Mathematical analysis and numerical methods for pricing some pension plans and mortgages

Autora: María del Carmen Calvo Garrido

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Director: Carlos Vázquez Cendón

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Mathematical analysis and numerical methods for pricing some pension plans and mortgages

AUTORA:

María del Carmen Calvo Garrido

DIRECTOR:

Carlos Vázquez Cendón

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El abajo firmante hace constar que es director de la Tesis Doctoral titulada **"Mathematical analysis and numerical methods for pricing some pension plans and mortgages"** desarrollada por María del Carmen Calvo Garrido, cuya firma también se incluye, dentro del programa de doctorado **"Métodos Matemáticos y Simulación Numérica en Ingeniería y Ciencias Aplicadas"** en el Departamento de Matemáticas (Universidade da Coruña), dando su consentimiento para su presentación y posterior defensa.

The undersigned hereby certifies that he is supervisor of the Thesis entitled **"Mathematical analysis and numerical methods for pricing some pension plans and mortgages"** developed by María del Carmen Calvo Garrido, whose signature is also included, inside the Ph.D Program **"Mathematical Methods and Numerical Simulation in Applied Sciences and Engineering"** at the Department of Mathematics (University of A Coruña), consenting to its presentation and posterior defense.

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Director:

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Doctoranda:

María del Carmen Calvo Garrido

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Table of Contents

List of Tables

List of Figures

Abstract

The main objective of this thesis concerns to the mathematical analysis and numerical solution of some partial differential equation (PDE) models for pricing defined benefit pension plans and fixed-rate mortgages.

In pension plans, the benefits at retirement depend on the average salary. When early retirement is not allowed a Cauchy problem associated with a degenerated Kolmogorov operator is posed, otherwise a complementarity problem arises. Under both alternatives, we obtain the existence and uniqueness of solution in appropriate functional spaces. If we incorporate jumps in the salary, then the Cauchy and complementarity problems are associated with a partial integro-differential operator (PIDE).

In fixed-rate mortgages, we consider the possibility of prepayment and default. The underlying stochastic factors are the interest rate and the value of the collateral. The mortgage valuation requires solving a sequence of linked free boundary problems. The values of the insurance and coinsurance are the solution of respective sequences of linked Cauchy problems.

For the numerical solution a Lagrange-Galerkin method for PDEs discretization is proposed, which is also combined with an active set iterative method based on an augmented Lagrangian formulation for inequality constraints. The integral term in the PIDE operator is treated with appropriate numerical integration techniques.

Resumen

El objetivo principal de esta tesis se centra en el análisis matemático y la solución numérica de algunos modelos de ecuaciones en derivadas parciales (EDPs) para valorar planes de pensiones con beneficios definidos e hipotecas a tipo de interés fijo.

En los planes de pensiones, los beneficios por jubilación dependen del salario medio. Cuando no se permite jubilación anticipada, se plantea un problema de Cauchy asociado con un operador de Kolmogorov degenerado, en otro caso aparece un problema de complementariedad. Bajo ambas alternativas, obtenemos la existencia y unicidad de solución en los espacios funcionales adecuados. Además, si incorporamos saltos en el salario, entonces los problemas de Cauchy y de complementariedad están asociados a un operador integro-diferencial.

En las hipotecas a tipo de interés fijo, consideramos las opciones de amortización anticipada e impago. Los factores estocásticos subyacentes son el tipo de interés y el valor del colateral. La valoración del contrato requiere resolver una serie de problemas de frontera libre enlazados. Los valores del seguro y de la fracción de pérdida potencial no cubierta por dicho seguro son la solución de una serie de problemas de Cauchy enlazados.

Para obtener una solución numérica, se propone un método de Lagrange-Galerkin para la discretización de las EDPs, que se combina con un método iterativo de conjunto activo basado en una formulación de tipo lagrangiana aumentada para las restricciones de desigualdad. El término integral en el operador integro-diferencial se trata con técnicas de integración numérica apropiadas.

Resumo

O obxectivo principal desta tese céntrase na análise matemática e a solución numérica dalgúns modelos de ecuacións en derivadas parciais (EDPs) para valorar plans de pensións con beneficios definidos e hipotecas a tipo de xuro fixo.

Nos plans de pensións, os beneficios por xubilación dependen do salario medio. Cando non se permite xubilación anticipada, formúlase un problema de Cauchy asociado cun operador de Kolmogorov dexenerado, noutro caso aparece un problema de complementariedade. Baixo as dúas alternativas, obtemos a existencia e unicidade de solución nos espazos funcionais axeitados. Ademais, se incorporamos saltos no salario, entón os problemas de Cauchy e de complementariedade están asociados cun operador integro-diferencial.

Nas hipotecas a tipo de xuro fixo, consideramos as opcións de amortización anticipada e incumprimento. Os factores estocásticos subxacentes son o tipo de xuro e o valor do colateral. A valoración do contrato require resolver unha serie de problemas de fronteira libre enlazados. Os valores do seguro e da fracción de perda potencial non cuberta polo devandito seguro son a solución dunha serie de problemas de Cauchy enlazados.

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Introduction

In this work we study some mathematical models for pricing specific financial derivatives. More precisely, the mathematical modeling, the analysis and numerical solution of some pension plans and mortgages is analyzed. A general classification of these products can be established according to some of their intrinsic features. Actually, among all the variety of these derivatives, we focus on the study of defined benefit pension plans and fixed-rate mortgages.

The dynamic hedging methodology introduced in the seventies of the last century by Black and Scholes [11] and Merton [45] for the case of European vanilla options has been extended since then to price more complex derivatives [68]. In the present thesis, this methodology is applied to obtain the partial differential equations that govern the valuation of defined benefit pension plans and fixed-rate mortgages. From the mathematical point of view the price of both products can be obtained as the solution of problems associated with degenerated parabolic operators.

In the first part of this work, we focus on the study of pension plans with defined benefits. In this kind of pension plans, benefits at retirement are fixed or determined by a set formula involving some labor-related factors of the employee. In this work, it is assumed that this formula depends on the average salary of the worker during certain number of years. Actually, the main aim is to obtain the value of the pension plan, which is understood as the value of the liability that the promoter has to set aside to guarantee future promised payments to the member of the plan.

The value of the plan depends on the salary of the member of the plan as the main underlying variable. In a first approach, we assume that the salary follows a stochastic dynamics which mainly consists in a geometric Brownian motion, the trajectories of which are continuous. However, in certain situations (such as crisis or bubbles in some economic sectors) abrupt changes in the salary may appear, so that the consideration of stochastic models only driven by Brownian motion results to be not enough realistic. In this new setting, the consideration of a jump-diffusion process for the salary evolution would be more appropriate. For this purpose, we will also consider the case when the salary presents discontinuous paths with a finite number of jumps following a Poisson distribution.

Jump-diffusion models proposed by Merton in [46] or more recently by Kou in [39] fit properly some market data or sudden abrupt changes in the underlying factor. In this work, we also consider the case when salary dynamics follows a Merton jump-diffusion model.

On the other hand, at retirement date the benefits received by the member of the plan are supposed to be indicated to the average salary of a certain number of years. In this sense, there is a certain analogy between this kind of pension plans and Asian options. In particular, it is suitable to introduce a new variable representing the accumulated salary during these years as a previous step to apply Ito formula and dynamic hedging methodology. Once both classical tools are used, the defined benefit pension plans valuation problem can be written in terms of a Cauchy problem associated with a degenerated Kolmogorov operator when early retirement is not allowed. Assuming that benefits at retirement depend on the average salary this model was first introduced by Sherris and Shen in [61] based in alternative arguments. If we add the possibility of early retirement then a complementarity problem arises. If the benefits at retirement depend only on the salary at the retirement date (that is, on the final salary instead of on the average salary) and early retirement is allowed, in [26] Friedman and Shen introduce the corresponding one factor complementarity model and obtain the existence of solution as well as some qualitative properties of the free boundary.

However, in the present work we are interested in pension plans indicated to the average salary. Thus, in order to develop the mathematical analysis of the Cauchy problem associated with the Kolmogorov operator for the case without early retirement opportunity we take into account the analogy with Asian options with European feature. As far as the study of the existence of solution is concerned, Barucci, Polidoro and Vespri proved the existence and uniqueness of solution for European Asian options in [6]. In order to obtain the corresponding results for defined benefit pension plans we mainly extend the results for homogeneous Kolmogorov equations to non homogeneous ones.

For the mathematical analysis of the complementarity problems governing the value of pension plans with early retirement opportunity, the main departure point is the paper [47] by Pascucci and Monti, where the existence and regularity of the strong solution to the obstacle problem associated with the pricing of American Asian options with arithmetic average is analyzed. Thus, in the present thesis we mainly extend some theorems proved in [47] and [24].

Concerning the numerical solution of the mathematical models to price pension plans, in this work we propose a high order Lagrange-Galerkin method for time and space discretization, which is based on the one initially introduced by Rui and Tabata in [57] for a convection-diffusion equation with constant coefficients and later extended to other degenerated convection-diffusion-reaction PDE problems in [7] and [8]. This method is appropriate in convection dominated problems, as it is the case of European Asian option pricing problem [9] or the valuation of the derivatives studied in this thesis. In the case of pension plans when early retirement is allowed or to deal with the prepayment option in the mortgages, this Lagrange-Galerkin discretization scheme is combined with an Augmented Lagrangian Active Set algorithm (ALAS) to treat the non linearities associated with the inequality constraints in the free complementarity problems modeling the valuation of these derivatives.

Moreover, in the presence of jumps in the underlying salary, the integral term in the PIDE governing pension plans value is discretized explicitly in time, thus entering the second member of the discrete problem. The value of this integral term is approximated by using the composite trapezoidal rule.

In order to compare the numerical solution of the PDE and PIDE models with alternative pricing tools, several Monte Carlo simulation techniques have been implemented (see [27], as an example of general reference with financial applications in view). First, a crude Monte Carlo technique to obtain the confidence intervals for the value of the pension plan when early retirement is not allowed has been carried out to illustrate that the numerical results with PDE model always belong to the confidence interval. Additionally, a Monte Carlo algorithm proposed by Longstaff and Schwartz in [42] for American options is adapted in this thesis to obtain the value of the plan when early retirement is allowed. In the presence of jumps in the salary, a suitable Monte Carlo technique for jump-diffusion models has been developed.

In the second part of this thesis we address the pricing of some fixed rate mortgages. In order to fix the ideas, we define a mortgage as a contract in which the borrower obtains funds from a bank or a financial institution using a house as a collateral. In this part of the work, we focus on fixed-rate mortgages with the options of prepayment and default, where the fixed rate satisfies an equilibrium condition in order to guarantee the absence of arbitrage. More precisely, following [37] and [30], at origination the value of the contract jointly with the value of the insurance that the lender can have on the loan as a protection against borrower default and any arrangement fee, must be equal to the value of the loan to the borrower. Thus, the main objective is to obtain the value of the mortgage to the lender and also the value of other components of the contract, such as the insurance and the fraction of the loss not covered by the insurance (coinsurance). As in the work [60], the value of the contract and the other components depend on the house price and the interest rate, which are the underlying stochastic factors. We consider that the house price dynamics follows an stochastic differential equation, the stochastic part of which is governed by a Brownian motion. Among all the possible stochastic models to describe the dynamics of the interest rate, we assume the classical Cox-Ingersoll-Ross (CIR) process [18]. In contrast with Vasicek model [65], CIR model guarantees that the interest rates are positive under certain assumptions on the parameters and also satisfies the desired mean reversion property.

The value of the contract is the present value to the lender of the monthly scheduled payments from the borrower to the lender, without considering the insurance that the lender can have on the loan as a protection in case of borrower default.

As in [60], we assume the possibility of borrower default and prepayment. More precisely, we consider that prepayment can happen at any time and that default only can occur at payment dates, so that the pricing problem can be also understood as a sequence of linked options with early exercise, one for each month. Moreover, as we have mentioned before, at origination the fixed-rate of the contract must satisfy an equilibrium condition, otherwise there would be arbitrage. In order to obtain the value of this rate a Newton method with discrete derivative (secant method) is implemented.

From the mathematical point of view, a dynamic hedging technique can be applied leading to a PDE which governs the valuation of any asset depending on the house price and the interest rate. In particular, the mortgage valuation problem with prepayment and default options can be posed as a sequence of free boundary problems where the final condition for a month comes from the value at the same date of the following month. On the other hand, in order to obtain the insurance and coinsurance values, a sequence of linked Cauchy problems is posed where also the final condition for a month comes from the value at the same date of the following month.

Concerning the numerical solution of the mortgage pricing models, we propose the use of analogous numerical techniques to the ones in pension plans. More precisely, for each month the complementarity problem related to the mortgage value and the Cauchy problem related to the insurance price are solved with the numerical techniques used in pension plans with and without early retirement, respectively. Once we arrive at origination, the iterative process updates the interest rate to solve again the mortgage and insurance problems until an equilibrium rate is achieved by using a secant method to solve the equilibrium equation. Once the equilibrium rate is approximated, the coinsurance value is obtained by solving the corresponding sequence of Cauchy problems.

Note that the proposed model and numerical methods exhibit a good behaviour in small volatility regimes (convection dominated problems), where the work [60] proposes the use of first order PDEs obtained from an asymptotic expansion technique. Also the proposed complete model and methods result to be suitable in larger volatility regimes, where the asymptotic model proposed in [60] would require higher order terms of the asymptotic expansion, thus increasing the complexity and computational cost.

The outline of this thesis is as follows.

Part I, which consists of three chapters, deals with pension plans pricing problems. First a brief introduction to pension plans is given, mainly describing the characteristics of pension plans and possible classifications in terms of their intrinsic features.

In Chapter 1 the mathematical model to price defined benefit pension plans without early retirement is posed as a Cauchy problem associated with a degenerated Kolmogorov operator. At the beginning of the chapter, the stochastic differential equation which governs the dynamic of the salary and a new variable representing the cumulative salary are introduced. Then, the PDE model governing the valuation of the plan without early retirement is derived by using Ito lemma and a dynamic hedging methodology. Next, the mathematical analysis to obtain the existence and uniqueness of solution for the model and the numerical methods applied to solve the problem are described. Finally, some examples showing the numerical results obtained by solving the PDE problem and by implementing the alternative Monte Carlo simulation are presented.

In Chapter 2, we introduce the complementarity problem that arises when the option of early retirement is allowed. We also analyze the existence of solution for the model and we implement appropriate methods to obtain a numerical solution. Finally, as in the previous chapter, we present and compare numerical results obtained by solving the PDE problem and by implementing the Longstaff-Schwartz technique. Also, a comparison between the cases with and without early retirement and the aspect of the early retirement boundary are shown.

In Chapter 3, we consider the possibility of jumps in the underlying salary by using a Merton jump-diffusion model. Next, the corresponding problems associated with a partial integro-differential operator without and with early retirement are posed. Then, the numerical techniques for these models are briefly described and the corresponding results with PIDE solvers and suitable Monte Carlo simulation techniques for jump-diffusion processes are shown at the end of the chapter.

Part II deals with mortgage contracts and contains one chapter. In a brief introduction we first point out several characteristics of a mortgage contract and we describe its main components. Moreover, we state a classification of this derivative attending to some of its features.

In Chapter 4, we state the mathematical model that governs the valuation of fixed-rate mortgages with prepayment and default options and in addition to the mathematical model to obtain the value of the other components of the contract such as the insurance and the coinsurance. Thus, we derive the PDE model to price these assets and we introduce the obstacle problem that arises when prepayment is allowed and the Cauchy problems that govern the valuation of the other components of the contract. Finally, we describe how to solve the models by using numerical techniques and we present some of the results obtained.

Part I

Pension plans

Introduction to pension plans

Pension plans are an important aspect of finance in countries with advanced economies, and they affect the retirement decisions taken by their populations. Pension plans are generally classified as either defined contributions plans or defined benefit plans [12].

In the first case, at retirement the employee receives an annuity whose value depends on the investment earnings and the total contributions of both employer and employee to the pension plan account of the employee. Moreover, sometimes the employee can decide among the possible investments of the account bearing the risk.

In defined benefit plans, the pension at retirement is determined by a fixed amount or by a set formula that can involve several labor-related factors, such as the number of years of service, the salary or the average salary and this amount does not depend on investment returns. The employer must contribute enough annually to cover the value of the benefits that members of the plan earned that year. In addition, sometimes, in order to compensate for any investment losses, the employers may have to make extra contributions. In some defined benefit pension plans, if an employer does not pay the required contributions, the employer can be penalized and if an employer is suffering financial problems he/she is allowed to pay the contribution in future years. Moreover, there exists some kind of protection around these plans, for this purpose certain organisms guarantee the payment of the benefits to the employee.

The first private pension plan in the United States was established in 1875 by the American Express Company and was soon followed by pensions provided by utilities, banking and manufacturing companies [53]. Almost all of the early pension plans were traditional defined benefit pension plans that paid workers a specific monthly benefit at retirement.

As it is indicated in [53], defined benefit pension plans offer workers a number of advantages such as:

- workers are promised a specific benefit at retirement determined by the contract,
- workers can know in advance what benefits they will receive,
- the benefits of workers are certain and not dependent on investment returns,
- employers bear the risk and are responsible for providing the retirement benefits and the benefits are not dependent on the amount of salary workers are able to contribute.

Employees can start receiving pension benefits when they reach the normal retirement age set by their pension plan. However, in many pension plans workers have the early retirement opportunitiy once they have completed a certain number of years of service or they have reached a given age. But the monthly benefits that they would receive if they choose the early retirement option would be lower than if they retire at a normal age because they will be paid the same amount during more time.

A traditional defined benefit pension plan pays the salary at retirement multiplied by a number of years and by an accrued rate, with the resulting amount being paid as a monthly pension or a lump sum. Final Average Pay (FAP) schemes, for which the average salary over the last years before retirement determines the benefit, are most common in the United States. In the case of defined benefit pension plans, the employer bears the liability, which is the amount of money that the employer has to set aside to fund the employee's retirement pension in the future. The value of this liability is roughly what we refer to in this work as the value of the pension plan. Sometimes, however, pension plans also include integration with social security (integrated plans). In typical integrated plans, the pension benefit at retirement is a fraction of the average salary minus the social security benefit.

In this first part of this thesis we study defined benefit pension plans. Thus, our main objective is to obtain the value of the plan to the employee considering that retirement benefits depend on the average salary of the member of the plan during a certain number of years. First, in Chapter 1, we deal with the case in which early retirement is not allowed. Chapter 2 is devoted to pension plans with the possibility of early retirement. In both cases, we present the PDE problem, we analyze the existence of solution and we solve the mathematical models with appropriate numerical methods. Finally, in Chapter 3 we take into account the possibility of jumps in the underlying salary.

Chapter 1

Pension plans without early retirement

1.1 Introduction

In this chapter we use a mathematical approach to obtain the value of a defined benefit pension plan based on average salary without early retirement. This price is understood as the value of the liabilities of the plan with an active member. More precisely, we assume that the amount received by the employee depends on the average salary corresponding to a certain number of years before retirement. The departure point in our modeling approach is the consideration that the salary is stochastic, so that the pension plan can be handled as an option on the average salary. The dynamic hedging methodology in option pricing can then be adapted to state a partial differential equation (PDE) model. Indeed, some features appear that are also found in Asian options and bond pricing PDE models [68]. In [61], the same kind of models are stated for pension plans depending on the salary at retirement or on the average salary by using a risk-neutral probability approach. Some relatee previous works in insurance can be found, for example in [49], [62] and [69]

Additionally, we state the existence and uniqueness of solutions by using the methodology developed by [6] for this kind of Kolmogorov equations [50]. This methodology is mainly based on [6] for the existence of sub and supersolutions and [3] for uniqueness.

Moreover, we provide a numerical method for solving the PDE model by using the techniques developed in [9] for Asian options pricing models. The numerical analysis of the proposed characteristics Crank-Nicolson time discretization was addressed in [7]. Furthermore, in [8], its combination with Lagrange finite elements is studied and appropriate different quadrature formulas required in the practical implementation of the method for the fully discretized problem are considered. Both of these works were applied to the general (possibly degenerated) convection-diffusion-reaction equation under certain assumptions. We note that these assumptions are not fully verified by the PDE that we are dealing with. This explains why the predicted orders of convergence are not achieved in the academic example presented here, for which the exact solution can be analytically obtained. For the particular case of pension plans, the results obtained using this Crank-Nicolson Lagrange-Galerkin method are compared with a Monte Carlo simulation.

This chapter is organized as follows. In Section 2 the mathematical model is posed in terms of a final value problem associated with a Kolmogorov equation. In Section 3 the mathematical analysis of the model is developed to obtain the existence and uniqueness of solution. Section 4 contains the description of the numerical methods, which are applied in a bounded computational domain after a localization procedure and a variational formulation. In Section 5 some examples are presented to illustrate the performance of the proposed numerical method.

Most of the results in this chapter are included in the reference [14].

1.2 Mathematical modeling

Let us denote the age of entry of a member in the pension plan by t_0 and the time since the entry by t, so that $t = 0$ corresponds to the recruitment date of a person aged t_0. As the pension plan is indexed to the salary of the member, let us denote by

$S_{t_0,t}$ (or simply by S_t) the wage at time t of a member entered the pension plan with an age t_0. Following [61], we assume that S_t is governed by the stochastic differential equation (SDE)

$$dS_t \;=\; \alpha(t, S_t)\, dt \;+\; \sigma(t, S_t)\, dZ_t, \tag{1.1}$$

where:

- the salary growth rate α depends on the time t since the entry into the plan, the current salary S_t and the age at entry t_0,
- σ denotes the volatility of the salary,
- Z_t is a Wiener process.

The model that we are using assumes that uncertainty about the salary only depends on the volatility, and that it follows a diffusion model. This kind of evolution could correspond to an employee having a variable part of his/her salary (perhaps related to his/her bonus or the firm benefits). We also point out that in real situations some sudden events could produce abrupt changes in the salary. In that case, a jump-diffusion model turns out to be more appropriate, so that the SDE (1.1) could be replaced by the following, for example:

$$dS_t \;=\; \alpha(t, S_t)\, dt \;+\; \sigma(t, S_t)\, dZ_t \;+\; d\left(\sum_{i=1}^{N_t} Y_i\right), \tag{1.2}$$

where N_t denotes a Poisson process with parameter $\widetilde{\lambda}$ and (Y_i) is a sequence of square integrable, identically distributed random variables, so that Z_t, N_t and (Y_i) are independent. For the case $\sigma = 0$ the model would include the pure jump case. As it is discussed in depth in Chapter 3, model (1.2) leads to a partial integro-differential equation, so that appropriate numerical techniques to cope with jumps have to be applied.

Let us denote by $v_t = V(t, S_t; t_0)$ the value at time t of the benefits payable to the member of the plan when he/she is aged $t_0 + t$ and the salary is S_t. In this section we pose the mathematical model in terms of a PDE to obtain V, when the retirement benefits depend on the continuous arithmetic average of the salary during the last n_y years, and early retirement is not allowed.

Moreover, the payment from the fund is assumed to occur in case of the death of a member or cancellation of the plan (withdrawal) to move to another plan. Therefore, we assume three possible states of a member of the plan: active (a), dead (d) and withdrawn (w). It is considered that retirement occurs at the final age of service in the pension fund. The transition intensities from active membership to death or cancellation are denoted by $\mu_d(t;t_0)$ and $\mu_w(t;t_0)$, respectively. From an actuarial standpoint, these intensities are understood as forces of decrement acting at time t on a member aged t_0+t and can be expressed in terms of the corresponding transition probabilities from one state to another.

In classic actuarial mathematics, the value of the benefits of the pension plan, v, is understood as the value of the provision that the plan promoter should reserve in order to guarantee the contractual obligations to the active member. By using actuarial arguments, when assuming deterministic benefits paid to the fund and ignoring any contributions due to stochastic salary evolution, the variation of the value of the retirement benefits from time t to time $t+dt$ is given by Thiele's differential equation (see [13], for example):

$$dv_t = r(t)v_t dt - \sum_{i=d,w} \mu_i(t;t_0)(A_i(t,S_t;t_0) - v_t)dt, \tag{1.3}$$

where $r(t)$ is the deterministic risk-free interest rate and $A_i(t,S_t;t_0)$ denotes the deterministic benefit paid by the fund in case of death ($i=d$) or withdrawal ($i=w$). Note that the difference $A_i(t,S_t;t_0) - v_t$ represents the sum-at-risk associated with decrement i, so that

$$\sum_{i=d,w} \mu_i(t;t_0)(A_i(t,S_t;t_0) - v_t)dt \tag{1.4}$$

denotes the expected value of the payments from the fund in the interval $[t,t+dt]$.

Following Section 5 in [61], we assume that

$$A_i(t,S;t_0) = \alpha_i S, \quad \alpha_i \geq 0,\ i=d,w,$$

so that the death and withdrawal benefits are a constant multiple of the salary, and that the transition intensities $\mu_i(t;t_0) = \mu_i$ are nonnegative constants.

As we are assuming that retirement benefits depend on the continuous arithmetic average of the salary, in an analogous way to the case of Asian options [68] we introduce the following variable representing the cumulative function of the salary of the last n_y years before T_r:

$$I_t = \int_0^t g(\tau, S_\tau)d\tau, \tag{1.5}$$

with

$$g(t,S) = \begin{cases} 0 & if \quad 0 \leq t < T_r - n_y \\ h(S) & if \quad T_r - n_y \leq t \leq T_r, \end{cases} \tag{1.6}$$

where $T_r > n_y$ denotes the retirement date and h is appropriately chosen. In this work we consider the particular case $h(S) = k_1 S$, with a given accrual constant $k_1 > 0$. As in the case of Asian options, the variation of I in the interval $[t, t+dt]$:

$$dI_t = I_{t+dt} - I_t = \int_t^{t+dt} g(\tau, S_\tau)d\tau = g(t, S_t)dt. \tag{1.7}$$

For simplicity, hereafter we drop the dependence on the entry age t_0 in all functions (in particular, the value of the pension plan is denoted by $v_t = V(t, S_t, I_t; t_0) = V(t, S_t, I_t)$).

Next, we can apply Itô's lemma (see [32]) jointly with the Thiele differential equation (1.3) to obtain the variation of V from t to $t + dt$

$$\begin{aligned} dv_t &= \left(\partial_t V + \alpha(t,S_t)\partial_S V + g(t,S_t)\partial_I V + \frac{1}{2}\sigma(t,S_t)^2\partial_{SS}V\right)dt + \sigma(t,S)\partial_S V dZ_t \\ &\quad + \left(\sum_{i=d,w}\mu_i(A_i(t,S_t) - v_t)\right) dt, \end{aligned} \tag{1.8}$$

where the first two terms on the right-hand side are associated with the stochastic variation of the salary while the third term is related to the expected payments in $(t, t + dt)$ due to death or withdrawal.

In [61] the PDE model is obtained by arguing that the risk-adjusted expected change in the liabilities value, after taking into account the benefits cashflows from the fund related to possible death or withdrawal, should be equal to the risk-free interest rate. We apply the dynamic hedging methodology to deduce the PDE model. For this purpose, we consider two pension plans, with values V_j $(j = 1, 2)$ and that pay the quantities $A_d^j(t, S)$ and $A_w^j(t, S)$ in the cases of death and withdrawal, respectively. Moreover, we assume that the intensities μ_i are independent of j. Therefore, the variation of each plan V_j from t to $t + dt$ satisfies the corresponding equation (1.8). At this point, we proceed in a similar manner to the dynamic hedging methodology used in the case of bonds [68]. Thus, by buying one unit of plan V_1 and selling Δ units of plan V_2, the value of the resulting portfolio is

$$\Pi = V_1 - \Delta V_2 \,.$$

Note that the variation of the portfolio value between t and $t + dt$ is given by

$$d\Pi = dV_1 - \Delta dV_2 = (\dots)\ dt \ + \ \sigma\left(\partial_S V_1 - \Delta \partial_S V_2\right)\, dZ_t, \tag{1.9}$$

where $(\dots)$ contains the drift term. Therefore, Π turns out to be risk-free for the following choice:

$$\Delta = \left(\partial_S V_1 / \partial_S V_2\right). \tag{1.10}$$

Moreover, for this choice of Δ, the variation of the risk-free portfolio is given by

$$\begin{aligned} d\Pi \;\; = \;\; & \Bigg[\partial_t V_1 + \frac{1}{2}\sigma^2 \partial_{SS} V_1 + g\partial_I V_1 + \sum_{i=d,w} \mu_i (A_i^1 - V_1) - \\ & -\Delta\left(\partial_t V_2 + \frac{1}{2}\sigma^2 \partial_{SS} V_2 + g\partial_I V_2 + \sum_{i=d,w} \mu_i (A_i^2 - V_2)\right)\Bigg]\, dt. \end{aligned} \tag{1.11}$$

By using the arbitrage free assumption, this variation is also given by $d\Pi = r\Pi dt$. So, we obtain the identity

$$\begin{aligned} & (\partial_S V_1)^{-1}\left(rV_1 - \partial_t V_1 - g\partial_I V_1 - \frac{1}{2}\sigma^2 \partial_{SS} V_1 - \sum_{i=d,w} \mu_i (A_i^1 - V_1)\right) \\ = \; & (\partial_S V_2)^{-1}\left(rV_2 - \partial_t V_2 - g\partial_I V_2 - \frac{1}{2}\sigma^2 \partial_{SS} V_2 - \sum_{i=d,w} \mu_i (A_i^2 - V_2)\right). \end{aligned} \tag{1.12}$$

Note that (1.12) holds for any considered pair of pension plans. Then, we can introduce the quantity

$$\beta(t,S,I) = (\partial_S V)^{-1}\left(rV - \partial_t V - g\partial_I V - \frac{1}{2}\sigma^2\partial_{SS}V - \sum_{i=d,w}\mu_i(A_i - V)\right). \quad (1.13)$$

So, by reordering the terms in (1.13) we obtain the following PDE that governs the value of the benefits of the pension plan:

$$\partial_t V + \beta\,\partial_S V + g\,\partial_I V + \frac{1}{2}\sigma^2\,\partial_{SS}V - (\mu_d + \mu_w + r)\,V \;=\; -\mu_d A_d - \mu_w A_w, \quad (1.14)$$

which is initially posed on the unbounded domain $(0,T_r)\times\Omega$, with $\Omega = (0,+\infty)\times(0,+\infty)$.

Assuming that at retirement date, T_r, the owner of the pension plan receives a fraction of the average salary during the last n_y years, equation (1.14) is completed with the final condition

$$V(T_r,S,I) = \frac{a}{n_y}\,I, \quad (1.15)$$

where $a \in (0,1)$ is a given constant.

Remark 1.2.1. *Note that function β can be related to the market price of the risk associated with uncertainty about the salary. More precisely, if we introduce:*

$$\lambda = \frac{\alpha-\beta}{\sigma}\,, \quad (1.16)$$

then equation (1.8) can be equivalently written as:

$$dv_t = (rv_t + \lambda\sigma\partial_S V)\,dt + \sigma\partial_S V dZ_t, \quad (1.17)$$

so that:

$$dv_t - rv_t dt = \sigma\partial_S V(\lambda dt + dZ_t). \quad (1.18)$$

Therefore, the extra reward of the pension plan is λdt per assumed risk unit. Then λ can be understood as the market price of the risk associated with uncertainty about the salary.

1.3 Mathematical analysis

So far, the mathematical model for the value of the pension plan has been posed as a Cauchy problem associated with the backward-in-time equation (1.14) jointly with the final condition (1.15).

In this section we study the existence and uniqueness of solution. In what follows we assume that σ and β are proportional to the salary, so that $\sigma(t,S) = \sigma S$ and $\beta(t,S,I) = \theta S$, where $\theta > 0$ and $\sigma > 0$ are given constants. The assumption on σ implies that salary volatility increases with salary, which is a reasonable argument, mainly in the private sector. Additionally the joint assumption on the expressions of both σ and β implies that the market price of risk is a constant parameter when a lognormal evolution for salaries is considered (i.e. $\alpha(t,S) = \alpha S$). Moreover, these assumptions allow for the use of the techniques developed in this section to obtain the existence and uniqueness of solution by appropriately transforming the PDE. Nevertheless, the numerical methods described in the next section could be applied without the need of these simplifying assumptions.

Thus, let us consider the following Kolmogorov operator:

$$\mathcal{L}[V] = \partial_t V + \theta S \partial_S V + g(t,S)\partial_I V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V - (r + \mu_d + \mu_w)V, \tag{1.19}$$

where V denotes a function defined on the domain $(0,T_r) \times \Omega$. Thus, we consider the Cauchy problem

$$\left\{ \begin{array}{rcll} \mathcal{L}[V] & = & f & for \quad (t,S,I) \in (0,T_r) \times \Omega, \\ V(T_r,S,I) & = & \dfrac{a}{n_y} I & for \quad (S,I) \in \Omega, \end{array} \right. \tag{1.20}$$

where $f = -(\mu_d \alpha_d + \mu_w \alpha_w)S$.

In order to study the existence of solution we introduce the following change of variables and unknown

$$y_1 = S, \quad y_2 = \frac{1}{2}\sigma^2 I, \quad \tau = \frac{1}{2}\sigma^2 (T_r - t),$$

$$\phi(\tau, y_1, y_2) = y_1^m \exp(q\tau) V \left(T_r - \frac{2\tau}{\sigma^2}, y_1, \frac{2y_2}{\sigma^2} \right),$$

and the following parameters

$$m = \frac{\theta}{\sigma^2},\ q = m^2 - m + \frac{2(r + \mu_d + \mu_w)}{\sigma^2},\ T = \frac{1}{2}\sigma^2(T_r - 0).$$

After the previous changes, the Cauchy problem (1.20) can be written in terms of the new unknown as:

$$\begin{cases} \mathcal{L}_1[\phi] &= F \quad for \quad (\tau, y_1, y_2) \in (0, T) \times \Omega, \\ \phi(0, .) &= \Lambda \quad for \quad (y_1, y_2) \in \Omega, \end{cases} \tag{1.21}$$

where operator $\mathcal{L}_1$ is defined as:

$$\mathcal{L}_1[\phi] = \partial_\tau \phi - y_1^2\, \partial_{y_1^2} \phi - \overline{g}(\tau, y_1)\, \partial_{y_2} \phi, \tag{1.22}$$

for any function ϕ defined in $(0, T) \times \Omega$, with:

$$\overline{g}(\tau, y_1) = \begin{cases} k_1 y_1 \quad if \quad \tau \leq \dfrac{1}{2}\sigma^2 n_y\,, \\ \\ 0 \qquad if \quad \tau > \dfrac{1}{2}\sigma^2 n_y\,. \end{cases} \tag{1.23}$$

The second member function in (1.21) is given by:

$$F(\tau, y_1, y_2) = \frac{2}{\sigma^2}(\mu_d \alpha_d + \mu_w \alpha_w) \exp(q\tau) y_1^{m+1}, \tag{1.24}$$

and the initial condition is:

$$\Lambda(y_1, y_2) = \frac{2a}{\sigma^2 n_y}\, y_1^m y_2. \tag{1.25}$$

Next, we analyse the existence of solution for (1.21). As indicated in [6], the presence of the coefficient y_1^2 in the second order term leads to a kind of degeneracy in the equation that cannot be easily avoided by the usual logarithmic transformation in classical linear Kolmogorov equations. If the change $y_1 = \log(S)$ is applied instead of $y_1 = S$ then the exponential term $k_1 \exp(y_1)$ will appear in the expression (1.23) of $\overline{g}$ for $\tau \leq \sigma^2 n_y / 2$ and the degeneracy problem is shifted to the first-order term coefficient. Nevertheless, in [44] for the case of Asian options the logarithmic change is applied and the consequent existence results are obtained from the results appearing in [28]. We follow the ideas of [6], thereby avoiding the use of the logarithmic change in variable S.

In order to state the existence of solution, for $p \geq 1$ we introduce the following functional space related to the solution of problem (1.21):

$$\mathcal{S}^p_{loc}(\Omega) = \{u \in L^p_{loc}(\Omega)/\, \partial_\tau - y_1\partial^2_{y_1^2} - \overline{g}\partial_{y_2} \in L^p_{loc}(\Omega)\}, \tag{1.26}$$

and, for $\alpha \in (0,1)$, we consider the Hölder spaces $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$, $\mathcal{C}^{1,\alpha}_{\mathcal{L}_1}(\Omega)$ and $\mathcal{C}^{2,\alpha}_{\mathcal{L}_1}(\Omega)$ defined by the norms (see [24], for example):

$$|| \, u \, ||_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} = \sup_\Omega | \, u \, | + \sup_{z,y\in\Omega, z\neq y} \frac{| \, u(z) - u(y) \, |}{|| \, z^{-1} \circ y \, ||^\alpha}, \tag{1.27}$$

$$|| \, u \, ||_{\mathcal{C}^{1,\alpha}_{\mathcal{L}_1}(\Omega)} = || \, \partial_{y_1} u \, ||_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)}, \tag{1.28}$$

$$|| \, u \, ||_{\mathcal{C}^{2,\alpha}_{\mathcal{L}_1}(\Omega)} = || \, \partial^2_{y_1^2} u \, ||_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} + || \, \partial_{y_2} u - \partial_\tau u \, ||_{\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)} \, . \tag{1.29}$$

Note that the notation $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$ associates the definition to the particular form of the operator $\mathcal{L}_1$ and that $u \in \mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$ implies that u is Hölder continuous in the usual sense. Moreover, some embedding theorems recalled in [24] state the relationships between spaces $\mathcal{S}^p_{loc}(\Omega)$ and $\mathcal{C}^\alpha_{\mathcal{L}_1}(\Omega)$.

In order to state the existence of solution, we first present a result concerning the existence of solutions for boundary-value problems associated with second-order operators with non-negative characteristic form [50]. More precisely, in [43] the following result is stated for the problem

$$\left\{ \begin{array}{rcll} \mathcal{L}_1[u] & = & G & in \quad Q, \\ u & = & H & on \quad \partial Q, \end{array} \right. \tag{1.30}$$

where Q is an open bounded set in R^3.

Theorem 1.3.1. *Let Q be an open bounded set of R^3 such that $\overline{Q} \subset \{y_1 \neq 0\}$ and let $G \in C(Q)$, $H \in C(\partial Q)$. Then the problem (1.30) has a classical solution $u \in C^{2,\alpha}(Q) \cap C(\overline{Q})$.*

Note that although hypothesis $\overline{Q} \subset \{y_1 \neq 0\}$ is not introduced in [43], it is assumed anyway as we cannot guarantee that the coefficient associated with the second-order term is bounded from below by some positive constant (see [6]). Moreover, although only the homogeneous case $G = 0$ is addressed in [43], the extension to $G \in C(\partial Q)$ is straightforward. In fact, a nonhomogeneous case is stated in Theorem 4.2 of [56]. Next, as in [6] , we recall a sufficient condition for the regularity of the points at the boundary (see Theorems 6.1 and 6.3 in [43]) to be used in problem (1.30).

Proposition 1.3.1. *Let Ω be an open set of R^3 such that $\overline{\Omega} \subset \{y_1 \neq 0\}$ and let $(t^0, y_1^0, y_2^0) \in \partial\Omega$. Let us consider the problem (1.21). If there exists an outer normal vector $\nu = (\nu_t, \nu_{y_1}, \nu_{y_2})$ such that one of*

1. *$\nu_{y_1} \neq 0$ or*
2. *$\nu_{y_1} = 0$, but $y_1^0 \nu_{y_2} - \nu_t > 0$ and there exists a positive constant δ such that $(y_1^0)^2\delta^2 \leq y_1^0 \nu_{y_2} - \nu_t$ and that*

$$\{(t, y_1, y_2) \in R^3 / (t - t_0 - \delta^2\nu_t)^2 + \delta^2(y_1 - y_1^0)^2 + (y_2 - y_2^0 - \delta^2\nu_{y_2})^2 \leq \delta^4\} \subset R^3 - \Omega$$

is satisfied, then (t^0, y_1^0, y_2^0) is a regular point.

Next, we introduce the concepts of supersolution and subsolution associated with problem (1.21), which is posed on the unbounded domain $(0, T) \times \Omega$.

Definition 1.3.1. *A supersolution:*

$$\overline{\phi} \in \mathcal{S}^p_{loc}((0, T) \times \Omega) \cap \mathcal{C}((0, T) \times R^2)$$

of problem (1.21) is a function satisfying:

$$\begin{cases} \mathcal{L}_1[\overline{\phi}] & \geq & F & for & (\tau, y_1, y_2) \in (0, T) \times \Omega, \\ \overline{\phi}(0, .) & \geq & \Lambda & for & (y_1, y_2) \in \Omega. \end{cases} \tag{1.31}$$

Moreover, a subsolution $\underline{\phi}$ to problem (1.21) is defined simple by considering the reverse inequalities in (1.31).

In the following proposition, we shall obtain a supersolution and a subsolution to problem (1.21).

Proposition 1.3.2. *For $\alpha_1 \geq 3$ and $\alpha_2 \geq 1$, let:*

$$\overline{\phi}(\tau, y_1, y_2) = \gamma y_1^m y_2 \exp(\widetilde{q}\tau) + \frac{2}{\sigma^2}(\mu_d\alpha_d + \mu_w\alpha_w) y_1^{m+1} \exp((q+\widetilde{q})\tau) + \gamma k_1 y_1^{m+1} \exp(\widetilde{q}\tau), \tag{1.32}$$

where:

$q = m^2 - m + \frac{2(r+\mu_d+\mu_w)}{\sigma^2}$, $\widetilde{q} = m^2 + (\alpha_1 - 1)m + \alpha_2$ *and* $\gamma = \dfrac{2a}{\sigma^2 n_y}$

Then, the function $\overline{\phi}$ is a supersolution to problem (1.21). Moreover, the function $\underline{\phi} = 0$ is a subsolution to problem (1.21).

Proof.

Clearly, $\underline{\phi} = 0$ is a subsolution to problem (1.21).

In order to state the supersolution properties, first we note that for $\tau = 0$ we obtain

$$\bar{\phi}(0, y_1, y_2) = \gamma y_1^m y_2 + (\mu_d\alpha_d + \mu_w\alpha_w)\frac{2}{\sigma^2} y_1^{m+1} + \gamma k_1 y_1^{m+1} \geq \gamma y_1^m y_2 \tag{1.33}$$

So, as:

$$(\mu_d\alpha_d + \mu_w\alpha_w)\frac{2}{\sigma^2} y_1^{m+1} \geq 0, \quad \gamma k_1 y_1^{m+1} \geq 0$$

the second inequality in (1.31) is satisfied.

Next, in order to verify the first inequality, we calculate:

$$\begin{aligned} \mathcal{L}_1[\bar{\phi}] \;&=\; \widetilde{q}\gamma y_1^m y_2 \exp(\widetilde{q}\tau) + \frac{2}{\sigma^2}(\mu_d\alpha_d + \mu_w\alpha_w) y_1^{m+1}(q+\widetilde{q}) \exp((q+\widetilde{q})\tau) \\ &\quad + \widetilde{q} k_1 y_1^{m+1} \gamma \exp(\widetilde{q}\tau) - \gamma m(m-1) y_1^m y_2 \exp(\widetilde{q}\tau) \\ &\quad - \frac{2}{\sigma^2}(\mu_d\alpha_d + \mu_w\alpha_w) m(m+1) y_1^{m+1} \exp((q+\widetilde{q})\tau) \\ &\quad - k_1 m(m+1)\gamma y_1^{m+1} \exp(\widetilde{q}\tau) - \bar{g}(\tau, y_1)\gamma y_1^m \exp(\widetilde{q}\tau) \end{aligned} \tag{1.34}$$

Then, after some easy computations, we get

$$\begin{aligned}\mathcal{L}_1[\bar{\phi}] &= (\widetilde{q} - m(m-1))\,\gamma y_1^m y_2 \exp(\widetilde{q}\tau) \\ &+\frac{2}{\sigma^2}(\mu_d\alpha_d + \mu_w\alpha_w)\,(q+\widetilde{q} - m(m+1))\,y_1^{m+1}\exp((q+\widetilde{q})\tau) \\ &+ [\widetilde{q}y_1k_1 - k_1y_1m(m+1) - \bar{g}(\tau,y_1)]\,\gamma y_1^m \exp(\widetilde{q}\tau) \qquad (1.35)\end{aligned}$$

Now, note first that $\alpha_1 \geq 3$ and $\alpha_2 \geq 1$ implies that:

$$\widetilde{q} - m(m-1) = m^2 + (\alpha_1 - 1)m + \alpha_2 - m^2 + m \geq 0$$

Secondly, the conditions on α_1 and α_2 jointly with inequalities $m > 0$ and $\frac{2(r+\mu_d+\mu_w)}{\sigma^2} > 0$ imply that:

$$m^2 + (\alpha_1 - 1)m + \alpha_2 \geq 0$$

$$m^2 + (\alpha_1 - 3)m + \frac{2(r+\mu_d+\mu_w)}{\sigma^2} + \alpha_2 \geq 1$$

Thus, we have $q+\widetilde{q}-m(m+1) \geq 1$ and $\widetilde{q} \geq 0$. On the other hand, as $(\alpha_1-2)m+\alpha_2 \geq 1$, then $\widetilde{q} - m(m+1) - 1 \geq 0$, and therefore:

$$\widetilde{q}y_1k_1 - k_1y_1m(m+1) - k_1y_1 \geq 0$$

Next, by using that the product of k_1 and y_1 is an upper bound for $\bar{g}(\tau, y_1)$, we have:

$$\widetilde{q}y_1k_1 - k_1y_1m(m+1) - \bar{g}(\tau,y_1) \geq \widetilde{q}y_1k_1 - k_1y_1m(m+1) - k_1y_1 \geq 0$$

Therefore, the inequalities:

$$\widetilde{q} - m(m-1) \geq 0, \quad q+\widetilde{q} - m(m+1) \geq 1, \quad \widetilde{q}y_1k_1 - k_1y_1m(m+1) - \bar{g}(\tau,y_1) \geq 0$$

hold, so that first inequality in (1.31) is satisfied and the proof is concluded. □

.

Next, we adapt the theorem appearing in [6] to our case, in order to state the following theorem.

Theorem 1.3.2. *Let $\Lambda \in \mathcal{C}(\Omega)$ and $F \in \mathcal{C}((0,T) \times \Omega)$. Let $\overline{\phi}$ and $\underline{\phi}$ be a super and a subsolution of problem (1.21), respectively, such that $\underline{\phi} \leq \overline{\phi}$ in $(0,T) \times \Omega$. Then, there exists a classical solution ϕ to problem (1.21), such that*

$$\underline{\phi} \leq \phi \leq \overline{\phi} \quad in \quad (0,T) \times \Omega \tag{1.36}$$

Proof.

First, we define a sequence of initial-boundary-value problems posed on bounded open subsets of $(0,T) \times \Omega$, Ω_k, such that $\Omega_k \subset \Omega_{k+1}$ and $\cup \Omega_k = (0,T) \times \Omega$. More precisely, we consider the sequence of bounded sets:

$$\Omega_k = (0,T) \times (1/(k+1), k+1) \times (1/(k+1), k+1) \quad \text{with} \quad k \in N$$

and we consider the sequence of cut-off functions $\chi_k : (0,+\infty) \times (0,+\infty) \to R$, $\chi_k \in C(\Omega)$, such that:

$$\chi_k(y_1, y_2) = \begin{cases} 0 & if\ (y_1,y_2) \notin (1/(k+1), k+1) \times (1/(k+1), k+1) \\ 1 & if\ (y_1,y_2) \in (1/k, k) \times (1/k, k) \end{cases} \tag{1.37}$$

and $0 \leq \chi_k(y_1,y_2) \leq 1$, otherwise. In terms of these functions, we define:

$$\underline{\Lambda}_k(\tau, y_1, y_2) = \chi_k(y_1,y_2)\Lambda(y_1,y_2) + (1 - \chi_k(y_1,y_2))\underline{\phi}(\tau, y_1, y_2). \tag{1.38}$$

By Theorem 1.3.1, there exists a classical solution, $u_k \in C^{2,\alpha}(\Omega_k) \cap C(\overline{\Omega}_k)$, to problem:

$$\begin{cases} \mathcal{L}_1[u_k] & = & F & in & \Omega_k \\ u_k & = & \underline{\Lambda}_k & on & \partial\Omega_k \end{cases} \tag{1.39}$$

Moreover, from the maximum principle, it follows that

$$\underline{\phi} \leq u_k \leq \overline{\phi} \quad in \quad \Omega_k \tag{1.40}$$

Note that functions u_k are not defined outside Ω_k. Next, we use the same arguments as in [6] to build a solution for problem (1.21). For this purpose, let us introduce the sets:

$$D_k = \{(t, y_1, y_2) \in \Omega_k / T/(3k) < t < T/(1 - 1/(3k))\} \tag{1.41}$$

so that:

$$(0,T)\times\Omega = \cup_{k\in N} D_k$$

Note that from (1.40) it follows that the sequence $\{u_k\}$ is bounded in $\overline{D}_1$ and from Proposition 4.1 in [6] it is also equicontinuous. Thus, following the Ascoli-Arzela theorem, there exists a subsequence that converges uniformly to some function $v_1 \in C(\overline{D}_1)$. Moreover, v_1 is a classical solution of (1.21) in D_1 and $\underline{\phi} \le v_1 \le \overline{\phi}$ in D_1. Next, we can apply the same argument to the previous subsequence on the set $\overline{D}_2$, thereby obtaining a new subsequence that converges to $v_2 \in C(\overline{D}_2)$, so that v_2 is the solution of (1.21) in D_2, verifies that $\underline{\phi} \le v_2 \le \overline{\phi}$ in D_2 and v_2 coincides with v_1 in D_1. The argument can continue by induction and we can define a limit function u as follows: for $(t, y_1, y_2) \in (0,T)\times\Omega$ we choose a natural number n such that $(t, y_1, y_2) \in D_n$ and define $u(t, y_1, y_2) = v_n(t, y_1, y_2)$. In this way, u is well defined and verifies equation (1.21) and $\underline{\phi} \le u \le \overline{\phi}$ in $(0,T)\times\Omega$.

It now remains to prove that the function u verifies the boundary condition at $\tau = 0$. For this purpose, we verify that for any $(y_1^0, y_2^0) \in R^+ \times R^+$, we have:

$$\lim_{(t,y_1,y_2)\to(0,y_1^0,y_2^0)} u(\tau, y_1, y_2) = \Lambda(y_1^0, y_2^0) \tag{1.42}$$

Note that the fact that $u_k(0, y_1^0, y_2^0) = \Lambda(y_1^0, y_2^0)$ for $(0, y_1^0, y_2^0) \in \partial\Omega_k$ does not guarantee the same result for function u at the boundary. Nevertheless, the use of a standard argument of barrier functions in the proof of Proposition 1.3.1 provides an estimate of the rate of convergence as (t, y_1, y_2) tends to $(0, y_1^0, y_2^0)$, which is uniform with respect to k, thus allowing to obtain (1.42).

Remark 1.3.3. *Note that, by construction, $u_{k+1} \ge \underline{\Lambda}_k$ so that $\{u_k\}$ is an increasing sequence. On the other hand, if we define*

$$\overline{\Lambda}_k(\tau, y_1, y_2) = \chi_k(y_1, y_2)\Lambda(y_1, y_2) + (1 - \chi_k(y_1, y_2))\overline{\phi}(\tau, y_1, y_2) \tag{1.43}$$

and we consider it as the boundary condition for problem (1.39). We then obtain a decreasing sequence of solutions $\{v_k\}$ converging, uniformly on compact sets, to a *solution v of the problem (1.21) that verifies $\underline{\phi} \leq v \leq \overline{\phi}$. In the case where uniqueness can be proved, both solutions u and v coincide.*

Theorem 1.3.4. *There exists a classical solution ϕ to problem (1.21). Moreover, the solution verifies:*

$$0 \leq \phi \leq \bar{\phi} = \gamma y_1^m y_2 \exp(\widetilde{q}\tau) + \frac{2}{\sigma^2}(\mu_d \alpha_d + \mu_w \alpha_w) y_1^{m+1} \exp((q+\widetilde{q})\tau) + \gamma k_1 y_1^{m+1} \exp(\widetilde{q}\tau),$$

with:

$q = m^2 - m + \frac{2(r+\mu_d+\mu_w)}{\sigma^2}$, $\widetilde{q} = m^2 + 2m + 1$ *and* $\gamma = \dfrac{2a}{\sigma^2 n_y}$

The proof of theorem 1.3.4 straightly follows from Theorem 1.3.2 and Proposition 1.3.2 by taking $\alpha_1 = 3$ and $\alpha_2 = 1$.

Remark 1.3.5. *Note that the choice of super and subsolutions implies that any solution satisfies $\phi(\tau, 0, y_2) = 0$.*

In order to study the uniqueness of solution, we directly apply the following result taken from [3] to the particular case of problem (1.21). This result has already been used in [29] for Black-Scholes equations associated with one factor models.

Theorem 1.3.6. *Let $\Omega \subset R^n$ be an arbitrary unbounded open domain and let L be the differential operator:*

$$L = \sum_{i,j=1}^{n} a_{ij}(\tau, y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n} b_i(t, y)\partial_{y_i} + c - \partial_\tau\,,$$

defined in $(0, T) \times R^n$, such that the coefficients of L satisfy:

$$\sum_{i,j=1}^{n} a_{ij}(\tau, y)\xi_i\xi_j \leq 0, \quad \textit{for all} \quad (\tau, y) \in [0, T] \times \Omega,\ \xi \in R^n,$$

and:

$$\mid a_{ij} \mid \leq A(\mid y \mid^2 + 1), \quad \mid b_i \mid \leq B(\mid y \mid^2 + 1)^{1/2}, \quad \mid c \mid \leq C$$

in $[0,T] \times \Omega$ *for some positive constants* A*,* B *and* C*. If* u *is a classical solution of* $Lu \leq 0$ *in* $[0,T] \times \Omega$*, such that:*

$$u \geq 0, \quad \textit{for } (\tau, y) \in \{[0,T] \times \partial\Omega\} \cup \{\{0\} \times \partial\Omega\}$$

and:

$$u(\tau, y) \leq -M \exp\{k \log(\mid y \mid^2 + 1) + 1\}^2 \tag{1.44}$$

in $[0,T] \times \Omega$ *for some positive constant* M *and* k*, then* $u \geq 0$ *in* $[0,T] \times \Omega$*.*

The following result follows directly from the previous theorems and existence results.

Corollary 1.3.1. *There exists an unique classical solution of Problem (1.21) such that (1.44) is satisfied.*

1.4 Numerical solution

In this section we introduce the numerical method for solving the problem (1.20). First, we point out some difficulties in the numerical solution. On one hand, the spatial domain Ω is unbounded. Due to this fact, as in the localization technique used in previous section, domain truncation and boundary conditions are proposed. Note that the particular localization procedure used in the previous section is not practical for numerical purposes. On the other hand, the diffusion matrix is strongly degenerated. So, we propose a combination of the Crank-Nicolson characteristics method for the time discretization and piecewise quadratic finite elements method for the spatial discretization on the bounded domain. In the literature we can find different applications of the classical first order method of characteristics for the solution of financial problems (see [20] and [66], for example). Recently, a higher order Crank-Nicolson characteristic methods for general convection-diffusion-reaction equations (eventually degenerated) have been proposed and analyzed numerically in [7] and [8]. Furthermore, they have been successfully applied to price Asian options in [9] and to the ratchet-cap pricing problem in [63]. In this section we apply it to the particular pension plan pricing problem. Although this problem does not satisfy all the hypothesis required in [9] to obtain a second order Lagrange-Galerkin scheme, in practice, good numerical results are obtained. It is important to note that, due to the specific expression of the PDE, in the Lagrangian step the characteristic curves associated with the convection term in the equation can be computed exactly, thus avoiding the use of appropriate ode solvers to approximate the position of the basis point of the characteristics.

1.4.1 Localization procedure and formulation in a bounded domain

First, we consider a problem posed in a sufficiently large spatial bounded domain, so that the solution in the region of financial interest is not affected by the truncation of the unbounded domain and the required boundary conditions (localization procedure). This procedure was analyzed in [35] for vanilla options and Dirichlet boundary conditions. For this purpose, let us introduce the notation

$$x_0 = t, \quad x_1 = S \quad \text{and} \quad x_2 = I, \tag{1.45}$$

and let us consider both x_1^∞ and x_2^∞ be large enough real numbers suitably chosen and let

$$\Omega^* = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty),$$

with $x_0^\infty = T_r$. Then, let us denote the Lipschitz boundary by $\Gamma^* = \partial\Omega^*$, such that $\Gamma^* = \bigcup_{i=0}^{2}(\Gamma_i^{*,-} \bigcup \Gamma_i^{*,+})$, where we use the notation

$$\Gamma_i^{*,-} = \{(x_0, x_1, x_2) \in \Gamma^* / x_i = 0\}$$

$$\Gamma_i^{*,+} = \{(x_0, x_1, x_2) \in \Gamma^* / x_i = x_i^\infty\}$$

for $i = 0, 1, 2$.

Then, the PDE in problem (1.20) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij}\,\frac{\partial^2 V}{\partial x_i x_j} + \sum_{j=0}^{2} b_j\,\frac{\partial V}{\partial x_j} + b_0\,V = f_0, \tag{1.46}$$

where the data involved is defined as follows

$$B(x_0,x_1,x_2) = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b}(x_0,x_1,x_2) = (b_j) = \begin{pmatrix} 1 \\ \theta x_1 \\ g(t,x_1) \end{pmatrix}, \tag{1.47}$$

$$b_0(x_0,x_1,x_2) = -(r+\mu_d+\mu_w), \quad f_0(x_0,x_1,x_2) = -(\mu_d\alpha_d + \mu_w\alpha_w)x_1. \tag{1.48}$$

Thus, following [50], in terms of the normal vector to the boundary pointing inwards Ω^*, $\vec{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ^*:

$$\Sigma^0 = \{x \in \Gamma^* / \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0\}, \tag{1.49}$$

$$\Sigma^1 = \Gamma^* - \Sigma^0, \tag{1.50}$$

$$\Sigma^2 = \left\{ x \in \Sigma_0 / \sum_{i=0}^{2}\left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j}\right) m_i < 0 \right\}. \tag{1.51}$$

As indicated in [50], the boundary conditions at $\Sigma^1 \cup \Sigma^2$ for the so-called first boundary valued problem associated with (1.46) are required. Note that $\Sigma^1 = \Gamma_1^{*,+}$ and $\Sigma^2 = \Gamma_0^{*,+} \cup \Gamma_2^{*,+}$. Therefore, in addition to the final condition in (1.20) on $\Gamma_0^{*,+}$, we propose the following conditions:

$$\frac{\partial V}{\partial x_1} = 0 \quad \text{on} \quad \Gamma_1^{*,+}, \tag{1.52}$$

$$\frac{\partial V}{\partial x_2} = \frac{a}{n_y} \quad \text{on} \quad \Gamma_2^{*,+}. \tag{1.53}$$

At this point, the question of existence of the solution for the previous problem in the localized domain and the convergence to the solution in the unbounded domain arises. In general, existence is gained for Dirichlet boundary conditions obtained from the initial condition (value at the time horizon in the original financial variables formulation). In [35] a particular study with Dirichlet conditions for the European multiasset vanilla option has been developed. In [33], the existence, uniqueness and convergence for multiasset American options is addressed. Moreover, in [5] the use of Dirichlet and Neumann boundary conditions deduced from the payoff function in the framework of viscosity solutions was analyzed. Taking into account the analogies with European-style Asian options, we follow the ideas in [38] and [9], so that we impose Neumann boundary conditions obtained from the payoff. In this way, we avoid to impose that the solution matches the initial condition at the new boundaries of the bounded domain. We note that the convergence analysis to the solution in the unbounded domain is an open problem. In view of the numerical results appearing in a forthcoming section of this work, numerical evidence of convergence is obtained.

Next, in order to state the problem (1.20) as an equivalent initial-boundary-value problem in divergence form, we distinguish the time and space bounded domains and introduce the following change of time variable and the notation for spatial-like variables

$$\tau = T_r - t, \quad x_1 = S, \quad x_2 = I, \tag{1.54}$$

in addition to the notation related to the spatial domain: $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ and $\Gamma = \bigcup_{i=1}^{2} (\Gamma_i^- \bigcup \Gamma_i^+)$, with $\Gamma_i^- = \{(x_1, x_2) \in \Gamma / x_i = 0\}$ and $\Gamma_i^+ = \{(x_1, x_2) \in \Gamma / x_i = x_i^\infty\}$ for $i = 1, 2$.

Problem (1.20) is then replaced by the following:

Find $V : [0, T_r] \times \Omega \to \mathbb{R}$, such that

$$\partial_\tau V - Div(A \nabla V) + \vec{v} \cdot \nabla V + lV = f \qquad \text{in } (0, T_r) \times \Omega\,, \tag{1.55}$$

$$V(0, .) = \varphi \qquad \text{in } \Omega\,, \tag{1.56}$$

$$\frac{\partial V}{\partial x_1} = g_1 \qquad \text{on } (0, T_r) \times \Gamma_1^+\,, \tag{1.57}$$

$$\frac{\partial V}{\partial x_2} = g_2 \qquad \text{on } (0, T_r) \times \Gamma_2^+\,, \tag{1.58}$$

where the data involved is defined as follows

$$A(x_1,x_2)=\begin{pmatrix} \frac{1}{2}\sigma^2x_1^2 & 0 \\ 0 & 0 \end{pmatrix},\quad \vec{v}(x_1,x_2)=\begin{pmatrix} (\sigma^2-\theta)x_1 \\ -g(T_r-\tau,x_1) \end{pmatrix},\tag{1.59}$$

$$l(\tau,x_1,x_2)=(r+\mu_d+\mu_w),\quad f(\tau,x_1,x_2)=(\mu_d\alpha_d+\mu_w\alpha_w)x_1,\tag{1.60}$$

$$\varphi(x_1,x_2)=\frac{a}{n_y}x_2,\quad g_1(\tau,x_1,x_2)=0,\quad g_2(\tau,x_1,x_2)=\frac{a}{n_y}.\tag{1.61}$$

Figure 1.1 shows the qualitative behaviour of the velocity field, $\vec{v}$, at the boundaries for different parameter cases and times. Note that the velocity field either enters the domain or is tangential at Γ_2^+, while either it is tangential or it points outwards the domain at Γ_2^-. Also, both the diffusion matrix and the velocity field vanish at Γ_1^-. The previously discussed requirements of boundary conditions is closely related to inflow boundaries. Also note that the velocity field is not continuous with respect to the time variable.

1.4.2 Time discretization

The method of characteristics is used for the time discretization and it is included in the more general setting of upwinding methods, which take into account the local direction of the flux. More precisely, it is based on a finite difference scheme for the discretization of the material derivative, i. e., the time derivative along the characteristic lines of the convective part of the equation. In this section we will also introduce the variational formulation for the time discretized problem.

First, we define the characteristics curve through $\mathbf{x}=(x_1,x_2)$ at time $\bar{\tau}$, $X_e(\mathbf{x},\bar{\tau};s)$, which verifies:

$$\partial_s X_e(\mathbf{x},\bar{\tau};s)=\vec{v}(X_e(\mathbf{x},\bar{\tau};s)),\ X_e(\mathbf{x},\bar{\tau};\bar{\tau})=\mathbf{x}.\tag{1.62}$$

For $N>1$ let us consider the time step $\Delta\tau=T_r/N$ and the time mesh points $\tau^n=n\Delta\tau$, $n=0,\frac{1}{2},1,\frac{3}{2},\ldots,N$. The material derivative approximation by characteristics method is given by:

$$\frac{DV}{D\tau}=\frac{V^{n+1}-V^n\circ X_e^n}{\Delta\tau},$$

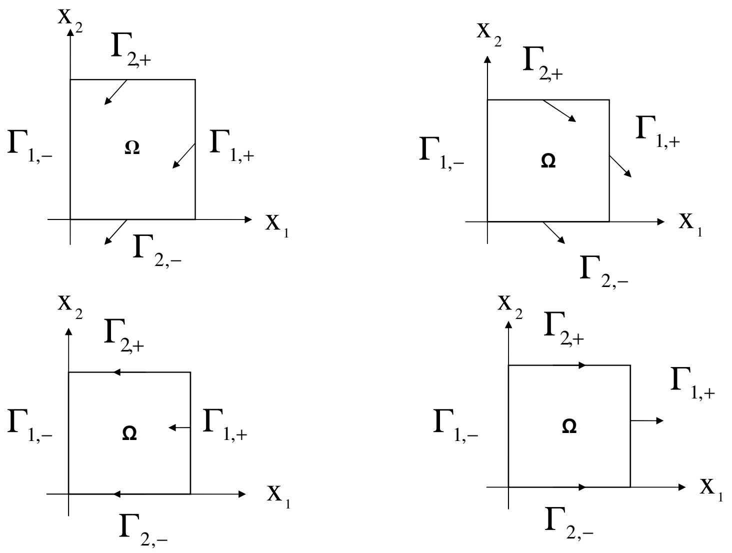

Figure 1.1: Bounded domain and velocity field, $\vec{v}$, at the boundaries: On the left, the case $\theta > \sigma^2$ for $\tau \leq n_y$ (top) and $\tau > n_y$ (bottom). On the right, the same for the case $\theta \leq \sigma^2$

where $X_e^n(\mathbf{x}) := X_e(\mathbf{x}, \tau^{n+1}; \tau^n)$. In view of the expression of the velocity field and the continuous function g given by expressions (1.59) and (1.6), respectively, the components of $X_e^n(\mathbf{x})$ can be analytically computed. More precisely, we distinguish the following two main cases:

- If $\theta \neq \sigma^2$ then $[X_e^n]_1(\mathbf{x}) = x_1 \exp((\theta - \sigma^2)\Delta\tau)$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & if \quad n\Delta\tau > n_y, \\ \\ \dfrac{k_1 x_1}{\sigma^2 - \theta}\big(1 - \exp((\theta - \sigma^2)\Delta\tau)\big) + x_2 & if \quad n\Delta\tau \leq n_y, \end{cases}$$

- If $\theta = \sigma^2$ then $[X_e^n]_1(\mathbf{x}) = x_1$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & if \quad n\Delta\tau > n_y, \\ \\ k_1 x_1 \Delta\tau + x_2 & if \quad n\Delta\tau \leq n_y. \end{cases}$$

Next, we consider a Crank-Nicolson scheme around $(X_e(\mathbf{x}, \tau^{n+1}; \tau), \tau)$ for $\tau = \tau^{n+\frac{1}{2}}$. So, for $n = 0, \ldots, N-1$, the time discretized equation can be written as:

Find V^{n+1} such that:

$$\begin{aligned} \frac{V^{n+1}(\mathbf{x}) - V^n(X_e^n(\mathbf{x}))}{\Delta\tau} - \frac{1}{2} Div(A\nabla V^{n+1})(\mathbf{x}) - \frac{1}{2} Div(A\nabla V^n)(X_e^n(\mathbf{x})) + \\ \frac{1}{2}(l\, V^{n+1})(\mathbf{x}) + \frac{1}{2}(l\, V^n)(X_e^n(\mathbf{x})) &= \\ \frac{1}{2} f^{n+1}(\mathbf{x}) + \frac{1}{2} f^n(X_e^n(\mathbf{x})). \end{aligned} \tag{1.63}$$

In order to obtain the variational formulation of the semidiscretized problem, we multiply equation (1.63) by a suitable test function, integrate in Ω, use the classical Green formula and the following one (see Lemma 3.4 in [48]):

$$\begin{aligned} \int_\Omega Div(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x}) d\mathbf{x} &= \int_\Gamma (\mathbf{F}_e^n)^{-T}(\mathbf{x})\mathbf{n}(x) \cdot (\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x}) d\mathbf{x} \\ &\quad - \int_\Omega (\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x})) \cdot \nabla\Psi(\mathbf{x}) d\mathbf{x} \\ &\quad - \int_\Omega Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x}) d\mathbf{x}. \end{aligned} \tag{1.64}$$

Note that in the present case we have

$$\int_\Omega Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x}) d\mathbf{x} = 0.$$

After these steps, we can write a variational formulation for the semidiscretized problem as follows:

Find $V^{n+1} \in H^1(\Omega)$ such that, $\forall \Psi \in H^1(\Omega)$:

$$
\begin{aligned}
&\int_\Omega V^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega (\mathbf{A}\nabla V^{n+1})(\mathbf{x})\nabla\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega lV^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} = \\
&\int_\Omega V^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_\Omega (\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\nabla\Psi(\mathbf{x})d\mathbf{x} - \\
&\frac{\Delta\tau}{2}\int_\Omega lV^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Gamma \widetilde{g}^n(\mathbf{x})\Psi(\mathbf{x})dA_\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Gamma_1^+} \bar{g}_1{}^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_\mathbf{x} + \\
&\frac{\Delta\tau}{2}\int_\Omega f^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega f^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}\,,
\end{aligned}
\tag{1.65}
$$

where $\mathbf{F}_e^n = \nabla X_e^n$ can be analytically computed, $\bar{g}_1^{n+1}(\mathbf{x}) = g_1^{n+1}(\mathbf{x})a_{11}(\mathbf{x}) = 0$ and

$$
\widetilde{g}^n(\mathbf{x}) = \begin{cases}
0 & \textit{on} \quad \Gamma_1^- \\[2ex]
-\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & \textit{on} \quad \Gamma_2^- \\[2ex]
\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & \textit{on} \quad \Gamma_2^+ \\[2ex]
\left[(\mathbf{F}_e^n)^{-T}\right]_{11}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))g_1^n(X_e^n(\mathbf{x})) & \textit{on} \quad \Gamma_1^+
\end{cases}
\tag{1.66}
$$

Remark 1.4.1. *Note that once the characteristics method for time discretization has been applied, only boundary condition (1.57), which corresponds to the boundary with nonvanishing diffusive term, is used to obtain the variational formulation. Note that condition (1.58) is mainly motivated by the velocity field entering the domain. Nevertheless, at each time step, the upwinded total derivative term is part of the second member of the discretized equation.*

Remark 1.4.2. *As a result of the application of the characteristics method we need to evaluate functions at the points $X_e^n(\mathbf{x})$, which are obtained by upwinding in the trajectories of the velocity field. Some technical interpolation skills are required when these points are placed outside the domain, the idea being to use the information at the boundaries. For the points that enter the domain through Γ_1^+ we use boundary condition (1.57), while for those ones entering through Γ_2^+ we use (1.58).*

1.4.3 Finite elements discretization

As we mention at the beginning of the section, we use the Crank-Nicolson characteristics method for the time discretization jointly with finite elements for spatial discretization. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. More precisely, let us define the finite elements space V_h by

$$V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) : \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{1.67}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

For a more general equation, under suitable assumptions on the data, in [8] the method is analyzed and proved to be unconditionally stable in case of exact integration of the integral terms. Also, for academic cases of constant coefficients convection-diffusion and pure convection equations the study of the different quadrature formulas to compute the involved integral terms is carried out. Thus, in case of trapezoidal or Simpson formulas for the pure convection equation in one spatial dimension stability can be proved. In the presence of an additional diffusive term the stability region is smaller for lower Peclet numbers, so these formulas result to be convenient for convection dominated problems. These results can be extended to higher spatial dimensions when product of one dimensional finite element spaces and quadrature formulas are considered. Note that the piecewise quadratic finite elements over quadrangular meshes are a particular case of product finite element spaces. Noting that they do not correspond to the analyzed academic cases, in all presented examples in this work we use a Simpson quadrature formula to approximate all the integral terms appearing in the fully discretized problem.

1.4.4 Monte Carlo simulation

In this section, the use of a Monte Carlo simulation technique applied to the particular case of pension plans valuation is described. First, we consider the case in which early retirement is not allowed and in the following chapter we will extend this simulation technique to take into account the early exercise option.

We will assume that the evolution of the salary under the risk neutral measure Q is described by the following stochastic differential equation:

$$dS_t = \beta(t, S_t)dt + \sigma(t, S_t)dZ_t^Q, \qquad (1.68)$$

where Z_t^Q denotes a Wiener process under this measure, $\beta(t, S) = \theta S$ and $\sigma(t, S) = \sigma S$. We will suppose that the salary at initial time is known.

In valuing pension plans, the payoff depends on the cumulative variable I. So, in a similar way to the case of Asian options, it is necessary to simulate paths over multiple dates in order to approximate the integral that appears in expression (1.5). Moreover, the pension plan value is given by the following expectation (see Chapter 9 in [52], for example):

$$V(t, S_t, I_t) = E_Q\left[e^{-(r+\mu_d+\mu_w)(T_r-t)}\frac{a}{n_y}I - \int_t^{T_r} e^{-(r+\mu_d+\mu_w)(u-t)}f(u, S_u)du\right], \qquad (1.69)$$

where we also have to use a discrete approximation of the integral which appears in the expression of the expectation concerning the second member f in equation (1.20). For this purpose, we consider a set of fixed points $t = t_0 < t_1 < ... < t_m = T_r$, with T_r the date at which the payoff is received. If we denote $S(t) = S_t$, the simulated salary $S(t_{j+1})$ from $S(t_j)$ is derived as follows (see [27], for example):

$$S(t_{j+1}) = S(t_j)exp([\theta - \frac{1}{2}\sigma^2](t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j}W_{j+1}), \qquad (1.70)$$

where $W_1, ..., W_m$ are independent standard normal random variables. This relies on the fact that $Z^Q(t_{j+1}) - Z^Q(t_j)$ has mean 0 and standard deviation $\sqrt{t_{j+1} - t_j}$. In order to reduce the discretization error the number of time steps m must be suitably large.

1.5 Numerical results

First, we consider an academic test with known analytical solution. More precisely, the appropriate data is imposed, so that the solution is given by

$$V^e(\tau, \mathbf{x}) = \exp\left(\frac{\tau x_1 x_2}{10^4}\right), \qquad (\tau, \mathbf{x}) \in (0, 40) \times \Omega,$$

with $\Omega = (0, 40) \times (0, 40)$, for the choice

$$f(\tau, \mathbf{x}) = \begin{cases} \exp\left(\frac{\tau x_1 x_2}{10^4}\right)\left(p(\tau, \mathbf{x}) - k_1 x_1^2 \tau 10^{-4}\right) & if \quad \tau \leq n_y \\ \\ \exp\left(\frac{\tau x_1 x_2}{10^4}\right) p(\tau, \mathbf{x}) & if \quad \tau > n_y \end{cases} \tag{1.71}$$

with $p(\tau, \mathbf{x}) = 10^{-4} x_1 x_2 - 10^{-4}\sigma^2 \tau x_1 x_2 - 0.5 \times 10^{-8} \sigma^2 \tau^2 x_1^2 x_2^2 + 10^{-4}(\sigma^2 - \theta)\tau x_1 x_2 + l$. Initial and boundary conditions data in (4.34), (1.57) and (1.58) are provided by the exact solution.

The computed $l^\infty((0, 40); l^2(\Omega))$ errors for different meshes and numbers of time steps for two different set of parameters are shown in Tables 1.1 and 1.2. The only differences in the data are the value of parameters μ_w ($\mu_w = 0$ in Table 1.1 and $\mu_w = 0.2$ in Table 1.2) and k_1 ($k_1 = 0.25$ in Table 1.1 and $k_1 = 0.5$ in Table 1.2). The number of nodes and elements of the referred quadratic finite element meshes are shown in Table 1.3. The qualitative and quantitative results for both set of parameters are very close. For a sufficiently fine fixed mesh in space, a first order convergence in time is clearly observed. If the mesh in space is not sufficiently fine the first order convergence appears until the spatial error dominates the total error. For a fixed, sufficiently fine mesh in time, a second order convergence in space is illustrated. Also if the time mesh is not sufficiently fine, then the second order converges appears until the time error dominates the total error. For the academic cases analyzed in [8] a second order convergence both in space and time is obtained. We note that the academic cases in [8] correspond to constant coefficient equations and certain assumptions on the velocity field. In fact, all theoretical results stated in [7, 8] assume that the velocity field is continuous with respect to the time variable, which is not the case in the present example.

NT	Mesh 12	Mesh 24	Mesh 48	Mesh 96
100	4.695867×10^{-2}	4.675261×10^{-2}	4.671738×10^{-2}	4.676544×10^{-2}
1000	6.088001×10^{-3}	5.517705×10^{-3}	5.423675×10^{-3}	5.420425×10^{-3}
10000	2.592423×10^{-3}	6.473885×10^{-4}	5.812531×10^{-4}	5.483266×10^{-4}
100000	1.575346×10^{-3}	3.958598×10^{-4}	1.112673×10^{-4}	5.526974×10^{-5}

Table 1.1: Errors for different meshes and numbers of time steps (NT) at time $\tau = 40$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.25$, $\mu_d = 0.025$ and $\mu_w = 0$ are considered

NT	Mesh 12	Mesh 24	Mesh 48	Mesh 96
100	7.172828×10^{-2}	7.159433×10^{-2}	7.153603×10^{-2}	7.138368×10^{-2}
1000	1.116242×10^{-2}	1.068611×10^{-2}	1.065671×10^{-2}	1.065913×10^{-2}
10000	2.230513×10^{-3}	1.191936×10^{-3}	1.115101×10^{-3}	1.080565×10^{-3}
100000	1.547804×10^{-3}	4.176562×10^{-4}	1.117329×10^{-4}	1.020686×10^{-4}

Table 1.2: Errors for different meshes and numbers of time steps (NT) at time $\tau = 40$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$ and $\mu_w = 0.2$ are considered

After the previous academic test, the results for the first real data set are shown in Table 1.4. More precisely, at the point $(S, I) = (25, 20)$ the values for different meshes to illustrate the convergence are indicated. These results have been compared with those ones obtained by Monte Carlo simulation. For the computation of 99% confidence intervals associated with the Monte Carlo technique, 250 time steps per year and 50000 paths have been consider. The computed results show that the numerical solution of the PDE model belongs to the confidence interval associated with the simulation technique. The corresponding confidence interval at this point is indicated in the caption of the table.

	Number of elements	**Number of nodes**
Mesh 12	144	625
Mesh 24	576	2401
Mesh 48	2304	9409
Mesh 96	9216	37249

Table 1.3: FEM meshes data

NT	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
100	2.805271	2.805411	2.805429	2.805436
1000	2.800663	2.800788	2.800796	2.800798
10000	2.800294	2.800313	2.800319	2.800321
100000	2.800248	2.800276	2.800274	2.800273

Table 1.4: Retirement benefits at time $t = 0$ and at mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with Monte Carlo simulation is [2.700880, 2.800527].

It is also important to illustrate the effect of truncation and introduction of boundary conditions in the obtained values at the financially interesting region. For this purpose, we consider that S represents salaries in thousands of unit currencies (for example, $S = 1$ corresponds to $1,000$ euros or dollars). Moreover, we note that average salary is actually given by

$$\overline{S} = \frac{1}{n_y} \int_{T_r - n_y}^{T_r} S(\tau) d\tau = \frac{I}{k_1 n_y}. \tag{1.72}$$

With this in mind, in Table 1.5 we represent the obtained pension plan values for the salaries $S = 1.2$, 2.4, 4.8 (for example, $S = 1.2$ corresponds to 1,200 currency units) and the average salaries $\overline{S} = 1$ and 2 (corresponding to the values $I = 15$ and 30, respectively). Note the small influence of the location of the boundaries of the truncated domain in the obtained value. The last row of this table shows the 99% confidence intervals obtained by using Monte Carlo simulation and related to the previously considered salaries.

	$(S,I) = (1.2, 15)$	$(S,I) = (2.4, 30)$	$(S,I) = (4.8, 30)$
$\Omega = (0,40) \times (0,40)$	0.133451	0.270426	0.546684
$\Omega = (0,80) \times (0,80)$	0.133371	0.266911	0.535248
$\Omega = (0,160) \times (0,160)$	0.133376	0.266756	0.533692
$\Omega = (0,320) \times (0,320)$	0.133375	0.266751	0.533469
Monte Carlo	[0.132986, 0.133653]	[0.265971, 0.270887]	[0.531943, 0.548774]

Table 1.5: Retirement benefits value at time $t = 0$ for different domains when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

Table 1.6 shows the behaviour of the pension plan value at time $t = 0$ in terms of the different involved parameters in the model and for different (S, I) coordinates. First note that, taking expression (1.72) into account, the number of years affects to the actual average salary. This explains the choice of coordinate I in the case of $n_y = 15$, in order to maintain the same value of $\overline{S}$ than in the case of $n_y = 30$. The corresponding results at time $t = 0$ obtained by Monte Carlo simulation are indicated in Table 1.7. In this case, with Monte Carlo, only the salary at time $t = 0$ is known, the value of the salary and cumulative salary are simulated at different dates along the path. The value of the pension plan does not depend on the cumulative salary I at origination because we only take into account the salary from time $T_r - n_y$. First, note that in both cases, with the PDE and with Monte Carlo, increasing volatility leads to a small increase in the benefit of pension plan value. The same occurs with increasing value of a. As expected, an increase in risk free interest rates leads to lower benefit values. The number of years has almost not influence on the obtained values, as we are maintaining the value of the resulting average salary. Table 1.8 and Table 1.9 show the behaviour of the pension plan value at time $t = 38$, the results in the first one are obtained with the PDE model and the second ones with Monte Carlo simulation. The salary and cumulative salary at time $t = 38$ are known and we simulate the values from this date. In this case the value of the cumulative function

I up to this time is considered and influences the value obtained with Monte Carlo simulation. In order to compare the behaviour at time $t = 0$ and $t = 38$ we present Figures 1.2 and 1.3. Note the influence of two factors (salary and average salary) in the second case, while in the first case, the value of the average salary has a very small influence on the pension plan value. This is because the time $t = 0$ is before the initial date ($T_r - n_y = 10$) which is used to compute the average salary that enters in the payoff function.

n_y	σ	r	a	$(S, I) = (1.2, 15)$	$(S, I) = (1.2, 22.5)$	$(S, I) = (2.4, 30)$
30	0.1	0.025	0.75	0.133451	0.133598	0.270426
			0.95	0.133488	0.133665	0.271463
		0.075	0.75	0.109117	0.109176	0.220036
			0.95	0.109124	0.109185	0.220478
	0.2	0.025	0.75	0.133636	0.133838	0.270412
			0.95	0.133736	0.134004	0.271634
		0.075	0.75	0.109187	0.109265	0.219801
			0.95	0.109217	0.109319	0.220308
				$(S, I) = (1.2, 7.5)$	$(S, I) = (1.2, 11.25)$	$(S, I) = (2.4, 15)$
15	0.1	0.025	0.75	0.133384	0.133394	0.266849
			0.95	0.133402	0.133415	0.266907
		0.075	0.75	0.109098	0.109114	0.218547
			0.95	0.109102	0.109143	0.218623
	0.2	0.025	0.75	0.133404	0.133419	0.266844
			0.95	0.133431	0.133448	0.266924
		0.075	0.75	0.109103	0.109105	0.218208
			0.95	0.109107	0.109111	0.218223

Table 1.6: Retirement benefits value at time $t = 0$ for different (S, I) points and parameter values

n_y	σ	r	a	$S = 1.2$	$S = 2.4$
30	0.1	0.025	0.75	[0.132986, 0.133653]	[0.265971, 0.270887]
			0.95	[0.133344, 0.133800]	[0.269689, 0.271899]
		0.075	0.75	[0.108926, 0.109262]	[0.217852, 0.220523]
			0.95	[0.109009, 0.109349]	[0.218018, 0.220698]
	0.2	0.025	0.75	[0.132630, 0.133681]	[0.265259, 0.271161]
			0.95	[0.133257, 0.134199]	[0.266513, 0.272398]
		0.075	0.75	[0.108729, 0.109419]	[0.217458, 0.220838]
			0.95	[0.108941, 0,109642]	[0.217882, 0.221283]
				$S = 1.2$	$S = 2.4$
15	0.1	0.025	0.75	[0.132993, 0.133651]	[0.265986, 0.266901]
			0.95	[0.133354, 0.133809]	[0.266707, 0.267618]
		0.075	0.75	[0.108927, 0.109263]	[0.217854, 0.218555]
			0.95	[0.109010, 0.109350]	[0.218021, 0.218700]
	0.2	0.025	0.75	[0.132637, 0.133688]	[0.265273, 0.267176]
			0.95	[0.133266, 0.134208]	[0.266531, 0.268417]
		0.075	0.75	[0.108730, 0.109420]	[0.217460, 0.218840]
			0.95	[0.108942, 0.109643]	[0.217885, 0.219286]

Table 1.7: The 99% confidence intervals with Monte Carlo simulation at time $t = 0$ for different salaries and parameter values

n_y	σ	r	a	$(S, I) = (1.2, 15)$	$(S, I) = (1.2, 22.5)$	$(S, I) = (2.4, 30)$
30	0.1	0.025	0.75	0.29442368	0.40814817	0.58884736
			0.95	0.36005233	0.50410336	0.72010465
		0.075	0.75	0.26883814	0.37174032	0.53767628
			0.95	0.32822141	0.45856416	0.65644281
	0.2	0.025	0.75	0.29442379	0.40814829	0.58884759
			0.95	0.36005247	0.50410351	0.72010495
		0.075	0.75	0.26883824	0.37174042	0.53767649
			0.95	0.32822153	0.45856429	0.65644307
				$(S, I) = (1.2, 7.5)$	$(S, I) = (1.2, 11.25)$	$(S, I) = (2.4, 15)$
15	0.1	0.025	0.75	0.31308212	0.42680661	0.62616424
			0.95	0.38368635	0.52773738	0.76737271
		0.075	0.75	0.28572099	0.38862318	0.57144199
			0.95	0.34960635	0.47994911	0.69921269
	0.2	0.025	0.75	0.31308234	0.42680684	0.62616465
			0.95	0.38368664	0.52773766	0.76737322
		0.075	0.75	0.28572121	0.38862338	0.57144237
			0.95	0.34960661	0.47994937	0.69921317

Table 1.8: Retirement benefits value at time $t = 38$ for different (S, I) points and parameter values

n_y	σ	r	a	$(S, I) = (1.2, 15)$	$(S, I) = (1.2, 22.5)$	$(S, I) = (2.4, 30)$
30	0.1	0.025	0.75	[0.294336, 0.294456]	[0.408061, 0.408180]	[0.588672, 0.588912]
			0.95	[0.359979, 0.360108]	[0.504030, 0.504159]	[0.719958, 0.720216]
		0.075	0.75	[0.268779 0.268890]	[0.371681, 0.371792]	[0.537557, 0.537780]
			0.95	[0.328140, 0.328260]	[0.458482, 0.458603]	[0.656279, 0.656521]
	0.2	0.025	0.75	[0.294248, 0.294489]	[0.407973, 0.408214]	[0.588497, 0.588979]
			0.95	[0.359904, 0.360164]	[0.503955, 0.504215]	[0.719808, 0.720327]
		0.075	0.75	[0.268718, 0.268943]	[0.371621, 0.371845]	[0.537437, 0.537886]
			0.95	[0.328062, 0.328305]	[0.458405, 0.458648]	[0.656124, 0.656610]
				$(S, I) = (1.2, 7.5)$	$(S, I) = (1.2, 11.25)$	$(S, I) = (2.4, 15)$
15	0.1	0.025	0.75	[0.302969, 0.313124]	[0.426694, 0.426848]	[0.625938, 0.626248]
			0.95	[0.383589, 0.383762]	[0.527640, 0.527813]	[0.761770 0.767523]
		0.075	0.75	[0.285645, 0.285788]	[0.388547, 0.388690]	[0.571290, 0.571577]
			0.95	[0.349498, 0.349659]	[0.479841, 0.480001]	[0.698996, 0.699317]
	0.2	0.025	0.75	[0.312856, 0.313168]	[0.426581, 0.426892]	[0.625712, 0.626335]
			0.95	[0.383489, 0.383837]	[0.527540, 0.527888]	[0.766977, 0.767674]
		0.075	0.75	[0.285568, 0.285857]	[0.388470, 0.388759]	[0.571136, 0.571713]
			0.95	[0.349394, 0.349718]	[0.479737, 0.480061]	[0.698789, 0.699436]

Table 1.9: The 99% confidence intervals with Monte Carlo simulation at time $t = 38$ for different (S,I) and parameter values

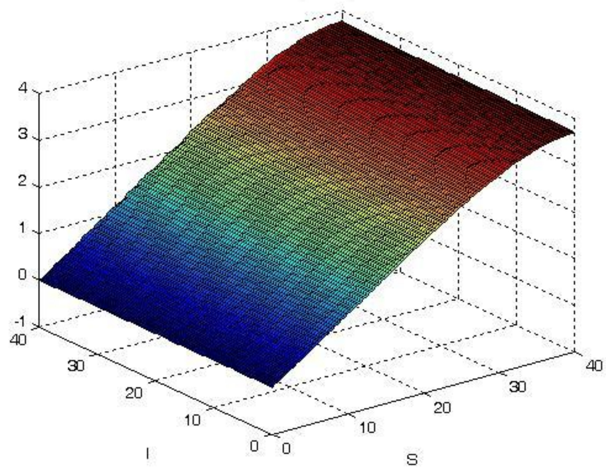

Figure 1.2: Retirement benefits at time $t = 0$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

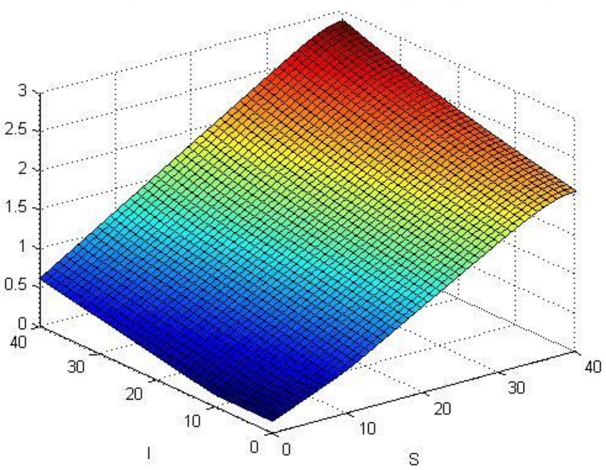

Figure 1.3: Retirement benefits at time $t = 38$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

Chapter 2

Pension plans allowing early retirement

2.1 Introduction

In the previous chapter, we posed a mathematical model to obtain the value of a defined benefit pension plan where it is assumed that the amount of money received by the employee depends on the average salary corresponding to a certain number of years before retirement.

In the present chapter, we incorporate the option of early retirement, which requires the application of similar modeling tools as those ones used for pricing American options in quantitative finance. Therefore, a free boundary problem formulation naturally arises, the free boundary being an optimal retirement boundary between the region where it is optimal to retire and the region where it is optimal to continue working.

In a similar way to the case without early retirement, the departure point in the modeling is the consideration that the salary is stochastic, so that the pension plan can be handled as an option on the average salary. Then, the dynamic hedging methodology in option pricing can be adapted to state a PDE model in the same way as for Asian options with American style. In [61], different models are stated for pension plans depending on the salary at retirement or on the average salary by using a risk neutral probability approach in case without early retirement. More recently, in [17] and [26] the case of early retirement when the benefit of the plan depends on the salary at retirement date is rigorously analyzed and existence, uniqueness and the properties of the optimal retirement boundary are obtained. In this work we address the more complex case with retirement payments based on the average salary. In this case the Black-Scholes model contains two spatial-like variables corresponding to two underlying stochastic factors instead of one.

We first decompose the problem into two time regions: one corresponding to the time range to average the salary and the other previous to the initial averaging time. For the first time region, the existence of solution is proved by using the methodology developed in [47] and [51] to study obstacle problems associated with hypoelliptic equations of Kolmogorov type. The regularity of the solution is also analyzed. In the second time region a classical one factor Black-Scholes equation is posed, involving a non-homogeneous term.

Moreover, for the PDE discretization we use the same numerical method than in the previous chapter and the non-linearities associated with the inequality constraints in the complementarity formulation due to early retirement are treated by means of the recently introduced Augmented Lagrangian Active Set (ALAS) method [36]. Among the different possible alternatives (projected relaxation, penalization, other duality techniques, ...) this method results to be efficient and robust (see for example, [10] for a comparison with a duality algorithm in an Asian option pricing problem of American style).

This chapter is organized as follows. In Section 2 everything concerning the mathematical model is proposed. In Section 3 the mathematical analysis tools allow to study the existence and regularity of solution. Section 4 contains the description of the different involved numerical techniques. In Section 5 some examples are presented to illustrate the performance of the proposed numerical method and the behaviour of the solution and the optimal retirement boundary. Also the numerical results are

compared with the confidence intervals obtained by the Longstaff-Schwartz simulation technique (see [42]).
Most of the contents presented in this chapter are included in reference [15].

2.2 Mathematical modeling

As we mention in the previous chapter, following [61] we assume that the salary S_t is governed by the stochastic differential equation

$$dS_t = \alpha(t, S_t)dt + \sigma(t, S_t)dZ_t,$$

where:

- the salary growth rate α depends on the time t since the entry into the plan, the current salary S_t and the age at entry t_0,
- σ denotes the volatility of the salary,
- Z_t is a Wiener process.

Note that the model assumes that uncertainty on the salary only depends on the volatility and follows a diffusion model although in real situations some sudden events could produce abrupt changes in the salary. In that case, a jump-diffusion model results to be more appropriate, as mentioned in the previous chapter and discussed in Chapter 3. In this work we consider a pension plan indicated to average salary during certain number of years before retirement date. Moreover, we incorporate the option of early retirement with a certain penalization. So, let us denote by $v_t = V(t, S_t; t_0)$ the value at time t of the benefits payable to the member of the plan when he/she is aged $t_0 + t$ and the salary is S_t.

In this section we pose the mathematical model in terms of a complementarity problem associated with a PDE to obtain V, when the retirement benefits depend on the continuous arithmetic average of the salary during the last n_y years before retirement date T_r. Additionally, the early retirement after a given date T_0, such that

$T_r - n_y < T_0 < T_r$, is allowed. Early retirement option has been analyzed in [26] for the plans indexed to the final salary.

Moreover, it is also assumed the existence of payment of benefits from the fund in case of death of a member or canceling the plan (withdrawal) to move to another plan. Therefore, we assume three possible states of a member of the plan: active (a), death (d) and withdrawn (w). It is considered that retirement occurs at the final age of service in the pension fund unless the involved early retirement option is exercised. The transition intensities from active membership to death or cancellation are denoted by $\mu_d(t;t_0)$ and $\mu_w(t;t_0)$, respectively. In the actuarial approach these intensities are understood as forces of decrement acting at time t on a member aged t_0+t and can be expressed in terms of the corresponding transition probabilities from one state to another.

In a similar way to the case without early retirement, by using actuarial arguments, when assuming deterministic benefits paid to the fund and ignoring any contributions, the variation of the value of the retirement benefits is given by the following Thiele's differential equation (see [13], for example):

$$dv_t = r(t)v_t dt - \sum_{i=d,w} \mu_i(t;t_0)\left(A_i(t,S_t;t_0) - v_t\right)dt, \tag{2.1}$$

where $r(t)$ is the deterministic time dependent risk-free interest rate and $A_i(t,S_t;t_0)$ denotes the deterministic benefit paid by the fund in case of death ($i=d$) or withdrawal ($i=w$). Note that the difference $A_i(t,S_t;t_0) - v_t$ represents the sum-at-risk associated with the corresponding decrement i, so that

$$\sum_{i=d,w} \mu_i(t;t_0)(A_i(t,S_t;t_0) - v_t)dt \tag{2.2}$$

denotes the instantaneous expected value of the payments from the fund.

Following Section 5 in [61], we assume that

$$A_i(t,S;t_0) = \alpha_i S, \quad \alpha_i \geq 0, \quad i = d, w,$$

so that the death and withdrawal benefits are a constant multiple of the salary, and that the transition intensities $\mu_i(t;t_0) = \mu_i$ are nonnegative constants.

As we are considering that retirement benefits depend on the continuous arithmetic average of the salary, analogously to the case of Asian options, we introduce the following variable representing the cumulative function of the salary since the last n_y years before T_r:

$$I_t = \int_0^t g(\tau, S(\tau))d\tau, \tag{2.3}$$

with

$$g(t,S) = \begin{cases} 0 & \text{if} \quad 0 \le t < T_r - n_y \\ h(S) & \text{if} \quad T_r - n_y \le t \le T_r, \end{cases} \tag{2.4}$$

where h is appropriately chosen. Specifically, in this work we consider the particular case $h(S) = k_1 S$, where the accrual constant k_1 is positive, that is

$$k_1 > 0. \tag{2.5}$$

For simplicity, hereafter we drop the dependence on the entry age "t_0" in all functions (in particular, the value of the plan is denoted by $v_t = V(t, S_t, I_t; t_0) = V(t, S_t, I_t)$). Then we can apply Itô's lemma (see [32]) to V jointly with Thiele differential equation (2.1) to obtain

$$\begin{aligned} dv_t = & \left(\partial_t V + \alpha(t,S_t)\partial_S V + g(t,S_t)\partial_I V + \frac{1}{2}\sigma(t,S_t)^2 \partial_{SS} V\right) dt + \sigma(t,S_t)\partial_S V dZ_t \\ & + \left(\sum_{i=d,w} \mu_i (A_i(t,S_t) - v_t)\right) dt, \end{aligned}$$

where the first two terms on the right hand side are associated with the stochastic variation of the salary while the third term is related to the expected payments due to death or withdrawal.

In [61] the PDE for the case without early retirement is obtained by arguing that the risk-adjusted expected change in the liabilities value, after allowing the benefits cash flows from the fund, should be equal to the risk-free interest rate. Furthermore, in the previous chapter a dynamic hedging methodology to deduce the PDE for the same case without early retirement option is applied. In both cases the following PDE for the value of the benefits of the pension plan

$$\partial_t V + \beta\, \partial_S V + g\, \partial_I V + \frac{1}{2}\sigma^2\, \partial_{SS} V - (\mu_d + \mu_w + r)\, V = -\mu_d A_d - \mu_w A_w, \tag{2.6}$$

is obtained, where $\beta = \alpha - \lambda\sigma$ is related to the market price of risk associated with the uncertainty of the salary, here denoted by λ. Assuming that at the retirement date T_r the owner of the pension plan receives a fraction of the average salary during last n_y years, the pension plan pricing model for the case without early retirement is defined by equation (2.6) jointly with the final condition

$$V(T_r, S, I) = \frac{a}{n_y} I, \tag{2.7}$$

where $a \in (0, 1)$ is a given accrual constant.

In the previous chapter the existence, uniqueness and regularity of the solution for the case without early retirement option are analyzed. Moreover, the proposed numerical method to solve the PDE problem (2.6)-(2.7) and some numerical examples are described.

If early retirement is allowed, it seems reasonable to assume that a member of the plan would retire when he/she maximizes the benefits of retirement among all possible dates (optimal stopping times) to retire. The financial argument results to be very similar to the one used when pricing financial products including the option of early exercise (American options, callable bonds, Bermudan products,...). On the other hand, from the pension plan manager viewpoint it seems reasonable to penalize the benefits received by the member when he/she decides to retire before the expected retirement date. Thus, we assume that the member will receive at the early retirement date t, the quantity given by

$$\Psi(t, S, I) = \begin{cases} 0 & \text{if } t < T_0, \\ \left(1 - \dfrac{T_r - t}{T_r - T_0}\right) \dfrac{aI}{t - (T_r - n_y)} & \text{if } t \geq T_0. \end{cases} \tag{2.8}$$

Note that retirement is not allowed before T_0 so that we assume the member receives nothing when retiring before that date. As in the case without early retirement, under the risk neutral measure Q, the stochastic evolution of the salaries is given by

$$dS_t = \beta(t, S_t)dt + \sigma(t, S_t)dZ_t^Q,$$

where Z^Q denotes a Wiener process under this measure.

By using analogous arbitrage free arguments to the ones argued in early exercise financial products (for instance, see Theorem 11.7 in [52] or Theorem 5.3.2 in [40], the related PDE problem can be written in terms of a complementarity problem as follows:

$$\begin{cases} \max\{\mathcal{L}V - f, \Psi - V\} = 0, & \text{in } (0, T_r) \times \mathbb{R}_+^2, \\ V(T_r, S, I) = \dfrac{a}{n_y} I, & (S, I) \in \mathbb{R}_+^2, \end{cases} \tag{2.9}$$

where

$$\mathcal{L}V = \partial_t V + \beta \partial_S V + g \partial_I V + \frac{\sigma^2 S^2}{2} \partial_{SS} V - (r + \mu_d + \mu_w) V,$$
$$f(t, S, I) = -(\mu_d \alpha_d + \mu_w \alpha_w) S.$$

Note that the previous complementarity problem can be characterized as a free boundary problem where not only the value function V is obtained, but also the early retirement region ($V = \Psi$) and the no retirement region ($V > \Psi$), as well as the free boundary (optimal retirement boundary) separating both regions at each time instant are identified.

The value of the pension plan benefits can also be expressed in probabilistic terms as an Snell envelope by the following formula (see Chapter 11 in [52], for example):

$$V(t, S_t, I_t) = \sup_{\tau \in \mathcal{T}(t, T_r)} E_Q \left[e^{-(r+\mu_d+\mu_w)(\tau - t)} \Psi(\tau, S_\tau, I_\tau) - \int_t^\tau e^{-(r+\mu_d+\mu_w)(u-t)} f(u, S_u) du \right], \tag{2.10}$$

where $\mathcal{T}(t, T_r)$ denotes the set of stopping times between t and T_r and E_Q denotes the expected value under the measure Q.

2.3 Mathematical analysis

In the previous section, the mathematical model for the benefit of the pension plan with the early retirement option has been posed as the Cauchy linear complementarity problem (2.9). In this section we study the existence and regularity of solutions.

In what follows we assume that σ and β are proportional to the salary, so that $\sigma(t,S) = \sigma S$ and $\beta(t,S,I) = \theta S$, where $\theta > 0$ and $\sigma > 0$ are given constants. The assumption on σ implies that the salary volatility increases with salary, which is a reasonable argument, mainly in the private sector. Additionally, the joint assumption on the expressions of both σ and β implies that the market price of risk is a constant parameter when a log-normal evolution for salaries is considered (that is, when $\alpha(t,S) = \alpha S$).

Next, having in view that the function g in (2.4) vanishes when $t < T_r - n_y$, in order to analyze the existence of solutions, we decompose (2.9) into two different problems:

- **Problem 1:**

  Find a function V_1, defined in $[T_r - n_y, T_r] \times \mathbb{R}_+^2$, such that

$$\begin{cases} \max\{\mathcal{L}_1 V_1 - f, \Psi - V_1\} = 0, & \text{in } (T_r - n_y, T_r) \times \mathbb{R}_+^2, \\ V_1(T_r, S, I) = \dfrac{a}{n_y} I, & (S,I) \in \mathbb{R}_+^2, \end{cases} \tag{2.11}$$

  where

$$\mathcal{L}_1 V = \partial_t V + \theta S \partial_S V + k_1 S \partial_I V + \frac{\sigma^2 S^2}{2} \partial_{SS} V - k_2 V, \qquad f(t,S,I) = -k_3 S, \tag{2.12}$$

  with

$$k_2 = r + \mu_d + \mu_w, \qquad k_3 = \mu_d \alpha_d + \mu_w \alpha_w. \tag{2.13}$$

- **Problem 2:**

  For each $I > 0$, find a function $V_2(\cdot,\cdot;I)$ defined in $[0, T_r - n_y] \times \mathbb{R}_+$, such that

$$\begin{cases} \mathcal{L}_2 V_2(\cdot,\cdot;I) = f, & \text{in } (0, T_r - n_y) \times \mathbb{R}_+, \\ V_2(T_r - n_y, S; I) = V_1(T_r - n_y, S, I), & S \in \mathbb{R}_+, \end{cases} \tag{2.14}$$

  where

$$\mathcal{L}_2 V = \partial_t V + \theta S \partial_S V + \frac{\sigma^2 S^2}{2} \partial_{SS} V - k_2 V, \tag{2.15}$$

  and f and k_2 are as in (2.12)-(2.13).

Problem 1 consists of a complementarity problem associated with the degenerate parabolic operator $\mathcal{L}_1$. Note that the obstacle function Ψ vanishes for $t \leq T_0$ and therefore in this time range the inequality constraint reduces to the positivity of the solution, which is in turn guaranteed by the positivity of the final condition and the maximum principle. Problem 2 consists of a parametrized family of one-dimensional Cauchy problems (associated with the standard non-homogeneous Black-Scholes equation with linear second member) depending on I as a parameter through the final condition.

In the next subsections we discuss the existence of solutions for both problems.

2.3.1 Mathematical analysis of Problem 1.

We refer to operator $\mathcal{L}_1$ in Problem 1 as an *ultra-parabolic* operator because only the first order derivative w.r.t. I appears, while the solution is a function of the spatial variables S, I and the time variable t. Recently, in the study of the arithmetic American Asian option pricing problem, Monti and Pascucci in [47] proved the existence and uniqueness of the solution of a homogeneous problem which is similar to Problem 1. In this section, we generalize those results to include (2.11).

As a first step, we construct a supersolution for (2.11).

Definition 2.3.1. *A function $\bar{U} \in C^2((T_r - ny, T_r) \times \mathbb{R}^2_+) \cap C((T_r - ny, T_r] \times \mathbb{R}^2_+)$ such that*

$$\mathcal{L}_1 \bar{U} \leq f \quad \textit{and} \quad \bar{U} \geq \Psi \quad \textit{in } (T_r - n_y, T_r) \times \mathbb{R}^2_+, \tag{2.16}$$

is called a supersolution to problem (2.11).

We can obtain a super-solution in two different ways: either we perform the same changes of variables that in the previous chapter or we work in the original variables and try to obtain the same kind of supersolution used in [47]. In the present work we follow the latter approach.

Proposition 2.3.1. *If γ and δ are suitably large constants then the function $\bar{U}$ defined by*

$$\bar{U}(t,S,I) = \gamma e^{-\delta t}\left(S + \sqrt{S^2+I^2}\right) \tag{2.17}$$

is a super-solution to problem (2.11).

Proof. We set

$$\bar{U} = \bar{U}_1 + \bar{U}_2\,,$$

where

$$\bar{U}_1(t,S,I) = \gamma e^{-\delta t}\sqrt{S^2+I^2} \quad \text{and} \quad \bar{U}_2(t,S,I) = \gamma e^{-\delta t} S.$$

Recalling that $f(t,S,I) = -k_3 S$, the thesis follows from the following points:

i) there exists $\delta_0 > 0$ such that

$$\mathcal{L}_1 \bar{U}_1 \le 0 \tag{2.18}$$

for any $\gamma \ge 0$ and $\delta \ge \delta_0$. Indeed we have

$$\mathcal{L}_1 \bar{U}_1(t,S,I) = \frac{\gamma e^{-t\delta}}{2\left(I^2+S^2\right)^{3/2}} W(S,I)$$

where

$$W(S,I) := \sigma^2 S^2 I^2 - 2\left(I^2+S^2\right)\left(S^2(k_2+\delta-\theta) + I^2(k_2+\delta) - k_1 IS\right).$$

Therefore $\mathcal{L}_1 \bar{U}_1 \le 0$ if and only if $W(S,I) \le 0$. By the elementary inequality

$$|SI| \le \frac{S^2+I^2}{2},$$

we have

$$W(S,I) \le \left(\frac{S^2+I^2}{2}\right)^2 \sigma^2 - 2\left(S^2+I^2\right)\left((k_1+k_2+\delta-\theta)S^2 + (k_1+k_2+\delta)I^2\right).$$

Therefore $W(S,I) \le 0$ if and only if

$$\frac{S^2+I^2}{4}\sigma^2 - 2\left((k_1+k_2+\delta-\theta)S^2 + (k_1+k_2+\delta)I^2\right) \le 0,$$

which is true if δ is sufficiently large;

ii) there exists γ_0 such that

$$\mathcal{L}_1\bar{U}_2(t,S,I) + k_3 S \le 0, \qquad (t,S,I) \in (T_r - n_y, T_r) \times \mathbb{R}^2_+, \tag{2.19}$$

for any $\gamma \ge \gamma_0$ and $\delta \ge \max\{\delta_0, \theta - k_2\}$. More precisely, inequality (2.19) follows from the identity

$$\mathcal{L}_1\bar{U}_2(t,S,I) + k_3 S = \left(\gamma e^{-\delta t}\left(\theta - k_2 - \delta\right) + k_3\right) S;$$

iii) finally, for any $\delta \ge 0$ there exists $\gamma \ge \gamma_0$ such that

$$\bar{U}(t,S,I) \ge \gamma e^{-\delta T_r} I \ge \Psi(t,S,I), \quad (t,S,I) \in [T_r - n_y, T_r] \times \mathbb{R}^2_+.$$

$\square$

Existence and regularity of solutions to (2.11) are more delicate matters. Indeed, on the one hand operator $\mathcal{L}_1$ is not uniformly parabolic so the classical theory does not apply; on the other hand, it is well-known that even in the standard case of uniformly parabolic operators, a complementarity problem does not admit classical solutions. Following [47], we study problem (2.11) in the framework of hypoelliptic equations of Kolmogorov type. More precisely, we consider an operator in $\mathbb{R}^3$ of the form

$$L = \bar{a}(t,S,I)\partial_{SS} + Y, \tag{2.20}$$

where

$$Y = \partial_t + \theta S \partial_S + k_1 S \partial_I \tag{2.21}$$

is the first order part of $\mathcal{L}_1$. By the results in [55] and [23], it is known that under the assumption (2.5) (i.e. $k_1 > 0$) and if the coefficient $\bar{a}$ is a smooth function such that

$$\frac{1}{\mu} \le \bar{a} \le \mu \qquad \text{on } \mathbb{R}^3, \tag{2.22}$$

for some positive constant μ, then operator L has a global fundamental solution which can be estimated by Gaussian functions. Note that $\mathcal{L}_1$ is *locally* of the form (2.20) since the function $\bar{a}(t,S,I) = \frac{\sigma^2 S^2}{2}$ verifies the non-degeneracy condition (2.22) on compact subsets of $\mathbb{R} \times \mathbb{R}^2_+$.

The obstacle problem for a general class of degenerate parabolic operators including (2.20) was first studied by Di Francesco, Pascucci and Polidoro in [24] who proved the existence of *strong solutions*: specifically, for any domain Ω of $\mathbb{R}^3$ and $p \geq 1$, we introduce the Sobolev-Stein spaces

$$\mathcal{S}^p(\Omega) = \{U \in L^p(\Omega) \mid \partial_S U,\, \partial_{SS} U,\, YU \in L^p(\Omega)\}$$

endowed with the semi-norm

$$\|U\|_{\mathcal{S}^p} = \|U\|_{L^p} + \|\partial_S U\|_{L^p} + \|\partial_{SS} U\|_{L^p} + \|YU\|_{L^p}.$$

If $u \in \mathcal{S}^p(\Omega')$ for any compact subset $\Omega' \subseteq \Omega$, then we write $\mathcal{S}^p_{\text{loc}}(\Omega)$.

Definition 2.3.2. *A strong solution to problem (2.11) is a function $U \in \mathcal{S}^1_{loc} \cap C((T_r - n_y, T_r] \times \mathbb{R}^2_+)$ which satisfies the differential inequality a.e. in $(T_r - n_y, T_r) \times \mathbb{R}^2_+$ and the final condition in the pointwise sense.*

The main result of this section in the following

Theorem 2.3.1. *Problem* (2.11) *admits a strong solution U which belongs to $\mathcal{S}^p_{loc}$ for any $p \geq 1$ and satisfies the inequality*

$$U \leq \bar{U} \tag{2.23}$$

where $\bar{U}$ is the supersolution in (2.17). *Moreover, U coincides with the Snell envelope in (2.10).*

Proof. Let $D_\rho(x_1, x_2)$ denote the Euclidean ball centered at $(x_1, x_2) \in \mathbb{R}^2$, with radius ρ. We consider the sequence of domains $O_n = D_n\left(n + \frac{1}{n}, 0\right) \cap D_n\left(0, n + \frac{1}{n}\right)$ covering $\mathbb{R}^2_+$. For any $n \in \mathbb{N}$, the cylinder $H_n = (T_r - n_y, T_r) \times O_n$ is a $\mathcal{L}_1$*-regular domain* in the sense that there exists a barrier function at any point of the parabolic boundary $\partial_P H_n := \partial H_n \setminus (\{0\} \times O_n)$ (cf. Remark 3.1 in [24]). Since $\mathcal{L}_1$ satisfies condition (2.22) on H_n for any n, then by Theorem 3.1 in [24] we have: for any $n \in \mathbb{N}$, there exists a strong solution $U_n \in \mathcal{S}^p_{\text{loc}}(H_n) \cap C(H_n \cup \partial_P H_n)$ to problem

$$\begin{cases} \max\{\mathcal{L}_1 U - f, \Psi - U\} = 0 & \text{in } H_n, \\ u|_{\partial_P H_n} = \frac{aI}{n_y}. \end{cases} \tag{2.24}$$

Moreover, the following estimate holds: for every $p \geq 1$ and $H \subset\subset H_n$ there exists a positive constant C, only depending on $H, H_n, p, \|\Psi\|_{L^\infty(H_n)}$ such that

$$\|U_n\|_{\mathcal{S}^p(H)} \leq C. \tag{2.25}$$

Next we consider a sequence of cut-off functions $\chi_n \in C_0^\infty(\mathbb{R}^2_+)$, such that $\chi_n = 1$ on O_{n-1}, $\chi_n = 0$ on $\mathbb{R}^2_+ \setminus O_n$ and $0 \leq \chi_n \leq 1$. We set

$$\Psi_n(t, S, I) = \chi_n(S, I)\Psi(t, S, I) + (1 - \chi_n(S, I))\bar{U}(t, S, I)\,,$$

where $\bar{U}$ is the supersolution in (2.17), and we denote by U_n the strong solution to (2.24) with $\Psi = \Psi_n$. By the comparison principle we have $\Psi \leq U_n \leq U_{n+1} \leq \bar{U}$ and therefore, by estimate (2.25), for every $p \geq 1$ and $H \subset\subset H_n$ we have $\|U_n\|_{\mathcal{S}^p(H)} \leq C$ for some constant C depending on H but not on n. Then we can pass to the limit as $n \to \infty$, on compact subsets of $(T_r - n_y, T_r) \times \mathbb{R}^2_+$, to get a strong solution of $\max\{\mathcal{L}_1 U - f, \Psi - U\} = 0$. A standard argument based on barrier functions shows that $U(t, \cdot)$ is continuous up to $t = T_r$ and attains the final datum. The stochastic representation of the solution (2.10) can be proved as in Theorem 1 in [47], thus leading to uniqueness of solution. □

Next, we briefly discuss the regularity properties of the strong solution U of problem (2.11). For greater convenience, we put $x = (S, I)$ and, using the matrix notation, we rewrite the vector field Y in (2.21) as

$$Y = \langle Bx, \nabla_x \rangle + \partial_t$$

where

$$B = \begin{pmatrix} \theta & 0 \\ k_1 & 0 \end{pmatrix}$$

and ∇_x is the gradient in the variables x. As observed by Lanconelli and Polidoro [41], in the case that the coefficient $\bar{a}$ in (2.20) is constant, then operator L is invariant[1] w.r.t. the left translations in the group law

$$(\tau, \xi) * (t, x) = (\tau + t, x + e^{tB}\xi),$$

where

$$e^{tB} = \begin{pmatrix} e^{t\theta} & 0 \\ \frac{k_1(e^{t\theta}-1)}{\theta} & 1 \end{pmatrix}.$$

In addition, if $\theta = 0$ then L is also homogeneous of degree two[2] w.r.t the dilations group

$$\delta_\lambda(t, S, I) = (\lambda^2 t, \lambda S, \lambda^3 I), \qquad \lambda > 0.$$

It turns out that the operations just defined are naturally designed for the study of the optimal regularity properties of the solution in Theorem 2.3.1. Indeed, let us first recall the following embedding theorem proved in [24].

Theorem 2.3.2. *Let O, Ω be bounded domains of $\mathbb{R}^3$ such that $O \subset\subset \Omega$ and $p > 8$. There exists a positive constant c, only dependent on B, Ω, O and p, such that*

$$\|u\|_{C_B^{1,\alpha}(O)} \leq c\|u\|_{\mathcal{S}^p(\Omega)}, \qquad \alpha = 1 - \frac{6}{p}, \tag{2.26}$$

for any $u \in \mathcal{S}^p(\Omega)$. In (2.26), the anisotropic Hölder space $C_B^{1,\alpha}$ is defined in terms

---

[1] L is left-$*$-invariant if

$$LU\left((\tau, \xi) * (t, x)\right) = (LU)\left((\tau, \xi) * (t, x)\right).$$

[2] L is δ_λ-homogeneous of degree two if

$$LU\left(\delta_\lambda(t, x)\right) = \lambda^2 (LU)\left(\delta_\lambda(t, x)\right).$$

of the following norms[3]*:*

$$\|U\|_{C_B^{0,\alpha}(\Omega)} = \sup_{\Omega}|U| + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega\\(t,x)\neq(\tau,\xi)}} \frac{|U(t,x)-U(\tau,\xi)|}{\|(\tau,\xi)^{-1}*(t,x)\|_B^{\alpha}},$$

$$\|U\|_{C_B^{1,\alpha}(\Omega)} = \|U\|_{C_B^{0,\alpha}(\Omega)} + \|\partial_S U\|_{C_B^{0,\alpha}(\Omega)} + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega\\(t,x)\neq(\tau,\xi)}} \frac{|U(t,x)-U(\tau,\xi)-(S-S')\partial_S U(\tau,\xi)|}{\|(\tau,\xi)^{-1}*(t,x)\|_B^{1+\alpha}},$$

where $\|\cdot\|_B$ *is the* δ_λ*-homogeneous norm in* $\mathbb{R}^3$ *defined by*

$$\|(t,S,I)\|_B = |t|^{\frac{1}{2}} + |S| + |I|^{\frac{1}{3}}.$$

As a consequence of Theorem 2.3.2, the strong solution of problem (2.11) belongs locally to the space $C_B^{1,\alpha}$ for any $\alpha < 1$. Actually, sharp interior regularity results for a class of problems including (2.11) were proved in [25]: it turns out that the solution of (2.11) belongs to the class $\mathcal{S}^{\infty}_{\mathrm{loc}}$ and this is the optimal regularity for this kind of problem.

We remark explicitly that, for any *bounded* domain Ω, there exists a positive constant c_Ω such that

$$\begin{aligned}\|(\tau,\xi)^{-1}*(t,x)\|_B &= \|(t-\tau,x-\xi)+(0,(\mathrm{Id}_2-e^{(t-\tau)B})\xi)\|_B\\ &\le c_\Omega|(t-\tau,x-\xi)|^{\frac{1}{3}}, \qquad (\tau,\xi),(t,x)\in\Omega,\end{aligned}$$

where Id_2 is the identity matrix in $\mathbb{R}^2$. In particular, we have

$$C_B^{0,\alpha}(\Omega) \subseteq C^{0,\frac{\alpha}{3}}(\Omega)$$

where $C^{0,\alpha}$ denotes the standard Euclidean Hölder space.

Moreover, if $U \in C_B^{1,\alpha}(\Omega)$ then $U, \partial_S U \in C_B^{0,\alpha}(\Omega)$ and also

$$|U((t,x)*(\tau,0)) - U(t,x)| = |U((t+\tau, e^{\tau B}x)) - U(t,x)| \le c_\Omega|\tau|^{\frac{1+\alpha}{2}}$$

which implies the Hölder regularity of order $\frac{1+\alpha}{2}$ *along the integral curves of* Y. As a matter of fact, Y can be identified with the vector field $Y(t,x) = (1, Bx)$ and

[3] *We use the notation* $x = (S,I)$ *and* $\xi = (S',I')$.

$\gamma(\tau) := (t+\tau, e^{\tau B}x)$ is the integral curve of Y starting from (t, x), that is the solution of the problem

$$\frac{d\gamma(\tau)}{d\tau} = Y(\gamma(\tau)), \qquad \gamma(0) = (t, x).$$

Note however that the $C_B^{1,\alpha}$-regularity of U does not imply the existence of the Euclidean derivative $\partial_I U$: roughly speaking, ∂_I can be recovered by commuting ∂_S and Y

$$[\partial_S, Y] = \partial_S Y - Y\partial_S = \theta\partial_S + k_1\partial_I,$$

and therefore intrinsically it has to be considered as a third order derivative. For this reason, in the numerical solution of the problem we adopt the natural approach of using a semi-Lagrangian method for time discretization, that mainly consists of a finite differences scheme along the integral curves of the convective part Y of the equation (cf. Subsection 2.4.1).

Concerning Problem 2, up to a logarithmic change of variable, it consists of a Cauchy problem for a non-homogeneous heat equation with a continuous final datum which, by (2.23), has exponential growth at infinity. Thus, standard results guarantee existence and uniqueness of a classical solution V_2. Moreover, continuous dependence results imply that V_2 inherits the regularity in the I variable from the final datum.

2.4 Numerical solution

In our numerical approach, the methods have been designed to solve the problem without distinguishing the time region where actually only one factor is acting from the region where the PDE is governed by the two stochastic factors (Problem 1). Thus, the numerical solution of the complementarity problem formulation (2.9) is addressed. Although this approach involves a bit more computational cost it simplifies the computational code.

In order to enumerate the numerical techniques, the main difficulties and the way to overcome them numerically are briefly outlined. First, as in the case without early exercise opportunity, a localization technique is used to cope with the initial formulation in an unbounded domain. Also, as the diffusive term is strongly degenerate so that the PDE can be understood as an example of extreme convective dominated case, we propose a Crank-Nicolson characteristics time discretization scheme combined with piecewise quadratic Lagrange finite element method. For the inequality constraints associated with the early retirement option, we propose a mixed formulation and the use of an Augmented Lagrangian Active Set technique.

Therefore, we first consider a problem posed in a sufficiently large spatial bounded domain, so that the solution in the region of financial interest is not affected by the truncation of the unbounded domain and the required boundary conditions (localization procedure). For this purpose, as in the case without early retirement treated in the previous chapter, we first introduce the change of variable in time and notation

$$\tau = T_r - t, \quad x_1 = S, \quad x_2 = I. \tag{2.27}$$

Let us consider x_1^∞ and x_2^∞ be large enough real numbers and let $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ be the computational bounded domain. Moreover, let $\partial\Omega = \bigcup_{i=1}^2 (\Gamma_i^- \bigcup \Gamma_i^+)$, with $\Gamma_i^- = \{(x_1, x_2) \in \partial\Omega \mid x_i = 0\}$ and $\Gamma_i^+ = \{(x_1, x_2) \in \partial\Omega \mid x_i = x_i^\infty\}$ for $i = 1, 2$.

The iterative algorithms we propose for the numerical solution of (2.9) are based on the Lagrangian formulation of the complementarity problem. This approach mainly consists on replacing the inequality in (2.9) by an identity in terms of an appropriate Lagrange variable or multiplier to be denoted by P. This mixed formulation classically appears when dealing with duality methods for solving obstacle problems (see [22], for example). Then, problem (2.9) admits an equivalent mixed formulation. Moreover, after the domain truncation, this mixed formulation in the unbounded domain can be approximated by the following one in the bounded domain:

Find V and $P : [0, T_r] \times \Omega \longrightarrow R$, satisfying the partial differential equation

$$\partial_\tau V - Div\,(A\nabla V) + \vec{v} \cdot \nabla V + lV + P = f \quad \text{in } (0, T_r) \times \Omega, \tag{2.28}$$

the complementarity conditions

$$V \geq \overline{\Psi}, \quad P \leq 0, \quad (V - \overline{\Psi})P = 0 \quad \text{in } (0, T_r) \times \Omega, \tag{2.29}$$

the initial condition

$$V(0, x_1, x_2) = \frac{a}{n_y} x_2 \quad \text{in} \quad \Omega, \tag{2.30}$$

and the boundary conditions

$$\frac{\partial V}{\partial x_1} = g_1 \qquad \text{on } (0, T_r) \times \Gamma_1^+ , \tag{2.31}$$

$$\frac{\partial V}{\partial x_2} = g_2 \qquad \text{on } (0, T_r) \times \Gamma_2^+ , \tag{2.32}$$

where the involved data is defined as follows

$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}(\tau, x_1, x_2) = \begin{pmatrix} (\sigma^2 - \theta) x_1 \\ -g(T_r - \tau, x_1) \end{pmatrix}, \tag{2.33}$$

$$l(\tau, x_1, x_2) = r + \mu_d + \mu_w, \quad f(\tau, x_1, x_2) = (\mu_d \alpha_d + \mu_w \alpha_w) x_1, \tag{2.34}$$

$$\overline{\Psi}(\tau, x_1, x_2) = \Psi(T_r - \tau, x_1, x_2) \tag{2.35}$$

$$g_1(\tau, x_1, x_2) = 0, \quad g_2(\tau, x_1, x_2) = \frac{a}{n_y}. \tag{2.36}$$

Note that the presence of the boundary conditions at the new boundaries of the bounded domain are related to the expression of the velocity field and the diffusion matrix. This issue has been widely discussed in the previous chapter.

2.4.1 Lagrange-Galerkin discretization

Concerning the numerical methods, in the present subsection we mainly describe the solution of the free-boundary aspect related to the early retirement possibility. Nevertheless, we summarize the description of the second order Lagrange-Galerkin method that has been analyzed in [7, 8] for time-space discretization. Its application to the pension plan model without early retirement is described in detail in the previous chapter.

The method of characteristics is based on a finite differences scheme for the discretization of the material derivative, i.e. the time derivative along the characteristic lines of the convective part of the equation [54]. The material derivative operator is given by:

$$\frac{D}{D\tau} = \partial_\tau + \vec{v} \cdot \nabla.$$

For a brief description of the method, we first define the characteristics curve $X_e(\mathbf{x}, \bar{\tau}; s)$ through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, i.e. the integral path associated with the vector field $\vec{v}$ through $\mathbf{x}$, which verifies

$$\partial_s X_e(\mathbf{x}, \bar{\tau}; s) = \vec{v}(X_e(\mathbf{x}, \bar{\tau}; s)), \quad X_e(\mathbf{x}, \bar{\tau}; \bar{\tau}) = \mathbf{x}. \tag{2.37}$$

For $N > 1$ let us consider the time step $\Delta\tau = T_r/N$ and the time mesh-points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by the characteristics method is given by

$$\frac{DV}{D\tau} = \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau},$$

where $X_e^n(\mathbf{x}) := X_e(\mathbf{x}, \tau^{n+1}; \tau^n)$. In view of the expression of the velocity field and the continuous function g given by (2.33) and (2.4) respectively, the components of $X^n(\mathbf{x})$ can be analytically computed. More precisely, we distinguish the following two cases:

- if $\theta \neq \sigma^2$ then $[X_e^n]_1(\mathbf{x}) = x_1 \exp((\theta - \sigma^2)\Delta\tau)$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ \\ \dfrac{k_1 x_1}{\sigma^2 - \theta}\left(1 - \exp((\theta - \sigma^2)\Delta\tau)\right) + x_2 & \text{if} \quad n\Delta\tau \leq n_y; \end{cases}$$

- if $\theta = \sigma^2$ then $[X_e^n]_1(\mathbf{x}) = x_1$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ \\ k_1 x_1 \Delta\tau + x_2 & \text{if} \quad n\Delta\tau \leq n_y. \end{cases}$$

Next, we consider a Crank-Nicolson scheme around $(X_e(\mathbf{x},\tau^{n+1};\tau),\tau)$ for $\tau = \tau^{n+\frac{1}{2}}$. So, for $n = 0,\ldots,\ N-1$, the time discretized equation for the case without early retirement ($P = 0$) can be written as:

- *Find* V^{n+1} *such that*

$$\frac{V^{n+1}(\mathbf{x}) - V^n(X_e^n(\mathbf{x}))}{\Delta\tau} - \frac{1}{2}Div(A\nabla V^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla V^n)(X_e^n(\mathbf{x})) + \frac{1}{2}(l\,V^{n+1})(\mathbf{x}) + \frac{1}{2}(l\,V^n)(X_e^n(\mathbf{x})) = \frac{1}{2}f^{n+1}(\mathbf{x}) + \frac{1}{2}f^n(X_e^n(\mathbf{x})). \tag{2.38}$$

In the previous chapter an appropriate variational formulation for the previously time discretized problem has been stated. Moreover, for the spatial discretization a piecewise quadratic finite elements space can be used. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T,\mathcal{Q}_2,\Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T denotes the subset of nodes of the element T. More precisely, let us define the finite elements space V_h by

$$V_h = \{\phi_h \in \mathcal{C}^0(\overline{\Omega}) \mid \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{2.39}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$. Indeed, either Simpson or trapezoidal rules to approximate the different integral terms appearing in the fully discretized problem can be used.

So, if N_{dof} denotes the dimension of the finite elements space, in the case without early retirement we have to solve the linear system with N_{dof} unknowns

$$M_h V_h^n = b_h^{n-1}, \tag{2.40}$$

at each time step, where matrix M_h does not depend on time. Note that the dependence on time of the PDE operator is restricted to the velocity and this term has been discretized by characteristics method, which combined with the previously described Crank-Nicolson scheme leads to a time-independent matrix.

2.4.2 Augmented Lagrangian Active Set (ALAS) algorithm

The Augmented Lagrangian Active Set (ALAS) algorithm proposed in [36] is here applied to the fully discretized in time and space mixed formulation (2.28)-(2.29). More precisely, after this discretization the discrete problem can be written in the form:

$$M_h V_h^n + P_h^n = b_h^{n-1}, \tag{2.41}$$

with the discrete complementarity conditions

$$V_h^n \geq \overline{\Psi}_h^n, \quad P_h^n \leq 0, \quad \left(V_h^n - \overline{\Psi}_h^n\right) P_h^n = 0, \tag{2.42}$$

where P_h^n denotes the vector of the multiplier values and $\overline{\Psi}_h^n$ denotes the vector of the nodal values defined by function $\overline{\Psi}$.

The basic iteration of the ALAS algorithm consists of two steps. In the first one the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step, a reduced linear system associated with the inactive part is solved. Thus, we use the algorithm for unilateral problems, which are based on the augmented Lagrangian formulation.

First, for any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} := \{1, 2, \ldots N_{\mathrm{dof}}\}$, let us denote by $[M_h]_{\mathcal{I}\mathcal{I}}$ the principal minor of matrix M_h and by $[M_h]_{\mathcal{I}\mathcal{J}}$ the co-diagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, for each time t_n, the ALAS algorithm computes not only V_h^n and P_h^n but also a decomposition $N = \mathcal{J}^n \cup \mathcal{I}^n$ such that

$$\begin{aligned}
M_h V_h^n + P_h^n &= b_h^{n-1}, & & \\
[P_h^n]_j + \beta \left[V_h^n - \overline{\Psi}\right]_j &\leq 0, & \quad & \forall j \in \mathcal{J}^n, \\
[P_h^n]_i &= 0, & & \forall i \in \mathcal{I}^n,
\end{aligned} \tag{2.43}$$

for a given positive parameter β. In the above equations, $\mathcal{I}^n$ and $\mathcal{J}^n$ are the *inactive* and the *active* sets at time t_n respectively. More precisely, the iterative algorithm builds sequences $\left\{V_{h,m}^n\right\}_m$, $\left\{P_{h,m}^n\right\}_m$, $\{\mathcal{I}_m^n\}_m$ and $\{\mathcal{J}_m^n\}_m$, converging to V_h^n, P_h^n, $\mathcal{I}^n$ and $\mathcal{J}^n$, by means of the following steps:

1. Initialize $V^n_{h,0} = \overline{\Psi}^n_h$ and $P^n_{h,0} = \min\{b^n_h - M_h V^n_{h,0}, 0\} \leq 0$. Choose $\beta > 0$. Set $m = 0$.

2. Compute

$$\begin{aligned} Q^n_{h,m} &= \min\left\{0, P^n_{h,m} + \beta\left(V^n_{h,m} - \overline{\Psi}^n_{h,m}\right)\right\}, \\ \mathcal{J}^n_m &= \left\{j \in \mathcal{N}, \left[Q^n_{h,m}\right]_j < 0\right\}, \\ \mathcal{I}^n_m &= \{i \in \mathcal{N}, \left[Q^n_{h,m}\right]_i = 0\}. \end{aligned}$$

3. If $m \geq 1$ and $J^n_m = J^n_{m-1}$ then convergence is achieved. Stop.

4. Let V and P be the solution of the linear system

$$\begin{aligned} &M_h V + P = b^{n-1}_h, \\ &P = 0 \text{ on } \mathcal{I}^n_m \text{ and } V = \overline{\Psi}^n_{h,m} \text{ on } \mathcal{J}^n_m. \end{aligned} \tag{2.44}$$

   Set $V^n_{h,m+1} = V$, $P^n_{h,m+1} = \min\{0, P\}$, $m = m + 1$ and go to Step 2.

It is important to note that, instead of solving the full linear system in (4.52), for $\mathcal{I} = \mathcal{I}^n_m$ and $\mathcal{J} = \mathcal{J}^n_m$ the following reduced one on the inactive set is solved:

$$\begin{aligned} [M_h]_{\mathcal{I}\mathcal{I}}\,[V]_{\mathcal{I}} &= \left[b^{n-1}\right]_{\mathcal{I}} - [M_h]_{\mathcal{I}\mathcal{J}}\left[\overline{\Psi}\right]_{\mathcal{J}}, \\ [V]_{\mathcal{J}} &= \left[\overline{\Psi}\right]_{\mathcal{J}}, \\ P &= b^{n-1} - M_h V. \end{aligned} \tag{2.45}$$

In [36], it is proved the convergence of the algorithm in a finite number of steps for a Stieltjes matrix (i.e. a real symmetric positive definite matrix with negative off-diagonal entries, cf. [64]) and a suitable initialization (the same we consider in this paper). They also proved that $\mathcal{I}_m \subset \mathcal{I}_{m+1}$. Nevertheless, a Stieltjes matrix can be only obtained for linear elements but never for the here used quadratic elements because we have some positive off-diagonal entries coming from the stiffness matrix (actually we use a lumped mass matrix). However, we have obtained good results by using ALAS algorithm with quadratic finite elements.

2.5 Numerical results

In this section we present some numerical results to illustrate the performance of the proposed numerical methods. For this purpose, we have considered the following model parameters:

$$\sigma = 0.1,\ \theta = 0.025,\ r = 0.025,\ a = 0.75,\ T_r = 40,\ T_0 = 15,$$
$$n_y = 30,\ k_1 = 0.5,\ \mu_d = 0.025,\ \mu_w = 0.2,\ \alpha_d = 1,\ \alpha_w = 0.$$

The bounded computational domain is defined by the values $x_1^\infty = 40$ and $x_2^\infty = 40$. Concerning the quadrangular finite element meshes data, in Table 2.1 the number of nodes and elements of the referred meshes in forthcoming tables and figures are shown. Moreover, we set $\beta = 10000$ in the ALAS algorithm.

	N. Elem	**N. Nodes**
Mesh 12	144	625
Mesh 24	576	2401
Mesh 48	2304	9409
Mesh 96	9216	37249

Table 2.1: FEM meshes data

Also different numbers of time steps have been considered in different tests, thus for given values of S and I, Table 2.2 illustrates the convergence of the proposed numerical methods as soon as the mesh is refined in time and space for $t = 38$ and $(S, I) = (25, 20)$. Note that for all times the point $(S, I) = (25, 20)$ is located at the region where early retirement is not optimal. For the same point, in Tables 2.3 and 2.4 the results for $t = T_r - n_y = 10$ and $t = 0$, respectively, also illustrate the convergence with mesh refinement in time and space.

The results obtained by the numerical solution of the PDE formulation have been compared with those ones obtained by a Monte Carlo simulation based technique. Among the different possible alternatives, the technique proposed by Longstaff and

Schwartz has been adapted (see [42], for details). The Longstaff-Schwartz(LS) method is mainly based on the construction of a regression of the value function. As indicated in [31], for example, the L-S technique results to be one of the most competitive. We also acknowledge the interest of the technique proposed in [31], which is based on the previous computation of the early retirement boundary. However, the computational cost turn to be very large when dealing with long term products as the here treated pension plans. For the computation of 99% confidence interval associated with the LS technique, 250 time steps per year and 50000 paths have been considered. A quadratic regression polynomial has been used. In all forthcoming Tables, the computed results show that the numerical solution of the PDE model belongs to the confidence interval associated with the LS simulation technique. The corresponding confidence intervals are indicated in the caption of Tables 2.2 to 2.4

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	1.703567	1.703345	1.703326	1.703321
2500	1.703773	1.703519	1.703491	1.703488
5000	1.703871	1.703615	1.703581	1.703577
10000	1.703923	1.703663	1.703626	1.703622

Table 2.2: Retirement benefits at time $t = 38$ at the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [1.699157, 1.704596].

Figure 2.1 displays the computed retirement benefits at $t = 38$ for 10000 time steps and Mesh 96 from Table 2.1, the information about the numerical solution is completed with Figure 2.2 showing the multiplier. Thus, the region where the multiplier is negative coincides with the region where is optimal to exercise the early retirement option. Moreover, Figure 2.3 shows the early retirement (coincidence) region in red and the non early retirement (non coincidence) region in blue, the curve separating both region being the optimal retirement boundary (free boundary). Note that for a given large enough value of the average salary, there exists a critical

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.857282	2.857427	2.857433	2.857441
2500	2.858981	2.859121	2.859124	2.859123
5000	2.859983	2.860038	2.860039	2.860038
10000	2.860583	2.860498	2.860497	2.860496

Table 2.3: Retirement benefits at time $t = 10$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.847081, 2.878122].

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.800543	2.800665	2.800674	2.800683
2500	2.800349	2.800472	2.800479	2.800481
5000	2.800288	2.800366	2.800373	2.800374
10000	2.800295	2.800314	2.800319	2.800321

Table 2.4: Retirement benefits at time $t = 0$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.700476, 2.80057].

salary below which the optimal strategy is early retirement. Figure 2.4 shows the computed retirement benefits at $t = T_r - n_y = 10$ for 10000 time steps and Mesh 96, as expected the early retirement region is empty and we have obtained that the computed multiplier is equal to zero.

The evolution in time of the optimal retirement boundary is depicted in Figure 2.5. More precisely, the critical salary for early retirement is represented as a function $\bar{S} = \bar{S}(t, I)$ of time t and the cumulative salary I. The points placed below the graph of the function correspond to early retirement region while those ones above the graph represent the no early retirement region. The figure also illustrate the treatment of the condition that before $T_0 = 15$ early retirement is not allowed (a flat region defined by S=0 is observed).

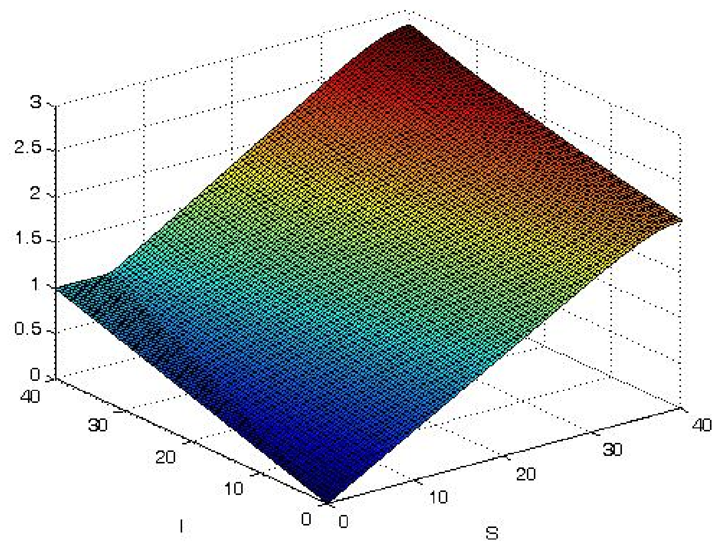

Figure 2.1: Retirement benefits at time $t = 38$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

Table 2.5 shows some numerical results obtained for the pension plan benefits V and the multiplier P. More precisely, for different values of (S, I), first rows show the computed values of V with Monte Carlo simulation (MC) and PDE model (PDE) when early retirement of the pension member is not allowed. Forth and fifth rows indicate the value of V with Longstaff-Schwartz (LS) and PDE when early retirement is allowed. Las row shows the computed multiplier P in the PDE numerical solution. As expected, the value of V is always greater in case of allowing early retirement. Also note that only the point $(S, I) = (4, 10)$ belongs to the region where it is optimal not to retire as indicated by the null value of the multiplier. The other points belong to the early retirement region, as illustrated by the fact that the value of pension plan matches the early retirement value and the no zero value of the multiplier.

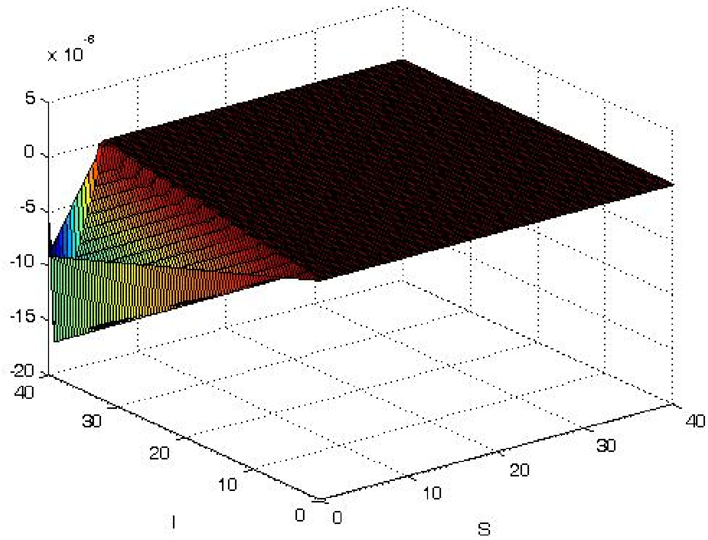

Figure 2.2: Multiplier value at time $t = 38$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

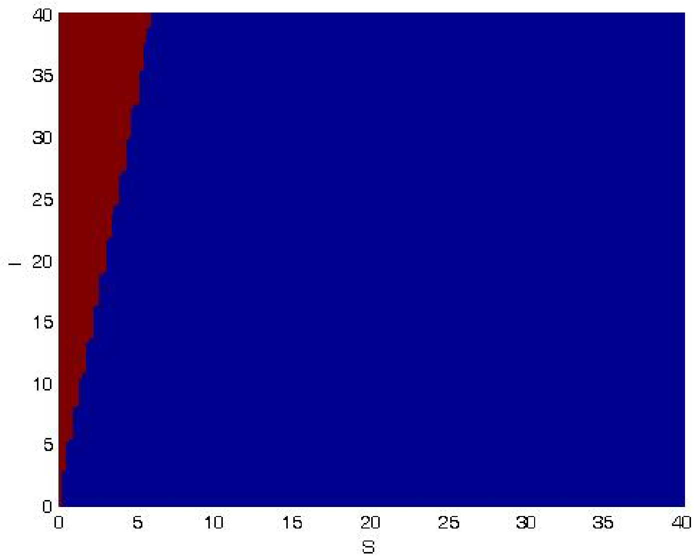

Figure 2.3: Free boundary at time $t = 38$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

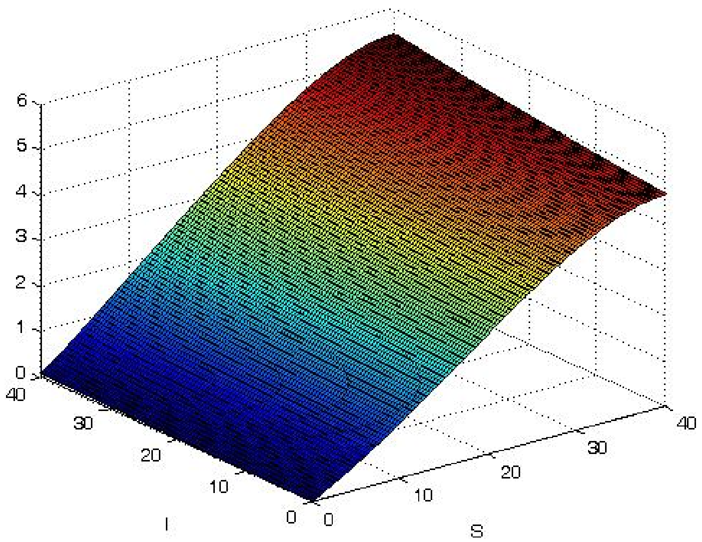

Figure 2.4: Retirement benefits at time $t = 10$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

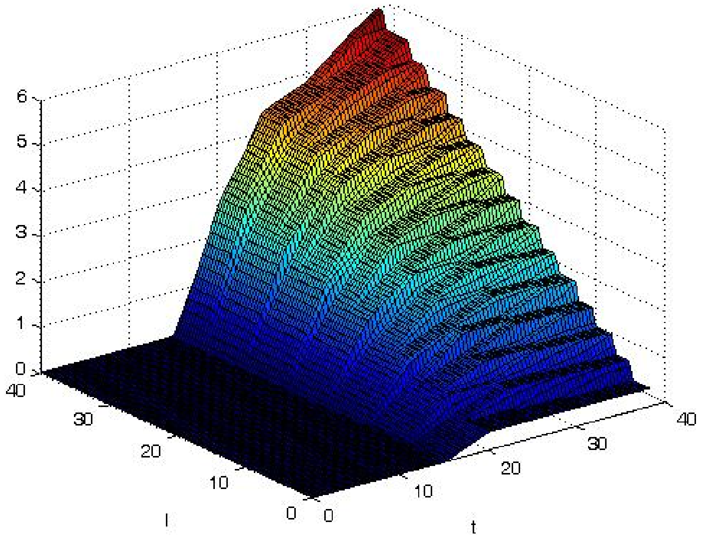

Figure 2.5: Optimal retirement boundary $S = \overline{S}(t, \overline{I}(t))$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$	$(S,I)=(4,10)$
V without ER(MC)	(0.294336, 0.294456)	(0.408061, 0.408180)	(0.588672, 0.588912)	(0.374293, 0.375106)
V without ER(PDE)	0.29442368	0.40814817	0.58884736	0.37488062
V with ER(LS)	(0.369643, 0.369643)	(0.554464, 0.554464)	(0.739286, 0.739286)	(0.374518, 0.374919)
V with ER(PDE)	0.36964285	0.55446428	0.73928571	0.37488181
P with ER(PDE)	-3.4539×10^{-6}	-6.9085×10^{-6}	-6.9078×10^{-6}	0

Table 2.5: Retirement benefits without and with early retirement (ER) at different points when time $t = 38$ and the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. Computed values with (PDE), Monte Carlo (MC) and Longstaff-Schwartz (LS) and Multiplier (P).

Chapter 3

Pricing pension plans under jump-diffusion models

3.1 Introduction

Sometimes the diffusion processes proposed by Black and Scholes in [11] or by Merton in [45] do not fit properly market data or certain situations, such as an abrupt change in the value of the underlying. However, jump-diffusion models later proposed by Merton in [46] and Kou [39] result to be more appropriate to describe these situations.

Following this idea, in this chapter we consider the possibility of jumps in the underlying salary. Actually, we assume that the salary dynamics can be modelled by means of a Merton jump-diffusion process. In this new setting, the value of the pension plan can be obtained as the solution of an initial boundary value problem if early retirement is not allowed or as the solution of a complementarity problem when there exists the early retirement opportunity. Due to the presence of jumps, both problems are related to an integro-differential operator of parabolic type.

There are several papers dealing with the numerical valuation of different financial derivatives when the underlying follows a jump-diffusion processes. For example, in [21] an implicit method is developed for the numerical solution of option pricing models where jumps in the underlying following a Merton model are assumed and where the integral term is computed using a fast Fourier transform (FFT) method. The application of this scheme for the valuation of American Asian Options under jump diffusion is addressed in [20]. In [1] the value of a European option with jumps in the underlying is obtained by solving Merton and Kou jump-diffusion models. We notice that when using an implicit scheme, the integral term leads to a dense matrix and efficient algorithms are required to solve the dense system, as the one recently implemented in [58] for treating the complementarity problem related to American options, also under both jump-diffusion models.

In order to solve the partial integro-differential equation (PIDE) that arises in the presence of jumps in the salary, we propose a Lagrange-Galerkin discretization for the time and space discretizations, combined with an Augmented Lagrangian Active Set (ALAS) algorithm for solving the inequalities associated with the free boundary problem and with the explicit treatment of the integral term [19]. This explicit scheme for the integral term maintains the same matrix as in the pure diffusion case and modifies the second member of the linear system associated with the discretized problem. The results from the application of these numerical methods are compared with the ones obtained by implementing the Monte Carlo simulation (see [27], for example) for the case without early retirement and the Longstaff-Schwartz technique proposed in [42] for the case with early retirement.

This chapter is organized as follows. In Section 2 a jump-diffusion model for pension plans is posed. In Section 3 appropriate numerical methods are applied to find a solution to the PIDE problem that arises in the presence of jumps and Monte Carlo simulation techniques for jumps are proposed. Finally, in Section 4 the obtained numerical results obtained by using both procedures are presented.

3.2 Processes with jumps

3.2.1 Poisson process

The Poisson process is a stochastic process with discontinuous paths and it is used to construct more complex jump processes (see [52] or [19], for example). For this purpose, we consider a sequence $(\tau_n)_{n\geq 1}$ of independent random variables with a exponential distribution, with parameter $\widetilde{\lambda} > 0$, that is:

$$\tau_n \sim Exp_{\widetilde{\lambda}}, \quad n \geq 1.$$

We consider a model where jumps occur randomly and τ_n denotes the time distance between the n-th jump and the previous one. So, for any n $\in \mathbb{N}$,

$$T_n = \sum_{k=1}^{n} \tau_k \tag{3.1}$$

denotes the time of the n-th jump. We remark that

$$E[T_n - T_{n-1}] = E[\tau_n] = \frac{1}{\widetilde{\lambda}}, \quad n \in \mathbb{N}$$

where the intensity parameter $\widetilde{\lambda}$ is the number of expected jumps in a unit time interval. Taking into account the previous notation we introduce the concepts of Poisson and compound Poisson processes.

Definition 3.2.1. *A Poisson process with intensity $\widetilde{\lambda}$ is the process*

$$N_t = \sum_{n\geq 1} n\mathbf{1}_{[T_n,T_{n+1})}(t), \quad t \in \mathbb{R}_{\geq 0},$$

with T_n defined as in (3.1).

The Poisson process N_t counts the number of jumps (only non-negative integer values) between 0 and t. By usual convention the trajectories of N are continuous from the right:

$$N_t = N_{t+} = \lim_{s\,\downarrow\, t} N_s, \quad t \geq 0$$

Proposition 3.2.1. *Let $(N_t)_{t\geq 0}$ be a Poisson process. Then, the following properties are satisfied:*

1. *For any $t > 0$, N_t is almost surely finite.*

2. *For any ω, the sample path $t \to N_t(\omega)$ is piecewise constant and increases by jumps of size 1.*

3. *The sample paths $t \to N_t(\omega)$ are right continuous with finite left limits (cadlag).*

4. *For any $t > 0$, $N_{t-} = N_t$ with probability 1.*

5. *(N_t) is continuous in probability:*

$$\forall t > 0, \quad N_s \underset{s \to t}{\overset{\mathbb{P}}{\to}} N_t$$

6. *For any $t > 0$, N_t follows a Poisson distribution with parameter $\widetilde{\lambda} t$*

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(N_t = n) = e^{(-\widetilde{\lambda} t)} \frac{(\widetilde{\lambda} t)^n}{n!}.$$

7. *The characteristic function of N_t is given by*

$$E[e^{iuN_t}] = exp\{\widetilde{\lambda} t(e^{iu} - 1)\}, \quad \forall u \in \mathbb{R}.$$

8. *(N_t) has independent increments: for any $t_1 < ... < t_n, N_{t_n} - N_{t_{n-1}}, ..., N_{t_2} - N_{t_1}, N_{t_1}$ are independent random variables.*

9. *The increments of N are homogeneous: for any $t > s$, $N_t - N_s$ has the same distribution as N_{t-s}.*

10. *(N_t) has the Markov property:*

$$\forall t > s, \quad E[f(N_t)|N_u, u \leq s] = E[f(N_t)|N_s].$$

All these properties have been proved in [19].

Definition 3.2.2. *Let N be a Poisson process with intensity $\widetilde{\lambda}$ and assume that $Y = (Y_i)$ is a sequence of independent and identically distributed random variables in $\mathbb{R}^d$ with distribution η, i.e. $Y_i \sim \eta$ for $i \geq 1$, and which are independent of N. The compound Poisson process is defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i \quad t \geq 0. \tag{3.2}$$

From the definition of compound Poisson process we can easily deduce that:

1. The sample paths of X are cadlag piecewise constant functions.

2. The jump times $(T_i)_{i\geq 1}$ can be expressed as partial sums of independent exponential random variables with parameter $\widetilde{\lambda}$.

3. The jump sizes $(Y_i)_{i\geq 1}$ are independent and identically distributed with law η.

3.2.2 A jump-diffusion model for pension plans

As we have pointed out in Chapter 1, in some cases of abrupt changes in the salary, Brownian motion is not appropriate enough to describe the evolution of this underlying factor and it is necessary to adopt a jump-diffusion model. More precisely, we assume that the salary follows the following stochastic differential equation:

$$dS_t \;=\; \alpha(t, S_t)\, dt \;+\; \sigma(t, S_t)\, dZ_t \;+\; d\left(\sum_{i=1}^{N_t} Y_i\right), \tag{3.3}$$

where $(N_t)_{t\geq 0}$ denotes a Poisson process with parameter $\widetilde{\lambda}$ and (Y_i) is a sequence of square integrable, independent and identically distributed random variables, so that Z_t, N_t and (Y_i) are independent. Moreover, in order to completely define the model, we must also specify the distribution of jump sizes $\nu(y)$. For this purpose, we will consider Merton model [46], so that (Y_i) are taken from the log-normal distribution $(LN(\mu, \gamma^2))$, with the density

$$\nu(y) = \frac{1}{y\gamma\sqrt{2\pi}} exp\left(-\frac{(\log y - \mu)^2}{2\gamma^2}\right), \tag{3.4}$$

where μ is the mean jump size and γ is the standard deviation of the jump size. There are other possibilities that could be taken into account, such as Kou model [39], in which case the set (Y_i) exhibits the log-double-exponential density

$$\nu(y) = \begin{cases} q\alpha_2 y^{\alpha_2-1}, & y < 1 \\ p\alpha_1 y^{-\alpha_1-1}, & y \geq 1, \end{cases} \tag{3.5}$$

where p, q, α_1 and α_2 are positive constants such that $p+q=1$ and $\alpha_1 > 1$.

Partial integral differential equation (PIDE)

In Chapter 1, a PDE model for pricing pension plans without early retirement in the absence of jumps in the salary is posed by using a dynamic hedging technique. In the case of a jump-diffusion process for the salary, if we denote the value of the pension plan by $V(t, S_t, I_t)$, then the function V solves the following partial integral differential equation (PIDE):

$$\begin{aligned} &\partial_t V + \beta\,\partial_S V + g\,\partial_I V + \tfrac{1}{2}\sigma^2\,\partial_{SS} V - \,(\mu_d + \mu_w + r)\,V \\ &+ \int_0^\infty \widetilde{\lambda}\,[V(t, Sy, I) - V(t, S, I) - S(y-1)\partial_S V(t, S, I)]\,\nu(y)dy \\ &\qquad = -\mu_d A_d - \mu_w A_w, \end{aligned} \tag{3.6}$$

posed in the unbounded domain $(0, T_r) \times \mathbb{R}_+^2$ and with the same final condition considered in Chapter 1:

$$V(T_r, S, I) = \frac{a}{n_y}\,I, \tag{3.7}$$

where $a \in (0, 1)$ is a given constant.

As we pointed out before, $\nu(y)$ denotes the probability density function of the jump amplitude Y_i. Thus, we have

$$\int_0^\infty \nu(y)dy = 1.$$

Moreover, as we consider the Merton model probability density given by (3.4), we can obtain the expectation

$$E[Y_i] = \int_0^\infty y\nu(y)dy = e^{\mu+\gamma^2/2}.$$

Therefore, PIDE (3.6) can be written as

$$\partial_t V + (\beta - \widetilde{\lambda}\kappa S)\,\partial_S V + g\,\partial_I V + \tfrac{1}{2}\sigma^2\,\partial_{SS} V - \left(\mu_d + \mu_w + r + \widetilde{\lambda}\right) V$$
$$+\widetilde{\lambda}\int_0^\infty V(t, Sy, I)\nu(y)dy = -\mu_d A_d - \mu_w A_w, \tag{3.8}$$

where $\kappa = e^{\mu+\gamma^2/2} - 1$. Note that there is a new integral term in the equation due to the presence of jumps. This term makes the PIDE more difficult to solve that the corresponding PDE. In a forthcoming section we show how to discretize this integral in order to find a numerical solution of the PIDE problem.

As in the absence of jumps in the salary, in what follows we assume that $\beta(t, S, I) = \theta S$ and $\sigma(t, S) = \sigma S$.

Next, we introduce the integro-differential operator

$$\begin{aligned}\mathcal{L}_j V =& \partial_t V + (\theta - \widetilde{\lambda}\kappa)S\partial_S V + g\partial_I V + \frac{\sigma^2 S^2}{2}\partial_{SS} V - \left(r + \mu_d + \mu_w + \widetilde{\lambda}\right) V\\ &+ \widetilde{\lambda}\int_0^\infty V(t, Sy, I)\nu(y)dy,\end{aligned}$$

so that for the case without early retirement we will consider the PIDE problem

$$\begin{cases} \mathcal{L}_j V = f, & \text{in } (0, T_r)\times\mathbb{R}_+^2, \\ V(T_r, S, I) = \dfrac{a}{n_y} I, & (S, I) \in \mathbb{R}_+^2, \end{cases} \tag{3.9}$$

while if we incorporate the possibility of early retirement, under a jump-diffusion process for the salary, the complementarity problem (2.9) turns out to be

$$\begin{cases} \max\{\mathcal{L}_j V - f, \Psi - V\} = 0, & \text{in } (0, T_r)\times\mathbb{R}_+^2, \\ V(T_r, S, I) = \dfrac{a}{n_y} I, & (S, I) \in \mathbb{R}_+^2, \end{cases} \tag{3.10}$$

where

$$f(t, S, I) = -(\mu_d\alpha_d + \mu_w\alpha_w)S$$

and the expression for the obstacle Ψ is given by equation (2.8).

3.3 Numerical solution

In Chapter 1, we propose a Lagrange-Galerkin method to discretize the PDE model that appears when the valuation of pension plans based on average salary without early retirement is considered. Then, we include the early retirement option and, in Chapter 2, this numerical method is combined with an augmented Lagrangian active set technique. In this chapter, we extend these numerical techniques including the explicit computation of the integral term in order to solve the PIDE model associated with the jump-diffusion process. The results obtained with these numerical methods are compared with the ones obtained by using Monte Carlo techniques.

The PIDE is initially posed also on an unbounded domain. So, as in the case without jumps, we approximate it by a bounded doamain formulation and also introduce boundary conditions. Note that in the PIDE case, the domain of integration in the integral term also need to be localized. For this purpose, we introduce the following changes in time and space variables and notations

$$\tau = T_r - t, \quad x_1 = S, \quad x_2 = I, \quad \bar{x}_1 = log(x_1), \quad \eta = log(y). \tag{3.11}$$

Let us keep considering x_1^∞ and x_2^∞ be large enough real numbers and let $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ be the computational bounded domain.

Moreover, let $\partial\Omega = \bigcup_{i=1}^{2}(\Gamma_i^- \bigcup \Gamma_i^+)$, with $\Gamma_i^- = \{(x_1, x_2) \in \partial\Omega \mid x_i = 0\}$ and $\Gamma_i^+ = \{(x_1, x_2) \in \partial\Omega \mid x_i = x_i^\infty\}$ for $i = 1, 2$.

Then, we write problems (3.9) and (3.10) in the bounded domain. In the case of the complementarity problem, in order to apply the iterative algorithm we propose, we replace the inequality by an identity and we incorporate the Lagrange multiplier P as a new unknown of the problem, identically to the case with no jumps.

Thus, the equivalent problem in the bounded domain for the case without early retirement is: Find $V : [0, T_r] \times \Omega \to \mathbb{R}$, such that

$$\begin{gathered} \partial_\tau V - Div(A\nabla V) + \vec{v} \cdot \nabla V + lV - \\ \widetilde{\lambda} \int_{\eta min}^{\eta max} \bar{V}(\tau, \bar{x}_1 + \eta, x_2)\bar{\nu}(\eta) d\eta = f \quad \text{in } (0, T_r) \times \Omega, \end{gathered} \tag{3.12}$$

while for the case with early retirement it can be written as:
Find V and $P : [0, T_r] \times \Omega \longrightarrow R$, satisfying the partial differential equation

$$\begin{aligned} &\partial_\tau V - Div\,(A\nabla V) + \vec{v}\cdot\nabla V + lV - \\ \widetilde{\lambda}\int_{\eta min}^{\eta max} &\bar{V}(\tau, \bar{x}_1+\eta, x_2)\bar{\nu}(\eta)d\eta + P = f \quad \text{in } (0,T_r)\times\Omega \end{aligned} \tag{3.13}$$

the complementarity conditions

$$V \geq \overline{\Psi}, \quad P \leq 0, \quad (V - \overline{\Psi})P = 0 \quad \text{in } (0, T_r) \times \Omega. \tag{3.14}$$

Remark 3.3.1. *Note that the differential term of the PIDE is computed in the domain* $[0, x_1^\infty] \times [0, x_2^\infty]$*, using the discrete grid* $0 = x_{1_0}, x_{1_1}, \cdots, x_{1_q} = x_1^\infty$.
Since $log(x_{1_0}) = -\infty$*, we choose* $\eta min = log(x_{1_1})$ *and* $\eta max = log(x_{1_q})$ *as it is proposed in [21].*

In both cases, the function $\bar{\nu}$ is given by

$$\bar{\nu}(\eta) = \frac{1}{\gamma\sqrt{2\pi}} exp\left(-\frac{(\eta-\mu)^2}{2\gamma^2}\right), \tag{3.15}$$

the initial and boundary conditions are

$$V(0, x_1, x_2) = \frac{a}{n_y} x_2 \quad \text{in} \quad \Omega, \tag{3.16}$$

$$\frac{\partial V}{\partial x_1} = g_1 \quad \text{on } (0, T_r) \times \Gamma_1^+, \tag{3.17}$$

$$\frac{\partial V}{\partial x_2} = g_2 \quad \text{on } (0, T_r) \times \Gamma_2^+ \tag{3.18}$$

and the involved data in both cases is defined as follows

$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}(\tau, x_1, x_2) = \begin{pmatrix} (\sigma^2 - \theta + \widetilde{\lambda}\kappa)x_1 \\ -g(T_r - \tau, x_1) \end{pmatrix}, \tag{3.19}$$

$$l(\tau, x_1, x_2) = r + \mu_d + \mu_w + \widetilde{\lambda}, \quad f(\tau, x_1, x_2) = (\mu_d\alpha_d + \mu_w\alpha_w)x_1, \tag{3.20}$$

$$\overline{\Psi}(\tau, x_1, x_2) = \Psi(T_r - \tau, x_1, x_2), \quad g_1(\tau, x_1, x_2) = 0, \tag{3.21}$$

$$g_2(\tau, x_1, x_2) = \frac{a}{n_y}. \tag{3.22}$$

3.3.1 Time discretization

In order to discretize the material derivative we apply the method of characteristics. Thus, a similar way to the problem without jumps, we first define the characteristics curve $X_e(\mathbf{x}, \bar{\tau}; s)$ through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, i.e. the integral path associated with the vector field $\vec{v}$ through $\mathbf{x}$, which verifies

$$\partial_s X_e(\mathbf{x}, \bar{\tau}; s) = \vec{v}(X_e(\mathbf{x}, \bar{\tau}; s)), \quad X_e(\mathbf{x}, \bar{\tau}; \bar{\tau}) = \mathbf{x}. \tag{3.23}$$

For $N > 1$ let us consider the time step $\Delta\tau = T_r/N$ and the time mesh-points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by the characteristics method is given by

$$\frac{DV}{D\tau} = \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau},$$

where $X_e^n(\mathbf{x}) := X_e(\mathbf{x}, \tau^{n+1}; \tau^n)$. In view of the expression of the velocity field and the continuous function g given by (3.19) and (1.6) respectively, the components of $X_e^n(\mathbf{x})$ can be analytically computed. More precisely, we distinguish the following two cases:

- if $(\sigma^2 - \theta + \widetilde{\lambda}\kappa) \neq 0$ then $[X^n]_1(\mathbf{x}) = x_1 \exp((\theta - \sigma^2 - \widetilde{\lambda}\kappa)\Delta\tau)$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ \dfrac{k_1 x_1}{\sigma^2 - \theta + \widetilde{\lambda}\kappa}\Big(1 - \exp((\theta - \sigma^2 - \widetilde{\lambda}\kappa)\Delta\tau)\Big) + x_2 & \text{if} \quad n\Delta\tau \leq n_y; \end{cases}$$

- if $(\sigma^2 - \theta + \widetilde{\lambda}\kappa) = 0$ then $[X^n]_1(\mathbf{x}) = x_1$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ k_1 x_1 \Delta\tau + x_2 & \text{if} \quad n\Delta\tau \leq n_y. \end{cases}$$

Next, we consider a Crank-Nicolson scheme around $(X(\mathbf{x}, \tau^{n+1}; \tau), \tau)$ for $\tau = \tau^{n+\frac{1}{2}}$ in order to discretize the differential term, while the integral term is computed explicitly in time by using the solution obtained at the preceding iteration. In this way, the system matrix results to be the same as in the case without jumps (block pentadiagonal for piecewise quadratic finite elements).

So, for $n = 0, \ldots, N-1$, the time discretized equation for the case without early retirement ($P = 0$) can be written as:

- *Find* V^{n+1} *such that*

$$\frac{V^{n+1}(\mathbf{x}) - V^n(X_e^n(\mathbf{x}))}{\Delta\tau} - \frac{1}{2}Div(A\nabla V^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla V^n)(X_e^n(\mathbf{x})) + \frac{1}{2}(l\,V^{n+1})(\mathbf{x}) + \frac{1}{2}(l\,V^n)(X^n(\mathbf{x})) - \widetilde{\lambda}\int_{\eta min}^{\eta max} \bar{V}^n(\bar{x}_1+\eta, x_2)\bar{\nu}(\eta)d\eta = \frac{1}{2}f^{n+1}(\mathbf{x}) + \frac{1}{2}f^n(X_e^n(\mathbf{x})).$$

Remark 3.3.2. *Note that we have a grid in the spatial coordinates* (x_1, x_2) *and, at each time step* $n+1$*, we have previously computed at time step* n *the finite element approximation of* V *at the grid points of the spatial mesh. So, in order to approximate the value of* $\bar{V}^n(\bar{x}_1+\eta, x_2) = V^n(e^{\bar{x}_1+\eta}, x_2)$ *we perform a quadratic interpolation procedure.*

In order to apply finite elements, first we obtain the variational formulation of the semidiscretized problem in a similar way to the case without jumps: we multiply equation (3.24) by a suitable test function, integrate in Ω, use the classical Green formula and the following one (see Lemma 3.4 in [48]):

$$\begin{aligned}\int_\Omega Div(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} \;&=\; \int_\Gamma (\mathbf{F}_e^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}\\ &\quad - \int_\Omega (\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\cdot\nabla\Psi(\mathbf{x})d\mathbf{x}\\ &\quad - \int_\Omega Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}.\end{aligned} \tag{3.24}$$

Note that in the present case we have

$$\int_\Omega Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} = 0.$$

After these steps, we can write a variational formulation for the semidiscretized problem as follows:

$$
\begin{aligned}
&\text{Find } V^{n+1} \in H^1(\Omega) \text{ such that, } \forall \Psi \in H^1(\Omega) :\\
&\int_{\Omega} V^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega}(\mathbf{A}\nabla V^{n+1})(\mathbf{x})\nabla\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega} lV^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} =\\
&\int_{\Omega} V^{n}(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_{\Omega}(\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^{n})(X_e^n(\mathbf{x}))\nabla\Psi(\mathbf{x})d\mathbf{x} -\\
&\frac{\Delta\tau}{2}\int_{\Omega} lV^{n}(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Gamma}\widetilde{g}^n(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \frac{\Delta\tau}{2}\int_{\Gamma_{1,+}}\bar{g}_1{}^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} +\\
&\frac{\Delta\tau}{2}\int_{\Omega} f^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega} f^{n}(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} +\\
&\Delta\tau\widetilde{\lambda}\int_{\Omega}\int_{\eta min}^{\eta max}\bar{V}^n(\bar{x}_1+\eta, x_2)\bar{\nu}(\eta)d\eta\Psi(\mathbf{x})d\mathbf{x}\,,
\end{aligned}
\tag{3.25}
$$

where $\mathbf{F}_e^n = \nabla X_e^n$ can be analytically computed, $\bar{g}_1^{n+1}(\mathbf{x}) = g_1^{n+1}(\mathbf{x})a_{11}(\mathbf{x}) = 0$ and

$$
\widetilde{g}^n(\mathbf{x}) = \begin{cases}
0 & on \quad \Gamma_{1,-}\\[2ex]
-\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & on \quad \Gamma_{2,-}\\[2ex]
\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & on \quad \Gamma_{2,+}\\[2ex]
\left[(\mathbf{F}_e^n)^{-T}\right]_{11}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))g_1^n(X_e^n(\mathbf{x})) & on \quad \Gamma_{1,+}
\end{cases}
\tag{3.26}
$$

3.3.2 Finite elements discretization

As we have already mention, we use finite elements for space discretization. For this purpose and in a similar way to the case without jumps, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. More precisely, let us define the finite elements space V_h by

$$
V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) : \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{3.27}
$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

3.3.3 Augmented Lagrangian Active Set (ALAS) algorithm

In the presence of jumps and in order to treat the non-linearities associated with the inequality constraints in the complementarity formulation when early retirement is allowed, we implement, as in the case without jumps, the ALAS algorithm proposed in [36] and explained in section 2.4.2.

3.3.4 Numerical integration of the integral term

In order to approximate the additional integral term arising in the PIDE associated with the jump-diffusion model, we propose the use of a classical composite trapezoidal rule with $m+1$ points in the following way:

$$\begin{aligned}&\int_{\eta min}^{\eta max} \bar{V}^n(\bar{x}_1+\eta, x_2)\bar{\nu}(\eta)d\eta \approx \tfrac{h}{2}\Big[\bar{V}^n(\bar{x}_1+\eta min, x_2)\bar{\nu}(\eta min)\\ &+\bar{V}^n(\bar{x}_1+\eta max, x_2)\bar{\nu}(\eta max) + 2\textstyle\sum_{j=1}^{m-1}\bar{V}^n(\bar{x}_1+k_j, x_2)\bar{\nu}(k_j)\Big]\end{aligned} \tag{3.28}$$

where $k_j = \eta min + jh$ for $j = 1, ..., m-1$ and $h = \frac{\eta max - \eta min}{m}$.

3.3.5 Monte Carlo

In order to describe a Monte Carlo simulation technique in the context of jump-diffusion processes in this section, we first describe how to simulate the underlying salary with jumps (see [27], for example). Once we obtain the salary, Monte Carlo method is applied in a similar way to the case without jumps. For the case with early retirement opportunity, a Longstaff-Schwartz [42] algorithm is implemented analogously to the case without jumps too.

In the absence of jumps, note that the evolution of the salary under the risk neutral measure Q was given by expression (1.68).

If $\kappa = E[Y_i] - 1 = e^{\mu+\gamma^2/2} - 1$ then the process

$$\sum_{i=1}^{N_t} Y_i - \widetilde{\lambda}\kappa t$$

is a martingale. Having this previous property in view and assuming Merton model for jumps, the stochastic differential equation that describes the risk neutral dynamics of the salary can be written in the form

$$dS_t \;=\; \hat{\beta}(t, S_t)dt \;+\; \sigma S_t dZ_t^Q \;+\; d\left(\sum_{i=1}^{N_t} Y_i\right), \tag{3.29}$$

where Z^Q denotes a Wiener process under this measure and $\hat{\beta}(t, S) = (\theta - \widetilde{\lambda}\kappa)S$. In the case without early retirement the value of the plan is given by the expectation (1.69), while when early retirement is allowed the value of the plan is written in probabilistic terms as the Snell envelope (2.10).

3.3.6 Simulating at fixed dates

In order to simulate the jump-diffusion model for the salary, we mainly follow one of the approaches proposed in [27]. More precisely, we simulate the process at a fixed set of dates $0 = t_0 < t_1 < ... < t_m = T_r$ without explicitly distinguishing the effect of the jump and diffusion terms. Thus, we denote $S(t) = S_t$ just to accomodate a bit the readership of the algorithm and simulate $S(t)$ at the times $t_1, \ldots, t_m$ in the following way:

$$S(t_{j+1}) = S(t_j)\exp([\theta - \frac{1}{2}\sigma^2 - \widetilde{\lambda}\kappa](t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j}W_{j+1}) \prod_{i=N(t_j)+1}^{N(t_{j+1})} Y_i, \tag{3.30}$$

where $W_1, ..., W_m$ are independent standard normal random variables. By convention, the product over i is equal to 1 if $N(t_{j+1}) = N(t_j)$. Instead of directly simulating $S(t)$ from the previous representation, we introduce the logarithmic salary $X(t) = log(S(t))$ and implement

$$X(t_{j+1}) = X(t_j) + (\theta - \frac{1}{2}\sigma^2 - \widetilde{\lambda}\kappa)(t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j}W_{j+1}) \sum_{i=N(t_j)+1}^{N(t_{j+1})} logY_i, \tag{3.31}$$

We point out that with this change of variable we replace products with sums and now we have to sample $log(Y_i)$ which is faster than sampling Y_i. Then, we can exponentiate the simulated values of the $X(t_i)$ to produce samples of the $S(t_i)$.

A sketch of the algorithm for implementing (3.30) from t_j to t_{j+1} comprises the following steps:

1. Generate $W \sim N(0,1)$

2. Generate $N \sim Poisson(\widetilde{\lambda}(t_{j+1}-t_j))$; if $N=0$, set $M=0$ and go to step 4

3. Assuming $logY_i \sim N(\mu,\gamma^2)$ and $\sum_{i=1}^{n} logY_i \sim N(\mu n, \gamma^2 n) = \mu n + \gamma\sqrt{n}N(0,1)$, then, generate $W_2 \sim N(0,1)$; set $M = \mu N + \gamma\sqrt{N}W_2$

4. Set

$$X(t_{j+1}) = X(t_j) + (\theta - \frac{1}{2}\sigma^2 - \widetilde{\lambda}\kappa)(t_{j+1}-t_j) + \sigma\sqrt{t_{j+1}-t_j}W + M.$$

3.4 Numerical results

In this section, some numerical results corresponding to a jump-diffusion model for the salary are shown.

First, we consider a pension plan without the early retirement option. In this case, we present the results obtained by solving the PIDE model and by implementing the Monte Carlo simulation technique based on the techniques described in the previous section.

Next, we incorporate the early retirement option and the hereafter presented results correspond to solving the complementarity problem associated with the PIDE and with the Longstaff-Schwartz technique [42].

The number of nodes and elements of the quadratic finite element meshes are indicated in Table 3.1. All Monte Carlo simulations presented in this section have been performed by using 7000 time steps per year and 50000 paths.

In all the following examples, the retirement date $T_r = 40$, the number of subintervals in the composite trapezoidal rule $m = 50$ and the bounded computational domain defined by the values $x_1^\infty = 40$ and $x_2^\infty = 40$ have been considered. Particularly, for the jump-diffusion model we considered the data estimated in [2] for the underlying associated with European call options:

$$\widetilde{\lambda} = 0.1, \quad \mu = -0.90 \quad and \quad \gamma = 0.45. \tag{3.32}$$

In future work, clearly this data need to be obtained from real data corresponding to salary evolution in some way, being the estimation from historical data the a priori easiest way to obtain these parameters.

3.4.1 Pension plans without early retirement

In this section we show the results obtained when the early retirement option is not allowed. In Table 3.2 the computed values of the pension plan at origination and mesh point $(S, I) = (25, 20)$ for different meshes and time steps are indicated in order to illustrate the convergence.

Next, in Table 3.3 for different parameters we show the computed values at time $t = 0$ obtained by solving the PIDE problem for the same salaries and average salaries considered in Chapter 1. The corresponding values with the Monte Carlo technique are those ones appearing in Table 3.4. As in the absence of jumps, note that only the salary at time $t = 0$ is known, so that the salary and cumulative salary values are simulated at the different dates along the path. As expected, the value of the pension plan does not depend on the cumulative salary I at origination because we only take into account the average salary from time $t = T_r - n_y$.

The behaviour of the plan at time $t = 38$ is shown in Tables 3.5 and 3.6. The first values are obtained with de PIDE model and the second ones with Monte Carlo simulation. As in the case without jumps, at $t = 38$, the salary and the cumulative salary are known and we simulate the values from this date. In this case the value of the variable I up to this time is considered and influences the value obtained with

Monte Carlo. In order to obtain the solution of the PIDE model, Mesh 96 and 100000 time steps have been considered.

	Number of elements	**Number of nodes**
Mesh 12	144	625
Mesh 24	576	2401
Mesh 48	2304	9409
Mesh 96	9216	37249

Table 3.1: FEM meshes data

NT	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
100	2.603778	2.604122	2.604051	2.604056
1000	2.598576	2.599026	2.599109	2.599123
10000	2.598018	2.598408	2.598447	2.598455
100000	2.598004	2.598389	2.598421	2.598422

Table 3.2: Retirement benefits under a jump diffusion process without early retirement at time $t = 0$ and at mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$, are considered. The 99% confidence interval with Monte Carlo simulation is [2.582251, 2.651059].

3.4.2 Pension plans with early retirement option

In this section we show the results obtained when the early retirement option is allowed. For this purpose, we have considered the following model parameters:

$$\sigma = 0.1,\ \theta = 0.025,\ r = 0.025,\ a = 0.75,\ T_r = 40,\ T_0 = 15,$$
$$n_y = 30,\ k_1 = 0.5,\ \mu_d = 0.025,\ \mu_w = 0.2,\ \alpha_d = 1,\ \alpha_w = 0$$

and we set $\beta = 10000$ in the ALAS algorithm.

n_y	σ	r	a	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$
30	0.1	0.025	0.75	0.133059	0.133312	0.269216
			0.95	0.133132	0.133407	0.269375
		0.075	0.75	0.108654	0.108919	0.219359
			0.95	0.108859	0.109011	0.219732
	0.2	0.025	0.75	0.133175	0.133553	0.269321
			0.95	0.133393	0.133871	0.269483
		0.075	0.75	0.108771	0.108929	0.219532
			0.95	0.108947	0.109049	0.219941
				$(S,I)=(1.2,7.5)$	$(S,I)=(1.2,11.25)$	$(S,I)=(2.4,15)$
15	0.1	0.025	0.75	0.132617	0.132694	0.266294
			0.95	0.132745	0.133003	0.266357
		0.075	0.75	0.108549	0.108675	0.218003
			0.95	0.108974	0.109102	0.218342
	0.2	0.025	0.75	0.132691	0.132867	0.266344
			0.95	0.132853	0.132986	0.266414
		0.075	0.75	0.108697	0.108812	0.218058
			0.95	0.109021	0.109094	0.218192

Table 3.3: Retirement benefits under a jump diffusion process without early retirement at time $t=0$ for different (S,I) points and parameter values

Table 3.7 illustrates the convergence of the method in the presence of jumps as soon as the mesh is refined in time and space at $t=38$ and the mesh point $(S,I)=(25,20)$. Note that for all times the point $(S,I)=(25,20)$ is located at the region where early retirement is not optimal. For the same point, in Tables 3.8 and 3.9 the results for $t=T_r-n_y=10$ and $t=0$, respectively, also illustrate the convergence with mesh refinement in time and space. Moreover, the obtained results by solving the PIDE have been compared with those ones of Monte Carlo simulation techniques. More precisely, we have implemented the Longstaff-Schwartz technique and the corresponding confidence intervals are indicated in the caption.

Finally, Table 3.10 shows a comparison between the values obtained with both techniques, solving the PIDE and with Monte Carlo and if early retirement is allowed or not. In the first rows the values at different points with Monte Carlo simulation

n_y	σ	r	a	$S = 1.2$	$S = 2.4$
30	0.1	0.025	0.75	[0.132289, 0.133327]	[0.264578, 0.269653]
			0.95	[0.132695, 0.133722]	[0.265391, 0.269945]
		0.075	0.75	[0.108458, 0.109211]	[0.216916, 0.219422]
			0.95	[0.108811, 0.109291]	[0.217682, 0.219982]
	0.2	0.025	0.75	[0.132368, 0.133663]	[0.264736, 0.269526]
			0.95	[0.132813, 0.134197]	[0.265626, 0.269795]
		0.075	0.75	[0.108479, 0.109381]	[0.216959, 0.219661]
			0.95	[0.108727, 0,109624]	[0.217455, 0.219948]
				$S = 1.2$	$S = 2.4$
15	0.1	0.025	0.75	[0.132296, 0.133333]	[0.264591, 0.266667]
			0.95	[0.132704, 0.133731]	[0.265408, 0.267462]
		0.075	0.75	[0.108459, 0.109212]	[0.216918, 0.218424]
			0.95	[0.108842, 0.109292]	[0.217301, 0.218692]
	0.2	0.025	0.75	[0.132375, 0.133570]	[0.264749, 0.266840]
			0.95	[0.132721, 0.134206]	[0.265642, 0.268412]
		0.075	0.75	[0.108480, 0.109402]	[0.217457, 0.218663]
			0.95	[0.108629, 0.109526]	[0.217697, 0.218951]

Table 3.4: The 99% confidence intervals with Monte Carlo simulation under a jump diffusion process at time $t = 0$ for different salaries and parameter values

(MC) and PIDE model (PIDE) when early retirement is not allowed. Then forth and fifth rows indicate the value of V with Longstaff-Schwartz technique (LS) and PIDE model (PIDE) when early retirement is allowed. Last row shows the computed multiplier in the PIDE solution. As in the case without jumps, the value of V is always greater in case of allowing early retirement and also, the point $(S, I) = (4, 10)$ belongs to the region where early retirement is not optimal as indicated by the null value of the multiplier. The other points belong to the early retirement region and the value of the pension plan matches the early retirement value and the value of the multiplier is different from zero. In order to obtain these results solving the PIDE model the Mesh 96 and 10000 time steps are considered.

n_y	σ	r	a	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$
30	0.1	0.025	0.75	0.293409	0.406986	0.586518
			0.95	0.358895	0.502986	0.718043
		0.075	0.75	0.267828	0.370626	0.535824
			0.95	0.327221	0.457491	0.654372
	0.2	0.025	0.75	0.293479	0.407026	0.586697
			0.95	0.358923	0.503005	0.718192
		0.075	0.75	0.267915	0.370697	0.535897
			0.95	0.327251	0.457518	0.654429
				$(S,I)=(1.2,7.5)$	$(S,I)=(1.2,11.25)$	$(S,I)=(2.4,15)$
15	0.1	0.025	0.75	0.311833	0.424796	0.622203
			0.95	0.382331	0.525624	0.763118
		0.075	0.75	0.284386	0.386681	0.567829
			0.95	0.348298	0.478046	0.695367
	0.2	0.025	0.75	0.311902	0.422867	0.622293
			0.95	0.382419	0.525735	0.763386
		0.075	0.75	0.284417	0.386872	0.577897
			0.95	0.348392	0.478139	0.695471

Table 3.5: Retirement benefits value under a jump diffusion process without early retirement at time $t = 38$ for different (S, I) points and parameter values

n_y	σ	r	a	$(S, I) = (1.2, 15)$	$(S, I) = (1.2, 22.5)$	$(S, I) = (2.4, 30)$
30	0.1	0.025	0.75	[0.293242, 0.293526]	[0.406966, 0.407250]	[0.586484, 0.587052]
			0.95	[0.358811, 0.359114]	[0.502862, 0.503165]	[0.717622, 0.718228]
		0.075	0.75	[0.267718 0.267983]	[0.370620, 0.370885]	[0.535436, 0.535965]
			0.95	[0.326974, 0.327259]	[0.457317, 0.457601]	[0.653948, 0.654517]
	0.2	0.025	0.75	[0.293241, 0.293597]	[0.406965, 0.407321]	[0.586482, 0.587193]
			0.95	[0.358754, 0.359133]	[0.502805, 0.503184]	[0.717507, 0.718266]
		0.075	0.75	[0.267678, 0.268008]	[0.370580, 0.370911]	[0.535355, 0.536017]
			0.95	[0.326922, 0.327277]	[0.457265, 0.457620]	[0.653844, 0.654554]
				$(S, I) = (1.2, 7.5)$	$(S, I) = (1.2, 11.25)$	$(S, I) = (2.4, 15)$
15	0.1	0.025	0.75	[0.311551, 0.311918]	[0.424586, 0.424967]	[0.621720, 0.622484]
			0.95	[0.381244, 0.382421]	[0.525347, 0.525773]	[0.762592 0.763443]
		0.075	0.75	[0.283279, 0.284619]	[0.386546, 0.386897]	[0.567287, 0.567989]
			0.95	[0.347937, 0.348316]	[0.477730, 0.478123]	[0.694775, 0.695560]
	0.2	0.025	0.75	[0.311550, 0.312009]	[0.422586, 0.425055]	[0.621723, 0.622660]
			0.95	[0.381281, 0.382805]	[0.525332, 0.525856]	[0.762562, 0.763611]
		0.075	0.75	[0.283583, 0.284515]	[0.386485, 0.386917]	[0.567165, 0.578030]
			0.95	[0.347393, 0.348397]	[0.477736, 0.478220]	[0.694786, 0.695754]

Table 3.6: The 99% confidence intervals with Monte Carlo simulation under a jump diffusion process at time $t = 38$ for different salaries and parameter values

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	1.695962	1.695863	1.695855	1.696236
2500	1.696683	1.696559	1.696543	1.696944
5000	1.697076	1.696938	1.696919	1.697326
10000	1.697285	1.697128	1.697107	1.697112

Table 3.7: Retirement benefits under a jump diffusion process with early retirement at time $t = 38$ at the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [1.692632, 1.704589].

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.756395	2.758438	2.758443	2.758448
2500	2.758234	2.759291	2.759294	2.759297
5000	2.758651	2.760128	2.760134	2.760137
10000	2.760221	2.760392	2.760413	2.760414

Table 3.8: Retirement benefits under a jump diffusion process with early retirement at time $t = 10$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.725933, 2.794409].

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.598371	2.598801	2.598848	2.598867
2500	2.598153	2.598582	2.598627	2.598671
5000	2.598032	2.598463	2.598507	2.598513
10000	2.598019	2.598408	2.598447	2.598455

Table 3.9: Retirement benefits under a jump diffusion process with early retirement at time $t = 0$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.589137, 2.667215].

	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$	$(S,I)=(4,10)$
V without ER(MC)	(0.293242, 0.293526)	(0.406966, 0.407250)	(0.586484, 0.587052)	(0.374073, 0.374953)
V without ER$(PIDE)$	0.293409	0.406986	0.586518	0.374628
V with ER(LS)	(0.369643, 0.369643)	(0.554464, 0.554464)	(0.739286, 0.739286)	(0.374264, 0.374965)
V with ER$(PIDE)$	0.369643	0.554464	0.739286	0.374635
P with ER$(PIDE)$	-1.645011×10^{-5}	-2.264008×10^{-5}	-2.304657×10^{-5}	0

Table 3.10: Retirement benefits under a jump diffusion process without and with early retirement (ER) at different points when time $t = 38$ and the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. Computed values with (PIDE), Monte Carlo (MC) and Longstaff-Schwartz (LS) and Multiplier (P).

Part II

Mortgages

Introduction to mortgage contracts

A mortgage is a financial contract between a lender (usually a bank or a financial institution) and a borrower in which the borrower obtains funds from the lender using a risky asset (in this case the property, a house for example) as a collateral. This contract is composed by a set of components such as:

- the property itself being financed
- the legal document that the lender issues to guarantee the payment of the debt which may include restrictions on the use or disposal of the property on behalf of the lender
- the person who borrows the money and also has an ownership interest in the property
- the lender that is usually a bank or other financial institution, although can also be some investors,
- the initial amount of the loan. This value could be equal to the value of the house but usually is less than it in order to reduce the risk
- the interest rate for use of the lender's money
- and the final repayment of the amount outstanding, which may be at the end of the scheduled term or the borrower can generally have the option to prepay the outstanding balance before the maturity, in this last case acting in a similar way to callable bonds or American options.

We can classify the mortgage into different types taking into account several factors that characterize the contract, such as the interest rate, the term of the loan, the payment amount and frequency and the prepayment option. Concerning the interest rate, it can be fixed for the life of the loan or floating.

The term of the loan is the number of years that the loan will be repaid. Some mortgage loans may have no amortization, or require the full repayment of the remaining balance at a certain date. Related to the amount paid per period and the frequency of payments, this amount may change or the borrower can have the option to increase or decrease the scheduled amount paid. The borrower can also have the option to prepay the outstanding balance before the maturity of the loan. Some types of mortgages limit or restrict prepayment of all or a portion of the loan, or require payment of a penalty to the lender for prepayment. The borrower also have the option to default. In this case the lender will lose the future promised payments unless there exists an insurance on the loan that covers a fraction of this loss. The lender is responsible for contracting this insurance and paying for it.

Taking into account the interest rate, we can distinguish between Fixed-Rate Mortgages (FRM) and Adjustable-Rate Mortgages (ARM). In the first case, the interest rate the borrower has to pay and the periodic payments are fixed for the live of the loan. Nevertheless, in the second case the interest rate is floating and according to a specific index (LIBOR or EURIBOR, for example). This interest rate is fixed for some period (annually, for example) and after which it is periodically adjusted according to the chosen index.

In the second part of this thesis, which comprises Chapter 4, we deal with contracts of the first type with monthly payments in which the fixed rate is the equilibrium rate and needs to be adjusted by using an iterative process. Moreover, we will consider that the borrower has the options to prepay the loan and/or default, in this last case future promised payments from the borrower to the lender are lost unless the lender has an insurance on the loan.

Chapter 4

Fixed Rate Mortgages (FRM)

4.1 Introduction

As we have mentioned in the previous introduction, a mortgage is a financial contract in which the borrower obtains funds (usually from a bank or a financial institution) by using a risky asset (in this case a house) as a collateral. The value of this contract depends on the house price and the interest rate, as underlying factors. In order to pay the mortgage, monthly payments from the borrower to the lender are considered so that cancelation occurs when at the maturity of the loan the debt is totally paid. Thus, the mortgage value is understood as the present value of the borrower scheduled monthly payments without including the insurance the lender can have on the loan. Moreover, in the present work the possibilities of the remaining mortgage value prepayment and borrower default are also considered. Prepayment can occur at any time during the life of the loan (analogously to the exercise in American options) while default only can happen at any monthly payment date. In fact, at each monthly payment date the borrower decides either to make the payment or default if the house value is less than the mortgage price. Thus, if we consider both prepayment and default option, the pricing problem is equivalent to a sequence of linked American options, one for each month. Moreover, starting from the final mortgage value at last month, the final mortgage value at the end of each month is obtained from the mortgage value for the corresponding next month at the same date.

At origination, the contract must be in equilibrium, which is achieved if the value of the mortgage to the lender plus the insurance against default is equal to the amount of money lent to the borrower, otherwise the contract would not be arbitrage free. This equilibrium provides the fixed rate of the loan.

In order to obtain the value of the contract and other components (such as insurance and coinsurance), option pricing methodology can be applied and leads to a sequence of backward in time partial differential equation (PDE). The problem is divided in monthly intervals where the final condition for a given month comes from the value at the same date of the following month. Additionally, the option of prepayment leads to free boundary problem formulations.

In [34] the properties of the free boundary are studied for the case in which default is not allowed so that the problem is much simpler as there is only a stochastic factor (the interest rate). Moreover, in order to solve backwards the PDE several numerical methods need to be provided. For example, in [37] and [4] explicit finite-differences schemes have been used. In [60] a semi-implicit Crank-Nicolson finite-difference scheme to discretize the PDE and a projected successive over-relaxation (PSOR) method to solve the complementarity problem (associated with prepayment feature) have been applied. Moreover, a technique based on the application of singular perturbation theory in order to speed up the calculation is also established. Basically, for small volatilities, the higher order terms in the PDE are neglected and the first order PDE is analytically solved. Finally, a comparison between the two methods is presented. However, some differences between the solution of first order PDE and the numerical solution of the original PDE are observed and some comments about the need of using higher order terms in the asymptotic expansion are pointed out, specially in scenarios with higher volatilities. In [67] the inclusion of higher order terms for European and American options is discussed and the corresponding PDE problems require numerical methods of the same complexity of those ones applied to the original problem.

In this work, we numerically solve the original equation by proposing the PDE discretization with the techniques developed in [9] for Asian options and more recently applied to pension plans in [14] and [15]. More precisely, we use a characteristics method to discretize first order terms and a Crank-Nicolson scheme that evaluates the functions at the previous time step in the basis of the characteristics, which consists on a different approach from the one proposed in [60]. These methods are particularly well suited for convection dominated problems, as those ones appearing in the case of small volatilities. If we neglect second order terms, then we recover the perturbation based solution proposed in [60]. The numerical analysis of the proposed characteristics Crank-Nicolson time discretization, the fully discretized problem when combined with Lagrange finite elements and the use of numerical integration formulas has been addressed in [7] and [8]. Both papers are applied to general convection-diffusion-reaction equations under certain assumptions. Furthermore, the non-linearities associated with the inequality constraints in the complementarity formulation due to prepayment are treated by means of the recently introduced Augmented Lagrangian Active Set (ALAS) method [36].

This chapter is organized as follows. In Section 2, first we state the mathematical model by describing the stochastic variables and deriving the PDE that governs the valuation of the mortgage components. Then, we establish the final, payment date and equilibrium conditions, as well as other characteristics of the contract. Also, the free boundary problem associated with prepayment option is presented. Section 3 contains the description of the numerical techniques. Some numerical results are presented in Section 4.

Most of the results in this chapter are included in the reference [16].

4.2 Mathematical modeling

4.2.1 Stochastic economic framework

A mortgage can be treated as a derivative financial product, for which the underlying state variables are the house price and the term structure of interest rates.

The value of the house at time t, H_t, is assumed to follow the standard log-normal process (see [45]), that satisfies the following stochastic differential equation:

$$dH_t = (\mu - \delta)H_t dt + \sigma_H H_t dX_t^H, \tag{4.1}$$

where

- μ is the instantaneous average rate of house-price appreciation,
- δ is the 'dividend-type' per unit service flow provided by the house,
- σ_H is the house-price volatility,
- and X_t^H is the standardized Wiener process for house price.

Note that this process has an absorbing barrier at zero, more precisely if H_t reaches at any time the value zero, it remains zero thereafter. The dividend-type parameter δ is associated with the benefits of owning the house (usage, hiring, ...). The previous model does not take into account possible jumps in the house price, which would require the use of jump-diffusion models.

Deriving the risk-neutral process for house price by changing to a risk neutral probability measure involves replacing the expected drift term $\mu - \delta$ by $\mu - \delta - \lambda\sigma$, where λ represents the market price of risk associated with the uncertainty of the house price [30]. Using risk neutrality arguments, $\mu - \lambda\sigma$ is equal to the risk-free interest rate r_t. So, by substituting this expression in equation (4.1), we obtain

$$dH_t = (r_t - \delta)H_t dt + \sigma_H H_t dX_t^H. \tag{4.2}$$

The other source of uncertainty, the interest rate r_t at time t, is assumed to be stochastic and its evolution can be modeled with the following classical Cox-Ingersoll-Ross (CIR) process [18],

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dX_t^r, \tag{4.3}$$

where

- κ is the speed of adjustment in the mean reverting process,
- θ is the long term mean of the short-term interest rate (steady state spot rate),
- σ_r is the interest-rate volatility parameter,
- and X_t^r is the standardized Wiener process for interest rate.

Note that the CIR model is mean-reverting. Moreover, if $2\kappa\theta \geq \sigma_r^2$ and $r_0 > 0$ then zero is a natural reflecting barrier and negative interest rates cannot be achieved. In [34] a Vasicek model is considered so that negative interest rates can be obtained.

Wiener processes, X_t^H and X_t^r can be assumed to be correlated according to $dX_t^H dX_t^r = \rho dt$, where ρ is the instantaneous correlation coefficient.

4.2.2 Statement of the mortgage pricing PDE problem

The price of any asset whose value is a function of house price H_t, interest rate r_t and time t is a stochastic process, $F_t = F(t, H_t, r_t)$, where F is a smooth enough function. Then, by using the dynamic hedging methodology as it is proposed in [59], the function F is the solution of a certain PDE problem. Here, it is assumed that the house price evolution is described by equation (4.2) and the interest rate dynamics is governed by equation (4.3). So, we can apply Itô's Lemma (see [32], for example) to obtain the variation of F_t, dF_t, from time t to $t + dt$ for small dt. Hereafter, we suppress the dependence on t in order to simplify notation:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial H}dH + \frac{\partial F}{\partial r}dr + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F}{\partial r^2}\right) dt \tag{4.4}$$

At this point, we build a portfolio Π by buying one unit of the asset F_1 with maturity T_1 and selling Δ_2 and Δ_1 units of the asset F_2 with maturity T_2 and of the underlying asset H, respectively. Thus,

$$\Pi = F_1 - \Delta_2 F_2 - \Delta_1 H \tag{4.5}$$

Note that the variation of the portfolio value between t and $t + dt$ is given by:

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 dH, \tag{4.6}$$

where Δ_1 and Δ_2 are constant in $[t, t + dt]$. As it is the case of dividends in options on assets, δ acts over the price of the underlying asset H reducing its value by δH over a time interval $[t, t + dt]$. So, the portfolio must change by an amount $-\delta H \Delta_1 dt$ during this time interval. Thus, the variation in the value of the portfolio is

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1(dH + \delta H dt). \tag{4.7}$$

Moreover, Π turns out to be risk-free for the following choice:

$$\Delta_2 = \frac{\partial F_1/\partial r}{\partial F_2/\partial r}, \quad \Delta_1 = \frac{\partial F_1}{\partial H} - \Delta_2 \frac{\partial F_2}{\partial H} \tag{4.8}$$

So, for this choice of Δ, the variation of the risk-free portfolio is given by:

$$\begin{aligned} d\Pi \;=\; & \left[\frac{\partial F_1}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2}\right) - \delta H \frac{\partial F_1}{dH}\right. \\ & \left. - \frac{\partial F_1/\partial r}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2}\right) - \delta H \frac{\partial F_2}{dH}\right)\right] dt. \end{aligned}$$

By using the arbitrage-free assumption, this variation is also given by $d\Pi = r\Pi dt$. Thus, we obtain the identity:

$$\begin{aligned} & \frac{1}{\partial F_1/\partial r}\left(\frac{\partial F_1}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2} + (r - \delta)H\frac{\partial F_1}{\partial H} - rF_1\right) \\ & = \frac{1}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2} + (r - \delta)H\frac{\partial F_2}{\partial H} - rF_2\right). \end{aligned}$$

The left hand side of the equality is a function of T_1 but not of T_2 and the right side is a function of T_2 but not T_1. This is only possible if both sides are independent of maturity date, so that

$$\frac{1}{\partial F/\partial r}\left(\frac{\partial F}{\partial t}+\frac{1}{2}\sigma_H^2H^2\frac{\partial^2 F}{\partial H^2}+\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H\partial r}+\right.$$
$$\left.+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 F}{\partial r^2}+(r-\delta)H\frac{\partial F}{\partial H}-rF\right) = a(t,H,r), \qquad (4.9)$$

where it is convenient to write $a(t,H,r)=-\kappa(\theta-r)$, which is a standard procedure in the literature (see [37], [4], for example).

So, by reordering the terms in (4.9) we obtain the following PDE that governs the valuation of any asset depending on house price and interest rate, in particular the fixed-rate mortgages.

$$\frac{\partial F}{\partial t}+\frac{1}{2}\sigma_H^2H^2\frac{\partial^2 F}{\partial H^2}+\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H\partial r}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 F}{\partial r^2}+$$
$$+(r-\delta)H\frac{\partial F}{\partial H}+\kappa(\theta-r)\frac{\partial F}{\partial r}-rF = 0. \qquad (4.10)$$

4.2.3 Mortgage contract

In the fixed-rate mortgage we are considering, the loan is repaid by a sequence of equal monthly payments at given dates T_m, $m=1,...,M$. Moreover, assuming $T_0=0$, let $\Delta T_m=T_m-T_{m-1}$ the duration of month m. Thus, assuming that M is the number of months, c is the fixed contract rate and $P(0)$ is the initial loan (i.e. the principal at $t=T_0=0$), the fixed mortgage payment (MP) is given by formula:

$$MP=\frac{(c/12)(1+c/12)^M P(0)}{(1+c/12)^M-1}, \qquad (4.11)$$

For $m=1,...,M$, the unpaid loan just after the (m-1)th payment is given by

$$P(m-1)=\frac{((1+c/12)^M-(1+c/12)^{m-1})P(0)}{(1+c/12)^M-1}, \qquad (4.12)$$

If $t_m=t-T_{m-1}$ denotes the time elapsed at month m (which starts at $t=T_{m-1}$), we introduce $\tau_m=\Delta T_m-t_m$ as the time until T_m. This change of time variable transforms equation (4.10) into another one associated with an initial value problem. More precisely, the mortgage value to the lender during month m, $V(\tau_m,H,r)$, without including the insurance the lender has on the loan, satisfies the PDE

$$
\begin{aligned}
-\frac{\partial F}{\partial \tau_m} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + & \\
+(r-\delta)H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - rF \quad &= \quad 0 \qquad (4.13)
\end{aligned}
$$

for $0 \leq \tau_m \leq \Delta T_m$, $0 \leq H < \infty$, $0 \leq r < \infty$. We clarify a certain abuse of notation: if $\overline{F}$ denotes the solution of (4.10) and F the solution of (4.13) then $F(\tau_m, H, r) = \overline{F}(T_m - \tau_m, H, r)$.

We will take into account the prepayment and default options. On one hand the option to default on the mortgage that can only happen at payment dates when the borrower does not pay the monthly amount MP, and on the other hand the option to prepay the mortgage, which can be exercised at any time during the life of the loan. If the borrower decides to fully amortize the mortgage at time τ_m, this person should pay the lender the total debt payment $TD(\tau_m)$, which includes an early termination penalty and is defined as follows:

$$TD(\tau_m) = (1+\Psi)(1+c(\Delta T_m - \tau_m))P(m-1), \qquad (4.14)$$

where Ψ is the prepayment penalty factor.

Thus, at each payment date, the borrower must decide whether to pay the required monthly payment or default and hand over the house to the lender. The option to prepay gives the borrower the right to exercise the prepayment at any time during the lifetime of the mortgage (American feature).

The mortgage pricing problem starts from the value of the mortgage at maturity $(t = T_M)$, which just before the last payment is given by

$$V(\tau_M = 0, H, r) = \min(MP, H) \qquad (4.15)$$

while at the other payment dates, it is given by

$$V(\tau_m = 0, H, r) = \min(V(\tau_{m+1} = \Delta T_{m+1}, H, r) + MP, H), \qquad (4.16)$$

where $1 \leq m \leq M-1$.

If the borrower defaults, which occurs when the mortgage value is equal to the house value, the lender will lose the promised future payments. Then, the lender might have taken an insurance against default which would cover a fraction of the loss associated with default. This asset has no value for the borrower. Actually, it is part of the lender's portfolio, as indicated in [60] this asset adds to the lender's position in the contract. In order to obtain the value of the insurance to the lender, denoted by $I(\tau_m, H, r)$, we must solve equation (4.13) with suitable payment date conditions. In order to pose them, we assume that in case of default the insurer accepts to pay a fraction γ of the currently unpaid balance to the lender up to a maximum indemnity or cap, Γ. By taking this into account, depending if default occurs or not, the insurance value at the maturity of the loan is

$$I(\tau_M = 0, H, r) = \begin{cases} \min(\gamma(MP - H), \Gamma) & \text{(Default)} \\ \\ 0 & \text{(No default)} \end{cases} \tag{4.17}$$

At earlier payment dates, the value of the insurance is

$$I(\tau_m = 0, H, r) = \begin{cases} \min(\gamma[TD(\tau_m = 0) - H], \Gamma) & \text{(Default)} \\ \\ I(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases} \tag{4.18}$$

where $1 \leq m \leq M - 1$.

The fraction of the potential loss not covered by the insurance is referred as the coinsurance. At each payment date, the coinsurance is the difference between the values of the potential loss and the insurance coverage. In this case, in order to price the coinsurance, $CI(\tau_m, H, r)$, equation (4.13) must be solved again with suitable conditions. At maturity, the value of the coinsurance is

$$CI(\tau_M = 0, H, r) = \begin{cases} \max((1 - \gamma)(MP - H), (MP - H) - \Gamma) & \text{(Default)} \\ \\ 0 & \text{(No default)} \end{cases} \tag{4.19}$$

At earlier payment dates, the value of the coinsurance is

$$CI(\tau_m = 0, H, r) = \begin{cases} \max((1-\gamma)[TD(\tau_m = 0) - H], [TD(\tau_m = 0) - H] - \Gamma) & \text{(Default)} \\ \\ CI(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases} \tag{4.20}$$

where $1 \leq m \leq M - 1$.

4.2.4 Arbitrage free condition

At the time of origination, the value of the contract together with the insurance and any upfront points must be the same to the lender as the value of the loan to the borrower. Thus, arbitrage is avoided and the contract is fair for both parts. Formally,

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1-\xi)P(0), \tag{4.21}$$

where $\xi P(0)$ is the value of the upfront points, understood as an arrangement fee. The arrangement fee, the prepayment penalty Ψ and whether or not the lender holds an insurance are specified in the contract. So, this equation contains only one free parameters, the contract rate c. It is necessary to find the value of the interest rate c which satisfies the equilibrium condition (4.21) and ensures that the contract is fair and arbitrage free. It can be obtained by using an iterative method for nonlinear equations.

Arbitrage equilibrium analysis

In order to give an idea of the equilibrium mortgage contract rate, different contracts are considered (see [37]):

- Basic contract: in this simple case the arrangement fee $\xi = 0$ and no insurance is charged. So, equation (4.21) reduces to

  $$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = P(0). \tag{4.22}$$

  The arbitrage condition requires that $(H_{initial}, r_{initial})$ be a point in state space where immediate prepayment is an optimal strategy. For all values of $c > \hat{c}$ the point $(H_{initial}, r_{initial})$ is in fact in the interior of the prepayment region. Since the borrower simultaneously takes the loan and pays it off on the right of $\hat{c}$, no equilibrium is observed when $c > \hat{c}$ and $\hat{c}$ is not really a valid solution because the borrower is indifferent between prepayment and continuation. The practise of loaning less than the full value of the house in order to reduce the risk of the loan is a standard one. In our case when $P(0) = H$ no equilibrium could exists, since it implies that default would also be an optimal strategy and it is not possible because the borrower could earn the flow of service on the house until the first payment becomes due.

- Contract with points: if an arrangement fee, ξ, is introduced into the equation, the equilibrium equation has this expression:

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1 - \xi)P(0) \tag{4.23}$$

In this case, the equilibrium contract rate c is $c_1 < \hat{c}$. Then, the problem of a continuum of values satisfying equation (4.22) is removed. Now, the point $(H_{initial}, r_{initial})$ is in the interior of the continuation region.

- Contract with insurance: now we consider the case where insurance can have value, but upfront points are no charged ($\xi = 0$). The expression for the equilibrium condition in this case is as follows:

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = P(0) \tag{4.24}$$

Now, there is an isolated equilibrium when $c = c_2$ such that, $c_1 < c_2 < \hat{c}$ as well as the continuum, $c \geq \hat{c}$. At these latter values immediate prepayment is the optimal strategy, so insurance has no value, and in the other case at c_2 insurance has positive value.

- Full contract: this is the general case with insurance and upfront points. The equilibrium equation is this case is (4.21). There is an unique value of $c = c_3$ which satisfies the equation. Therefore, it is necessary that $c_3 \leq c_1$ and $c_3 \leq c_2$.

4.2.5 The free boundary problem

Let us consider the following linear operator,

$$\begin{aligned} \mathcal{L}V &= \partial_{\tau_m} V - \frac{1}{2}\sigma_H^2 H^2 \partial_{HH} V - \rho\sigma_H\sigma_r H\sqrt{r}\partial_{Hr} V - \frac{1}{2}\sigma_r^2 r \partial_{rr} V \\ &\quad -(r-\delta)H\partial_H V - \kappa(\theta - r)\partial_r V + rV. \end{aligned} \tag{4.25}$$

So, the free boundary problem associated with the valuation of the mortgage contract, can be reduced to the linear complementarity problem:

$$\mathcal{L}V \leq 0, \quad (TD(\tau_m) - V(\tau_m, H, r)) \geq 0, \quad (\mathcal{L}V)(TD(\tau_m) - V(\tau_m, H, r)) = 0. \tag{4.26}$$

The option to prepay can be exercised at any time during the lifetime of the contract (it is American in type). If $V = TD$ then it is optimal for the borrower to prepay, otherwise $\mathcal{L}V = 0$ and we are inside the continuation region.

4.3 Numerical solution

In order to obtain a numerical approach of the value of the contract at origination, we need to solve a free boundary problem for each month to obtain the value of the mortgage during that month, jointly with an additional initial value problem when the lender holds an insurance. Once we know the value at origination of the contract and the insurance, the equilibrium condition (4.21) is checked to find the interest rate for which the contract is arbitrage free. For this purpose, a Newton-like method is implemented. By using the equilibrium rate, we solve another initial value problem to obtain the coinsurance. For the numerical solution of the PDE, we propose a Crank-Nicolson characteristics time discretization scheme combined with quadratic Lagrange finite element method. Thus, first a localization technique is used to cope with the initial formulation in an unbounded domain. For the inequality constraints associated with the complementarity problem, we propose a mixed formulation and an augmented Lagrangian active set technique.

4.3.1 Localization procedure and formulation in a bounded domain

In this section we replace the unbounded domain by a bounded one and determine the required boundary conditions. For this purpose, we introduce the notation:

$$x_0 = \tau_m, \quad x_1 = \frac{H}{H_\infty} \quad \text{and} \quad x_2 = \frac{r}{r_\infty}, \tag{4.27}$$

where both H_∞ and r_∞ are sufficiently large suitably chosen real numbers. Let $\Omega = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$, with $x_0^\infty = \Delta T_m$, $x_1^\infty = x_2^\infty = 1$. Then, let us denote the Lipschitz boundary by $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{i=0}^{2} (\Gamma_i^- \cup \Gamma_i^+)$, where:

$$\Gamma_i^- = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = 0\},\ \Gamma_i^+ = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\},\ i = 0, 1, 2.$$

Then, the PDE in problem (4.13) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 F}{\partial x_i x_j} + \sum_{j=0}^{2} b_j \frac{\partial F}{\partial x_j} + b_0 F = f_0, \tag{4.28}$$

where the involved data are defined as follows:

$$B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma_H^2 x_1^2 & \frac{1}{2}\rho x_1 \sqrt{x_2/r_\infty}\,\sigma_H \sigma_r \\ 0 & \frac{1}{2}\rho x_1 \sqrt{x_2/r_\infty}\,\sigma_H \sigma_r & \frac{1}{2}\sigma_r^2 x_2/r_\infty \end{pmatrix}, \tag{4.29}$$

$$\vec{b} = (b_j) = \begin{pmatrix} -1 \\ (x_2 r_\infty - \delta) x_1 \\ \kappa(\theta - x_2 r_\infty)/r_\infty \end{pmatrix}, \quad b_0 = -x_2 r_\infty, \quad f_0 = 0. \tag{4.30}$$

Thus, following [50], in terms of the normal vector to the boundary pointing inward Ω, $\vec{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma^0 = \left\{ x \in \Gamma / \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \quad \Sigma^1 = \Gamma - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

As indicated in [50] the boundary conditions at $\Sigma^1 \bigcup \Sigma^2$ for the so-called first boundary value problem associated with (4.28) are required. Note that $\Sigma^1 = \Gamma_1^+ \bigcup \Gamma_2^+$ and $\Sigma^2 = \Gamma_0^-$. Therefore, in addition to an initial condition depending on the payment date Γ_0^- (see section 4.2.3), we impose the following Neumann conditions:

$$\frac{\partial F}{\partial x_1} = 0 \quad on \quad \Gamma_1^+, \tag{4.31}$$

$$\frac{\partial F}{\partial x_2} = 0 \quad on \quad \Gamma_2^+. \tag{4.32}$$

Next, taking into account the new variables we write the equation (4.13) in divergence form in the bounded domain. As in [60], we consider the case $\rho = 0$. Thus, the initial-boundary value problem for the insurance and coinsurance can be written in the form: Find $J : [0, \Delta T_m] \times \Omega \to \mathbb{R}$ such that

$$\frac{\partial J}{\partial \tau_m} + \vec{v} \cdot \nabla J - Div(A \nabla J) + lJ = f \qquad \text{in } (0, \Delta T_m) \times \Omega\,, \tag{4.33}$$

$$\frac{\partial J}{\partial x_1} = g_1 \qquad \text{on } (0, \Delta T_m) \times \Gamma_1^+\,, \tag{4.34}$$

$$\frac{\partial J}{\partial x_2} = g_2 \qquad \text{on } (0, \Delta T_m) \times \Gamma_2^+\,, \tag{4.35}$$

where $J = I, CI$ and the appropriate initial condition for each month is given by the equations (4.17) and (4.18) when we are pricing the insurance and by the equations (4.19) and (4.20) in the case of valuing the coinsurance.

Furthermore, for the complementarity problem associated with the mortgage value during montn m, we can pose the following mixed formulation:

Find $V : [0, \Delta T_m] \times \Omega \to \mathbb{R}$ satisfying the partial differential equation

$$\frac{\partial V}{\partial \tau_m} + \vec{v} \cdot \nabla V - Div(A \nabla V) + lV + P = f \qquad \text{in } (0, \Delta T_m) \times \Omega\,, \tag{4.36}$$

the complementarity conditions

$$V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \;\; \text{in } (0, \Delta T_m) \times \Omega \tag{4.37}$$

the boundary conditions

$$\frac{\partial V}{\partial x_1} = g_1 \qquad \text{on } (0, \Delta T_m) \times \Gamma_1^+\,, \tag{4.38}$$

$$\frac{\partial V}{\partial x_2} = g_2 \qquad \text{on } (0, \Delta T_m) \times \Gamma_2^+ \tag{4.39}$$

and the initial condition for each month, given by the equations (4.15) or (4.16).

For both problems, the involved data is defined as follows

$$A = \begin{pmatrix} \frac{1}{2}\sigma_H^2 x_1^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 \frac{x_2}{r_\infty} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta)x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix} \tag{4.40}$$

$$l = x_2 r_\infty, \quad f = 0,\ g_1 = 0,\ g_2 = 0. \tag{4.41}$$

Next, the qualitative behaviour of the velocity field on the boundaries is studied:

- On boundary Γ_1^-, since $x_1 = 0$ then

  $$\vec{v} = \left(0, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty)\right)/r_\infty\right),$$

  so the velocity field is tangential to the boundary.

- On boundary Γ_2^-, since $x_2 = 0$ then

  $$\vec{v} = \left((\sigma_H + \delta)x_1, \left(\frac{1}{2}\sigma_r^2 - \kappa\theta\right)/r_\infty\right),$$

  so as $\sigma_r \leq \sqrt{2\kappa\theta}$ the velocity field either points outward the domain or it is tangential to the boundary.

- On boundary Γ_1^+, since $x_1 = 1$ then

  $$\vec{v} = \left(\sigma_H - r_\infty x_2 + \delta, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty)\right)/r_\infty\right),$$

  so if $(\sigma_H^2 + \delta) < r_\infty x_2$ the velocity field enters the domain, otherwise it points outward the domain.

- On boundary Γ_2^+, since $x_2 = 1$ then

  $$\vec{v} = \left((\sigma_H - r_\infty + \delta)x_1, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - r_\infty)\right)/r_\infty\right),$$

  so if $\frac{1}{2}\sigma_r^2 < \kappa(\theta - r_\infty)$ the velocity field enters the domain, otherwise it points outward the domain.

4.3.2 Time discretization

First, we define the characteristics curve through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}_m$, $X(\mathbf{x}, \bar{\tau}_m; s)$, which satisfies:

$$\frac{\partial}{\partial s} X(\mathbf{x}, \bar{\tau}_m; s) = \vec{v}(X(\mathbf{x}, \bar{\tau}_m; s)),\ X(\mathbf{x}, \bar{\tau}_m; \bar{\tau}_m) = \mathbf{x}. \tag{4.42}$$

For $N > 1$ let us consider the time step $\Delta\tau_m = \Delta T_m / N$ and the time mesh points $\tau_m^n = n\Delta\tau_m$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by characteristics method is given by:

$$\frac{DF}{D\tau_m} = \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau_m},$$

where $F = CI$, I , V and $X^n(\mathbf{x}) := X(\mathbf{x}, \tau_m^{n+1}; \tau_m^n)$. In view of the expression of the velocity field the components of $X^n(\mathbf{x})$ can be analytically computed:

$$\begin{aligned} X_1^n(\mathbf{x}) &= x_1 \exp\left(-\left(\sigma_H^2 + \delta + \frac{\sigma_r^2}{2\kappa} - \theta\right)\Delta\tau_m\right) \times \\ & \quad \exp\left(\left(\frac{-x_2 r_\infty}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} + \frac{\theta}{\kappa}\right)(\exp(-\kappa\Delta\tau_m) - 1)\right) \\ X_2^n(\mathbf{x}) &= \left(-\frac{\sigma_r^2}{2\kappa r_\infty} + \frac{\theta}{r_\infty}\right)(1 - \exp(-\kappa\Delta\tau_m)) + x_2 \exp(-\kappa\Delta\tau_m) \end{aligned}$$

Next, we consider a Crank-Nicolson scheme around $(X(\mathbf{x}, \tau_m^{n+1}; \tau_m), \tau_m)$ for $\tau_m = \tau_m^{n+\frac{1}{2}}$. So, for $n = 0, \ldots, N-1$, the time discretized equation for $F = I$, CI, V and $P = 0$ can be written as follows:

Find F^{n+1} such that:

$$\begin{aligned} \frac{F^{n+1}(\mathbf{x}) - F^n(X^n(\mathbf{x}))}{\Delta\tau_m} - \frac{1}{2} Div(A\nabla F^{n+1})(\mathbf{x}) - \frac{1}{2} Div(A\nabla F^n)(X^n(\mathbf{x})) + & \\ \frac{1}{2}(l\, F^{n+1})(\mathbf{x}) + \frac{1}{2}(l\, F^n)(X^n(\mathbf{x})) &= 0. \end{aligned} \tag{4.43}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply equation (4.43) by a suitable test function, integrate in Ω, use the classical

Green formula and the following one ([9]):

$$\begin{aligned}\int_\Omega Div(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} \;&=\; \int_\Gamma (\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}\\ &-\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\cdot\nabla\Psi(\mathbf{x})d\mathbf{x}\\ &-\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}\end{aligned} \tag{4.44}$$

Note that, in the present case, we have:

$$Div((\nabla X^n)^{-T}(\mathbf{x})) = \begin{pmatrix} 0 \\ \dfrac{r_\infty}{\kappa}(1-\exp(\kappa\Delta\tau_m)) \end{pmatrix}. \tag{4.45}$$

After these steps, we can write a variational formulation for the semi-discretized problem as follows:

Find $F^{n+1}\in H^1(\Omega)$ such that, for all $\Psi\in H^1(\Omega)$:

$$\begin{aligned}\int_\Omega F^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Omega(\mathbf{A}\nabla F^{n+1})(\mathbf{x})\nabla\Psi(\mathbf{x})d\mathbf{x}\, +&\\ +\frac{\Delta\tau_m}{2}\int_\Omega lF^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} \;&=\\ \int_\Omega F^n(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau_m}{2}\int_\Omega(\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\nabla\Psi(\mathbf{x})d\mathbf{x}-&\\ -\frac{\Delta\tau_m}{2}\int_\Omega lF^n(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Gamma \widetilde{g}^n(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}}+&\\ +\frac{\Delta\tau_m}{2}\int_{\Gamma_{1+}}\bar{g}_1^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \frac{\Delta\tau_m}{2}\int_{\Gamma_{2+}}\bar{g}_2^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}}-&\\ -\frac{\Delta\tau_m}{2}\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}&\end{aligned} \tag{4.46}$$

where ∇X^n can be analytically computed, $\bar{g}_1(\mathbf{x}) = g_1(\mathbf{x})a_{11}(\mathbf{x}) = 0$, $\bar{g}_2(\mathbf{x}) = g_2(\mathbf{x})a_{22}(\mathbf{x}) =$

0 and

$$\widetilde{g}^n(\mathbf{x}) := \begin{cases} -\left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x})a_{22}(X^n(\mathbf{x}))\frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on \quad \Gamma_1^- \\ 0 & on \quad \Gamma_2^- \\ \left[(\nabla X^n)^{-T}\right]_{22}(\mathbf{x})a_{22}(X^n(\mathbf{x}))g_2^n(X^n(\mathbf{x})) & on \quad \Gamma_2^+ \\ \left[(\nabla X^n)^{-T}\right]_{11}(\mathbf{x})a_{11}(X^n(\mathbf{x}))g_1^n(X^n(\mathbf{x}))+ & \\ +\left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x})a_{22}(X^n(\mathbf{x}))\frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on \quad \Gamma_1^+ \end{cases} \tag{4.47}$$

4.3.3 Finite elements discretization

For the spatial discretization we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. More precisely, let us define the finite elements space F_h by

$$V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) : \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{4.48}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of piecewise continuous functions on $\bar{\Omega}$.

4.3.4 Augmented Lagrangian Active Set (ALAS) algorithm

The Augmented Lagrangian Active Set (ALAS) algorithm proposed in [36] is here applied to the fully discretized in time and space mixed formulation (4.36)-(4.37). More precisely, after this discretization, the discrete problem can be written in the form:

$$M_h V_h^n + P_h^n = b_h^{n-1}, \tag{4.49}$$

with the discrete complementarity conditions

$$V_h^n \leq TD_h^n, \quad P_h^n \geq 0, \quad (TD_h^n - V_h^n)\, P_h^n = 0, \tag{4.50}$$

where P_h^n denotes the vector of the multiplier values and TD_h^n denotes the vector of the nodal values defined by function TD.

The basic iteration of the ALAS algorithm consists of two steps. In the first one the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step a reduced linear system associated with the inactive part is solved. We use the algorithm for unilateral problems, which is based on the augmented Lagrangian formulation.

First, for any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} := \{1, 2, \ldots N_{dof}\}$, let us denote by $[M_h]_{\mathcal{I}\mathcal{I}}$ the principal minor of matrix M_h and by $[M_h]_{\mathcal{I}\mathcal{J}}$ the co-diagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, for each mesh time τ_{m_n}, the ALAS algorithm computes not only V_h^n and P_h^n but also a decomposition $N = \mathcal{J}^n \cup \mathcal{I}^n$ such that

$$\begin{aligned}
M_h V_h^n + P_h^n &= b_h^{n-1}, \\
[P_h^n]_j + \beta\, [V_h^n - TD]_j &> 0 \qquad \forall j \in \mathcal{J}^n, \\
[P_h^n]_i &= 0 \qquad \forall i \in \mathcal{I}^n,
\end{aligned} \tag{4.51}$$

for a given positive constant β. In the above, $\mathcal{I}^n$ and $\mathcal{J}^n$ are, respectively, the *inactive* and the *active* sets at time τ_{m_n}. More precisely, the iterative algorithm builds sequences $\left\{V_{h,k}^n\right\}_k$, $\left\{P_{h,k}^n\right\}_k$, $\{\mathcal{I}_k^n\}_k$ and $\{\mathcal{J}_k^n\}_k$, converging to V_h^n, P_h^n, $\mathcal{I}^n$ and $\mathcal{J}^n$, by means of the following steps:

1. Initialize $V_{h,0}^n = TD_h^n$ and $P_{h,0}^n = \max(b_h^n - M_h V_{h,0}^n, 0) \geq 0$. Choose $\beta > 0$. Set $k = 0$.

2. Compute

$$\begin{aligned}
Q_{h,k}^n &= \max\left\{0, P_{h,k}^n + \beta\left(V_{h,k}^n - TD_{h,k}^n\right)\right\}, \\
\mathcal{J}_k^n &= \left\{j \in \mathcal{N}, \left[Q_{h,k}^n\right]_j > 0\right\}, \\
\mathcal{I}_k^n &= \{i \in \mathcal{N}, \left[Q_{h,k}^n\right]_i = 0\}.
\end{aligned}$$

3. If $k \geq 1$ and $J_k^n = J_{k-1}^n$ then convergence is achieved. Stop.

4. Let V and P be the solution of the linear system

$$\begin{aligned} & M_h V + P = b^{n-1}, \\ & P = 0 \text{ on } \mathcal{I}_k^n \text{ and } V = TD \text{ on } \mathcal{J}_k^n. \end{aligned} \tag{4.52}$$

   Set $V_{h,k+1}^n = V, P_{h,k+1}^n = \max\{0, P\}, k = k+1$ and go to 2.

It is important to note that, instead of solving the full linear system in (4.52), for $\mathcal{I} = \mathcal{I}_k^n$ and $\mathcal{J} = \mathcal{J}_k^n$ the following reduced one on the inactive set is solved:

$$\begin{aligned} [M_h]_{\mathcal{I}\mathcal{I}} [V]_{\mathcal{I}} &= [b^{n-1}]_{\mathcal{I}} - [M_h]_{\mathcal{I}\mathcal{J}} [TD]_{\mathcal{J}}, \\ [V]_{\mathcal{J}} &= [TD]_{\mathcal{J}}, \\ P &= b^{n-1} - M_h V. \end{aligned} \tag{4.53}$$

In [36], it is proved the convergence of the algorithm in a finite number of steps for a Stieltjes matrix (i.e., a real symmetric positive definite matrix with negative off-diagonal entries [64]) and a suitable initialization (the same we consider in this work). They also proved that $\mathcal{I}_k \subset \mathcal{I}_{k+1}$. Nevertheless, a Stieltjes matrix can be only obtained for linear elements but never for the here used quadratic elements because we have some positive off-diagonal entries coming from the stiffness matrix (actually we use a lumped mass matrix). However, we have obtained good results by using ALAS algorithm with quadratic finite elements.

4.3.5 Iterative method for the arbitrage free equation

In order to obtain the interest rate which satisfies the equilibrium condition (4.21), a Newton method with discrete derivative (secant method) is implemented (see [59]) to solve $f(c) = 0$, where f is defined to balance the equilibrium condition in the form

$$f(c) = V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) - (1 - \xi) P(0)$$

Starting from an initial value c_0 and Δ_0 the initial increment in c_0. Then the value of the mortgage components involved in the equilibrium condition are calculated with c_0. Next, we check if $f(c_0)$ is less than a given tolerance, if this condition is not satisfied we set $c_1 = c_0 + \Delta_0$ and repeat the process. At iteration i, we check if $f(c_i)$ is less than a tolerance, if it is not the case we compute

$$\Delta_i = -\frac{\Delta_{i-1} f(c_i)}{f(c_i) - f(c_{i-1})}, \quad i \geq 1 \tag{4.54}$$

and update $c_{i+1} = c_i + \Delta_i$ until the convergence criterium is fulfilled.

4.4 Numerical results

In order to obtain the solution of the fixed rate mortgage valuation problem we need to specify a set of parameters, related to the economic environment, contract characteristics and insurance. All of them, based on the existent literature (see [4] and [60]), are shown in Table 4.1. Moreover, concerning the numerical methods employed to solve the problem, we consider the parameters collected in Table 4.2.

Economic framework	
Steady state spot rate, θ	10 %
Speed of reversion, κ	25 %
House service flow, δ	7.5%
Correlation coefficient, ρ	0
Contract specifications	
Initial value of the house, $H_{initial}$	100000€
Ratio of the loan to value	95 %
Initial estimate for contract rate, c_0	10%
Prepayment penalty, Ψ	5%
Insurance	
Guaranteed fraction of total loss, γ	80%
Cap, Γ	$20\% H_{initial}$

Table 4.1: Fixed parameters in the mortgage valuation model

Computational domain	
H_∞	200000€
r_∞	40 %
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
Time discretization	
Time steps per month	30
ALAS algorithm	
Parameter β	10000

Table 4.2: Numerical resolution parameters

In Tables 4.3, 4.4, 4.5 and 4.6 the influence of different parameters (such as interest rate and house price volatilities, loan maturity, spot interest rate and arrangement fee) in the contract rate, mortgage value and insurance and coinsurance is shown.

If we increase the life of the loan the equilibrium interest rate, the insurance and coinsurance increase, however the value of the mortgage decreases as expected.

Otherwise the effect of increasing the volatilities reduces the value of the mortgage and increases the values of the insurance and the coinsurance. This variation in the volatilities also produces and increment in the contract fixed rate.

Figures 4.1 to 4.3 illustrate the values at origination of the contract, insurance and coinsurance when the arrangement fee is equal to 0.5% and the early exercise penalty takes the value of 5%. We consider the fixed parameters of the model shown in Table 4.1. In this case the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%. Moreover, Figure 4.4 shows the prepayment (coincidence) region in red and the non early exercise (non coincidence) region in blue, the curve separating both regions is the optimal retirement boundary (free boundary). The prepayment region coincides with high house prices and low interest rates because default is unlikely at high house values so the borrower is willing to prepay at high interest rates.

Finally, Table 4.7 shows the results for a case with higher volatility in the house price (20%). We notice that as soon as volatility becomes higher, although it results much cheaper from the computational point of view, neglecting second order terms in the PDE as proposed with the perturbation method in [60] can produce very inaccurate prices. On the other hand, the increase in volatility produces a decrease in the mortgage value and an increase in the insurance as expected.

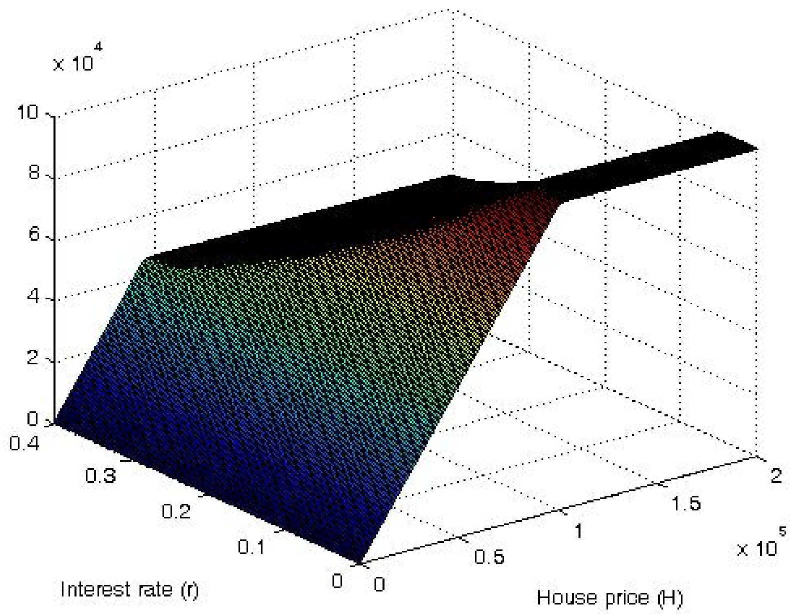

Figure 4.1: Mortgage value at origination

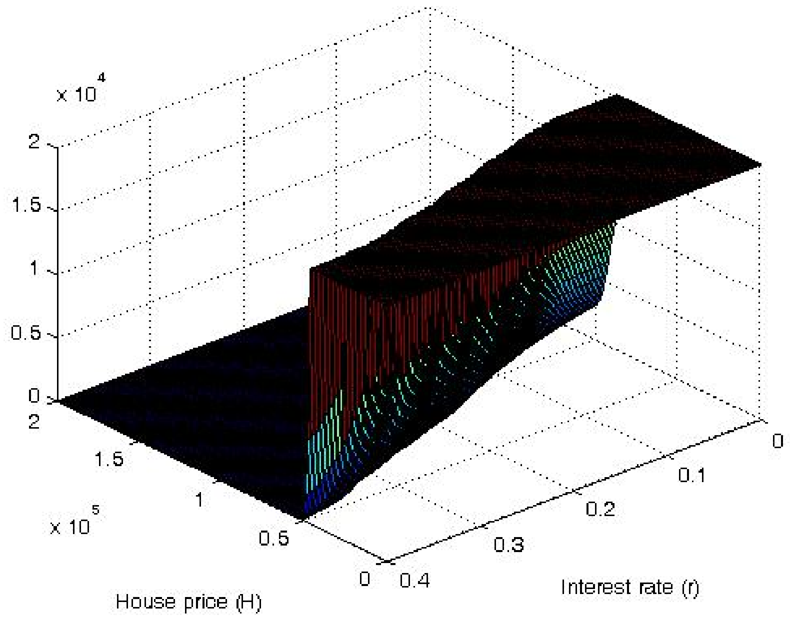

Figure 4.2: Insurance value at origination

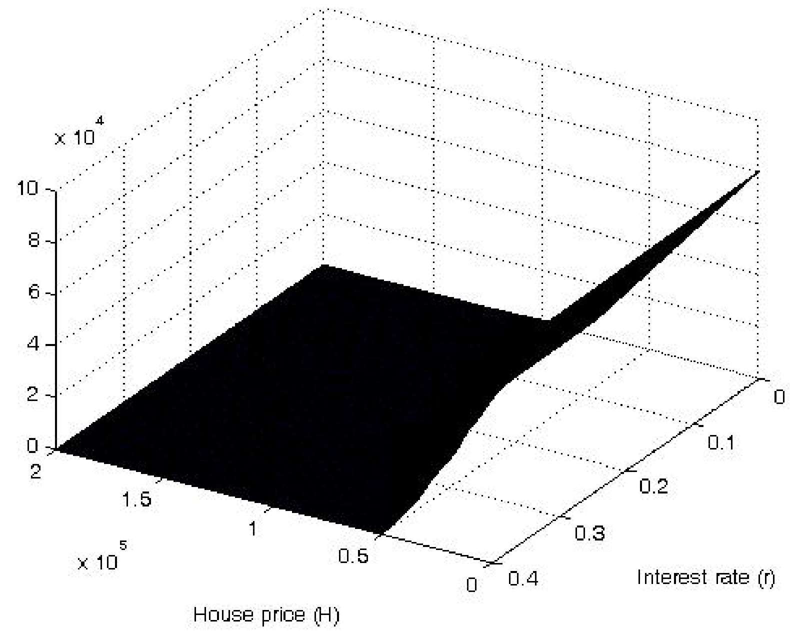

Figure 4.3: Coinsurance value at origination

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.0839%	94549	449	112
		0.5%	8.9911%	94116	410	103
		1%	8.8992%	93663	386	96
		1.5%	8.8119%	93230	345	86
	10%	0%	10.0782%	94656	343	84
		0.5%	9.9696%	94208	317	79
		1%	9.8634%	93764	288	72
		1.5%	9.7579%	93316	260	66
	12%	0%	11.1662%	94691	309	76
		0.5%	11.0389%	94274	249	62
		1%	10.9203%	93870	181	45
		1.5%	10.8006%	93422	154	38
25	8%	0%	9.2605%	93961	1039	260
		0.5%	9.1876%	93549	974	243
		1%	9.1158%	93117	933	233
		1.5%	9.0453%	92677	899	225
	10%	0%	10.1258%	94314	685	171
		0.5%	10.0369%	93878	646	162
		1%	9.9440%	93417	632	158
		1.5%	9.8551%	92970	604	151
	12%	0%	11.1585%	94536	464	116
		0.5%	11.0462%	94126	399	101
		1%	10.9219%	93667	382	94
		1.5%	10.8111%	93240	337	85

Table 4.3: Contract rate, mortgage contract, insurance and coinsurance values for $\sigma_r = 5\%$, $\sigma_H = 5\%$ and different contract specifications

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.3028%	94390	609	152
		0.5%	9.1741%	93959	566	141
		1%	9.0484%	93523	526	131
		1.5%	8.9184%	93064	511	128
	10%	0%	10.5172%	94506	494	124
		0.5%	10.3544%	94065	459	115
		1%	10.1925%	93621	429	107
		1.5%	10.0424%	93196	378	95
	12%	0%	11.8193%	94610	389	97
		0.5%	11.6207%	94161	364	91
		1%	11.4324%	93723	327	81
		1.5%	11.2617%	93270	305	76
25	8%	0%	9.5142%	93778	1222	306
		0.5%	9.3969%	93315	1209	302
		1%	9.2833%	92847	1202	300
		1.5%	9.1746%	92387	1187	296
	10%	0%	10.6232%	94102	898	224
		0.5%	10.4877%	93688	836	209
		1%	10.3441%	93235	815	203
		1.5%	10.2052%	92780	795	198
	12%	0%	11.8641%	94344	655	163
		0.5%	11.6778%	93885	639	159
		1%	11.4993%	93430	620	155
		1.5%	11.3534%	93017	557	138

Table 4.4: Contract rate, mortgage contract, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 5\%$ and different contract specifications

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.0078%	92650	2350	587
		0.5%	8.9084%	92242	2282	571
		1%	8.8132%	91845	2205	551
		1.5%	8.7195%	91446	2129	532
	10%	0%	10.0154%	92984	2015	503
		0.5%	9.8983%	92565	1960	490
		1%	9.7861%	92154	1896	474
		1.5%	9.6801%	91748	1826	456
	12%	0%	11.1181%	93270	1730	432
		0.5%	10.9775%	92849	1676	418
		1%	10.8459%	92427	1622	405
		1.5%	10.7241%	92015	1559	389
25	8%	0%	9.2191%	91407	3594	898
		0.5%	9.1386%	90991	3533	882
		1%	9.0585%	90565	3484	870
		1.5%	8.9818%	90144	3430	857
	10%	0%	10.0815%	91997	3003	751
		0.5%	9.9881%	91590	2934	733
		1%	9.9022%	91204	2845	711
		1.5%	9.8104%	90778	2797	699
	12%	0%	11.1048%	92532	2468	624
		0.5%	10.9742%	92090	2434	608
		1%	10.8564%	91675	2376	592
		1.5%	10.7423%	91239	2235	584

Table 4.5: Contract rate, mortgage contract, insurance and coinsurance values for $\sigma_r = 5\%$, $\sigma_H = 10\%$ and different contract specifications

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.2331%	92591	2409	602
		0.5%	9.1019%	92182	2343	586
		1%	8.9759%	91779	2271	568
		1.5%	8.8473%	91358	2217	554
	10%	0%	10.4358%	92933	2066	517
		0.5%	10.2713%	92508	2017	504
		1%	10.1134%	92086	1963	490
		1.5%	9.9619%	91662	1914	479
	12%	0%	11.7276%	93237	1762	440
		0.5%	11.5309%	92801	1724	431
		1%	11.3515%	92376	1674	418
		1.5%	11.1841%	91943	1632	408
25	8%	0%	9.4344%	91298	3701	926
		0.5%	9.3221%	90862	3663	917
		1%	9.2165%	90434	3615	906
		1.5%	9.1125%	90001	3574	896
	10%	0%	10.5161%	91935	3065	766
		0.5%	10.3746%	91492	3033	758
		1%	10.2381%	91049	3001	750
		1.5%	10.1078%	90608	2966	740
	12%	0%	11.7368%	92498	2502	626
		0.5%	11.5608%	92048	2476	619
		1%	11.3896%	91582	2468	616
		1.5%	11.2423%	91149	2426	606

Table 4.6: Contract rate, mortgage contract, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 10\%$ and different contract specifications

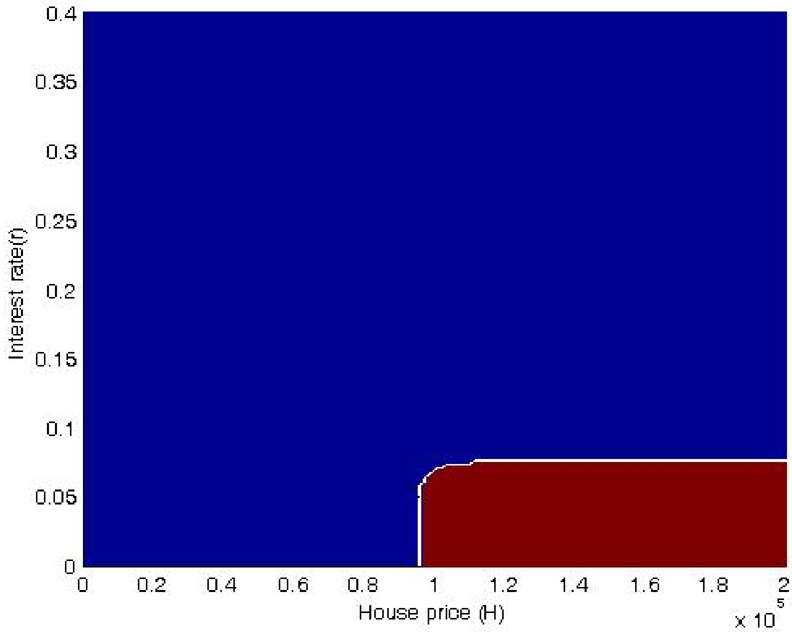

Figure 4.4: Free boundary at origination

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.3117%	87941	7059	2036
		0.5%	9.1721%	87538	6987	2050
		1%	9.0397%	87150	6900	2068
		1.5%	8.9103%	86760	6815	2078
	10%	0%	10.4659%	88460	6540	1780
		0.5%	10.3083%	88068	6457	1788
		1%	10.1506%	87664	6386	1790
		1.5%	10.0025%	87270	6305	1799
	12%	0%	11.7052%	88964	6036	1591
		0.5%	11.5177%	88550	5975	1587
		1%	11.3434%	88146	5904	1585
		1.5%	11.1749%	87738	5837	1584
25	8%	0%	9.6966%	86212	8788	3612
		0.5%	9.5921%	85829	8696	3668
		1%	9.4888%	85446	8604	3712
		1.5%	9.3908%	85074	8501	3754
	10%	0%	10.6486%	86814	8186	2860
		0.5%	10.5207%	86410	8115	2914
		1%	10.3964%	86008	8042	2981
		1.5%	10.2766%	85608	7967	3022
	12%	0%	11.7152%	87422	7578	2310
		0.5%	11.5638%	87010	7515	2348
		1%	11.4219%	86606	7444	2392
		1.5%	11.2794%	86186	7389	2412

Table 4.7: Contract rate, mortgage contract, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 20\%$ and different contract specifications

Conclusions

The objective of this work has been to contribute to the modeling, mathematical analysis and numerical solution of pricing problems related to defined benefit pension plans and fixed rate mortgages. These models are mainly posed in terms of PDEs. Once the models have been posed, the mathematical analysis tools allow to obtain the existence and uniqueness of solution in certain cases. For the effective numerical computation of pension plans and mortgages values, different sets of adequate numerical techniques for each problem have been considered. The algorithms that develop the numerical methods have been implemented and provide illustrative numerical results in agreement with the expected ones. More precisely, in the first part of this work the dynamic hedging methodology provides a PDE model associated with a Kolmogorov equation that governs the value of a defined benefit pension plan depending on the average salary and without early retirement opportunity. This methodology allows to interpret the price of this liability by using the framework of option pricing theory. Once the PDE model has been posed, the existence of a solution can be obtained by extending previously used tools in the literature for arithmetic Asian options pricing models. Also, the uniqueness of solutions can be obtained. Moreover, a Crank-Nicolson Lagrange-Galerkin method for the numerical solution is proposed so that the numerical results can be discussed in terms of the different model parameters. Moreover, these results are in agreement with the confidence intervals obtained by using a Monte Carlo simulation technique.

When the possibility of early retirement is taken into account, the pension plan pricing model can be formulated in terms of an obstacle problem associated with a

Kolmogorov equation during the early retirement opportunity time interval. From the probabilistic approach, the value of the plan is given in terms of the Snell envelope of a function depending on time, salary and average salary. Again, a rigorous mathematical analysis of the PDE problem allows us to obtain the existence, uniqueness and certain regularity properties of the solution. In order to compute the value of the pension plan and the early retirement (free) boundary, the Lagrange-Galerkin method proposed for the case without early retirement is here combined with an Augmented Lagrangian active set method based on mixed formulation. Numerical examples illustrate the performance of the numerical techniques for the PDE problem and the expected properties of the solution. Also the numerical results with the PDE formulation are in agreement with those ones obtained from the probabilistic approach when compared with the Longstaff-Schwartz simulation technique.

Moreover, if we assume that the salary follows a jump-diffusion process of Merton type then a PIDE problem arises and appropriate models are posed for the cases with and without early retirement. In this setting, the numerical methods proposed in the absence of jumps are applied jointly with the explicit treatment in time of additional integral term in the PIDE. After using an appropriate quadrature formula to discretize this term, the numerical results obtained by means of the PIDE approach are successfully compared with the ones computed with the corresponding Monte Carlo simulation techniques.

As an extension of the previous work in defined benefit pension plans, the rigorous mathematical analysis of the PIDE model analogously to the PDE one seems a natural forthcoming objective to be considered. From the practical point of view, the calibration of the parameters appearing in the models by means of either historical or market data seems a very challenging future objective.

In the second part of this thesis, we first review the statement of the PDE model for pricing fixed-rate mortgages with prepayment and default options. Note that the problems associated with mortgage, insurance and coinsurance pricing are of the same nature as the ones in pension plans, although the conditions through different months and the determination of the equilibrium rate make the rigorous formulation more complex. Next, the numerical techniques proposed for the different problems associated with pension plans are adapted for solving the PDE problems associated with mortgage, insurance and coinsurance pricing, as well as to identify the optimal prepayment boundary (free boundary). The equilibrium interest rate of the loan is obtained by a kind of secant method for a nonlinear equilibrium equation, each iteration requiring the complete solution of the coupled mortgage and insurance problems from loan maturity to origination. Numerical results illustrate the performance of the proposed numerical techniques and show the expected qualitative behaviour of the mortgage, insurance and coinsurance values, as well as the optimal prepayment boundary that separates the prepayment and non prepayment regions. The proposed set of numerical techniques is also suitable for the case of larger volatilities, where the use of perturbation techniques would require the consideration of higher order terms in the asymptotic expansion, thus increasing the complexity of the model equations and the computational cost.

The obtained results in the second part of the thesis also open the possibility of future work in different lines. For example, following the analogy with the contents in the part devoted to pension plans, the consideration of jump-diffusion models for the collateral price seems an interesting problem, which is also motivated by the recent bubble and crisis phenomena of the real state sector in many countries. From the theoretical point of view, the mathematical analysis of the model seems another challenging objective. From the practical aspect, again the calibration of the parameters in the model would require the analysis of historical data or the use of information about mortgages implicit in the market.

As a final conclusion, the present works tries to contribute to the understanding and rigorous analysis of some complex mathematical models arising in problems from quantitative finance, as well as to the appropriate selection of the suitable numerical methods to solve these models, which allows us to develop efficient software toolboxes to price some defined benefit pension plans and fixed rate mortgages.

Resumen extenso

En este trabajo se estudian modelos para valorar algunos derivados financieros específicos. En concreto, se aborda el modelado matemático, el análisis y la resolución numérica de algunos planes de pensiones y contratos hipotecarios. Se puede establecer una clasificación general de estos productos en base a algunas de sus características intrínsecas. De hecho, entre toda la variedad que existe de estos derivados, nos centraremos en el estudio de hipotecas a tipo de interés fijo y planes de pensiones con beneficios definidos.

La metodología de cobertura dinámica, introducida en los años setenta del siglo pasado por Black y Scholes [11] y Merton [45] para el caso de opciones vainilla europeas, se aplicó desde entonces a derivados más complejos [68]. En esta tesis, se aplica esta técnica para obtener las ecuaciones en derivadas parciales que gobiernan la valoración de planes de pensiones con beneficios definidos e hipotecas a tipo de interés fijo. Desde el punto de vista matemático, el precio de ambos productos se puede obtener como la solución de problemas asociados con operadores parabólicos degenerados.

Por un lado, en relación a los planes se pensiones, de manera general, se pueden clasificar en planes de pensiones con contribuciones definidas y planes de pensiones con beneficios definidos [12]. En el primer caso, cada miembro del plan tiene una cuenta individual, que se financia con sus contribuciones y con las contribuciones del patrocinador del plan. Estas contribuciones se invierten y el resultado de dichas inversiones se ingresa de nuevo en la cuenta de cada trabajador. En la fecha de jubilación el trabajador recibe una anualidad, cuyo valor depende de las ganancias de las inversiones realizadas y de las contribuciones totales de ambos, empresario y empleado, a la cuenta del empleado. Además, en algunas ocasiones el trabajador puede decidir las posibles inversiones a realizar, corriendo así con parte del riesgo. En los planes de pensiones con beneficios definidos, la pensión en el momento de jubilación viene determinada por una cantidad fija o por una fórmula establecida que puede involucrar a varios factores relacionados con la vida laboral del empleado, tales como el número de años de servicio, el salario o el salario medio. Esta cantidad no depende del resultado de las inversiones realizadas.

Anualmente, el empresario debe realizar las contribuciones necesarias para poder hacer frente al pago de los beneficios que un miembro del plan recibiría en ese año. Además, los empresarios pueden tener que realizar contribuciones adicionales por varias razones, como por ejemplo para contrarrestar las pérdidas sufridas en alguna de las inversiones. En algunos planes con beneficios definidos, el empresario puede ser penalizado si no realiza las contribuciones requeridas y puede retrasar el pago de estas contribuciones a años futuros en caso de que esté atravesando por problemas económicos. Además, existen ciertos organismos que garantizan el pago de los beneficios al empleado en caso de incumplimiento por parte del empresario.

En la parte de este trabajo dedicada a los planes de pensiones, nos centraremos en el estudio de los planes de pensiones con beneficios definidos. De este modo, el principal objetivo es obtener el valor de los beneficios por jubilación que recibirá el miembro del plan, entendiéndolo como el valor de la reserva con la que debe contar el patrocinador del plan para hacer frente a los pagos futuros prometidos.

Consideraremos que el valor del plan depende del salario del miembro del mismo como variable subyacente. Como primera aproximación, asumiremos que la dinámica de este factor es estocástica y gobernada únicamente por un movimiento Browniano geométrico, cuyas trayectorias son continuas. Sin embargo, en algunas situaciones (tales como las de crisis o burbujas en algunos sectores), pueden aparecer cambios bruscos en el salario, de modo que la consideración de este tipo de modelos estocásticos no es suficientemente realista y sería más apropiado considerar un modelo de difusión con saltos para la evolución del salario. Por este motivo, también tendremos en cuenta la posibilidad de que el salario tenga una trayectoria discontinua con un número finito de saltos siguiendo una distribución de Poisson.

Los modelos de difusión con saltos, propuestos por Merton en [46] o más recientemente por Kou en [39], se ajustan de manera más adecuada a algunos datos de mercado o situaciones con repentinos cambios bruscos en el subyacente. En este trabajo suponemos que la dinámica del salario en presencia de saltos se modela con el proceso de salto-difusión de Merton.

En la fecha de jubilación, los beneficios recibidos por un miembro del plan están indicados al salario medio de cierto número de años. En este sentido, hay una analogía entre planes de pensiones y opciones asiáticas, por lo que es necesario introducir una nueva variable representando el salario acumulado durante estos años, como paso previo a la aplicación de la fórmula de Ito y la metodología de cobertura dinámica. Una vez usadas ambas herramientas, cuando no se permite jubilación anticipada, el problema de valoración de planes de pensiones con beneficios definidos se puede escribir en términos de un problema de Cauchy asociado a un operador de Kolmogorov degenerado. Suponiendo que los beneficios por jubilación dependen del salario medio, este modelo fue introducido por Sherris y Shen en [61] basándose en argumentos alternativos. Si tenemos en cuenta la posibilidad de jubilación anticipada, aparece un problema de complementariedad asociado también al mismo operador de Kolmogorov degenerado. Cuando los beneficios en el instante de jubilación dependen únicamente del salario en esa fecha, en [26], Friedman y Shen introducen el modelo y describen el estudio de existencia de solución, así como algunas propiedades cualitativas de la frontera libre.

Sin embargo, en el presente trabajo estamos interesados en planes de pensiones basados en el salario medio. De este modo, para desarrollar el análisis matemático del problema de Cauchy, asociado con el operador de Kolmogorov para el caso sin jubilación anticipada, tendremos en cuenta su analogía con las opciones asiáticas de estilo europeo. En cuanto al estudio de existencia de solución, Barucci, Polidoro y

Vespri demostraron la existencia y unicidad de solución para opciones asiáticas de tipo europeo en [6]. Para obtener los resultados correspondientes para planes de pensiones con beneficios definidos, fundamentalmente extendemos los resultados para ecuaciones de Kolmogorov homogéneas al caso no homogéneo.

Para el análisis matemático de los problemas de complementariedad que gobiernan la valoración de planes de pensiones con opción de jubilación anticipada, el principal punto de partida es el artículo [47] de Pascucci y Monti, donde se prueba la existencia y regularidad de la solución fuerte del problema de obstáculo asociado con el problema de valoración de opciones asiáticas de tipo americano sobre la media aritmética. Así, básicamente en esta tesis se extendieron algunos de los teoremas demostrados en [47] y [24].

En relación a la resolución numérica de los modelos para valorar los planes de pensiones, primero debemos señalar que nos encontramos con varias dificultades. Por un lado, el dominio no está acotado en las direcciones de las variables espaciales. Para solucionar este problema truncaremos el dominio e impondremos condiciones de contorno adecuadas. Por otro lado, la matriz de difusión es degenerada y por ello utilizaremos un método de características de alto orden. De manera más precisa, en este trabajo proponemos un método de Lagrange-Galerkin de alto orden para la discretización espacial y temporal. Más concretamente, utilizaremos el método de características de alto orden para la discretización en tiempo y elementos finitos cuadráticos para la discretización en espacio. Dicho método se basa en el introducido inicialmente por Rui y Tabata en [57] para una ecuación de convección-difusión con coeficientes constantes. Más tarde fue extendido a otros problemas de convección-difusión-reacción (incluso degenerados) en [7] y [8]. Este método resulta adecuado en problemas con convección dominante, como es el caso de la valoración de opciones asiáticas de tipo europeo o de los productos derivados que se estudian en esta tesis. En el caso de planes de pensiones, cuando se permite la opción de jubilación anticipada, este esquema numérico se combina con el algoritmo iterativo de tipo Augmented

Lagrangian Active Set (ALAS) propuesto en [36] para tratar las no linealidades asociadas con las restricciones de desigualdad en los problemas de frontera libre, que modelan la valoración de estos derivados.

Además, para el caso de planes de pensiones con presencia de saltos en el salario, el término integral en el operador integro-diferencial se discretiza de manera explícita en tiempo, entrando en el segundo miembro del problema discreto. El valor de esta integral se aproxima usando la regla del trapecio compuesto.

En este trabajo, también se implementó una técnica de simulación de tipo Monte Carlo (véase [27], una referencia general con aplicaciones financieras) para obtener el valor del plan de pensiones cuando no se permite la opción de jubilación anticipada. Además, para obtener el valor del plan cuando la jubilación anticipada está permitida, se desarrolló el algoritmo propuesto para opciones americanas por Longstaff y Schwartz en [42]. Para dichas implementaciones, suponemos que el salario bajo una medida de riesgo neutro sigue una ecuación diferencial estocástica y, al igual que en el caso de las opciones asiáticas, es necesario simular caminos sobre múltiples fechas para aproximar la integral que aparece en la definición de la variable que representa al salario acumulado. Para aplicar Monte Carlo en presencia de saltos en el salario, se optó por simular el salario en un conjunto fijo de fechas sin tener en cuenta explícitamente el efecto de los términos de salto y difusión [27].

Por otro lado, en la segunda parte de esta tesis, se aborda el problema de valoración de hipotecas a tipo de interés fijo.

Una hipoteca es un contrato financiero a través del cual el prestatario obtiene fondos, normalmente de un banco o una institución financiera, usando la propiedad como garantía (colateral). Los principales componentes del contrato son:

- la propiedad que se financia,
- el documento legal emitido por el prestamista para asegurar el pago de la deuda, que puede contener restricciones sobre el uso y disposición de la propiedad por parte del prestamista,

- la persona que toma prestado el dinero y que está interesada en poseer la propiedad,
- el prestamista, que es normalmente un banco u otra institución financiera, aunque también podrían ser inversores,
- la cantidad inicial prestada, que puede coincidir con el valor total de la casa o ser menor para reducir el riesgo,
- el tipo de interés por disponer del dinero del prestamista
- y el pago final de la deuda, que puede ser al final del plazo temporal establecido o el prestatario puede tener la opción de amortizar de manera anticipada antes de vencimiento la cantidad adeudada, de manera similar por ejemplo al caso de bonos con opción de recompra (*callable*).

Podemos establecer diferentes tipos de contratos teniendo en cuenta algunas de las características de una hipoteca como, por ejemplo, el tipo de interés, el número de años en los que se pagará el crédito, la cantidad a pagar y la frecuencia de pago, o si existe o no la posibilidad de amortización anticipada. Teniendo en cuenta el tipo de interés, se pueden considerar hipotecas a tipo de interés fijo (Fixed-Rate Mortgages(FRM)) o hipotecas a tipo de interés variable (Adjustable-Rate Mortgages(ARM)). En el primer caso, el tipo de interés que el prestatario tiene que pagar y los pagos periódicos son fijos durante toda la vida del contrato. Sin embargo, en el segundo caso el tipo de interés es variable y de acuerdo con un índice concreto (LIBOR o EURIBOR, por ejemplo). Este tipo de interés se mantiene fijo durante cierto periodo (por ejemplo, un año) después del cual se ajusta de manera periódica según el índice elegido.

En este trabajo estudiamos las hipotecas a tipo de interés fijo, en las cuales dicho tipo satisface una condición de equilibrio que garantiza la ausencia de arbitraje. Esta condición consiste en que, al inicio del contrato, la suma del valor de la hipoteca junto con el seguro y cualquier tipo de comisión establecida entre el prestatario y el prestamista, debe ser igual a la cantidad prestada. Así, el tipo de interés del contrato se ajustará mediante un proceso iterativo. Además, consideraremos que la persona que toma prestado el dinero tiene las opciones de amortización anticipada del crédito y de impago de la deuda, en este último caso se perderán los pagos futuros acordados entre el prestatario y el prestamista, a no ser que este último tenga contratado algún seguro sobre el crédito que lo proteja en estas situaciones. El objetivo principal es obtener el valor del contrato para el prestamista, así como el valor de otros componentes del contrato, como son el seguro y la fracción de la pérdida potencial no cubierta por el seguro.

El valor del contrato y del resto de componentes depende del precio de la casa y del tipo de interés como variables estocásticas subyacentes. Consideramos que la dinámica del precio de la casa sigue una ecuación diferencial estocástica gobernada por un movimiento Browniano geométrico. Entre todos los posibles modelos para describir la dinámica del tipo de interés, supondremos que su evolución está determinada por el modelo de Cox-Ingersoll-Ross(CIR) [18]. En contraposición con el modelo de Vasicek [65], el modelo de CIR garantiza que los tipos de interés son positivos bajo ciertas hipótesis sobre los parámetros y también satisface la deseada propiedad de reversión a la media.

El valor del contrato es el valor presente para el prestamista de los pagos mensuales acordados del prestatario al prestamista, sin tener en cuenta el seguro sobre el crédito que el prestamista puede tener como medio de protección, en caso de que el prestatario no cumpla con sus pagos.

En este trabajo consideraremos la opción de amortización anticipada por parte del prestatario y la posibilidad de que este mismo incumpla sus pagos. De manera más concreta, si tenemos en cuenta que la opción de amortización anticipada puede ocurrir en cualquier momento durante la vida del contrato y que el impago solo puede pasar en fechas de pago, el problema de valoración se traduce en una sucesión de opciones americanas enlazadas, una para cada mes.

Además, como se menciona anteriormente, al inicio del contrato el tipo de interés del mismo debe satisfacer cierta condición de equilibrio, de no ser así, existiría arbitraje. Para la obtención de dicho tipo de interés se implementó un método de Newton con aproximación discreta de la derivada (método de la secante).

Mediante la técnica de cobertura dinámica se deduce la ecuación en derivadas parciales para obtener el valor de cualquier activo que dependa del precio de la casa y del tipo interés. En particular, el problema de valoración de una hipoteca con opciones de amortización anticipada e incumplimiento por parte del prestatario se plantea como una sucesión de problemas de frontera libre donde la condición final para un mes es el valor del contrato obtenido en el mes siguiente, de modo que la opción de amortización anticipada se puede tratar como un problema de frontera libre. Por otro lado, para obtener el valor del seguro y de la fracción de la pérdida no cubierta por el seguro se debe resolver una sucesión de problemas de Cauchy enlazados entre sí, donde de nuevo la condición final para un mes es el valor obtenido en el mes siguiente.

En cuanto a la solución numérica de los modelos de valoración de contratos hipotecarios, proponemos el uso de técnicas análogas a las utilizadas en los planes de pensiones. Para ser más precisos, para cada mes se resuelve el problema de complementariedad relacionado con el valor de la hipoteca aplicando las técnicas numéricas descritas para los planes de pensiones con jubilación anticipada y, además, se resuelve el problema de Cauchy asociado al el valor del seguro con las técnicas numéricas usadas para los planes de pensiones sin jubilación anticipada. Una vez que llegamos al inicio del contrato, se actualiza el valor del tipo de interés y se resuelven de nuevo ambos problemas hasta obtener el tipo de equilibrio. Una vez ajustado el tipo de interés, se obtiene el valor de la fracción de la pérdida no cubierta por el seguro, resolviendo la correspondiente sucesión de problemas de Cauchy.

Nótese que los modelos y métodos numéricos propuestos muestran un buen comportamiento cuando las volatilidades son pequeñas (en problemas de convección dominante), donde el trabajo [60] propone el uso de ecuaciones en derivadas parciales de primer orden obtenidas mediante una técnica de desarrollos asintóticos. También, el modelo y los métodos propuestos en esta tesis resultan adecuados cuando las volatilidades son altas, situación en la que los modelos propuestos en [60] requieren del uso de términos de mayor orden de los desarrollos asintóticos, incrementando así la complejidad y el coste computacional.

El esquema de esta memoria es el siguiente:

La Parte I, que consta de tres capítulos, está dedicada a los planes de pensiones. En primer lugar damos una breve introducción a los planes de pensiones, describiendo sus características principales y posibles clasificaciones en término de sus propiedades intrínsecas. También se explica en qué consisten los planes de pensiones con beneficios definidos, que serán en los que se centra este trabajo.

En el Capítulo 1 se plantea el modelo para valorar planes de pensiones con beneficios definidos y sin jubilación anticipada como un problema de Cauchy asociado con un operador de Kolmogorov degenerado. Para ello, al inicio del capítulo, se presenta la ecuación diferencial estocástica que gobierna la dinámica del salario y, de manera similar al caso de opciones asiáticas, se introduce una nueva variable que representa el salario acumulado durante cierto número de años antes de la fecha de jubilación. A continuación, utilizando la técnica de cobertura dinámica, se deduce la ecuación en derivadas parciales (EDP) que rige la valoración del plan y se plantea dicho problema de valoración como un problema de Cauchy asociado a un operador de Kolmogorov degenerado. Además, se realiza un estudio de la existencia y unicidad de solución del modelo y se describen los métodos numéricos aplicados. Finalmente, se analizan y comparan algunos resultados numéricos obtenidos resolviendo la EDP y mediante la técnica de simulación de Monte Carlo.

El Capítulo 2 también está dedicado a planes de pensiones y en este caso, como se permite la opción de jubilación anticipada, la valoración de este tipo de planes se formula como un problema de obstáculo de manera similar al caso de opciones americanas. A continuación, se analiza la existencia de solución de dicho problema de complementariedad y se implementan los métodos numéricos apropiados para obtener una solución numérica de dicho problema. Por último, presentamos algunos resultados numéricos, así como una comparación entre los casos con y sin jubilación anticipada y el aspecto de la frontera libre. Estos resultados se obtuvieron resolviendo la EDP y mediante la técnica de simulación de Monte Carlo propuesta por Longstaff y Schwartz.

En el Capítulo 3 consideramos la posibilidad de que haya saltos en el salario. En primer lugar, introducimos el modelo de salto-difusión que describe la dinámica del salario. A continuación, planteamos el modelo matemático asociado con un operador integro-diferencial que gobierna la valoración de planes de pensiones bajo dicho proceso de salto-difusión, con y sin jubilación anticipada. Finalmente, obtenemos una solución del modelo usando esquemas numéricos apropiados y se muestran algunos resultados. Además de la resolución de la ecuación integro-diferencial, se describe como simular el salario con saltos y la implementación de Monte Carlo para este caso.

La Parte II, constituida por un capítulo, se centra en el estudio de las hipotecas, por lo que en primer lugar señalamos algunas de las características de un contrato hipotecario y describimos los principales componentes de dicho contrato. Además, establecemos una clasificación de este derivado teniendo en cuenta sus características.

En el Capítulo 4 establecemos el modelo matemático que gobierna la valoración de hipotecas a tipo de interés fijo, con amortización anticipada y posibilidad de impago por parte del prestatario. También planteamos los modelos matemáticos para valorar otros dos componentes del contrato: el seguro que el prestamista puede tener sobre el crédito y la fracción de pérdida potencial que no cubre el seguro. Para ello, en primer lugar, introducimos las ecuaciones diferenciales estocásticas que gobiernan la dinámica del precio de la casa y del tipo de interés. A continuación, derivamos la EDP para valorar la hipoteca y sus componentes, presentando el problema de obstáculo que surge cuando se permite la amortización anticipada y los problemas de Cauchy que gobiernan la valoración de los otros componentes del contrato. Finalmente, describimos como resolver dichos problemas usando técnicas numéricas apropiadas y presentamos algunos de los resultados obtenidos.

Por último, mencionamos que la mayor parte de los resultados indicados en el

Capítulo 1 aparecen recogidos en la referencia [14], los incluidos en el Capítulo 2 se encuentran en artículo [15] y los recogidos en el Capítulo 4 aparecen en [16].

Resumo extenso

Neste traballo estúdanse modelos para valorar algúns derivados financeiros específicos. En concreto, abórdase o modelado matemático, a análise e a resolución numérica dalgúns plans de pensións e contratos hipotecarios. Pódese establecer unha clasificación xeral destos produtos tendo en conta algunhas das súas características intrínsecas. De feito, entre toda a variedade que existe destos derivados, centrarémonos no estudo de hipotecas a tipo de xuro fixo e plans de pensións con beneficios definidos.

A metodoloxía de cobertura dinámica, introducida nos anos setenta do pasado século por Black e Scholes [11] e Merton [45] para o caso de opcións vainilla europeas, aplicouse dende entón a derivados mais complexos [68]. Nesta tese, aplícase esta técnica para obter as ecuacións en derivadas parciais que gobernan a valoración de plans de pensións con beneficios definidos e hipotecas a tipo de xuro fixo. Dende o punto de vista matemático o prezo de ámbolos dous produtos pódese obter como a solución de problemas asociados con operadores parabólicos dexenerados.

Por unha banda, en relación aos plans de pensións, de maneira xeral, pódense clasificar en plans de pensións con contribucións definidas e plans de pensións con beneficios definidos [12]. No primeiro caso, cada membro do plan ten unha conta individual, que se financia coas súas contribucións e coas contribucións do patrocinador do plan. Estas contribucións invírtense e o resultado de ditos investimentos ingrésase de novo na conta de cada traballador. Na data de xubilación o traballador recibe unha anualidade cuxo valor depende das ganancias dos investimentos realizados e das contribucións totais de ambos, empresario e empregado, á conta do empregado. Ademais, nalgunhas ocasións o traballador pode decidir os posibles investimentos a realizar, correndo desde xeito con parte do risco. Nos plans de pensións con beneficios definidos, a pensión no momento de xubilación ven determinada por unha cantidade fixa ou por unha fórmula establecida que pode involucrar varios factores relacionados coa vida laboral do empregado, tales como o número de anos de servicio, o salario ou o salario medio. Esta cantidade non depende do resultado dos investimentos realizados.

Anualmente, o empresario debe realizar as contribucións necesarias para poder facer fronte ao pagamento dos beneficios que un membro do plan recibiría nese ano. Ademais, os empresarios poden ter que realizar contribucións adicionais por varias razóns, como por exemplo para contrarrestar as pérdas sufridas nalgún dos investimentos. Nalgúns plans con beneficios definidos, o empresario pode ser penalizado no caso de non realizar as contribucións requiridas e pode atrasar o pagamento destas contribucións a anos futuros cando estea a atravesar por problemas económicos. Ademais, existen certos organismos que garanten o pagamento dos beneficios ao empregado en caso de incumprimento por parte do empresario.

Na parte deste traballo dedicado aos plans de pensións, centrarémonos no estudo dos plans de pensións con beneficios definidos. Deste modo, o principal obxectivo é obter o valor dos beneficios por xubilación que recibirá o membro do plan, entendéndoo como o valor da reserva coa que debe contar o patrocinador do plan para facer fronte aos pagamentos futuros prometidos.

Consideraremos que o valor do plan depende do salario do membro do mesmo como variable subxacente. Como primeira aproximación, asumiremos que a dinámica deste factor é estocástica e gobernada unicamente por un movemento Browniano xeométrico, cuxas traxectorias con continuas. Non obstante, nalgunhas situacións (tales como as de crise ou burbullas nalgúns sectores), poden aparecer cambios bruscos no salario, de xeito que a consideración deste tipo de modelos estocásticos non é suficientemente realista e seria máis apropiado considerar un modelo de difusión con saltos para a evolución do salario. Por este motivo, tamén teremos en conta a posibilidade de que o salario teña unha traxectoria discontinua cun número finito de saltos seguindo unha distribución de Poisson.

Os modelos de difusión con saltos, propostos por Merton en [46] ou máis recentemente por Kou en [39], axústanse de xeito máis axeitado a algúns datos de mercado ou situacións con repentinos cambios bruscos no subxacente. Neste traballo supoñemos que a dinámica do salario en presenza de saltos se modela co proceso de salto-difusión de Merton.

Na data de xubilación, os beneficios recibidos por un membro do plan están indicados ao salario medio de certo número de anos. Neste sentido, hai unha analoxía entre plans de pensións e opcións asiáticas e é necesario introducir unha nova variable representando o salario acumulado durante estes anos, como paso previo á aplicación da fórmula de Ito e á metodoloxía de cobertura dinámica. Unha vez usadas ambas ferramentas, cando non se permite xubilación anticipada, o problema de valoración de plans de pensións con beneficios definidos pódese escribir en termos dun problema de Cauchy asociado a un operador de Kolmogorov dexenerado. Supoñendo que os beneficios por xubilación dependen do salario medio, este modelo foi introducido por Sherris e Shen en [61] baseándose en argumentos alternativos. Se temos en conta a posibilidade de xubilación anticipada, aparece un problema de complementariedade asociado tamén ao mesmo operador de Kolmogorov dexenerado. Cando os beneficios no instante de xubilación dependen unicamente do salario nesa data, en [26], Friedman e Shen introducen o modelo e describen o estudo de existencia de solución, así como algunhas propiedades cualitativas da fronteira libre.

Non obstante, neste traballo estamos interesados en plans de pensións baseados no salario medio. Deste xeito, para levar a cabo a análise matemática do problema de Cauchy, asociado ao operador de Kolmogorov para o caso sen xubilación anticipada, teremos en conta a súa analoxía coas opcións asiáticas de estilo europeo. En canto ao estudo de existencia de solución, Barucci, Polidoro e Vespri demostraron a existencia e unicidade de solución para opcións asiáticas europeas en [6]. Para obter os resultados correspondentes para plans de pensións con beneficios definidos, fundamentalmente estendemos os resultados para ecuacións de Kolmogorov homoxéneas ao caso non homoxéneo.

Para a análise matemática dos problemas de complementariedade que gobernan a valoración de plans de plans de pensións con opción de xubilación anticipada o principal punto de partida é o artigo de Pascucci e Monti, onde se proba a existencia e regularidade da solución forte do problema de obstáculo asociado ao problema de valoración de opcións asiáticas de tipo americano sobre a media aritmética. Deste xeito, básicamente nesta tese estenderonse algúns dos teoremas demostrados en [47] e [24].

En relación á resolución numérica dos modelos para valorar os plans de pensións, primeiro debemos salientar que nos atopamos con varias dificultades. Por un lado, o dominio non está acoutado nas direccións das variables espaciais. Para solucionar este problema truncaremos o dominio e impoñeremos condicións de contorno axeitadas. Por outro lado, a matriz de difusión é dexenerada e por iso utilizaremos un método de características de alta orde. De xeito máis preciso, neste traballo propoñemos un método de Lagrange-Galerkin de alta orde para a discretización espacial e temporal. Máis concretamente, utilizaremos o método de características de alta orde para a discretización en tempo e elementos finitos cuadráticos para a discretización en espazo. O devandito método baséase no introducido inicialmente por Rui e Tabata en [57] para unha ecuación de convección-difusión con coeficientes constantes. Máis tarde estendeuse a outros problemas de convección-diffusión-reacción (incluso dexenerados) en [7] e [8]. Este método adoita ser axeitado en problemas con convección dominante, como é o caso da valoración de opcións asiáticas de tipo europeo ou dos produtos derivados que se estudan nesta tese. No caso de plans de pensións, cando se permite a opción de xubilación anticipada, este esquema numérico combínase co algoritmo iterativo de tipo Augmented Lagrangian Active Set (ALAS) proposto en [36] para tratar as non linealidades asociadas coas restricións de desigualdade nos problemas de fronteira libre, que modelan a valoración destes derivados.

Ademais, para o caso de plans de pensións con presenza de saltos no salario, o termo integral no operador integro-diferencial discretízase de maneira explícita en tempo, entrando no segundo membro do problema discreto. O valor desta integral aproxímase usando a regra do trapecio composto.

Neste traballo, tamén se implementou unha técnica de simulación de tipo Monte Carlo (véxase [27], unha referencia xeral con aplicacións financeiras) para obter o valor do plan de pensións cando non se permite a opción de xubilación anticipada. Ademais, para obter o valor do plan cando a xubilación anticipada está permitida, desenvolveuse o algoritmo proposto para opcións americanas por Longstaff e Schwartz en [42]. Para as devanditas implementacións, supoñemos que o salario baixo unha medida de risco neutro segue unha ecuación diferencial estocástica e, ao igual que no caso das opcións asiáticas, é necesario simular camiños sobre múltiples datas para aproximar a integral que aparece na definición da variable que representa ao salario acumulado. Para aplicar Monte Carlo en presenza de saltos no salario, optouse por simular o salario nun conxunto fixo de datas sen ter en conta explicitamente o efecto dos termos de salto e difusión [27].

Por outro lado, na segunda parte desta tese, abórdase o problema de valoración de hipotecas a tipo de xuro fixo.

Unha hipoteca é un contrato financeiro a través do cal o prestameiro obtén fondos, normalmente dun banco ou dunha institución financeira, usando a propiedade como garante (colateral). Os principais compoñentes do contrato son:

- a propiedade que se financia,
- o documento legal emitido polo prestamista para asegurar o pagamento da débeda, que pode conter restricións sobre o uso e disposición da propiedade por parte do prestamista,
- a persoa que toma prestado o diñeiro e que está interesada en posuír a propiedade,
- o prestamista, que é normalmente un banco ou outra institución financeira aínda que tamén poderían ser investidores,
- a cantidade inicial prestada, que pode coincidir co valor total da casa ou ser menor para reducir o risco,

- o tipo de xuro por dispoñer do diñeiro do prestamista
- e o pagamento final da débeda, que pode ser ao final do prazo temporal establecido ou o prestameiro pode ter a opción de amortizar de xeito anticipado antes de vencemento a cantidade debida, de xeito similar por exemplo ao caso de bonos con opción de recompra (*callable*).

Podemos establecer diferentes tipos de contratos tendo en conta algunhas das características dunha hipoteca como, por exemplo, o tipo de xuro, o número de anos nos que se pagará o crédito, a cantidade a pagar e a frecuencia de pagamento, ou se existe ou non a posibilidade de amortización anticipada. Tendo en conta o tipo de xuro, pódense considerar hipotecas a tipo de xuro fixo (Fixed-Rate Mortgages(FRM)) ou hipotecas a tipo de xuro variable (Adjustable-Rate Mortgages(ARM)). No primeiro caso, o tipo de xuro que o prestameiro ten que pagar e os pagamentos periódicos son fixos durante toda a vida do contrato. Non obstante, no segundo caso o tipo de xuro é variable e de acordo cun índice concreto (LIBOR ou EURIBOR, por exemplo). Este tipo de xuro mantense fixo durante certo período (por exemplo, un ano) despois do cal se axusta de xeito periódico segundo o índice elixido.

Neste traballo estudamos as hipotecas a tipo de xuro fixo, nas cales o devandito tipo satisfai unha condición de equilibrio que garante a ausencia de arbitraxe. Esta condición consiste en que, ao inicio do contrato, a suma do valor da hipoteca xunto co seguro e calquera tipo de comisión establecida entre o prestamista e o prestameiro, debe ser igual á cantidade prestada. Así, o tipo de xuro do contrato axustarase mediante un proceso iterativo. Ademais, consideraremos que a persoa que toma prestado o diñeiro ten as opcións de amortización anticipada do crédito e de non pagamento da débeda, en cuxo caso se perderán os pagamentos futuros acordados entre o prestameiro e o prestamista, a non ser que este último teña contratado algún seguro sobre o crédito que o protexa nestas situacións. O obxectivo principal é obter o valor do contrato para o prestamista, así como o valor doutros compoñentes do contrato, como son o seguro e a fracción da perda potencial non cuberta polo seguro.

O valor do contrato e do resto de compoñentes depende do prezo da casa e do tipo de xuro como variables estocásticas subxacentes. Consideramos que a dinámica do prezo da casa segue unha ecuación diferencial estocástica gobernada por un movemento Browniano xeométrico. Entre todos os posibles modelos para describir a dinámica do tipo de xuro, supoñeremos que a súa evolución está determinada polo modelo de Cox-Ingersoll-Ross(CIR) [18]. En contraposición co modelo de Vasicek [65], o modelo de CIR garante que os tipos de xuro son positivos baixo certas hipóteses sobre os parámetros e que se satisfai a propiedade de reversión á media.

O valor do contrato é o valor presente para o prestamista dos pagamentos mensuais acordados do prestameiro ao prestamista, sen ter en conta o seguro sobre o crédito que o prestamista pode ter como medio de protección, en caso de que o prestameiro non cumpra cos seus pagamentos.

Neste traballo consideraremos a opción de amortización anticipada por parte do prestameiro e a posibilidade de que este mesmo incumpra os seus pagamentos. De xeito máis concreto, se temos en conta que a opción de amortización anticipada pode acontecer en calquera momento durante a vida do contrato e que o incumprimento só pode pasar en datas de pagamento, o problema de valoración tradúcese nunha sucesión de opcións americanas enlazadas, unha para cada mes.

Ademais, como se menciona anteriormente, ao inicio do contrato o tipo de xuro do mesmo debe satisfacer certa condición de equilibrio, de non ser así, existiría arbitraxe. Para a obtención do devandito tipo de xuro implementouse un método de Newton coa aproximación discreta da derivada (método da secante).

Mediante a técnica de cobertura dinámica dedúcese a ecuación en derivadas parciais para obter o valor de calquera activo que dependa do prezo da casa e do tipo de xuro. En particular, o problema de valoración dunha hipoteca con opcións de amortización anticipada e incumprimento por parte do prestameiro formúlase como unha sucesión de problemas de fronteira libre, onde a condición final para un mes é o valor do contrato obtido no mes seguinte, de xeito que a opción de amortización anticipada se pode tratar como un problema de fronteira libre. Por outro lado, para obter o valor do seguro e da fracción da perda non cuberta polo seguro débese resolver unha sucesión de problemas de Cauchy enlazados entre si, onde de novo a condición final para un mes é o valor obtido no mes seguinte.

En canto a solución numérica dos modelos de valoración de contratos hipotecarios, propoñemos o uso de técnicas análogas ás utilizadas nos plans de pensións. Sendo mais precisos, para cada mes resolvemos o problema de complementariedade asociado co valor da hipoteca aplicando ás técnicas numéricas descritas para os plans de pensións con xubilación anticipada e ademais resolvemos o problema de Cauchy asociado ao valor do seguro coas técnicas numéricas usadas para os plans de pensións sen xubilación anticipada. Unha vez chegados ao inicio do contrato, actualizamos o valor do tipo de xuro e resolvemos de novo os dous problemas ata atopar o tipo de equilibrio. Unha vez axustado o tipo de xuro, obtense o valor da fracción da perda non cuberta polo seguro, resolvendo a correspondente sucesión de problemas de Cauchy.

Nótese que os modelos e métodos numéricos propostos mostran un bon comportamento cando as volatilidades son pequenas (en problemas de convección dominante), onde o traballo [60] propón o uso de ecuacións en derivadas parciais de primeira orde obtidas mediante a técnica de desenvolvementos asintóticos. Tamén, o modelo e os métodos propostos nesta tese, resultan axeitados cando as volatilidades son altas, situación na cal os modelos propostos en [60] requiren do uso de termos de maior orde dos desenvolvementos asintóticos, incrementando así a complexidade e o custo computacional.

O esquema desta memoria é o seguinte:

A Parte I, que consta de tres capítulos, está dedicada aos plans de pensións. En primeiro lugar damos unha breve introdución aos plans de pensións, describindo as súas características principais e posibles clasificacións en función das súas propiedades intrínsecas. Tamén se explica en que consisten os plans de pensións con beneficios definidos, que serán nos que se centra este traballo.

No Capítulo 1 formúlase o modelo para valorar plans de pensións con beneficios definidos e sen xubilación anticipada como un problema de Cauchy asociado cun ope-rador de Kolmogorov dexenerado. Para iso, ao inicio do capítulo, preséntase a ecuación diferencial estocástica que goberna a dinámica do salario e, de xeito similar ao caso de opcións asiáticas, introdúcese tamén unha nova variable que representa o salario acumulado durante certo número de anos antes da data de xubilación. A continuación, utilizando a técnica de cobertura dinámica, dedúcese a ecuación en derivadas parciais (EDP) que rexe a valoración do plan e formúlase o devandito problema de valoración como un problema de Cauchy asociado a un operador de Kolmogorov dexenerado. Ademais, realízase un estudo da existencia e unicidade de solución do modelo e descríbense os métodos numéricos aplicados. Finalmente, analízanse e compáranse algúns resultados numéricos obtidos resolvendo a EDP e mediante a técnica de simulación de Monte Carlo.

O Capítulo 2 tamén está dedicado a plans de pensións e neste caso, como se permite a opción de xubilación anticipada, a valoración deste tipo de plans formúlase como un problema de obstáculo de xeito similar ao caso de opcións americanas. A continuación, analízase a existencia de solución do devandito problema de complementariedade e impleméntanse métodos numéricos apropiados para obter unha solución numérica do devandito problema. Por último, presentamos algúns resultados numéricos, así como unha comparación entre os casos con e sen xubilación anticipada e o aspecto da fronteira libre. Estes resultados obtivéronse resolvendo a EDP e mediante a técnica de simulación de Monte Carlo proposta por Longstaff e Schwartz.

No Capítulo 3 consideramos a posibilidade de que haxa saltos no salario. En primeiro lugar, introducimos o modelo de salto-difusión que describe a dinámica do salario. A continuación, formulamos o modelo matemático asociado cun operador integro-diferencial que goberna a valoración de plans de pensións baixo o devandito proceso de salto-difusión, con e sen xubilación anticipada. Finalmente, obtemos unha solución do modelo usando esquemas numéricos apropiados e móstranse algúns resultados. Ademais da resolución da ecuación integro-diferencial, descríbese como simular o salario con saltos e a implementación de Monte Carlo para este caso.

A Parte II, constituída por un capítulo, céntrase no estudo das hipotecas, polo que en primeiro lugar sinalamos algunhas das características dun contrato hipotecario e describimos os principais compoñentes do devandito contrato. Ademais, establecemos unha clasificación deste derivado tendo en conta algunhas das súas características.

No Capítulo 4 establecemos o modelo matemático que goberna a valoración de hipotecas a tipo de xuro fixo, con amortización anticipada e posibilidade de incumprimento por parte do prestameiro. Tamén plantexamos os modelos matemáticos para valorar outros dous compoñentes do contrato: o seguro que o prestamista pode ter sobre o crédito e a fracción de perda potencial que non cobre o seguro. Para iso, en primeiro lugar, introducimos as ecuacións diferenciais estocásticas que gobernan a dinámica do prezo da casa e do tipo de xuro. A continuación, derivamos a EDP para valorar a hipoteca e os seus compoñentes, presentando o problema de obstáculo que xorde cando se permite a amortización anticipada e os problemas de Cauchy que gobernan a valoración dos outros compoñentes do contrato. Finalmente, describimos como resolver os devanditos problemas usando técnicas numéricas apropiadas e presentamos algúns dos resultados obtidos.

Para rematar, mencionamos que a meirande parte dos resultados indicados no Capítulo 1 aparecen recollidos na referencia [14], os incluidos no Capítulo 2 atópanse no artigo [15] e os recollidos no Capítulo 4 aparecen en [16].

Bibliography

[1] A. Almendral and C. W. Oosterlee, Numerical Valuation of options with jumps in the underlying, Applied Numerical Mathematics, 53 (2005), 1-18.

[2] L. Andersen and J. Andreasen, Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing, Review of Derivatives Research, 4 (2000), 231-262.

[3] D. G. Aronson and P. Besala, Uniqueness of solution for parabolic equations with unbounded coefficients, Colloquium Mathematicum, 18 (1967), 125-135.

[4] J. A. Azevedo-Pereira, D. P. Newton and D. A. Paxson, UK Fixed Rate Repayment Mortgage and Mortgage Indemnity Valuation, Real Estate Economics, 30 (2002), 185-211.

[5] G. Barles, C. Daher and M. Romano, Convergence of numerical schemes for parabolic equations arising in finance theory, Mathematical Models and Methods in Applied Sciences, 5 (1995), 125-143.

[6] E. Barucci, S. Polidoro and V. Vespri, Some results on partial differential equations and Asian options, Mathematical Models and Methods in Applied Sciences, 21 (2001), 475-497.

[7] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part I: Time discretization, SIAM Journal on Numerical Analysis, 44 (2006), 1829-1853.

[8] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part II: Fully discretized scheme and quadrature formulas, SIAM Journal on Numerical Analysis, 44 (2006), 1854-1876.

[9] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical solution of variational inequalities for pricing Asian options by higher order Lagrange-Galerkin methods, Applied Numerical Mathematics, 56 (2006), 1256-1270.

[10] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Comparison of two algorithms to solve a fixed-strike Amerasian options pricing problem, in International Series in Numerical Mathematics, 154 (2006), 95-106.

[11] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political. Economy, 81 (1973), 637-654.

[12] Z. Bodie, Pensions as retirement income insurance, Journal of Economic Literature, 18, (1990), 28-49.

[13] N. L. Jr. Bowers, H. V. Gerber, J. C. Nickman and C. J. Nesbitt, Actuarial Mathematics, Society of Actuaries, Schaumburg, Illinois (1997).

[14] M. C. Calvo-Garrido and C. Vázquez, Pricing pension plans based on average salary without early retirement: PDE modelling and numerical solution, Journal of Computational Finance, 16, (2012), 111-140.

[15] M. C. Calvo-Garrido, A. Pascucci and C. Vázquez, Mathematical analysis and numerical methods for pricing pension plans allowing early retirement, SIAM Jounal of Applied Mathematic, 73 (2013), 1747-1767.

[16] M. C. Calvo-Garrido and C. Vázquez, A new numerical method for pricing Fixed-Rate Mortgages with prepayment and default options, accepted for publication in International Journal of Computer Mathematics (2013).

[17] E. Chevalier, Optimal early retirement near the expiration of a pension plan, Finance and Stochastics, 10 (2006), 204-221.

[18] J.C. Cox, J.E. Ingersoll and S.A. Ross, A theory of the term structure of the interest rates, Econometrica, 53 (1985), 385-407.

[19] R. Cont and P. Tankov, Financial Modelling With Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, 2004.

[20] Y. D'Halluin, P. A. Forsyth and G. Labahn, A semi-Lagrangian approach for American Asian Options Under Jump Diffusion, SIAM Journal on Scientific Computing, 27 (2005), 315-345.

[21] Y. D'Halluin, P. A. Forsyth and K.R. Vetzal, Robust numerical methods for contingent claims under jump diffusion processes, IMA Journal of Numerical Analysis, 25 (2005), 87-112.

[22] P. Faurre, Analyse numérique: Notes d'optimisation, École Polytechnique, Palaiseau, 1988.

[23] M. Di Francesco and A. Pascucci, On a class of degenerate parabolic equations of Kolmogorov type, AMRX Applied Mathematics Research eXpress, 3 (2005), 77-116.

[24] M. Di Francesco, A. Pascucci and S. Polidoro, The obstacle problem for a class of hypoelliptic ultraparabolic equations, Proceedings Royal Society of London Serie A, 464 (2008), 155-176.

[25] M. Frentz, K. Nystrom, A. Pascucci and S. Polidoro, Optimal regularity in the obstacle problem for Kolmogorov operators related to American Asian options, Mathematische Annalen, 347 (2010), 805-838.

[26] A. Friedman and W. Shen, A variational inequality approach to financial valuation of retirement benefits based on salary, Finance and Stochastics, 6 (2002), 273-302.

[27] P. Glasserman, Monte Carlo methods in financial engineering, Springer, Berlin, 2003.

[28] T.G. Gencev, Ultraparabolic equations, Soviet Mathematics Doklady., 4 (1963), 979-982.

[29] A. Gleit, Valuation of general contingent claims, Journal of Financial Economics, 6 (1978), 71-87.

[30] J. E. Hilliard, J. B. Kau and V. C. Slawson, Valuing Prepayment and Default in a Fixed-Rate Mortgage: A Bivariate Binomial Options Pricing Technique, Real Estate Economics, 26 (1998), 431-468.

[31] A. Ibañez and F. Zapatero, Monte Carlo valuation of American options through computation of the optimal exercise boundary, Journal of Financial and Quantitative Analysis, 39 (2004), 253-275.

[32] K. Itô, On Stochastic Differential Equations, Memoirs American Mathematical Society, 4 (1951), 1-51.

[33] P. Jaillet, D. Lamberton and B. Lapeyre, Variational inequalities and the pricing of American option, Acta Applicanda Mathematicae, 21 (1990), 263-289.

[34] L. Jiang, B. Bian and F. Yi, A parabolic variational inequality arising from the valuation of fixed rate mortgages, European Journal of Applied Mathematics, 16 (2005), 361-383

[35] R. Kangro and R. Nicolaides, Far field boundary conditions for Black-Scholes equations, SIAM Journal Numerical Analysis, 38 (2000), 1357-1368.

[36] T. Kärkkäinen, K. Kunisch and P. Tarvainen, Augmented Lagrangian active set methods for obstacle problems, Journal of Optimization Theory and Applications, 119 (2003), 499-533.

[37] J. B. Kau, D. C. Keenan, W. J. Muller III and J. F. Epperson, The Valuation at Origination of Fixed-Rate Mortgages with Default and Prepayment, Journal of Real Estate Finance and Economics, 11 (1995), 5-36

[38] A. Kemma and T. Vorst, A pricing method for options based on average values, Journal Banking and Finance, 14 (1990), 113-129.

[39] S. G. Kou, A jump-diffusion model for option pricing, Management Science, 48 (2002), 1086-1101.

[40] D. Lamberton and B. Lapeyre, Introduction to Stochastic Calculus Applied to Finance, Second Edition, Chapman & HallCRC Financial Mathematics Series, 2007.

[41] E. Lanconelli and S. Polidoro, On a class of hypoelliptic evolution operators, Rendiconti del Seminario Matematico Università e Politecnico di Torino, 52 (1994), 29-63.

[42] F. Longstaff and E. Schwartz, Valuing American options by simulation: a simple least-squares approach, Review of Financial Studies, 14 (2001), 113-147.

[43] M. Manfredini, The Dirichlet problem for a class of ultraparabolic equations, Advances in Differential Equations, 2 (1997), 831-866.

[44] M.D. Marcozzi, On the valuation of Asian options by Variational Methods, SIAM Journal of Scientific Computing, 24 (2003), 1124-1140.

[45] R. C. Merton, Theory of rational option pricing, The Bell Journal of Economics and Management Science, 4 (1973), 141-183.

[46] R. C. Merton, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 3 (1976), 125-144.

[47] L. Monti and A. Pascucci, Obstacle problem for arithmetic Asian options, Comptes Rendus de l'Académie des sciences Paris, Series I, 347 (2009), 1443-1446.

[48] M. R. Nogueiras, Numerical analysis of second order Lagrange-Galerkin schemes. Application to option pricing problems, PhD Thesis, University of Santiago de Compostela, 2005.

[49] R. Norberg, Bonus in life insurance: principles and prognoses in a stochastic environment, Working paper N° 142, Laboratory of Actuarial Mathematics, University of Copenhagen, (2009) 40pp.

[50] O. A. Oleǐnik and E. V. Radkevič, Second Order Equations with Nonnegative Characteristic Form, American Mathematical Society, Providence, Rhode Island, Plenum Press, New York-London, 1973.

[51] A. Pascucci, Free boundary and optimal stopping problems for American Asian options, Finance and Stochastics, 12 (2008), 21-41.

[52] A. Pascucci, PDE and Martingale Methods in Option Pricing, Bocconi & Springer Series, Berlin, 2011.

[53] Pension Benefit Guaranty Corporation. U.S. Government Agency, A Predictable, Secure Pension for Life: Defined Benefit Pensions, 2000.

[54] O. Pironneau, On the transport-diffusion algorithm and its application to Navier-Stokes equation, Numerische Mathematik, 38 (1982), 309-332.

[55] S. Polidoro, On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type, Le Matematiche, 49 (1994), 53-105.

[56] S. Polidoro and M. Di Francesco, Schauder estimates, Harnack inequality and lower bound for Kolmogorov type operators in non-divergence form, Advances in Differential Equations, 358 (2006), 4873-4893.

[57] H. Rui and M. Tabata, A second order characteristics finite element scheme for convection-diffusion problems, Numerische Mathematik, 92 (2002), 161-177.

[58] S. Salmi and J. Toivanen, An iterative method for pricing American options under jump-diffusion models, Applied Numerical Mathematics, 61 (2011), 821-831.

[59] N. J. Sharp, Advances in Mortgage Valuation: an Option-Theoretic Approach. PhD Thesis, School of Mathematics, The University of Manchester, (2006).

[60] N. J. Sharp, D. P. Newton and P. W. Duck, An Improved Fixed-Rate Mortgage Valuation Methodology with Interacting Prepayment and Default Options, Journal of Real Estate Finance and Economics, 19 (2008), 49-67.

[61] M. Sherris and W. Shen, Financial valuation of retirement benefits based on salary, Paper presented at the Quantitative Methods in Finance Conference, Sydney,(1999).

[62] D. C. Shimko, The equilibrium valuation of risky discrete cash flows in continuous time, Journal of Finance, 44 (1989), 1373-1383.

[63] M. Suárez-Taboada and C. Vázquez, Numerical solution of a PDE model for a ratchet-cap pricing with BGM interest rate dynamics, Applied Mathematics and Computation, 218 (2012), 5217-5230.

[64] R. S. Varga, Matrix iterative analysis, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1962.

[65] O. Vasicek, An equilibrium characterization of the term structure, Journal of Financial Economics, 5 (1977), 177-188.

[66] C. Vázquez, An upwind numerical approach for an American and European option pricing model, Applied Mathematics and Computation, 97 (1998), 273-286.

[67] M. Widdicks, P.W. Duck, D. Andricopoulos and D.P. Newton, The Black-Scholes equation revisited: asymptotic expansions and singular perturbations, Journal of Mathematical Finance, 15 (2005), 373-392.

[68] P. Wilmott, J. Dewynne and S. Howison, Option pricing. Mathematical Models and Computation, Oxford Financial Press, Oxford, 1993.

[69] H. Wolthius, Life Insurance Mathematics: The Markovian Model, Caire Education Series, Volume 2. Caire, Brussels.