in} \quad \Omega, \\ u = T,\ k(x)\nabla u \cdot n = \Phi & \text{on} \quad \Gamma_d, \\ u = t,\ k(x)\nabla u \cdot n = \varphi & \text{on} \quad \Gamma_u\,. \end{cases} \tag{3.2}$$

The question is to reconstruct numerically the pair (φ, t).

However, in practical problems data are not expected to be compatible, since data errors may occur from measurement discretization and modelling errors. The ill-posedness in Hadamard's sense arises —dramatically— when one tries to approximate a given data (Φ, T): it is possible to approach it as closely as desired on Γ_d by traces of a single harmonic function, the "surprise" being a hectic behavior of this function on the remaining part of the boundary, see Hadamard's example [25] (reported in subsection 1.2). In this section, we resort to a rough but usually efficient regularizing techniques consisting of solving the ill-posed problem iteratively and choosing a suitable stopping criteria which determines an optimal solution [19]. The introduction of two distinct fields, each of them meeting only one of the overspecified data, avoid the need of a regularisation procedure for the resolution of the data completion problem. Using separately the two boundary conditions on Γ_d has also been used in the algorithm proposed by Kozlov *et al* [34], where again no regularization procedure is cast into the resolution method.

We will restrict ourselves to situations where the boundary $\partial\Omega$ consists of two closed manifolds of class C^2 such that $\partial\Omega = \Gamma_d \bigcup \Gamma_u$ and $\Gamma_d \bigcap \Gamma_u = \emptyset$. However, all the results stated thereafter are also true in the case of less smooth boundaries and when Γ_d and Γ_u

have contact points. As already mentioned, the pairs of compatible data are dense in the set of all possible data pairs. For this known result we refer to the book by Fursikov [23].

Lemma 2. *For T and Φ data:*

1. *For a fixed T in $H^{1/2}(\Gamma_d)$, the set of data Φ for which there exists a function u in $H^1(\Omega)$ satisfying the Cauchy problem (3.1) is everywhere dense in $H^{-1/2}(\Gamma_d)$.*

2. *For a fixed Φ in $H^{-1/2}(\Gamma_d)$, the set of data T for which there exists a function u in $H^1(\Omega)$ satisfying the Cauchy problem (3.1) is everywhere dense in $H^{1/2}(\Gamma_d)$.*

Proof: Let us prove the first assertion. The second one can be obtained by the same arguments. It is sufficient to prove the result for $T = 0$. Let $u \in H^1(\Omega)$ satisfying the problem:

$$\begin{cases} \nabla \cdot k(x)\nabla u = 0 & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_d, \\ k(x)\nabla u \cdot n = \Phi & \text{on} \quad \Gamma_u \end{cases} \tag{3.3}$$

Assume, now, that the first assertion fails. Let R be the subspace of $H^{-1/2}(\Gamma_d)$ consisting of fluxes Φ compatible with the Dirichlet data 0 (i.e; the fluxes Φ for which (3.3) is solvable). $\bar{R}$, the closure in $H^{-1/2}(\Gamma_d)$, is a proper subspace of R in $H^{-1/2}(\Gamma_d)$. Therefore there exists a non-vanishing continuous linear form l on $H^{-1/2}(\Gamma_d)$ ($l \in R$ in $H^{1/2}(\Gamma_d)$) such that:

$$\langle l, \Phi \rangle = 0\,, \quad \forall \;\, \Phi \in R\,. \tag{3.4}$$

Consider, now, the mixed well-posed following direct problem:

$$\begin{cases} \nabla \cdot k(x)\nabla v = 0 & \text{in} \quad \Omega, \\ v = l, & \text{on} \quad \Gamma_d, \\ k(x)\nabla v \cdot n = 0 & \text{on} \quad \Gamma_u \end{cases} \tag{3.5}$$

Applying the second Green's formula to the fields v and u we get:

$$\int_{\partial\Omega} vk(x)\nabla u \cdot n - uk(x)\nabla v \cdot n = 0\,. \tag{3.6}$$

The integral is to be understood in the duality sense. Exploiting the boundary data, we obtain:

$$\int_{\Gamma_u} vk(x)\nabla u \cdot n = 0\,. \tag{3.7}$$

Let us consider a field $\chi \in \mathcal{C}^\infty(\Gamma_u)$, the well-posed mixed problem:

$$\begin{cases} \nabla\cdot k(x)\nabla\mathcal{Z} = 0 & \text{in} \quad \Omega, \\ \mathcal{Z} = 0, & \text{on} \quad \Gamma_d, \\ k(x)\nabla\mathcal{Z}\cdot n = \chi & \text{on} \quad \Gamma_u \end{cases} \tag{3.8}$$

has a unique solution in $H^1(\Omega)$. From (3.7) one gets:

$$\int_{\Gamma_u} \chi\cdot v = 0, \;\; \forall\chi \;\; \text{in} \;\; \mathcal{C}^\infty(\Gamma_u) \tag{3.9}$$

and therefore v satisfies the following homogenous Cauchy problem:

$$\begin{cases} \nabla\cdot k(x)\nabla v = 0 & \text{in} \quad \Omega, \\ v = 0, & \text{on} \quad \Gamma_u, \\ k(x)\nabla v\cdot n = 0 & \text{on} \quad \Gamma_u\,. \end{cases} \tag{3.10}$$

Then by the Holmgren theorem $v = 0$, which leads to $l \equiv 0$. □

Minimization problem

In this section we formulate the previous inverse problem as a minimization one. For each given $(\tau, \eta) \in H^{1/2}(\Gamma_i)\times H^{-1/2}(\Gamma_i)$, we consider two mixed well-posed problems. The first one is called the Dirichlet problem (with Dirichlet condition on Γ_d)

$$(P_D)\begin{cases} \nabla\cdot k(x)\nabla u_D &= 0 \qquad \text{in } \Omega \\ u_1 &= T \qquad \text{on } \Gamma_d \\ k(x)\nabla u_D\cdot n + \alpha u_D &= \eta + \alpha\tau \quad \text{on } \Gamma_u \end{cases}$$

where α is a scalar parameter.

The second one is called the Neumann problem (with Neumann condition on Γ_d)

$$(P_N)\begin{cases} \nabla\cdot k(x)\nabla u_N &= 0 \qquad \text{in } \Omega \\ k(x)\nabla u_N\cdot n + \beta u_N &= \eta + \beta\tau \quad \text{on } \Gamma_u \\ k(x)\nabla u_N\cdot n &= \Phi \qquad \text{on } \Gamma_d \end{cases}$$

where β is a scalar parameter.

Then, the unknown data (t, φ) can be characterized as the solution of the following minimization problem: $(\varphi, t) = \arg\min_{\eta,\tau} E_{\alpha,\beta}(\eta, \tau)$ where $E_{\alpha,\beta}$ is the following energy-like error functional defined on $H^{1/2}(\Gamma_u)\times H^{-1/2}(\Gamma_u)$ by:

$$E(\eta,\tau) = \int_\Omega (k(x)\nabla u_D - k(x)\nabla u_N).(\nabla u_D - \nabla u_N)\,. \tag{3.11}$$

For the problems (P_D) and (P_N), the introduced parameters (α, β) permit us to specify different types of boundary conditions on Γ_u. We will treat the minimization problem using one of the following conditions:

- The Neumann-Dirichlet case which will be denoted by ND. It corresponds to $\alpha = 0$ (i.e. (P_D) with Neumann boundary condition on Γ_u) and $\beta = +\infty$ (i.e. (P_N) with Dirichlet boundary condition on Γ_u).
- The Dirichlet-Dirichlet case which will be denoted by DD. It corresponds to $\alpha = \beta = +\infty$. In this case, the first order optimality condition leads to the variational form of the Steklov-Poincaré operator.
- The Neumann-Neumann case which will be denoted by NN. It corresponds to $\alpha = \beta = 0$. It describes the so-called dual Steklov-Poincaré operator.

3.2 An energy-like error functional

The energy-like error derivative

Using the properties of the u_D and u_N, it is straightforward to derive an alternative expression of the E_{ND} functional:

$$E_{ND}(\eta,\tau) = \int_{\Gamma_i} (\eta - k(x)\nabla u_N.n)(u_D - \tau) + \int_{\Gamma_c} (k(x)\nabla u_D.n - \Phi)(T - u_N)\,. \tag{3.12}$$

This expression shows that the error between the two fields u_D and u_N can be expressed equivalently by an integral involving only the boundary of the domain Ω with respect to the pair (η, τ), it is easy to evaluate the gradient of the error functional:

Lemma 3. *For a pair* (η, τ)

$$\frac{\partial E_{ND}(\eta,\tau)}{\partial \eta}.\psi = \int_{\Gamma_i} [u_D(\eta) - \tau]\,\psi + \int_{\Gamma_d} k(x)\nabla u_1^*(\psi)\cdot n\,[T - u_N(\tau)] \tag{3.13}$$

$$\frac{\partial E_{ND}(\eta,\tau)}{\partial \tau}.h = \int_{\Gamma_u} [k(x)\nabla u_N(\tau)\cdot n - \eta]\,h + \int_{\Gamma_d} [\Phi - k(x)\nabla u_D(\eta)\cdot n]\,u_2^*(h) \tag{3.14}$$

for all $(h,\psi) \in H^{1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$, *and where* u_1^* *and* u_2^* *solve:*

$$\begin{cases} \nabla\cdot k(x)\nabla u_1^* = 0 & \text{in} \quad \Omega, \\ u_1^* = 0 & \text{on} \quad \Gamma_d, \\ k(x)\nabla u_1^*\cdot n = \psi & \text{on} \quad \Gamma_u \end{cases} \tag{3.15}$$

$$\begin{cases} \nabla\cdot k(x)\nabla u_2^* = 0 & \text{in} \quad \Omega, \\ u_2^* = h & \text{on} \quad \Gamma_u, \\ k(x)\nabla u_2^*\cdot n = 0 & \text{on} \quad \Gamma_d\,. \end{cases} \tag{3.16}$$

Problems (3.15) *and* (3.16) *depend on directions* ψ *and* h.

The Lagrangian derivative: adjoint state

The components of the gradient can be computed in a more efficient way by using the adjoint state method, which makes it possible to evaluate the gradient in any direction using only the determination of two adjoint fields v_1 and v_2. We consider the following Lagrangian:

$$L(u_D,u_N,v_1,v_2,\lambda,\mu;\eta,\tau)=\int_\Omega k(x)(\nabla u_D-\nabla u_N)^2+\int_\Omega k(x)\nabla u_D\nabla v_1-\int_{\Gamma_u}\eta v_1 \\ +\int_\Omega k(x)\nabla u_N\nabla v_2-\int_{\Gamma_d}\Phi v_2+\int_{\Gamma_u}[\mu v_2+\lambda(u_N-\tau)] \quad (3.17)$$

and the following spaces:

$$\begin{aligned} V_1 &= \{v\in H^1(\Omega)\,/\,v|_{\Gamma_d}=T\} \\ V_1^0 &= \{v\in H^1(\Omega)\,/\,v|_{\Gamma_d}=0\}\,. \end{aligned}$$

Then $(u_N,u_N,v_1,v_2,\lambda,\mu)\in V_1\times H^1(\Omega)\times V_1^0(\Omega)\times H^1(\Omega)\times H^{-1/2}(\Gamma_u)\times H^{-1/2}(\Gamma_u)$.

Lemma 4. *For any (η,τ) and the above defined fields, it follows:*

$$E_{ND}(\eta,\tau)=L(u_D,u_N,v_1,v_2,\lambda,\mu;\eta,\tau). \quad (3.18)$$

The gradient of E_{ND} can be obtained from the partial derivatives of L with respect to η and τ, that is:

$$\frac{\partial E_{ND}(\eta,\tau)}{\partial\eta}.\psi=-\int_{\Gamma_u}2v_1\psi \quad (3.19)$$

$$\frac{\partial E_{ND}(\eta,\tau)}{\partial\tau}.h=-\int_{\Gamma_u}2(\eta-k\nabla u_N.n-k\nabla v_2.n)h \quad (3.20)$$

where:

$$\begin{cases} \nabla\cdot k(x)\nabla v_1=0 & in \quad \Omega \\ v_1=0 & on \quad \Gamma_d \\ k(x)\nabla v_1\cdot n=k(x)\nabla u_N\cdot n-\eta & on \quad \Gamma_u \end{cases} \quad (3.21)$$

$$\begin{cases} \nabla\cdot k(x)\nabla v_2=0 & in \quad \Omega \\ v_2=0 & on \quad \Gamma_u \\ k(x)\nabla v_2\cdot n=k\nabla u_D\cdot n-\Phi & on \quad \Gamma_d\,. \end{cases} \quad (3.22)$$

We opted for this Lagrangian form, where boundary conditions are incorporated explicitly, in order to ensure that the underlying functions spaces are fixed and do not depend on the unknown field τ.

Revisiting Kozlov-Maz'ya-Fomin's algorithm

The method is related to that introduced by Kozlov in [34] and widely numerically experimented (see [43] and references therein). In that approach, the data completion problem is solved on the basis of an alternating iterative procedure, where successive solutions of well-posed mixed boundary value problems for the original equation are computed. The method has been proved to be convergent. Our approach generalizes that of [34]: as shown below, the KMF's method can be viewed as the energy-like error functional minimization by an alternating procedure in the φ and t directions. Therefore, solving the Cauchy system (3.2) is achieved when the data completion (φ, t) leads to the same field $u_D = u_N$ in Ω. Basically, the iterative data completion procedure of [34] is derived from these observations. It can be summarized as follows: starting from an initial guess of the flux φ on Γ_u, this guess is iteratively corrected by solving alternately problems of form P_D and P_N, where at each iteration the appropriate boundary data results from the solution of the previously solved boundary value problems. A sequence of well-posed mixed problems is generated as follows: u^{2j+1} solves P_D with t replaced by u^{2j}, while u^{2j+2} solves P_N with φ replaced by $k(x)\nabla u^{2j+1} \cdot n$. Reverting to our energy-like error functional, the link with KMF's algorithm is revealed by the following proposition:

Proposition 3. *The KMF's algorithm can be interpreted as an alternating-direction minimization method for the energy-like error functional E. More precisely:*

- *Step $2j+1$ of KMF algorithm: u^{2j+1} is characterized by:*

$$u^{2j+1} = u_N(\tau^{2j+1}) \Longleftrightarrow \tau^{2j+1} = \arg\min E_{ND}(\eta^{2j}, \tau) \textit{ with } \eta^{2j} = \nabla u^{2j}.n|_{\Gamma_u}$$

- *Step $2j+2$ of KMF algorithm: u^{2j+2} leads to:*

$$u^{2j+2} = u_D(\eta^{2j+2}) \Longleftrightarrow \eta^{2j+2} = \arg\min E_{ND}(\eta, \tau^{2j+1}) \textit{ with } \tau^{2j+1} = u^{2j+1}|_{\Gamma_u}$$

The convergence of KMF's algorithm is proved for a compatible data pair.

3.3 The first order optimality condition

We derive here the first order optimality condition for the previous minimization problem. In the case of compatible data we have the following result.

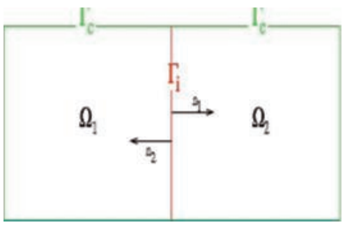
Theorem 6. *When (T, Φ) is a compatible pair, the minimum of $E_{\alpha,\beta}$ is reached when:*

$$\begin{aligned} u_D &= u_N + Const \quad on \quad \Gamma_u, \\ k(x)\nabla u_D \cdot n &= k(x)\nabla u_N . n \quad on \quad \Gamma_u. \end{aligned} \tag{3.23}$$

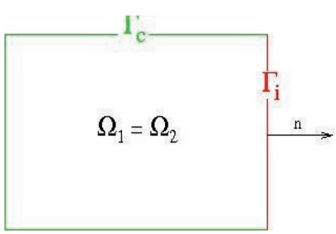
Remark 5. Link with the Domain Decomposition tools: *When dealing with the domain decomposition tools to solve P_D and P_N on the subdomain Ω_1 and Ω_2 respectively, the parallel solution P_D and P_N would give the value u_D and u_N of u on each subdomain and these values would satisfy the compatibility condition: continuity of the fields and of the normal derivative across Γ_i*

$$\begin{aligned} u_D &= u_N \quad on \quad \Gamma_u, \\ k(x)\nabla u_D \cdot n &= -k(x)\nabla u_N \cdot n \quad on \quad \Gamma_u. \end{aligned} \tag{3.24}$$

(the sign $-$ is related to the change of unit normal orientation).



Whereas, in the missing boundary data recovering (Ω_1 and Ω_2 being equal to the domain



Ω), we resort here to a fictition domain decomposition process.

Proof: We derive the first optimality conditions in the three considered cases. (i) *The Neumann-Dirichlet case:* the problems (P_D) and (P_N) are considered using, respectively, Neumann and Dirichlet conditions on Γ_u. For each given $(\tau,\eta) \in H^{1/2}(\Gamma_i) \times H^{-1/2}(\Gamma_i)$, we have the following mixed well-posed problems

$$(P_D)\begin{cases} \nabla\cdot k(x)\nabla u_D^\eta &= 0 \quad \text{in } \Omega \\ u_D^\eta &= T \quad \text{on } \Gamma_d \\ k(x)\nabla u_D^\eta\cdot n &= \eta \quad \text{on } \Gamma_u \end{cases} \qquad (P_N)\begin{cases} \nabla\cdot k(x)\nabla u_N^\tau &= 0 \quad \text{in } \Omega \\ k(x)\nabla u_N^\tau\cdot n &= \Phi \quad \text{on } \Gamma_d \\ u_N^\tau &= \tau \quad \text{on } \Gamma_u\,. \end{cases}$$

Using the Green formula, the partial derivative of $E_{ND} = E_{0,+\infty}$ with respect to τ is given by

$$\begin{aligned} \frac{\partial E_{ND}}{\partial\tau}(w) &= \int_\Omega 2\nu k(x)\nabla(u_N^\tau - u_D^\eta).\nabla r_N^w\,dx \\ &= \int_{\partial\Omega} k(x)\nabla(r_N^w).n\,(u_N^\tau - u_D^\eta)\,ds, \end{aligned}$$

where r_N^w is the solution to

$$\begin{cases} \nabla\cdot k(x)\nabla r_N^w &= 0 \quad \text{in } \Omega \\ k(x)\nabla(r_N^w)\cdot n &= 0 \quad \text{on } \Gamma_d \\ r_N^w &= w \quad \text{on } \Gamma_u. \end{cases}$$

Since $k(x)\nabla(r_N^w)\cdot n = 0$ on Γ_d, we get:

$$\frac{\partial E_{ND}}{\partial\tau}(w) = \int_{\Gamma_u} k(x)\nabla(r_N^w)\cdot n\,(u_N^\tau - u_D^\eta)\,ds \quad \forall\, w\in H^{1/2}(\Gamma_u). \tag{3.25}$$

In a similar way we derive the partial derivative of E_{ND} with respect to η

$$\begin{aligned} \frac{\partial E_{ND}}{\partial\eta}(h) &= \int_\Omega 2k(x)\nabla(u_D^\eta - u_N^\tau)\cdot\nabla r_D^h\,dx \\ &= \int_{\Gamma_u} k(x)\nabla(u_D^\eta - u_N^\tau)\cdot n\,r_D^h\,ds\,, \quad \forall\, h\in H^{-1/2}(\Gamma_u), \end{aligned} \tag{3.26}$$

where r_D^h is the solution to

$$\begin{cases} \nabla\cdot k(x)\nabla r_D^h &= 0 \quad \text{in } \Omega \\ r_D^h &= 0 \quad \text{on } \Gamma_d \\ k(x)\nabla(r_D^h)\cdot n &= h \quad \text{on } \Gamma_u. \end{cases}$$

Consider the Steklov-Poincaré operator

$$\begin{aligned} S_N \;:\; H^{1/2}(\Gamma_u) &\longrightarrow H^{-1/2}(\Gamma_u) \\ w &\longrightarrow k(x)\nabla(r_N^w)\cdot n\,. \end{aligned} \tag{3.27}$$

One can observe that the kernel $N(S_N)$ and the range $R(S_N)$ of the operator S_N are defined by

$$N(S_N) = \mathbb{R} \text{ and } R(S_N) = H^{-1/2}(\Gamma_u).$$

Then, it follows that $S_N : H^{1/2}(\Gamma_u)/N(S_N) \longrightarrow H^{-1/2}(\Gamma_u)$ is an isomorphism. Consequently, the equation (3.25) implies the first condition of Theorem 6:

$$u_N - u_D = \text{Const} \quad \text{on} \quad \Gamma_u.$$

For the second condition, we introduce the dual Steklov-Poincaré operator:

$$\begin{array}{cccc} S_D^{-1} : & H^{-1/2}(\Gamma_u) & \longrightarrow & H^{1/2}(\Gamma_u) \\ & h & \longrightarrow & r_D^h. \end{array} \tag{3.28}$$

From the fact that (T, Φ) is a compatible pair one can deduce that S_D^{-1} is an isomorphism. Then the equality

$$k(x)\nabla(u_D^\eta) \cdot n = k(x)\nabla(u_N^\tau) \cdot n \quad \text{on } \Gamma_u,$$

follows immediately from the equation (3.26).

(ii) *The Dirichlet-Dirichlet case:* We consider the problems (P_D) and (P_N) with Dirichlet conditions on Γ_u. Then, we have

$$(P_D) \left\{ \begin{array}{rll} \nabla \cdot k(x)\nabla(u_D^v) & = 0 & \text{in } \Omega \\ u_D^v & = T & \text{on } \Gamma_d \\ u_D^v & = v & \text{on } \Gamma_u, \end{array} \right. \qquad (P_N) \left\{ \begin{array}{rll} \nabla \cdot k(x)\nabla(u_N^v) & = 0 & \text{in } \Omega \\ k(x)\nabla(u_N^v) \cdot n & = \Phi & \text{on } \Gamma_d \\ u_N^v & = v & \text{on } \Gamma_u, \end{array} \right.$$

for each $v \in H^{1/2}(\Gamma_u)$.

Using the Green formula, we derive

$$E_{+\infty,+\infty}(v) = E_{DD}(v) = 1/2 \int_\Omega k(x)\nabla(u_D^v - u_N^v).\nabla(u_D^v - u_N^v).$$

The partial derivative of E_{DD} with respect to v is given by:

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_\Omega k(x)\nabla(u_D^v(v) - u_N^v(v)) \cdot \nabla(r_D^h - r_N^h) \quad \forall h \in H^{1/2}(\Gamma_u),$$

where r_D^h and r_N^h are respectively the solution to

$$\left\{ \begin{array}{rll} \nabla \cdot k(x)\nabla r_D^h & = 0 & \text{in } \Omega \\ r_D^h & = 0 & \text{on } \Gamma_d \\ r_D^h & = h & \text{on } \Gamma_u, \end{array} \right. \qquad \left\{ \begin{array}{rll} \nabla \cdot k(x)\nabla r_N^h & = 0 & \text{in } \Omega \\ k(x)\nabla(r_N^h) \cdot n & = 0 & \text{on } \Gamma_d \\ r_N^h & = h & \text{on } \Gamma_u\,. \end{array} \right.$$

From the weak formulation of the last problems we obtain:

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_{\partial\Omega} k(x)\nabla(u_D^v(v) - u_N^v(v)) \cdot n \cdot (r_D^h) + \int_{\partial\Omega} (u_D^v(v) - u_N^v(v))k(x)\nabla(r_N^h) \cdot n\,.$$

Using the fact that $r^h_{D\Gamma_d} = 0$ and $k(x)\nabla(r_N^h) \cdot n_{\Gamma_d} = 0$, we get

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_{\Gamma_u} k(x)\nabla(u_D^v(v) - u_N^v(v)) \cdot n \cdot h\,, \quad \forall h \in H^{1/2}(\Gamma_u).$$

The second condition of Theorem 6 follows immediately from the last equation.

(iii) *The Neumann-Neumann case:* Here we impose Neumann conditions on Γ_u. This case corresponds to the so-called dual Cauchy-Steklov-Poincaré operator. We consider the following mixed boundary value problems:

$$(P_D)\begin{cases} \nabla \cdot k(x)\nabla u_D^\eta &= 0 \quad \text{in } \Omega \\ k(x)\nabla(u_D^\eta) \cdot n &= \eta \quad \text{on } \Gamma_u \\ u_D^\eta &= T \quad \text{on } \Gamma_d \end{cases} \qquad (P_N)\begin{cases} \nabla \cdot k(x)\nabla u_N^\eta &= 0 \quad \text{in } \Omega \\ k(x)\nabla(u_N^\eta) \cdot n &= \eta \quad \text{on } \Gamma_u \\ k(x)\nabla(u_N^\eta) \cdot n &= \Phi \quad \text{on } \Gamma_d\,. \end{cases}$$

The Theorem 6 is fulfilled when $u_D^\eta = u_N^\eta + const$, which can be expressed by:

$$E_{NN}(g) = \int_\Omega k(x)\nabla(u_D^\eta - u_N^\eta) \cdot \nabla(u_D^\eta - u_N^\eta)\,.$$

The partial derivative of E_{NN} with respect to η can be written as:

$$\frac{\partial E_{NN}}{\partial \eta}(h) = \int_\Omega k(x)\nabla(u_D^\eta(\eta) - u_N^\eta(\eta)) \cdot \nabla(r_D^h - r_N^h)\,, \ \ \forall\, h \in H^{-1/2}(\Gamma_u),$$

where r_D^h and r_N^h are respectively the solution to

$$\begin{cases} \nabla \cdot k(x)\nabla r_D^h &= 0 \quad \text{in } \Omega \\ k(x)\nabla(r_D^h) \cdot n &= h \quad \text{on } \Gamma_u \\ r_D^h &= \quad \text{on } \Gamma_d \end{cases} \qquad \begin{cases} -\nabla \cdot k(x)\nabla r_N^h &= 0 \quad \text{in } \Omega \\ k(x)\nabla(r_N^h) \cdot n &= 0 \quad \text{on } \Gamma_d \\ k(x)\nabla(r_N^h) \cdot n &= h \quad \text{on } \Gamma_u\,. \end{cases}$$

Using Green formula and the fact that $r_D^h = 0$ and $k(x)\nabla(r_N^h) \cdot n = 0$ on Γ_d, we obtain

$$\frac{\partial E_{NN}}{\partial \eta}(h) = \int_{\Gamma_u} (u_D^\eta - u_N^\eta) \cdot h\,, \quad \forall h \in H^{-1/2}(\Gamma_u)$$

which implies the first condition of (3.23).

□

The interfacial operators

In this section we introduce interfacial operators. For each case, we rephrase the first order optimality condition, described in the previous section, in terms of an interfacial operator.

1. The Neumann-Dirichlet case: The solutions u_D^η and u_N^τ can be decomposed as

$$u_D^\eta = u_D^0 + r_D^\eta \text{ and } u_N^\tau = u_N^0 + r_N^\tau.$$

Then, the equalities (3.23) can be rewritten as

$$\begin{cases} r_N^\tau - r_D^\eta &= u_D^0 - u_N^0 & \text{on } \Gamma_u \\ k(x)\nabla(r_N^\tau)\cdot n - k(x)\nabla(r_D^\eta)\cdot n &= k(x)\nabla(u_D^0)\cdot n - k(x)\nabla(u_N^0)\cdot n & \text{on } \Gamma_u. \end{cases}$$

Using the definitions of the fields r_N^τ and r_D^η, we deduce the following interfacial system satisfied by (τ, η)

$$\begin{cases} \tau - S_D^{-1}(\eta) &= u_D^0 - u_N^0 & \text{on } \Gamma_u \\ -S_N(\tau) + \eta &= k(x)\nabla(u_N^0)\cdot n - k(x)\nabla(u_D^0)\cdot n & \text{on } \Gamma_u, \end{cases}$$

which can be written as:

$$S\begin{pmatrix} \tau \\ \eta \end{pmatrix} = \chi,$$

where $\chi = \begin{pmatrix} u_D^0 - u_N^0 \\ k(x)\nabla(u_N^0)\cdot n - k(x)\nabla(u_D^0)\cdot n \end{pmatrix}$ only depends on the data (T, Φ) and

$$S = \begin{pmatrix} I & -S_D^{-1} \\ -S_N & I \end{pmatrix}.$$

2. The Dirichlet-Dirichlet case: We decompose the solutions u_D^v and u_N^v as

$$u_D^v = u_D^0 + r_D^v \quad \text{and} \quad u_N^v = u_N^0 + r_N^v\,.$$

According to the previous theorem, when the minimum is reached we have

$$\begin{cases} u_D^v &= u_N^v \quad \text{on} \quad \Gamma_u, \\ k(x)\nabla(u_N^v)\cdot n &= k(x)\nabla(u_D^v)\cdot n \quad \text{on} \quad \Gamma_u. \end{cases}$$

The first condition is always fulfilled. The second one reads

$$k(x)\nabla(r_D^v)\cdot n - k(x)\nabla(r_N^v)\cdot n = -(k(x)\nabla(u_D^0)\cdot n - k(x)\nabla(u_N^0)\cdot n) \quad \text{on} \quad \Gamma_u\,.$$

This identity amounts to the requirement that v satisfies the Steklov-Poincaré type equation:

$$S(v) = \chi \quad \text{on} \quad \Gamma_u, \tag{3.29}$$

where $\chi = -(k(x)\nabla(u_D^0) \cdot n - k(x)\nabla(u_N^0)) \cdot n$, and S is the Stokes-Cauchy-Steklov-Poincaré operator formally defined by

$$S(v) = (S_D - S_N)(v) = k(x)\nabla(r_D^v) \cdot n - k(x)\nabla(r_N^v) \cdot n\,.$$

3. The Neumann-Neumann case:

   In this case the first relation in (3.23) gives

$$u_D^\eta = u_N^\eta \ \text{ and } S(\eta) = \chi \quad \text{on} \quad \Gamma_u,$$

   where $\chi = -(u_D^0 - u_N^0)$, and S is defined by $S(\eta) = S_D^{-1} - S_N^{-1} = r_D^\eta - r_N^\eta$.

3.4 Numerical illustration

We deal with a two dimensional problem defined in a cross section Ω. The domain Ω, the accessible boundary Γ_a and the inaccessible boundary Γ_u are depicted in Figure 5.

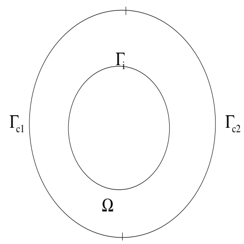


Figure 5: Domain Ω for the test problem.

Aming to validate the proposed approach, we consider here the identification of the temperature and the heat flux on the inner circle Γ_u from an over-specified data on the outer circle Γ_a. The data is generated from numerical solution (i.e. Synthetic data). The numerical simulation are run under the Freefem++ software environment [26]; it is a free software based on the Finite Element Method. The domain Ω is discretized using an uniform mesh with 100 nodes on Γ_u and 200 nodes on Γ_a.
We reconstructed the unknown data on Γ_u in the case Dirichlet-Dirichlet.
The stopping criteria is $E(\lambda) \leq \varepsilon$, where ε is a given tolerance level.

This test deals with the reconstruction of singular data. We note $k = \dfrac{1}{\sqrt{x^2+y^2}}$.
Synthetic data are generated by solving the following forward problem:

$$\begin{cases} -div(k\nabla u) = 0 & \text{in} \quad \Omega \\ k\frac{\partial u}{\partial n} = 1 & \text{on} \quad \Gamma_{c1} \\ k\frac{\partial u}{\partial n} = -1 & \text{on} \quad \Gamma_{c2} \\ u = RE(\frac{1}{z-a}) & \text{on} \quad \Gamma_u \end{cases} \tag{3.30}$$

where $z = x + iy$ and taking the restriction of this solution and its normal derivative on Γ_d, we obtain the Cauchy data Φ and T in problem (1.1).
We choose a source point a in the vinicity of the inner boundary.
In Table 3, we represent some numerical results for various values of the parameter a. As can be seen from this table, the number of iterations increases when the value of a approaches of Γ_u.
In Figure 6, the reconstruction of the Dirichlet data are presented, in, for $a = 0.5$, $a = 0.8$

a	nr. of nodes	nr.elements	ε	nr. iterations
0.5	680	1210	10^{-3}	2267
0.8	680	1210	10^{-3}	15020
0.9	680	1210	10^{-3}	32227

Table 3: Number of iterations at different values of the parameter a.

and $a = 0.9$ on Γ_u. The agreement in both these reconstructions are reasonable and stable. We observe that the recovered solution matches quite well the exact one. Note that there are some problems to actually capture the high peak of this solution.

In the previous test, the data were the exact values of the synthetic solution. However, in practice, it is necessary to consider noisy data. Here, we still consider the singular test case but the input boundary Dirichlet data T has been perturbed as follows:

$$T = T + \sigma w$$

where σ denotes the noise level relative to $\|T\|_{L^2(\Gamma_d)}$, and w is a random function generated by Freefem++. We illustrate the results with various levels for $a = 0.8$.
In Figures 7 and 8, we plot the exact and the recovered solution still on Γ_u.

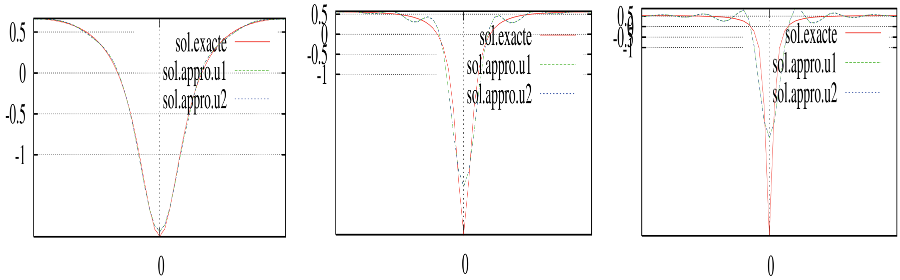


Figure 6: Reconstruction of Dirichlet data

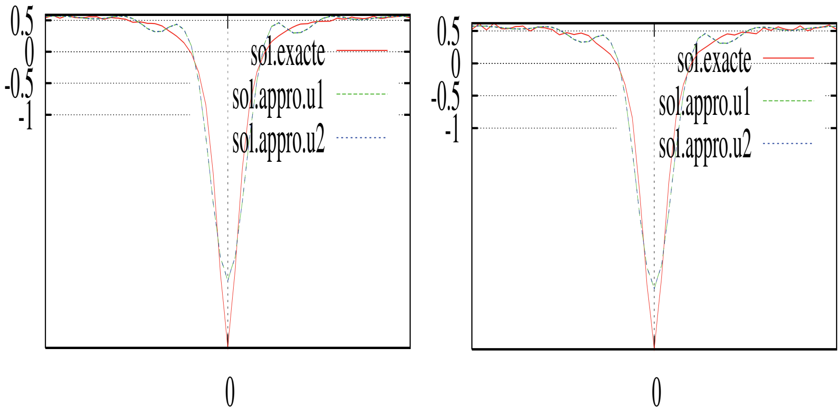


Figure 7: Reconstruction of Dirichlet data with a noise of 2% (left) and 4% (right).

Aknowledgments

The author wishes to warmly thank Stéphane Andrieux, Amel Ben Abda, Slim Chaabane, Chokri Elhechmi and Michel Kern whose works and results are the matter of this short course. Many thanks also to Badreddine Rjaibi and Imen Ben Saad who helped in the presentation of these results.

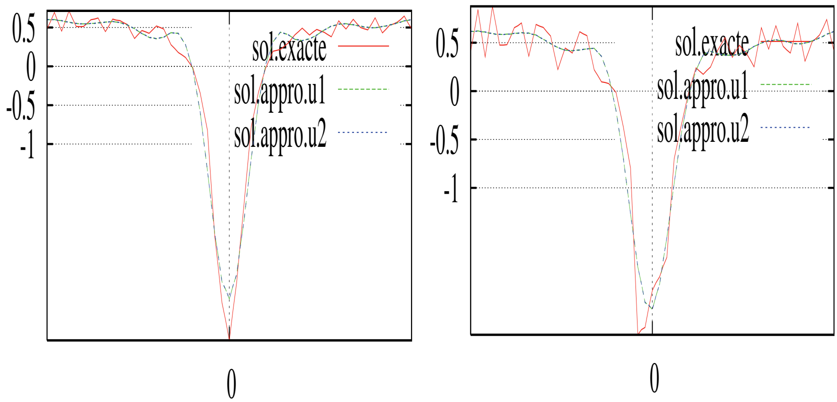


Figure 8: Reconstruction of Dirichlet data with a noise of 10% (left) and 20% (right).

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