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ESSAYS ON INDIVIDUAL AND COLLECTIVE DECISION MAKING

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siendo directores los Dres. D. Ignacio García Jurado y D. José Carlos Rodríguez Alcantud, obtuvo la máxima calificación de SOBRESALIENTE CUM LAUDE.

A mis cuatro hijas:

María, Elena, Alba y Raquel.

Gracias.

Preface

Some years ago, when I got my degree in Mathematics I did not Know much about Decision Theory. My job in the School of Business at the University of Salamanca put me in contact with this area. Some time later I began to give some practical lessons of the subject "Decisión y Juegos" with José Manuel Gutiérrez Díez and I got more and more interested in both Decision Theory and Game Theory. José Manuel introduced me to Ignacio García Jurado (thank you very much, Manuel, for that and also for your help during all these years) and from then Ignacio has been not only my advisor, but also the person that has helped me in making many decisions about my future as a researcher. Ignacio, I will never be able to thank you enough for your help and patience.

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I am also grateful to all my colleagues at the Faculty of Economics and Business at University of Salamanca. I should mention many of them that have played and important role in different times in my life, but I hope they understand that I only mention those that have been nearer to me in the last few years. Thank you to Bernardo (without you I would not be here, you know), Federico, Mercedes and Aurora for the enjoyable conversations during the daily coffee breaks. And many thanks to Inocencia, our officemate, because she is always with me whenever I have an administrative work, and whenever I need someone to speak to.

Of course, the most important support has come from my parents during all my life. *Soy todo lo que soy gracias a vosotros*. Thank you also to my brothers, Ramón and Laura, because the path with you has been always easier.

Finally, I have to render many thanks to my four daughters that fill up my life of happiness and that have given to me the necessary willpower to finish this work.

And for finishing, I have to say that although there is only one signature in the thesis, this is not entirely true. Carmelo, my husband, is my coauthor. He has been always with me from the 9th of July 1994, when we got married, and nothing in my life would have been possible without him since then. Carmelo, both of us know that this is not my thesis; both of us know that this is OUR thesis.

María Dolores García Sanz
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Contents

Preface iii

Contents v

Notations vii

INTRODUCTION 1

SHORT BIBLIOGRAPHY . . . 7

1 Egalitarian evaluation of infinite utility streams **7**

1.1 Introduction . . . 8
1.2 Notation and Preliminaries . . . 10
1.3 Some relationships and other auxiliary results . . . 12
1.4 Existence of RNS and SP Social Welfare Functions . . . 13
1.5 Existence of HE and SP Social Welfare Functions . . . 15
1.6 Conclusions . . . 18
1.7 Bibliography . . . 22

2 Ranking opportunity sets.
A characterization of an advised choice **25**

2.1 Introduction . . . 26
2.2 Notation and Preliminaries . . . 29
2.3 Choice in different times . . . 30
2.4 The case of an advised choice . . . 44
2.5 Conclusions and future research . . . 59
2.6 Bibliography . . . 61

3 Rational Choice by two sequential criteria **65**

3.1 Introduction . . . 66
3.2 Definitions and properties of rationality . . . 69
3.3 Rationality properties of a compound choice function . . . 72
3.4 Choice functions rational by two sequential criteria . . . 92
3.5 Conclusions and future research . . . 103
3.6 Bibliography . . . 105

4 Cooperation in Markovian queueing models **109**
4.1 Introduction 110
4.2 Basic Markovian models 111
4.3 Cooperation under preemptive priority 115
4.4 The basic Markovian model with expected times in the queue 123
4.5 Conclusions and future research 130
4.6 Bibliography 132

Resumen en Castellano **137**
BIBLIOGRAFÍA DEL RESUMEN 155

Notations

This thesis consists of independent chapters and for this reason all of them are self-contained. It is possible that some of the notation is introduced in more than one chapter. Anyway, the following symbols and notation are common for all the chapters.

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}^*$	The set of natural numbers including 0
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The set of real numbers
$\varnothing$	The empty set
2^N	The set of all subsets of N
$T \subseteq S$	T is a subset of S
$T \subset S$	T is a subset of S and T is not equal to S
$T \times S$	The cartesian product of T and S
$\|S\|$	The number of elements of S
$\mathcal{P}^*(X)$	the set of nonempty subsets of a set X
R	A binary relation on a set X
$\succcurlyeq$	A binary relation on $\mathcal{P}^*(X)$
$\square$	The end mark of a proof
$\diamond$	The end mark of an example
$\triangleleft$	The end mark of a remark

Let $x, y \in \mathbb{R}^{\mathbb{N}}$:

$x \geqslant y$	$x_i \geqslant y_i$ for each $i = 1, 2, ...$
$x \gg y$	$x_i > y_i$ for each $i = 1, 2, ...$
$x > y$	If $x \geqslant y$ and $x \neq y$

Introduction

This dissertation is devoted to the extensive field of decision-making, in both the individual and collective cases. It is organized in four independent chapters. The first three ones deal with different individual situations where a decision-maker has to decide. Chapters 1 and 2 have to do with ranking different alternatives of a set and Chapter 3 involves cases in which the decision-makers' preferences are represented as choice functions.

In Chapter 1 we deal with the problem of resolving distributional conflicts among an infinite and countable number of generations. In this context, economists are interested in postulating axioms of **equity among the generations** and **efficiency** in a variety of forms. The properties of efficiency under inspection are different versions of Pareto axiom. Equity is considered many times a synonym of anonymity, and this property is considered a suitable axiom for a social welfare function or relation.

We deal in this chapter with the Basu-Mitra (2003) approach to the problem of intergenerational social choice, and contribute to such approach by analyzing the impact of the structure of the feasible set of utilities on Banerjee's (2006) impossibility theorem. Here the properties under inspection are Weak Dominance and a weak equity postulate that was introduced in Asheim and Tungodden (2004), namely Hammond Equity for the Future (HEF). Banerjee (2006) proves that they are incompatible under the Basu-Mitra perspective when the feasible utilities are $[0, 1]$. Here we prove that if we consider the discrete domain $\mathbb{N}$ instead, then an explicit expression for a Paretian social welfare function that accounts for a strengthened form of Hammond Equity for the Future can be given.

In a similar line of inquiry we wonder whether different versions of the Hammond Equity postulate can be combined into a Paretian social welfare function. Both the continuous $[0, 1]$ and the discrete $\mathbb{N}^*$ instances are analyzed. The analysis is more complex than the previous case, and the range of situations is richer.In the continuous case all the results we obtain are impossibility results. In the discrete case, nonetheless we prove that HE is incompatible with any version of Strong Pareto, we obtain that there exist social choice functions that combine a version of HE with a weaker expression of Pareto postulate.

This chapter is based on Alcantud and García-Sanz (2008).

Chapter 2 is devoted to the problem of extending an agent's preference over a set of alternatives to a ranking of its nonempty subsets. We first consider a situation in which the decision is made in two or more different times (this question has been partially studied in Krause (2007) for the two-times case) and using the indirect-utility criterion characterized by Kreps (1979). We characterize a ranking defined over sequences of ordered subsets (elements of a direct product), each one from the set of alternatives in each time. This ranking consists on the application of the indirect-utility criterion to each coordinate in a lexicographic way.

Some rankings of subsets consisting on the lexicographic compositions of two criteria of ranking subsets have been studied by different authors. Some of these compositions are completely characterized using suitable axioms. We address the interested reader to Barberá et al. (2004).

In a second section we consider a model where we rank the subsets of a set of alternatives also applying the indirect-utility criterion, but now we consider the possibility of having an adviser for those cases where such criterion produces ties. This adviser has not an individual binary relation defined on the set of alternatives, but rather for any subset of alternatives he selects some "focal elements" (those that he prefers the most), that we represent by a choice function.

Chapter 3 deals with the rationality of choice functions. In Suzumura (1983) we can find a survey of the characterization theorems considering the classical concept of rational choice functions defined on different domains. Here we adopt and extend this classical concept: we also consider as rational the behavior of a decision-maker that applies sequentially such type of choices. This behavior is represented as the composition of different choice functions in an established order. Aizerman and Aleskerov (1995) also consider this kind of choice behavior and Kalai et al. (2002) study the rationality of a choice function by multiple rationales when the choice is a sigle element in the set of alternatives and applying all the rationales simultaneously at each instance. We follow the line initiated by Manzini and Mariotti (2007) who consider the sequential rationality of a choice function by the application of different rationales in a fixed order, and specifically the case of two rationales. They restrict their study to the case of single-valued choice functions. We think that it is interesting to analyze the problem in terms of set-valued choice functions and this is the case we consider along Chapter 3.

We focus on how the compound function of two choice functions behaves and specifically which properties verified by the two initial choice functions carry over to the compound function. Aizerman and Aleskerov (1995) study some instances of this problem for properties of choice functions defined over domains which contain all the finite and nonempty subsets of a set of alternatives. We add to this study with the analysis of some other properties in domains of the same kind, and with some properties of choice functions defined on arbitrary domains.

Finally we obtain some results of rationality in the classical sense for a compound choice function defined on different domains and then, for the cases in which the domain contains all the finite and nonempty subsets of the set of alternatives, we give a complete characterization of a choice function that is rational by two sequential criteria in terms of two testable necessary and sufficient conditions.

In Chapter 4 we focus on problems where at least two agents are implied in such a way that the decisions they take affect the others agents' outcomes.

It is not difficult to find problems of different social or economic situations where different agents (players) appear with different points of view. Game Theory surged in the 20th century as a branch of Mathematics that considers this class of situations. As these problems are very common in the real life, Game Theory has achieved a high impact in different branches of knowledge such as biology, politics, psychology,..., and over all, economics.

The agents implied in a game problem have well-defined objectives, thus they act *rationally*, and at the same time, they take into account the knowledge or expectations of other decision-makers' behavior, thus they also act *strategically*.

In non-cooperative models, players can negotiate about what to do, but binding agreements are not possible.

Opposite, in cooperative models binding agreements are possible, and also side payments can be allowed.

On the other hand, operations research analyzes situations where a decision-maker faces an optimization problem guided by an objective function. Most fields within operations research have been approached from a game theoretical perspective, for the cases in which several decision makers interact in situations that can be modelled as optimization problems. Borm et al. (2001) provides a review of this topic.

One of the major branches within operations research is *queueing theory* (see, for example, Gross and Harris (1998)). Surprisingly enough, queueing models have rarely been approached from the point of view of cooperative game theory. González and Herrero (2004) is one of the scarce papers in which cooperation is analyzed in queueing models. In this last chapter of the thesis we also approach different queueing situations from that point of view. Thus we obtain different models of cost games and study if cooperation is or not a good option. For those cases where the answer is affirmative we propose and characterize an allocation rule for distributing the costs.

Recently we became aware of the paper of Yu et al. (2008) that also deals with cooperation in queueing systems. They optimize the capacity of the system (in the models we study the capacity is given by the average times that the clients spend in the system or in the queue) and they do not fix maximum values for the times that the clients spend in it.

This chapter is based on García-Sanz et al. (2008).

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Chapter 1

Egalitarian evaluation of infinite utility streams: analysis of some Pareto efficient axiomatics

Contents

1.1 Introduction **8**

1.2 Notation and Preliminaries **10**

1.3 Some relationships and other auxiliary results **12**

1.4 Existence of RNS and SP Social Welfare Functions **13**

1.5 Existence of HE and SP Social Welfare Functions **15**

1.5.1 "Continuous" domain 15

1.5.2 "Discrete" domain 16

1.6 Conclusions **18**

1.7 Bibliography **22**

1.1 Introduction

The problem of "evaluating a stream" emerges from some economic problems which have the common characteristic of "not having a natural termination date". For instance, the optimization of the economic growth with streams (of consumption, for example) which extend over an infinite future.

We deal in this chapter with the problem of resolving distributional conflicts among an infinite and countable number of generations.

In this context, economists are interested in postulating axioms of **equity among the generations** and **efficiency** in a variety of forms.

The properties of efficiency are materialized as different versions of the Pareto axiom. The equity is considered many times a synonym of anonymity, and the anonymity property or the finite anonymity are considered suitable axioms to be verified for a social welfare function or relation. Hammond (1976) postulates another equity condition, the Hammond Equity, that establishes that when comparing distributions that give all, except two generations, the same benefits "any order-preserving change which diminishes inequality of utilities between the conflicting generations is socially preferable". Recently Asheim and Tungodden (2004a) have introduced a variation of the Hammond Equity postulate, the "Hammond equity for the future" property. This property is a one sided equity condition that states that "a sacrifice of the present generation leading to an equal gain for all future generations is weakly desirable if the present remains better off than the future" and it is considered instead of the anonymity property.

Another factor we have to take in consideration when dealing with the problem of ranking infinite horizon intergenerational streams is **the domain** for the utility levels associated with each period (equal across generations). Moreover it is more realistic to make some restrictions over the domains we are dealing with. For example it seems reasonable to consider that the human perception is not unlimited, so that a natural restriction to impose is a discrete domain. It is also natural that the utilities have a well-defined smallest unit (as happens when they measure monetary amounts).

In this field the researcher has a natural tendency to try and avail himself with an explicit numerical expression associated with each infinite stream. But as suspected by Ramsey (1928), respecting the equal treatment of all generations poses some intrinsic incompatibilities for the efficiency that can be assured. Discounting future generation's endowments is common place but it does not treat all generations alike. The Rawlsian criterion $\mathbf{W}_R(\mathbf{x}) = inf\ \{x_i : i = 1, 2, 3, ...\}$ is 'more ethical' in the sense that it is not influenced by the position that each generation occupies, but it violates very weak versions of Pareto efficiency. Thus an alternative approach to the resolution of the aggregation problem postulates the existence of social welfare relations instead.

The analysis of inter-generational equity in the context of aggregating infinite utility streams started with the aforementioned work of Ramsey (1928) who established a conjecture about the difficulty of aggregating infinite streams respecting an inter-generational equity. Following Koopmans (1960), Diamond (1965) proves the impossibility of having a Paretian welfare function, continuous in the sup norm, and that treats all generations equally. The domain for the utilities he considers is the unit interval. Epstein (1986), Shinotsuka (1998), Fleurbaey and Michel (2003), Suzumura and Shinotsuka (2007) and Sakai (2003, 2006) also obtain impossibility results in the same line. Basu and Mitra (2003) prove that the impossibility result from Diamond survives without the restriction of continuity and without any topological consideration or domain restriction.

Despite this negative situation Svenson (1980) makes a positive contribution by giving a non-constructive proof of the existence of a social welfare order over the set of infinite streams of utilities verifying the Pareto condition and some requirements of equity (anonymity). He considers a stronger topology than the one used in Diamond (1965) and the same domain (the unit interval). Bossert et al. (2004) provide a strengthening of Svenson possibility result and Hara et al. (2007) contains some other impossibility results in this same line of inquiry. Basu and Mitra (2007) generalize the Svenson possibility result for a general domain for the utilities and at the same time they establish the relation among these results in the sense that a social welfare order satisfying Pareto axiom and anonymity can not be representable [1].

In this sense Basu and Mitra (2007) consider the possibility of weakening the Pareto axiom and obtain that it is possible to combine Anonymity and a weak form of the Pareto postulate, labelled Weak Dominance, in a social welfare function irrespective of the domain for the utility levels.

Nevertheless, for some other sets of axioms the structure of the domain is crucial and in the same work Basu and Mitra prove that in their original impossibility result, if the Strong Pareto axiom is replaced by the Weak Pareto axiom the impossibility result remains. But if the domain considered for the utility levels changes to be $\mathbb{N}^* = \{0, 1, 2, \ldots\}$, then the impossibility result becomes a possibility one.

The inspection of Paretian social welfare functions that agree with other equity axioms has little tradition in the literature. Asheim and Tungodden (2004a) obtain another impossibility result for a social welfare order with domain of utilities the unit interval

[1] An order $\succcurlyeq$ on a set X is representable if there exists a real function f on X such that $x \succcurlyeq y \Leftrightarrow f(x) \geqslant f(y)$.

when they deal with the weak equity condition named "Hammond Equity for the Future" (HEF) and some other postulates. The HEF axiom expresses a weak preference for profiles where the sacrifice of the present generation makes all future generations better off by a constant utility amount. It is introduced in that paper as a weak form of the Hammond Equity postulate.

Asheim et al. (2007) prove that it is impossible to aggregate infinite utility streams in an upper semi-continuous binary relation that satisfies a weak version of Weak Dominance and HEF. The domain they consider is any $\mathbf{Y}$ such that $[0,1] \subseteq \mathbf{Y} \subseteq \mathbb{R}$.

Banerjee (2006) considers the Hammond Equity for the Future property in the case of social welfare functions (instead of orders) and obtains impossibility when the domain is the unit interval and the function verifies the Weak Dominance axiom.

We are not aware of any other work that studies the compatibility of equity axioms other than anonymity with a Paretian function.

We prove in section 1.4 that under the conditions in Banerjee's theorem the impossibility also turns into possibility when we consider the domain of utilities $\mathbb{N}^*$ instead of the unit interval and with a stronger form of the Hammond Equity for the Future. Moreover we give a constructive proof, thus an *explicit* expression for a Strongly Paretian social welfare function that accounts for a strengthened form of Hammond Equity for the Future can be given.

In a similar line of inquiry we wonder whether different versions of the Hammond Equity postulate can be combined into a Paretian social welfare function. Both the continuous $[0,1]$ and the discrete $\mathbb{N}^*$ instances are analyzed. Such problem is investigated in section 1.5.

This chapter is based in Alcantud and García-Sanz (2008).

1.2 Notation and Preliminaries

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. We adopt the usual notation for such utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. By $(y)_{con}$ we mean the constant sequence $(y, y, ...)$, and $= (x, (y)_{con})$ holds for $(x, y, y, y, ...)$. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$. In this paper we are concerned with two sets of axioms of different nature on SWFs. Firstly we introduce some consequentialist equity axioms.

Axiom 1a (*Hammond Equity*, also HE). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

Axiom 1b (*Hammond Equity* -Lauwers' version-, also HE(L)). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j \geqslant y_j \geqslant y_k \geqslant x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

Axiom 1c (*Hammond Equity* (a), also HE(a)). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

Axiom 1d (*Hammond Equity* (b), also HE(b)). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j = y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

Axioms 1a to 1d above are variations of a common equity principle: when there is a conflict between two generations, every other generation being as well off, the stream where the least favoured generation is better off must be weakly preferred.

The precise meaning of the term "conflict" produces different formal requirements.

We also discuss some implications of the following axiom, that was introduced in Asheim and Tungodden (2004a).

Axiom 2 (*Hammond Equity for the Future*, also HEF). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ $(x_1 > y_1 > y > x)$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

HEF states the following ethical restriction on the ranking of streams where the level of utility is constant from the second period on and the present generation is better-off than the future: if the sacrifice by the present generation conveys a higher utility for all future generations, then such trade off is weakly preferred. Asheim and Tungodden (2004a) and Asheim et al. (2007), Section 4.3, explain that it is a very weak equity condition –under certain consistency requirements on the social preferences "condition HEF is much weaker and more compelling than the standard 'Hammond Equity' condition"– that can be endorsed both from an egalitarian and utilitarian point of view.

Notation. In all the axioms above, when $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$ is requested in place of $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$ we refer to HE$^+$, ... , HEF$^+$.

As a reinforcement of HEF$^+$ we introduce a consequentialist equity axiom in the spirit of Lauwers' (1998) Non-Substitution property. It captures the following ethical principle: a large but finite improvement in the first generation can never compensate a sustained improvement for all future generations.

Axiom 2′ (*Restricted Non-Substitution, also RNS*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ with $y > x$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Of course, in addition we intend to account for some kind of efficiency. In this sense the stronger axiom we deal with is the following.

Axiom 3 (*Strong Pareto, also SP*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

The next efficiency axiom is implied by Strong Pareto.

Axiom 4 (*Monotonicity, also MON*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Other axioms that are succesively weaker versions of Strong Pareto follow.

Axiom 5 (*Weak Pareto, also WP*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and either $\mathbf{x} \gg \mathbf{y}$ or there is $j \in \mathbb{N}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Axiom 6 (*Dominance, also D*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ then (a) if there is $j \in \mathbb{N}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$, and ($b$) if $\mathbf{x} \gg \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Axiom 7 (*Weak Dominance, also WD*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

1.3 Some relationships and other auxiliary results

We have mentioned that RNS implies HEF$^+$. Besides RNS is stronger than Weak Non-Substitution (cf., Asheim et al., 2006) and in the presence of either WD or MON, RNS implies Non-Substitution (cf., Lauwers, 1998).

Our analysis below is simplified by recalling that the Rawlsian criterion $\mathbf{W}_R(\mathbf{x}) = inf\,\{x_i\,:\,i = 1, 2, 3, ...\}$ satisfies a reinforced version of MON (but not WD), generic Anonymity (i.e., it attaches the same values to all the permutations of a given stream), HEF$^+$, and all four versions of Hammond Equity (Axioms 1a, 1b, 1c, and 1d) that we have stated. It does not agree with (Restricted, Weak) Non-Substitution. However a modified version, namely $\mathbf{W}_{FR}(\mathbf{x}) = inf\,\{x_i : i = 2, 3, ...\}$, does satisfy MON and RNS.

Remark 1.1. Some trivial relationships among requirements we have defined follow. Of course HE(L) $\Rightarrow$ HE(a) $\Rightarrow$ HE(b). Besides HE(a)$=$ $HE + HE(b)$ $\Rightarrow$ HE, and the converse is true under either WD or MON in well-established instances like l_∞, the set of bounded real-valued infinite sequences, or $\mathbf{X} = Y^{\mathbb{N}}$ with $\mathbf{Y} \subseteq \mathbb{R}$ order-dense. ◁

We now recall other relationships between HE and HEF under Monotonicity or Dominance.

Lemma 1.1. *Any HE or HE(b) and Monotonic SWF satisfies HEF. Also, if $\mathbf{X} = l_\infty$ or $\mathbf{X} = Y^{\mathbb{N}}$ with $\mathbf{Y} \subseteq \mathbb{R}$ order-dense, then HE(b) plus D entail HEF.*

Proof. Asheim et al. (2006), Proposition 3, states a result alike the first statement for social welfare relations. Its proof is direct and can be mimicked here.

Suppose now that $\mathbf{W}$ is a SWF on either $\mathbf{X} = l_\infty$ or $\mathbf{X} = Y^{\mathbb{N}}$ with $\mathbf{Y} \subseteq \mathbb{R}$ order-dense, and also that $\mathbf{W}$ agrees with HE(b) and D. In order to check that $\mathbf{W}$ satisfies HEF too, take $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ with $x_1 > y_1 > y > x$. There is z such that $y > z > x$. Define $\mathbf{z} = (z, z, x, x, x, ...)$, thus $\mathbf{y} \gg \mathbf{z}$ and by D we obtain $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$. Because $x_1 > z = z_1 = z_2 > x = x_2$ and $x_i = z_i$ for each $i \geqslant 2$, HE(b) yields $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$ thus $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$. □

Remark 1.2. One can readily check that HE(L) is incompatible with Weak Dominance in virtually any useful instance of $\mathbf{X}$. ◁

We end this section with a technical result that is used later on.

Lemma 1.2. *Suppose that* $\mathbf{W} : \mathbf{Y}^{\mathbb{N}} \longrightarrow \mathbb{R}$ *satisfies HE (resp., HE(b)) and MON.*

(a) If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ *are such that* $x_j > y_j > y_k > x_k$ *(resp.,* $x_j > y_j = y_k > x_k$*) for some* $j, k \in \mathbb{N}$ *and* $y_t \geqslant x_t$ *when* $j \neq t \neq k$*, then* $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

(b) If we further assume $y_s > x_s$ *for some* $j \neq s \neq k$ *and SP, then* $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Proof. Pick $\mathbf{z} \in \mathbf{X}$ such that $z_t = x_t$ when $j \neq t \neq k$, $z_j = y_j$, $z_k = y_k$.

Using MON we obtain $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$. If case ($b$) holds then SP entails $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{z})$. In each instance the conclusion follows because HE (resp., HE(b)) yields $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$. □

1.4 Existence of RNS and SP Social Welfare Functions

Basu and Mitra, 2003, Theorem 1 states that no SWF is Strongly Paretian and Equitable or Anonymous [2].

If we replace Anonymity by Hammond Equity for the Future, then Banerjee (2006) proves that the impossibility of making these requirements compatible with an SWF remains when $\mathbf{Y} = [0, 1]$ even though we only require Weak Dominance instead of Strong Pareto. We proceed to analize the situation when $\mathbf{Y} = \mathbb{N}^*$.

Theorem 1.1. *There are SWF's on* $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$*, where* $\mathbf{Y} = \{0, 1, 2, ...\}$*, that satisfy both RNS and Strong Pareto.*

[2]The *Anonymity* axioms states that a finite permutation of a utility stream produces a utility stream with the same social utility.

Proof. Our proof is constructive: we give an explicit expression for an SWF on $\mathbf{X}$ that satisfies RNS and SP.

Recall that the application

$$\psi(n) = \frac{n}{1+n} \text{ for each } n \in \mathbb{N}^*$$

maps $\mathbb{N}^*$ into $[0,1)$ and satisfies: $m < n$ if and only if $\psi(m) < \psi(n)$ for every possible m, n (Bridges and Mehta, 1996, p. 30).

Now for any $\mathbf{x} = (x_1, x_2, x_3, ...) \in \mathbf{X}$ let

$$\mathcal{W}(\mathbf{x}) = \psi(x_1) + x_2 + \sum_{i=3}^{+\infty} \frac{\psi(x_i)}{2^i} \tag{1.1}$$

This expression produces a well-defined function because $\psi(x_i) \in [0,1)$ when $i = 3, 4, ...$ The properties of ψ permit to prove that $\mathcal{W}$ is Strongly Paretian. In order to check that $\mathcal{W}$ satisfies RNS, take x_1, x_1' and $y' > y$ elements from Y. Because

$$\mathcal{W}(x_1', (y')_{con}) = \psi(x_1') + y' + \frac{\psi(y')}{4}$$

$$\mathcal{W}(x_1, (y)_{con}) = \psi(x_1) + y + \frac{\psi(y)}{4}$$

we deduce that the difference

$$\mathcal{W}(x_1', (y')_{con}) - \mathcal{W}(x_1, (y)_{con}) = (\psi(x_1') - \psi(x_1)) + (y' - y) + \frac{1}{4}(\psi(y') - \psi(y))$$

is strictly positive. The reason for such inequality is that $\psi(x_1') - \psi(x_1) > -1$, $y' - y \geqslant 1$, and $\psi(y') - \psi(y) > 0$. $\square$

The criterion given in Theorem 1.1 has other nice features, for example it must satisfy all four sensitivity properties recalled in Asheim at al. (2007), namely Strong Sensitivity, Sensitivity to the Present, Restricted Sensitivity, and Weak Sensitivity. A severe drawback is that it discriminates future generations. Besides, it contradicts the Independent Future postulate (cf., Fleurbaey and Michel, 2003) as the comparison between the streams $(0,1,0,0,0,...)$ and $(0,0,2,0,0,...)$ proves. As a consequence of the results in Section 1.5 below, it can not fulfil any version of HE either.

1.5 Existence of HE and SP Social Welfare Functions

As happens with the weaker HEF, the problem of combining the ethics that the Hammond Equity principle incorporates with efficiency under the Basu-Mitra approach depends on the domain of utility streams. In subsection 1.5.1 we show that the problem when the domain is $\mathbf{X} = [0,1]^{\mathbb{N}}$ has been ellucidated in part and we complete the corresponding study. An analysis of the case where $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$ with $\mathbf{Y} = \mathbb{N}^*$ is performed in subsection 1.5.2. Even though we have recalled the standard defense of this framework, we emphasize that we do not argue in order to endorse any concrete structure for such domain.

1.5.1 "Continuous" domain

The next consequence of Lemma 1.1 follows immediately after Banerjee (2006), Theorem 1.

Corollary 1.1. *There is no Dominant SWF on* $[0,1]^{\mathbb{N}}$ *that satisfies any of the axioms* 1a *to* 1d.

Proof. We have argued that if a Dominant SWF satisfies any of the axioms 1a to 1d then it satisfies HE(b). But Lemma 1.1 ensures that such SWF must satisfy HEF, which is impossible by virtue of Banerjee (2006), Theorem 1. $\square$

Despite this Corollary, one may wonder if there exist SFWs that are both HE(b) and WD when $\mathbf{X} = [0,1]^{\mathbb{N}}$. We now show that the answer to this latter question is negative, thus *no version of the Hammond Equity postulate under inspection is compatible with a Dominant SWF* when the domain is $[0,1]^{\mathbb{N}}$. This conclusion is unsurprising since HEF is much weaker than HE in the presence of either MON or D, and HEF is already incompatible with WD under the Basu-Mitra approach.

Proposition 1.1. *There are not SWFs on* $\mathbf{X} = [0,1]^{\mathbb{N}}$ *that satisfy both HE(b) and WD.*

Proof. We proceed by contradiction. Let $\mathbf{W} : [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ be HE(b) and WD. For each $0 < x < 1$ we let $L(x) := \mathbf{W}(x,x,0,0,....)$ and $R(x) := \mathbf{W}(\frac{1+x}{2},x,0,0,....)$. Then $I(x) := (L(x), R(x))$ is nonempty because $\mathbf{W}$ is WD.

Besides, $\frac{1}{2} > y > x > 0$ implies $I(x) \cap I(y) = \varnothing$:

$$L(y) = \mathbf{W}(y,y,0,0,....) > \mathbf{W}(\frac{1+x}{2},x,0,0,....) = R(x)$$

by application of HE(b) to $\frac{1+x}{2} > y > x$. This is impossible because an uncountable number of different rational numbers are assigned. $\square$

Remark 1.3. Observe that the arguments in this section apply to l_∞ as well. $\lhd$

1.5.2 "Discrete" domain

Now we wonder if it is possible to combine any version of Hammond Equity with WD (or stronger axioms) under the Basu-Mitra perspective when $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$ and $\mathbf{Y} = \{0, 1, 2, ...\}$.

In Theorem 1.2 we show that the answer to that question is in the negative when SP and either HE, HE(L), HE(a) or HE(b) is required. In fact in order to reach such negative conclusion we only need that Y has enough elements as to make the Hammond Equity principle meaningful. Observe that according to Remark 1.1 it suffices to deal with HE(b) and HE.

Theorem 1.2. *There are not SWFs on $\mathbf{X} = Y^{\mathbb{N}}$, where $|Y| \geqslant 3$ (resp., $|Y| \geqslant 4$), that satisfy both HE(b) (resp., HE) and SP.*

Proof. We first prove that HE and SP can not be displayed by any $\mathbf{W}$ on $\mathbf{X}$.

We use a standard construction to produce a suitable uncountable collection $\{E_i\}_{i \in I}$ of infinite proper subsets of $\mathbb{N}$. We request that $\forall i, j \in I \; [\, i < j \Rightarrow E_i \subsetneq E_j$ and $E_j - E_i$ is infinite$\,]$. We also need that there is an index $q \in E_i$ for all index $i \in I$. In order to justify that such collection exists, we take $\{r_1, r_2, ...\}$ an enumeration of the rational numbers in $(0, 1)$ and set $E(i) = \{n \in \mathbb{N} : r_n < i\}$ for each $i \in I = (r_1, 1)$ in order that $q = r_1 \in E(i)$ for each $i \in I$.

In order to ease the notation we assume without loss of generality that $\{0, 1, 2, 3\} \subseteq Y$. Let us define the following two utility streams associated with each $i \in I$:

$$r(i)_p = \begin{cases} 1 & \text{if} \quad p \in E_i, p \neq q \\ 3 & \text{if} \quad p = q \\ 0 & \quad \text{otherwise} \end{cases}$$

$$l(i)_p = \begin{cases} 1 & \text{if} \quad p \in E_i, p \neq q \\ 2 & \text{if} \quad p = q \\ 0 & \quad \text{otherwise} \end{cases}$$

By SP, the open interval $(\mathbf{W}(l(i)), \mathbf{W}(r(i)))$ is not empty.

We intend to check that $j < i \Rightarrow \mathbf{W}(l(i)) > \mathbf{W}(r(j))$, which is impossible because an uncountable number of distinct rational numbers would be obtained. Let us fix $k \in E_i - E_j$. We claim that Lemma 1.2 (b) applies to coordinates q and k of $l(i)$ and $r(j)$. Observe that $3 = r(j)_q > 2 = l(i)_q > 1 = l(i)_k > 0 = r(j)_k$. Also, when $q \neq p \neq k$ we have: $l(i)_p = r(j)_p$ when either $p \in E_i \cap E_j$ or $p \notin E_i \cup E_j$, and $l(i)_p = 1 > 0 = r(j)_p$ for every $p \in E_i$, $p \notin E_j$ (recall that there are an infinite number of elements in $E_i - E_j$). This ends the argument for the case HE plus SP.

In order to prove that HE(b) and SP are incompatible in the current assumptions, we assume that $\{0,1,3\} \subseteq Y$ in order to ease the algebra. Then we mimick the argument with the following variation of the $l(i)$'s streams (and the $r(i)$'s remaining the same):

$$l(i)_p = \begin{cases} 1 & \text{if} \quad p \in E_i \\ 0 & \quad \text{otherwise} \end{cases}$$

Observe that we now appeal to the appropriate variation of Lemma 1.2 to conclude the argument. $\square$

Despite this negative result, Theorem 1.3 below assures that WP can be combined with HE$^+$ / HE(a)$^+$ / HE(b)$^+$ even in the presence of Anonymity. In order to prove it we state the following auxiliary result.

Lemma 1.3. *The function $\nu(n) = \sum_{i=0,1,\ldots,n} \frac{1}{2^i}$ $(n = 0,1,2,\ldots)$ is strictly increasing in n and satisfies: $x > y_2 \geqslant y_1 > z \Rightarrow \nu(y_1) - \nu(z) > \nu(x) - \nu(y_2)$.*

Proof. Fix $x > y_2 \geqslant y_1 > z$. Some straightforward computations yield

$$\nu(y_1) - \nu(z) = \frac{1}{2^{z+1}}(1 + \frac{1}{2} + \ldots + \frac{1}{2^{y_1 - z - 1}}) \geqslant \frac{1}{2^{z+1}}$$

and

$$\nu(x) - \nu(y_2) = \frac{1}{2^{y_2+1}}(1 + \frac{1}{2} + \ldots + \frac{1}{2^{x - y_2 - 1}}) < \frac{1}{2^{y_2}}$$

since

$$1 + \frac{1}{2} + \ldots + \frac{1}{2^{x - y_2 - 1}} < 2.$$

Because $y_2 \geqslant z + 1$ the conclusion follows. $\square$

Theorem 1.3. *There are SWFs on $\mathbf{X} = Y^{\mathbb{N}}$, where $Y = \{0,1,2,\ldots.\}$, that satisfy both HE(a)$^+$, Anonymity, and WP.*

Proof. We closely follow Mitra and Basu's proof in (2007) that there are WP and Anonymous SFWs on $\mathbf{X} = Y^{\mathbb{N}}$. The binary relation on $\mathbf{X}$ given by $\mathbf{x} \sim \mathbf{y}$ if and only if $x_i = y_i$ eventually is an equivalence relation. The equivalence class of $\mathbf{x}$ is denoted by $[\mathbf{x}]_\sim$. We select an element $g([\mathbf{x}]_\sim)$ from each equivalence class $[\mathbf{x}]_\sim$ in the quotient set $\frac{\mathbf{X}}{\sim}$. For simplicity we write $g^{\mathbf{x}} = g([\mathbf{x}]_\sim)$, and as usual $g^{\mathbf{x}} = (g_1^{\mathbf{x}}, g_2^{\mathbf{x}}, \ldots)$. Thus when $\mathbf{x}, \mathbf{y}$ satisfy that $x_i = y_i$ eventually one has $g^{\mathbf{x}} = g^{\mathbf{y}}$.

Let us denote $A_N(\mathbf{x}) = \nu(x_1) + ... + \nu(x_N) - (\nu(g_1^{\mathbf{x}}) + ... + \nu(g_N^{\mathbf{x}}))$ for each $N \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{X}$, and consider the function $h(\mathbf{x}) = lim_{N\to\infty}(A_N(\mathbf{x}))$, which is well defined because $A_N(\mathbf{x})$ is eventually constant (for any fixed $\mathbf{x}$). Then h is clearly Anonymous and Weakly Dominant. We now prove that h satisfies HE(a)$^+$.

If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, our construction entails $g^{\mathbf{x}} = g^{\mathbf{y}}$. Therefore there is an index N_0 such that $A_N(\mathbf{y}) - A_N(\mathbf{x}) = \nu(y_j) - \nu(x_j) + \nu(y_k) - \nu(x_k)$ for each $N > N_0$. Now Lemma 1.3 yields $A_N(\mathbf{y}) - A_N(\mathbf{x}) > 0$ whenever $N > N_0$ and thus $h(\mathbf{y}) > h(\mathbf{x})$.

Finally, we define the SWF that satisfies our requirements by the expression:

$$\mathbf{W}(\mathbf{x}) = \frac{1}{2} \cdot \frac{h(\mathbf{x})}{1 + |h(\mathbf{x})|} + min\{x_1, x_2, ...\}$$

It is clear that $\mathbf{W}$ is Anonymous because so is h. By mimicking Mitra and Basu's argument, we can check that it is WP: the key point is that

$$H(t) := \frac{1}{2} \cdot \frac{t}{1+|t|} \quad \text{is strictly increasing, with values in } \left(-\frac{1}{2}, \frac{1}{2}\right)$$

and thus whenever $\mathbf{y} \gg \mathbf{x}$ because $min\{y_1, y_2, ...\} \geqslant min\{x_1, x_2, ...\} + 1$ we always get $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$. In order to prove that $\mathbf{W}$ is HE(a)$^+$, let us select $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. Now $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$ is enforced due to the following two inequalities: $h(\mathbf{y}) > h(\mathbf{x})$ as was proved above, thus $H(h(\mathbf{y})) > H(h(\mathbf{x}))$ because H is strictly increasing; and $min\{y_1, y_2, ...\} \geqslant min\{x_1, x_2, ...\}$. □

1.6 Conclusions

Banerjee (2006) argued that " ... a compromise that generates a possibility of ranking infinite utility streams is open to debate and does not necessarily call for abandoning the appealing equity postulate, Hammond Equity for the Future." Here we provide support for such possibility: if the feasible set of utilities is contained in $\mathbb{N}^*$ then it is possible to strengthen HEF to RNS even if we require the full force of the Pareto postulate under the Basu-Mitra approach. Our argument is constructive and a explicit criterion has been provided. Evidences like the discouraging Zame (2007, Theorem 4′) –which implies that no Weak Paretian and Anonymous welfare relation can be "explicitly described"– make this feature especially valuable.

Now we have more arguments to contribute to the following debate: in combining equity and Pareto-efficiency under the Basu-Mitra position, what is the influence of the choice of the set of feasible utilities? If we are bound by the HEF/RNS ethics we conclude that the set of feasible utilities is the determinant factor: when it is $[0,1]$ even the weakest possible combination ends in impossibility, but when it is $\mathbb{N}^*$ there is an explicit criterion that accounts for their strongest versions. It was known that this is not the case when Anonymity is the equity principle under inspection: we can not assure that a given structure produces compatibility or incompatibility without considering the amount of Pareto-efficiency we want to reach. In addition, we have proved that if we are interested in imposing the HE spirit instead then the appeal to $[0,1]$ as the set of feasible utilities determines incompatibility, while the appeal to $\mathbb{N}^*$ does not (and the other factors must be examined: namely, the precise form of the HE postulate and the version of the Pareto axiom in use).

The following tables gather some of the results that have served us to motivate our discussion, and permit to compare differences in the approaches when we vary the feasible utilities.

Table 1. Summary of results for domains of utility streams $Y^{\mathbb{N}}$ under Anonymity

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Non-existence $\star$	Non-existence
WP	Existence †	Non-existence
D	Existence	Non-existence $\diamond$
WD	Existence	Existence ‡

Statement $\star$ is proven in Basu and Mitra (2003). All †, ‡ and $\diamond$ appear in Basu and Mitra (2007). The other statements in the table derive from $\diamond$ and †.

In each of the four cases where compatibility is guaranteed, one can try to identify the class of groups of permutations for which extended anonymity (or $\mathcal{Q}$-Anonymity as introduced by Mitra and Basu, 2007) is compatible with the respective efficiency axiom under the Basu-Mitra approach. We have not pursued this topic yet.

Table 2. Summary of results for domains of utility streams $Y^{\mathbb{N}}$ under RNS

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Existence $\star$	Non-existence
WP	Existence	Non-existence
D	Existence	Non-existence
WD	Existence	Non-existence $\diamond$

Banerjee (2006) proves that $\diamond$ holds even if RNS is weakened to HEF. Statement $\star$ is justified in Theorem 1.1, where an explicit SWF that satisfies RNS and SP is provided. The other statements in the table derive from them.

This results add to Asheim et al. (2007), where incompatibilities of HEF with the Pareto postulate are obtained under continuity assumptions.

Table 3. Summary of results for domains of utility streams $Y^{\mathbb{N}}$ under different versions of HE

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Non-existence $\star$	Non-existence
WP	Depends on version †	Non-existence
D	Depends on version	Non-existence
WD	Depends on version	Non-existence $\diamond$

With respect to HE(L), all the combinations in the table are impossible as is stated in Remark 1.2. Unless otherwise stated, the statements in the table concern all the other variations of the Hammond Equity postulate.

Proposition 1.1 conveys statement $\diamond$ irrespective of the version of HE that we require. The statements above $\diamond$ are now trivial. Case $\star$ is non-existence for all the versions of Hammond Equity that we have dealt with by Theorem 1.2 plus Remark 1.1. Combination † and the instances below it produce non-existence for HE(L), but even if Anonymity is imposed we can combine HE$(a)^+$ and WP/D/WD into an explicit SWF when $\mathbf{Y} = \mathbb{N}^*$ (cf. Theorem 1.3).

We recall that the Rawlsian criterion proves that in tables 1 and 3 above, existence is guaranteed when the efficiency axiom requested is MON (irrespective of $\mathbf{X}$ and the equity axiom). As for table 2, the discrete case is trivial because in that case we can even obtain SP, and $\mathbf{W}_{FR}(\mathbf{x}) = inf\,\{x_i : i = 2, 3, ...\}$ satisfies MON and RNS as mentioned before.

This latter table permits us to contribute to de debate in Banerjee (2006) , regarding how demanding are the properties of representability and continuity (in the adequate sense) for a social welfare ordering. We thus narrow the focus to the continuous case of our study. Bossert, Sprumont and Suzumura (2007) prove that there exist social welfare orders that satisfy HE and Weak Pareto (which implies existence when HEF replaces HE, because HE and WP together imply the HEF property in this context). We also have that there are no social orders satisfying the postulates of HEF, WP and a kind of continuity (Roemer and Suzumura (2007), chapter 4), thus we have the same impossibility result when combining HE, WP and continuity in a social welfare order. Moreover we have proved that there exist no social welfare functions satisfying HE and any version of Pareto efficiency on these domains. In particular no social welfare orderings satisfying HEF and WP is representable. Thus we can conclude that continuity (in the sense of Roemer and Suzumura (2007)) and representability are equally demanding in this setting.

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Chapter 2

Ranking opportunity sets. A characterization of an advised choice

Contents

2.1 Introduction **26**

2.2 Notation and Preliminaries **29**

2.3 Choice in different times **30**

2.3.1 Choice in two different times 30

2.3.2 Choice in n different times 37

2.4 The case of an advised choice **44**

2.4.1 A characterization of the ranking of subsets denoted by $\succcurlyeq_{\mathcal{C}}$ . . 52

2.5 Conclusions and future research **59**

2.6 Bibliography **61**

2.1 Introduction

This chapter concerns rankings of subsets of a set of alternatives, also named opportunity sets. The standard model of a situation of this class considers an agent's preference ordering over a set of alternatives and extend it to a ranking of its nonempty subsets. In the literature about this topic there are many interpretations for these kinds of problems and different axioms are considered suitable depending on the specific context.

A simple example where an agent ranks subsets of alternatives, already cited by Kreps (1979), concerns ranking menus in a restaurant. The individual will choose a meal, but initially he has to select a menu from which he will choose his meal later. We can mention some other contexts where problems of this kind are found, such as voting situations (the selection of a committee), matchings and assignments (admission of groups of students in colleges, hire several workers,...), coalition formations (an agent has to assign different values to the different subsets of colleagues for making a coalition), and so on.

In all these situations the elements are feasible alternatives and the subsets are "opportunity sets" (or menus) of feasible alternatives. There are different criteria to rank these opportunity sets that can be applied considering the particular characteristics of the situation we are dealing with. We can cite some of such situations in order to illustrate and motivate the chapter.

- Choice under complete uncertainty. In these situations a decision can lead to different consequences and the decision-maker has not the possibility of assigning probabilities to them. So making a decision is equivalent to the ranking of the sets of consequences. Different criteria can be considered suitable to be applied here: the maxmin criterion (pessimistic), the minmax criterion (optimistic),...

- Freedom of choice and preference for flexibility. In this kind of contexts the decision-maker attaches value not only to the quality of the choice, but also to the degree of freedom he gets from it. For example, it is usual to prefer a situation where the decision-maker makes the choice better than another one in which the decision-maker is forced to select an alternative, even if the final option is the same in both cases. It is also frequent that the decision-maker prefers those subsets with a greater number of alternatives because (for example) all the alternatives are attractive for him, or because he does not know if he is going to change his preferences before making the final choice,...

- Limited rationality. In this sense, in some decision-situations, one may tend to concentrate on certain “focal” elements or features ignoring the rest of elements or available information.

We mention especially here the *indirect-utility criterion* because it is the one we use in this chapter. It is applied when the quality of the final choice of the agent is all that matters: those subsets with better “best elements” are preferred.

There are many authors that have made an axiomatic approach of this topic: they select desirable criteria for a situation and seek to identify the rankings satisfying different combinations of them.

Fishburn (1972) pioneered the study of preferences of voters over sets of alternatives. Kannai and Peleg (1984) obtain an impossibility result for an order over the set of subsets of a set verifying two attractive axioms. Some other impossibility results are obtained weakening these axioms or considering some others (Barberá and Pattanaik (1984), Fishburn (1984), Holzman (1984)). Bossert (1989) characterizes a preorder with the Kannai-Peleg axioms and a neutrality property also used by other authors. Nehring and Puppe (1996) extend an order over the set of elements to a ranking of its nonempty subsets based on principles of independence and continuity. They characterize rankings that depend on the maximal and minimal elements of the different subsets of alternatives only. Bossert et al. (2000) and Arlegi (2003) approach the problem in the framework of choice under complete uncertainty. They characterize four decision rules not focused on worst or best outcomes only and with intuitive justifications in terms of “limited rationality”.

This context has been extended to incorporate the value of freedom of choice (Bossert et al. (1994), Puppe (1996), Pattanik and Xu (2000) and Xu (2004)). Dutta and Sen (1996) and Alcalde-Unzu and Ballester (2005) among others characterize utilitarian rules, and Alcantud and Arlegi (2008) axiomatize a meaningful family of additively representable rankings of sets. Some other models have served to study these problems, an extensive survey of which can be found in Barberá et al. (2004).

On the other hand, either individual or social decision making may be based in multiple criteria that can be applied in more than one stage. For example, we can think about a family deciding a place for holidays or how to distribute its income (different criteria for parents and children). In many situations of this kind we give preference to a criterion over another one, using a second criterion only for breaking ties produced by the application of a first one: lexicographical order. Different procedures of choice for such lexicographic applications of multiple criteria have been considered in Houy and

Tademuna (2007). We deal with the lexicographical application of different criteria to finite sequences of subsets of different sets of alternatives.

Lexicographic compositions of two criteria of ranking subsets have been studied by different authors. Some of these compositions are completely characterized using suitable axioms. We address the interested reader to Barberá et al. (2004). In this chapter we also characterize two different criteria for ranking subsets based on lexicographic compositions in the line of those in Barberá et al.

So, our approach in this work follows the line of choice under limited rationality that determines the ranking of the subsets looking only at their best alternatives. Such a model is the germ of the indirect-utility approach characterized by Kreps (1979): $A \succcurlyeq B \Leftrightarrow \max(A) R \max(B)$, where R is an individual complete preorder on X (a finite set of objects) that is not necessarily a linear order.

In section 2.3 we study a choice in different times. Both the sets of alternatives and the decision-maker criteria can be different in the different times we deal with. We define a ranking for sequences of ordered subsets (elements of a direct product), each one from the set of alternatives in each time. We first study the case of two different times, and thus we have pair of subsets of alternatives, and then we generalize it to the case of n different times. The ranking we deal with is defined using the indirect-utility criterion applied in each coordinate and with a lexicographical order.

This question has already been studied in Krause (2008) for the case of two different times. He uses 5 axioms for its characterization including a neutrality one and a technical axiom called "simple time discounting". He uses a slightly different notation from ours and supposes that for all the subsets that include alternatives from both times "it is natural" to assume its equivalence with a two-element subset, one from each time, because he is concerned with the indirect utility context only. In this sense he expresses the axioms for the set of two-element subsets. He also uses complete preorders defined over the sets of alternatives in both choice times.

In this chapter we characterize this criterion with only 3 axioms in a model where the preorders over the sets of alternatives are not fixed, in the line of Kreps (1979), and separating the study from the line in Krause that considers fixed preorders over both sets of alternatives. We also generalize and characterize the criterion for the case of choices in n different times.

In section 2.4 we also rely on the indirect-utility criterion for ranking the subsets of X, but we change the model considering the possibility of having an adviser. This adviser has not an individual binary relation defined on X, but rather for any $S \subseteq X$ he selects some "focal elements" (those that he prefers the most), that we represent by a choice function $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$ such that $\varnothing \neq \mathcal{C}(S) \subseteq S$. Then, when we compare two subsets we take into account this adviser only for those subsets that after the first ranking (that of the indirect-utility criterion) are in the same indifference class. We then apply the indirect-utility criterion too, but now to those subsets previously selected by the adviser.

We call the ranking defined under these characteristics "ranking associated to a choice function". It is a complete preorder.

Afterwards we consider the next question:

Given a ranking of subsets of a finite set X, $\succcurlyeq$, which is a complete preorder (and therefore there exists a complete preorder on X trivially induced by $\succcurlyeq$: $aRb \Leftrightarrow \{a\} \succcurlyeq \{b\}$), when is this observed ranking a ranking associated to a choice function? We give a complete answer to this question in terms of two testable conditions.

This last situation we deal with can be considered a choice in two different stages where both the sets of alternatives and the decision-maker criteria are the same. Nevertheless we want to remark that the ranking associated to a choice function is not a particular case of the ranking in two times. Though the sets of alternatives are the same in both times (this is a particular case of our first situation), in the second stage we are not applying a criterion to the subsets of the set of alternatives, but to smaller subsets of them that include the alternatives previously selected by an adviser.

2.2 Notation and Preliminaries

Along this chapter we denote by X a finite set of objects (or alternatives) and by $\mathcal{P}^*(X)$ the set of nonempty subsets of X.

A binary relation on X, $R \subseteq X \times X$, is interpreted as a preference relation of an agent, that is xRy (or $(x, y) \in R$) if and only if the element $x \in X$ is considered at least as good as the element $y \in X$. This relation produces in a natural way a strict relation P and an indifference relation I :

$$xPy \Leftrightarrow \{xRy \text{ and not } yRx\}$$

and

$$xIy \Leftrightarrow \{xRy \text{ and } yRx\}.$$

Definition 2.1. *Let X be a finite set of alternatives and R a binary relation on X. R is a total or complete preorder if it is complete and transitive.*

Definition 2.2. *Given a total preorder R on a finite set X, a* ***best element*** *for R is an element $x \in X$ such that*

$$xRx' \text{ for all } x' \in X.$$

According to this definition a best element has not to be unique (it is unique when R is a linear relation, i.e. it satisfies the antisymmetric property) and all the best elements are indifferent among them.

Abusing notation, $\max(A)$ stands for a best element of a subset $A \subseteq X$.

Definition 2.3. *Let X be a set and $\mathcal{D}$ a nonempty domain of nonempty subsets of X. A choice function is an application $\mathcal{C} : \mathcal{D} \to \mathcal{P}^*(X)$ such that $\mathcal{C}(S) \subseteq S$.*

Along this chapter $\mathcal{D} = \mathcal{P}^*(X)$.

2.3 Choice in different times

In this section we study a choice situation in which a decision maker makes his choice in different times and he can have different sets of alternatives in each time and different binary relations on these sets of alternatives. Citing Krause (2008) “in reality agents typically make a sequence of choices over time from a corresponding sequence of opportunity sets, rather than a single once-and-for-all choice from a single opportunity set”. We can think for example in ordering different menus in a restaurant today for having lunch, and the same tomorrow in a different restaurant with different menus (the set of alternatives changes and perhaps our criterion for ranking meals too).

We divide this section in two subsections. In subsection 2.3.1 we formalize the case in which the DM considers two different decision times. We introduce the axioms we deal with, their independence and interpretation and the characterization theorem for a two-times criterion. In subsection 2.3.2 we generalize the criterion, the axioms and the characterization theorem to the case of n times. Nevertheless the proofs for this general case are not very different to the ones for the two-times case we include them.

2.3.1 Choice in two different times

We first introduce further notation.

- X_1 and Y_2 are finite and nonempty sets of alternatives. They are available in times 1 and 2 respectively. X_1 and Y_2 can be equal or different sets.

- R_1 and R_2 are transitive and complete binary relations defined on X_1 and Y_2 respectively. They capture equal or different decision-maker's criteria. Their respective strict relations are denoted by P_1 and P_2 and their respective indifferent relations by I_1 and I_2.
- The space of alternatives is $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, and we denote a complete preorder defined on it by $\succcurlyeq$.

Let us first introduce some axioms on a complete preorder $\succcurlyeq$ over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$. As Proposition 2.1 below proves, they have important implications on the structure of $\succcurlyeq$.

Axiom 1 (A1). Independence of fixed coordinates.

a) Given $(A, B), (A', B) \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, then

$$(A, B) \succcurlyeq (A', B) \Rightarrow (A, B') \succcurlyeq (A', B') \text{ for all } B' \in \mathcal{P}^*(Y_2).$$

b) Given $(A, B), (A, B') \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, then

$$(A, B) \succcurlyeq (A, B') \Rightarrow (A', B) \succcurlyeq (A', B') \text{ for all } A' \in \mathcal{P}^*(X_1).$$

Axiom 2 (A2). Extension Robustness for both coordinates.

a) Given $(A, B), (A', B) \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, then

$$(A, B) \succcurlyeq (A', B) \Rightarrow (A, B) \sim (A \cup A', B).$$

b) Given $(A, B), (A, B') \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, then

$$(A, B) \succcurlyeq (A, B') \Rightarrow (A, B) \sim (A, B \cup B').$$

Axiom A1 states that the relation by $\succcurlyeq$ between two pairs of subsets remains the same when one of the coordinates is fixed in both pairs, whatever the fixed coordinate is.

Axiom A2 extends the Extension Robustness property introduced by Kreps (1979). Such property establishes that adding a subset A' that is at most as good as a given subset A to A leads to a set that is indifferent to A itself. Axiom A2 establishes this property for each coordinate when the other one is fixed.

The next proposition is crucial in the subsequent analysis. As we have explained before, it supposes a key difference with Krause's position.

Proposition 2.1. *Let $\succcurlyeq$ be a complete preorder defined over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$, that verifies axioms (A1) and (A2). Then for any $(A, B) \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ there exist $a \in A$ and $b \in B$ such that $(A, B) \sim (\{a\}, \{b\})$. Moreover the elements $a \in A$ and $b \in B$ verify that $(\{a\}, B) \succcurlyeq (\{a_i\}, B)$ for all $B \subseteq Y_2$ and for all $a_i \in A$, and $(A, \{b\}) \succcurlyeq (A, \{b_j\})$ for all $A \subseteq X_1$ and for all $b_j \in B$.*

Proof. Let us denote $A = \{a_1, \ldots, a_n\}$ and consider

$$(\{a_1\}, B), (\{a_2\}, B), \ldots, (\{a_n\}, B) \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2).$$

As we know that $\succcurlyeq$ is complete and transitive, we can suppose without loss of generality

$$(\{a_1\}, B) \succcurlyeq (\{a_2\}, B) \succcurlyeq \ldots \succcurlyeq (\{a_n\}, B).$$

Taking into account again that $\succcurlyeq$ is transitive and applying axiom A2, part *a*), successively we obtain

$$(\{a_1\}, B) \sim (\{a_1, a_2\}, B) \sim \ldots \sim (\{a_1, \ldots, a_n\}, B) = (A, B).$$

If we now take $B = \{b_1, \ldots, b_m\}$ we can suppose in the same way that

$$(\{a_1\}, \{b_1\}) \succcurlyeq (\{a_1\}, \{b_2\}) \succcurlyeq \ldots \succcurlyeq (\{a_1\}, \{b_m\})$$

and again by the transitive property and the application of axiom A2, part *b*), successively we obtain

$$(\{a_1\}, \{b_1\}) \sim (\{a_1\}, \{b_1, b_2\}) \sim \ldots \sim (\{a_1\}, \{b_1, \ldots, b_m\}) = (\{a_1\}, B).$$

By transitivity of the indifference relation we conclude

$$(A, B) \sim (\{a_1\}, \{b_1\}).$$

This means that $a = a_1$ and $b = b_1$ satisfy our first claim. The second claim has been established above for the set B. Axiom A1, *a*) extends it for any other subset $B' \subseteq Y_2$.

We also have above that $(\{a_1\}, \{b_1\}) \succcurlyeq (\{a_1\}, \{b_i\})$ for all $b_i \in B$. Applying now axiom A1, *b*) we obtain that $(A, \{b_1\}) \succcurlyeq (A, \{b_i\})$ for all $b_i \in B$ and for all $A \subseteq X_1$, which proves the last claim in the proposition. □

Moreover these axioms are independent as we can observe in the next examples.

Example 2.1. Let us consider the complete preorder defined on $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ by

$$(A,B) \succcurlyeq (A',B') \Leftrightarrow \begin{cases} |A| > |A'|, \text{ or} \\ |A| = |A'| \text{ and } |B| \geqslant |B'| \end{cases}$$

This ranking satisfies A1, but it does not satisfy A2, as it can be immediately proved by considering, for example, A and A' different singletons. $\diamond$

Example 2.2. Let us consider a complete preorder R defined on X. The ranking over $\mathcal{P}^*(X) \times \mathcal{P}^*(X)$ defined as

$$(A,B) \succcurlyeq (A',B') \Leftrightarrow \begin{cases} \max(A) P \max(A'), \text{ or} \\ \max(A) I \max(A') \text{ and } \max(A \cup B) R \max(A' \cup B') \end{cases}$$

is a complete preorder that satisfies A2, but it does not satisfy A1.

Indeed:

To prove property A2, part a), we have to consider $(A,B) \succcurlyeq (A',B)$ and then conclude that $(A,B) \sim (A \cup A', B)$.

From $(A,B) \succcurlyeq (A',B)$ we have two possibilities:

i) $\max(A) P \max(A') \Rightarrow \max(A) I \max(A \cup A')$ and $\max(A \cup B) I \max(A \cup A' \cup B)$ which implies that $(A,B) \sim (A \cup A', B)$.

ii) $\max(A) I \max(A')$ and $\max(A \cup B) R \max(A' \cup B)$ thus $\max(A) I \max(A \cup A')$ and $\max(A \cup B) I \max(A \cup A' \cup B)$ which also leads to $(A,B) \sim (A \cup A', B)$.

For part b) we take $(A,B) \succcurlyeq (A,B')$ and we have to prove that $(A,B) \sim (A, B \cup B')$.

As we have that $\max(A) I \max(A)$, we obtain $\max(A \cup B) R \max(A \cup B')$ and then we have that $\max(A \cup B) I \max(A \cup B \cup B')$ which concludes that $(A,B) \sim (A, B \cup B')$.

The assertion about property A1 can be proved considering, for example, $A = \{a\}, A' = \{a'\}, B = \{b\}$ and $B' = \{b'\}$ being $aPb'Pa'Pb$. In this case $(A,B) \succcurlyeq (A,B')$, but $(A',B') \succ (A',B)$. $\diamond$

Let us now define a preorder over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ associated with the respective criteria R_1 and R_2 over X_1 and Y_2. It is a criterion based on the indirect-utility assessment in both coordinates and we denote it by $\succcurlyeq_{U \times U}$.

Definition 2.4. *Let X_1 and Y_2 be finite sets of alternatives and R_1 and R_2 complete preorders defined on X_1 and Y_2 respectively. The indirect-utility criterion in two times associated with R_1*

and R_2, $\succcurlyeq_{U\times U}$, over $\mathcal{P}^*(X_1)\times\mathcal{P}^*(Y_2)$ is defined by:

$$(A,B)\succcurlyeq_{U\times U}(A',B')\Leftrightarrow\begin{cases} a) & \max(A)P_1\max(A'),\ or\\ b) & \max(A)I_1\max(A')\ and\ \max(B)R_2\max(B')\end{cases}$$

This ranking is a complete preorder that satisfies both axioms A1 and A2 as it can be easily proved. Moreover it also verifies the next axiom that implies that whenever a pair is strictly preferred to another one with the second coordinates being equal, it does not matter the subsets we have in this second place. That is, any pairs of subsets with the same first elements maintain the same order when comparing them, whatever the subsets in the second coordinate are.

Axiom 3 (A3). First component wins. If $(A,B)\succ(A',B)$, then $(A,B')\succ(A',B'')$ for any $B',B''\subseteq Y_2$.

Let us give now some examples to prove the independence among A3 and A1 and A2.

The ranking in Example 2.2 satisfies axiom A3, but it does not satisfy axiom A1.

The ranking defined in the next example 2.3 verifies A1 and A2 and it does not verify A3, as it can be easily proved.

Example 2.3. Let X_1 and Y_2 be finite sets of alternatives and R_1 and R_2 complete preorders defined on X_1 and Y_2 respectively. We define the ranking $\succcurlyeq$ over $\mathcal{P}^*(X_1)\times\mathcal{P}^*(Y_2)$ by

$$(A,B)\succcurlyeq(A',B')\Leftrightarrow\begin{cases}\max(B)P_2\max(B'),\ \text{or}\\ \max(B)I_2\max(B')\ \text{and}\ \max(A)R_1\max(A')\end{cases}$$

$\diamond$

The ranking in the Example 2.1 is a complete preorder that satisfies A3 and A1 and it does not satisfy A2.

Our main theorem in this subsection proves that the criterion given by Definition 2.4 is characterized by axioms A1, A2 and A3.

Theorem 2.1. *A complete preorder $\succcurlyeq$ over $\mathcal{P}^*(X_1)\times\mathcal{P}^*(Y_2)$ verifies A1, A2 and A3 if and only if there exist complete preorders R_1 over X_1 and R_2 over Y_2 such that $\succcurlyeq=\succcurlyeq_{U\times U}$.*

Proof. We have already mentioned that the ranking $\succcurlyeq_{U\times U}$ associated with R_1 and R_2 satisfies axioms A1, A2 and A3.

Let us prove now that if $\succcurlyeq$ is a complete preorder over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ verifying axioms A1, A2 and A3, there exist complete preorders R_1 and R_2 over X_1 and Y_2 respectively such that "$\succcurlyeq = \succcurlyeq_{U\times U}$".

Let us define R_1 over X_1 as

$$xR_1x' \Leftrightarrow (\{x\}, B) \succcurlyeq (\{x'\}, B) \ \ \forall B \subseteq Y_2$$

and R_2 over Y_2 as

$$yR_2y' \Leftrightarrow (A, \{y\}) \succcurlyeq (A, \{y'\}) \ \ \forall A \subseteq X_1.$$

Nevertheless the definitions of R_1 and R_2 are given for all $B \subseteq Y_2$ and for all $A \subseteq X_1$, it is enough to say that xR_1y if and only if there exists a subset $B \subseteq Y_2$ such that $(\{x\}, B) \succcurlyeq (\{x'\}, B)$, and the same for R_2. Applying then that $\succcurlyeq$ satisfies A1, we obtain the relation for all $A \subseteq X_1$ and for all $B \subseteq Y_2$ as it is established.

The relations R_1 and R_2 are obviously complete preorders because $\succcurlyeq$ is a complete preorder.

Now let us denote by $\succcurlyeq_{U\times U}$ the indirect-utility criterion in two times over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ associated with R_1 and R_2. We prove that $\succcurlyeq_{U\times U} = \succcurlyeq$.

We know that for every $(A, B) \in \mathcal{P}^*(X_1) \times \mathcal{P}^*(Y_2)$ there exist elements $a \in A, b \in B$, given by Proposition 2.1, such that $(A, B) \sim (\{a\}, \{b\})$, $(\{a\}, B) \succcurlyeq (\{a_i\}, B)\ \forall a_i \in A$ and for all $B \subseteq Y_2$ and $(A, \{b\}) \succcurlyeq (A, \{b_j\})\ \forall b_j \in B$ and for all $A \subseteq X_1$.

Let us see that it is also true that $(A, B) \sim_{U\times U} (\{a\}, \{b\})$.

Indeed we know that $\succcurlyeq_{U\times U}$ satisfies axioms A1 and A2, and then there must exist $\bar{a} \in A$ and $\bar{b} \in B$ such that $(A, B) \sim_{U\times U} (\{\bar{a}\}, \{\bar{b}\})$. As it must be also true (Proposition 2.1) that

$$(\{\bar{a}\}, B) \succcurlyeq_{U\times U} (\{a_i\}, B),\ \forall a_i \in A \text{ and } \forall B \subseteq Y_2$$

we have that this is also true if $a_i = a$ and then we have two possibilities.

a) $\bar{a}P_1a \Leftrightarrow (\{\bar{a}\}, B) \succ (\{a\}, B)$, which is not possible because $(\{a\}, B) \succcurlyeq (\{a_i\}, B)$ for all $a_i \in A$ and for all $B \subseteq Y_2$.

b) $\bar{a}I_1a$ and $\max(B) R_2 \max(B)$.

In this case, as we have $\max(B) I_2 \max(B)$ we obtain that $(\{\bar{a}\}, B) \sim_{U\times U} (\{a\}, B)$ for all B. In particular if $B = \{\bar{b}\}$ we obtain that $(\{\bar{a}\}, \{\bar{b}\}) \sim_{U\times U} (\{a\}, \{\bar{b}\})$.

Analogously we obtain that $(A, \{b\}) \sim_{U\times U} (A, \{\bar{b}\})$ for all A. Thus, if $A = \{a\}$ we have $(\{a\}, \{b\}) \sim_{U\times U} (\{a\}, \{\bar{b}\}) \sim_{U\times U} (\{\bar{a}\}, \{\bar{b}\})$ and then $(A, B) \sim_{U\times U} (\{a\}, \{b\})$.

Now let $(A,B) \sim (\{a\},\{b\})$ and $(A',B') \sim (\{a'\},\{b'\})$ be such that $(A,B) \succcurlyeq_{U\times U} (A',B')$. We have to prove that $(A,B) \succcurlyeq (A',B')$.

There are two possibilities to consider:

(i) aP_1a', thus applying the definition of R_1 we have that $(\{a\},B) \succ (\{a'\},B)$ for all $B \in Y_2$. In particular we obtain that

$$(\{a\},\{b\}) \succ (\{a'\},\{b\})$$

and applying axiom A3

$$(\{a\},\{b\}) \succ (\{a'\},B') \ \ \forall B'.$$

Taking $B' = \{b'\}$ we conclude

$$(\{a\},\{b\}) \succ (\{a'\},\{b'\}).$$

(ii) aI_1a' and bR_2b'. Then we have

$$(\{a\},B) \sim (\{a'\},B) \ \ \forall B$$

and

$$(A,\{b\}) \succcurlyeq (A,\{b'\}) \ \ \forall A.$$

In particular if $A = \{a\}$ and $B = \{b'\}$ we obtain that

$$(\{a\},\{b\}) \succcurlyeq (\{a\},\{b'\}) \sim (\{a'\},\{b'\})$$

and we conclude.

To end, we have to prove that if $(A,B) \succcurlyeq (A',B')$, then it must be $(A,B) \succcurlyeq_{U\times U} (A',B')$.

If this is not true, $(\{a'\},\{b'\}) \succ_{U\times U} (\{a\},\{b\})$ and we have again two possibilities:

(i) $a'P_1a$ thus $(\{a'\},B) \succ (\{a\},B) \ \ \forall B$, and in particular if $B = \{b'\}$ we have

$$(\{a'\},\{b'\}) \succ (\{a\},\{b'\}).$$

Applying A3 we obtain

$$(\{a'\},\{b'\}) \succ (\{a\},\{b\})$$

against the hypothesis.

(ii) $a' I_1 a$ and $b' P_2 b$ and then
$$(\{a'\}, B) \sim (\{a\}, B) \;\; \forall B$$
and
$$(A, \{b'\}) \succ (A, \{b\}) \;\; \forall A.$$
Taking $A = \{a'\}$ and $B = \{b\}$ we obtain
$$(\{a'\}, \{b'\}) \succ (\{a'\}, \{b\}) \sim (\{a\}, \{b\})$$
which also contradicts the hypothesis and concludes the proof.

□

2.3.2 Choice in n different times

In this subsection we generalize the indirect-utility criterion in two times to the case in which a DM considers n different times. He can have different sets of alternatives in each time and different binary relations defined on these sets of alternatives.

Our DM ranks subsets of alternatives in different times and we formalize this situation considering a ranking over the direct product of all the sets of nonempty subsets of alternatives in each period
$$X = \mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$$
where X_i stands for the finite set of alternatives in time i and R_i for the binary relation defined on X_i.

Next definition formalizes the criterion consisting on the lexicographical application of the indirect-utility criterion to the elements of X.

Definition 2.5. *The indirect-utility criterion in n times associated with $R_1, \ldots, R_n$.*
$$(A_1, \ldots, A_n) \succcurlyeq_U (B_1, \ldots, B_n) \textit{ if and only if}$$

- $\max(A_i) I_i \max(B_i)$ *for all* $i = 1, \ldots, n$, *or*
- *There exists* $j \in \{1, \ldots, n\}$ *such that* $\max(A_i) I_i \max(B_i)$ *for all* $i = 1, \ldots, j-1$ *and* $\max(A_j) P_j \max(B_j)$.

This criterion verifies the next axioms, that generalize the ones we have used in the previous section, as it can be easily proved.

Axiom 1 (GA1). Independence of fixed coordinates.

If $(A_1,\ldots,A_n) \succcurlyeq (B_1,\ldots,B_n)$ where $A_i = B_i$ for all $i \in S \subseteq N$, then it must be $(A'_1,\ldots,A'_n) \succcurlyeq (B'_1,\ldots,B'_n)$ where $A'_i = B'_i$ for all $i \in S$ and $A'_j = A_j$ and $B'_j = B_j$ for all $j \in N \setminus S$.

This axiom establishes that when we compare two elements in X with some coordinates being equal, the relation between them does not change if we change such coordinates in such way that they remain equal in both elements.

Next axiom enunciates the strong robustness property (Kreps (1979)) in each coordinate when the others remain the same.

Axiom 2 (GA2). Strong Robustness in each coordinate.

If $(A_1,\ldots,A_i,\ldots,A_n) \succcurlyeq (A_1,\ldots,A'_i,\ldots,A_n)$ then it must be $(A_1,\ldots,A_i,\ldots,A_n) \sim (A_1,\ldots,A_i \cup A'_i,\ldots,A_n)$.

Next we proceed as in the previous section, when we dealt with two decision periods, in order to characterize the indirect-utility criterion in n times.

Proposition 2.2. *Let $\succcurlyeq$ be a complete preorder defined over $X = \mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$, that verifies axioms (GA1) and (GA2). Then for any $(A_1,\ldots,A_n) \in \mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$ there exist $a_1 \in A_1,\ldots,a_n \in A_n$ such that $(A_1,\ldots,A_n) \sim (\{a_1\},\ldots,\{a_n\})$. Moreover the elements $a_i \in A_i$ verify that $(A'_1,\ldots,\{a_i\},\ldots,A'_n) \succcurlyeq (A'_1,\ldots,\{a'_i\},\ldots,A'_n)$ for all $A'_j \subseteq X_j,\ j \in N \setminus \{i\}$, and for all $a'_i \in A_i$.*

Proof. Let us take $(A_1,\ldots,A_n) \in X$ and $A_1 = \{a^1_1, a^1_2,\ldots,a^1_{p_1}\}$. As we know that $\succcurlyeq$ is complete, we can suppose without loss of generality that

$$(\{a^1_1\},A_2,\ldots,A_n) \succcurlyeq (\{a^1_2\},A_2,\ldots,A_n) \succcurlyeq \ldots \succcurlyeq (\{a^1_{p_1}\},A_2,\ldots,A_n).$$

Taking into account that $\succcurlyeq$ is transitive and applying axiom GA2 successively we obtain

$$(\{a^1_1\},A_2,\ldots,A_n) \sim (\{a^1_1,a^1_2\},A_2,\ldots,A_n) \sim \ldots \sim (\{a^1_1,\ldots,a^1_n\},A_2,\ldots,A_n)$$

an then

$$(\{a^1_1\},A_2,\ldots,A_n) \sim (A_1,\ldots,A_n).$$

If we now take $A_2 = \{a_1^2, \ldots, a_{p_2}^2\}$ we can suppose in the same way that

$$(\{a_1^1\}, \{a_1^2\}, A_3, \ldots, A_n) \succcurlyeq (\{a_1^1\}, \{a_2^2\}, A_3, \ldots, A_n) \succcurlyeq \ldots \succcurlyeq (\{a_1^1\}, \{a_{p_2}^2\}, A_3, \ldots, A_n)$$

and applying again axiom GA2 successively (and the transitive property) we obtain

$$(\{a_1^1\}, \{a_1^2\}, A_3, \ldots, A_n) \sim (\{a_1^1\}, \{a_1^2, a_2^2\}, A_3, \ldots, A_n) \sim \ldots \sim$$

$$\sim (\{a_1^1\}, \{a_1^2, \ldots, a_{p_2}^2\}, A_3, \ldots, A_n) = (\{a_1^1\}, A_2, \ldots, A_n).$$

By transitivity of the indifference relation we conclude

$$(\{a_1^1\}, \{a_1^2\}, A_3, \ldots, A_n) \sim (A_1, \ldots, A_n).$$

Let us now suppose (induction hypothesis) that

$$(\{a_1^1\}, \{a_1^2\}, \ldots, \{a_1^{i-1}\}, A_i, \ldots, A_n) \sim (A_1, \ldots, A_n)$$

and $A_i = \{a_1^i, \ldots, a_{p_i}^i\}$.

We can suppose that

$$(\{a_1^1\}, \{a_1^2\}, \ldots, \{a_1^{i-1}\}, \{a_1^i\}, A_{i+1}, \ldots, A_n) \succcurlyeq (\{a_1^1\}, \{a_1^2\}, \ldots, \{a_1^{i-1}\}, \{a_2^i\}, A_{i+1}, \ldots, A_n) \succcurlyeq$$

$$\succcurlyeq \ldots \succcurlyeq (\{a_1^1\}, \{a_1^2\}, \ldots, \{a_1^{i-1}\}, \{a_{p_i}^i\}, A_{i+1}, \ldots, A_n).$$

Applying the transitive property and axiom GA2 successively we obtain

$$(\{a_1^1\}, \{a_1^2\}, \ldots, \{a_1^i\}, A_{i+1}, \ldots, A_n) \sim (A_1, \ldots, A_n).$$

Therefore we can conclude by induction that

$$(A_1, \ldots, A_n) \sim (\{a_1^1\}, \ldots, \{a_1^n\}).$$

This means that $a_1 = a_1^1, \ldots, a_n = a_1^n$ satisfy our first claim. The second claim has been established above for the first coordinate and the sets $A_2, \ldots, A_n$. Axiom GA1 extends it for any subsets $A_2' \subseteq X_2, \ldots, A_n' \subseteq X_n$.

We also have above that $(\{a_1^1\}, \{a_1^2\}, A_3, \ldots, A_n) \succcurlyeq (\{a_1^2\}, \{a_j^2\}, A_3, \ldots, A_n)$ for all $a_j^2 \in A_2$. Applying now axiom GA1 again we obtain that $(A_1', \{a_1^2\}, A_3', \ldots, A_n') \succcurlyeq (A_1', \{a_j^2\}, A_3', \ldots, A_n')$ for all $a_j^2 \in A_2$ and for all $A_1' \subseteq X_1, A_3' \subseteq X_3, \ldots, A_n' \subseteq X_n$ which proves the second claim in the Proposition 2.2 for the second coordinate. In the same way we conclude the claim for any coordinate $i \in N$. $\square$

Next examples prove the independence of the GA1 and GA2 axioms.

Example 2.4. Let us consider the complete preorder $\succcurlyeq$ defined on $\mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$ by

$$(A_1, \ldots . A_n) \succcurlyeq (B_1, \ldots, B_n) \Leftrightarrow \begin{cases} i)\ |A_i| = |B_i|\ \forall i = 1, \ldots, n, \text{ or} \\ ii)\ \exists j \in \{1, \ldots, n\} \text{ such that } |A_i| = |B_i|\ \forall i = 1, \ldots, j-1 \\ \text{ and } |A_j| > |B_j| \end{cases}$$

$\succcurlyeq$ satisfies GA1, but it does not satisfy GA2. $\diamond$

Next example gives a complete preorder satisfying GA2 and not satisfying GA1.

Example 2.5. Let us consider a complete preorder R defined on X and the ranking over $\mathcal{P}^*(X) \times \ldots \times \mathcal{P}^*(X)$ defined as

i) $(A_1, \ldots, A_n) \succ (B_1, \ldots, B_n) \Leftrightarrow \exists j \in \{1, \ldots, n\}$ such that $\max(A_1 \cup \ldots \cup A_i) I$ $\max(B_1 \cup \ldots \cup B_i)\ \forall i = 1, \ldots, j-1$ and $\max(A_1 \cup \ldots \cup A_j) P \max(B_1 \cup \ldots \cup B_j)$

ii) $(A_1, \ldots, A_n) \sim (B_1, \ldots, B_n) \Leftrightarrow \max(A_1 \cup \ldots \cup A_i) I \max(B_1 \cup \ldots B_i)$ for all $i = 1, \ldots, n$.

GA2 is satisfied because if

$$(A_1, \ldots, A_i, \ldots, A_n) \succcurlyeq (A_1, \ldots, A_i', \ldots, A_n)$$

we have that

$$\max(A_1 \cup \ldots \cup A_i) R \max(A_1 \cup \ldots \cup A_i')$$

and then

$$\max(A_1 \cup \ldots \cup A_i) I \max(A_1 \cup \ldots \cup A_i \cup A_i')$$

which leads to

$$(A_1, \ldots, A_i, \ldots, A_n) \sim (A_1, \ldots, A_i \cup A_i', \ldots, A_n)$$

In order to prove that GA1 is not satisfied let us take $a_1, a_2, b_1, b_2 \in X$ such that $a_1 P b_2 P b_1 P a_2$. We have

$$(\{a_1\}, \{a_2\}, A_3, \ldots, A_n) \succcurlyeq (\{a_1\}, \{b_2\}, A_3, \ldots, A_n)$$

but

$$(\{b_1\}, \{b_2\}, A_3, \ldots, A_n) \succ (\{b_1\}, \{a_2\}, A_3, \ldots, A_n)$$

because $\max(\{b_1\} \cup \{b_2\}) = b_2 P \max(\{b_1\} \cup \{a_2\}) = b_1$. $\diamond$

Moreover the ranking given in definition 2.5 also satisfies the next axiom.

Axiom 3 (GA3). Impatience.

$$(A_1, \ldots, A_i, \ldots, A_n) \succ (A_1, \ldots, A_i', \ldots, A_n) \Rightarrow$$

$$(A_1, \ldots, A_i, \ldots, A_n) \succ (A_1, \ldots, A_i', A_{i+1}', \ldots, A_n')$$

for all $A_{i+1}' \subseteq X_{i+1}, \ldots, A_n' \subseteq X_n$.

This last axiom implies that whenever an element is strictly preferred to another one with only a different coordinate, then it does not matter which subsets we have in all the coordinates after the different one. The element that was preferred goes on being strictly preferred to those elements with the same coordinates until the different one, whatever other subsets are in the rest of them.

It is immediate to prove that the rankings given in the examples 2.4 and 2.5 above satisfy GA3. In order to finish the proof of the independence of this axiom and axioms GA1 and GA2 we give in the next example 2.6 a ranking that obviously satisfies GA1 and GA2, but not GA3.

Example 2.6. Let $X_1, \ldots, X_n$ be finite sets of alternatives, $R_1, \ldots, R_n$ complete preorders defined respectively over $X_1, \ldots, X_n$ and $X = \mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$.

If $(A_1, \ldots, A_n), (B_1, \ldots, B_n) \in X$ we define

$$(A_1, \ldots, A_n) \succcurlyeq (B_1, \ldots, B_n) \text{ if and only if}$$

- $\max(A_i) I_i \max(B_i)$ for all $i = 1, \ldots, n$, or
- There exists $j \in \{1, \ldots, n\}$ such that $\max(A_i) I_i \max(B_i)$ for all $i = n, \ldots, n-j+1$ and $\max(A_j) P_j \max(B_j)$.

$\diamond$

Next theorem characterizes the indirect-utility criterion in n times.

Theorem 2.2. *A complete preorder $\succcurlyeq$ over $\mathcal{P}^*(X_1) \times \mathcal{P}^*(X_2) \times \ldots \times \mathcal{P}^*(X_n)$ verifies GA1, GA2 and GA3 if and only if there exist complete preorders R_i over X_i, $i = 1, \ldots, n$, such that $\succcurlyeq = \succcurlyeq_U$, where $\succcurlyeq_U$ is the indirect-utility criterion associated with $R_i, i = 1, \ldots, n$.*

Proof. We have already mentioned that the ranking $\succcurlyeq_U$ given by Definition 2.5 satisfies axioms GA1, GA2 and GA3.

Let us prove now that if $\succcurlyeq$ is a complete preorder over $\mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$ verifying axioms GA1, GA2 and GA3, there exist complete preorders R_i over $X_i, \forall i = 1, \ldots, n$ such that "$\succcurlyeq = \succcurlyeq_U$".

Let us define R_i over X_i, just as the generalization of R_1 and R_2 in Theorem 2.1:

$$x_i R_i y_i \Leftrightarrow (A_1, \ldots, \{x_i\}, \ldots, A_n) \succcurlyeq (A_1, \ldots, \{y_i\}, \ldots, A_n)$$

for all $A_1 \subseteq X_1, \ldots, A_{i-1} \subseteq X_{i-1}, A_{i+1} \subseteq X_{i+1}, \ldots, A_n \subseteq X_n$.

Nevertheless the definition of R_i is given for all $A_j \subseteq X_j$, $j \in N \setminus \{i\}$, it can be defined considering that $x_i R_i y_i$ if and only if there exists a subset $A_j \subseteq X_j$ for each $j \in N \setminus \{i\}$ such that $(A_1, \ldots, \{x_i\}, \ldots, A_n) \succcurlyeq (A_1, \ldots, \{y_i\}, \ldots, A_n)$. Applying then axiom GA1 we conclude that the relation is verified for all $A_j \subseteq X_j$, $j \in N \setminus \{i\}$ as the definition establishes.

Now let us denote by $\succcurlyeq_U$ the total preorder over $\mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$ we have called "indirect-utility criterion in n times" associated with $R_1, \ldots, R_n$. We prove that $\succcurlyeq_U = \succcurlyeq$.

We know that for each $(A_1, \ldots, A_n) \in \mathcal{P}^*(X_1) \times \ldots \times \mathcal{P}^*(X_n)$ there exist elements $a_i \in A_i$, for all $i = 1, \ldots, n$ as in Proposition 2.2 such that $(A_1, \ldots, A_n) \sim (\{a_1\}, \ldots, \{a_n\})$, $(A_1, \ldots, \{a_i\}, \ldots, A_n) \succcurlyeq (A_1, \ldots, \{a'_i\}, \ldots, A_n)$ for all $a'_i \in A_i$, and for all $A_j \subseteq X_j$, $j \in N \setminus \{i\}$.

Let us see that it is also true that $(A_1, \ldots, A_n) \sim_U (\{a_1\}, \ldots, \{a_n\})$.

Indeed we know that $\succcurlyeq_U$ satisfies axioms GA1 and GA2, thus there must exist $\bar{a}_i \in A_i$, $i = 1, \ldots, n$, such that $(A_1, \ldots, A_n) \sim_U (\{\bar{a}_1\}, \ldots, \{\bar{a}_n\})$. As it must be also true (Proposition 2.2) that

$$(A_1, \ldots, \{\bar{a}_i\}, \ldots, A_n) \succcurlyeq_U (A_1, \ldots, \{a'_i\}, \ldots, A_n), \ \forall a'_i \in A_i \text{ and } \forall A_j \subseteq X_j, \ j \in N \setminus \{i\}$$

we have that this is also true if $a'_i = a_i$ and then we must consider two possibilities.

a) $\bar{a}_i P_i a_i \Leftrightarrow (A_1, \ldots, \{\bar{a}_i\}, \ldots, A_n) \succ (A_1, \ldots, \{a_i\}, \ldots, A_n)$, which is not possible because $(A_1, \ldots, \{a_i\}, \ldots, A_n) \succcurlyeq (A_1, \ldots, \{a'_i\}, \ldots, A_n) \quad \forall a'_i \in A_i$ and for all

$A_j \subseteq X_j,\ j \in N \setminus \{i\}$.

b) $\bar{a}_i I_i a_i$ and $\max(A_j) P_j \max(A_j)$.

This case is not possible because we have that $\max(A_j) I_j \max(A_j)$ for all $j = i + 1, \ldots, n$. Then we obtain that $(A_1, \ldots, \{\bar{a}_i\}, \ldots, A_n) \sim_U (A_1, \ldots, \{a_i\}, \ldots, A_n)$ for all $A_j \subseteq X_j, j \in N \in \setminus \{i\}$.

Let us now consider the elements $(A_1, \ldots, A_n) \sim (\{a_1\}, \ldots, \{a_n\})$ and $(A'_1, \ldots, A'_n) \sim (\{a'_1\}, \ldots, \{a'_n\})$ such that $(A_1, \ldots, A_n) \succcurlyeq_U (A'_1, \ldots, A'_n)$. We have to prove that then $(A_1, \ldots, A_n) \succcurlyeq (A'_1, \ldots, A'_n)$.

There are two possibilities to consider:

(i) There exists $i \in N$ such that $a_j I_j a'_j$ for all $j = 1, \ldots, i-1$ and $a_i P_i a'_i$.

Applying the definition of the binary relation R_i we have that $(A_1, \ldots, \{a_i\}, \ldots, A_n) \succ (A_1, \ldots, \{a'_i\}, \ldots, A_n)$ for all $A_j \subseteq X_j, j \in N \setminus \{i\}$. In particular we obtain that

$$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a'_i\}, \ldots, \{a_n\}).$$

Because of axiom GA3 we have

$$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a'_i\}, A'_{i+1}, \ldots, A'_n)\ \ \forall A'_j \subseteq X_j, j = i+1, \ldots, n.$$

Taking $A'_j = \{a'_j\}\ \forall j = i+1, \ldots, n$ we conclude

$$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a_{i-1}\}, \{a'_i\}, \ldots, \{a'_n\}).$$

As we have $a_j I_j a'_j$ for all $j = 1, \ldots, i-1$, if we apply the definition of $R_j,\ j = 1, \ldots, i-1$ we have that

$$(\{a_1\}, \ldots, \{a_{i-1}\}, \{a'_i\}, \ldots, \{a'_n\}) \sim (\{a'_1\}, \ldots, \{a'_n\})$$

and by transitivity

$$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a'_1, \ldots, \{a'_n\}).$$

(ii) $a_i I_i a'_i$ for all $i = 1, \ldots, n$.

In this case we have

$$(\{a_1\}, \ldots \{a_n\}) \sim (\{a'_1\}, \ldots, \{a'_n\})$$

and we conclude.

To end, we have to prove that whenever $(A_1, \ldots, A_n) \succcurlyeq (A'_1, \ldots, A'_n)$, it must be $(A_1, \ldots, A_n) \succcurlyeq_U (A'_1, \ldots, A'_n)$.

If not, $(\{a'_1\}, \ldots, \{a'_n\}) \succ_U (\{a_1\}, \ldots, \{a_n\})$ and then we have that there exists $i \in \{1, \ldots, n\}$ such that $a'_j I_j a_j$ for all $j = 1, \ldots, i-1$ and $a'_i P_i a_i$. Therefore

$$(\{a'_1\}, \ldots, \{a'_i\}, A_{i+1}, \ldots, A_n) \succ (\{a'_1\}, \ldots, \{a'_{i-1}\}, \{a_i\}, A_{i+1}, \ldots, A_n)$$

for all $A_j \subseteq X_j,\ j = i+1, \ldots, n$.

In particular if we take $A_j = \{a'_j\},\ j = i+1, \ldots, n$, we have

$$(\{a'_1\}, \ldots, \{a'_n\}) \succ (\{a'_1\}, \ldots, \{a_i\}, \ldots, \{a'_n\}).$$

Applying now axiom GA3 we obtain

$$(\{a'_1\}, \ldots, \{a'_n\}) \succ (\{a'_1\}, \ldots, \{a'_{i-1}\}, \{a_i\}, \ldots, \{a_n\})$$

As we know that

$$(\{a'_1\}, \ldots, \{a'_{i-1}\}, \{a_i\}, \ldots, \{a_n\}) \sim (\{a_1\}, \ldots, \{a_n\})$$

because $a'_j I_j a_j$ for all $j = 1, \ldots, i-1$ we conclude

$$(\{a'_1\}, \ldots, \{a'_n\}) \succ (\{a_1\}, \ldots, \{a_n\})$$

against the hypothesis. □

2.4 The case of an advised choice

In the same way that in the previous section, along this one we also consider a complete preorder defined on a finite set X and extend it to a ranking of opportunity sets in two steps. Again we appeal to the indirect-utility criterion.

Besides, for each complete preorder $\succcurlyeq$ on $\mathcal{P}^*(X)$ we consider the complete preorder R on X naturally induced by it and defined as

$$xRy \Leftrightarrow \{x\} \succcurlyeq \{y\}, \ \ \forall x, y \in X.$$

The criterion of choice we consider in this section works as follows. In a first step, we apply the indirect-utility criterion directly to the subsets we are comparing, just as in the classical way; and we use a second step to break ties only. In such a case we apply again the indirect-utility criterion, but now to smaller subsets of each initial one. These smaller subsets include some elements of the initial ones that we call "focal elements" and that have been previously selected by "an adviser". Therefore the situation is different from the one in the previous section. There, both the set of alternatives and the preorders R_1 and R_2 defined on them could be different in the two different times in which the choice took place. Now we consider the same set of alternatives and the same preorder defined on it in the two different times. The only difference between these two different choice times are the subsets to which we apply the indirect-utility criterion.

Before formalizing our criterion we introduce some examples from the real world to motivate it.

Suppose that a couple is going to marry and they want to select a wedding list from different shops. Each shop has its own catalogue with its wedding list and they have to select one of them. In order to do that the woman ranks the catalogues looking only to some concrete elements she considers indispensable in a wedding list and comparing all these elements from the different catalogues. Then, her couple selects from each catalogue in her best class of preference those presents that he considers indispensable in his wedding list. Then she compares these subsets ranking them only looking to the elements in them she likes the most. After having selected the catalogue, their relatives and friends will decide which presents in the catalogue they buy.

Another example is a family making a decision about the place to spend some holiday days. The parents select some places after considering how much they can pay, how far the places are, and so on. Afterwards, in order to make the final choice they ask their children opinion. Then they probably have smaller subsets of places for choosing.

The next definition formalizes this criterion for ranking subsets.

Definition 2.6. *Let X be a finite set of alternatives and R a complete preorder on X, and let $\mathcal{C}$ be a choice function on X. We define the ranking of subsets of X associated with R and $\mathcal{C}$ as:*

$$A \succcurlyeq_{\mathcal{C}} B \text{ if and only if}$$

a) $\max(A) P \max(B)$, *or*

b) $\max(A) I \max(B)$ *and* $\max(\mathcal{C}(A)) R \max(\mathcal{C}(B))$

for all $A, B \subseteq X$.

Remark 2.1. The ranking $\succcurlyeq_{\mathcal{C}}$ is a total preorder on the set of nonempty subsets of X because R is a total preorder on X. $\lhd$

Let us introduce now an analytical example of this criterion for ranking subsets.

Example 2.7. Consider $X = \{a, b, b', c, c\}$ and R a complete preorder defined on X satisfying

$$aPbIb'PcPd$$

where P and I are the strict and the indifference relations associated to R.

Let $\mathcal{C}$ be the choice function defined on X given by:

$$\begin{array}{rcc}
\mathcal{C}(S) & = & S \text{ for all } S \text{ with } |S| = 2 \\
\mathcal{C}(\{a,b,b'\}) & = & \{a,b,b'\} \\
\mathcal{C}(\{a,b,c\}) & = & \{c\} \\
\mathcal{C}(\{a,b,d\}) & = & \{a,b,d\} \\
\mathcal{C}(\{a,b',c\}) & = & \{c\} \\
\mathcal{C}(\{a,b',d\}) & = & \{d\} \\
\mathcal{C}(\{a,c,d\}) & = & \{d\} \\
\mathcal{C}(\{b,b',c\}) & = & \{b,c\} \\
\mathcal{C}(\{b,b',d\}) & = & \{d\} \\
\mathcal{C}(\{b,c,d\}) & = & \{c,d\} \\
\mathcal{C}(\{b',c,d\}) & = & \{c\} \\
\mathcal{C}(\{a,b,b',c\}) & = & \{a,b,c\} \\
\mathcal{C}(\{a,b,b',d\}) & = & \{d\} \\
\mathcal{C}(\{a,b,c,d\}) & = & \{d\} \\
\mathcal{C}(\{a,b',c,d\}) & = & \{c,d\} \\
\mathcal{C}(\{b,b',c,d\}) & = & \{c,d\} \\
\mathcal{C}(\{a,b,b',c,d\}) & = & \{d\}
\end{array}$$

Let us now construct the ranking $\succcurlyeq_{\mathcal{C}}$ associated with $\mathcal{C}$. We obtain:

$$\{a\} \sim_{\mathcal{C}} \{a,b\} \sim_{\mathcal{C}} \{a,b'\} \sim_{\mathcal{C}} \{a,c\} \sim_{\mathcal{C}} \{a,d\} \sim_{\mathcal{C}} \{a,b,b'\} \sim_{\mathcal{C}} \{a,b,d\} \sim_{\mathcal{C}} \{a,b,b',c\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{a,b,c\} \sim_{\mathcal{C}} \{a,b',c\} \sim_{\mathcal{C}} \{a,b',c,d\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{a,b',d\} \sim_{\mathcal{C}} \{a,c,d\} \sim_{\mathcal{C}} \{a,b,b',d\} \sim_{\mathcal{C}} \{a,b,c,d\} \sim_{\mathcal{C}} \{a,b,b',c,d\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{b\} \sim_{\mathcal{C}} \{b'\} \sim_{\mathcal{C}} \{b,b'\} \sim_{\mathcal{C}} \{b,c\} \sim_{\mathcal{C}} \{b,d\} \sim_{\mathcal{C}} \{b',c\} \sim_{\mathcal{C}} \{b',d\} \sim_{\mathcal{C}} \{b,b',c\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{b,c,d\} \sim_{\mathcal{C}} \{b',c,d\} \sim_{\mathcal{C}} \{b,b',c,d\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{b,b',d\} \succ_{\mathcal{C}}$$

$$\succ_{\mathcal{C}} \{c\} \sim_{\mathcal{C}} \{c,d\}\} \succ_{\mathcal{C}} \{d\}.$$

$\diamond$

Of course the criterion stated in Definition 2.6 is not universal: there are complete preorders that can not be represented in this form. The next example illustrates this statement.

Example 2.8. Let $X = \{a,b\}$. We consider the complete preorder on $\mathcal{P}^*(X)$ defined by

$$\{a,b\} \succ \{a\} \succ \{b\}.$$

Then $\succcurlyeq$ can not be represented as a ranking associated to a choice function $\mathcal{C}$ and the complete preorder R on X induced by $\succcurlyeq$ and that is defined as

$$aRb \text{ and } \neg(bRa) \Leftrightarrow aPb.$$

Indeed:

A choice function $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$ can be defined in the three different forms:

$a)$ $\mathcal{C}(\{a,b\}) = \{a\}$,

$b)$ $\mathcal{C}(\{a,b\}) = \{b\}$, or

$c)$ $\mathcal{C}(\{a,b\}) = \{a,b\}$.

In all of them $\mathcal{C}(\{a\}) = \{a\}$ and $\mathcal{C}(\{b\}) = \{b\}$.

From definitions $a)$ and $c)$ we obtain that the best element in $\mathcal{C}(\{a,b\})$ is a in both cases and thus

$$\{a,b\} \sim_{\mathcal{C}} \{a\}.$$

From definition $b)$ the best element in $\mathcal{C}(\{a,b\})$ is b and then

$$\{a\} \succ_{\mathcal{C}} \{a,b\}.$$

Therefore $\succcurlyeq \neq \succcurlyeq_{\mathcal{C}}$ for any possible choice function $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$. $\diamond$

Our goal in this section is to characterize the ranking introduced by Definition 2.6: we want to identify the behavior of a decision maker that uses it. In this sense we analyze some properties that this criterion satisfies. Before introducing them, we deal with a bit of notation. The next subsection 2.4.1 completes the analysis.

Remember that X denotes a finite set of alternatives, and $\succcurlyeq$ a ranking of subsets of X which is a complete preorder. Using the usual decomposition in mutually disjoint indifference classes by $\succcurlyeq$ we can write (and so it is considered along the whole section)

$$X = X_1 \cup \ldots \cup X_n$$

in such a way that $x_i, x_i' \in X_i \Leftrightarrow \{x_i\} \sim \{x_i'\}$, and $\{x_1\} \succ \{x_2\} \succ \cdots \succ \{x_n\}$ for all $x_i \in X_i,\ i = 1, \ldots, n$. Or in terms of the complete preorder R induced by $\succcurlyeq$ over the set of alternatives X, all the elements in X_i are indifferent by R and at the same time we have $x_1 P x_2 P \ldots P x_n$ for all $x_i \in X_i$ and for all $i = 1, \ldots, n$.

We can consider the restriction of the preorder $\succcurlyeq$ to any domain of nonempty subsets of X. Let $\mathcal{P}'(X)$ be one such domain and let us consider the restriction of $\succcurlyeq$ to it.

Definition 2.7. *Let $A_1, \ldots, A_p$ be elements in $\mathcal{P}'(X)$. A chain of strict preferences*

$$A_1 \succ A_2 \succ \ldots \succ A_{p-1}$$

is maximum for A_p in $\mathcal{P}'(X)$ with respect to the complete preorder $\succcurlyeq$ if it verifies

$$A_1 \succ A_2 \succ \ldots \succ A_{p-1} \succ A_p$$

and also that if $A \in \mathcal{P}'(X)$ verifies that $A \succ A_p$, then it must be $A \sim A_i$ for some $i \in \{1, \ldots, p-1\}$.

It is obvious from Definition 2.7 that if two subsets in $\mathcal{P}'(X)$ have the same maximum chain, they have to be indifferent by $\succcurlyeq$. In fact the next proposition holds.

Proposition 2.3. *Let A be a subset in $\mathcal{P}'(X)$ and let us suppose that*

$$A_1 \succ \ldots \succ A_k$$

and

$$B_1 \succ \ldots \succ B_r$$

are two maximum chains for A in $\mathcal{P}'(X)$. Then it must be $r = k$ and $B_i \sim A_i$ for all $i = 1, \ldots, k$.

Proof. We know that $A_1 \succ \ldots \succ A_k$ is a maximum chain for A in $\mathcal{P}'(X)$ and $B_i \succ A$ for all $i = 1, \ldots, r$, so for every $i = 1, \ldots, r$ it is forceful that $B_i \sim A_j$ for some $j = 1, \ldots, k$. Because of $B_i \nsim B_j$ whenever $i \neq j$, it must be $k \geqslant r$. The same argument applied to the fact that $B_1 \succ \ldots \succ B_r$ is a maximum chain for A and $A_l \succ A$ for all $l = 1, \ldots, k$ yields us to $k \leqslant r$. Thus we obtain that $r = k$.

Moreover it must be $B_i \sim A_i$ for all $i = 1, \ldots, k$ because if $B_i \sim A_j$ and $i > j$ we obtain that

$$B_1 \succ B_2 \succ \ldots \succ B_i \succ A_{j+1} \succ \ldots \succ A_k$$

is a chain for A and it has more than k elements because $i > j$ which contradicts the fact that $A_1 \succ \ldots \succ A_k$ is a maximum chain for A. The same argument in case that $i < j$ concludes the proof. □

Definition 2.8. *A chain of strict preferences $A_1 \succ \ldots \succ A_k$ is maximum in $\mathcal{P}'(X)$ with respect to the complete preorder $\succcurlyeq$ if any subset in $\mathcal{P}'(X)$ is indifferent by $\succcurlyeq$ to any of the subsets in the chain.*

Then we obtain that every set of alternatives in $\mathcal{P}'(X)$ with a maximum chain with k elements must be indifferent to the subset that appears in the place $k+1$ in a maximum chain in $\mathcal{P}'(X)$ as the next corollary states.

Corollary 2.1. *If $A \in \mathcal{P}'(X)$ has a maximum chain with k subsets of $\mathcal{P}'(X)$ and*

$$A_1 \succ A_2 \succ \ldots \succ A_p$$

is a maximum chain in $\mathcal{P}'(X)$ with respect to the complete preorder $\succcurlyeq$, then $A \sim A_{k+1}$.

Let us now introduce two new properties for a ranking of subsets defined on X.

Property P1. If A and B are subsets of the finite set X and there exists an element $a \in A$ such that for every $b \in B$ is $\{a\} \succ \{b\}$, then $A \succ B$.

We denote by $[X_i]$ the set of subsets of X with best elements with respect to the complete preorder R in X_i.

Property P2. For any $i = 1, \ldots, n$ we can find subindexes $i_1, \ldots, i_p$ such that $i = i_1 < i_2 < \ldots < i_p \leqslant n$ and for any maximum chain in $[X_i]$

$$A_1 \succ A_2 \succ \ldots \succ A_p$$

we have that $A \cap X_{i_k} \neq \varnothing$ for all $A \in [X_i]$ verifying that $A \sim A_k$.

Remark 2.2. We obtain from P2 that in a maximum chain in $[X_i]$, it must be $A_1 \sim X_i$ because, in other case, $X_i \sim A_k$ with $k > 1$, thus X_i should verify that $X_i \cap X_{i_k} \neq \varnothing$ with $i_k \neq i$, which is impossible. ◁

Property P1 is fairly intuitive. If a single subset of a set A is strictly preferred to all the single subsets of a set B, the set A is strictly preferred to the set B.

Property P2 requests that when a set A with best elements in a class i verifies that X_i (a set with all its elements in the class i) is strictly preferred to it, A must contain elements from another class worse than class i. In fact, all the subsets in $[X_i]$ with a maximum chain of order 2 in $[X_i]$ must contain elements in X_{i_2} for some $i_2 > i_1 = i$. Those subsets in $[X_i]$ with a maximum chain of order 3 must contain elements in X_{i_3} for some $i_3 > i_2 > i_1 = i$, and so on those subsets in $[X_i]$ with a maximum chain in $[X_i]$ of order k must contain elements in some X_{i_k} with $n \geqslant i_k > \ldots > i_3 > i_2 > i_1 = i$. We have that $A \cap X_i \neq \varnothing$ for all $A \in [X_i]$. Then if $A \in [X_i]$ has a chain with no elements in $[X_i]$ (there are no elements in $[X_i]$ strictly preferred to A), that is $A \sim A_1$, then $A \cap X_i \neq \varnothing$, and A can also have elements in other X_j's with $i < j$ or not.

Now we consider the ranking of subsets associated to a complete preorder R and a choice function $\mathcal{C}$ defined on a set X given in Definition 2.6 and denoted by $\succcurlyeq_{\mathcal{C}}$.

It is obvious that $\succcurlyeq_{\mathcal{C}}$ verifies P1.

Indeed, if there exists $a \in A$ such that $\{a\} \succ_{\mathcal{C}} \{b\}\ \forall b \in B \Rightarrow \max(A) P \max(B) \Rightarrow A \succ_{\mathcal{C}} B$.

In Example 2.7 above we can observe that every subset with best element a is strictly preferred to those subsets with best elements b or b', which at the same time are strictly preferred to those subsets with best element c, that are strictly preferred to the subset $\{d\}$ (the only one with the best element d).

Let us now prove that $\succcurlyeq_{\mathcal{C}}$ also satisfies P2.

Let us consider any $i = 1, \ldots, n$ and let

$$X_i \sim_{\mathcal{C}} A_1 \succ_{\mathcal{C}} A_2 \succ_{\mathcal{C}} \ldots \succ_{\mathcal{C}} A_p$$

be a maximum chain in $[X_i]$ with respect to $\succcurlyeq_{\mathcal{C}}$.

As we know that $\max(A_j) \in X_i$ for all $A_j \in [X_i]$ we obtain

$$\max(\mathcal{C}(A_1)) P \max(\mathcal{C}(A_2)) P \ldots P \max(\mathcal{C}(A_p)).$$

This fact implies the existence of subindexes $i = i_1 < i_2 < \ldots < i_p \leqslant n$ such that $\max(\mathcal{C}(A_j)) \in X_{i_j}$.

We now prove that these subindexes $i_1, \ldots, i_p$ verify property P2.

Let us take a subset $A \in [X_i]$ with a maximum chain in $[X_i]$ with k elements. We have (Corollary 2.1)

$$A_1 \succ_{\mathcal{C}} A_2 \succ_{\mathcal{C}} \ldots \succ_{\mathcal{C}} A_k \succ_{\mathcal{C}} A$$

and

$$A \sim_{\mathcal{C}} A_{k+1}.$$

Then the definition of $\succcurlyeq_{\mathcal{C}}$ leads us to

$$\max(\mathcal{C}(A)) I \max(\mathcal{C}(A_{k+1}))$$

and

$$\max(\mathcal{C}(A_{k+1})) \in X_{i_{k+1}}$$

which let us conclude that

$$\max(\mathcal{C}(A)) \in X_{i_{k+1}} \text{ and thus } A \cap X_{i_{k+1}} \neq \varnothing.$$

Following the Example 2.7 above we can illustrate this theoretical part. There we have that $X = X_1 \cup X_2 \cup X_3 \cup X_4 = \{a\} \cup \{b, b'\} \cup \{c\} \cup \{d\}$ and

$$[X_1] = \{\{a\}, \{a,b\}, \{a,b'\}, \{a,c\}, \{a,d\}, \{a,b,b'\}, \{a,b,c\}, \{a,b,d\}, \{a,b',c\}, \{a,b',d\},$$

$$\{a,c,d\}, \{a,b,b',c\}, \{a,b,b',d\}, \{a,b,c,d\}, \{a,b',c,d\}, \{a,b,b',c,d\}\}$$

$$[X_2] = \{\{b\}, \{b'\}, \{b,b'\}, \{b,c\}, \{b,d\}, \{b',c\}, \{b',d\}, \{b,b',c\}, \{b,b',d\}, \{b,c,d\},$$

$$\{b',c,d\}, \{b,b',c,d\}\}$$

$$[X_3] = \{\{c\}, \{c,d\}\}$$

$$[X_4] = \{\{d\}\}.$$

Thus, in $[X_1]$ we have a maximum chain

$$\{a\} \succ_{\mathcal{C}} \{a,b,c\} \succ_{\mathcal{C}} \{a,b,c,d\}$$

among other possibilities.

In this chain $i_1 = 1$. Now, if we take all the subsets in $[X_1]$ indifferent to $\{a,b,c\}$ we can observe that all of them have elements in X_2, thus we can choose $i_2 = 2 > 1$, and in X_3, thus we can also select $i_2 = 3 > 1$. Now, considering all the subsets in $[X_1]$ indifferent to $\{a,b,c,d\}$ we observe that they all have elements in X_4, but not in X_3, hence we can only choose $i_3 = 4 > i_2 = 2$ or $3 > i_1 = 1$. These subindexes verify P2 for $[X_1]$.

In $[X_2]$

$$\{b\} \succ_{\mathcal{C}} \{b,c,d\} \succ_{\mathcal{C}} \{b,b',d\}$$

is a maximum chain among other possibilities.

In this chain we have that $i_1 = 2$. Now we consider all the subsets in $[X_2]$ that are indifferent to $\{b,c,d\}$ and observe that they all have elements in X_3 and X_4, thus we can choose $i_2 = 3$ or $i_2 = 4$. Nevertheless we know that the maximum chain in $[X_2]$ has three elements therefore if we choose $i_2 = 4$ we have not any possibility for i_3. Thus $i_2 = 3$ and then $i_3 = 4$.

In $[X_3]$ there is a maximum chain with an only element, for Example X_3, thus we only have $i_1 = 3$ for $[X_3]$ and the same for $[X_4]$ thus $i_1 = 4$ in this last class of subsets.

2.4.1 A characterization of the ranking of subsets denoted by $\succcurlyeq_{\mathcal{C}}$

In this subsection we solve the problem under inspection. Our primitive concepts are the finite set X and a ranking of subsets of X, namely $\succcurlyeq$, which is a complete preorder on $\mathcal{P}^*(X)$. The question now is if we can define a choice function $\mathcal{C}$ over the set X in such a way that the ranking of subsets associated with it (and the preorder R trivially induced by $\succcurlyeq$) coincides with the given ranking. We give a complete answer to this question in terms of suitable properties.

We have already mentioned above that not any complete preorder on $\mathcal{P}^*(X)$ can be represented as a complete preorder associated with a choice function (Example 2.8). We also mentioned some examples of the real world where this kind of behavior takes place. The next example shows that we can also find cases in which different choice functions define the same ranking, so that even if we take into account different advisers, the final choice can be the same.

Example 2.9. Let $X = \{a, b\}$ and $\succcurlyeq$ a ranking of subsets of X defined as

$$\{a\} \sim \{a, b\} \succ \{b\}.$$

If we consider $\mathcal{C}(\{a, b\}) = \{a\}$, then $\succcurlyeq_{\mathcal{C}} = \succcurlyeq$.

But the choice function $\mathcal{C}'(\{a, b\}) = \{a, b\}$ also verifies that $\succcurlyeq_{\mathcal{C}'} = \succcurlyeq$. ◇

Therefore in cases where it is possible to find a choice function $\mathcal{C}$ in such a way that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$, this choice function may not be unique. In order to propose a canonical expression for a solution to our problem we first prove the next lemma.

Lemma 2.1. *Let X be a finite set of objects and $\succcurlyeq$ a complete preorder on $\mathcal{P}^*(X)$. If we can define different choice functions $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ on $\mathcal{P}^*(X)$ such that $\succcurlyeq = \succcurlyeq_{\mathcal{C}_i}$ for all $i = 1, \ldots, r$, then the choice function given by $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$[1] also verifies that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$.*

Proof. If $A, B \subseteq X$ and $A \succcurlyeq_{\mathcal{C}_i} B$ for all $i = 1, \ldots, r$, we have:

a) $\max(A) P \max(B)$, or

b) $\max(A) I \max(B)$ and $\max(\mathcal{C}_i(A)) R \max(\mathcal{C}_i(B))$, for all $i = 1, \ldots, r$.

If case *a*) happens for all $A, B \subseteq X$ such that $A \succcurlyeq_{\mathcal{C}_i} B$, then the definition of $\mathcal{C}$ is irrelevant i.e., for any choice function $\mathcal{C}$ we obtain that $A \succcurlyeq_{\mathcal{C}} B$.

Then we have to deal with the case *b*) in which it happens that there exist $A, B \subseteq X$ such that $A \succcurlyeq_{\mathcal{C}_i} B$ for all $i = 1, \ldots, r$ and such that $\max(A) I \max(B)$. Therefore $\max(\mathcal{C}_i(A)) R \max(\mathcal{C}_i(B))$ for all $i \in \{1, \ldots, r\}$.

We have

$$\mathcal{C}(A) = \mathcal{C}_1(A) \cup \ldots \cup \mathcal{C}_r(A) \text{ and } \mathcal{C}(B) = \mathcal{C}_1(B) \cup \ldots \cup \mathcal{C}_r(B),$$

thus

$$\max(\mathcal{C}(A)) = \max(\{\max(\mathcal{C}_1(A)), \ldots, \max(\mathcal{C}_r(A))\})$$

and

$$\max(\mathcal{C}(B)) = \max(\{\max(\mathcal{C}_1(B)), \ldots, \max(\mathcal{C}_r(B))\}).$$

As we have that $\max(\mathcal{C}_i(A)) R \max(\mathcal{C}_i(B))$ for all $i = 1, \ldots, r$, we conclude that

$$\max(\mathcal{C}(A)) R \max(\mathcal{C}(B))$$

[1] $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_r)(S) = \mathcal{C}_1(S) \cup \mathcal{C}_2(S) \cup \ldots \cup \mathcal{C}_r(S)$, $\forall S \subseteq X$.

which implies that $A \succcurlyeq_{\mathcal{C}} B$. □

This lemma let us deal with a **canonical choice function** for a ranking of subsets $\succcurlyeq$. This canonical choice function is an application

$$\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X) \text{ such that } \succcurlyeq_{\mathcal{C}} = \succcurlyeq$$

maximum in the sense that if there exists another choice function

$$\mathcal{C}' : \mathcal{P}^*(X) \to \mathcal{P}^*(X) \text{ verifying } \succcurlyeq_{\mathcal{C}'} = \succcurlyeq$$

then

$$\mathcal{C}'(S) \subseteq \mathcal{C}(S) \text{ for all } S \subseteq X.$$

Now we suppose that we have a complete preorder $\succcurlyeq$ over $\mathcal{P}^*(X)$ such that there exists a choice function $\mathcal{C}$ on X such that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$. We look for a canonical expression for this choice function.

Recall that

$$X = X_1 \cup \ldots \cup X_n$$

in such a way that all the elements in X_i are indifferent by R and at the same time we have $x_1 P x_2 P \cdots P x_n$ for all $x_i \in X_i$. We associate with each complete preorder $\succcurlyeq$ on $\mathcal{P}^*(X)$ that satisfies property P2 a choice function according to the following recursive definition for the subindexes that P2 assures.

Let us consider $[X_i]$ for any $i = 1, \ldots, n$ and let

$$A_1 \succ A_2 \succ \ldots \succ A_p$$

be a maximum chain in $[X_i]$ with respect to $\succcurlyeq$, so that P2 assures that we can find subindexes $i = i_1 < i_2 < \ldots < i_p \leqslant n$ such that $A_k \cap X_{i_k} \neq \varnothing$.

We take the first subindex $i_1(i) = i$, so that all the subsets S_1 in $[X_i]$ indifferent to $X_i \sim A_1$ verify $S_1 \cap X_i \neq \varnothing$. We denote this subindex by $i_1(i)$ because we want to remark that it depends on the domain $[X_i]$ from X where we are.

Let us now consider all the subsets S_2 in $[X_i]$ indifferent to A_2. We denote $i_2(i)$ the minimum subindex $l \in \{i+1, \ldots, n\}$ that verifies that $S_2 \cap X_l \neq \varnothing$ for all $S_2 \sim A_2$.

Now we denote $i_3(i)$ the minimum subindex $l \in \{i_2(i)+1, \ldots, n\}$ that verifies that $S_3 \cap X_l \neq \varnothing$ for all $S_3 \sim A_3$.

Recursively we obtain $i = i_1(i) < i_2(i) < \ldots < i_p(i) \leqslant n$ such that $S_k \cap X_{i_k(i)} \neq \varnothing$ for all $S_k \in [X_i]$ indifferent to A_k, and with all the $i_r(i), r = 1, \ldots, p$ as minimum as possible verifying the necessary condition of non empty intersection.

Adding up the different maximum chains obtained by this way for every $[X_i]$ we obtain a maximum chain in X with respect to $\succcurlyeq$

$$X_1 \sim A_{i_1(1)} \succ \ldots \succ A_{i_p(1)} \succ X_2 \sim A_{i_1(2)} \succ \ldots \succ A_{i_p(2)} \succ \ldots \succ X_n \sim A_{i_1(n)} \tag{2.1}$$

Then every subset of X must be indifferent to one of the subsets in this chain. The next expression gives us the canonical choice function associated to $\succcurlyeq$.

Definition 2.9. *Let S be a subset of X with a maximum chain with respect to $\succcurlyeq$ as given by equation (2.1). Let us suppose that $S \sim A_{i_r(k)}$, that is, S is a subset with its best elements in X_k and indifferent to the subset in place r in the maximum chain in X_k. Then*

$$\mathcal{C}(S) = S \cap (X_{i_r(k)} \cup X_{i_r(k)+1} \cup \ldots \cup X_n).$$

Remark 2.3. The definition of $\mathcal{C}$ can be given in many different forms. In fact, when $S \sim A_{i_r(k)}$ any definition of $\mathcal{C}(S)$ such that

$$\mathcal{C}(S) \cap X_{i_r(k)} \neq \varnothing \text{ and } S \cap (X_1 \cup \ldots \cup X_{i_r(k)-1}) = \varnothing$$

verifies that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$. We choose the one given in Definition 2.9 by virtue of Lemma 2.1.$\lhd$

The next theorem is our main result in this section and it states the characterization of the complete preorders on $\mathcal{P}^*(X)$ that can be represented as a preorder associated with a choice function over $\mathcal{P}^*(X)$ through properties P1 and P2.

Theorem 2.3. *Let X be a finite set of objects and $\succcurlyeq$ a complete preorder on $\mathcal{P}^*(X)$. There exists a choice function $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$ such that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$ if and only if $\succcurlyeq$ verifies properties P1 and P2.*

Proof. We have already mentioned that the complete preorder $\succcurlyeq_{\mathcal{C}}$ given in Definition 2.6 verifies properties P1 and P2.

Now, let us suppose that we have a complete preorder $\succcurlyeq$ on $\mathcal{P}^*(X)$ that verifies properties P1 and P2 thus we can define the canonical choice function associated with $\succcurlyeq$, $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$, as the one given in Definition (2.9).

Finally we check that $\succcurlyeq = \succcurlyeq_{\mathcal{C}}$.

In the first place, we suppose that $A \succcurlyeq B$, and we prove that $A \succcurlyeq_{\mathcal{C}} B$.

If $A \succcurlyeq B$ there must exist $a \in A$ such that $\{a\} \succcurlyeq \{b\}$ for all $b \in B$, because in other case, for all $a \in A$ there would exist $b \in B$ verifying that $\{b\} \succ \{a\}$. Then there would exist $b' \in B$ and $a' \in A$ such that $\{b'\} \succ \{a'\} \succcurlyeq \{a\}$ for all $a \in A$. Applying now that $\succcurlyeq$ verifies property P1 we conclude that $B \succ A$.

Then, if $\{a\} \succ \{b\}$ for all $b \in B$ the definition of $\succcurlyeq_{\mathcal{C}}$ yields $A \succ_{\mathcal{C}} B$.

Therefore, let us suppose that there exists $b \in B$ such that $\{a\} \sim \{b\} \Leftrightarrow aIb$. This means that $\max(A) I \max(B)$.

Case 1. If $a \in X_i$ and $A \sim X_i$, we would have that $\mathcal{C}(A) = A$, thus

$$\max(\mathcal{C}(A)) I \max(A) I \max(B) R \max(\mathcal{C}(B))$$

and this takes us to

$$A \succcurlyeq_{\mathcal{C}} B.$$

Case 2. $a \in X_i$ and $X_i \succ A$. As we know that $A \succcurlyeq B$ it is also true that $X_i \succ B$.

Now because of property P2 we have subindexes $i_1, \ldots, i_p$ such that $i = i_1 < i_2 < \ldots < i_p \leqslant n$ and in such way that if

$$A_1 \succ \ldots \succ A_k$$

is a maximum chain for $A \in [X_i]$, it is $A \cap X_{i_{k+1}} \neq \varnothing$.

Now two possibilities arise:

i) If both A and B have a maximum chain with k elements we would obtain $A \sim A_{k+1} \sim B$, thus $\mathcal{C}(A) = A \cap (X_{i_{k+1}} \cup \ldots \cup X_n)$ and $\mathcal{C}(B) = B \cap (X_{i_{k+1}} \cup \ldots \cup X_n)$, which implies that $\max(\mathcal{C}(A)) I \max(\mathcal{C}(B))$ and therefore $A \sim_{\mathcal{C}} B$.

ii) If A and B have different maximum chains, the one for A has fewer subsets from $[X_i]$ than the one for B because $A \succcurlyeq B$. Thus, if the maximum chain for A is of order k ($A \sim A_{k+1}$), and the maximum chain for B is of order l ($B \sim A_{l+1}$), then $\mathcal{C}(A) = A \cap (X_{i_{k+1}} \cup \ldots \cup X_n)$ and $\mathcal{C}(B) = B \cap (X_{i_{l+1}} \cup \ldots \cup X_n)$ where $i_{k+1} < i_{l+1}$ which implies $\max(\mathcal{C}(A)) P \max(\mathcal{C}(B))$ and then $A \succ_{\mathcal{C}} B$.

To finish the proof we need to check that if $A \succcurlyeq_{\mathcal{C}} B$ then $A \succcurlyeq B$, which is equivalent to prove that

$$A \succ B \Rightarrow A \succ_{\mathcal{C}} B.$$

In case that $A \succ B$, we have that A and B have different maximum chains and we obtain as in case 2 *ii*) above that $A \succ_{\mathcal{C}} B$, which concludes the proof. $\square$

Remark 2.4. We want to remark that although we have used in the proof of Theorem 2.3 the definition of $\mathcal{C}$ given by Definition 2.9 the precise construction of the subindexes is not relevant in the proof. The only relevant issue is that they form a nondecreasing sequence. For this reason the theorem can be proved from any set of subindexes verifying proposition P2.

In this connection recall that Example 2.9 has illustrated a related fact: different advisers' choice may produce the same ranking when the decision maker's ordering remains unchanged. Example 2.10 below emphasizes this feature of the model. $\triangleleft$

We also want to remark that in Example 2.8 property P2 is not verified. In fact the relation $\{a,b\} \succ \{a\}$ would imply that the subset $\{a\}$ should contain elements of at least two different indifference classes by R, which is obviously impossible.

We finish recalling the ranking given in the Example 2.7. Now we investigate if we can "recover" the adviser's choice through the explicit construction given by the proof of Theorem 2.3.

Example 2.10. Recall that we have $X = \{a\} \cup \{b,b'\} \cup \{c\} \cup \{d\} = X_1 \cup X_2 \cup X_3 \cup X_4$.

Now, let us suppose that we observe the next ranking on $\mathcal{P}^*(X)$:

$$\{a\} \sim \{a,b\} \sim \{a,b'\} \sim \{a,c\} \sim \{a,d\} \sim \{a,b,b'\} \sim \{a,b,d\} \sim \{a,b,b',c\} \succ$$

$$\succ \{a,b,c\} \sim \{a,b',c\} \sim \{a,b',c,d\} \succ$$

$$\succ \{a,b',d\} \sim \{a,c,d\} \sim \{a,b,b',d\} \sim \{a,b,c,d\} \sim \{a,b,b',c,d\} \succ$$

$$\succ \{b\} \sim \{b'\} \sim \{b,b'\} \sim \{b,c\} \sim \{b,d\} \sim \{b',c\} \sim \{b',d\} \sim \{b,b',c\} \succ$$

$$\succ \{b,c,d\} \sim \{b',c,d\} \sim \{b,b',c,d\} \succ$$

$$\succ \{b,b',d\} \succ$$

$$\succ \{c\} \sim \{c,d\} \succ \{d\}.$$

Because it is the ranking $\succcurlyeq_{\mathcal{C}}$ associated with $\mathcal{C}$ in Example 2.7 it is a complete preorder that agrees with properties P1 and P2.

Let us now construct the choice function $\mathcal{C}'$ as in Theorem 2.3.

In $[X_1]$ we choose the maximum chain

$$\{a\} \succ \{a,b,c\} \succ \{a,b,c,d\}.$$

In $[X_2]$

$$\{b\} \succ \{b,c,d\} \succ \{b,b',d\}.$$

In $[X_3]$ we have a maximum chain with an only element $\{c\}$, and the same happens in $[X_4]$ where we have the maximum chain $\{d\}$.

Adding up all these chains we have the maximum chain in X

$$\{a\} \succ \{a,b,c\} \succ \{a,b,c,d\} \succ \{b\} \succ \{b,c,d\} \succ \{b,b',d\} \succ \{c\} \succ \{d\}.$$

For those subsets $S \subseteq X$ that are indifferent to $\{a\}$ we define $\mathcal{C}'(S) = S \cap X = S$. We have $i_1(1) = 1$.

For those subsets S that are indifferent to $\{a,b,c\}$ we can observe that the minimum subindex j verifying that $j > 1$ and $S \cap X_j \neq \varnothing$ is $j = 2$. Then we take $i_2(1) = 2$ and define $C'(S) = S \cap (X_2 \cup X_3 \cup X_4)$. Thus

$$\mathcal{C}'(\{a,b,c\}) = \{b,c\},\ \mathcal{C}'(\{a,b',c\}) = \{b',c\} \text{ and } \mathcal{C}'(\{a,b',c,d\}) = \{b',c,d\}$$

definitions that are different from the definitions of the initial $\mathcal{C}$ in the Example 2.7.

Let us consider now those subsets in $[X_1]$ indifferent to $\{a,b,c,d\}$. All of them have elements in X_4, but not in X_3, thus $i_3(1) = 4$ and $\mathcal{C}'(S) = S \cap X_4$. Then we have

$$\mathcal{C}'(\{a,b',d\}) = \{d\},\ \mathcal{C}'(\{a,c,d\}) = \{d\},\ \mathcal{C}'(\{a,b,b',d\}) = \{d\},\ \mathcal{C}'(\{a,b,c,d\}) = \{d\}$$

and $\mathcal{C}'(\{a,b,b',c,d\}) = \{d\}$.

Let us continue now with the subsets in $[X_2]$.

For those subsets S in this class indifferent to $\{b\}$ we define $\mathcal{C}'(S) = S$ (we have $i_1(2) = 2$).

Observing the sets S in $[X_2]$ that are indifferent to $\{b,c,d\}$ we have that all of them have elements in X_3 and in X_4 (3,4>2), and then $i_2(2) = 3$ or $i_2(2) = 4$. As we have to consider the minimum of these subindexes we take $i_2(2) = 3$, and then $i_3(2) = 4$.

Therefore we have

$$\mathcal{C}'(\{b,c,d\}) = \{c,d\}, \;\; \mathcal{C}'(\{b',c,d\}) = \{c,d\}, \;\; \mathcal{C}'(\{b,b',c,d\} = \{c,d\}$$

and

$$\mathcal{C}'(\{b,b',d\}) = \{d\}.$$

In $[X_3]$ we have an only subindex $i_1(3) = 3$ and then

$$\mathcal{C}'(\{c,d\}) = \{c,d\}.$$

We can check that the definition of $\mathcal{C}'$ does not coincide with the definition of $\mathcal{C}$ in the Example 2.7. $\diamond$

Remark 2.5. The ranking in the example above does not verify P1 if we have, for example $\{b,c\} \succ \{a,b,c\}$.

It does not verify P2 if we change the relation for, for instance, the subset $\{a,c\}$ and

$$\{a\} \succ \{a,c\} \succ \{a,b,c\} \succ \{a,b,c,d\}$$

is a maximum chain for $\succcurlyeq$ in $[X_1]$, and the other relations being the same as above.

In this case we can not find the subindexes $i_1 = 1, \ldots, i_p \leqslant 4$ verifying property P2 for $[X_1]$. $\triangleleft$

2.5 Conclusions and future research

In this chapter we have dealt with rankings of subsets of a set of alternatives in different times by using the indirect-utility criterion. In section 2.3 we have considered the possibility of having different sets of alternatives and different complete preorders defined on them. We have characterized the ranking of subsets in different times comparing sequences of subsets, one of every set of alternatives, using the indirect-utility criterion in each coordinate and applying it in a lexicographical way.

In section 2.4 we have considered a two-period selection, but now with an only set of alternatives. We rank the subsets applying the indirect-utility criterion too, but now, in those cases where this produces ties we take into account an adviser that selects, for any subset of alternatives, a smaller subset of them which we represent with a choice function. Then the decision-maker applies the indirect-utility criterion to these smaller subsets. We characterize this ranking of subsets via two properties.

In the case we use an adviser we want to consider in the future the case in which the choice function that represents it verifies some rationality properties. We intend to focus our interest on advisers that behave in a "sensible" way.

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Chapter 3

Rational choice by two sequential criteria

Contents

3.1 Introduction **66**

3.2 Definitions and properties of rationality **69**

3.3 Rationality properties of a compound choice function **72**

3.3.1 Choice functions on domains containing all the finite and nonempty subsets of the set of alternatives 75

3.3.2 Choice functions defined on arbitrary domains 82

3.4 Choice functions rational by two sequential criteria **92**

3.5 Conclusions and future research **103**

3.6 Bibliography **105**

3.1 Introduction

In many decision problems the decision-maker's preference is represented by a choice function rather than a binary relation. Moreover, choice theory has a very important factor of applicability, and its results underlie some economic, psychological, sociological,... models, which awards to its study an extra appeal. The description of individual choices by means of a binary relation is very common. The choice functions consisting of the selection, for every set of alternatives, of those elements that are best preferred outcomes for a binary relation is considered a "reasonable" choice. The interesting question is the converse: given a choice function, is there a single underlying binary relation from which our choice function is derived as we have detailed above? If the answer is positive we say that the choice function is rational or that there exists a binary relation that rationalizes it. Different methods for axiomatizing "rational choice" have been developed on the basis of choice functions generated by binary relations and optimization criteria that are referred to as "revealed preferences". In Suzumura (1983) we can find a survey of the characterization of this classical concept of rational choice functions that has been extensively studied for different authors among we can cite Arrow (1959), Richter (1966), Wilson (1970), Sen (1971),...

Non-classical choice mechanisms have also been considered for other authors such as Aizerman and Malishevski (1981). In this line of research, Nehring (1996) gives a first contribution to the problem of the existence of maximal elements for non-binary choice functions. Other results in the same line are in Tian and Zhou (1995), Rodríguez Palmero and García Lapresta (2002) and Alcantud (2002, 2006). Some extensions of the classical notion of rationality are Gaertner and Xu (2004) who give a concept of rationality based on the classical one but taking into account that some available alternatives can have a "degree of availability" because the decision-maker can consider the choice procedure unacceptable or some of these alternatives are forbidden for any law; or Bossert and Suzumura (2007) who "introduce a model of choice where external norms are taken into consideration" and such model includes the traditional model as a particular case.

The reasons of the interest in choice theory are diverse. We can cite, among others, the next ones.

- Many problems of decision theory, applied mathematics,... are based on the choice of the "best" options in some sense from each given set of possibilities.

- Many economic and social models examine questions of individual choice. Also in psychological phenomena, the idea of describing an individual behavior in terms of choosing the best options is a very attractive topic.

- Political problems also deal with models of this kind for formalizing different concepts of individual voting choice and when the option that maximizes the individual utility also maximizes collective utility.

As expected, the study of rationality of a decision-maker is very extensive in the literature. Given the choice of an agent, the literature of "revealed preference" tries to explain this behavior.

In this regard we can find different results about the possible rationality of a choice function depending on the satisfaction of some suitable properties. The meaning of "rationality" has had different interpretations and we deal with the one that identifies a rational choice function with the optimization of a binary relation irrespective of the properties that this binary relation verifies. Nevertheless the literature on rational choice functions also deals vastly with questions about which properties the binary relation that rationalizes a rational choice function verifies: acyclicity, transitivity, quasi-transitivity,... We also include some results in this line.

Moreover the possible rationality of a choice function does not only depend upon the properties that it verifies, but also upon the domain it is defined on. The problem of choice implies the definition of the set of options and its relevant subsets. The possible presence of restrictions over the subsets is essential in the formal model. Sen (1971), Bandyopadhyay and Sengupta (1991) among others deal with choice functions defined on domains that contain all the finite and nonempty subsets of a universal set of alternatives. Nevertheless Sen (1971) remarks that "while it is not required that the domain includes all infinite sets as well, nothing would of course be affected in the results and the proofs even if all infinite sets are included in the domain", and that "it is not really necessary that even all finite sets be included in the domain. All the results and proofs would continue to hold even if the domain includes all pairs and triples but not all finite sets". In chapter 2 of Suzumura (1983) we can find a survey of this topic including the case with a domain consisting on an arbitrary family of nonempty subsets of an arbitrary nonempty universal set of alternatives. More recently Bossert et al (2006) "develop new necessary conditions for choice functions defined on arbitrary domains to be rationalized by binary relations that are quasi-transitive or acyclic", and give a new sufficient condition for a choice function to be rationalized by an acyclic binary relation.

Another aspect to bear in mind in the problems of decision is the possibility of having different criteria for making a choice. If that is the case, we can give priority to some of them over the others and apply them in a sequential way, which we call "sequential choice". In case that we have two criteria applied in a sequential we say that the choice is made by two sequential criteria. This means that we compose two choice functions in an established order (Aizerman and Aleskerov (1995) also consider this kind of choice behavior and name this operation "superposition" of choice functions). So despite we follow the classical rationality requirements we admit a mechanism for making a decision that is rather natural and logical, but that can produce a choice that is not rational in a classical sense: the composition of choice functions. So our DMs consider rational not only a choice function that is derived from a single binary relation, but also the sequential application of such type of choices. Examples of elections of this type are rather frequent: selecting people applying criteria that reduce successively the set of alternatives, selecting places or hotels for holidays (for example, we eliminate first those that are too far, then those that are too expensive, and so on),...

Some authors have also studied similar aspects of choice theory: Kalai et al. (2002) study the rationality of a choice function by multiple binary relations when the choice is an only element in the set of alternatives and applying all the relations simultaneously at each set of alternatives. Houy (2007) includes the study of the order of the criteria, if it affects or not to the final choice, and different procedures of choice for such lexicographic applications of multiple criteria have been considered in Houy and Tademuna (2007). We follow the line initialized by Manzini and Mariotti (2007) who consider the sequential rationality of a choice function by the application of different binary relations in a fixed order, and specifically the case of two relations, but they also limit themselves to the case of single-valued choice functions. We think that it is interesting to analyze the problem in terms of set-valued choice functions and this is the case we consider in this chapter.

We study how the compound function of two choice functions behaves, that is which of the properties verified by the two initial choice functions are also verified by the compound function. Aizerman and Aleskerov (1995) make this study for some properties and choice functions always defined over domains which contain all the finite and nonempty subsets of a set of alternatives. We add to this study with the analysis of some other properties in domains of the same kind, and with some properties of choice functions defined on arbitrary domains. From the results of this study and the different rationality characterization theorems we obtain some corollaries establishing the rationality of a choice function that is obtained as the composition of other two functions that are rational in a certain sense.

For those problems in which the domain contains all the finite and nonempty subsets of the sets of alernatives, we consider a choice function that does not verify the properties of rationality that the different rationality theorems demand. We wonder when we can find two rational choice functions (verifying the demanded properties) such that the choice made by the sequential application of these two functions coincides with the choice made by the decision-maker and that we had observed. When the answer is positive we say that the choice function is "rational by two sequential criteria".

We give a complete characterization of a choice function that is rational by two sequential criteria in terms of two testable necessary and sufficient conditions.

The rest of the chapter is organized al follows. In section 3.2 we set our notation and recall different properties of rationality that afterwards will appear in the rationality theorems. Section 3.3 is devoted to the analysis of the rationality properties that are preserved under the operation of composition. In section 3.4 we characterize the class of choice functions that are rational by two sequential criteria, and we conclude with some final remarks in section 3.5.

3.2 Definitions and properties of rationality

In this section we set the notation and introduce different properties of rationality for a choice function that are usual in the literature about this topic.

Definition 3.1. *Let X be a set and $\mathcal{D}$ a nonempty domain of nonempty subsets of X. A choice function is an application $\mathcal{C} : \mathcal{D} \to \mathcal{P}(X)$ such that $\mathcal{C}(S) \subseteq S$ and $\mathcal{C}(S) \neq \varnothing$ for all $S \in \mathcal{D}$.*

Let $\mathcal{C}$ be a choice function on a domain $\mathcal{D}$.

(i) $\mathcal{C}$ verifies the *Chernoff condition* if for any $S, T \in \mathcal{D}$ such that $S \subseteq T$ we have

$$\mathcal{C}(T) \cap S \subseteq \mathcal{C}(S)$$

which is equivalent to

$$\forall S, T \in \mathcal{D},\ \mathcal{C}(S \cup T) \subseteq \mathcal{C}(S) \cup \mathcal{C}(T)$$

whenever $S, T \in \mathcal{D} \Rightarrow S \cup T \in \mathcal{D}$ (Sertel and der Bellen (1979)).

(ii) $\mathcal{C}$ satisfies the *Arrow's axiom*, if for any $S, T \in \mathcal{D}$ such that $S \subseteq T$ then

$$\mathcal{C}(T) \cap S = \mathcal{C}(S)$$

Remark 3.1. It is obvious that Arrow's axiom is stronger than the Chernoff condition. $\triangleleft$

(iii) $\mathcal{C}$ satisfies the *Concordance condition* if for all $S, T \in \mathcal{D}$ such that $S \cup T \in \mathcal{D}$

$$\mathcal{C}(S \cup T) \supseteq \mathcal{C}(S) \cap \mathcal{C}(T).$$

The next proposition is obtained directly from these definitions.

Proposition 3.1. *A choice function $\mathcal{C}$ satisfies the Chernoff condition and the Concordance property if and only if it satisfies:*
For all $S, T \in \mathcal{D}$ such that $S \cup T \in \mathcal{D}$,

$$\mathcal{C}(S) \cap \mathcal{C}(T) \subseteq \mathcal{C}(S \cup T) \subseteq \mathcal{C}(S) \cup \mathcal{C}(T).$$

(iv) $\mathcal{C}$ satisfies the *property* γ (Sen 1971) if for any collection of subsets $\{M_i\}_{i \in I}$ in the domain $\mathcal{D}$ such that $\cup_{i \in I} M_i \in \mathcal{D}$, it is true that

$$x \in \mathcal{C}(M_i)\ \forall i \in I \Rightarrow x \in \mathcal{C}\left(\cup_{i \in I} M_i\right).$$

This property generalizes the Concordance property. Moreover, if we consider a domain $\mathcal{D}$ that contains all the subsets with two elements of the set of alternatives and apply this property to them, we obtain a weaker property also named the *Direct Condorcet Property* in the literature and that we call the "binariness property".

(v) $\mathcal{C}$ satisfies the *Binariness property* if for any $S \in \mathcal{D}$ and $x \in S$ we have:

$$x \in \mathcal{C}(\{x, y\}), \forall y \in S \Rightarrow x \in \mathcal{C}(S)$$

where $\mathcal{D}$ must contain all the subsets of the set of alternatives with two elements.

Remark 3.2. If a choice function $\mathcal{C}$ verifies Arrow's axiom, then $\mathcal{C}(\bigcup_{i \in I} M_i) \cap M_i = \mathcal{C}(M_i)$, and from here we obtain that $\mathcal{C}$ verifies property γ and therefore the binariness property[1]. $\triangleleft$

(vi) $\mathcal{C}$ satisfies the property of *Independence of Irrelevant Alternatives (IIA)*[2] if for any $S, T \in \mathcal{D}$ such that $S \subseteq T$ we have

[1]The domain must be closed under the union of sets.
[2]In Aizerman and Aleskerov (1995) this property is called "independence of outcast of options".

$$\mathcal{C}(T) \subseteq S \Rightarrow \mathcal{C}(S) = \mathcal{C}(T)$$

or equivalently

$$S' \subseteq T \setminus \mathcal{C}(T) \text{ and } T \setminus S' \in \mathcal{D} \Rightarrow \mathcal{C}(T \setminus S') = C(T)$$

(vii) $\mathcal{C}$ satisfies the *Superset property* if for all $S, T \in \mathcal{D}$

$$S \subseteq T \text{ and } \mathcal{C}(T) \subseteq \mathcal{C}(S) \Rightarrow \mathcal{C}(S) = \mathcal{C}(T).$$

It is clear that when a choice function satisfies the IIA property it also verifies the superset property, but both properties are not equivalent.

Remark 3.3. If a choice function satisfies Arrow's axiom and we consider $S, T \in \mathcal{D}$ such that $S \subseteq T$ and $\mathcal{C}(T) \subseteq S$, then we conclude that $\mathcal{C}(S) = \mathcal{C}(T)$, which means that the Arrow's axiom implies the IIA property and thus the superset property.$\lhd$

Definition 3.2. *Let X be a set of alternatives and $\mathcal{D}$ a nonempty domain of nonempty subsets of X. Let $\mathcal{C}$ be a choice function on $\mathcal{D}$. We define the binary relations on X denoted by $R_{\mathcal{C}}$ and $R^*_{\mathcal{C}}$ as:*

- $(x, y) \in R_{\mathcal{C}} \Leftrightarrow$ *there exists $S \in \mathcal{D}$ such that $x, y \in S$ and $x \in \mathcal{C}(S)$.*
- $(x, y) \in R^*_{\mathcal{C}} \Leftrightarrow$ *there exists $S \in \mathcal{D}$ such that $x, y \in S$, $x \in \mathcal{C}(S)$ and $y \in S \setminus \mathcal{C}(S)$.*

Definition 3.3. *Let $\{x_1, \ldots, x_t\}$, with $t \geqslant 2$, be a finite sequence in a set X and $\mathcal{C}$ a choice function on a domain $\mathcal{D}$. $\{x_1, \ldots, x_t\}$ is an H-cycle of order t if it verifies:*

$$(x_1, x_2) \in R^*_C, (x_i, x_{i+1}) \in R_C \ (\forall i = 2, 3, \ldots, t-1) \text{ and } (x_t, x_1) \in R_C.$$

Definition 3.4. *Let $\{x_1, \ldots, x_t\}$, with $t \geqslant 2$, be a finite sequence in a set X and $\mathcal{C}$ a choice function on a domain $\mathcal{D}$. $\{x_1, \ldots, x_t\}$ is an SH-cycle of order t if it verifies:*

$$(x_1, x_2) \in R_C, (x_i, x_{i+1}) \in R^*_C \ (\forall i = 2, 3, \ldots, t-1) \text{ and } (x_t, x_1) \in R^*_C.$$

Definition 3.5. *A finite sequence $\{x_1, \ldots, x_t\}$, with $t \geqslant 2$, in a set X in such a way that we have a choice function $\mathcal{C}$ on a domain $\mathcal{D}$, is a cycle of order t if it verifies:*

$$(x_i, x_{i+1}) \in R^*_C \ (\forall i = 1, 2, 3, \ldots, t-1) \text{ and } (x_t, x_1) \in R^*_C.$$

If a cycle exists, then an SH-cycle exists and then an H-cycle exists, but the converses are not true.

(viii) $\mathcal{C}$ satisfies *Houthakker's axiom of revealed preference* if there exists no H-cycle of any order.

(ix) $\mathcal{C}$ satisfies the *strong axiom of revealed preference* if there exists no SH-cycle of any order.

Remark 3.4. Houthakker's axiom of revealed preference implies the strong axiom of revealed preference. $\lhd$

(x) $\mathcal{C}$ satisfies the *weak axiom of revealed preference* if for any $x, y \in X$, if $(x, y) \in R^*_C$, then $(y, x) \notin R_C$. Equivalently, there exists no H-cycle of order 2 (therefore no SH-cycle of order 2).

(xi) $\mathcal{C}$ is acyclic if it has not cycles of any order.

Definition 3.6. *Let $\mathcal{C}$ be a choice function on a space of alternatives, and let $(S_1, \ldots, S_t)$ be a finite sequence of sets in such a space. We say that this sequence is $\mathcal{C}$-related if and only if*

$$\forall i \in \{1, \ldots, t-1\}, \;\; S_i \cap \mathcal{C}(S_{i+1}) \neq \varnothing \text{ and } S_t \cap \mathcal{C}(S_1) \neq \varnothing$$

(xii) $\mathcal{C}$ satisfies *the Hansson's axiom of revealed preference* if for any $\mathcal{C}$-related sequence of sets it is $S_i \cap \mathcal{C}(S_{i+1}) = \mathcal{C}(S_i) \cap S_{i+1}$ for all $i = 1, 2, \ldots, t-1$.

The next proposition establishes some equivalences and implications among some of the properties introduced above. They are useful when we deal with the rationality theorems we state in the sections bellow.

Proposition 3.2 (Suzumura (1983), page 26)**.**

- *Houthakker's axiom of revealed preference and Hansson's axiom of revealed preference are equivalent.*
- *The strong axiom of revealed preference is implied by Houthakker's axiom and implies the weak axiom of revealed preference.*
- *The weak axiom of revealed preference implies Arrow's axiom.*

3.3 Rationality properties of a compound choice function

As we have already mentioned in the introduction we admit as rational a choice made by the sequential application of different choice functions that are derived from different binary relations. In this sense we first need to consider choice functions that are rational in a certain sense and study if their compound choice function is well-behaved with respect to rationality. More technically we study the preservation of properties of choice functions under the operation of composition. In this sense it is worth investigating the properties of the choice function that we obtain as the composition of other two choice functions belonging to a particular domain.

In this section we make this study and also recall some classical theorems of rationality that are present in the literature about this topic. We derive from them some results of rationality of a compound choice function.

As the conditions of rationality for a choice function depend on the domain they are defined on, we divide this section in two subsections. In the first one we deal with choice functions defined on domains that contain all the finite and nonempty subsets of the set of alternatives, and in the second one we consider choice functions defined on arbitrary domains.

First of all we set our framework.

X denotes a general set of alternatives. A binary relation on X, $R \subseteq X \times X$, is interpreted as a preference relation of an agent, that is xRy (or $(x,y) \in R$) if and only if the element $x \in X$ is considered at least as good as the element $y \in X$. This relation produces in a natural way a strict relation P and an indifference relation I on X defined as:

$$xPy \Leftrightarrow \{xRy \text{ and not } yRx\}$$

and

$$xIy \Leftrightarrow \{xRy \text{ and } yRx\}.$$

Let us recall now some properties of a binary relation that are relevant in different contexts of choice theory.

Definition 3.7. *Let X be a set of alternatives an R a binary relation defined on X.*

- *R is* reflexive *if xRx for all $x \in X$.*
- *R is* transitive *if whenever xRy and yRz it is xRz where x, y, z are arbitrary elements of X.*
- *R is* complete *if for any $x, y \in X$ it is xRy or yRx.*
- *R is an ordering if it is transitive and complete.*

- *R is* quasi-transitive *if it is not transitive, but the strict relation derived from it is transitive.*
- *R is acyclic if for any finite sequence of alternatives $\{x_1, \ldots, x_t\}$ such that*

$$(x_1, x_2) \in P(R), (x_2, x_3) \in P(R), \ldots, (x_{t-1}, x_t) \in P(R)$$

one has $(x_t, x_1) \notin P(R)$, where $P(R)$ is the strict relation derive from R.

The next definitions respectively formalize: i) the concept of "rational choice function", and ii) the idea of sequential application of two criteria of decision-making as a compound function.

Definition 3.8. *A choice function $\mathcal{C}$ on $\mathcal{D}$ (see Definition 3.1) is rational if there exists a binary relation R on X such that $\mathcal{C}(S) = \{x \in S : \forall y \in S,\ (x, y) \in R\}$, for any set of alternatives $S \in \mathcal{D}$. That is, the choice is made by the optimization of a preference relation.*

When the preference relation R is an ordering, we speak of complete or full rationality, or we say that the choice function $\mathcal{C}$ is completely or full rational.

If the preference relation R is quasi-transitive, we say that the choice function $\mathcal{C}$ is quasi-transitive rational.

If the relation R is acyclic, we say that the choice function $\mathcal{C}$ is acyclic rational.

Definition 3.9. *Let X be a set of alternatives and*

$$\mathcal{C}_1 : \mathcal{D} \to \mathcal{P}(X) \text{ and } \mathcal{C}_2 : \mathcal{D}' \to \mathcal{P}(X)$$

two choice functions with domains $\mathcal{D}$ and $\mathcal{D}'$ respectively in such a way that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. We define the composition of $\mathcal{C}_1$ with $\mathcal{C}_2$ and write

$$\mathcal{C}_2 \circ \mathcal{C}_1 : \mathcal{D} \to \mathcal{P}(X)$$

as the choice function $\mathcal{C}$ such that:

$$A \in \mathcal{D} \Rightarrow \mathcal{C}(A) = (\mathcal{C}_2 \circ \mathcal{C}_1)(A) = \mathcal{C}_2(\mathcal{C}_1(A)).$$

As we have mentioned above our aim in this section is the inspection of the rationality properties of the composition of two choice functions. The analysis depends both on the properties of each factor and the structure of the domain they are defined on. We proceed considering different classes of domains in two different subsections.

3.3.1 Choice functions on domains containing all the finite and nonempty subsets of the set of alternatives

In this subsection we deal with domains $\mathcal{D}$ that contains all the finite and nonempty sets of the set of alternatives. Some results are also true if we consider domains containing only all the subsets with two and three elements of the set of alternatives (see Sen (1971) and Suzumura (1983)), but for the sake of simplicity we refrain from distinguishing these situations.

The order in our exposition is the following. We state the rationality characterization theorems, beginning with the result that characterizes the full rationality and followed by the subsequent theorem for the quasi-transitivity rationality. Then the theorem that characterizes acyclicity rationality ($\Leftrightarrow$ rationality) is presented. After each case we study the possible preservation of the properties involved by the operation of composition and when possible, we state the corresponding theorem of rationality of the compound function. We consider not only compound functions of rational choice functions in the same sense (both full, quasi-transitive or acyclic rational at the same time), but also when one of them is, for example, full rational, and the other only quasi-transitive rational, etc. We give examples for those cases in which the properties verified by the initial functions are not verified by the compound function of them.

To begin with we present the characterization theorem for full rational choice functions.

Theorem 3.1 (Arrow 1959). *A choice function $\mathcal{C}$ over $\mathcal{D}$ is full rational if and only if it satisfies Arrow's axiom.*

We now recall some results about the properties verified by a compound choice function that Aizerman and Aleskerov (1995) have stated.

Theorem 3.2. *Let us denote by* ***K, H, C, O*** *(as in Aizerman and Aleskerov (1995)) the domains of choice functions that verify Arrow's axiom, the Chernoff condition, the Concordance property and the IIA property respectively. Then:*

- *The domain* ***K*** *is closed unconditionally under the composition of choice functions. Opposite, none of the domains* ***H, C, O, H∩C, H∩O, C∩O, H∩C∩O*** *is closed unconditionally with respect to the composition of choice functions.*

- *If $\mathcal{C}_1$ ∈**K**, then if $\mathcal{C}_2$ ∈**H, C, O, H∩C, H∩O, C∩O** or **H∩C∩O**, we have that $\mathcal{C}_2 \circ \mathcal{C}_1$ ∈**H, C, O, H∩C, H∩O, C∩O** or **H∩C ∩O** respectively.*

  These results are not true if we interchange $\mathcal{C}_1$ and $\mathcal{C}_2$ above.

Remark 3.5. Though the compound function of two choice functions that verify the Chernoff condition does not necessarily verifies such property, we prove below (Remark 3.9) that such compound function satisfies a weaker property that we call property P. $\lhd$

From Theorems 3.1 and 3.2 we conclude that the choice function that results of the composition of two full rational choice functions is also full rational in the classical sense as is established in the next corollary.

Corollary 3.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both full rational (or, which is equivalent, satisfy Arrow's axiom), then $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is full rational.*

In Suzumura (1983) and Sen (1971) we find the next equivalences that establish the full rationality of a choice function with some other properties different to (but equivalent to in these domains) Arrow's axiom.

Theorem 3.3. *Let $\mathcal{C}$ be a choice function on a choice space containing all the finite and nonempty subsets of a set of alternatives X. The next properties of $\mathcal{C}$ are mutually equivalent:*

i) *Full rationality.*

ii) *Houthakker's axiom of revealed preference.*

iii) *Strong axiom of revealed preference.*

iv) *Weak axiom of revealed preference.*

v) *Arrow's axiom.*

From Theorems 3.1, 3.2 and 3.3 we obtain the next result that establishes the full rationality of the composition of two choice functions when both of them verify full rationality.

Corollary 3.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined on choice spaces $\mathcal{D}$ and $\mathcal{D}'$ (respectively) that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. Then we have:*

i) *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are full rational, then $\mathcal{C}_2 \circ \mathcal{C}_1$ is full rational.*

ii) *If $\mathcal{C}_1$ and $\mathcal{C}_2$ verify Houthakker's axiom of revealed preference, then $\mathcal{C}_2 \circ \mathcal{C}_1$ also verifies it, and so it is full rational.*

iii) *If $\mathcal{C}_1$ and $\mathcal{C}_2$ verify the strong axiom of revealed preference, then $\mathcal{C}_2 \circ \mathcal{C}_1$ also verifies it, and so it is full rational.*

iv) *If $\mathcal{C}_1$ and $\mathcal{C}_2$ verify the weak axiom of revealed preference, then $\mathcal{C}_2 \circ \mathcal{C}_1$ also verifies it, and so it is full rational.*

Let us continue with the same study, but now for quasi-transitive rational choice functions. We begin recalling the characterization theorem for this case due to Blair (1976).

Theorem 3.4 (Blair et al. 1976, p. 367). *A choice function $\mathcal{C}$ on a space that contains all the finite and nonempty subsets of the set of alternatives is quasi-transitive rational if and only if it verifies the properties: Chernoff condition, superset and binariness.*

We know (see Theorem 3.2) that if $\mathcal{C}_1$ verifies the Arrow's axiom and $\mathcal{C}_2$ verifies the Chernoff condition, the choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ satisfies the Chernoff condition too. We now study what happens when we pick $\mathcal{C}_2$ satisfying the superset property or the binariness property and $\mathcal{C}_1$ fulfils the Arrow's axiom or weaker properties. We obtain that both properties are preserved in such situation as the next propositions state.

Proposition 3.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions respectively defined on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. If $\mathcal{C}_1$ verifies the Arrow's axiom and $\mathcal{C}_2$ the superset property, then the choice function $\mathcal{C}_2 \circ \mathcal{C}_1$ verifies the superset property.*

Proof. Let us suppose that S and T are subsets of alternatives in $\mathcal{D}$ and

$$S \subseteq T \text{ and } (\mathcal{C}_2 \circ \mathcal{C}_1)(T) \subseteq (\mathcal{C}_2 \circ \mathcal{C}_1)(S).$$

We have to prove that $(\mathcal{C}_2 \circ \mathcal{C}_1)(S) = (\mathcal{C}_2 \circ \mathcal{C}_1)(T)$.

$\mathcal{C}_1$ verifies the Arrow's axiom so that $\mathcal{C}_1(T) \cap S = \mathcal{C}_1(S)$, and then $\mathcal{C}_1(S) \subseteq \mathcal{C}_1(T)$.

By assumption we have

$$\mathcal{C}_2\left(\mathcal{C}_1(T)\right) \subseteq \mathcal{C}_2\left(\mathcal{C}_1(S)\right)$$

and also that $\mathcal{C}_2$ verifies the superset property. We can conclude then

$$(\mathcal{C}_2 \circ \mathcal{C}_1)(S) = (\mathcal{C}_2 \circ \mathcal{C}_1)(T).$$

□

Proposition 3.4. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two domains containing all the finite and nonempty subsets of the sets of alternatives. Let $\mathcal{C}_1$ be a choice function defined on $\mathcal{D}$ that verifies the binariness property and the Chernoff condition, and moreover $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. If $\mathcal{C}_2$ is a choice function defined on $\mathcal{D}'$ that verifies the binariness property, then the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ verifies the binariness property.*

Proof. We have to prove that for any $S \in \mathcal{D}$

$$x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x,y\}), \forall y \in S \Rightarrow x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(S).$$

From the assumptions we have that $x \in \mathcal{C}_1(\{x,y\})\ \forall y \in S$, and we know that $\mathcal{C}_1$ verifies the binariness property, thus we obtain $x \in \mathcal{C}_1(S)$.

Since $\mathcal{C}_2$ verifies the binariness property too we have

$$x \in \mathcal{C}_2(\{x,z\})\ \forall z \in \mathcal{C}_1(S) \Rightarrow x \in \mathcal{C}_2(\mathcal{C}_1(S)).$$

Then we are done if we prove that for any $z \in \mathcal{C}_1(S)$, $x \in \mathcal{C}_2(\{x,z\})$ holds true. Indeed:

$z \in \mathcal{C}_1(S)$ implies, because of $\mathcal{C}_1$ verifies the Chernoff condition, $z \in \mathcal{C}_1(\{x,z\})$, for all $x \in S$.

Because $x \in \mathcal{C}_1(\{x,y\})$ for all $y \in S$, we can conclude

$$\mathcal{C}_1(\{x,z\}) = \{x,z\} \text{ for any } z \in \mathcal{C}_1(S) \text{ and } x \in S.$$

From $x \in \mathcal{C}_2(\mathcal{C}_1(\{x,y\}))\ \forall y \in S$ we obtain, in particular

$$x \in \mathcal{C}_2(\mathcal{C}_1(\{x,z\})) = \mathcal{C}_2(\{x,z\})\ \forall z \in \mathcal{C}_1(S),$$

and from this and the fact that $\mathcal{C}_2$ verifies the binariness property, we obtain $x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$, which concludes the proof. □

From Theorem 3.4 and Propositions 3.3 and 3.4 we obtain the next result that establishes the quasi-transitive rationality of the compound choice function of a full rational choice function with a quasi-transitive one.

Corollary 3.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. If $\mathcal{C}_1$ is full rational and $\mathcal{C}_2$ is quasi-transitive rational, then the choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is quasi-transitive rational.*

Proof. $\mathcal{C}_1$ verifies the Chernoff condition and the binariness property because it verifies the Arrow's axiom and those properties are implied by it (see Remarks 3.1 and 3.2). From Theorem 3.2 we know that $\mathcal{C}_2 \circ \mathcal{C}_1$ satisfies the Chernoff condition and from Proposition 3.3 we have that $\mathcal{C}_2 \circ \mathcal{C}_1$ verifies the superset property. As Proposition 3.4 establishes that $\mathcal{C}_2 \circ \mathcal{C}_1$ also verifies the binariness property, we conclude that it is quasi-transitive rational. □

We do not need to study the composition in the reverse order ($\mathcal{C}_1$ verifying the superset property or the binariness property, and $\mathcal{C}_2$ verifying the Arrow's axiom) because we know that in this case the Chernoff condition is not preserved, thus the compound function can be neither full rational nor quasi-transitive rational.

Next we study the composition of two quasi-transitive choice functions. In this case the compound choice function has not to be necessarily quasi-transitive rational.

Proposition 3.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ choice functions defined respectively on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. Let us suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are quasi-transitive rational choice functions. Then $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ does not necessarily verify the superset property thus it is not necessarily a quasi-transitive rational choice function.*

The next example proves this proposition.

Example 3.1. Let us consider the set $X = \{x, y, z, t\}$. We define the next choice function on $\mathcal{P}^*(X)$:

$$
\begin{array}{lllllll}
\mathcal{C}_1(\{x,y\}) & = & \{x,y\} & & & & \\
\mathcal{C}_1(\{x,z\}) & = & \{x,z\} & \mathcal{C}_1(\{x,y,z\}) & = & \{x,y,z\} & \\
\mathcal{C}_1(\{x,t\}) & = & \{t\} & \mathcal{C}_1(\{x,y,t\}) & = & \{y,t\} & \\
\mathcal{C}_1(\{y,z\}) & = & \{y,z\} & \mathcal{C}_1(\{x,z,t\}) & = & \{z,t\} & \mathcal{C}_1(\{x,y,z,t\}) = \{y,z,t\}. \\
\mathcal{C}_1(\{y,t\}) & = & \{y,t\} & \mathcal{C}_1(\{y,z,t\}) & = & \{y,z,t\} & \\
\mathcal{C}_1(\{z,t\}) & = & \{z,t\} & & & &
\end{array}
$$

$\mathcal{C}_1$ verifies the property γ (and therefore the binariness property), the Chernoff condition and the superset property.

We define now $\mathcal{C}_2$ on $\mathcal{P}^*(X)$ according to:

$$\begin{array}{lll} \mathcal{C}_2(\{x,y\}) = \{x,y\} & & \\ \mathcal{C}_2(\{x,z\}) = \{x,z\} & \mathcal{C}_2(\{x,y,z\}) = \{x,y,z\} & \\ \mathcal{C}_2(\{x,t\}) = \{x,t\} & \mathcal{C}_2(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}_2(\{x,y,z,t\} = \{x,y,z\}. \\ \mathcal{C}_2(\{y,z\}) = \{y,z\} & \mathcal{C}_2(\{x,z,t\}) = \{x,z\} & \\ \mathcal{C}_2(\{y,t\}) = \{y,t\} & \mathcal{C}_2(\{y,z,t\}) = \{y,z\} & \\ \mathcal{C}_2(\{z,t\}) = \{z\} & & \end{array}$$

$\mathcal{C}_2$ also verifies the property γ, the Chernoff condition and the superset property.

Then we have that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is given by

$$\begin{array}{lll} \mathcal{C}(\{x,y\}) = \{x,y\} & & \\ \mathcal{C}(\{x,z\}) = \{x,z\} & \mathcal{C}(\{x,y,z\}) = \{x,y,z\} & \\ \mathcal{C}(\{x,t\}) = \{t\} & \mathcal{C}(\{x,y,t\}) = \{y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{y,z\} \\ \mathcal{C}(\{y,z\}) = \{y,z\} & \mathcal{C}(\{x,z,t\}) = \{z\} & \\ \mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,z\} & \\ \mathcal{C}(\{z,t\}) = \{z\} & & \end{array}$$

and it does not verify the superset property because $\mathcal{C}(\{x,z,t\}) = \{z\} \subsetneq \mathcal{C}(\{x,z\})$. ◇

From Proposition 3.4 we obtain the weaker result that states that the compound choice function of two quasi-transitive rational choice functions verifies the binariness property.

Proposition 3.6. *Let C be a choice function on a domain containing all the finite subsets of a set of alternatives X. If C is the compound function of two quasi-transitive rational choice functions, then C verifies the binariness property.*

Finally we set the conditions for a choice function to be acyclic rational or, which is equivalent, rational.

Theorem 3.5 (Blair et al. 1976)**.** *A choice function on a domain containing all the finite and nonempty subsets of a set of alternatives X is acyclic rational ($\Leftrightarrow$ is rational (Suzumura (1983), page 35)[3]) if and only if it verifies the Chernoff condition and the binariness property.*

Using again Proposition 3.4 and Theorem 3.2 we conclude directly from Theorem 3.5 the acyclic rationality of a compound choice function that results of the composition of a full rational choice function with an acyclic rational choice function.

[3]The same text establishes that this property is not true for choice functions on general domains.

Corollary 3.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined respectively on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$. Let $\mathcal{C}_1$ be full rational and let $\mathcal{C}_2$ be acyclic rational. Then, the compound choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is acyclic rational (or equivalently it is rational).*

Moreover if we compound two quasi-transitive rational choice functions we obtain a choice function that is not necessarily acyclic rational, because it may not verify the Chernoff condition as the next example proves.

Example 3.2. Let us consider the choice functions defined by the expressions

$$\begin{array}{rclrcl}
\mathcal{C}_1(\{x,y\}) & = & \{x,y\} & & & \\
\mathcal{C}_1(\{x,z\}) & = & \{x,z\} & \mathcal{C}_1(\{x,y,z\}) & = & \{x,y\} \\
\mathcal{C}_1(\{x,t\}) & = & \{x,t\} & \mathcal{C}_1(\{x,y,t\}) & = & \{x,y,t\} \\
\mathcal{C}_1(\{y,z\}) & = & \{y\} & \mathcal{C}_1(\{x,z,t\}) & = & \{x,z,t\} \\
\mathcal{C}_1(\{y,t\}) & = & \{y,t\} & \mathcal{C}_1(\{y,z,t\}) & = & \{y,t\} \\
\mathcal{C}_1(\{z,t\}) & = & \{z,t\} & & &
\end{array} \quad \text{and } \mathcal{C}_1(\{x,y,z,t\}) = \{x,y,t\}$$

and

$$\begin{array}{rclrcl}
\mathcal{C}_2(\{x,y\}) & = & \{x,y\} & & & \\
\mathcal{C}_2(\{x,z\}) & = & \{x,z\} & \mathcal{C}_2(\{x,y,z\}) & = & \{x,y,z\} \\
\mathcal{C}_2(\{x,t\}) & = & \{x,t\} & \mathcal{C}_2(\{x,y,t\}) & = & \{x,y,t\} \\
\mathcal{C}_2(\{y,z\}) & = & \{y,z\} & \mathcal{C}_2(\{x,z,t\}) & = & \{x,z\} \\
\mathcal{C}_2(\{y,t\}) & = & \{y,t\} & \mathcal{C}_2(\{y,z,t\}) & = & \{y,z\} \\
\mathcal{C}_2(\{z,t\}) & = & \{z\} & & &
\end{array} \quad \text{and } \mathcal{C}_2(\{x,y,z,t\} = \{x,y,z\}$$

that verify the binariness property (in fact, the stronger property γ), the Chernoff condition and the superset property.

Then the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is given by

$$\begin{array}{rclrcl}
\mathcal{C}(\{x,y\}) & = & \{x,y\} & & & \\
\mathcal{C}(\{x,z\}) & = & \{x,z\} & \mathcal{C}(\{x,y,z\}) & = & \{x,y\} \\
\mathcal{C}(\{x,t\}) & = & \{x,t\} & \mathcal{C}(\{x,y,t\}) & = & \{x,y,t\} \\
\mathcal{C}(\{y,z\}) & = & \{y\} & \mathcal{C}(\{x,z,t\}) & = & \{x,z\} \\
\mathcal{C}(\{y,t\}) & = & \{y,t\} & \mathcal{C}(\{y,z,t\}) & = & \{y,t\} \\
\mathcal{C}(\{z,t\}) & = & \{z\} & & &
\end{array} \quad \text{and } \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\}$$

and it does not verify the Chernoff condition because

$$\{x, z, t\} \subseteq \{x, y, z, t\}$$

but

$$\mathcal{C}(\{x, y, z, t\}) \cap \{x, z, t\} = \{x, t\} \nsubseteq \{x, z\} = \mathcal{C}(\{x, z, t\}).$$

$\diamond$

This example entails that the composition of two acyclic rational choice functions may not be an acyclic rational choice function.

Proposition 3.7. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions respectively defined on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty subsets of a set of alternatives X, and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, that verify the Chernoff condition and the binariness property, then the compound function $\mathcal{C}_2 \circ \mathcal{C}_1$ may not verify the Chernoff condition (Example 3.2), thus it may not be acyclic rational ($\Leftrightarrow$ rational). That means that the compound choice function of two acyclic rational ($\Leftrightarrow$ rational) choice functions is not necessarily rational.*

From Proposition 3.4 we can conclude the weaker result that states that when we compound two acyclic rational choice functions we obtain a choice function that verifies the binariness property (just as in the case of two quasi-transitive choice functions, as Proposition 3.6 establishes).

Proposition 3.8. *Let $\mathcal{C}$ be a choice function over a domain containing all the finite and nonempty subsets of a set of alternatives X. If $\mathcal{C}$ is the compound function of two acyclic rational (rational) choice functions, then $\mathcal{C}$ verifies the binariness property.*

3.3.2 Choice functions defined on arbitrary domains

Throughout this subsection we do not impose any restriction on the domains under inspection. In particular, they may not contain all the finite and nonempty subsets of the set of alternatives.

The organization of this subsection is alike the previous one. First, we state the characterization theorems of rationality for choice functions. Then we consider two choice functions verifying different properties of rationality and study when the compound function of them verifies such properties. When it is possible we state the corresponding result of rationality of the compound choice function.

We begin with an analysis of full rationality. We make use of the following characterization.

Theorem 3.6 (Suzumura 1977). *A choice function over a domain $\mathcal{D}$ is full rational if and only if it satisfies Houthakker's axiom of revealed preference.*

Now we proceed to study the behavior of the compound choice function of two full rational choice function on these domains. First of all we prove that the condition of full rationality is preserved when we compound two choice functions that verify it.

Proposition 3.9. *If two choice functions $\mathcal{C}_1$, defined on an arbitrary domain $\mathcal{D}$, and $\mathcal{C}_2$, defined on an arbitrary domain $\mathcal{D}'$ such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, satisfy Houthakker's axiom of revealed preference, then the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ satisfies Houthakker's axiom of revealed preference.*

Proof. Let us suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ have not H-cycles and that nevertheless $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ has an H-cycle. This implies that there exists a finite sequence of alternatives $\{x_1, \ldots, x_t\}$ with $t \geqslant 2$ such that:

$$(x_1, x_2) \in R_C^*, (x_i, x_{i+1}) \in R_C \;\; i = 2, \ldots, t-1 \text{ and } (x_t, x_1) \in R_C.$$

So we obtain:

$$(x_1, x_2) \in R_C^* \Rightarrow \exists S \in \mathcal{D} : x_1, x_2 \in S, x_1 \in \mathcal{C}(S) \text{ and } x_2 \in S \setminus \mathcal{C}(S) \tag{3.1}$$

$$(x_i, x_{i+1}) \in R_C \Rightarrow \exists S_i \in \mathcal{D} : x_i, x_{i+1} \in S_i \text{ and } x_i \in \mathcal{C}(S_i) \; \forall i = 2, \ldots, t-1 \tag{3.2}$$

and

$$(x_t, x_1) \in R_C \Rightarrow \exists S_t \in \mathcal{D} : x_t, x_1 \in S_t \text{ and } x_t \in \mathcal{C}(S_t) \tag{3.3}$$

From equation (3.1) we have:

(a) $x_1 \in \mathcal{C}_2(\mathcal{C}_1(S)) \Rightarrow x_1 \in \mathcal{C}_1(S)$, and

(b)
$$x_2 \notin \mathcal{C}_2(\mathcal{C}_1(S)) \Rightarrow \begin{cases} \text{Case 1:} & x_2 \notin \mathcal{C}_1(S), \text{ or} \\ \text{Case 2:} & x_2 \in \mathcal{C}_1(S) \setminus \mathcal{C}_2(\mathcal{C}_1(S)) \end{cases}$$

Case 1: $(x_1, x_2) \in R_{C_1}^*$.

From equation (3.2) we conclude that for all $i = 2, \ldots, t-1$ there exists $S_i \in \mathcal{D}$ such that $x_i, x_{i+1} \in S_i$ and $x_i \in \mathcal{C}_1(S_i)$, thus $(x_i, x_{i+1}) \in R_{C_1}, \forall i = 2, \ldots, t-1$. From (3.3) we have that there exists $S_t \in \mathcal{D}$ such that $x_1, x_t \in S_t$ and $x_t \in \mathcal{C}_1(S_t)$ thus $(x_t, x_1) \in R_{C_1}$. Therefore we obtain that $\mathcal{C}_1$ has an H-cycle, and we conclude.

Case 2: $(x_1, x_2) \in R_{C_1}, (x_2, x_1) \in R_{C_1}$ and $(x_1, x_2) \in R^*_{C_2}$ (because $x_1 \in \mathcal{C}_2(\mathcal{C}_1(S))$ and $x_2 \in \mathcal{C}_1(S) \setminus \mathcal{C}_2(\mathcal{C}_1(S))$ where $S \in \mathcal{D}$ and then $\mathcal{C}_1(S) \in \mathcal{D}'$). Now we deal with two different possibilities:

i) If $x_{i+1} \in C_1(S_i)\, \forall i = 2, \ldots, t-1$ (and recalling that $x_i, x_{i+1} \in S_i$ with $S_i \in \mathcal{D}$) we obtain $(x_i, x_{i+1}) \in R_{C_2}, i = 2, \ldots, t-1$, because $x_i \in \mathcal{C}_2(\mathcal{C}_1(S_i))$ as equation (3.2) establishes. If we also have $x_1 \in \mathcal{C}_1(S_t)$, equation (3.3) implies that $x_t \in \mathcal{C}_2(\mathcal{C}_1((S_t))$, and we conclude the existence of an H-cycle for $\mathcal{C}_2$, which ends the argument.

ii) If $x_{i+1} \notin \mathcal{C}_1(S_i)$ for some $i = 2, \ldots, t-1$, we obtain that $(x_i, x_{i+1}) \in R^*_{C_1}$ and

$$(x_i, x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})$$

is an H-cycle of $\mathcal{C}_1$ which finishes the proof.

In case that it is $x_1 \notin \mathcal{C}_1(S_t)$, $(x_t, x_1, \ldots, x_{t-1})$ is an H-cycle for $\mathcal{C}_1$ and the proof also concludes.

□

Thus we can establish the full rationality of a compound choice function of two full rational choice functions over arbitrary domains.

Corollary 3.5. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions defined on arbitrary domains $\mathcal{D}$ and $\mathcal{D}'$ respectively and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, and both of them satisfy Houthaker's axiom of revealed preference, then the compound choice function $\mathcal{C}_2 \circ \mathcal{C}_1$ is full rational. Equivalently: if two choice functions are full rational, their compound function is also full rational.*

Because of the equivalence between the Houthaker's axiom and the Hansson's axiom we also have the same result for this property.

Corollary 3.6. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions defined on arbitrary domains $\mathcal{D}$ and $\mathcal{D}'$ respectively such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, and both of them satisfy the Hansson's axiom of the revealed preference (which is equivalent to saying that both functions satisfy the Houtaker's axiom of the revealed preference, and then equivalent to saying that both functions are full rational), then the compound choice function $\mathcal{C}_2 \circ \mathcal{C}_1$ is full rational.*

Let us now continue with the case of quasi-transitive rational choice functions, for which we have the next result.

Theorem 3.7 (Suzumura 1983, p. 50)**.** *A choice function $\mathcal{C}$ on a space of choice is quasi-transitive rational if it satisfies the strong axiom of revealed preference.*

When we study if the sufficient condition of quasi-transitive rationality (the strong axiom of revealed preference) is preserved by the composition of two choice functions (defined on arbitrary domains) we obtain that the answer is negative, as the next proposition establishes.

Proposition 3.10. *If two choice functions $\mathcal{C}_1$, defined on an arbitrary domain $\mathcal{D}$, and $\mathcal{C}_2$, defined on an arbitrary domain $\mathcal{D}'$ such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, satisfy the strong axiom of revealed preference, then the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ does not necessarily satisfy the strong axiom of revealed preference.*

Next example proves this proposition.

Example 3.3. Let $X = \{x, y, z, t\}$ and the domain of choice $\mathcal{D} = \{\{x, y\}, \{y, z\}, \{z, t\}, \{x, t\}\}$ and over it we define a choice function $\mathcal{C}_1$ as

$$\begin{array}{rcl} \mathcal{C}_1(\{x,y\}) & = & \{x\} \\ \mathcal{C}_1(\{y,z\}) & = & \{y,z\} \\ \mathcal{C}_1(\{z,t\}) & = & \{z,t\} \\ \mathcal{C}_1(\{x,t\}) & = & \{t\} \end{array}$$

Now we consider the domain $\mathcal{D}' = \{\{x, y\}, \{y, z\}, \{z, t\}, \{x, t\}, \{x\}, \{t\}\}$ and over it we define a choice function $\mathcal{C}_2$ as

$$\begin{array}{rcl} \mathcal{C}_2(\{x,y\}) & = & \{x,y\} \\ \mathcal{C}_2(\{y,z\}) & = & \{y\} \\ \mathcal{C}_2(\{z,t\}) & = & \{z\} \\ \mathcal{C}_2(\{x,t\}) & = & \{x,t\} \\ \mathcal{C}_2(\{x\}) & = & \{x\} \\ \mathcal{C}_2(\{t\}) & = & \{t\} \end{array}$$

We have that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$ thus we can define over $\mathcal{D}$ the choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ that is given by

$$\begin{array}{rcl} \mathcal{C}(\{x,y\}) & = & \{x\} \\ \mathcal{C}(\{y,z\}) & = & \{y\} \\ \mathcal{C}(\{z,t\}) & = & \{z\} \\ \mathcal{C}(\{x,t\}) & = & \{t\} \end{array}$$

As we can observe, $\mathcal{C}_1$ and $\mathcal{C}_2$ have not any SH-cycles therefore both of them verify the strong axiom of revealed preference, but $\{x, y, z, t\}$ is an SH-cycle for $\mathcal{C}$. $\diamond$

Therefore if we compose two choice functions that verify the strong axiom of revealed preference, we obtain a choice function that may not be quasi-transitive rational.

We also have that if $\mathcal{C}_1$ (over an arbitrary domain $\mathcal{D}$) has not SH-cycles (which implies that it is quasitransitive rational) and $\mathcal{C}_2$ (over an arbitrary domain $\mathcal{D}'$ such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) is full rational (it verifies the Houthaker's axiom of revealed preference), then the choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ does not necessarily satisfy the strong axiom of revealed preference, thus it is not necessarily quasi-transitive rational or full rational.

This is proved considering in Example 3.3 above, the choice function $\mathcal{C}_2'$ defined as

$$\begin{array}{rcl} \mathcal{C}_2'(\{x,y\}) & = & \{x,y\} \\ \mathcal{C}_2'(\{y,z\}) & = & \{y\} \\ \mathcal{C}_2'(\{z,t\}) & = & \{z\} \\ \mathcal{C}_2'(\{x,t\}) & = & \{x\} \\ \mathcal{C}_2'(\{x\}) & = & \{x\} \\ \mathcal{C}_2'(\{t\}) & = & \{t\} \end{array}$$

in place of $\mathcal{C}_2$. In this case $\{x, y, z, t\}$ is also an SH-cycle for $\mathcal{C}_2' \circ \mathcal{C}_1$.

If we consider the reverse statement, that is, $\mathcal{C}_1$ verifies Houthaker's axiom of revealed preference (it is full rational) and $\mathcal{C}_2$ verifies the strong axiom of revealed preference (which implies that it is quasi-transitive rational), then the compound choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ may not verify the Houthaker's axiom of revealed preference and then it is not necessarily full rational as the next example shows.

Example 3.4. Let $X = \{x, y, z, t\}$ and the domain of choice $\mathcal{D} = \{\{x, y\}, \{x, z\}, \{z, t\}, \{x, y, t\}\}$ and define a choice function $\mathcal{C}_1$ on $\mathcal{D}$ by

$$\begin{array}{rcl} \mathcal{C}_1(\{x,y\}) & = & \{x\} \\ \mathcal{C}_1(\{x,z\}) & = & \{x,z\} \\ \mathcal{C}_1(\{z,t\}) & = & \{z,t\} \\ \mathcal{C}_1(\{x,y,t\}) & = & \{x,t\} \end{array}$$

Now we consider the domain $\mathcal{D}' = \{\{x, y\}, \{x, z\}, \{z, t\}, \{x, t\}, \{x, y, t\}, \{x\}\}$ and define a choice function $\mathcal{C}_2$ on $\mathcal{D}'$ by

$$\begin{array}{rcl} \mathcal{C}_2(\{x,y\}) & = & \{x\} \\ \mathcal{C}_2(\{x,z\}) & = & \{x,z\} \\ \mathcal{C}_2(\{z,t\}) & = & \{z,t\} \\ \mathcal{C}_2(\{x,t\}) & = & \{t\} \\ \mathcal{C}_2(\{x,y,t\}) & = & \{t\} \\ \mathcal{C}_2(\{x\}) & = & \{x\} \end{array}$$

We have that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$ thus we can define the choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ over $\mathcal{D}$ by

$$\begin{array}{rcl} \mathcal{C}(\{x,y\}) & = & \{x\} \\ \mathcal{C}(\{x,z\}) & = & \{x,z\} \\ \mathcal{C}(\{z,t\}) & = & \{z,t\} \\ \mathcal{C}(\{x,y,t\}) & = & \{t\} \end{array}$$

As we can observe, $\mathcal{C}_1$ has no H-cycles (thus it has not SH-cycles, either) and $\mathcal{C}_2$ has not any SH-cycles (it has H-cycles), but $\{t,x,z\}$ is an H-cycle for $\mathcal{C}$, thus it does not verify Houthaker's axiom of revealed preference and therefore it is not full rational. $\diamond$

Nevertheless, we have a positive result when we compound a full rational choice function with a choice function that has not any SH-cycle (and then, it is quasi-transitive rational). The result is a quasi-transitive rational choice function, as the next proposition establishes.

Proposition 3.11. *Let $\mathcal{C}_1$ be a choice function defined on an arbitrary domain $\mathcal{D}$ that has not any H-cycle ($\Leftrightarrow$ is full rational) and $\mathcal{C}_2$ a choice function defined on a domain $\mathcal{D}'$ such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$ and such that it has not any SH-cycle (which implies that $\mathcal{C}_2$ is quasi-transitive rational). Then the compound choice function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ has not any SH-cycle, thus it is quasi-transitive rational.*

Proof. Let us suppose that $\mathcal{C}$ has an SH-cycle, hence there exists a finite sequence of alternatives $\{x_1, \ldots, x_t\}$ such that

$$(x_1, x_2) \in R_C, (x_i, x_{i+1}) \in R_C^* \;\; i = 2, \ldots, t-1 \text{ and } (x_t, x_1) \in R_C^*$$

and we have the following assertions.

There exists $S_1 \in \mathcal{D}$ such that $x_1, x_2 \in S_1$ and $x_1 \in \mathcal{C}(S_1)$.

For any $i = 2, \dots, t-1$ there exists $S_i \in \mathcal{D}$ such that $x_i, x_{i+1} \in S_i$, $x_i \in \mathcal{C}(S_i)$ and $x_{i+1} \in S_i \setminus \mathcal{C}(S_i)$.

There exists $S_t \in \mathcal{D}$ such that $x_t, x_1 \in S_t$, $x_t \in \mathcal{C}(S_t)$ and $x_1 \in S_t \setminus \mathcal{C}(S_t)$.

From these facts we obtain

$$x_1, x_2 \in \mathcal{C}_1(S_1), \text{ and } x_i \in \mathcal{C}_1(S_i) \ \forall i = 2, \dots, t.$$

Moreover, $x_{i+1} \in S_i \setminus \mathcal{C}(S_i)$ for $i = 2, \dots, t-1$ and $x_1 \in S_t \setminus \mathcal{C}(S_t)$.

Now if $x_{i+1} \notin \mathcal{C}_1(S_i)$ for some $i = 2, \dots, t-1$, then $\{x_i, x_{i+1}, \dots, x_t, x_1, \dots, x_{i-1}\}$ is an H-cycle for $\mathcal{C}_1$ which is against the hypothesis, and the same holds for $\{x_t, x_1, \dots, x_{i-1}\}$ if $x_1 \notin \mathcal{C}_1(S_t)$. Therefore

$$x_{i+1} \in \mathcal{C}_1(S_i) \ \forall i = 1, 2, \dots t-1, \text{ and } x_1 \in \mathcal{C}_1(S_t).$$

From here we conclude that $\{x_1, x_2, \dots, x_t\}$ is an SH-cycle for $\mathcal{C}_2$ which produces a contradiction and finishes the proof. □

We now continue with the case of acyclic rational choice functions. First we state the characterization theorem for this type of rational choice functions.

Theorem 3.8 (Suzumura 1983, p. 51)**.** *A choice function defined on an arbitrary domain is acyclic rational if it satisfies the weak axiom of revealed preference and R^*_C is acyclic.*

Our next proposition studies the preservation of the sufficient properties for a choice function to be acyclic rational by the operation of composition. The weak axiom of revealed preference is preserved, however the acyclicity of R^*_C is not.

Proposition 3.12. *a) If $\mathcal{C}_1$ (defined on an arbitrary domain $\mathcal{D}$) and $\mathcal{C}_2$ (defined on an arbitrary domain $\mathcal{D}'$ such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) are two choice functions that verify the weak axiom of the revealed preference, then the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ also satisfies this axiom.*

b) If $R^*_{C_1}$ and $R^*_{C_2}$ are acyclic, then R^*_C may not be acyclic, where $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$.

Proof. *a*) Because $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the weak axiom of revealed preference we have that none of them has H-cycles of order two, that is:

$$\text{If } (x, y) \in R^*_{C_1} \Rightarrow (y, x) \notin R_{\mathcal{C}_1}.$$

$$\text{If } (x,y) \in R^*_{\mathcal{C}_2} \Rightarrow (y,x) \notin R_{\mathcal{C}_2}.$$

Let us now suppose that $\mathcal{C}$ has an H-cycle of order two, thus there exist alternatives x, y such that $(x,y) \in R^*_C$ and $(y,x) \in R_C$. Then

$$(x,y) \in R^*_C \Rightarrow \exists S \in \mathcal{D} : x, y \in S, x \in \mathcal{C}(S) \text{ and } y \in S \setminus \mathcal{C}(S).$$

Therefore

$$x \in \mathcal{C}(S) = \mathcal{C}_2(\mathcal{C}_1(S)) \Rightarrow x \in \mathcal{C}_1(S). \tag{3.4}$$

Moreover

$$(y,x) \in R_C \Rightarrow \exists S' \in \mathcal{D} : x, y \in S' \text{ and } y \in \mathcal{C}(S') = \mathcal{C}_2(\mathcal{C}_1(S'))$$

and therefore

$$y \in C_1(S') \Rightarrow (y,x) \in R_{C_1}. \tag{3.5}$$

From $y \notin \mathcal{C}_2(\mathcal{C}_1(S))$, two cases arise:

(a) If $y \notin \mathcal{C}_1(S)$ we conclude, because in that case we have by (3.4) that $(x,y) \in R^*_{C_1}$, and also by (3.5) that $(y,x) \in R_{C_1}$, which contradicts the fact that $\mathcal{C}_1$ verifies the weak axiom of revealed preference.

(b) If $y \in \mathcal{C}_1(S)$, as $y \notin \mathcal{C}_2(\mathcal{C}_1(S))$ we obtain that $(x,y) \in R^*_{\mathcal{C}_2}$ by (3.4). Moreover we have that $x, y \in S'$ and $y \in \mathcal{C}_2(\mathcal{C}_1(S'))$, and then

b.1) If $x \in \mathcal{C}_1(S')$ we finish, because in that case $(y,x) \in R_{\mathcal{C}_2}$, thus $\mathcal{C}_2$ does not verify the weak axiom of revealed preference.

b.2) If $x \notin \mathcal{C}_1(S')$ we obtain from (3.5) that $(y,x) \in R^*_{C_1}$, but $\mathcal{C}_1$ verifies the weak axiom of revealed preference and this implies $(x,y) \notin R_{C_1}$ which contradicts the fact that $x \in \mathcal{C}_1(S)$ and $y \in S$.

b) The next example proves this part.

Example 3.5. Let us consider the choice functions on the set of finite and nonempty subsets of the set $\{x_1, x_2, x_3\}$ given by:

$$\begin{array}{c} \mathcal{C}_1(\{x_1,x_2\}) = \{x_1\} \\ \mathcal{C}_1(\{x_1,x_3\}) = \{x_1,x_3\} \\ \mathcal{C}_1(\{x_2,x_3\}) = \{x_2\} \\ \mathcal{C}_1(\{x_1,x_2,x_3\} = \{x_1,x_2,x_3\} \end{array} \Rightarrow \begin{array}{c} (x_1,x_2) \in R^*_{C_1} \\ (x_1,x_3) \in R_{C_1} \text{ and } (x_3,x_1) \in R_{C_1} \\ (x_2,x_3) \in R^*_{C_1} \\ (x_2,x_1) \in R_{C_1} \text{ and } (x_3,x_2) \in R_{C_1} \end{array}$$

$$\begin{array}{rcl} \mathcal{C}_2(\{x_1,x_2\}) = \{x_1\} & & (x_1,x_2) \in R^*_{C_2} \\ \mathcal{C}_2(\{x_1,x_3\}) = \{x_3\} & \Rightarrow & (x_3,x_1) \in R^*_{C_2} \\ \mathcal{C}_2(\{x_2,x_3\}) = \{x_2,x_3\} & & (x_2,x_3) \in R_{C_2} \text{ and } (x_3,x_2) \in R_{C_2} \\ \mathcal{C}_2(\{x_1,x_2,x_3\} = \{x_1,x_2,x_3\} & & (x_2,x_1) \in R_{C_2} \text{ and } (x_1,x_3) \in R_{C_2}. \end{array}$$

$R^*_{C_1}$ and $R^*_{C_2}$ are acyclic. Let us consider the compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$, defined as:

$$\begin{array}{rcl} \mathcal{C}(\{x_1,x_2\}) = \{x_1\} & & (x_1,x_2) \in R^*_C \\ \mathcal{C}(\{x_1,x_3\}) = \{x_3\} & \Rightarrow & (x_3,x_1) \in R^*_C \\ \mathcal{C}(\{x_2,x_3\}) = \{x_2\} & & (x_2,x_3) \in R^*_C. \\ \mathcal{C}(\{x_1,x_2,x_3\} = \{x_1,x_2,x_3\} & & \end{array}$$

We have that (x_1,x_2,x_3) is a cycle for R^*_C and so $\mathcal{C}$ is not acyclic. ◇

□

As we have shown above, when we compound two choice functions $\mathcal{C}_1$ and $\mathcal{C}_2$ (respectively defined on general domains $\mathcal{D}$ and D' such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) verifying the weak axiom of revealed preference and such that $R^*_{\mathcal{C}_1}$ and $R^*_{\mathcal{C}_2}$ are acyclic, we can obtain a choice function that is not acyclic rational.

Contrary to this discouraging situation we have a positive result when we compound a full rational choice function with another one that verifies the weaker axiom of revealed preference and that $R^*_{\mathcal{C}}$ is acyclic. In such a case we obtain an acyclic rational choice function as the next proposition proves.

Proposition 3.13. *Let $\mathcal{C}_1$ be a full rational choice function on an arbitrary domain $\mathcal{D}$ ($\Leftrightarrow \mathcal{C}_1$ has not H-cycles) and $\mathcal{C}_2$ a choice function on an arbitrary domain $\mathcal{D}'$ (such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) that verifies the weak axiom of revealed preference and such that $R^*_{\mathcal{C}_2}$ is acyclic (which implies that $\mathcal{C}_2$ is acyclic rational). Then $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is also acyclic rational.*

Proof. It is clear from Proposition 3.12 that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ verifies the weak axiom of revealed preference.

Let us see that $R^*_{\mathcal{C}}$ is acyclic.

Otherwise, there exists a finite set of alternatives $\{x_1, \dots, x_t\}$ such that

$$(x_i, x_{i+1}) \in P(R^*_{\mathcal{C}}) \; \forall i = 1, \dots, t-1 \text{ and } (x_t, x_1) \in P(R^*_{\mathcal{C}})$$

where $P(R^*_{\mathcal{C}})$ is the strict preference relation associated to $R^*_{\mathcal{C}}$.

From this we obtain that for all $i = 1, \dots, t-1$ there exists $S_i \in \mathcal{D}$ such that $x_i, x_{i+1} \in S_i$, $x_i \in \mathcal{C}(S_i)$ and $x_{i+1} \in S_i \setminus \mathcal{C}(S_i)$, and there exists $S_t \in \mathcal{D}$ such that $x_t, x_1 \in S_t$, $x_t \in \mathcal{C}(S_t)$ and $x_1 \in S_t \setminus \mathcal{C}(S_t)$.

Then we have

$$x_i \in \mathcal{C}_2(\mathcal{C}_1(S_i)) \text{ and } x_{i+1} \notin \mathcal{C}_2(\mathcal{C}_1(S_i)) \ \forall i = 1, \dots, t-1$$

and

$$x_t \in \mathcal{C}_2(\mathcal{C}_1(S_t)) \text{ and } x_1 \notin \mathcal{C}_2(\mathcal{C}_1(S_t)).$$

If $x_{i+1} \notin \mathcal{C}_1(S_i)$ for some $i = 1, \dots, t-1$, then $\{x_i, x_{i+1}, \dots, x_t, x_1, \dots, x_{i-1}\}$ is an H-cycle for $\mathcal{C}_1$ against the hypothesis, and the same would be true for $\{x_t, x_1, \dots, x_{t-1}\}$ if $x_1 \notin \mathcal{C}_1(S_t)$. Therefore $x_i, x_{i+1} \in \mathcal{C}_1(S_i)$ for all $i = 1, \dots, t-1$, and in the same way $x_t, x_1 \in \mathcal{C}_1(S_t)$.

We conclude that $(x_i, x_{i+1}) \in R^*_{\mathcal{C}_2}$ for all $i = 1, \dots, t-1$, and $(x_t, x_1) \in R^*_{\mathcal{C}_2}$. Because $\mathcal{C}_2$ verifies the weak axiom of revealed preference we obtain that $(x_{i+1}, x_i) \notin R^*_{\mathcal{C}_2}$ for all $i = 1, \dots, t-1$ and $(x_1, x_t) \notin R^*_{\mathcal{C}_2}$. This implies that $(x_i, x_{i+1}) \in P(R_{\mathcal{C}_2^*})$ for all $i = 1, \dots, t-1$ and $(x_t, x_1) \in P(R_{\mathcal{C}_2^*})$ which is not possible because $R^*_{\mathcal{C}_2}$ is acyclic, and the proof concludes. □

Remark 3.6. Proposition 3.13 remains true if we interchange the properties of $\mathcal{C}_1$ and $\mathcal{C}_2$ and reinforce Houthaker's axiom to the strong axiom of revealed preference. We omit the proof because it is essentially the same as the one for Proposition 3.13 above. ◁

Finally we state the conditions for a choice function over an arbitrary domain to be rational.

Theorem 3.9 (Suzumura 1983, p. 51)**.** *A choice function $\mathcal{C}$ is rational if it satisfies the weak axiom of revealed preference.*

Because of Proposition 3.12 a) we conclude that the composition of two choice functions that verify the weak axiom of revealed preference is a rational choice function too.

Corollary 3.7. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions (defined on arbitrary domains $\mathcal{D}$ and $\mathcal{D}'$ respectively such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) that satisfy the weak axiom of revealed preference, then the compound choice function $\mathcal{C}_2 \circ \mathcal{C}_1$ is rational.*

As the strong axiom of revealed preference is implied by Houthakker's axiom and implies the weak axiom of revealed preference (Proposition 3.2) we obtain that when two choice functions verify either Houthakker's axiom, the strong axiom of revealed preference or the weak axiom of revealed preference, then their compound function is rational.

3.4 Choice functions rational by two sequential criteria

This far we have considered choice functions that are rational in some way depending on the properties that they verify and the domains they are defined over.

Besides we have focused on the composition of two choice functions and studied when the properties verified by two choice functions are transferred to the compound function of them. As corollaries we have obtained some results that include different conditions for a compound choice function to be rational.

Now we take the reverse position. Along this section we consider the problem of the decomposition of a given choice function with respect to some rational (in the classical sense) choice functions. That is, the choice made by the decision-maker for any set of alternatives is our primitive concept. We investigate if such behavior can be explained by the sequential application of two "rational" choice functions in an established order: given a choice function, we wonder if there exist two (full, acyclic, quasi-transitive,...) rational choice functions such that their compound function produces the choice that we had observed. Of course the case when the primitive choice function is already "rational" has been studied before.

Our main finding is Theorem 3.11 that characterizes the choice functions that can be expressed as the composition of two rational choice functions using two testable properties. One of these properties is property γ and the other one is introduced below (see Definition 3.12).

Along this section we always deal with choice functions defined over domains that contain all the finite and nonempty sets of the set of alternatives.

First we introduce the "upper and lower approximations of a choice function" (Aizerman and Aleskerov (1995)), that intend to approximate a non rational choice function by a rational one. To that purpose we introduce the next definitions.

Definition 3.10. *Let $\mathcal{C}$ be a choice function on $\mathcal{D}$. The upper approximation of $\mathcal{C}$ in the class of choice functions on $\mathcal{D}$ satisfying certain properties is another choice function $\mathcal{C}^u$ on $\mathcal{D}$ in that class such that $\mathcal{C}(S) \subseteq \mathcal{C}^u(S)$ for any $S \in \mathcal{D}$, and such that if $\bar{\mathcal{C}}$ is another choice function on $\mathcal{D}$ in the same class and verifying that $\mathcal{C}(S) \subseteq \bar{\mathcal{C}}(S)$ for all $S \in \mathcal{D}$, it must be $\mathcal{C}^u(S) \subseteq \bar{\mathcal{C}}(S)$ for all $S \in \mathcal{D}$.*

Definition 3.11. *Let $\mathcal{C}$ be a choice function on $\mathcal{D}$. The lower approximation of $\mathcal{C}$ in the class of choice functions on $\mathcal{D}$ satisfying certain properties is another choice function $\mathcal{C}^l$ on $\mathcal{D}$ such that $\mathcal{C}^l(S) \subseteq \mathcal{C}(S)$ for any $S \in \mathcal{D}$, and such that if $\bar{\mathcal{C}}$ is another choice function on $\mathcal{D}$ in the same class and verifying that $\bar{\mathcal{C}}(S) \subseteq \mathcal{C}(S)$ for all $S \in \mathcal{D}$, it must be $\bar{\mathcal{C}}(S) \subseteq \mathcal{C}^l(S)$ for all $S \in \mathcal{D}$.*

In both definitions we say that the functions $\mathcal{C}^u$ and $\mathcal{C}^l$ are in a class Q if they verify the properties of all the functions in such class.

The next result (Theorem 5.15 in Aizerman and Aleskerov (1995)) settles which choice functions can be approximated by choice functions verifying the properties γ and Chernoff.

Theorem 3.10. *For any choice function $\mathcal{C}$ there exists an upper approximation that verifies the Chernoff condition and the binariness property (in fact it verifies property γ) and this function is given by*

$$\mathcal{C}^u(S) = \{x \in S : \forall y \in S \text{ there exists } S' \in \mathcal{D} \text{ such that } x, y \in S' \text{ and } x \in \mathcal{C}(S')\}.$$

Moreover if the choice function $\mathcal{C}$ verifies the binariness property, it also has a lower approximation that is defined as

$$\mathcal{C}^l(S) = \{x \in S : x \in \mathcal{C}(\{x, y\}),\ \forall y \in S\}$$

and that also satisfies the Chernoff condition and the binariness property.

We stress the fact that for any choice function $\mathcal{C}$ on a domain $\mathcal{D}\mathcal{D}$ verifying the binariness property, the choice functions $\mathcal{C}^l$ and $\mathcal{C}^u$ defined above satisfy that for all $S \in \mathcal{D}$

$$\mathcal{C}^l(S) \subseteq \mathcal{C}(S) \subseteq \mathcal{C}^u(S).$$

Thus $\mathcal{C}^u(S) \neq \varnothing$ for all $S \in \mathcal{D}$ (remember that we have $\mathcal{C}(S) \neq \varnothing$ for all $S \in \mathcal{D}$ (Definition 3.1)). By virtue of Theorem 3.5 we conclude that $\mathcal{C}^u$ is rational.

The case of $\mathcal{C}^l$ deserves an special remark.

Remark 3.7. The choice function that we have denoted $\mathcal{C}^l$ verifies the binariness property and also the Chernoff condition, but it must not verify that $\mathcal{C}^l(S) \neq \varnothing$ for all $S \in \mathcal{D}$ (see Example 3.8 bellow). Theorem 3.5 is a characterization theorem for choice functions $\mathcal{C}$ satisfying that $\mathcal{C}(S) \neq \varnothing$ for all $S \in \mathcal{D}$, thus when $\mathcal{C}^l$ verifies such property we can apply that theorem and conclude that $\mathcal{C}^l$ is also rational. Nevertheless, even when $\mathcal{C}^l(S)$ may be empty, we can find a binary relation R^l on the set of alternatives that rationalizes it.

This relation R^l is defined by

$$xR^l y \text{ if and only if } x \in \mathcal{C}(\{x,y\}).$$

Therefore $\mathcal{C}^l$ is always rational.

Moreover we can always assure that for any subset $S \in \mathcal{D}$ with two elements $\mathcal{C}^l(S) \neq \varnothing$ because $\mathcal{C}(S) \neq \varnothing$ for all $S \in \mathcal{D}$ and thus R^l is complete. $\lhd$

Now let us introduce a new property for a choice function $\mathcal{C}$ on $\mathcal{D}$.

Definition 3.12. *A choice function $\mathcal{C}$ satisfies the **Property** P if given two elements x, y of the set of alternatives and $S, T \in \mathcal{D}$, in such a way that*

$$\{x,y\} \subseteq S \subseteq T,$$

then

$$\text{if } \mathcal{C}(\{x,y\}) = \{x\} \text{ and } x \in \mathcal{C}(T), \text{ it must be } y \notin \mathcal{C}(S).$$

This property is weaker than the Chernoff condition and therefore whenever a choice function is rational it verifies property P. On the other hand this property allows for cyclical patterns even in pairwise comparisons.

The next example shows that if a choice function satisfies property P it must verify neither binariness property nor the Chernoff condition, which are the two necessary and sufficient conditions for a choice function over a domain containing all the finite and nonempty subsets of the set of alternatives to be rational.

Example 3.6. Let $\mathcal{C}$ be a choice function defined on the domain of nonempty subsets of $X = \{x,y,z,t\}$ given by

$$\begin{array}{rclrcl}
\mathcal{C}(\{x,y\}) & = & \{x,y\} & & & \\
\mathcal{C}(\{x,z\}) & = & \{x\} & \mathcal{C}(\{x,y,z\}) & = & \{y\} \\
\mathcal{C}(\{x,t\}) & = & \{x,t\} & \mathcal{C}(\{x,y,t\}) & = & \{x,y,t\} \\
\mathcal{C}(\{y,z\}) & = & \{y\} & \mathcal{C}(\{x,z,t\}) & = & \{x,t\} \\
\mathcal{C}(\{y,t\}) & = & \{y,t\} & \mathcal{C}(\{y,z,t\}) & = & \{y,t\} \\
\mathcal{C}(\{z,t\}) & = & \{z,t\} & & &
\end{array}
\quad \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\}$$

This function does not verify the binariness property because $x \notin \mathcal{C}(\{x,y,z\})$ and $x \in \mathcal{C}(\{x,y\})$ and $x \in \mathcal{C}(\{x,z\})$.

Moreover it does not verify the Chernoff condition because $\mathcal{C}(\{x,y,z,t\}) \cap \{x,y,z\}) = \{x,y\} \nsubseteq \mathcal{C}(\{x,y,z\})$.

Nonetheless $\mathcal{C}$ verifies property P. $\diamond$

Nevertheless we now prove that a choice function verifying property P can be written as the composition of two rational functions. We formalize this concept in the next definition.

Definition 3.13. *A choice function $\mathcal{C}$ on $\mathcal{D}$ is* rational by two sequential criteria *if there exist two rational choice functions $\mathcal{C}_1$ defined on $\mathcal{D}$ and $\mathcal{C}_2$ on $\mathcal{D}'$ with $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$, such that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$.*

Obviously, if $\mathcal{C}$ is a rational choice function then it is rational by two sequential criteria by considering, for example, $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_2$ the trivial choice function ($\mathcal{C}_2(S) = S$ for all $S \in \mathcal{D}'$).

We first prove a lemma that leads us to consider an initial choice function $\mathcal{C}$ that verifies property γ.

Lemma 3.1. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions (defined respectively on domains $\mathcal{D}$ and $\mathcal{D}'$ that contain all the finite and nonempty sets of the set of alternatives and such that $\mathcal{C}_1(\mathcal{D}) \subseteq \mathcal{D}'$) that verify the Chernoff condition and the binariness property, then the compound function $\mathcal{C}_2 \circ \mathcal{C}_1$ verifies property γ.*

Proof. Let $\{S_i\}_{i \in I}$ a collection of subsets of alternatives in $\mathcal{D}$. We must prove that

$$x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(S_i), \;\; \forall i \in I \Rightarrow x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(\cup_{i \in I} S_i).$$

We know that

$$x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(S_i), \;\; \forall i \in I \Rightarrow x \in \mathcal{C}_1(S_i) \;\; \forall i \in I.$$

Because $\mathcal{C}_1$ verifies the Chernoff condition we obtain that

$$x \in \mathcal{C}_1(\{x,y\}) \;\; \forall y \in S_i, \;\; \forall i \in I \Leftrightarrow x \in \mathcal{C}_1(\{x,y\}) \;\; \forall y \in \cup_{i \in I} S_i,$$

and $\mathcal{C}_1$ satisfies the binariness property, thus we have

$$x \in \mathcal{C}_1(\cup_{i \in I} S_i).$$

On the other hand we know that $\mathcal{C}_2$ verifies the Chernoff condition, and then

$$x \in \mathcal{C}_2(\mathcal{C}_1(S_i)) \;\; \forall i \in I \Rightarrow x \in \mathcal{C}_2(\{x,y\}) \;\; \forall y \in \mathcal{C}_1(S_i) \;\; \forall i \in I.$$

Thus applying that $\mathcal{C}_2$ verifies the binariness property we have

$$x \in \mathcal{C}_2(\cup_{i\in I}\mathcal{C}_1(S_i)).$$

Taking into account now that $\mathcal{C}_1$ verifies the Chernoff condition we have

$$\mathcal{C}_1(\cup_{i\in I}S_i) \subseteq \cup_{i\in I}\mathcal{C}_1(S_i).$$

Because

$$x \in \mathcal{C}_2(\cup_{i\in I}\mathcal{C}_1(S_i)) \cap \mathcal{C}_1(\cup_{i\in I}S_i),$$

and $\mathcal{C}_2$ verifies the Chernoff condition we conclude that

$$x \in \mathcal{C}_2(\mathcal{C}_1(\cup_{i\in I}S_i)) = (\mathcal{C}_2 \circ \mathcal{C}_1)(\cup_{i\in I}S_i)$$

which finishes the proof. □

As we have already mentioned, if we want to obtain a choice function that may not verify the necessary properties of rationality as the compound function of two rational ones, in such way that both of them must verify the Chernoff condition and the binariness property, we have to consider an initial choice function that verifies property γ. Our next theorem proves that this property together with property P characterize such functions.

Theorem 3.11. *Let $\mathcal{C}$ be a choice function on a domain $\mathcal{D}$ that contains all the finite and nonempty subsets of the set of alternatives. $\mathcal{C}$ is rational by two sequential criteria if and only if it verifies properties γ and P.*

Proof. We first prove that if $\mathcal{C}$ verifies properties P and γ, then it is rational by two sequential criteria.

Indeed let $\mathcal{C}$ be a choice function defined on a domain $\mathcal{D}$ satisfying the mentioned properties. Then we have by Theorem 3.10 that there exist an upper approximation and a lower approximation of it which verify the Chernoff condition and the binariness property (in fact they verify the stronger property γ), thus both of them are rational choice functions in the classical sense (see Theorem 3.5 and Remark 3.7). We consider these functions defined on the same domain $\mathcal{D}$ as $\mathcal{C}$ is defined on. Theorem 3.10 gives their respective expressions:

$$\mathcal{C}^u(S) = \{x \in S : \forall y \in S \text{ there exists } S' \in \mathcal{D} \text{ such that } x, y \in S' \text{ and } x \in \mathcal{C}(S')\}$$

and

$$\mathcal{C}^l(S) = \{x \in S : x \in \mathcal{C}(\{x, y\}),\ \forall y \in S\}.$$

Let us now prove that $\mathcal{C}_1 = \mathcal{C}^u$ and $\mathcal{C}_2 = \mathcal{C}^l$ satisfy $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$.

i) $\mathcal{C}(S) \subseteq (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$:

$x \in \mathcal{C}(S) \Rightarrow x \in \mathcal{C}_1(S)$, because we can select $S' = S$ for all $y \in S$.

Let us now suppose that $x \notin \mathcal{C}_2(\mathcal{C}_1(S))$.

In this case there exists $y \in \mathcal{C}_1(S)$ such that $\{y\} = \mathcal{C}(\{x, y\})$ because of the definition of $\mathcal{C}_2$.

From $y \in \mathcal{C}_1(S)$ we obtain that for all $s \in S$ there exists S_{ys} such that $y, s \in S_{ys}$ and $y \in \mathcal{C}(S_{ys})$.

As $\mathcal{C}$ verifies property γ we conclude that $y \in \mathcal{C}(\cup_{s \in S} S_{ys})$.

Then we have

$$\{x, y\} \subseteq S \subseteq \bigcup_{s \in S} S_{ys}$$

with $\{y\} = \mathcal{C}(\{x, y\})$ and $y \in \mathcal{C}(\cup_{s \in S} S_{ys})$.

Applying now that property P is verified by $\mathcal{C}$ we conclude that $x \notin \mathcal{C}(S)$, against the hypothesis.

Therefore we conclude that $x \in \mathcal{C}_2(\mathcal{C}_1(S))$.

ii) $\mathcal{C}(S) \supseteq (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$:
Select $x \in \mathcal{C}_2(\mathcal{C}_1(S))$, thus by the definition of $\mathcal{C}_2$ we have that $x \in \mathcal{C}(\{x, y\})$ for all $y \in \mathcal{C}_1(S)$. In particular $x \in \mathcal{C}(\{x, y\})$ for all $y \in \mathcal{C}(S)$ (because $\mathcal{C}(S) \subseteq \mathcal{C}_1(S)$). As we have that $\mathcal{C}$ verifies property γ we conclude that $x \in \mathcal{C}(\mathcal{C}(S)) \subseteq \mathcal{C}(S)$.

Conversely, let us prove that if $\mathcal{C}$ is the compound choice function of two rational choice functions, then $\mathcal{C}$ satisfies properties P and γ.

Let us denote by R_1 and R_2 the binary relations that rationalize $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, i.e.,

$$\mathcal{C}_1(S) = \{x \in S : (x, y) \in R_1,\ \forall y \in S\} = \mathcal{C}_{R_1}(S)$$

and

$$\mathcal{C}_2(S) = \{x \in S : (x, y) \in R_2,\ \forall y \in S\} = \mathcal{C}_{R_2}(S),$$

and $\mathcal{C}(S) = \mathcal{C}_2(\mathcal{C}_1(S)) = \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S))$.

Let us now prove that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ verifies property γ.

Let $\{S_i\}_{i\in I}$ be a collection of sets in the domain $\mathcal{D}$ such that $x \in \mathcal{C}(S_i)$ for all $i \in I$. We must prove that $x \in \mathcal{C}(\bigcup_{i\in I} S_i)$.

We have

$$x \in \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S_i)) \text{ for all } i \in I$$

and then

$$x \in \mathcal{C}_{R_1}(S_i) \ \ \forall i \in I$$

which implies

$$xR_1y \text{ for all } y \in S_i \ \ \forall i \in I \Rightarrow xR_1y \ \ \forall y \in \bigcup_{i\in I} S_i \Rightarrow x \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i).$$

If $x \notin \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(\bigcup_{i\in I} S_i))$, then there exists $z \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i)$ such that $\neg(xR_2z)$.

But on the other hand $x \in \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S_i))$ for all $i \in I$ and $z \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i) \subseteq \bigcup_{i\in I} \mathcal{C}_{R_1}(S_i)$ thus $z \in \mathcal{C}_{R_1}(S_i)$ for some $i \in I$ and thus xR_2z, which leads to a contradiction.

To end the proof we argue that $\mathcal{C}$ satisfies property P.

We take

$$\{x, y\} \subseteq S \subseteq T$$

such that

$$\{x\} = (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x, y\}) \text{ and } x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(T).$$

We prove that $y \notin \mathcal{C}(S)$.

Indeed let us now suppose that $y \in \mathcal{C}(S)$. Then we have

$(y, s) \in R_1$ for all $s \in S$ and $(y, s') \in R_2$ for all $s' \in S$ such that $s'R_1s$ for all $s \in S$.

As we also have that $x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(T)$ we obtain that

$(x, t) \in R_1$ for all $t \in T$.

We deduce that, in particular

i) $(y, x) \in R_1$ because $x \in S$ and $(x, y) \in R_1$ because $y \in T$, and

ii) $(y, x) \in R_2$ because xR_1t for all $t \in T$ and $S \subseteq T$.

Thus

$$\mathcal{C}_{R_1}(\{x, y\}) = \{x, y\} \text{ and } y \in \mathcal{C}_{R_2}(\{x, y\}) = \mathcal{C}(\{x, y\})$$

which contradicts the hypothesis. □

The next remark gives a different proof for the fact that the composition of two rational choice functions, both of them verifying that the choice is never empty, verifies properties P and γ without using the binary relations, but only with the properties verified by $\mathcal{C}_1$ and $\mathcal{C}_2$.

Remark 3.8. We can give a proof of the necessary condition of Theorem 3.11 when we suppose that both rational choice functions verify the condition of not making an empty choice for any subset of alternatives.

If $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are rational, they both verify the Chernoff condition and the binariness property. We conclude from Lemma 3.1 that $\mathcal{C}$ satisfies property γ.

Let us now prove that $\mathcal{C}$ satisfies property P.

We take

$$\{x, y\} \subseteq S \subseteq T$$

such that

$$\{x\} = (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x, y\}) \text{ and } x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(T).$$

We have to prove that $y \notin \mathcal{C}(S)$. We proceed by contradiction.

Let us suppose that

$$y \in \mathcal{C}(S) = (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$$

and therefore

$$y \in \mathcal{C}_1(S).$$

$\mathcal{C}_1$ verifies the Chernoff condition thus it must be $y \in \mathcal{C}_1(\{s, y\})$ for all $s \in S$, in particular $y \in \mathcal{C}_1(\{x, y\})$.

From $\{x\} = (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x, y\})$ we obtain that $x \in \mathcal{C}_1(\{x, y\})$ too, thus we conclude

$$\mathcal{C}_1(\{x, y\}) = \{x, y\}.$$

Then we have

$$\{x\} = (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x, y\}) = \mathcal{C}_2(\{x, y\}). \tag{3.6}$$

Moreover $x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(T) \Rightarrow x \in \mathcal{C}_1(T)$.

Because $\mathcal{C}_1$ satisfies the Chernoff condition it must be true that $x \in \mathcal{C}_1(S)$ and then

$$\{x, y\} \subseteq \mathcal{C}_1(S).$$

The facts that $y \in \mathcal{C}_2(\mathcal{C}_1(S))$ and $\mathcal{C}_2$ verifies the Chernoff condition entail $y \in \mathcal{C}_2(\{x, y\})$

which contradicts equation (3.6) and ends the argument. $\triangleleft$

Remark 3.9. In Remark 3.8 above we have not used the fact that the choice functions $\mathcal{C}_1$ and $\mathcal{C}_2$ verify the binariness property, and thus we have proved the next result:

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two choice functions that verify the Chernoff condition, then $\mathcal{C}_2 \circ \mathcal{C}_1$ satisfies property P.

Recall (Theorem 3.2) that when the two initial choice functions satisfy the Chernoff condition, the compound function of them must not verify it. We have reinforced the analysis by proving that at least it verifies the weaker property P. $\triangleleft$

As a corollary we can obtain the case of single-valued choice functions that has been studied by Manzini and Mariotti (2007). For such case property P becomes

$$\{x,y\} \subseteq S \subseteq T \text{ such that } \{x\} = \mathcal{C}(\{x,y\}) \text{ and } \{x\} = \mathcal{C}(T) \Rightarrow \{y\} \neq \mathcal{C}(S)$$

which is exactly the property that Manzini and Mariotti use (together with property γ) for characterizing "single-valued choice functions rationalized by two rationales". In fact they consider in property P the strict inclusions $\{x,y\} \subset S \subset T$, but when we deal with single-valued choice functions, $S = T$ implies obviously that $\mathcal{C}(T) = \{x\} \Rightarrow \mathcal{C}(S) \neq \{y\}$, and if $\{x,y\} = S$ then $\mathcal{C}(\{x,y\}) = \{x\} = \mathcal{C}(S) \Rightarrow \{y\} \neq \mathcal{C}(S)$. Moreover their approach is different in the sense that they consider that a choice function $\mathcal{C}$ is rational when there exists an acyclic binary relation P such that $\mathcal{C}(S) = \{x \in S | \nexists y \in S \text{ for which } (y,x) \in P\}$ instead of the definition of rational choice function that we use (see Definition 3.8), and they obtain asymmetric binary relations, property that is not verified by the relations that we obtain in the proof of Theorem 3.11.

The next example illustrates our result (Theorem 3.11) and at the same time proves that a choice function that verifies the two properties in the hypothesis of such Theorem 3.11 must not verify the Chernoff condition, thus it must not be a rational choice function.

Moreover we see that a choice function rational by two sequential criteria can be decomposed as the composition of two rational choice functions in more than one way.

Example 3.7. Let $X = \{x,y,z,t\}$ and we define the choice function $\mathcal{C}$ on $\mathcal{P}^*(X)$ as:

$$\begin{array}{lcl} \mathcal{C}(\{x,y\}) &=& \{x,y\} \\ \mathcal{C}(\{x,z\}) &=& \{x,z\} \\ \mathcal{C}(\{x,t\}) &=& \{x,t\} \\ \mathcal{C}(\{y,z\}) &=& \{y\} \\ \mathcal{C}(\{y,t\}) &=& \{y,t\} \\ \mathcal{C}(\{z,t\}) &=& \{z\} \end{array} \quad \begin{array}{lcl} \mathcal{C}(\{x,y,z\}) &=& \{x,y\} \\ \mathcal{C}(\{x,y,t\}) &=& \{x,y,t\} \\ \mathcal{C}(\{x,z,t\}) &=& \{x,z\} \\ \mathcal{C}(\{y,z,t\}) &=& \{y,t\} \end{array} \quad \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\}$$

This choice function satisfies property γ and property P. It does not satisfy the Chernoff condition ($\{x,z,t\} \subseteq \{x,y,z,t\}$ but $\mathcal{C}(\{x,y,z,t\}) \cap \{x,z,t\} = \{x,t\} \not\subseteq \{x,z\} = \mathcal{C}(\{x,z,t\})$).

The rational choice functions $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ are defined in this way following the line given in the proof of Theorem 3.11:

$$\begin{array}{lcl} \mathcal{C}_1(\{x,y\}) &=& \{x,y\} \\ \mathcal{C}_1(\{x,z\}) &=& \{x,z\} \\ \mathcal{C}_1(\{x,t\}) &=& \{x,t\} \\ \mathcal{C}_1(\{y,z\}) &=& \{y\} \\ \mathcal{C}_1(\{y,t\}) &=& \{y,t\} \\ \mathcal{C}_1(\{z,t\}) &=& \{z,t\} \end{array} \quad \begin{array}{lcl} \mathcal{C}_1(\{x,y,z\}) &=& \{x,y\} \\ \mathcal{C}_1(\{x,y,t\}) &=& \{x,y,t\} \\ \mathcal{C}_1(\{x,z,t\}) &=& \{x,z,t\} \\ \mathcal{C}_1(\{y,z,t\}) &=& \{y,t\} \end{array} \quad \mathcal{C}_1(\{x,y,z,t\}) = \{x,y,t\}$$

$$\begin{array}{lcl} \mathcal{C}_2(\{x,y\}) &=& \{x,y\} \\ \mathcal{C}_2(\{x,z\}) &=& \{x,z\} \\ \mathcal{C}_2(\{x,t\}) &=& \{x,t\} \\ \mathcal{C}_2(\{y,z\}) &=& \{y\} \\ \mathcal{C}_2(\{y,t\}) &=& \{y,t\} \\ \mathcal{C}_2(\{z,t\}) &=& \{z\} \end{array} \quad \begin{array}{lcl} \mathcal{C}_2(\{x,y,z\}) &=& \{x,y\} \\ \mathcal{C}_2(\{x,y,t\}) &=& \{x,y,t\} \\ \mathcal{C}_2(\{x,z,t\}) &=& \{x,z\} \\ \mathcal{C}_2(\{y,z,t\}) &=& \{y\} \end{array} \quad \mathcal{C}_2(\{x,y,z,t\}) = \{x,y\}$$

It is direct to prove that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$.

Nevertheless these choice functions $\mathcal{C}_1$ and $\mathcal{C}_2$ do not provide a unique solution to our problem as we can see by considering, for example, the choice function $\mathcal{C}_2'$ defined as follows.

$$\begin{array}{rclrcl}
\mathcal{C}_2'(\{x,y\}) & = & \{x,y\} & & & \\
\mathcal{C}_2'(\{x,z\}) & = & \{x,z\} & \mathcal{C}_2'(\{x,y,z\}) & = & \{x,y,z\} \\
\mathcal{C}_2'(\{x,t\}) & = & \{x,t\} & \mathcal{C}_2'(\{x,y,t\}) & = & \{x,y,t\} \\
\mathcal{C}_2'(\{y,z\}) & = & \{y,z\} & \mathcal{C}_2'(\{x,z,t\}) & = & \{x,z\} \\
\mathcal{C}_2'(\{y,t\}) & = & \{y,t\} & \mathcal{C}_2'(\{y,z,t\}) & = & \{y,z\} \\
\mathcal{C}_2'(\{z,t\}) & = & \{z\} & & &
\end{array} \text{ and } \mathcal{C}_2'(\{x,y,z,t\}) = \{x,y,z\}.$$

$\mathcal{C}_2'$ verifies property γ and the Chernoff condition. Some simple computations show that $\mathcal{C} = \mathcal{C}_2' \circ \mathcal{C}_1$. ◇

We finish with an example in which a single-valued choice function $\mathcal{C}$ is considered. Such function verifies properties γ and P and we construct $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Theorem 3.11 in such a way that $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$. In this example the choice function $\mathcal{C}_2$ verifies that $\mathcal{C}_2(S) = \varnothing$ for some $S \in \mathcal{D}$.

Example 3.8. Let $X = \{x,y,z\}$, $\mathcal{D} = \mathcal{P}^*(X)$ and $\mathcal{C}$ the choice function defined on $\mathcal{D}$ as

$$\begin{array}{rcl}
\mathcal{C}(\{x,y\}) & = & \{x\} \\
\mathcal{C}(\{x,z\}) & = & \{z\} \\
\mathcal{C}(\{y,z\}) & = & \{y\}
\end{array} \quad \mathcal{C}(\{x,y,z\}) = \{z\}.$$

$\mathcal{C}$ verifies the properties γ an P in a trivial way (and it does not verify the Chernoff condition) thus we can define $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Theorem 3.11. Their respective expressions are:

$$\begin{array}{rcl}
\mathcal{C}_1(\{x,y\}) & = & \{x\} \\
\mathcal{C}_1(\{x,z\}) & = & \{z\} \\
\mathcal{C}_1(\{y,z\}) & = & \{y,z\}
\end{array} \quad \mathcal{C}_1(\{x,y,z\}) = \{z\}$$

and

$$\begin{array}{rcl}
\mathcal{C}_2(\{x,y\}) & = & \{x\} \\
\mathcal{C}_2(\{x,z\}) & = & \{z\} \\
\mathcal{C}_2(\{y,z\}) & = & \{y\}
\end{array} \quad \mathcal{C}_2(\{x,y,z\}) = \varnothing.$$

$\mathcal{C}_1$ is rational because it verifies the Chernoff condition and the binariness property. $\mathcal{C}_2$ is rationalized by R_2 defined as

$$xR_2y \text{ and } \neg(yR_2x), zR_2x \text{ and } \neg(xR_2z), yR_2z \text{ and } \neg(zR_2y).$$

◇

3.5 Conclusions and future research

In this chapter we have considered the classical concept of rational choice function when we deal with a function that is rationalized by a single rational. Nevertheless we further consider as rational a decision-maker behavior that makes his choices applying different rational choice functions (in a classical sense) successively.

We have studied the choice functions that result when we compound choice functions verifying the different properties of rationality and analyze their behavior, that is we study which properties of rationality it verifies.

The following tables gather these results. In Table 1 we consider the case in which the choice functions are defined on domains containing all the finite and nonempty subsets of the set of alternatives. In Table 2 we summarize the case in which the domain is arbitrary. The notation we use is: A= Arrow's axiom, CH= Chernoff condition, C= Concordance property, IIA= Independence of Irrelevant Alternatives, SUP= Superset property, B= Binariness property, H= Houthakker's axiom of revealed preference, SR= Strong axiom of revealed preference, WA= Weak axiom of revealed preference, FR= full rational, QTR=quasi-transitive rational, AR=Acyclic rational and R= rational (for domains that contain all the finite and nonempty subsets of the set of alternatives AR=R). In both cases we have two tables, but the one on the right gathers the results obtained directly from the results on the table on the left and the rationality theorems.

Table 1. Results for domains containing all the finite and nonempty subsets of the set of alternatives.

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
A	A	A $\star$
A	CH	CH $\star$
A	C	C $\star$
A	IIA	IIA $\star$
A	SUP	SUP
B+CH	B	B

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
FR	FR	FR
FR	QTR	QTR
FR	R	R
QTR	QTR	B
R	R	B

The assertions $\star$ are proved in Aizerman and Aleskerov (1995). These authors also proved that Arrow's axiom is the only property unconditionally preserved by the composition of two choice functions in this list, and that it this not the case for properties C, CH and IIA. These are preserved when $\mathcal{C}_2$ satisfies the respective property and $\mathcal{C}_1$

satisfies Arrow's axiom. We have proved that the same is true for properties SUP and B and moreover that the binariness property is satisfied by $\mathcal{C}_2 \circ \mathcal{C}_1$ when it is satisfied by both $\mathcal{C}_1$ and $\mathcal{C}_2$ and $\mathcal{C}_1$ satisfies the Chernoff condition.

Table 2. Results for arbitrary domains.

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
H	H	H
H	SA	SA
H	WA+$R^*_{\mathcal{C}_2}$ acyclic	AR
WA	WA	WA

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
FR	FR	FR
FR	SA	QTR
FR	WA+$R^*_{\mathcal{C}_2}$ acyclic	AR
WA	WA	R

As far as we know this study does not appear in previous literature. We obtain in this case that the results are not very different from those in Table 1, bearing in mind that the conditions of rationality are only sufficient conditions except the case of FR in which H is a necessary and sufficient condition. The sufficient conditions for QTR (SA) and AR (WA and $R^*_{\mathcal{C}_2}$ acyclic) are not preserved when $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy them, but they are when they are satisfied by $\mathcal{C}_2$ and $\mathcal{C}_1$ is FR ($\Leftrightarrow$ satisfies H).

In the last part of this chapter we have dealt with the problem of the decomposition of a choice function by two choice functions that are rationalized by a single binary relation in the classical sense. We have obtained a characterization theorem for choice functions defined over domains that contain all the finite and nonempty subsets of the set of alternatives which is a common assumption to many related analysis.

For a future research we intend to extend this study to the case of the decomposition by more than two rational choice functions with weaker rationality properties. That should permit to check for a wider range of "rational" behavior.

Moreover the problem of the decomposition by two rational ones is not solved yet in the case of choice functions defined over arbitrary domains. A similar result to the one given in our characterization Theorem 3.9 for domains containing all the finite and nonempty subsets of the set of alternatives, or at least the identification of significant sufficient conditions as in such theorem would provide very interesting insights.

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Chapter 4

Cooperation in Markovian queueing models

Contents

4.1 Introduction **110**

4.2 Basic Markovian models **111**

4.3 Cooperation under preemptive priority **115**

4.4 The basic Markovian model with expected times in the queue **123**

4.4.1 The case $t_i^q = t^q$ for all $i \in N$ 129

4.5 Conclusions and future research **130**

4.6 Bibliography **132**

4.1 Introduction

The study of cooperation and competition in operational research models is a fruitful and challenging topic nowadays. Most fields within operations research are being approached from a game theoretical perspective, for the cases in which several decision makers interact in situations that can be modelled as optimization problems. Borm et al. (2001) provides a review of this topic.

One of the major branches within operations research is *queueing theory*. Competition in queueing models has been treated in many papers, a survey of which is Hassin and Haviv (2003) (for a survey in the control of queues, the reader is referred to Tadj and Choudhury (2005)). There are also a number of papers on cooperative issues in sequencing and scheduling (see, for instance, a review in Curiel et al. (2002) or other recent references such as Moulin and Stong (2002) and Maniquet (2003)). However, surprisingly enough, queueing models have rarely been approached from the point of view of cooperative game theory. González and Herrero (2004) is one of the scarce papers in which cooperation is analyzed in queueing models. It considers a Markovian situation in which several agents maintaining their own servers agree to cooperate and hold a common server for their populations. Each agent has specified a maximum value for the expected time in the system of the members of his population. The problem of how to allocate among the agents the cost of a common server, that fulfills the specification of each one, is dealt with, and applied to a cost sharing problem in the Spanish health system.

The study of cooperation in queueing models is a relevant issue which deserves the attention of game theorists and operation researchers. In many real world situations several providers of a particular service agree to maintain common servers which are available for all their populations: think of a group of banks which share a network of cash machines, a cluster of universities which hold one high-performance computer, or a set of hospitals keeping a joint blood bank. In all these situations questions like how to allocate the cost of the common servers or when a group of service providers should cooperate are really relevant and should be approached from a scientific point of view. This chapter is based on García-Sanz et al. (2008) which is devoted to deal with such questions in some Markovian models.

Moreover we have recently become aware of the paper of Yu et al. (2008) that also deals with cooperation in queueing systems. Nevertheless they do not consider different maximum values for the expected times in the system of the members of each populations, but the same for all of them, before and after cooperation.

The organization of this chapter is as follows. In Section 4.2 we set up our notation and analyze a variation of the model in González and Herrero (2004). In this model, each agent has a specification for the maximum time in the system and for the probability that one of his customers spends more than this maximum. In Section 4.3 we consider a new variation which allows for preemptive priority schemes to decrease the total cost. In this kind of problems it is usual to wonder how to share the earnings or costs of the grand coalition if all agents cooperate. So, in this context a rule for allocating the holding costs of the common server is introduced and axiomatically characterized. This rule can be easily computed and, moreover, provides core allocations. In section 4.4 we consider the model in Gonzalez and Herrero (2004), but in this case the agents are interested in the time in the queue instead of in the time in the system. In section 4.5 we conclude with some final comments.

4.2 Basic Markovian models

Consider a basic queueing system where customers arrive requiring a service, have to queue while the unique server is occupied, are selected from the queue by a certain discipline (i.e., a specification of the order in which they are selected), and leave the system after having been served. An $M/M/1$ model describes a system of this kind, when the arrivals occur according to a Poisson process with parameter λ (i.e., inter-arrival times are independent and identically distributed following an exponential distribution with mean $\frac{1}{\lambda}$), the service time follows an exponential distribution with average $\frac{1}{\mu}$, and the queue discipline is FCFS (first to come, first to be served). The steady state condition for this system is $\lambda < \mu$. From now on we only deal with $M/M/1$ systems in steady state. We assume that the reader is familiar with elementary issues of Markovian queues, more precisely, with the model $M/M/1$. Anyway, we briefly recall whenever needed some features in connection with that model (which is treated in deep, for instance, in Gross and Harris (1998)).

Consider a situation in which n agents run n $M/M/1$ systems which provide a similar service. Each agent $i \in N = \{1, \ldots, n\}$ runs his own queue and provides the service to his own population, λ_i, μ_i denoting the parameters characterizing agent i's $M/M/1$ system. Besides, each agent i wants that the average time that his customers spend in the system does not exceed a certain maximum value t_i. Moreover, the cost of maintaining a server is supposed to be a linear function of its efficiency, measured by the inverse of its expected service time (which, according to the properties of the exponential distribution, turns out to be the expected number of service completions per time unit), i.e. $c(i) = k\mu_i$, for all $i \in N$. Generally game theory deals with solutions which are invariant to scale changes, so we assume without loss of generality that $k = 1$. Now, since agents want to minimize the cost and since the expected time of a customer i in such an $M/M/1$ system is known to be $(\mu_i - \lambda_i)^{-1}$, then

$$t_i = \frac{1}{\mu_i - \lambda_i}$$

and thus

$$c(i) = \mu_i = \frac{1}{t_i} + \lambda_i.$$

González and Herrero (2004) considered the following question. How is the new situation if some agents agree to maintain one common server to attend their customers? They assume that this unique server should assure that the average time of a customer in the system is the lowest of the maximum admissible values for all the agents that make the arrangement (notice that this includes a feasibility assumption which guarantees that it is possible to ensure the desired service rate at the common server). If we take S the coalition of these agents and denote $t^S := \min\{t_i : i \in S\}$ and $\lambda_S := \sum_{i\in S} \lambda_i$, the cost of the unique server is

$$c(S) = \frac{1}{t^S} + \lambda_S. \tag{4.1}$$

Notice that $\sum_{i\in S} c(i) \geq c(S)$, so sharing the server in this way leads to a cost reduction. Equation (4.1) defines a cost TU-game (N, c). Remember that a cost TU-game is a pair (N, c), where N is a finite set of agents and c is the characteristic function, which assigns for every $S \subseteq N$ a real number $c(S)$ that indicates the cost of a particular project for the agents in coalition S, being $\mathcal{C}(\varnothing) = 0$ by convention. It is common to identify the game (N, c) with its characteristic function.

González and Herrero (2004) observe that c defined by (4.1) is the sum of an additive game plus an airport game. So, c is a concave game and its core is known to be the convex hull of the marginal contribution vectors. Moreover, its Shapley value $\Phi(c)$ can be easily computed and provides core allocations. For details on concave games and on airport games the reader can consult Owen (1995).

In the following we extend the model in (4.1) to deal not only with expected times. We consider the case where every agent i needs to guarantee for each of his customers that his time in the system will be smaller than or equal to a critical value ω_i with a sufficiently high probability $1 - \alpha_i$. In this case the cost of the unique server for coalition S is given in the following proposition.

Proposition 4.1. *In the conditions above, the cost of a common server which fulfills the conditions of the agents in S is given by:*

$$\hat{c}(S) = \lambda_S + \max_{i \in S}\{\frac{-\ln \alpha_i}{\omega_i}\}. \tag{4.2}$$

Proof. Let us denote by $\mathcal{W}_i$ the random variable "time in the system spent by a customer of type i". Therefore, for all $i \in N$, the condition

$$P(\mathcal{W}_i \leq \omega_i) \geq 1 - \alpha_i$$

must hold. It is a well-known result that the time that a customer spends in an $M/M/1$ system with parameters λ and μ follows an exponential distribution with mean $\frac{1}{\mu-\lambda}$. So, i will maintain a server with expected service time μ_i such that

$$P(\mathcal{W}_i \leq \omega_i) = 1 - e^{-(\mu_i - \lambda_i)\omega_i} = 1 - \alpha_i,$$

which implies that

$$\ln \alpha_i = -(\mu_i - \lambda_i)\omega_i$$

and thus

$$\mu_i = \lambda_i - \frac{\ln \alpha_i}{\omega_i}.$$

Now if a coalition S forms to maintain a common server which fulfills the specifications of all the agents, it should be satisfied that, for all $i \in S$,

$$P(\mathcal{W}_S \leq \omega_i) \geq 1 - \alpha_i,$$

where $\mathcal{W}_S$ is the random variable "time in the system spent by a customer of any agent in S". Then, the average service time μ of the server must satisfy for every $i \in S$

$$1 - e^{-(\mu - \sum_{i \in S} \lambda_i)\omega_i} \geq 1 - \alpha_i,$$

which implies that

$$\mu \geq \lambda_S - \frac{\ln \alpha_i}{\omega_i}$$

for all $i \in S$. So, for all $S \subset N$,

$$\hat{c}(S) = \lambda_S + \max_{i \in S}\{\frac{-\ln \alpha_i}{\omega_i}\}.$$

□

Equation (4.2) defines a cost TU-game $(N, \hat{c})$. We remark that our model includes as a particular case the cost game in González and Herrero (2004). Indeed, taking $\alpha_i = \frac{1}{e}$, for all $i \in N$, we obtain exactly the same game as in their paper. Moreover, we note again that $\hat{c}$ is the sum of an additive game plus an airport game which, once more, implies that $\hat{c}$ is concave, its core can be fully described and its Shapley value provides a specially noticeable core allocation. Following Littlechild and Owen (1973), the next corollary gives an explicit expression of the Shapley value in this context.

Corollary 4.1. *The Shapley value of the game $(N, \hat{c})$ is given by*

$$\Phi_{\pi(i)}(\hat{c}) = \frac{-\ln \alpha_{\pi(1)}}{n\omega_{\pi(1)}} + \frac{1}{n-1}\left(\frac{-\ln \alpha_{\pi(2)}}{\omega_{\pi(2)}} - \frac{-\ln \alpha_{\pi(1)}}{\omega_{\pi(1)}}\right) + \ldots +$$

$$+\frac{1}{n-i+1}\left(\frac{-\ln \alpha_{\pi(i)}}{\omega_{\pi(i)}} - \frac{-\ln \alpha_{\pi(i-1)}}{\omega_{\pi(i-1)}}\right) + \lambda_{\pi(i)}$$

for all $i \in N$, and where π is a permutation of N such that

$$\frac{-\ln \alpha_{\pi(1)}}{\omega_{\pi(1)}} \leq \frac{-\ln \alpha_{\pi(2)}}{\omega_{\pi(2)}} \leq \ldots \leq \frac{-\ln \alpha_{\pi(n)}}{\omega_{\pi(n)}}.$$

We finish this section with two remarks. The first has to do with the motivation of the new model treated here. The second is a technical comment.

Remark 4.1. The new model treated in this section is very natural and can be applied in many different scenarios, for instance in the cost sharing problem in the Spanish health system described in González and Herrero (2004). In fact, it is quite sensible to specify, for some specially delicate pathologies, a maximum value for the time in the system (with a high probability) instead of a maximum value for the expected time in the system. ◁

Remark 4.2. In this section we have considered queueing systems with an FCFS discipline. Actually, this assumption is only necessary to obtain expression (4.2) for $\hat{c}$. The expression for c given in this section is still valid if we simply assume that the system discipline satisfies the conservation law (see Kleinrock (1976) for details on the conservation law). ◁

4.3 Cooperation under preemptive priority

In this section we deal with the following question. Taking into account that the different players have different specifications for their populations, would it be helpful in order to diminish the cost of the common server that a priority scheme in the queue discipline is adopted?

We assume that the agents in N have agreed to run a common server to attend their customers. However, now we suppose that a priority scheme with n classes (one for each agent) has been established. In this section we always deal with priority schemes allowing preemption. Each class $i \in N$ corresponds to agent i, so it generates an expected number of clients per time unit λ_i, and it has a maximum value t_i for the expected waiting time in the system. We will moreover allow the use of *mixing* priority schemes. A mixing priority scheme (or *priority policy*) consists of multiplexing a finite set of priority schemes in such a way that each of them will operate during a desired percentage of time. The following theorem proves that in this context there always exists a priority policy whose associated cost is less than or equal to the cost of the FCFS system given in (4.1).

Theorem 4.1. *For any vector $(t_1, \ldots, t_n)$ of maximum expected waiting times in the system for the agents in N, there exists a priority policy that ensures these waiting times with a cost less than or equal to the one given by the approach in (4.1).*

Proof. Let us denote by $\Pi(N)$ the set of permutations of the finite set N. Let $\sigma \in \Pi(N)$ be an ordering of the n classes which establishes the priority scheme of the queue. Here $\sigma(i)$ represents the position which has been assigned to the class i. The smaller the position index, the higher priority associated to the class. It is well-known (see, for instance Gross and Harris (1998), page 233) that for any $\mu > \lambda_N$, the expected waiting time in the system for each class i under the priority scheme σ is

$$W_i(\sigma, \mu) = \frac{\mu}{(\mu - \sum\limits_{j:\sigma(j)<\sigma(i)} \lambda_j)(\mu - \sum\limits_{j:\sigma(j)\leq\sigma(i)} \lambda_j)}. \tag{4.3}$$

Notice that $W_i(\sigma, \mu)$ is a decreasing function of μ. We denote by $W(\sigma, \mu)$ the vector whose coordinates are given by (4.3) and by $\mathcal{F}(N, \mu)$ the set

$$\mathcal{F}(N, \mu) = conv \left\{ W(\sigma, \mu) \in \mathbb{R}^n : \sigma \in \Pi(N) \right\},$$

where *conv* stands for convex hull. We distinguish two cases.

Case 1. $t \in \mathcal{F}(N, \mu)$ for some $\mu > \lambda_N$.

Theorem 2 in Coffman and Mitrani (1980) established that $t = (t_1, \ldots, t_n)$ is achievable by some priority policy using a common server with a service rate μ if and only if $(t_1, \ldots, t_n) \in \mathcal{F}(N, \mu)$. Then, since t, λ and μ must satisfy the conservation law for queueing disciplines (see e.g. Kleinrock (1976), page 114) the following equation holds:

$$\sum_{i \in N} \frac{\lambda_i}{\lambda_N} t_i = \frac{1}{\mu - \lambda_N}. \tag{4.4}$$

Hence, the common service rate μ can be obtained solving equation (4.4). Its value is

$$\mu = \lambda_N + \frac{\lambda_N}{\sum_{i=1}^{n} \lambda_i t_i}. \tag{4.5}$$

Clearly,

$$\mu \leq \lambda_N + \frac{1}{t^N}, \tag{4.6}$$

so, in view of (4.1), the cost of the common server diminishes if a priority scheme is adopted under which the required vector $(t_1, \ldots, t_n)$ is in $\mathcal{F}(N, \mu)$.

Case 2. $t \notin \bigcup_{\mu > \lambda_N} \mathcal{F}(N, \mu)$.

From Lemma 2 in Coffman and Mitrani (1980), it is derived that any vector of expected waiting times in the system $(t_1, \ldots, t_n)$ with $t_i = t_j$ for every $i, j \in N$ belongs to the interior of $\mathcal{F}(N, \mu)$ for some μ. (Note that, in this case, $\mu = \lambda_N + \frac{1}{t^N}$.)

To prove the result, assume without loss of generality that $t_1 = t^N$. Let $l(t)$ be the line segment with extreme points t and $\hat{t} = (t_1, \ldots, t_1)$. The segment $l(t)$ is included in the halfspace $H^+ = \{x \in \mathbb{R}^n : \sum_{i \in N} \frac{\lambda_i}{\lambda_N} x_i \geq t^N\}$. Indeed, the hyperplane defining the halfspace H^+ contains $\hat{t}$ and its normal vector $(\frac{\lambda_1}{\lambda_N}, \ldots, \frac{\lambda_n}{\lambda_N}) \geq 0$. Thus, $\hat{t} + \mathbb{R}^n_+ \subset H^+$. Now, since clearly $l(t) \subset \hat{t} + \mathbb{R}^n_+$, the inclusion $l(t) \subset H^+$ follows.

The above construction proves that $l(t)$ intersects $\bigcup_{\mu > \lambda_N} \mathcal{F}(N, \mu)$ in a subsegment. All the points in that intersection, with the exception of $\hat{t}$, are attainable by priority policies with service rates smaller than $\frac{1}{t^N} + \lambda_N$, that corresponds to the policy attaining $\hat{t}$. (Notice that the service rate decreases while $\|W\|$ increases along the ray $\{x \in \mathbb{R}^n_+ : x_1 = x_2 = \ldots = x_n > 0\}$, see (4.3). An illustration can be found in Figure 2.)

Hence, any service rate μ^* associated with a point

$$t^* \in (l(t) \setminus \{\hat{t}\}) \cap \bigcup_{\mu > \lambda_N} \mathcal{F}(N, \mu)$$

satisfies the aspiration level given by t and with a service rate smaller than the one in

(4.1), namely $\frac{1}{t^N} + \lambda_N$. $\square$

Now we illustrate the result above for the two-classes situation. Here, the extreme points of the set $\mathcal{F}(N,\mu)$ are given by

$$(\frac{1}{\mu-\lambda_1}, \frac{\mu}{(\mu-\lambda_1)(\mu-\lambda_N)}), (\frac{\mu}{(\mu-\lambda_2)(\mu-\lambda_N)}, \frac{1}{\mu-\lambda_2}).$$

Of course, it must hold that $\mu > \lambda_N = \lambda_1 + \lambda_2$. Figure 1 illustrates this result where $\lambda_1 = \lambda_2 = 1$.

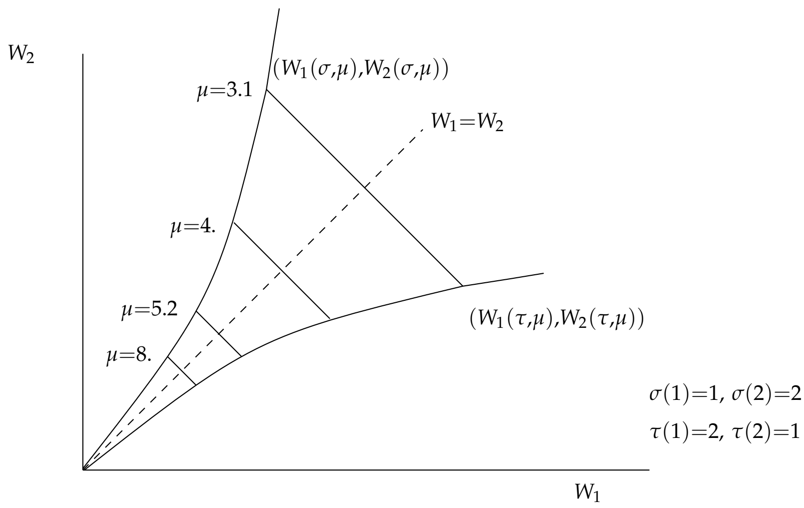

Figure 1: Vectors of achievable expected waiting times in the system.

According to Theorem 2 in Coffman and Mitrani (1980), any (t_1, t_2) which lies inside the region limited by the curves corresponding to the orderings σ and τ is achievable using a certain priority policy, by a server with common service rate

$$\mu = \lambda_1 + \lambda_2 + \frac{\lambda_1 + \lambda_2}{\lambda_1 t_1 + \lambda_2 t_2},$$

as it is derived from the conservation law (4.4).

Figure 2 displays the case where $(t_1, t_2) \notin \bigcup_{\mu > \lambda_N} \mathcal{F}(N,\mu)$. The cost associated with $\bar{t}$, according to Theorem 4.1, is less than or equal to $\frac{1}{t^N} + \lambda_N$.

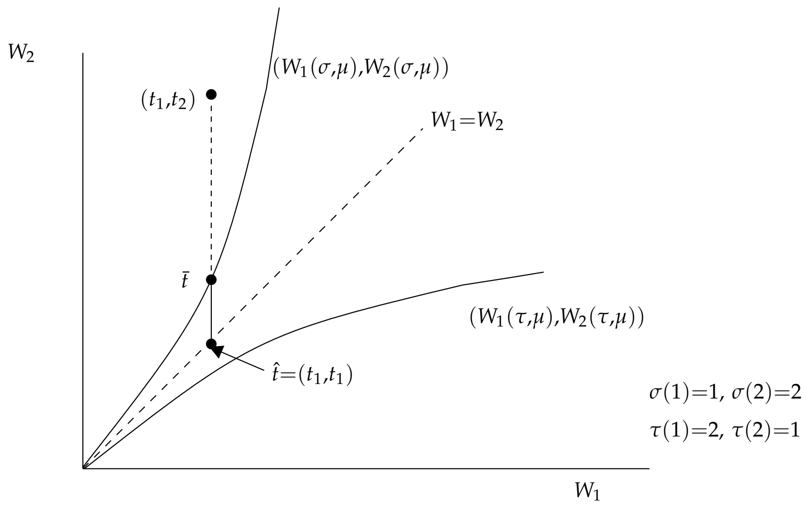

Figure 2: The case where $(t_1, t_2) \notin \mathcal{F}(N, \mu)$.

From now on we consider problems where the expected waiting time vector in the system $t = (t_i)_{i \in N}$ is achievable. (Notice that t being achievable implies that for any $S \subset N$ then $(t_i)_{i \in S}$ is achievable as well.) Let us denote by QS the set of queueing situations $(N, \{\lambda_i\}_{i \in N}, \{t_i\}_{i \in N})$ such that N is finite and $(t_i)_{i \in N} \in \mathcal{F}(N, \mu)$ with $\mu = \lambda_N + \frac{\lambda_N}{\sum_{i \in N} \lambda_i t_i}$. Then, the maintenance cost of a common server for any coalition $S \subset N$ is given by

$$\bar{c}(S) = \lambda_S + \frac{\lambda_S}{\sum_{i \in S} \lambda_i t_i}. \tag{4.7}$$

Notice that, as we have already remarked, $\bar{c}(N)$ is smaller than or equal to the total cost in González and Herrero's model.

The problem now is how to allocate $\bar{c}(N)$ among the agents. In order to do it, we consider the cost TU-game $(N, \bar{c})$ given by equation (4.7). Observe first that, for each $S \subset N$,

$$\bar{c}(S) - \sum_{i \in S} \bar{c}(\{i\}) = \bar{c}(S) - \sum_{i \in S} (\lambda_i + \frac{1}{t_i}) \leq 0.$$

In the class of queueing situations QS, an allocation rule f is a function which associates to each $(N, \{\lambda_i\}_{i \in N}, \{t_i\}_{i \in N}) \in QS$ a non-negative vector in $\mathbb{R}^N$, denoted by $f(N, \{\lambda_i\}_{i \in N}, \{t_i\}_{i \in N})$, such that the sum of its components equals $\bar{c}(N)$. We define the *proportional allocation rule*, denoted by φ^p, as

$$\varphi_i^p(N, \{\lambda_i\}_{i \in N}, \{t_i\}_{i \in N}) = \lambda_i + \frac{\lambda_i}{\sum_{j \in N} \lambda_j t_j}. \tag{4.8}$$

According to this rule, each agent $i \in N$ pays an additive part λ_i plus a splitting of $\frac{\lambda_N}{\sum\limits_{j\in N}\lambda_j t_j}$ proportional to λ_i. Notice that

$$\lambda_i + \frac{\lambda_i}{\sum\limits_{j\in N}\lambda_j t_j} = \frac{\lambda_i}{\lambda_N}\bar{c}(N)$$

so φ^p can also be said to allocate to each agent a splitting of $\bar{c}(N)$ proportional to λ_i.

An important property for an allocation rule f is that it provides *core allocations*. In this context this means that, for every $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ and every $S \subset N$,

$$\sum_{i\in S} f_i(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \leq \bar{c}(S)$$

or, in words, that the allocation of the total cost $\bar{c}(N)$ is acceptable for every coalition $S \subset N$. The following proposition shows that the proportional allocation rule in fact provides core allocations.

Proposition 4.2. *φ^p provides core allocations.*

Proof. For each coalition $S \neq \emptyset$, the difference $\sum\limits_{i\in S}\varphi_i^p(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) - \bar{c}(S)$ is

$$\sum_{i\in S}\frac{\lambda_i}{\sum\limits_{j\in N}\lambda_j t_j} - \sum_{i\in S}\frac{\lambda_i}{\sum\limits_{j\in S}\lambda_j t_j} \leq 0.$$

□

An obvious consequence of Proposition 4.2 is that each game $\bar{c}$ associated with an element of QS is totally balanced. The following example shows that $\bar{c}$ needs not to be concave and also that the Shapley value of $\bar{c}$ may fall outside its core.

Example 4.1. Take $N = \{1, 2, 3\}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and $t_1 = 1$, $t_2 = 47.29$ and $t_3 = 53.71$. After some algebra, it is easy to check that $(N, \{\lambda_i\}_{i\in n}, \{t_i\}_{i\in n}) \in QS$. Then we have:

- $\bar{c}(\{1\}) = 2$, $\bar{c}(\{2\}) = 1.021$, $\bar{c}(\{3\}) = 1.019$,
- $\bar{c}(\{1, 2\}) = 2.041$, $\bar{c}(\{1, 3\}) = 2.037$, $\bar{c}(\{2, 3\}) = 2.02$,
- $\bar{c}(\{1, 2, 3\}) = 3.029$.

Consider $S = \{1\} \subset T = \{1,2\}$ and $i = 3$. Then

$$\bar{c}(S \cup \{i\}) - \bar{c}(S) < \bar{c}(T \cup \{i\}) - \bar{c}(T),$$

so $\bar{c}$ is not a concave game. Moreover, if we compute the Shapley value of this game we obtain that $\Phi(\bar{c}) = (1.343, 0.845, 0.841)$, which is not a core allocation because $\Phi_1(\bar{c}) + \Phi_2(\bar{c}) = 2.188 > \bar{c}(\{1,2\}) = 2.041$. $\diamond$

In summary, φ^p is a reasonable allocation rule that (a) can be easily computed and (b) provides core allocations. So, this rule is our proposal for allocating the maintenance cost of the common server in this context. We finish the section providing an axiomatic characterization of this rule which shows that it has excellent properties from the point of view of the immunity to possible manipulations.

To start with, let us introduce two appealing properties for an allocation rule f defined on QS.

P1. Non advantageous reallocation. Let the queueing situations $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ and $(N, \{\tilde{\lambda}_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) \in QS$ be such that $\sum\limits_{i\in N} \lambda_i t_i = \sum\limits_{i\in N} \tilde{\lambda}_i \tilde{t}_i$ and $\lambda_N = \tilde{\lambda}_N$. Then

$$\sum_{i\in T} f_i(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = \sum_{i\in T} f_i(N, \{\tilde{\lambda}_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N})$$

for any $T \subset N$ with $\lambda_T = \tilde{\lambda}_T$.

The meaning of this property is that a rule should be invariant to reallocations of the parameters λ_i within any coalition T while keeping the total cost. This reallocation is one possible way in which a certain coalition can manipulate its parameters to obtain some advantage. Another possible way is performing artificial mergings or splittings. These manipulations are prevented by the next property. Before its introduction we need the following definition.

Definition 4.1. *Let $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ be such that $t_i = t$, for every $i \in N$. Then for each $S \subset N$, the S-manipulation of $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N})$ is the queueing situation $(N^S, \{\lambda_i\}_{i\in N^S}, \{t_i\}_{i\in N^S}) \in QS$ where*

- $N^S = (N \setminus S) \cup \{i_S\}$,
- $\lambda_{i_S} = \sum_{i\in S} \lambda_i$, *and*
- $t_{i_S} = t$.

Notice that, in these conditions, $\bar{c}(N) = \bar{c}(N^S)$ for every $S \subset N$. Now we present the second property.

P2. Non advantageous merging or splitting. Let $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ be such that $t_i = t$, for every $i \in N$. Then, for each $S \subset N$,

$$f_{i_S}(N^S, \{\lambda_i\}_{i\in N^S}, \{t_i\}_{i\in N^S}) = \sum_{i\in S} f_i(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}).$$

It is clear that the proportional allocation rule φ^p satisfies P1 and P2. Moreover, the next theorem shows that these two properties characterize the proportional allocation rule.

Theorem 4.2. *The proportional allocation rule φ^p is the unique allocation rule defined on QS which satisfies P1 and P2.*

Proof. We have already mentioned that φ^p satisfies P1 and P2. Let us check its uniqueness. Take an allocation rule f defined on QS which satisfies P1 and P2. Consider $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ and fix arbitrarily $j \in N$. We have to prove that

$$f_j(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = \lambda_j + \frac{\lambda_j}{\sum\limits_{i\in N} \lambda_i t_i}.$$

We define for all $i \in N$

$$\tilde{t}_i = \sum_{k\in N} \frac{\lambda_k}{\lambda_N} t_k = \tilde{t};$$

observe that P1 implies that

$$f_j(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) \tag{4.9}$$

(notice that P1 can be applied because $\sum_{i\in N} \lambda_i \tilde{t}_i = \sum_{i\in N} \lambda_i t_i$).
Now take the S-manipulation of $(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N})$ for $S = N \setminus \{j\}$. We know that:

- $\bar{c}(N^S) = \bar{c}(N), \quad (4.10)$
- $\bar{c}(N^S) = f_j(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i\}_{i\in N^S}) + f_{i_S}(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i\}_{i\in N^S}), \quad (4.11)$
- $\bar{c}(N) = f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) + \sum\limits_{k\neq j} f_k(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}). \quad (4.12)$

Since (4.10), (4.11), and (4.12) hold, and f satisfies P2, then

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) = f_j(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i\}_{i\in N^S}).$$

Hence, it is clear that $f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N})$ can be written as a function of λ_N, λ_j and $\tilde{t}_j$ (notice that $\tilde{t}_j = \tilde{t}$ does not really depend on j). So,

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) = F(\lambda_N, \lambda_j, \tilde{t}).$$

Suppose that F is linear in its second variable (we will prove below that this is actually true). Then,

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) = g(\lambda_N, \tilde{t})\lambda_j. \tag{4.13}$$

Thus

$$\begin{aligned} \bar{c}(N) &= \sum_{j\in N} f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i\}_{i\in N}) \\ &= g(\lambda_N, \tilde{t})\lambda_N, \end{aligned}$$

and so $g(\lambda_N, \tilde{t}) = \frac{\bar{c}(N)}{\lambda_N}$. Now, in view of (4.9) and (4.13), we get:

$$f_j(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = \frac{\bar{c}(N)}{\lambda_N}\lambda_j = \lambda_j + \frac{\lambda_j}{\sum\limits_{i\in N} \lambda_i t_i}.$$

So, to finish the proof we just need to check that F is linear in its second variable. Notice that we have a collection of functions

$$\{F(\alpha, \cdot, \beta) | \alpha, \beta \in (0, +\infty)\}$$

such that $F(\alpha, \cdot, \beta) : (0, \alpha] \longrightarrow [0, \alpha + \frac{1}{\beta}]$, for all $\alpha, \beta \in (0, +\infty)$. Let us take now $\alpha, \beta, x, y \in (0, +\infty)$ with $x + y \leq \alpha$. Then, there exists $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ where $t_i = \beta$ for every $i \in N$, $\lambda_1 = x$, and $\lambda_2 = y$. Define the S-manipulation of this problem for $S = \{1, 2\}$. Then, since f satisfies P2,

$$\begin{aligned} F(\alpha, x + y, \beta) &= f_{i_S}(N^S, \{\lambda_i\}_{i\in N^S}, \{t_i\}_{i\in N^S}) \\ &= f_1(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) + f_2(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \\ &= F(\alpha, x, \beta) + F(\alpha, y, \beta). \end{aligned}$$

So, for every $\alpha, \beta \in (0, +\infty)$, $F(\alpha, \cdot, \beta)$ is additive. Since $F(\alpha, \cdot, \beta)$ is also non-negative, it is clear that it is increasing. It is an easy exercise to prove that every additive, increasing function $h : (0, \alpha] \longrightarrow [0, \alpha + \frac{1}{\beta}]$ is also linear. This completes the proof. □

Finally we check that these two properties are independent.

(i) The rule f^1 which assigns to each queueing situation $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ the vector whose jth coordinate is given by

$$f_j^1(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = \frac{\bar{c}(N)}{|N|},$$

where $|N|$ is the number of agents in N, satisfies P1, but it does not satisfy P2.

(ii) The rule f^2 which assigns to each queueing situation $(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) \in QS$ the vector whose jth coordinate is given by

$$f_j^2(N, \{\lambda_i\}_{i\in N}, \{t_i\}_{i\in N}) = \frac{\lambda_j t_j}{\sum\limits_{i\in N} \lambda_i t_i} \bar{c}(N)$$

satisfies P2, but it does not satisfy P1.

4.4 The basic Markovian model with expected times in the queue

When we need a service and have to queue in order to get it, many of the times we are very worried about the time we have to spend in the queue, but not so much about the service time. For example, in the context of a health system that we have already mentioned above, it is reasonable to deal with times in the queue instead of times in the system, because the times that what we really want to diminish are the waiting times for a surgical intervention instead of the time of the surgery itself.

This section shows some difficulties when extending the results in section 4.2 to the case in which the agents are concerned with expected times *in the queue* instead of in the system. Surprisingly enough, this slight variation leads to a scenario in which sometimes it is better for the agents not to cooperate.

Consider again the same situation as in González and Herrero (2004) but in such a way that now each agent i has a maximum admissible value $t_i^q \in (o, +\infty)$ for the expected time of his customers in the queue. In a stationary $M/M/1$ system with parameters λ and μ, the average waiting time in the queue by a customer is given by

$$\frac{\lambda}{\mu(\mu - \lambda)}.$$

So, each i will choose a server with an expected service time μ_i such that

$$t_i^q \mu_i^2 - t_i^q \mu_i \lambda_i - \lambda_i = 0$$

which implies that

$$\mu_i = \frac{t_i^q \lambda_i \pm \sqrt{\left(t_i^q\right)^2 \lambda_i^2 + 4\lambda_i t_i^q}}{2t_i^q} = \frac{\lambda_i}{2} \pm \sqrt{\left(\frac{\lambda_i}{2}\right)^2 + \frac{\lambda_i}{t_i^q}}.$$

From the fact that $\mu_i > \lambda_i$ for all $i \in N$, it follows that only the positive square root is possible, and rewriting the expression, the cost of maintaining a server i in this situation is

$$c^q(i) = \frac{\lambda_i}{2} + \frac{\lambda_i}{2}\sqrt{1 + \frac{4}{\lambda_i t_i^q}}.$$

If coalition $S \subseteq N$ forms, all the agents in S assume that the common server should assure that the average time in the queue of a client is the lowest of all the maximum admissible values for all the agents that have made the arrangement, and then

$$c^q(S) = \frac{\lambda_S}{2} + \sqrt{\left(\frac{\lambda_S}{2}\right)^2 + \frac{\lambda_S}{t^{qS}}} = \frac{\lambda_S}{2} + \frac{\lambda_S}{2}\sqrt{1 + \frac{4}{\lambda_S t^{qS}}} \tag{4.14}$$

where $t^{qS} = \min_{i \in S}\{t_i^q\}$, and $\lambda_S = \sum_{i \in S} \lambda_i$.

The following example shows that in a situation like this, players may prefer not to cooperate.

Example 4.2. Take $N = \{1, 2\}$ and $\lambda_1 = \lambda_2 = 1$, $t_1^q = 100$, $t_2^q = 1$. Then:

- $c^q(N) = 1 + \sqrt{1 + \frac{2}{1}} = 1 + \sqrt{3}$.
- $c^q(1) + c^q(2) = \frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{100}} + \frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{1}} = 1 + \sqrt{0.26} + \sqrt{1.25}$.

Hence, $c^q(1) + c^q(2) < c^q(N)$. ◇

So, in the case that the agents are concerned with the time their customers spend in the queue, instead of with the time their customers spend in the system, maybe they will not have incentives to cooperate, at least under the conditions considered up to now. The next proposition gives a sufficient condition that makes cooperation to be a good option.

Proposition 4.3. *A sufficient condition in order that $\sum_{i \in S} c^q(i) \geq c^q(S)$ for a coalition $S \subset N$ is that*

$$\lambda_i t_i^q \leq \lambda_S t^{qS} \tag{4.15}$$

for all $i \in S$.

The proof is immediate if the terms $\sum_{i \in S} \frac{\lambda_i}{2} \sqrt{1 + \frac{4}{\lambda_i t_i^q}}$ and $\frac{\lambda_S}{2} \sqrt{1 + \frac{4}{\lambda_S t^{qS}}}$ are compared.

Taking $t_1^q = 4$ in the Example 4.2 above, one checks that the condition in Proposition 4.3 is not necessary.

The interpretation of condition (4.15) is clear. It says that the common server has to be able to take on more work than each one of the individual servers whilst maintaining expected sojourn time guaranties for the individual agents. In particular, this is true when the values t_i^q are homogeneous (notice that in Example 4.2 above t_1^q and t_2^q are strongly discrepant).

The problem now is how to allocate the total cost $c^q(N)$ in those cases in which condition 4.15 is satisfied, and therefore the agents have incentives to cooperate.

Let us denote by QS^q the set of queueing situations $(N, \{\lambda_i\}_{i \in N}, \{t_i^q\}_{i \in N})$ such that N is finite and $\lambda_i t_i^q \leq \lambda_S t^{qS}$ for all $i \in S$. If all the agents in N agree to maintain a common server, the cost they will have to assume is $c^q(N) = \frac{\lambda_N}{2} + \frac{\lambda_N}{2} \sqrt{1 + \frac{4}{\lambda_N t^{qN}}}$. In order to allocate this total cost among the agents, we consider the cost TU-game (N, c^q) with characteristic function given by equation 4.14.

Definition 4.2. *In the class of queueing situations QS^q, an allocation rule f is a function which associates to each $(N, \{\lambda_i\}_{i \in N}, \{t_i^q\}_{i \in N}) \in QS^q$ a non-negative vector in $\mathbb{R}^N$, denoted by $f(N, \{\lambda_i\}_{i \in N}, \{t_i^q\}_{i \in N})$, such that the sum of its components equals $c^q(N)$.*

The following example shows that this game is not necessarily a concave game.

Example 4.3. Take $N = \{1, 2, 3\}$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 4$, and $t_1^q = 7$, $t_2^q = 8$ and $t_3^q = 3$.

Then we have:

- $c^q(\{1\}) = 3,136634177$, $c^q(\{2\}) = 2,118033989$, $c^q(\{3\}) = 4,309401077$,
- $c^q(\{1, 2\}) = 5,138993315$, $c^q(\{1, 3\}) = 7,318813079$, $c^q(\{2, 3\}) = 6,31662479$,
- $c^q(\{1, 2, 3\}) = 9,32182538$.

Consider $S = \{1\} \subset T = \{1, 2\}$ and $i = 3$. Then

$$c^q(S \cup \{i\}) - c^q(S) < c^q(T \cup \{i\}) - c^q(T),$$

and this implies that c^q is not a concave game. ◇

So, the Shapley value is not necessarily a core allocation, and moreover the expression of this game makes it a bit difficult to calculate. Then we propose another rule in order to allocate the cost $c^q(N)$ when the sufficient condition (4.15) is true. We define a proportional allocation rule as

$$\varphi_i^q(c^q) = \frac{\lambda_i}{2} + \frac{\lambda_i}{2}\sqrt{1 + \frac{4}{\lambda_N t^{q^N}}} = \frac{\lambda_i}{\lambda_N} c^q(N)$$

This rule assigns to each agent i a splitting of $c^q(N)$ proportional to λ_i.

Nevertheless this value is not necessarily either a core allocation. For instance, if we calculate $\varphi^q(c^q)$ in the Example 4.3 above we obtain

$$\varphi^q(c^q) = (3,107275127, 2,071516751, 4,143033502),$$

which is not a core allocation because

$$\varphi_1^q(c^q) + \varphi_2^q(c^q) > \bar{c}^q(\{1,2\}).$$

The next proposition gives a necessary and sufficient condition for the proportional rule to be a core allocation.

Proposition 4.4. *A necessary and sufficient condition in order that*

$$\sum_{i \in S} \varphi_i^q(N, \{\lambda_i\}_{i \in N}, \{t_i^q\}_{i \in N}) \leqslant c^q(S)$$

is that

$$\lambda_S t^{q^S} \leqslant \lambda_N t^{q^N}$$

for all $S \subseteq N$.

The proof is immediate if the terms $\sum_{i \in S} \frac{\lambda_i}{2} + \sum_{i \in S} \frac{\lambda_i}{2}\sqrt{1 + \frac{4}{\lambda_N t^{q^N}}}$ and $c^q(S) = \frac{\lambda_S}{2} + \frac{\lambda_S}{2}\sqrt{1 + \frac{4}{\lambda_S t^{q^S}}}$ are compared.

This condition states that the common server for the whole coalition has to be able to take on more work than the common sever for any other coalition $S \subset N$. In particular it is true if all the values t_i^q are homogeneous, as it happened with the sufficient condition 4.3.

Besides this proportional rule we have proposed can be easily computed and verifies excellent properties from the point of view of the immunity to possible manipulations.

We go on providing an axiomatic characterization of it in a similar way as we characterize the proportional rule given by equation 4.2 in section 4.3.

To start with, let us redefine the two properties introduced in section 4.3 for characterizing the rule φ^p to the case of an allocation rule f defined on QS^q. We omit their intuitions because of the similarity with the properties P1 and P2 in section 4.3.

Q1. Non advantageous reallocation. Let the queueing situations $(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) \in QS^q$ and $(N, \{\tilde{\lambda}_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) \in QS^q$ be such that $\lambda_N = \tilde{\lambda}_N$ and $t^{q^N} = \tilde{t}^{q^N}$. Then

$$\sum_{i\in T} f_i(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) = \sum_{i\in T} f_i(N, \{\tilde{\lambda}_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N})$$

for any $T \subset N$ with $\lambda_T = \tilde{\lambda}_T$.

Definition 4.3. *Take $(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) \in QS^q$ where $t_i^q = t^q$ for all $i \in N$. For each subset $S \subset N$, the S-manipulation of $(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N})$ is the new queueing situation $(N^S, \{\lambda_i\}_{i\in N^S}, \{t_i^q\}_{i\in N^S}) \in QS^q$ where*

- $N^S = (N \setminus S) \cup \{i_S\}$,
- $\lambda_{i_S} = \sum_{i\in S} \lambda_i$, *and*
- $t_{i_S}^q = t^q$.

Notice that, in these conditions, $c^q(N) = c^q(N^S)$ for every $S \subset N$. Now we present the second property.

Q2. Non advantageous merging or splitting. Take $(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) \in QS^q$ where $t_i^q = t^q$ for every $i \in N$. Then, for each $S \subset N$,

$$f_{i_S}(N^S, \{\lambda_i\}_{i\in N^S}, \{t_i^q\}_{i\in N^S}) = \sum_{i\in S} f_i(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}).$$

It is clear that the proportional allocation rule φ^q satisfies Q1 and Q2. Moreover, the next theorem shows that these two properties characterize it.

Theorem 4.3. *The proportional allocation rule φ^q is the unique allocation rule defined on QS^q which satisfies Q1 and Q2.*

Proof. We have already mentioned that φ^q satisfies Q1 and Q2. Let us check its uniqueness. Take an allocation rule f defined on QS^q which satisfies Q1 and Q2. Consider

$(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) \in QS^q$ and fix arbitrarily $j \in N$. We have to prove that

$$f_j(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) = \frac{\lambda_j}{2} + \frac{\lambda_j}{2}\sqrt{1 + \frac{4}{\lambda_N t^{q^N}}}.$$

We define for all $i \in N$, $\tilde{t}_i{}^q = \min_{k\in N}\{t_k^q\}$. Then Q1 implies that

$$f_j^q(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) = f_j^q(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}). \tag{4.16}$$

Now take the S-manipulation of $(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N})$ for $S = N \setminus \{j\}$. We know that:

- $c^q(N^S) = c^q(N),$ (4.17)
- $c^q(N^S) = f_j(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i^q\}_{i\in N^S}) + f_{i_S}(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i^q\}_{i\in N^S}),$ (4.18)
- $c^q(N) = f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) + \sum_{k\neq j} f_k(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}).$ (4.19)

Since (4.17), (4.18), and (4.19) hold, and f satisfies Q2, then

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) = f_j(N^S, \{\lambda_i\}_{i\in N^S}, \{\tilde{t}_i^q\}_{i\in N^S}).$$

Hence, it is clear that $f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N})$ can be written as a function of λ_N, λ_j and $\tilde{t}_j^q$ (notice that $\tilde{t}_j^q = \tilde{t}^q$ does not really depend on j). So,

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) = F(\lambda_N, \lambda_j, \tilde{t}^q).$$

Just as in the proof of Theorem 4.2 we have that F is linear in its second variable (we only have to consider $F(\alpha, \cdot, \beta) : (0, \alpha] \longrightarrow [0, \frac{\alpha}{2} + \frac{\alpha}{2}\sqrt{1 + \frac{4}{\alpha\beta}}]$, for all $\alpha, \beta \in (0, +\infty)$). Then,

$$f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) = g(\lambda_N, \tilde{t}^q)\lambda_j. \tag{4.20}$$

Thus

$$\begin{aligned} c^q(N) &= \sum_{j\in N} f_j(N, \{\lambda_i\}_{i\in N}, \{\tilde{t}_i^q\}_{i\in N}) \\ &= g(\lambda_N, \tilde{t}^q)\lambda_N, \end{aligned}$$

and so $g(\lambda_N, \tilde{t}^q) = \frac{c^q(N)}{\lambda_N}$. Now, in view of (4.16) and (4.20), we get:

$$f_j(N, \{\lambda_i\}_{i\in N}, \{t_i^q\}_{i\in N}) = \frac{c^q(N)}{\lambda_N}\lambda_j = \frac{\lambda_j}{2} + \frac{\lambda_j}{2}\sqrt{1 + \frac{4}{\lambda_N t^{q^N}}}.$$

□

Moreover, also as in section 4.3 we have that these two properties are independent.

We conclude this section with the particular case in which all the agents fix the same maximum values for the expected times that their agents spend in the queue.

4.4.1 The case $t_i^q = t^q$ for all $i \in N$

In this section we consider different n parallel $M/M/1$ systems with parameters λ_i and μ_i, and all of them with the same maximum value for their customers waiting in the queue $t_i^q = t^q$.

In this case it is clear that the sufficient condition (4.15) for cooperation is true. Moreover it is also true that $\lambda_S t^q \leqslant \lambda_N t^q$, thus $\sum_{i\in S} \varphi_i(c^q) - c^q(S) \leqslant 0$ and therefore $\varphi^q(c^q)$ is a core allocation. In fact, under this condition the rule $\varphi^q(c^q)$ is characterized only by property Q2, as it is established in the next corollary.

Corollary 4.2. *The allocation rule φ^q defined over the subset of QS^q where $t_i^q = t^q$ for all $i \in N$ is the only allocation rule over this subset that verifies Q2.*

We also have that, under the condition of $t_i^q = t^q$ for every $i \in N$, the cost game c^q is concave as the next proposition states.

Proposition 4.5. *In the conditions above the cost game c^q is a concave game.*

Proof. We have to prove that for any $S, T \subseteq N$ such that $S \subseteq T$ and $i \notin T$

$$c^q(S \cup \{i\}) - c^q(S) \geqslant c^q(tT \cup \{i\}) - c^q(T).$$

We have

$$c^q(S \cup \{i\}) - c^q(S) = \sqrt{\frac{(\lambda_S + \lambda_i)^2}{4} + \frac{\lambda_S + \lambda_i}{t^q}} - \sqrt{\frac{\lambda_S^2}{4} + \frac{\lambda_S}{t^q}}.$$

Let us consider the real function

$$f(x) = \sqrt{\frac{(x + \lambda_i)^2}{4} + \frac{x + \lambda_i}{t^q}} - \sqrt{\frac{x^2}{4} + \frac{x}{t^q}}.$$

We have that $f'(x) < 0$. If we take $x = \lambda_S$ and $y = \lambda_T$ we have $\lambda_S \leqslant \lambda_T$ and we obtain

$$c^q(S \cup \{i\}) - c^q(S) = f(\lambda_S) \geqslant f(\lambda_T) = c^q(T \cup \{i\}) - c^q(T)$$

which concludes the proof. □

We obtain as a corollary that in this case the Shapley value is a core allocation too.

4.5 Conclusions and future research

In this chapter we have studied some queueing problems from the perspective of cooperative game theory. In this sense we have dealt with a situation in which different agents maintain their own servers to attend their own populations, and fix a maximum time for their clients in the system. Then we have considered the possibility of cooperating and maintaining a common server to attend all the populations with a maximum time in the system equal to the minimum of all the times that they had before cooperation. We obtain a cost game that is the sum of an additive game plus an airport game. Then the Shapley value is easy to calculate and always provides a core allocation.

Afterwards we deal with the possibility of reducing still more the cost of the common server introducing a priority discipline in the queue. We have proved that there exists a preemptive priority discipline that satisfies all the agents' time exigences before cooperation and with a smaller cost than with the discipline "first to come first to be served". The cost game we obtain in this case is not concave and we propose and characterize a rule to allocate the common cost different to the Shapley value and that is always a core allocation.

Finally we deal with the same situation of different parallel servers, but when they all consider a maximum value for the expected time that their clients spend in the queue, instead of in the system. Surprisingly enough, this apparently slight change leads to a very different situation: cooperation is not always a good option in the sense that the cost does not always diminish. We give a sufficient condition for cooperation to be a good option and propose and characterize an allocation rule for distributing the cost. We also give a necessary and sufficient condition for this allocation to stay in the core of the game.

For the future we have in mind to study some other allocation rules for the game of times in the queue that always stay in the core in case that cooperation is a good option.

We also want to study some other queueing situations from the point of view of cooperative game theory; for instance we would like to consider different models in which there is not a place where to queue and therefore the cost function has to take into account the losses of clients. The same models with different cost functions are other possibilities for future research.

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Resumen en Castellano

Resumen en castellano

La presente tesis se enmarca dentro del amplísimo campo de la Teoría de la Decisión. El individuo se enfrenta a un mercado de posibilidades en diferentes contextos y debe seleccionar una o varias alternativas entre todas las posibles, utilizando diferentes criterios de racionalidad, utilidad, etc.

El enfoque básico en los problemas de selección de alternativas se basa en el uso de *relaciones binarias*. Este modelo se encuentra ya expuesto en el seminal texto de Debreu "Theory of Value"(1959). Se trata de aproximar el problema explicitando cuál de entre cada dos opciones posibles es preferida a la otra, lo que matemáticamente se modeliza mediante relaciones binarias.

Los trabajos pioneros al respecto, como los de Edgeworth y Pareto, suponían la asociación de un valor numérico, "utilidad", para cada uno de los posibles resultados, de modo que cuanto mayor fuera la utilidad de un resultado más preferido se consideraba. Fueron los teoremas de Debreu (1959) y otros los que dieron condiciones que justificaban matemáticamente esta suposición.

Sin embargo, en este tipo de modelización se supone un comportamiento transitivo por parte del decisor que la experiencia demuestra que no es siempre real. Este aspecto fue ampliamente considerado por Arrow (1951). En la tesis abordamos el estudio de diferentes situaciones de elección en las que tratamos de "explicar" la preferencia revelada por parte del decisor en diferentes contextos.

Otro lenguaje también extensamente utilizado en la descripción de diversos problemas de elección es el de las *funciones de elección*. En el presente trabajo consideramos diversos aspectos sobre la racionalidad del comportamiento de un decisor que, de cada posible conjunto de alternativas a su alcance, selecciona un subconjunto (Aizerman (1985)).

Todo lo anterior se restringe a procesos de elección en los que no hay interacción entre individuos.

La rama de las Matemáticas que surge para estudiar las *situaciones conflictivas*, esto es, aquellas situaciones en las que varios agentes toman decisiones y el resultado final depende de las decisiones de todos, es la *teoría de juegos.*

Las primeras aportaciones a la teoría de juegos datan de principios del siglo XX con los trabajos de Zermelo (1913), Borel (1921) y von Neumann (1928). Sin embargo, se puede considerar que la teoría de juegos nace como disciplina científica en el año 1944 a partir de la publicación del libro *"Theory of Games and Economic Behavior"* de John von Neumann y Oskar Morgenstern. Posteriormente, en el año 1950, John Nash definió el concepto de equilibrio en juegos en forma estratégica.

La teoría de juegos se puede dividir en dos grandes áreas: *juegos no cooperativos* y *juegos cooperativos*. En los modelos no cooperativos los agentes no pueden tomar acuerdos vinculantes y se estudia cómo debe actuar cada uno de los jugadores para maximizar sus propios beneficios. En los modelos cooperativos los agentes sí pueden tomar acuerdos vinculantes, e incluso formar coaliciones, y el objetivo es repartir el beneficio o el coste resultante.

La tesis está organizada en cuatro capítulos independientes. Los tres primeros abordan diferentes situaciones de decisión individual en las que un decisor debe elegir entre varias alternativas u ordenarlas según sus preferencias. El último capítulo considera problemas en los que hay al menos dos agentes implicados. En concreto estudia distintos juegos cooperativos que surgen al representar diversas situaciones de teoría de colas. Presentamos a continuación un resumen de cada uno de ellos.

Egalitarian evaluation of infinite utility streams: analysis of some Pareto efficient axiomatics

En el primer capítulo tratamos de resolver conflictos de distribución entre un número infinito y contable de generaciones. En este contexto los economistas están tradicionalmente interesados en postular y combinar axiomas que garanticen un cierto trato equitativo entre las distintas generaciones, con axiomas de eficiencia.

Las propiedades de eficiencia se materializan en diferentes versiones del axioma de Pareto. La propiedad de equidad es a menudo considerada sinónima de la de "anonimato", que se señala como la adecuada para ser verificada por una función o relación de bienestar social (aparte de diferentes condiciones de continuidad). Hammond (1976) postula otra condición de equidad, la "Equidad de Hammond", que establece que al comparar distintas distribuciones que asignan los mismos beneficios a todas las generaciones excepto a dos, "cualquier cambio que disminuya las desigualdades entre las generaciones en conflicto preservando el orden entre ellas es socialmente preferible". Recientemente, Asheim y Tungodden (2004a) han introducido una variación de la propiedad de Hammond para comparaciones interpersonales: la propiedad de "Equidad de Hammond para el futuro". En ella se postula que "un sacrificio de la generación presente que supone una ganancia igual para todas las generaciones futuras es débilmente deseable siempre y cuando la presente generación continúe en mejor situación que las siguientes".

Otro factor a tener en cuenta en el problema de ordenar "cadenas de utilidad intergeneracional infinitas" es **el dominio** para los niveles de utilidad asociados a cada periodo (igual para todas las generaciones). En este contexto resulta más realista exigir ciertas cualificaciones a los dominios que se consideran. Por ejemplo, puesto que la percepción humana no es ilimitada, parece razonable que el dominio considerado sea discreto. También es una restricción natural que las utilidades tengan una unidad patrón (como ocurre cuando se miden cantidades monetarias).

En este tema existe una tendencia natural por parte del investigador a intentar encontrar una expresión numérica explícita asociada con cada cadena infinita de utilidades. Sin embargo, como ya sospechó Ramsey (1928), respetar el igual trato para todas las generaciones supone algunas incompatibilidades intrínsecas para asegurar la eficiencia. Descontar la dotación de la generación futura es lo usual para conseguirlo, pero obviamente no trata a todas las generaciones por igual. El criterio de Rawls, $\mathbf{W}_R(\mathbf{x}) = inf\{x_i : i = 1, 2, 3, ...\}$ es 'más ético' en el sentido de que no está influenciado por la posición que ocupe cada generación, pero aunque es monótono viola las versiones más débiles del axioma de eficiencia de Pareto. Por ello, una aproximación alternativa a la resolución del problema de agregación postula la existencia de relaciones de bienestar social.

La búsqueda de la equidad intergeneracional en el contexto de agregación de cadenas de utilidad infinitas se inició con el trabajo de Ramsey (1928), que establecía una conjetura sobre la dificultad de agregar dichas utilidades eficientemente respetando la misma. Diamond (1965) prueba la imposibilidad de conseguir una función de bienestar paretiana, continua respecto de la norma del supremo y que trate igual a todas las generaciones, cuando se considera el intervalo unidad como dominio para las utilidades. Por su parte Basu y Mitra (2003) prueban que este resultado de imposibilidad permanece sin la hipótesis de continuidad y sin ninguna restricción ni cualificación topológica ni del dominio.

Por su parte Svenson (1980) hace una demostración no constructiva de la existencia de un orden de bienestar social sobre el conjunto de cadenas infinitas de utilidades verificando la condición de Pareto y algún requerimiento de equidad (anonimato). Para ello considera una topología más fuerte que la usada en Diamond (1965) y el mismo dominio (el intervalo unidad). Bossert et al. (2004) proporcionan un resultado de posibilidad más fuerte que el de Svenson, y Hara et al. (2007) aportan algunos otros resultados de imposibilidad en la misma línea. Basu y Mitra (2003) amplían el resulado de posibilidad de Svenson para un dominio general de utilidades y, al mismo tiempo, establecen la relación entre estos resultados en el sentido de que un orden de bienestar social que satisface los axiomas de Pareto y anonimato no puede ser representable [1].

Asheim y Tungodden (2004) obtienen otro resultado de imposibilidad para un orden de bienestar social cuando el dominio de utilidades es el intervalo unidad, exigiendo la condición de equidad denominada "Equidad de Hamond para el futuro" (HEF) y otros postulados adicionales.

Asheim et al. (2007) prueban que es imposible agregar cadenas infinitas de utilidad con una relación binaria superiormente semicontimua que satisfaga Dominancia Débil y HEF. El dominio que consideran es cualquier Y tal que $[0, 1] \subseteq Y \subseteq \mathbb{R}$.

En este sentido Basu y Mitra (2007) consideran la posibilidad de debilitar el axioma de Pareto y obtienen que es posible combinar anonimato y una forma débil del postulado de Pareto (denominada Dominancia Débil) en una función de bienestar social, sea cual sea el dominio para los niveles de utilidad.

Por otro lado, en el estudio de ciertas combinaciones de axiomas la estructura del dominio es crucial. En el mismo trabajo, Basu y Mitra prueban que en su resultado original de imposibilidad, si el axioma fuerte de Pareto es sustituido por el axioma débil de Pareto el resultado de imposibilidad permanece. Pero si el dominio considerado para los niveles de utilidad fuera $\mathbb{N}^* = \{0, 1, 2, \ldots\}$, entonces el resultado de imposibilidad se transforma en uno de posibilidad.

La búsqueda de funciones de bienestar social que verifiquen otros axiomas de equidad no es frecuente en la literatura.

Banerjee (2006) considera la propiedad HEF en el caso de funciones de bienestar social (en lugar de órdenes) y obtiene otro resultado de imposibilidad cuando el dominio es el intervalo unidad y la función tiene que verificar el axioma de Dominancia Débil.

No conocemos ningún otro trabajo que estudie la compatiblidad de axiomas de equidad diferentes del de anonimato con una función paretiana.

Probamos en este capítulo que, bajo las condiciones del teorema de Banerjee, la imposibilidad también se transforma en posibilidad si el dominio de utilidades es $\mathbb{N}^*$ en lugar del intervalo unidad, cuando se exige un axioma más fuerte que el axioma HEF: el axioma de "no sustitución restringida (RNS)". Además, nuestra demostración es constructiva, de modo que obtenemos una expresión *explícita* para una función de bienestar social que verifica el axioma de Pareto Fuerte y RNS (más fuerte que el axioma HEF).

En una línea similar de investigación nos planteamos la existencia e interrelaciones de diferentes versiones del postulado de equidad de Hammond que puedan ser combinadas con funciones de bienestar social paretianas. Se consideran ambos dominios, el

[1] Un orden $\succcurlyeq$ sobre un conjunto X es representable si existe una función real f sobre X tal que $x \succcurlyeq y \Leftrightarrow f(x) \geqslant f(y)$.

discreto $\mathbb{N}^*$ y el continuo $[0, 1]$. En el caso continuo todos los resultados que se obtienen son de imposibilidad, mientras que en el discreto, si bien concluimos que el axioma fuerte de Pareto no puede combinarse con ninguna de las expresiones del axioma de equidad de Hammond, demostramos que sí es posible obtener funciones de bienestar social que satisfagan una expresión de tal axioma con una versión más débil del axioma de Pareto.

Este capítulo está basado en Alcantud y García-Sanz (2008).

Ranking opportunity sets. A characterization of an advised choice

Son numerosos también los problemas de decisión en los que se hace una selección de un subconjunto de alternativas previa a la elección final del agente. El capítulo 2 trata de la ordenación de tales subconjuntos de alternativas, también denominados "conjuntos de oportunidades". En el modelo estándar se considera una relación binaria definida por el agente sobre el conjunto de alternativas, que se extiende a una relación en el conjunto de subconjuntos no vacíos de dicho conjunto de alternativas. En la literatura sobre el tema existen numerosas interpretaciones para este tipo de problemas y se consideran adecuados distintos axiomas dependiendo de los contextos específicos.

Un ejemplo sencillo en el que un agente "ordena" subconjuntos de alternativas aparece citado ya en Kreps (1979), y consiste en la ordenación de diferentes menús en un restaurante: el individuo elegirá un plato, pero inicialmente ha de seleccionar un menú para ya más tarde quedarse con una única comida. Otros contextos en los que podemos encontrar situaciones de este tipo son diversos casos de votaciones como la selección de un comité, la admisión de un grupo de estudiantes en un colegio, la selección de un grupo de trabajadores, la formación de coaliciones (el agente debe asignar valor a los diferentes grupos de colegas con los que asociarse), etc.

En todas estas situaciones los elementos son posibles alternativas y los subconjuntos de posibles alternativas son "conjuntos de oportunidades" (o menús). Existen diferentes criterios para ordenar estos conjuntos, que pueden ser aplicados considerando las características particulares de la situación tratada. Citamos algunas de tales situaciones para ilustrar y motivar el capítulo.

- *Elección bajo incertidumbre total*. En estas situaciones una decisión puede llevar a diferentes consecuencias y el decisor no tiene posibilidad de asignar probabilidades a las mismas. Diferentes criterios pueden ser aplicados en estas situaciones: el criterio maxmin (pesimista), el minmax (optimista),...

- *Libertad de elección y preferencia por mayor flexibilidad*. En este criterio el decisor da valor no sólo a la calidad de la elección sino también al grado de libertad del que disfruta. Por ejemplo, es usual preferir una situación en la que el decisor selecciona por sí mismo un elemento que otra en la que es obligado a optar por una alternativa concreta, aunque la opción final sea la misma en los dos casos. También es frecuente que un decisor prefiera un subconjunto con más alternativas donde hacer su elección final que otro que contenga menos, porque (por ejemplo) todas las alternativas son atractivas para él, o porque no sabe si sus preferencias cambiarán en el futuro antes de que deba hacer una última elección.

- *Racionalidad limitada*. En ocasiones, en los procesos de elección se tiende a considerar sólo ciertos elementos "focales", o ciertos rasgos, ignorando el resto de elementos o de la información disponible.

  Mencionamos aquí especialmente el criterio de utilidad indirecta, que es el que utilizamos a lo largo del capítulo 2. Éste se aplica cuando sólo importa la calidad de la elección final del agente: se prefiere a aquellos conjuntos con "mejores maximales".

Muchos autores han realizado aproximaciones axiomáticas de estos problemas: seleccionan ciertos criterios deseables para situaciones concretas y buscan órdenes que satisfagan diferentes combinaciones de los mismos.

Fishburn (1972) fue pionero en considerar preferencias de votantes sobre conjuntos de alternativas. Kannai y Peleg (1984) obtienen un resultado de imposibilidad para un orden sobre el conjunto de subconjuntos de un conjunto verificando dos axiomas atractivos. Algunos otros resultados de imposibilidad se obtienen debilitando estos axiomas o considerando algunos otros (Barberá y Pattanaik (1984), Fishburn (1984), Holzman (1984)). Bossert (1989) caracteriza un quasi-orden con los axiomas de Kannai-Peleg y una propiedad de neutralidad también utilizada por otros autores. Nehring y Puppe (1996) extienden un orden sobre el conjunto de elementos a un ranking de sus subconjuntos no vacíos basado en principios de independencia y continuidad. Caracterizan así rankings que dependen sólo de los elementos máximo y mínimo de los diferentes subconjuntos de alternativas. Bossert et al. (2000) y Arlegi (2003) aproximan el problema en una situación de elección bajo incertidumbre completa, describiendo cuatro reglas de decisión que no se basan solamente en los peores y mejores elementos y que se justifican intuitivamente en términos de "racionalidad limitada".

Este contexto se amplía para incorporar el valor de la libertad de elección (Bossert et al. (1994), Puppe (1996), Pattanaik y Xu (2000) y Xu (2004)). Dutta y Sen (1996) y Alcalde-

Unzu y Ballester (2005), entre otros, caracterizan las reglas utilitarias, y Alcantud y Arlegi (2006) una familia significativa de rankings de conjuntos aditivamente representables. Algunos otros modelos han servido para estudiar estos problemas. En Barberá et al. (2004) puede encontrarse un amplio resumen al respecto.

Por otro lado, tanto la teoría de la decisión individual como la colectiva pueden basarse en múltiples criterios que pueden ser aplicados sucesivamente, todos a la vez, etc. Por ejemplo, podemos pensar en una familia decidiendo dónde ir de vacaciones o cómo distribuir su salario (probablemente los padres y los hijos tendrán diferentes criterios). En muchas situaciones de este tipo damos preferencia a un criterio sobre otro y usamos un segundo criterio sólo en caso de empate entre alternativas después de aplicar el primero (órdenes lexicográficos).

El estudio de composiciones lexicográficas de dos criterios para ordenar subconjuntos de alternativas ha sido realizado por diferentes autores. Algunas de tales composiciones están completamente caracterizadas usando los axiomas apropiados. Remitimos al lector interesado a Barberá et al. (2004).

Nuestra aproximación al respecto sigue la línea de elección bajo el modelo fundamental que determina el ranking de subconjuntos, considerando sólo las mejores alternativas de cada uno de ellos. En este sentido el modelo responde al principio de racionalidad limitada, de modo que el agente se concentra en ciertas alternativas "clave": el subconjunto de los mejores elementos. Tal modelo es el germen del criterio de utilidad indirecta caracterizado por Kreps (1979). Nosotros analizamos una situación en la que se supone que tenemos definido un preorden completo R sobre X (un conjunto finito de elementos) que no es necesariamente un orden lineal.

En una primera sección estudiamos una elección que se realiza en diferentes momentos del tiempo. Ambos, los conjuntos de alternativas y los criterios de decisión en cada momento, pueden ser diferentes. Definimos una relación binaria para tuplas de subconjuntos ordenados (elementos de un producto directo), cada uno del conjunto de alternativas disponibles en cada uno de los momentos de elección considerados. Comenzamos con una subsección en la que tratamos el problema con sólo dos momentos distintos de elección, de modo que tenemos parejas de subconjuntos de alternativas, y luego lo generalizamos al caso de n momentos diferentes. El ranking que aplicamos se define utilizando el criterio de utilidad indirecta ($A \succcurlyeq B \Leftrightarrow \text{máx}(A) R \,\text{máx}(B)$) aplicado a cada una de las coordenadas con un orden lexicográfico.

Esta cuestión ya ha sido estudiada en Krause (2008) para el caso de dos tiempos de elección diferentes. Él caracteriza este criterio mediante 5 entre los que incluye un axioma de neutralidad y otro axioma técnico llamado "simple time discounting". Krause utiliza una notación ligeramente diferente de la nuestra y supone que para cualquier subconjunto que incluya alternativas de ambos momentos de decisión "es natural" suponer su equivalencia con un subconjunto de dos elementos, uno de cada momento, dado que sólo se trata con el criterio de utilidad indirecta. En este sentido sólo expresa los axiomas para los subconjuntos de dos elementos. Utiliza también preórdenes completos definidos sobre los dos conjuntos de alternativas.

En el presente trabajo caractizamos el criterio definido, tanto para el caso de dos momentos de elección como para el caso general de n momentos diferentes, con sólo 3 axiomas y en un modelo donde los preórdenes sobre los conjuntos de alternativas no están fijados, en la línea de Kreps (1979).

En una segunda sección utilizamos también el criterio de utilidad indirecta para ordenar los subconjuntos de un conjunto de alternativas X, pero cambiamos el modelo anterior considerando la posibilidad de tener un "consejero". Este consejero no tiene definida una relación binaria sobre X, pero para cualquier subconjunto $S \subseteq X$ selecciona algunos "elementos fundamentales" (los que él prefiere), que representamos mediante una función de elección $\mathcal{C} : \mathcal{P}^*(X) \to \mathcal{P}^*(X)$ tal que a cada conjunto $S \in \mathcal{P}^*(X)$ le asigna un subconjunto no vacío de él mismo $\mathcal{C}(S) \subseteq S$. Así, en nuestro ranking de subconjuntos sólo recurriremos al consejero para aquellos subconjuntos que resulten ser indiferentes por nuestro primer criterio (el de la utilidad indirecta). Aplicamos entonces ese mismo criterio de utilidad indirecta, pero ahora a los subconjuntos previamente seleccionados por el consejero.

Al ranking de subconjuntos así definido lo denominamos "ranking asociado a una función de elección", que es un preorden completo.

Finalmente consideramos la cuestión siguiente. Dado un orden de subconjuntos de un conjunto finito X, $\succcurlyeq$, que es un preorden completo (de tal modo que existe un preorden completo sobre X inducido trivialmente por $\succcurlyeq$: $aRb \Leftrightarrow \{a\} \succcurlyeq \{b\}$), ¿cuándo es este orden observado el ranking asociado a una función de elección?. Probamos que esta pregunta tiene una respuesta positiva cuando $\succcurlyeq$ verifica dos propiedades concretas.

La situación de esta segunda sección puede ser considerada un elección en dos momentos del tiempo en los que los conjuinto de alternativas y los criterios de decisión (relación binaria sobre el conjunto de alternativas) son los mismos. Sin embargo queremos incidir en que el ranking asociado a una función de elección no es un caso particular de la primera situación, dado que en el segundo momento de decisión no aplicamos el criterio de utilidad indirecta a los subconjuntos del conjunto de alternativas sino a subconjuntos de cada uno de ellos previamente seleccionados por nuestro consejero.

Rational choice by two sequential criteria

En el capítulo 3 abordamos problemas en los que la preferencia del decisor se representa mediante una función de elección en lugar de una relación binaria. Esta rama de la teoría de la decisión tiene además un factor importante de aplicabilidad, de modo que sus resultados subyacen a diversos modelos económicos, sociológicos, psicológicos, etc., lo que proporciona a su estudio un atractivo añadido. Las funciones de elección que, para cada conjunto de alternativas, seleccionan aquellas que son consideradas las mejores respecto de una cierta relación binaria sobre el conjunto de alternativas, son consideradas "elecciones razonables". La cuestión más interesante radica en el estudio de cuándo, dada una función de elección, podemos garantizar la existencia de una relación binaria tal que la elección observada coincida con los mejores elementos por dicha relación. Si la respuesta es positiva decimos que la función de elección es racional o que existe una relación binaria que la racionaliza. Los diferentes métodos de axiomatización de una "elección racional" que han sido desarrollados sobre la base de funciones de elección generadas por relaciones binarias y criterios de optimización, quedan recogidos bajo la llamada teoría de la "preferencia revelada". En Suzumura (1983) encontramos una recopilación de las caracterizaciones de este concepto clásico de funciones de elección racionales, que ha sido ampliamente estudiado por diversos autores entre los que podemos citar a Arrow (1959), Richter (1966), Wilson (1970), Sen (1971),...

Algunos mecanismos de elección no clásicos han sido considerados por otros autores como Aizerman y Malishevski (1981). En esta línea, Nehring (1996) da una primera contribución al problema de la existencia de elementos maximales para funciones de elección "no binarias". En Tian y Zhou (1995), Rodríguez-Palmero y García-Lapresta (2002) y Alcantud (2002, 2006) aparecen resutados en ese mismo sentido. Gaertner y Xu (2004) presentan algunas extensiones de la noción clásica de racionalidad dando un concepto de la misma basado en el modelo clásico, pero teniendo en cuenta que algunas de las alternativas pueden tener un cierto "grado de disponibilidad". Éste se puede dar si el decisor puede, por ejemplo, considerar el proceso de elección inaceptable, o si alguna de las alternativas está prohibida por alguna ley. Bossert y Suzumura (2007) "presentan un modelo de elección donde se tienen en cuenta diversas normas externas", modelo que incluye al tradicional como un caso particular.

Las razones del interés de la teoría de la elección son diversas. Podemos citar entre otras las siguientes.

- Numerosos problemas de teoría de la decisión, matemática aplicada,... se basan en la elección de las "mejores" opciones en algún sentido de cada conjunto dado de posibilidades.

- Muchos modelos económicos y sociales examinan cuestiones de elección individual. También en fenómenos psicológicos la idea de describir el comportamiento individual en términos de elección de las mejores opciones es un tópico muy atractivo.

- Algunas cuestiones políticas tienen que ver con modelos de este tipo cuando se formalizan diferentes aspectos de elección individual y cuando la opción que maximiza la utilidad individal también maximiza la utilidad colectiva.

Como era de esperar, el estudio de la racionalidad de un decisor es amplio en la literatura. Podemos encontrar al respecto diferentes resultados de la posible racionalidad de una función de elección dependiendo de si satisface o no las propiedades adecuadas. El significado de "racionalidad" ha tenido diferentes interpretaciones y nosotros utilizamos la que identifica una función racional con la optimización de una relación binaria, independientemente de las propiedades que dicha relación binaria verifique. Sin embargo, la literatura sobre funciones de elección racionales también considera ampliamente la cuestión de qué propiedades verifica la relación binaria que racionaliza a una función de elección: aciclicidad, transitividad, quasi-transitividad,... Incluimos también en esta tesis algunos resultados en esta línea.

Además, la posible racionalidad de una función de elección no depende sólo de las propieda-
des que verifica, sino también del dominio sobre el que está definida. El problema de elección implica la definición del conjunto de opciones y sus subconjuntos. La presencia o no de restricciones sobre los subconjuntos es esencial en el modelo formal. Sen (1971), Bandyopadhyay y Sengupta (1991) entre otros consideran funciones de elección definidas sobre dominios que contienen todos los subconjuntos finitos y no vacíos de un conjunto universal de alternativas. Sin embargo Sen (1971) observa que, si bien no se requiere que el dominio incluya también a los subconjuntos infinitos, ningún resultado se vería afectado si tal cosa ocurriera, y también que para los resultados que él demuestra bastaría con que el dominio incluyera todos los subconjuntos de 2 y 3 elementos. En el capítulo 2 de Suzumura (1983) encontramos una recopilación de esta cuestión incluyendo el caso en el que el dominio consiste en una familia arbitraria de subconjuntos no vacíos de un conjunto universal de alternativas. Más recientemente Bossert et al. (2006) desarrollan nuevas condiciones necesarias para que funciones de elección sobre dominios arbitrarios sean racionalizadas por relaciones binarias quasi-transitivas o acíclicas, y dan una nueva condición suficiente para el caso de racionalidad acíclica.

Otro aspecto a tener en cuenta en los problemas de decision es la posibilidad de contar con varios criterios diferentes para decidir. Podemos, en tales casos, dar prioridad a algunos de ellos sobre los otros y aplicarlos de modo secuencial, lo que se denomina "elección secuencial". En caso de que tengamos dos criterios aplicados de manera secuencial decimos que la elección de hace mediante dos criterios secuenciales. Esto supone formalmente que componemos dos funciones de elección en un orden establecido (Aizerman y Aleskerov (1995) también consideran este tipo de comportamiento de elección y denominan a la operación "superposición" de funciones de elección). Así, aunque seguimos los requerimientos clásicos de racionalidad, admitimos un mecanismo para tomar decisiones (la composición de funciones de elección) que es bastante natural y lógico, aunque puede dar lugar a elecciones no racionales en sentido clásico. Nuestro decisor considera racionales no sólo las elecciones derivadas de una única relación binaria, sino también la aplicación secuencial de tal tipo de elecciones. Ejemplos de esta situación son bastante frecuentes: selección de personal aplicando criterios que reducen sucesivamente el conjunto de alternativas, selección de lugares y hoteles para vaciones (por ejemplo, eliminamos primero los que están demasiado lejos de la playa, luego los que son demasiado caros, y sucesivamente),...

Algunos autores han estudiado aspectos similares de teoría de la elección. Kalai et al. (2002) estudian la racionalidad de una función de elección por la aplicación de múltiples relaciones binarias cuando la elección es un único elemento del conjunto de alternativas y aplicando todas las relaciones simultáneamente. Houy (2007) también estudia si el orden de aplicación de los criterios afecta o no a la elección final. Nosotros seguimos la línea iniciada por Manzini y Mariotti (2007), que consideran la racionalidad secuencial de una función de elección por la aplicación de diferentes relaciones binarias en un orden fijo, específicamente en el caso de dos relaciones. Estos autores se restringen también al caso de funciones univaloradas. Nosotros creemos que es interesante el análisis del problema en términos de funciones de elección no univaloradas y es el caso que tratamos en el capítulo 3 de esta tesis.

Estudiamos cómo se comporta una función compuesta de otras dos, esto es, qué propiedades de las verificadas por las funciones iniciales verifica la función compuesta. Aizerman y Aleskerov (1995) hacen este estudio para algunas de las propiedades y para funciones de elección siempre definidas sobre dominios que contienen todos los subconjuntos finitos y no vacíos del conjunto de alternativas. Nosotros añadimos a esos restultados el análisis de algunas otras propiedades sobre dominios de la misma clase, y también el estudio de lo que ocurre con algunas propiedades para funciones de elección definidas sobre dominios arbitrarios. De los resultados que obtenemos concluimos algunos corolarios que establecen la racionalidad de una función de elección obtenida como composición de otras dos que son racionales en algún sentido.

Para aquellos problemas en los que el dominio contiene todos los conjuntos finitos y no vacíos del conjunto de alternativas, consideramos una función de elección que no verifica las propiedades de racionalidad que demandan los diferentes teoremas de elección racional. Nos preguntamos cuándo podemos encontrar dos funciones de elección racionales (que verifican las propiedades exigidas) y tales que la elección en dos pasos por estas dos funciones coincida con la realizada por el decisor, que habíamos observado. Cuando la respuesta es positiva decimos que la función de elección es "racional por dos criterios secuenciales".

Finalmente damos una caracterización completa de las funciones racionales por dos criterios secuenciales en términos de dos condiciones necesarias y suficientes contrastables.

Cooperation in Markovian queueing models

El cuarto y último capítulo de la tesis considera problemas en los que al menos dos agentes están implicados, de modo que las decisiones de cada uno afectan a los resultados de todos.

No resulta difícil encontrar situaciones sociales o económicas en las que coinciden distintos agentes y con distintos puntos de vista. La Teoría de Juegos es la rama de las Matemáticas que considera esta clase de situaciones.

Los agentes implicados en un problema de juegos tienen objetivos bien definidos, de forma que actúan *racionalmente*, y al mismo tiempo tienen en cuenta el conocimiento o expectativas del comportamiento de los otros decisores, de modo que también actúan *estratégicamente*.

En los modelos no cooperativos, los jugadores pueden negociar sobre qué hacer, pero no son posibles los acuerdos vinculantes. Por el contrario, en los modelos cooperativos sí lo son, y también los pagos laterales pueden estar permitidos.

Por otro lado, la Investigación Operativa analiza situaciones en las que el decisor se enfrenta a problemas de optimización guiado por una función objetivo.

El estudio de la cooperación y la competición en modelos de Investigación Operativa es un tema fructífero y atractivo hoy día. Muchos campos dentro de la Investigación Operativa, en los que varios agentes interactúan en situaciones que pueden ser modelizadas como un problema de optimización, han sido abordados desde la perspectiva de la teoría de juegos. Borm et al. (2001) proporciona una recopilación al respecto.

Una de las grandes ramas de la Investigación Operativa es la *teoría de colas*. La competición en modelos de colas ha sido considerada en munerosos artículos, un compendio de los cuales es Hassin and Haviv (2003) (para un resumen del tema de control de colas remitimos al lector a Tadj and Choudhury (2005)). Hay también varios artículos que tratan de aspectos cooperativos de secuenciación y planificación (véase, por ejemplo, una muestra en Curiel et al. (2002) o algunas otras referencias recientes como Moulin y Stong (2002) y Maniquet (2003)). En cualquier caso, de modo bastante sorprendente, los modelos de colas en raras ocasiones han sido tratados desde el punto de vista de la teoría de juegos cooperativos. González y Herrero (2004) es uno de los escasos artículos en los que se analiza la cooperación en modelos de colas. Se considera una situación de Markov en la que varios agentes que mantienen sus propios servidores se ponen de acuerdo para cooperar y mantener un servidor común que atienda a todas sus poblaciones. Cada agente especifica un valor máximo para el tiempo que sus clientes pasan en el sistema. Se estudia el problema de cómo distribuir el coste del servidor común que satisfaga las especificaciones de cada uno de los agentes, y se aplica a un problema de distribución de costes en el sistema de salud español. También más recientemente Yu et al. (2008) estudia modelos de cooperación en sistemas de colas. Estos autores no fijan sin embargo diferentes valores máximos para los tiempos en el sistema de los clientes, sino que es el mismo para todas las poblaciones tanto antes como después de la cooperación.

El estudio de la cooperación en modelos de colas es un tema relevante que puede suscitar el interés de teóricos de juegos y de especialistas en Investigación Operativa. En muchas situaciones del mundo real los proveedores de un servicio particular se ponen de acuerdo para mantener servidores en común que atiendan a todas sus poblaciones: pensemos en un grupo de bancos que comparten una red de cajeros automáticos, un grupo de universidades que comparten un ordenador de gran potencia, o un grupo de hospitales con un banco de sangre común. En todas estas situaciones son importantes cuestiones que deben ser enfocadas desde un punto de vista científico, como son la distribución del coste de los servidores comunes o cuándo es interesante cooperar para un grupo de proveedores o receptores de servicio.

En este capítulo, basado en García-Sanz et al. (2008), tratamos tales cuestiones en algunos modelos de Markov.

En un primer momento tratamos una variación del problema estudiado en González y Herrero (2004). En este modelo, cada agente especifica no sólo el valor máximo deseable del tiempo de sus clientes en el sistema sino también un valor máximo para la probabilidad de que sus clientes gasten más que ese tiempo, algo que creemos bastante razonable en estas situaciones. Por ejemplo, en el caso del sistema español de salud no sólo es razonable asignar un valor máximo para el tiempo que están los enfermos en el sistema (espera y atención), sino que la probabilidad de que dicho tiempo supere el máximo fijado debe ser baja. También estudiamos el caso bastante natural en el que el valor máximo se fija para el tiempo que un cliente debe pasar en la cola, en lugar de en el sistema. Son numerosos los ejemplos que podemos considerar en los que lo que realmente nos preocupa es el tiempo que pasamos esperando haciendo cola, y prácticamente nos es indiferente el tiempo real que se emplee en atendernos (pensemos, por ejemplo, en el caso de las listas de espera en los hospitales). Finalmente consideramos la posibilidad de que se puedan disminuir los costes del servidor común permitiendo una disciplina de prioridades en la cola diferente de la de "primero en llegar, primero en ser servido".

Además, en este tipo de problemas es usual plantearse cómo distribuir los costes producidos si se formara la gran coalición en la que todos los agentes cooperan. En cada caso proponemos una regla para distribuir tales costes. En el primero, el valor de Shapley, y en los casos en los que consideramos el tiempo de espera en la cola y la posibilidad de prioridades en la misma, proponemos una regla y la caracterizamos axiomáticamente.

Conclusiones y cuestiones abiertas

En el primer capítulo de esta tesis hemos estudiado la posibilidad de obtener un ranking de cadenas infinitas de utilidades sin abandonar el postulado de equidad conocido como "equidad de Hammond para el futuro" (HEF) además del axioma de Pareto. Proporcionamos una expresión explícita de un tal compromiso cuando el conjunto de posibles utilidades está contenido en $\mathbb{N}^*$. Los resultados de imposibilidad de Zame (2007, Theorem $4'$) –que suponen que ninguna relación de bienestar social que verifique las propiedades Pareto débil y anonimato puede ser "descrita explicitamente"– hace nuestro resultado especialmente valioso.

Además obtenemos argumentos para contribuir al debate sobre cuál es la influencia del dominio de utilidades cuando se combina equidad y eficiencia de Pareto en la aproximación de Basu y Mitra. Si nos basamos en la ética dada por las propiedades HEF/RNS concluimos que el dominio de utilidades es un factor determinante: si es $[0,1]$, incluso la más débil de las combinaciones posibles da lugar a un resultado de imposibilidad, pero cuando es $\mathbb{N}^*$ hay criterios explícitos para las versiones más fuertes. Este no era el caso cuando el Anonimato era el principio de equidad considarado: no podemos asegurar que una estructura dada produzca compatibilidad o incompatibilidad sin considerar el grado de eficiencia de Pareto que se pretende.

Además probamos que si imponemos HE, entonces el dominio $[0,1]$ determina incompatibilidad mientras que $\mathbb{N}^*$ no (deben examinarse los otro factores: la forma precisa del postulado HE y la versión del axioma de Pareto que se usa).

En la misma línea nos planteamos la existencia de diferentes versiones del axioma de Equidad de Hammond (HE) que pueden ser combinadas con funciones de bienestar social paretianas, tanto en el dominio discreto $\mathbb{N}^*$ como en el continuo $[0,1]$.

Las siguientes tablas incluyen algunos resultados que han servido para motivar nuestro trabajo, y que permiten comparar diferencias entre las distintas aproximaciones al cambiar el dominio de las utilidades.

Tabla 1. Resumen de resultados en dominios de cadenas de utilidad $Y^{\mathbb{N}}$ bajo Anonimato

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	No existe $\star$	No existe
WP	Existe $\dagger$	No existe
D	Existe	Non existe $\diamond$
WD	Existe	Existe $\ddagger$

La afirmación $\star$ está probada en Basu y Mitra (2003). Las marcadas con $\dagger$, $\ddagger$ y $\diamond$ aparecen en Basu y Mitra (2007). Las otras afirmaciones de la tabla se deducen de $\diamond$ y $\dagger$.

En cada uno de los cuatro casos donde la compatibilidad se garantiza, se puede intentar identificar los grupos de permutaciones para los que el anonimato extendido, (o $\mathcal{Q}$-Anonimato como está introducido en Mitra y Basu, 2007) es compatible con los axiomas respectivos de eficiencia en la aproximación de Basu y Mitra. No nos hemos propuesto aún este aspecto, que abordaremos en trabajos futuros.

Tabla 2. Resumen de resultados para dominios de cadenas de utilidad $Y^{\mathbb{N}}$ bajo RNS

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Existe $\star$	No existe
WP	Existe	No existe
D	Existe	No existe
WD	Existe	No existe $\diamond$

Banerjee (2006) prueba que $\diamond$ es cierta incluso si RNS es sustituida por la propiedad más débil HEF. En el capítulo 1 justificamos la afirmación $\star$ y proporcionamos una expresión explícita para una función de bienestar social que satisface HEF$^+$ y el axioma fuerte de Pareto. El resto de afirmaciones de la tabla se deducen de ellas.

Estos resultados se añaden a Asheim et al. (2007), donde se obtienen incompatibilidades de HEF con el postulado de Pareto bajo suposiciones de continuidad.

Con respecto al axioma denotado HE(L), todas las combinaciones en la tabla son imposibles. Salvo que se diga lo contrario, las afirmaciones en la tabla siguiente se refiren todas a las otras variaciones del postulado de Equidad de Hammond.

Tabla 3. Resumen de resultados para dominios de cadenas de utilidad $Y^{\mathbb{N}}$ bajo diferentes versiones de HE

	$Y = \mathbb{N}^*$	$Y = [0, 1]$
SP	No existe $\star$	No existe
WP	Depende de la versión ‡	No existe
D	Depende de la versión	No existe
WD	Depende de laversión	No existe $\diamond$

La afirmación $\diamond$ está demostrada en el capítulo 1 con independencia de la versión de HE que se utilice. Las afirmaciones por encima de $\diamond$ son entonces inmediatas. El caso $\star$ supone la no existencia para todas las versiones del axioma de Equidad de Hammond. La combinación de ‡ y las afirmaciones por debajo dan lugar a la no existencia para HE(L), pero incluso si se impone Anonimato podemos combinar HE(a)$^+$ y WP/D/WD y obtener una función de bienestar social cuando $Y = \mathbb{N}^*$.

Recordamos que el criterio Rawlsiano prueba que en las tablas 1 y 3 anteriores, la existencia está garantizada cuando el axioma de eficiencia requerido es Monotonía (con independencia de $\mathbf{X}$). También para la tabla 2 el caso discreto es trivial porque en tal caso podemos incluso obtener SP y $\mathbf{W}_{FR}(\mathbf{x}) = inf\,\{x_i : i = 2, 3, ...\}$ satisface MON y RNS como mencionamos anteriormente.

La segunda situación que hemos considerado (capítulo 2) tiene que ver con la extensión de un preorden completo definido sobre un conjunto de alternativas a un ranking (que también será un preorden completo) de los subconjuntos del conjunto de alternativas. Hemos estudiado situaciones en las que la decisión se toma en momentos diferentes del tiempo, aplicando en todos ellos el criterio de utilidad indirecta. Definimos un primer ranking en tiempos diferentes ordenando tuplas de subconjuntos de alternativas (cada uno del conjunto de alternativas disponibles en cada uno de los diferentes momentos de elección). Los conjuntos disponibles pueden ser iguales o distintos, representando la común situación en la que un decisor no tiene generalmente a su disposición las mismas posibilidades en los distintos momentos del tiempo en los que tiene que elegir. Además los preórdenes definidos sobre cada uno de estos conjuntos pueden también coincidir o no, dada la posibilidad de que se produzca un cambio en las preferencias de un decisor de un momento del tiempo a otro. Hemos caracterizado el ranking de tuplas de subconjuntos que consiste en la aplicación del criterio de utilidad indirecta, en orden lexicográfico, a cada una de las coordenadas, empezanco con la situación particular en la que sólo hay dos momentos diferentes para la toma de decisiones y generalizándolo después al caso de n momentos de elección diferentes. Además también tratamos una situación de ordenación de subconjuntos de un único conjunto de alternativas, sobre el que tenemos definido un preorden completo. En este caso consideramos, si hay indiferencia entre dos subconjuntos al aplicar el criterio de utilidad indirecta, la posibilidad de que un "consejero" seleccione de cada subconjunto unas cuantas alternativas "focales", sobre las que aplicar el criterio de utilidad indirecta. Para el futuro pensamos en la caracterización de otros tipos de criterios en dos o más tiempos utilizando rankings alternativos al de la utilidad indirecta. También en el caso de utilización de un consejero pensamos en la posibilidad de ser más exigentes y no permitir "cualquier consejo", sino que éste venga dado por una función de elección que verifique ciertas propiedades de racionalidad.

Y son precisamente cuestiones de racionalidad de funciones de elección el aspecto estudiado en el tercer capítulo. Establecemos diferentes resultados de racionalidad en sentido clásico para funciones de elección que se obtienen componiendo otras dos, tanto en el caso en el que el dominio de las funciones de elección verifica la restricción comúnmente aceptada de contener todos los subconjuntos finitos y no vacíos del conjunto de alternativas, como para aquellos casos en los que el dominio de las funciones de elección es arbitrario. Nosotros consideramos como racional a un decisor que obtiene su elección aplicando sucesivamente dos o más funciones de elección que son racionales en el sentido clásico (esto es, su elección resulta de optimizar una relación binaria sobre el conjunto de alternativas). Caracterizamos completamente las funciones de elección sobre dominios que contienen a los subconjuntos finitos y no vacíos del conjunto de alternativas que descomponen como composición de dos funciones de elección racionales en sentido clásico.

Dejamos pendiente para futuras investigaciones un resultado análogo para el caso de dominios arbitrarios o, al menos, la identificación de condiciones suficientes en la línea de los teoremas de racionalidad para las funciones de elección sobre dominios arbitrarios. La descomposición de una función de elección como composición de más de dos funciones racionales también resulta interesante y además permitiría calificar como “racional” un rango más amplio de comportamientos.

El cuarto y último capítulo de la tesis lo hemos dedicado al estudio de la cooperación en algunas situaciones de colas, mediante su aproximación teórica a través de la teoría de juegos cooperativos. Hemos estudiado cuándo la cooperación es interesante para los agentes en las diversas situaciones consideradas. En algunos casos en los que no resulta de entrada beneficiosa, damos condiciones suficientes para que lo sea. En todos los casos en los que la cooperación es atractiva hemos propuesto y caracterizado una regla para la distribución de los costes generados por la cooperación de todos los agentes.

Hay aún muchos modelos de colas que no se han considerado desde la perspectiva de la teoría de juegos cooperativos y que son atractivos para futuros trabajos. También puede resultar interesante estudiar nuestros modelos con diferentes funciones de coste.

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