ON THE EXISTENCE OF TOPOLOGIES COMPATIBLE WITH A GROUP DUALITY WITH PREDETERMINED PROPERTIES

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Dedicated to Sergey Antonyan on his 65^{th} birthday. We deeply appreciate his friendship.

Abstract. The paper deals with group dualities. A group duality is simply a pair (G, H) where G is an abstract abelian group and H a subgroup of characters defined on G. A group topology τ defined on G is *compatible* with the group duality (also called dual pair) (G, H) if G equipped with τ has dual group H.

A topological group (G, τ) gives rise to the natural duality $(G, G^\wedge)$, where $G^\wedge$ stands for the group of continuous characters on G. We prove that the existence of a g-barrelled topology on G compatible with the dual pair $(G, G^\wedge)$ is equivalent to the semireflexivity in Pontryagin's sense of the group $G^\wedge$ endowed with the pointwise convergence topology $\sigma(G^\wedge, G)$. We also deal with k-group topologies. We prove that the existence of k-group topologies on G compatible with the duality $(G, G^\wedge)$ is determined by a sort of completeness property of its Bohr topology $\sigma(G, G^\wedge)$ (Theorem 3.3).

For a topological abelian group (G, τ), denote by $G^\wedge := CHom(G, \mathbb{T})$ the group of all continuous characters on G. The weak topology associated to $G^\wedge$ is defined as the weakest topology on G for which all the elements of $G^\wedge$ are continuous. It is a group topology which will be denoted by τ^+ (or by $\sigma(G, G^\wedge)$ if the duality $(G, G^\wedge)$ is the prevailing point of view). Clearly, $\tau^+ \leq \tau$ and it is the bottom element in the duality $(G, G^\wedge)$. By its relationship with the Bohr compactification of (G, τ), τ^+ is called the Bohr topology of (G, τ). It is precompact and Hausdorff provided (G, τ) has sufficiently many continuous characters. The question of when a precompact and Hausdorff group topology on an abelian group is the Bohr topology corresponding to a locally compact group has been considered in [10], in [15] and recently in [17]. The present paper was originated by a thorough reading of [17].

More explicitly, the main question in [17] was: If (G, w) denotes a totally bounded abelian topological group (that is, precompact and Hausdorff), is there a locally compact topology on G, say τ, such that $\tau^+ = w$? If such τ exists, it can be said in categorical language that (G, w) is the Bohr reflection of (G, τ). The authors of [17] denote by $\mathcal{B}$ the class of all totally bounded abelian groups which are the Bohr reflection of a locally compact group. In the present paper we consider the question from another point of view. As a matter of fact a precompact Hausdorff topological group (G, w) is in $\mathcal{B}$ if there is a locally compact topology in the duality $(G, G^\wedge)$, where $G^\wedge$ denotes the character group of (G, w). Since in particular, every locally compact abelian group is g-barrelled, the question can be generalized to the following one:

Question 1. *Let (G, τ) be an abelian topological group. Under which conditions on G or $G^\wedge$ is there a g-barrelled topology in the duality $(G, G^\wedge)$?*

The g-barrelled groups were introduced in [7]. In Section 4 we formulate their definition, and we obtain a necessary and sufficient condition for a duality $(G, G^\wedge)$ to contain g-barrelled group topologies (Theorem 4.7).

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The existence of dualities $(G, G^\wedge)$ **without** g-barrelled topologies can be derived from deep results of the papers [2], [3] and [14], where the so called "Mackey problem for groups" is solved in the negative. More precisely, the authors of those papers present examples of topological groups (G, τ) such that, the supremum of the family of all the topologies on G which are locally quasi-convex and compatible with τ is not compatible with τ. The Mackey topology on a topological group (G, τ) is defined as the maximum - provided it exists - of all locally quasi-convex topologies on G compatible with τ ([19]). The above mentioned papers provide thus examples of locally quasi-convex groups without a Mackey topology, which makes evident the dissonance between the behaviour of locally convex spaces, and that of locally quasi-convex groups. We remind the reader that the Mackey-Arens Theorem asserts that for a fixed locally convex space (E, ρ), there exists a maximum in the family of all the topologies on E that are locally convex and compatible with ρ.

A g-barrelled **locally quasi-convex** topology μ on a group G is always the maximum of all the topologies which are locally quasi-convex and compatible with μ [7, 4.1]. Thus, whenever the existence of a g-barrelled locally quasi-convex topology in a fixed duality $(G, G^\wedge)$ can be guaranteed, it is unique and it is the Mackey topology. Consequently, if the Mackey topology for a topological group (G, τ) does not exist, the dual pair $(G, G^\wedge)$ does not contain either a g-barrelled topology on G. However, a Mackey topology may not be g-barrelled: these relationships, together with a grading of the Mackey property, are deeply analyzed in [11]. In Theorem 4.7 we give a necessary and sufficient condition for the existence of a g-barrelled topology in a group duality.

If in a given duality $(G, G^\wedge)$ there exists a locally quasi-convex g-barrelled, non locally compact topology ν, then $(G, \nu^+) \notin \mathcal{B}$. In this way a wealth of examples of groups which are not in $\mathcal{B}$ can be obtained, a complement to the results of [17].

The main results of the present paper are in Section 4. Under the mild condition that a duality $(G, G^\wedge)$ is separated, we prove (Theorem 4.7) that the existence of a g-barrelled topology μ on G such that $(G, \mu)^\wedge = G^\wedge$ is equivalent to the semireflexivity (in Pontryagin's sense) of the dual group $G^\wedge$ endowed with the pointwise convergence topology $\sigma(G^\wedge, G)$. If this holds, μ is precisely the compact-open topology $\tau_\mathcal{K}$ on G considered as the dual group of $(G^\wedge, \sigma(G^\wedge, G))$. Without requiring that $\tau_\mathcal{K}$ be compatible with the original topology of G, we characterize when $(G, \tau_\mathcal{K})$ is a g-barrelled group in the duality that it generates (Theorem 4.9).

In Section 3 we deal with the existence of a k-group topology in a general duality $(G, G^\wedge)$. The k-groups, defined by Noble in [20], constitute a class of abelian topological groups that includes the locally compact abelian ones. More generally, all the topological groups that are k-spaces (in the ordinary sense of this term for topological spaces) are k-groups. However there are k-groups in the sense of Noble that are not k-spaces (See 2.10). In Section 2 we clarify these notions and also recall the $k_\mathbb{T}$-groups introduced in [5]. The $k_\mathbb{T}$-groups are relevant because of their connection with completeness. In [6] they appear while proving that the Grothendieck Completeness Theorem, well known in the context of locally convex spaces, does not admit a natural generalization to locally quasi-convex groups.

We introduce the notion of $k_\mathbb{T}$-extension of a precompact group topology. According to the property of being or not a $k_\mathbb{T}$-group, the family $\mathcal{P}$ of precompact Hausdorff topologies on an abstract abelian group G can be split into the two well differentiated subfamilies:

(I) $\mathcal{P}_1$ formed by all those $w \in \mathcal{P}$ such that (G, w) is a $k_\mathbb{T}$-group. The elements $w \in \mathcal{P}_1$ give rise to dualities $(G, G^\wedge)$ that contain at least one k-group topology (Theorem 3.3), which in turn can be locally compact, or metrizable or none of them, as shown in Example 2.10.

(II) $\mathcal{P}_2$ formed by all those $w \in \mathcal{P}$ such that (G, w) is **not** a $k_\mathbb{T}$-group. The elements $w \in \mathcal{P}_2$ produce dualities $(G, G^\wedge)$ which do not contain a k-group topology (Theorem 3.3). Nevertheless, the $k_\mathbb{T}$-extension of each $w \in \mathcal{P}_2$ is a new precompact topology on G, which gives rise to a group duality that **contains** k-group topologies (Theorem 3.7). Further, w and its $k_\mathbb{T}$-extension produce the same family of compact subsets.

In the last section we develop some results about the duality generated by a discrete group. This is a particular case of dualities which contain a g-barrelled topology τ, such that (G,τ) has the Glicksberg property.

1. Notation and Remarks

If G is an abelian group, the set of all homomorphisms from G to $\mathbb{T}$ will be denoted by $Hom(G,\mathbb{T})$, where $\mathbb{T}$ is the unit complex circle. The elements of $Hom(G,\mathbb{T})$ are called *characters* and $Hom(G,\mathbb{T})$ has a group structure with respect to the pointwise operation. We shall write $Hom_p(G,\mathbb{T})$ to indicate that $Hom(G,\mathbb{T})$ is equipped with the pointwise convergence topology.

A *group duality* is a pair (G,H) where G is an abelian group and H is a subgroup of $Hom(G,\mathbb{T})$. If H separates points of G, the duality is said to be separated.

If (G,τ) is a topological abelian group, its *dual group* or *character group* $G^\wedge := CHom(G,\mathbb{T})$ is the set of all continuous characters of G. It is a subgroup of $Hom(G,\mathbb{T})$ and if it separates points of G, we say that (G,τ) is MAP (a shorthand for "maximally almost periodic").

Let $A\subseteq G$ and $B\subseteq G^\wedge$. The *polar* set of A is defined by

$$A^\triangleright=\{\chi\in G^\wedge\ :\ \forall x\in A\ \chi(x)\in\mathbb{T}_+\}$$

and the inverse polar of B is defined by

$$B^\triangleleft=\{x\in G\ :\ \forall \chi\in B\ \chi(x)\in\mathbb{T}_+\}$$

where $\mathbb{T}_+ := \{e^{2\pi it} : t\in[-\frac{1}{4},\frac{1}{4}]\}$.

A subset $A\subseteq G$ is *quasi-convex* if for every $x\in G\setminus A$ there exists an element $\phi\in A^\triangleright$ such that $\phi(x)\notin\mathbb{T}_+$. The *quasi-convex hull* of a subset $M\subseteq G$ is the smallest quasi-convex subset of G that contains M. It is straightforward to prove that it coincides with $M^{\triangleright\triangleleft}$; in particular M is quasi-convex if and only if $M=M^{\triangleright\triangleleft}$. The topological group (G,τ) is said to be *locally quasi-convex* if it admits a basis of neighborhoods of zero formed by quasi-convex subsets.

For a topological group (G,τ), the finest among all the locally quasi-convex topologies on G coarser than τ is the *locally quasi-convex modification of* τ. It will be denoted by $\mathcal{Q}\tau$ and has as a basis of 0-neighborhoods the family $\mathcal{B}=\{U^{\triangleright\triangleleft},\ U\in\mathcal{N}\}$, where $\mathcal{N}$ stands for the τ-neighborhood system of the neutral element.

Let (G,τ) be a topological group. A subset $S\subseteq G^\wedge$ is equicontinuous with respect to τ if and only if $S\subseteq U^\triangleright$ for some τ-neighborhood of zero U in G. This is the simplest formulation, for abelian topological groups and families of continuous characters, of the well-known notion of equicontinuous set of mappings in the context of uniform spaces.

For an abelian group G and a subgroup $\mathcal{L}$ of characters on G, $\sigma(G,\mathcal{L})$ will denote the weak topology on G with respect to the family $\mathcal{L}$. If we start with a topological group (G,τ), we will replace $\sigma(G,G^\wedge)$ by τ^+, whenever this symbol is easier to handle.

Symmetrically, $\sigma(G^\wedge,G)$ denotes the weak topology on $G^\wedge$ with respect to the evaluation mappings corresponding to the elements of G.

If the context is clear, $G^\wedge$ also denotes the character group **endowed with the compact-open topology**. The latter is the natural topology to deal with reflexivity in Pontryagin's sense, so we often use the term *Pontryagin dual* to underline that $G^\wedge$ carries the compact-open topology.

If τ_1 and τ_2 are group topologies on G we will say that they are *compatible* if $(G,\tau_1)^\wedge=(G,\tau_2)^\wedge$. For a dual pair (G,H) a topology τ on G is said to be *compatible with the duality* (G,H) or simply *to be in the duality* (G,H) if $(G,\tau)^\wedge=H$.

In the sequel it will be implicitly understood that all the groups considered are abelian. The terms "compact" or "precompact" do not include the Hausdorff property.

2. A short trip through k-spaces, k-groups and $k_{\mathbb{T}}$-groups

Although the notion of a k-space is well known, there is no uniformity in the literature whether it might be defined in the framework of Hausdorff spaces or simply in the context of topological spaces. Therefore we make precise our starting point and the properties which require further assumptions.

A topology τ on a set X is called a *k-topology* if the following condition holds:

Whenever $H \subseteq X$ is such that $H \cap K$ is τ-closed in K for every τ-compact subset K of X, then H is closed in τ.

If τ is a k-topology on X, the pair (X, τ) is called a k-space. As a matter of fact, there is a k-topology associated to each topology τ on a set X. It is commonly called the k-refinement of τ (in the literature, also the k-extension), and it is defined by its family of closed sets as follows:

Definition 2.1. The *k-refinement* $k(\tau)$ of a topology τ on a set X is defined by: $C \subset X$ is closed in $k(\tau)$ if $C \cap K$ is τ-closed in K, for every τ-compact subset $K \subset X$. The pair $(X, k(\tau))$ is also called the k-refinement of (X, τ).

Clearly $k(\tau)$ is well defined, it is a k-topology and $\tau \leq k(\tau)$. The equality holds if τ is already a k-topology.

The k-refinement $k(\tau)$ of a topology τ on a set X gives rise to the same compact subsets as τ. With the additional assumption that (X, τ) is Hausdorff, it holds that $k(\tau)$ is the finest topology on X with this property. For this reason some authors define the k-refinement only for a **Hausdorff** topology τ. In order to avoid confusion, we provide a proof of these facts.

Proposition 2.2. *Let (X, τ) be a topological space and let $k(\tau)$ be the k-refinement of τ. Then the following statements hold:*

(1) *τ and $k(\tau)$ give rise to the same compact subsets.*
(2) *τ and $k(\tau)$ induce the same topology on any compact $K \subset X$.*
(3) *For every topological space (Y, μ), a function $f : X \to Y$ such that the restriction $f_{|K}$ to any compact subset $K \subset X$ is continuous, is necessarily continuous with respect to $k(\tau)$.*

Furthermore, $k(\tau)$ is the finest among the topologies on X which satisfy the condition of (2) or (3). If (X, τ) is Hausdorff, $k(\tau)$ is also the finest among the topologies on X which have the same compact subsets as τ.

Proof. (1) Since $\tau \leq k(\tau)$, every $k(\tau)$-compact is τ-compact.
For the converse, fix a τ-compact subset $L \subset X$. In order to show that L remains $k(\tau)$-compact, take a cover $\mathcal{U} \subset k(\tau)$ of L. By the definition of the topology $k(\tau)$, for every $U \in \mathcal{U}$ the intersection $U \cap L$ is open in L. Since L is compact, the cover $\{U \cap L : U \in \mathcal{U}\}$ has a finite subcover, and so does $\mathcal{U}$.

(2) follows from (1) and the definition of $k(\tau)$.

Finally (3) follows from the equality $f^{-1}(D) \cap K = (f_{|K})^{-1}(D)$ for any $D \subset Y$ and $K \subset X$ and the definition of $k(\tau)$.

It is straightforward to prove that $k(\tau)$ is the finest topology on X satisfying (2) or (3).

In order to prove the last assertion, assume that μ is a topology on X which gives rise to the same compact subsets as τ. If $C \subset X$ is μ-closed, for every $K \subset X$ compact, $C \cap K$ is

μ-compact, and by the assumption, it is also τ-compact. Since (X,τ) is Hausdorff, $C\cap K$ is τ-closed. This holds for any τ-compact subset K, therefore C is $k(\tau)$-closed and $\mu\le k(\tau)$.
The requirement on the space X to be Hausdorff is used when claiming that every compact subset is closed. □

Two topologies τ_1 and τ_2 on a set X which have the same family of compact subsets may not induce the same topology on the compact subsets of both, as the following example shows. This is due to the unpleasant fact that in a non-Hausdorff space a compact subset is not necessarily closed

Example. Let $X=\{1/n, n\in\mathbb{N}\}\cup\{0\}$, let τ_1 be the topology on X induced by the Euclidean of $\mathbb{R}$ and τ_2 the topology whose open sets are all the subsets of X that do not contain $\{0\}$, together with the total set X. Clearly τ_1 and τ_2 have the same family of compact subsets: namely, every finite $F\subset X$, and every subset that contains $\{0\}$. However they do not induce the same topology on the compact subset $\{1/2n, n\in\mathbb{N}\}\cup\{0\}$.

On the other way round, if the assumption is that both topologies induce the same topology on the compact subsets of one of them, say τ_1, then $\mathcal{K}_1\subset\mathcal{K}_2$, where $\mathcal{K}_i$ are the respective families of compact subsets. It follows that $\mathcal{K}_1=\mathcal{K}_2$ under the assumption of coincidence of the induced topologies in each τ_i-compact, for $i\in\{1,2\}$.

Lemma 2.3. *Let $\mathcal{F}=\{\tau_i, i\in I\}$ be a family of Hausdorff topologies on a set X which give rise to the same compact subsets. The elements of $\mathcal{F}$ induce the same topology on the common compact subsets of X. If τ_1 is the supremum of $\mathcal{F}$, τ_1 has also the same family of compact subsets and $(\tau_1)_{|K}=(\tau_i)_{|K}$ for every compact $K\subset X$.*

Proof. In order to prove the first assertion, fix a τ_i-compact subset K. Since τ_i is Hausdorff, any $C\subset K$ is τ_i-closed if and only if it is τ_i-compact. Thus, by the assumption, the τ_i-closed subsets of K are the same for every $i\in I$.
Let us see now that every τ_i-compact is also τ_1-compact. As above, let $K\subset X$ be τ_i-compact. Pick a net $S:=\{x_j, j\in J\}$ in K. It has a τ_i-convergent subnet. Taking into account that the topologies $\tau_{i|K}$ coincide for all $i\in I$, without loss of generality we can assume that there exists $x\in K$ such that $x_j\xrightarrow{\tau_i}x$, $\forall i\in I$.
Let us prove that S also converges to x in τ_1. A basic τ_1-neighborhood of x has the form $V=\cap_{m=1}^n V_{i_m}$, where V_{i_m} is a neighborhood of x in τ_{i_m}. Since $x_j\xrightarrow{\tau_{i_m}}x$, for $m=1,\dots,n$, the net is eventually in $V_{i_1},\dots,V_{i_n}$. Now J is a directed set, therefore the net is also eventually in $V=\cap_{m=1}^n V_{i_m}$. As V was a basic arbitrary τ_1-neighborhood of x, it follows that $x_j\xrightarrow{\tau_1}x$. □

The k-refinement of a group topology may not be a group topology (Example 2.10). Around the 70's Noble defined the k-groups, a notion weaker than that of a k-space in the context of topological groups. The $k_{\mathbb{T}}$-groups were defined in [5] in the context of abelian topological groups. For the reader's convenience we state both definitions:

Definition 2.4. A topological group (G,τ) is *a k-group* if for every topological group (H,μ) and every homomorphism $f:G\to H$ the following holds:

If $f_{|K}$ is continuous for any compact $K\subset G$, then f is continuous.

For further use we write the following property, whose proof can be seen in [20, 1.1]:

Lemma 2.5. *A topological group (G,τ) is a k-group iff τ is the finest among all the* **group** *topologies on G that coincide with τ on the τ-compact subsets.*

To each topological group there corresponds a k-group structure, defined as follows:

For a topological group (G,τ), let $k_g(\tau)$ be the finest of all the group topologies on G that coincide with τ on every τ-compact subset $K\subset G$. Clearly $(G,k_g(\tau))$ is a k-group, which might be called the *k-group modification of* (G, τ). Also the topology $k_g(\tau)$ is called the k-group modification of τ. If (G, τ) is Hausdorff, then $k_g(\tau)$ is also the finest group topology of all those whose compact subsets are the τ-compact ones.

Observe that if a topological group (G, τ) is a k-space, it is in particular a k-group. However there are k-groups which are not k-spaces as shown in Example 2.10

Definition 2.6. A topological group G is *a* $k_{\mathbb{T}}$*-group* if every character $f : G \to \mathbb{T}$ such that its restriction $f_{|K}$ to any compact $K \subset G$ is continuous, must be continuous.

Every k-group is a $k_{\mathbb{T}}$-group, as a consequence of their definitions. The converse does not hold. A family of examples of $k_{\mathbb{T}}$-groups which are not k-groups can be modelled through the following proposition.

Proposition 2.7. *Let (G, τ) be a nonprecompact topological group. If (G, τ) is a k-group with Glicksberg property, then $(G, \sigma(G, G^\wedge))$ is a $k_{\mathbb{T}}$-group which is not a k-group.*

Proof. Let $f : G \to \mathbb{T}$ be a homomorphism such that $f_{|K}$ is continuous with respect to the topology induced by $\sigma(G, G^\wedge)$ on every compact subset $K \subset G$. From $\sigma(G, G^\wedge) < \tau$ we deduce that $f_{|K}$ is τ-continuous for every τ-compact subset K. Since (G, τ) is a k-group, we obtain that $f \in (G, \tau)^\wedge = (G, \sigma(G, G^\wedge))^\wedge$. Thus $(G, \sigma(G, G^\wedge))$ is a $k_{\mathbb{T}}$-group. Taking into account the inequality $\sigma(G, G^\wedge) < \tau$, Lemma 2.5 implies that it is not a k-group. □

For a Hausdorff topological group (G, τ) there might exist several compatible $k_{\mathbb{T}}$-group topologies on G with the same compact subsets as τ. On the other hand there is a unique k-group topology whose compact subsets are precisely the τ-compact subsets, namely $k_g(\tau)$. We will see below that the k-group modification $k_g(\tau)$ is compatible with τ if and only if (G, τ^+) is a $k_{\mathbb{T}}$-group.

For two group topologies on a group G, the property of "giving rise to the same compact subsets" is in some sense complementary to that of "being compatible", as expressed in the following proposition.

Proposition 2.8. *Let G be an abelian group and let τ_1, τ_2 be group topologies on G such that $\tau_1 \le \tau_2$.*

(1) *Assume τ_1 and τ_2 are compatible topologies.*
If (G, τ_2) is a $k_{\mathbb{T}}$-group, then (G, τ_1) is also a $k_{\mathbb{T}}$-group.
(2) *Assume τ_1 and τ_2 are Hausdorff and give rise to the same family of compact subsets.*
If (G, τ_1) is a $k_{\mathbb{T}}$-group, then also (G, τ_2) is a $k_{\mathbb{T}}$-group and both topologies are compatible.

In particular, if a topological group (G, τ) is a $k_{\mathbb{T}}$-group, then also (G, τ^+) is a $k_{\mathbb{T}}$-group.

Proof. (1) Denote by $\mathcal{K}_i$ the family of τ_i-compact subsets, $i \in \{1, 2\}$. From $\tau_1 \le \tau_2$ we obtain $\mathcal{K}_2 \subseteq \mathcal{K}_1$. Let $f : G \to \mathbb{T}$ be any homomorphism such that $f|_K$ is τ_1-continuous for all $K \in \mathcal{K}_1$. In particular, $f|_K$ is τ_2-continuous for all $K \in \mathcal{K}_2$. Since (G, τ_2) is a $k_{\mathbb{T}}$-group, then $f \in (G, \tau_2)^\wedge = (G, \tau_1)^\wedge$; therefore, (G, τ_1) is a $k_{\mathbb{T}}$-group.

(2) Let $f : G \to \mathbb{T}$ be any homomorphism such that $f|_K$ is τ_2-continuous for all $K \in \mathcal{K}_2 = \mathcal{K}_1$. By Lemma 2.3, $f|_K$ is also τ_1-continuous. Since (G, τ_1) is a $k_{\mathbb{T}}$-group, then $f \in (G, \tau_1)^\wedge \subseteq (G, \tau_2)^\wedge$, therefore (G, τ_2) is a $k_{\mathbb{T}}$-group. Further, it also holds $(G, \tau_2)^\wedge \subseteq (G, \tau_1)^\wedge$: in fact, every $f \in (G, \tau_2)^\wedge$ satisfies $f|_K$ is τ_2-continuous and by Lemma 2.3 τ_1-continuous, for all $K \in \mathcal{K}_2 = \mathcal{K}_1$. Thus, $f \in (G, \tau_1)^\wedge$ and τ_1 and τ_2 are compatible.

The last statement follows from (1), since τ^+ is compatible with τ. □

Remark 2.9. If a topological group (G, τ) has the Glicksberg property, then all the group topologies on G which lie between τ^+ and τ are simultaneously $k_{\mathbb{T}}$-group topologies or else none of them is a $k_{\mathbb{T}}$-group topology.

Next, we present an example of a k-group which is not a k-space. Its k-refinement is not even a topological group.

Example 2.10. A k-group whose k-refinement is not a topological group.

Claim 1. *The product $\mathbb{R}^{\mathbb{R}}$ of $\mathfrak{c}$ real lines is not a k-space. Nevertheless, it is a k-group.*

Proof. The proof of the first claim can be developed through the following hint provided in Kelley's book [18]. Let $A \subset \mathbb{R}^{\mathbb{R}}$ be defined by:

$x = (x_r)_{r\in\mathbb{R}} \in A \Leftrightarrow$ there exists $m \in \mathbb{Z}$ and $F \subset \mathbb{R}$ with $|F| \leq m$ and $m \geq 1$, such that:

$$\begin{cases} 1)\ x_r = m,\ \forall r \in \mathbb{R} \setminus F \\ 2)\ x_r = 0,\ \text{if}\ \ r \in F \end{cases}$$

It is easy to prove that $\overline{A} = A \cup \{0\}$, therefore A is not closed. However, for every compact $K \subset \mathbb{R}^{\mathbb{R}}$, $A \cap K$ is closed in K. Thus $\mathbb{R}^{\mathbb{R}}$ is not a k-space.

The fact that it is a k-group follows from [21, Theorem 5.7], where it is proved that the product of k-groups is a k-group. Obviously, each factor $\mathbb{R}$ is a k-space, therefore also a k-group. □

Claim 2. *Denote by π the product topology on $\mathbb{R}^{\mathbb{R}}$, and let $k(\pi)$ be its k-refinement. Then $(\mathbb{R}^{\mathbb{R}}, k(\pi))$ is not a topological group.*

Proof. As pointed out in Claim 1, $\mathbb{R}^{\mathbb{R}}$ endowed with the product topology π is not a k-space. If its k-refinement $k(\pi)$ were a group topology, $(\mathbb{R}^{\mathbb{R}}, k(\pi))$ would be a k-group, with the same compact subsets as $(\mathbb{R}^{\mathbb{R}}, \pi)$. Since $(\mathbb{R}^{\mathbb{R}}, \pi)$ is already a k-group, it must be $k(\pi) = \pi$. This contradicts the fact that π is not a k-space topology and therefore $\pi \neq k(\pi)$. □

3. The $k_{\mathbb{T}}$-extension of a precompact topology

The goal of this section is to determine conditions under which the existence of k-group topologies in a fixed duality $(G, G^\wedge)$ can be guaranteed. The $k_{\mathbb{T}}$-groups appear in this context because of the following result:

Lemma 3.1. [5, 6.1.5] *A topological group (G, τ) is a $k_{\mathbb{T}}$-group if and only if $k_g(\tau)$ is compatible with τ, where $k_g(\tau)$ is the k-group modification of τ.*

Proof. Let (G, τ) be a a $k_{\mathbb{T}}$-group. Then $\tau \leq k_g(\tau)$ implies $(G, \tau)^\wedge \subset (G, k_g(\tau))^\wedge$.
For the other inclusion, let $f : G \to \mathbb{T}$ be in $(G, k_g(\tau))^\wedge$. From the fact that $\tau \leq k_g(\tau) \leq k(\tau)$ and Proposition 2.2 it follows that τ and $k_g(\tau)$ admit the same family of compact sets and induce the same topology on them. Thus, $f_{|K}$ is τ-continuous for every τ-compact $K \subset G$. Since (G, τ) is a $k_{\mathbb{T}}$-group, f is τ-continuous and $f \in (G, \tau)^\wedge$.
Conversely, assume $(G, \tau)^\wedge = (G, k_g(\tau))^\wedge$. Let $f : G \to \mathbb{T}$ be a homomorphism such that $f_{|K}$ is $\tau_{|K}$-continuous for every compact $K \subset G$. By the definition of $k_g(\tau)$, f must be $k_g(\tau)$-continuous. Thus, $f \in (G, k_g(\tau))^\wedge = (G, \tau)^\wedge$ and (G, τ) is a $k_{\mathbb{T}}$-group. □

The $k_{\mathbb{T}}$-groups share with the k-groups the following property:

Lemma 3.2. [5, 6.1.6] *Let (G, τ) be a $k_{\mathbb{T}}$-group. Then $G^\wedge$ endowed with the compact-open topology is complete.*

The existence of $k_{\mathbb{T}}$-topologies in a fixed duality $(G, G^\wedge)$ is completely determined by the behaviour of the bottom topology $\sigma(G, G^\wedge)$ (Proposition 2.8). We prove next that also the existence of k-group topologies in $(G, G^\wedge)$ is determined by $\sigma(G, G^\wedge)$.

Theorem 3.3. *Let (G, τ) be a topological group, and $G^\wedge$ its character group. The following statements are equivalent:*

(1) *There is at least one k-group topology on G compatible with the duality $(G, G^\wedge)$.*
(2) *τ^+ is a $k_{\mathbb{T}}$-group topology.*

Proof. (1) $\Rightarrow$ (2) Assume that μ is a k-group topology in the duality $(G, G^\wedge)$. This means that $(G, \mu)^\wedge = G^\wedge$. In particular, μ is a $k_{\mathbb{T}}$-group topology and by (1) of Proposition 2.8, μ^+ is also a $k_{\mathbb{T}}$-group topology. Since μ is compatible with τ, $\tau^+ = \mu^+$ and the assertion follows.
(2) $\Rightarrow$ (1) Conversely, if (G, τ^+) is a $k_{\mathbb{T}}$-group, by Lemma 3.1 we obtain that $k_g(\tau^+)$ is a k-group topology on G compatible with τ^+. In other words, $k_g(\tau^+)$ is in the duality $(G, G^\wedge)$. $\square$

Remarks 3.4. (i) Observe that in a duality $(G, G^\wedge)$ there might be several compatible k-group topologies. For instance, in the duality $(c_0(\mathbb{T}), \mathbb{Z}^{(\mathbb{N})})$ presented and studied in [12], the maximum and the minimum of all the locally quasi-convex compatible topologies are both metrizable, thus both are compatible locally quasi-convex k-group topologies.
(ii) There is at most one **complete** metrizable locally quasi-convex topology on a group G with a fixed dual group $G^\wedge$. As proved in [7], such a topology is the Mackey topology on G for the corresponding duality $(G, G^\wedge)$.
(iii) For a fixed separated duality $(G, G^\wedge)$ let μ be a compatible topology on G, and let $\mathcal{K}_\mu$ be the set of μ-compact subsets of G. Assign to $\mathcal{K}_\mu$ the k-group topology it generates on G, say $k_g(\mu)$. Then, by Lemma 3.1, $k_g(\mu)$ is compatible with the duality $(G, G^\wedge)$, if and only if (G, μ) is a $k_{\mathbb{T}}$-group. Thus, the set of k-group topologies compatible with the duality $(G, G^\wedge)$ is in 1-1 correspondence with the set of families $\mathcal{K}_\mu$ for μ a $k_{\mathbb{T}}$-group topology in the dual pair $(G, G^\wedge)$.

Let us denote by $\mathcal{P}$ the family of precompact topologies on an abelian abstract group G, and by $\mathcal{P}_1$ the subfamily of those $w \in \mathcal{P}$ such that (G, w) is a $k_{\mathbb{T}}$-group. As proved in Theorem 3.3, the elements $w \in \mathcal{P}_2 := \mathcal{P} \setminus \mathcal{P}_1$ produce dualities $(G, G^\wedge)$ which do not contain k-group topologies. We next define the $k_{\mathbb{T}}$-extension of $w \in \mathcal{P}_2$, a sort of associated precompact topology which gives rise to a new duality which contains k-group topologies.

Notation. Denote by $\mathcal{M}$ the family of w-compact subsets of a precompact group (G, w). Let $\mathcal{H}$ be the set of all the characters $f \in Hom(G, \mathbb{T})$ such that $f_{|K}$ is w-continuous, $\forall K \in \mathcal{M}$.

Definition 3.5. Let (G, w) be a precompact group. The weak topology on G relative to $\mathcal{H}$, henceforth denoted $\tau_{\mathcal{H}}$ will be called *the $k_{\mathbb{T}}$-extension of w.*

Clearly $\tau_{\mathcal{H}}$ is a precompact topology on G, and $(G, \tau_{\mathcal{H}})^\wedge = \mathcal{H}$ by [7, 3.7]. Consequently, $G^\wedge := (G, w)^\wedge \subset \mathcal{H}$, and $w \leq \tau_{\mathcal{H}}$. Observe that the equality $\tau_{\mathcal{H}} = w$ implies that (G, w) is already a $k_{\mathbb{T}}$-group.

Some properties of the $k_{\mathbb{T}}$-extension.

Proposition 3.6. *Let (G, w) be a precompact group and let $\tau_{\mathcal{H}}$ be the $k_{\mathbb{T}}$-extension of w. Then,*

(1) *$\tau_{\mathcal{H}}$ and w give rise to the same family $\mathcal{M}$ of compact subsets of G. Further, $w_{|K} = (\tau_{\mathcal{H}})_{|K}$, for all $K \in \mathcal{M}$.*
(2) *$\tau_{\mathcal{H}}$ is a $k_{\mathbb{T}}$-group topology.*
(3) *$\tau_{\mathcal{H}}$ is the maximum in the family of all precompact topologies on G that coincide with w on the w-compact subsets of G.*
(4) *If w is Hausdorff, $\tau_{\mathcal{H}}$ is the maximum in the family of all precompact Hausdorff topologies on G with the same compact subsets as w.*

Proof. (1) From the fact $w \leq \tau_{\mathcal{H}}$, we only need to prove that a fixed w-compact subset $K \subset G$ is also $\tau_{\mathcal{H}}$-compact. To this end, pick a net $S := \{x_i, i \in I\}$ with range in K. Since K is w-compact, S has a w-convergent subnet. Without loss of generality, assume directly that $x_i \xrightarrow{w} x$. For every $f \in \mathcal{H}$ it holds $f(x_i) \longrightarrow f(x)$ in $\mathbb{T}$. Therefore, taking into account that $(G, \tau_{\mathcal{H}})$ is precompact with dual $\mathcal{H}$, $x_i \xrightarrow{\tau_{\mathcal{H}}} x$. Thus, K is also $\tau_{\mathcal{H}}$-compact and further induces on K the same topology as w.

(2) This is obvious from (1) together with the fact that $(G, \tau_{\mathcal{H}})^\wedge = \mathcal{H}$.

(3) Assume now that u is a precompact topology on G such that $u_{|K} = w_{|K}$ for every w-compact $K \subset G$. If $\mathcal{L} = (G, u)^\wedge$, every $f \in \mathcal{L}$ is clearly in $\mathcal{H}$. Therefore, $u \leq \tau_{\mathcal{H}}$.

(4) The assertion follows from (3) and Lemma 2.3.

□

For a topological group (G, τ) such that $\tau_{\mathcal{H}} \neq \tau^+$, the duality $(G, G^\wedge)$ does not admit any k-group topology, as seen in Theorem 3.3. However, $\tau_{\mathcal{H}}$ gives rise to a sort of "extended duality" which improves this lack, as shown next.

Theorem 3.7. *Let (G, τ) be a topological group, and $\tau_{\mathcal{H}}$ the $k_{\mathbb{T}}$-extension of τ^+. Then, $\tau_{\mathcal{H}}$ is a $k_{\mathbb{T}}$-group topology on G compatible with the duality $(G, \mathcal{H})$. Further, $\tau_{\mathcal{H}}$ and τ^+ give rise to the same family of compact subsets of G.*

Proof. By (2) of Proposition 3.6, $\tau_{\mathcal{H}}$ is a $k_{\mathbb{T}}$-group topology. Since $(G, \tau_{\mathcal{H}})^\wedge = \mathcal{H}$, according to Lemma 3.1, $k_g(\tau_{\mathcal{H}})$ is a k-group topology compatible with the duality $(G, \mathcal{H})$. The last assertion is proved by (1) in Proposition 3.6. □

The family $\mathcal{P}$ of precompact topologies on a fixed abelian group G, mentioned in the Introduction, offers the following picture:

- If (G, w) is a $k_{\mathbb{T}}$-group (that is, $w \in \mathcal{P}_1$), there exist k-group topologies on G compatible with the duality $(G, G^\wedge)$, where $G^\wedge = (G, w)^\wedge$. They might be locally compact or metrizable or none of them, as shown by the example $\mathbb{R}^{\mathbb{R}}$ (2.10).
- If (G, w) is not a $k_{\mathbb{T}}$-group (that is, $w \in \mathcal{P}_2$), the $k_{\mathbb{T}}$-extension of w, which we have called $\tau_{\mathcal{H}}$, gives rise to a **new** duality $(G, \mathcal{H})$, with k-group topologies. The latter are not compatible with w. In fact, if λ is one of them, $(G, \lambda)^\wedge = \mathcal{H} \neq (G, w)^\wedge$.

Proposition 3.8. *Let (G, w) be a precompact group with $w \in \mathcal{P}_2$, and let $\tau_{\mathcal{H}}$ be the $k_{\mathbb{T}}$-extension of w. Then, the Pontryagin dual of $(G, \tau_{\mathcal{H}})$ is complete and contains $(G, w)^\wedge$ as a topological subgroup. However, $(G, \tau_{\mathcal{H}})^\wedge$ may not be the completion of $(G, w)^\wedge$.*

Proof. By Lemma 3.2, $(G, \tau_{\mathcal{H}})^\wedge$ is complete. Clearly, $(G, w)^\wedge \subset (G, \tau_{\mathcal{H}})^\wedge$. By (1) of Proposition 3.6, w and $\tau_{\mathcal{H}}$ give rise to the same compact subsets, therefore the dual group $(G, w)^\wedge$ is a topological subgroup of $(G, \tau_{\mathcal{H}})^\wedge$, where both are considered with the compact-open topology.

The last assertion is obtained from the fact that $G^\wedge := (G, w)^\wedge$ may itself be complete and distinct from $(G, \tau_{\mathcal{H}})^\wedge = \mathcal{H}$, therefore non dense in $(G, \tau_{\mathcal{H}})^\wedge = \mathcal{H}$. The following example provides a proof of it. □

Example 3.9. A precompact group G whose $k_{\mathbb{T}}$-extension does not coincide with its completion.

We first state some auxiliary tools before the explicit example, described below in Claim 1.

Let $L^2[0,1]$ be the Hilbert space of square integrable functions on $[0,1]$, and let $L := L^2_{\mathbb{Z}}[0,1] \subset L^2[0,1]$ be the subgroup formed by all the almost everywhere integer valued functions, equipped with the induced topology. This group is considered in [1, Section 11], where a remarkable proof of the following fact is given:

(*) The Pontryagin dual of L is topologically isomorphic to the dual of $L^2[0,1]$, say

$$L^\wedge \approx (L^2[0,1])^\wedge$$

through the restriction mapping.

The following properties of L and $L^\wedge$ are needed for our argument:

(1) L is not Pontryagin reflexive. In fact, the natural mapping from $L \to L^{\wedge\wedge}$ is a non-surjective embedding. This derives from (*) and from the Pontryagin reflexivity of $L^2[0,1]$ as a Banach space. Thus, $L^{\wedge\wedge} \approx L^2[0,1]^{\wedge\wedge} \approx L^2[0,1]$.
(2) L is metrizable and complete, therefore its dual group $L^\wedge$ endowed with the compact-open topology $c(L^\wedge, L)$ is a k-space ([1], [8]).
(3) Every $\sigma(L^\wedge, L)$-compact subset of $L^\wedge$ is equicontinuous with respect to L. (For a proof see [7]. The term g-barrelled defines this property, see Section 4, 4.2).
(4) The natural mapping $L \to (L^\wedge, \sigma(L^\wedge, L))^\wedge$ is a topological isomorphism. (This derives from (3) plus the local quasi-convexity of L. A direct proof can be seen in [6].)
(5) $c(L^\wedge, L)$ and $\sigma(L^\wedge, L)$ induce the same topology on any $M \subset L^\wedge$ which is equicontinuous with respect to L. (This fact is well known).

Claim 1. *The precompact group $G := (L^\wedge, \sigma(L^\wedge, L))$ has the following properties:*

(i) *It is not a $k_{\mathbb{T}}$-group.*
(ii) *The $k_{\mathbb{T}}$-extension of $\sigma(L^\wedge, L)$ is precisely $\sigma(L^\wedge, L^{\wedge\wedge})$.*
(iii) *The group $(L^\wedge, \sigma(L^\wedge, L^{\wedge\wedge}))^\wedge$ is complete, but it is not the completion of $G^\wedge = (L^\wedge, \sigma(L^\wedge, L))^\wedge$.*

Proof. (i) follows from (ii), taking into account that $\sigma(L^\wedge, L) \neq \sigma(L^\wedge, L^{\wedge\wedge})$ (as stated in (1)).
(ii). Fix a character $f : (L^\wedge, \sigma(L^\wedge, L)) \to \mathbb{T}$ such that $f_{|K}$ is continuous for every $\sigma(L^\wedge, L)$-compact $K \subset L^\wedge$. Since $\sigma(L^\wedge, L) \leq c(L^\wedge, L)$, $f_{|K}$ is continuous with respect to $c(L^\wedge, L)_{|K}$. By (2), f is continuous with respect to the compact-open topology of $L^\wedge$, therefore $f \in L^{\wedge\wedge}$.
On the other hand, from $L^{\wedge\wedge} \approx L^2[0,1]^{\wedge\wedge}$ we deduce that every element in $L^{\wedge\wedge}$ is an evaluation $\tilde{x}$ for some $x \in L^2[0,1]$. Thus, the $k_{\mathbb{T}}$-extension of $\sigma(L^\wedge, L)$ is $\sigma(L^\wedge, L^{\wedge\wedge})$.
(iii) The completeness of $(L^\wedge, \sigma(L^\wedge, L^{\wedge\wedge}))^\wedge$ follows from Lemma 3.2.
By (4), $(L^\wedge, \sigma(L^\wedge, L))^\wedge$ is topologically isomorphic to L, and the latter is complete as stated in (2). Thus, $(L^\wedge, \sigma(L^\wedge, L))^\wedge$ is a complete (thus, closed) proper subgroup of $(L^\wedge, \sigma(L^\wedge, L^{\wedge\wedge}))^\wedge$.

4. On the existence of g-barrelled topologies in a group duality

The g-barrelled groups were introduced in [7]. They constitute a class of abelian topological groups which is, in some sense, the counterpart of the class of barrelled spaces, well-known objects in the theory of locally convex spaces. Before stating the definition of g-barrelled groups, we give convenient notation and provide elementary background to deal with them.

Notation 4.1. *For a topological group (G, τ), the symbol $\mathcal{K}$ will stand for the family of all $\sigma(G^\wedge, G)$-compact subsets of $G^\wedge$. The topology on G of uniform convergence on the members of $\mathcal{K}$ will be denoted by $\tau_{\mathcal{K}}$.*

The family $\{K^\triangleleft,\ K \in \mathcal{K}\}$ describes a basis of zero-neighborhoods for the topology $\tau_{\mathcal{K}}$ on G. Having a basis of quasi-convex sets, $\tau_{\mathcal{K}}$ is a locally quasi-convex topology. On the other hand, any locally quasi-convex topology ν on a group G is totally determined by the family $\mathcal{E}$ of all the equicontinuous subsets that it produces in its dual group $(G, \nu)^\wedge$. More precisely, ν is the topology of uniform convergence on the sets of $\mathcal{E}$, and $\{L^\triangleleft,\ L \in \mathcal{E}\}$ is a basis of zero neighborhoods for ν. A thorough study of this topic is done in [11].

Definition 4.2. [7] A topological group (G, τ) is *g-barrelled* if every $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ is equicontinuous with respect to τ. The term g-barrelled also applies to the topology τ.

The Hausdorff abelian topological groups that are locally compact, or complete metrizable, or pseudocompact, or locally pseudocompact, or precompact Baire bounded torsion are g-barrelled groups (see [7], [16], [13], [9]).

Remarks 4.3. (i) Local quasi-convexity is not required in the definition of a g-barrelled group. Nevertheless, if (G, τ) is a g-barrelled group, and $\mathcal{Q}\tau$ is the locally quasi-convex modification of τ, then $(G,\tau)^\wedge = (G,\mathcal{Q}\tau)^\wedge$ and $(G,\mathcal{Q}\tau)$ is g-barrelled and locally quasi-convex.

(ii) There is at most one g-barrelled locally quasi-convex topology on a topological group G which is compatible with the duality $(G,G^\wedge)$.

The statement contained in (i) follows from the fact that the equicontinuous subsets with respect to τ and $\mathcal{Q}\tau$ coincide ([11, Proposition 7.1]). The proof of (ii) is straightforward.

An interesting feature of locally quasi-convex g-barrelled groups is that they are topologically isomorphic to duals of precompact groups. For further use we express this property as a lemma.

Lemma 4.4. [4, 2.6] *Let (G,τ) be a locally quasi-convex, g-barrelled group. Then, the natural evaluation mapping $e : (G,\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge$ is a topological isomorphism.*

As we explained in the introduction, there are group dualities without g-barrelled topologies. Now the question is to find conditions on a topological group (G,τ) or in its dual $G^\wedge$, which imply the existence of g-barrelled topologies on G compatible with τ. By the remark (i) in 4.3, the question may be equivalently reformulated as follows: under which conditions is there a g-barrelled **locally quasi-convex** topology in a fixed duality $(G,G^\wedge)$?

Proposition 4.5. *Let (G,τ) be a MAP topological group. There exists a g-barrelled locally quasi-convex topology on G in the dual pair $(G,G^\wedge)$ if and only if $\tau_{\mathcal{K}}$ is compatible with τ.*

Proof. Observe first that all the topologies compatible with τ produce, in the common dual group $G^\wedge$, the same family of $\sigma(G^\wedge,G)$-compact subsets as τ. In other words, the family $\mathcal{K}$ is an invariant of the duality $(G,G^\wedge)$.

Assume now that $\tau_{\mathcal{K}}$ is compatible with τ. Clearly, every $K \in \mathcal{K}$ is equicontinuous with respect to $\tau_{\mathcal{K}}$. So, as indicated in Remark 4.3 (ii), $\tau_{\mathcal{K}}$ is the unique g-barrelled, locally quasi-convex topology on G which is in the dual pair $(G,G^\wedge)$.

For the converse implication, we prove first that the existence of a g-barrelled locally quasi-convex topology μ on G compatible with the pair $(G,G^\wedge)$ implies $\mu = \tau_{\mathcal{K}}$. Assume μ meets the mentioned requirements. Choose $V \subset G$ a quasi-convex neighborhood of zero in μ. Then $V^\triangleright$ is $\sigma(G^\wedge,G)$-compact, and therefore $V^{\triangleright\triangleleft}$ is a neighborhood of zero. From $V^{\triangleright\triangleleft} = V$, we obtain $\mu \le \tau_{\mathcal{K}}$. For the converse inequality, fix $K \in \mathcal{K}$. Since (G,μ) is g-barrelled, K is equicontinuous with respect to μ. Thus, it exists a μ-neighborhood of zero W such that $W \subset K^\triangleleft$. This implies that $\tau_{\mathcal{K}} \le \mu$. □

By remark (i) in 4.3, if $\tau_{\mathcal{K}}$ is not compatible, there are no g-barrelled topologies in the dual pair. Next we give a necessary and sufficient condition for $\tau_{\mathcal{K}}$ to be compatible with τ, or equivalently, to be compatible with $\sigma(G,G^\wedge)$. We recall that a topological group (G,τ) is semireflexive if the canonical mapping $\alpha : G \to G^{\wedge\wedge}$ is surjective.

Proposition 4.6. *Let (G,τ) be an abelian MAP topological group. The following statements are equivalent:*

(1) *$\tau_{\mathcal{K}}$ is compatible with $\sigma(G,G^\wedge)$, that is $(G,\tau_{\mathcal{K}})^\wedge = G^\wedge$.*
(2) *The group $(G^\wedge,\sigma(G^\wedge,G))$ is semireflexive.*

Proof. The proof is an easy consequence of the following argument. By Comfort-Ross Theorem, $(G^\wedge,\sigma(G^\wedge,G))^\wedge$ can be algebraically identified with G by means of the evaluation mapping $e : G \to (G^\wedge,\sigma(G^\wedge,G))^\wedge$, defined by $x \mapsto \widetilde{x} : \phi \mapsto \phi(x)$. On the other hand the topology for the Pontryagin dual $(G^\wedge,\sigma(G^\wedge,G))^\wedge$ is the topology of uniform convergence on the $\sigma(G^\wedge,G)$-compact subsets of $G^\wedge$. A zero neighborhood basis for $(G^\wedge,\sigma(G^\wedge,G))^\wedge$ is given by the family $\{K^\triangleright,\ K \in \mathcal{K}\}$, whilst a zero neighborhood basis for $(G,\tau_{\mathcal{K}})$ is given by the family $\{K^\triangleleft,\ K \in \mathcal{K}\}$. The direct and inverse polars can be identified since e is bijective and $e(K^\triangleleft) = K^\triangleright$. Thus, e is a topological isomorphism:

$$(G,\tau_{\mathcal{K}}) \overset{e}{\approx} (G^\wedge, \sigma(G^\wedge, G))^\wedge$$

Taking now duals in both sides we obtain:

$$(G,\tau_{\mathcal{K}})^\wedge \approx (G^\wedge, \sigma(G^\wedge, G))^{\wedge\wedge}$$

In order to prove that (1) $\Rightarrow$ (2), assume that $\tau_{\mathcal{K}}$ is compatible with τ, that is $(G,\tau_{\mathcal{K}})^\wedge = G^\wedge$. From the last isomorphism, it follows that $(G^\wedge, \sigma(G^\wedge, G))$ is semireflexive.
The implication (2)$\Rightarrow$ (1) also follows from the mentioned isomorphism. □

The results of Propositions 4.5 and 4.6 yield the following:

Theorem 4.7. *Let (G,τ) be a MAP topological group. The following assertions are equivalent:*

1. *There exists a g-barrelled topology on G compatible with the duality $(G, G^\wedge)$.*
2. *$(G^\wedge, \sigma(G^\wedge, G))$ is semireflexive.*
3. *The topology $\tau_{\mathcal{K}}$ on G is compatible with τ.*

For a MAP topological group (G,τ) which does not satisfy the conditions of the preceding theorem, it is natural to ask if $(G,\tau_{\mathcal{K}})$ can still be g-barrelled in the new duality it generates. We provide below a result in this line. First, recall the notion of *determined subgroup.*

Definition 4.8. A subgroup Y of an abelian topological group (X,τ) is said *to determine* X if the inclusion $i : (Y, \tau_{|Y}) \to (X,\tau)$ has a dual mapping $i^\wedge : (X,\tau)^\wedge \to (Y,\tau_{|Y})^\wedge$ which is a topological isomorphism. It is frequent to call Y a *determined subgroup of X.*

The above mentioned dual groups carry the compact-open topology, in other words they are Pontryagin duals. Clearly, the restriction mapping $i^\wedge$ is continuous without additional conditions on X or Y. If Y is dense in X, then $i^\wedge$ is monomorphism. Thus, the only specific property to be a determined subgroup is that the mapping $i^\wedge$ must be open. This is achieved if for every compact set $K \subset X$ there is a compact set $L \subset Y$ such that $i^\wedge(K^\triangleright) \supset L^\triangleright$. In what follows we relax this expression and simply say that $L^\triangleright \subset K^\triangleright$, which permits also to say that the compact-open topologies in $X^\wedge$ and $Y^\wedge$ coincide.

Theorem 4.9. *Let (G,τ) be a MAP topological group, and let $\mathcal{J} := (G,\tau_{\mathcal{K}})^\wedge$. The group $(G,\tau_{\mathcal{K}})$ is g-barrelled iff $(G^\wedge, \sigma(G^\wedge, G))$ determines $(\mathcal{J}, \sigma(\mathcal{J}, G))$.*

Proof. Clearly $G^\wedge \subset \mathcal{J} \subset Hom(G,\mathbb{T})$ and $(G^\wedge, \sigma(G^\wedge, G))$ is a topological subgroup of $(\mathcal{J}, \sigma(\mathcal{J}, G))$. Since G is MAP, $G^\wedge$ is dense in $Hom_p(G,\mathbb{T})$. Therefore $G^\wedge$ is also dense in $(\mathcal{J}, \sigma(\mathcal{J}, G))$ and their dual groups can be algebraically identified, which we simply write as an equality:

$$(G^\wedge, \sigma(G^\wedge, G))^\wedge = (\mathcal{J}, \sigma(\mathcal{J}, G))^\wedge$$

$\Rightarrow$) Assume that $(G,\tau_{\mathcal{K}})$ is g-barrelled. We must prove that the compact-open topology in $(G^\wedge, \sigma(G^\wedge, G))^\wedge$ and in $(\mathcal{J}, \sigma(\mathcal{J}, G))^\wedge$ coincide (the underlying set of both of them can be identified to G).

To this end, fix $K \subset \mathcal{J}$ a $\sigma(\mathcal{J}, G)$-compact subset. We must find a $\sigma(G^\wedge, G)$-compact subset $L \subset G^\wedge$, such that $L^\triangleright \subset K^\triangleright$. Since $(G,\tau_{\mathcal{K}})$ is g-barrelled, there exists a $\tau_{\mathcal{K}}$-zero neighborhood V such that $K \subset V^\blacktriangleright$ (the black triangle symbol indicates that the polar is taken in $\mathcal{J}$). By the definition of $\tau_{\mathcal{K}}$, $V \supset L^\triangleleft$ for some $L \subset G^\wedge$ which is $\sigma(G^\wedge, G)$-compact. Thus $K \subset L^{\triangleleft\blacktriangleright}$, and taking polars on both sides we obtain: $K^\triangleright \supset (L^{\triangleleft\blacktriangleright})^\triangleright$.

On the other hand $L^\triangleleft \subset G$ is quasi-convex in $\tau_{\mathcal{K}}$, therefore $L^\triangleleft = (L^\triangleleft)^{\blacktriangleright\triangleleft} = (L^{\triangleleft\blacktriangleright})^\triangleleft$.

Implementing this in the above expression, we get:

$$K^\triangleright \supset (L^{\triangleleft\blacktriangleright})^\triangleright = (L^\triangleleft)^{\blacktriangleright\triangleleft} = L^\triangleleft$$

Finally the inverse polar $L^\triangleleft$ can be identified with $L^\triangleright$, since $(G^\wedge, \sigma(G^\wedge, G))^\wedge = G$. Thus, we can simply write $K^\triangleright \supset L^\triangleright$, which proves that $(G^\wedge, \sigma(G^\wedge, G))$ determines $(\mathcal{J}, \sigma(\mathcal{J}, G))$.

$\Leftarrow$) In order to prove that $(G, \tau_{\mathcal{K}})$ is g-barrelled, fix now a $\sigma(\mathcal{J}, G)$-compact subset K of $\mathcal{J}$. By the assumption, there exists a $\sigma(G^\wedge, G)$-compact $L \subset G^\wedge$ such that $K^\triangleright \supset L^\triangleright$. Here the polars are taken in $(\mathcal{J}, \sigma(\mathcal{J}, G))^\wedge$ and $(G^\wedge, \sigma(G^\wedge, G))^\wedge$ respectively, but both dual groups are identified, as said above. Taking inverse polars with respect to $\mathcal{J}$ we can write: $K^{\triangleright\triangleleft} \subset L^{\triangleright\triangleleft}$, Thus:

$$K \subset K^{\triangleright\triangleleft} \subset L^{\triangleright\triangleleft} \subset L^{\triangleleft\triangleright}$$

Since $L^\triangleleft$ is a neighborhood of zero in $\tau_{\mathcal{K}}$, K is equicontinuous and therefore $(G, \tau_{\mathcal{K}})$ is g-barrelled. □

Concerning the last theorem, it arises the question whether the claim "$(G^\wedge, \sigma(G^\wedge, G))$ determines $(\mathcal{J}, \sigma(\mathcal{J}, G))$" is always true. We expect a negative answer, thus we formulate the open problem:

Question 2. Give an example of a topological group (G, τ) such that $(G^\wedge, \sigma(G^\wedge, G))$ does not determine $(\mathcal{J}, \sigma(\mathcal{J}, G))$, where $\mathcal{J} = (G, \tau_{\mathcal{K}})^\wedge$.

Denote by $\mathcal{B}$ the class considered in [17] of all precompact Hausdorff abelian groups which are the Bohr reflection of a locally compact group. Explicitly, $(G, w) \in \mathcal{B}$ if there exists a locally compact group topology τ on G such that $\tau^+ = w$. We end this section with two results which might complement the contents of [17]. The first one provides examples of groups which are not in $\mathcal{B}$. Loosely speaking, if a topological group (G, τ) gives rise to a duality which contains a locally quasi-convex g-barrelled non locally compact topology, then (G, τ^+) is not in $\mathcal{B}$.

Proposition 4.10. *Let (G, τ) be a topological group such that $\tau_{\mathcal{K}}$ is compatible with τ. If $(G, \tau_{\mathcal{K}})$ is not locally compact, then $(G, \tau^+) \notin \mathcal{B}$.*

Proof. This is an easy consequence of the uniqueness of a g-barrelled locally quasi-convex topology on G compatible with the duality $(G, G^\wedge)$. The topology $\tau_{\mathcal{K}}$ already meets these requirements. If G could be equipped with a locally compact topology μ, then (G, μ) would be a g-barrelled, locally quasi-convex group. Therefore μ cannot be compatible with τ. Thus, $\mu^+ \neq \tau^+$ and $(G, \tau^+) \notin \mathcal{B}$. □

Proposition 4.11. *The class $\mathcal{B}$ is not (countably) productive.*

Proof. Take a family $\{(G_i, w_i) \in \mathcal{B},\ i \in I\}$, whose members are non-compact and $|I| \geq \aleph_0$. By the definition of $\mathcal{B}$, for every $i \in I$ there exists a locally compact topology τ_i in G_i such that $(G_i, \tau_i^+) = (G_i, w_i)$. Observe that the product $G := \prod (G_i, w_i)$ is a precompact Hausdorff group. Further, the product topology $\prod w_i$ is the minimum of all the locally quasi-convex topologies on G compatible with the duality $(G, G^\wedge)$.
Since the product of g-barrelled groups is also g-barrelled ([4, 3.4]), $\prod \tau_i$ is a g-barrelled locally quasi-convex topology in the duality $(G, G^\wedge)$. Clearly (even if I is countable) $\prod \tau_i$ is not locally compact, and Proposition 4.10 applies. □

5. The family $\mathcal{D}_G$ of compatible topologies on a discrete group G

Let G be a group, δ the discrete topology on G and $\mathcal{D}_G$ the family of all group topologies on G compatible with δ. All the elements in $\mathcal{D}_G$ lie between δ^+ and δ and have $Hom(G, \mathbb{T})$ as character group. Glicksberg Theorem applied to (G, δ) yields that any topology $\tau \in \mathcal{D}_G$ has the same family of compact subsets as δ. Thus, the topologies in $\mathcal{D}_G$ give rise to the same dual group, algebraically and topologically. Namely: $(G, \tau)^\wedge = Hom_p(G, \mathbb{T})$, for all $\tau \in \mathcal{D}_G$.

We characterize now the family $\mathcal{D}_G$ in the class of MAP topological groups.

Proposition 5.1. *Let (G, τ) be a MAP group. The following statements are equivalent:*

1. $\tau \in \mathcal{D}_G$.
2. *$(G, \tau)^\wedge$ is a compact group and the τ-compact subsets of G are finite.*

(3) *$(G^\wedge, \sigma(G^\wedge, G))$ is a compact group.*

Proof. (1) $\Rightarrow$ (2). As said in the preceding comments, the τ-compact subsets of G are finite and $(G,\tau)^\wedge = Hom_p(G,\mathbb{T})$. Since $Hom_p(G,\mathbb{T})$ carries the pointwise convergence topology, it is a closed subgroup in the product $\mathbb{T}^G$. Therefore $Hom_p(G,\mathbb{T})$ is a compact Hausdorff group.

(2) $\Rightarrow$ (3). Clearly, if the τ-compact subsets are finite, the compact-open topology in $G^\wedge$ coincides with the pointwise convergence topology, thus $(G^\wedge, \sigma(G^\wedge, G)) = (G,\tau)^\wedge$ is compact.

(3) $\Rightarrow$ (1). By the assumption, $(G^\wedge, \sigma(G^\wedge, G))$ is a closed subgroup of the compact group $Hom_p(G,\mathbb{T})$. Assume by contradiction that $G^\wedge \neq Hom(G,\mathbb{T})$. Then, there exists a non-null continuous character on $Hom_p(G,\mathbb{T})$ which is null in $G^\wedge$. Since the continuous characters on $Hom_p(G,\mathbb{T})$ are precisely the evaluations on points of G, there must exist a non null $x \in G$ such that $\phi(x) = 1$ for all $\phi \in G^\wedge$. This contradicts the fact that (G,τ) is MAP. Therefore, it must be $G^\wedge = Hom(G,\mathbb{T})$ which proves (1). □

Corollary 5.2. *On an abstract group G, the family $\mathcal{D}_G$ does not contain any nondiscrete k-group topology. In particular, every metrizable nondiscrete group has discontinuous characters.*

Proof. Since all the topologies compatible with δ give rise to the same family of compact subsets, there is at most one k-group topology in $\mathcal{D}_G$. On the other hand, δ is already a k-group topology in $\mathcal{D}_G$. Thus the first claim is proved.
If μ is a metrizable nondiscrete group topology on G, it is a k-group topology. Therefore $\mu \notin \mathcal{D}_G$, which means that $(G,\mu)^\wedge \neq Hom(G,\mathbb{T})$. □

Corollary 5.3. *Every nondiscrete Mackey group (or g-barrelled group) G admits non-continuous characters.*

Proof. Let (G,μ) be a Mackey group. Then $(G,\mu)^\wedge \neq Hom(G,\mathbb{T})$, for otherwise μ would be compatible with δ and by the assumption $\mu = \delta$. If (G,μ) is g-barrelled, the previous argument can be applied to $(G,\mathcal{Q}\mu)$, which by Remark 4.3 (i) is a Mackey group, and admits the same character group as (G,μ). □

The property of "having finite compact subsets" is not sufficient to characterize the elements of $\mathcal{D}_G$ in the class of MAP topological groups. The next statement gives a related feature.

Proposition 5.4. *Let (G,τ) be a topological group whose compact subsets are finite. Then, (G,τ) is semireflexive.*

Proof. By the assumption, the Pontryagin dual of (G,τ) is $(G^\wedge, \sigma(G^\wedge, G))$. Since the character group of $(G^\wedge, \sigma(G^\wedge, G))$ is algebraically isomorphic to G, we have that $G^{\wedge\wedge}$ and G are isomorphic as groups. □

The more restrictive assumption that the $\sigma(G, G^\wedge)$-compact subsets of G are finite, yields g-barrelledness, as specified next:

Proposition 5.5. *Let (G,τ) be a topological group. The following statements are equivalent:*

(1) *The $\sigma(G, G^\wedge)$-compact subsets of G are finite.*
(2) *The Pontryagin dual of (G,τ) coincides with $(G^\wedge, \sigma(G^\wedge, G))$ and it is g-barrelled.*

Proof. (1) $\Rightarrow$ (2). Since every τ-compact subset of G is $\sigma(G, G^\wedge)$-compact, as in the proof of 5.4, we obtain that $(G^\wedge, \sigma(G^\wedge, G))$ is the Pontryagin dual of (G,τ). In order to prove that $X := (G^\wedge, \sigma(G^\wedge, G))$ is g-barrelled, fix $K \subset X^\wedge$ compact with respect to $\sigma(X^\wedge, X)$. Take into account that $X^\wedge$ and G are algebraically isomorphic and the topology $\sigma(X^\wedge, X)$ can be identified with $\sigma(G, G^\wedge)$. Thus, K can be considered a $\sigma(G, G^\wedge)$-compact subset of G. By (1) K is finite and therefore equicontinuous.

(2) ⇒ (1). Fix a $\sigma(G, G^\wedge)$-compact subset $L \subset G$. By the above mentioned identifications, L can be considered as a $\sigma(X^\wedge, X)$-compact subset of $X^\wedge$. Since $X = (G^\wedge, \sigma(G^\wedge, G))$ is g-barrelled, L is equicontinuous with respect to $\sigma(G^\wedge, G)$, which is a precompact topology. Therefore L is finite.

□

If $\tau \in \mathcal{D}_G$, clearly τ satisfies (1) and (2) in Proposition 5.5. The converse does not hold as the next example shows.

Example 5.6. A topological group G which is not in $\mathcal{D}_G$ and the $\sigma(G, G^\wedge)$-compact subsets are finite.

Let $\mathcal{L}$ be a second category subgroup of $\mathbb{T}$ and consider $\mathbb{T}$ as the group of characters on $\mathbb{Z}$. If $G := (\mathbb{Z}, \sigma(\mathbb{Z}, \mathcal{L}))$, clearly G is a precompact Hausdorff group such that $G^\wedge = \mathcal{L}$. The $\sigma(G^\wedge, G)$-topology on $\mathcal{L}$ coincides wih the topology induced on $\mathcal{L}$ as a subspace of $\mathbb{T}$. Thus, $\mathcal{L}$ is separable and $(G^\wedge, \sigma(G^\wedge, G))$ is g-barrelled (by [7, 1.6]). Since G satisfies (2) in Proposition 5.5, the $\sigma(\mathbb{Z}, \mathcal{L})$-compact subsets of $\mathbb{Z}$ are finite. Obviously, $\sigma(\mathbb{Z}, \mathcal{L}) \notin \mathcal{D}_G$.

Observe that G is not reflexive: its dual group is $\mathcal{L}$ and the bidual $G^{\wedge\wedge}$ is algebraically isomorphic to $\mathbb{Z}$, but the compact-open topology on $G^{\wedge\wedge}$ is discrete. In fact, being $\mathcal{L}$ dense in the metrizable compact group $\mathbb{T}$, it has the same dual as $\mathbb{T}$ algebraically and topologically. Since G is non-discrete, it is not topologically isomorphic to $G^{\wedge\wedge}$.

The starting group G is not g-barrelled, because it is countable and nondiscrete. Its Pontryagin dual $(G^\wedge, \sigma(G^\wedge, G))$ is g-barrelled, metrizable, noncompact, neither countably compact.

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