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ON THE INTRINSIC INSTABILITY OF THE ADVECTION–DIFFUSION EQUATION

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Abstract. *A new theory for the advective–diffusive phenomenon is described in this study and the causes for the failure of the conventional numerical methods for this problem are investigated.*

It is shown that Fick's law —the constitutive equation of the transport problem— is the cause of the appearance of oscillations in the numerical solutions of predominantly advective problems. Fick's law leads to the unreasonable result that mass can propagate at an infinite speed.

We propose a new formulation for the advective–diffusive problem by using a constitutive equation derived by M. Carlo Cattaneo in 1958 for thermodynamic and pure–diffusion problems. This new approach overcomes the problem of mass propagation at an infinite speed. It is also shown that the advective–diffusive problem is a wave–like problem. Hence, a pollutant diffuses like a wave in a fluid.

A detailed analysis of the new equations shows an important conclusion: a critical fluid velocity exists for each advective–diffusive problem. When fluid velocity is greater or equal than this critical speed the steady state problem is not anymore a well–posed problem and the transient problem is as well ill–posed if it is stated as a bounduary value problem. In this case we should formulate the advective–diffusive problem as an initial value problem. Furthermore, we propose stability conditions for the steady state advective–diffusive problem.

Several problems have been solved to check the good behaviour of the numerical solution of the new equations and the proposed stability conditions.

1 INTRODUCTION

The numerical resolution of fluid dynamic problems is quite difficult particularly at large fluid velocities [1]. The Finite Element Method has been successfully used in many engineering problems, but it has important disadvantages when it is used to solve fluid dynamic problems with significant convective terms. In these cases the reasons for the inaccurate solution are the nonlinear oscillatory nature of the Navier–Stokes equations. The oscillatory nature is inherent to the formulation and it persists if we attempt to solve a creeping flow problem —in this case we can neglect the nonlinear term— in a simple domain.

The advection–diffusion transport equation can be considered as the linear and scalar version of the shallow water equations [2] and it shows the oscillatory nature of the fluid dynamic problems. The numerical solution of this equation is quite complicated [3]. In recent years many stabilization techniques have been proposed for convection dominated transport problems [4, 5, 6], but they are not appropriate to solve three dimensional transient problems [7, 8, 9, 10].

In this paper we give an explanation to the oscillatory behaviour of the numerical solution of transport problems. In addition, we propose an stabilization technique. This stabilization technique is of a different nature as to the nowadays usual methods. We show that Fick's law —the constitutive equation of the transport problem— is the cause of the appearance of oscillations in the numerical solutions of predominantly advective problems. Fick's law leads to the unreasonable result that mass propagates at an infinite speed. The proposed stabilization technique consists in reformulating the transport problem by using Cattaneo's equation as the constitutive equation [12, 13]. This constitutive equation has been derived by M. Carlo Cattaneo in 1958 for thermodynamic and pure–diffusion problems. This new approach overcomes the problem of mass propagation at an infinite speed [11].

Taking all of this into account, the objectives of this paper are as follows: first, we shall prove that Fick's law leads to oscillations in the numerical solution of the advective-diffusive problems. And then we present a numerical stabilitation for the new formulation. In the first part of this study we review the classic transport problem formulation [1] and we prove that in this theory mass can propagate at an infinite speed. The next step will be to develop the formulation of the transport problem by using Cattaneo's equation and to study the changes in the corresponding solutions. Finally, several problems have been solved to check the stability of the new proposed equations.

2 FORMULATION OF THE TRANSPORT PROBLEM

2.1 General aspects and notation

We will assume that hydrodynamic equations are not coupled with transport equations. Hence, we can solve the hydrodynamic equations and subsequently solve the transport equations by using the density and velocity fields previously calculated. Thus, we will assume that velocity and density fields are known.

We use an Eulerian description of the motion. Let the vectorial functions $\boldsymbol{a}$, $\boldsymbol{q}$, $\boldsymbol{f}$ be fluid velocity, contaminant flux and contaminant source, respectively, and let the scalar functions ρ, u be fluid density and contaminant concentration. Finally, let the tensorial function $\widetilde{\boldsymbol{K}}$ be the diffusivity tensor. We suppose that the above functions are sufficiently smooth.

2.2 Classic formulation of the transport problem

The basic equations that describe the classic formulation of the transport problem are the following:

$$\rho\frac{\partial u}{\partial t} + \rho\,\boldsymbol{a}\cdot\mathbf{grad}(u) + \mathrm{div}(\rho\,\boldsymbol{q}) - \boldsymbol{f} = 0 \tag{1}$$

$$\boldsymbol{q} = -\rho\widetilde{\boldsymbol{K}}\mathbf{grad}(u) \tag{2}$$

where (1) is the equilibrium equation of the transport problem —mass pollutant conservation law— and (2) is the Fick's law —the constitutive equation of the advective–diffusive problem—. Moreover, $\boldsymbol{a}$ and ρ satisfy the hydrodynamic equations. We will show that the above formulation leads to mass propagation at an infinite speed. Consider an incompressible, homogeneous, isotropic and one-dimensional media (hence if $\boldsymbol{I}$ is the identity tensor, $\widetilde{\boldsymbol{K}} = \widetilde{k}\boldsymbol{I}$). We don't consider source terms. We suppose that the domain is long enough to be approximated as infinitely long, and we assume that the pollutant is added to the media as a rapid pulse. In this case, if we call $k = \rho\widetilde{k}$, we should solve the following problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \forall x \in \mathbb{R}, \quad t > 0,\\ &u(x, t = 0) = \delta(x), \quad \forall x \in \mathbb{R},\\ &\lim_{x\to\pm\infty} u(x,t) = 0, \quad t > 0. \end{aligned} \tag{3}$$

where $\delta(x)$ is the Dirac distribution. This problem can be solved by using a Fourier transform in the spatial coordinate. The solution of (3) is

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}, \quad \forall x \in \mathbb{R}, \quad t > 0. \tag{4}$$

If we fix a time $t = \tau_0 > 0$, we can define

$$\tilde{u}(x) = u(x,\tau_0) = \frac{1}{\sqrt{4\pi k\tau_0}} e^{-\frac{x^2}{4k\tau_0}} \tag{5}$$

which is the Gauss distribution function and hence $\tilde{u}(x) > 0, \quad \forall x \in \mathbb{R}$. The previous assertion implies that polluted fluid exists in the whole domain $\forall t > 0$. Moreover, at the initial time $u(x,0)=0 \quad \forall x \neq 0$, i.e., there is pollutant only in the origin of coordinates. Therefore, if we fix a generic point x_0, the following equation holds

$$u(x_0,\tau_0) > 0, \quad \forall \tau_0 > 0. \tag{6}$$

Hence, the mean velocity of the particles in $(x = x_0, t = t_0)$ is $v = x_0/\tau_0$ and this velocity is not bounded because the above assertion holds $\forall \tau_0 > 0$ and $\forall x_0 \in \mathbb{R}$. Figure 1 shows the solution of (3) for $t = 4$ and $k = 1$.

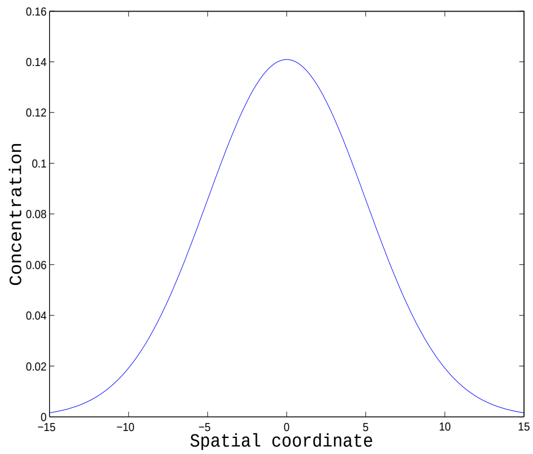

Figure 1: Solution of (3) for $t = 4$ and $k = 1$ (classic formulation of the transport problem).

2.3 Formulation of the transport problem by using Cattaneo's law

We will derive this formulation by substituting equation (2) by Cattaneo's law. This constitutive equation involves a tensorial function $\widetilde{\boldsymbol{\tau}}$. This mapping transforms each point $(\boldsymbol{x}, t)$ of the fluid path line into that point relaxation tensor. The coordinates of this relaxation tensor are specific diffusion process times. Up to now, Cattaneo's equation has been only used in non–advective thermal problems [12, 13]. Thus, we have to find Cattaneo's equation with convective term [11]. This equation has been derived by using a Lagrangian description and in Eulerian coordinates can be written as

$$\boldsymbol{q} + \widetilde{\boldsymbol{\tau}} \left(\frac{\partial \boldsymbol{q}}{\partial t} + \mathbf{grad}(\boldsymbol{q})\, \boldsymbol{a} \right) = -\rho \widetilde{\boldsymbol{K}} \mathbf{grad}(u) \tag{7}$$

The relations (7) and (1) are the basic equations for the transport problem described by using Cattaneo's law. In these equations $\boldsymbol{a}$ and ρ are solutions of the hydrodynamic equations. In order to compare the solutions of the classic formulation with the solutions of this new formulation we now solve a problem similar to (3). In this case we need two initial conditions because this problem involves second order derivatives with respect to the time. Then, we consider an incompressible, homogeneous, isotropic —hence, $\widetilde{\boldsymbol{K}} = \widetilde{k}\boldsymbol{I}$ and $\widetilde{\boldsymbol{\tau}} = \widetilde{\tau}\boldsymbol{I}$— one–dimensional and non–convective media. We call $k = \rho\widetilde{k}$ and $\tau = \rho\widetilde{\tau}$. With the above assumptions we can write this problem as [11]:

$$\begin{aligned}
&\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \forall x \in \mathbb{R}, \quad t > 0,\\
&u(x, t = 0) = \delta(x),\ \forall x \in \mathbb{R},\\
&\frac{\partial u}{\partial t}(x, t = 0) = 0,\ \forall x \in \mathbb{R},\\
&\lim_{x\to\pm\infty} u(x, t) = 0,\ t > 0.
\end{aligned} \tag{8}$$

We may solve (8) by using a Laplace and Fourier transform and we obtain:

$$u(x,t) = \begin{cases} \frac{1}{2}e^{-\frac{c^2}{2k}t}\left[\delta(|x| - ct) + \frac{c}{2k}I_0\left(\frac{c}{2k}\sqrt{c^2t^2 - x^2}\right) + \frac{c^2}{2k}t\frac{I_1\left(\frac{c}{2k}\sqrt{c^2t^2-x^2}\right)}{\sqrt{c^2t^2-x^2}}\right], & |x| \le ct\\ \\ 0, & |x| > ct \end{cases}$$

where I_0 and I_1 are the modified Bessel functions of the first kind of order 0 and 1 and c is the mass wave celerity defined by:

$$c = \sqrt{\widetilde{k}/\widetilde{\tau}} = \sqrt{k/\tau}. \tag{9}$$

We compare in figure 2 the solutions of (3) and (8). Clearly, if we use Cattaneo's equation a wave front exist which advances with a celerity c.

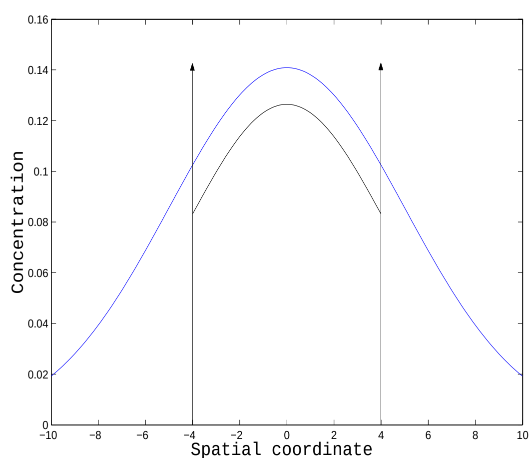

Figure 2: Comparison at time $t = 4$ between the solution of (3) —upper line— and the solution of (8) —lower line—. The parameters k and τ have a value of one.

3 STUDY OF THE TRANSPORT PROBLEM BY USING CATTANEO'S LAW

We have analyzed in previous sections the consequences of using Cattaneo's law in pure–diffusion problems. In the next sections we will consider the advective–diffusive problem. Our first step will be to undertake a theorical study of the one–dimensional problem in an incompressible, homogeneous and isotropic media. With these assumptions we can examine the physical problem in a bounded subdomain of $\mathbb{R}$ by solving the initial and boundary value problem:

$$\begin{aligned}
&\tau\frac{\partial^2 u}{\partial t^2}+2\tau a\frac{\partial^2 u}{\partial x\partial t}-(k-\tau a^2)\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0,\ \forall x\in[0,L],\quad t>0,\\
&u(x,0)=f(x),\ \forall x\in[0,L],\\
&\frac{\partial u}{\partial t}(x,0)=g(x),\ \forall x\in[0,L],\\
&u(0,t)=u_0(t),\quad t>0,\\
&u(L,t)=u_L(t),\quad t>0.
\end{aligned}\tag{10}$$

If we use the coordinate transformation given by:

$$\begin{aligned}
\xi&=x-at\\
\eta&=t,
\end{aligned}\tag{11}$$

we obtain

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \eta^2}-c^2\frac{\partial^2 u}{\partial \xi^2}=-\frac{1}{\tau}\frac{\partial u}{\partial \eta},\quad \forall\xi\in[-a\eta,L-a\eta],\quad \eta>0,\\
&u(\xi,0)=f(\xi),\quad \forall\xi\in[-a\eta,L-a\eta],\quad \eta>0,\\
&\frac{\partial u}{\partial \eta}(\xi,0)=g(\xi)+a\frac{df}{d\xi}(\xi),\quad \forall\xi\in[-a\eta,L-a\eta],\quad \eta>0\\
&u(\xi=-a\eta,\eta)=u_0(\eta),\quad \eta>0\\
&u(\xi=L-a\eta,\eta)=u_L(\eta),\quad \eta>0.
\end{aligned}\tag{12}$$

The solution of (12) is the concentration distribution as seen by an observer who moves with the fluid. In this reference system the boundary is defined by two parallel straight lines in the (ξ,η) plane. Therefore, the boundary has a constant length but it moves with velocity a. Our next step will be to show that problem (10) is not well–posed if $|a|>c$, being c the mass wave celerity defined by (9). We will prove this assertion only if $a>0$ —fluid moves in the positive sense of the spatial coordinate—. If $a<0$ the proof of the proposition is similar. With this hyphotesis ($a>0$), we have to show that (10) is not well–posed when $a\geq c$. Figure 3 shows a well–posed transport problem with $a<c$.

In this problem we consider two regions R_1 and R_2 of the whole domain. These regions are divided by the characteristic lines $\xi-c\eta=0$ and $\xi+c\eta=L$. In region R_1 we can

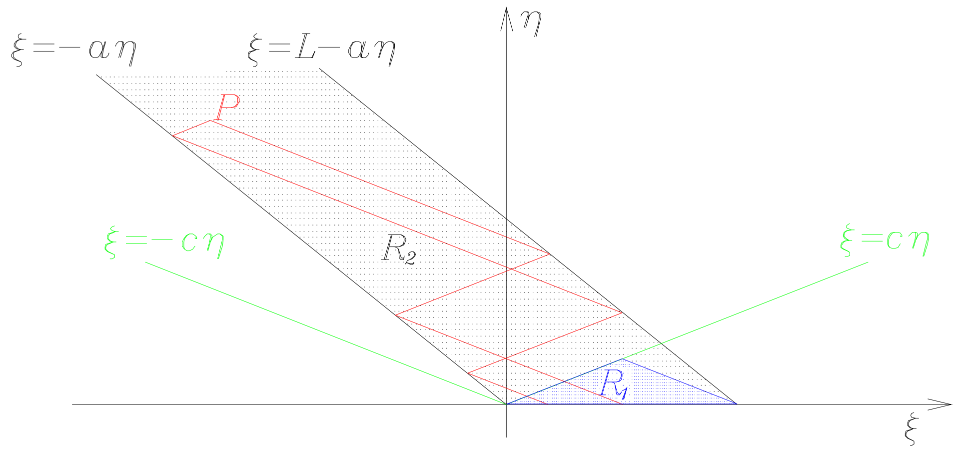

Figure 3: Example of a transport problem by using Cattaneo's law. This is a well–posed problem when it is stated as an initial and boundary value problem because fluid velocity is smaller than mass wave celerity c.

solve as in an infinite domain and get $u(\xi, \eta)$ as a function of the initial conditions. In region R_2 the solution is modified by the boundary conditions and, hence, the solution as in an infinite domain is not valid anymore. But at any point $P \in R_2$ we can give the solution of (10) as a function of the initial conditions and the prescribed values in the boundary points which intersect with the characteristic lines as shown in figure 3.

On the other hand, we show in figure 4 a problem with $a > c$. We will prove that this problem is not well–posed if it is stated as an initial and boundary value problem.

Let the set of points C be a subset of the boundary $\xi = L - a\eta$. The solution of (10) at a point of C —for example Q— is uniquely determined by the initial data. Thus, unless the prescribed values in the boundary are equal than the values obtained from the initial data, a global solution does not exist. Consequently, the problem is not a well–posed one in this case. From a numerical point of view, when $a \approx c$, $a < c$ the solution of (10) is oscillatory. This oscillation takes place between the solution determined by the initial data and the solution determined by the boundary conditions. Note that we obtain a double–valued solution in the downstream boundary. This is a logical result because it is not possible to transmit information towards upstream points from points on the downstream boundary. Thus the solution in downstream points is uniquely determined by the upstream flow. Since the solution of the transport problem by using Cattaneo's law are two pollution waves that propagate with celerities $a + c$ and $a - c$, we conclude that (10) is not well–posed as a boundary value problem when both waves move in the fluid direction —in this case $a > c$— and hence, all the pollutant particles move in the fluid direction. In this case, we should state the transport problem as an initial value problem.

This problem can easily be found in nature. For instance, a superficial wave problem in free surface flow or a sound wave problem in a fluid. In the first case the physical

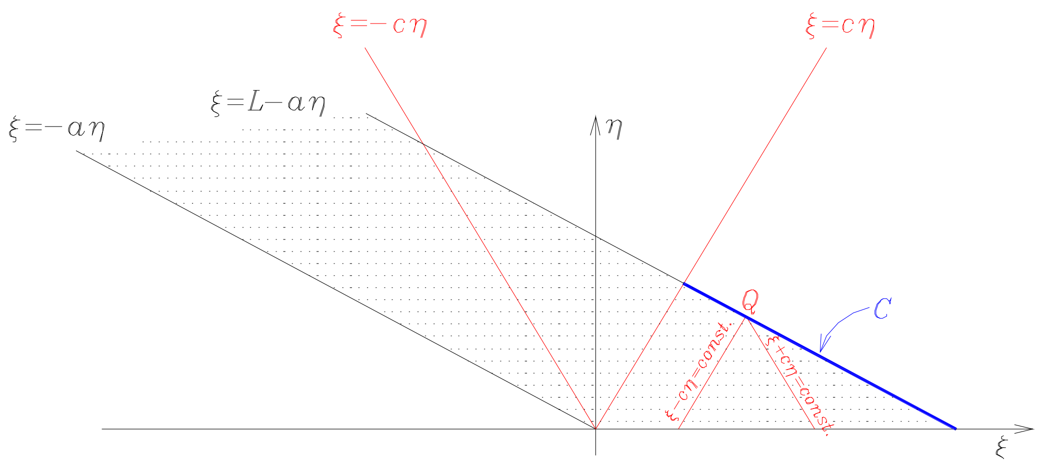

Figure 4: Example of a transport problem by using Cattaneo's law. This is an ill–posed problem when it is stated as an initial and boundary value problem because fluid velocity is greater than mass wave celerity c.

phenomenon is governed by the *Froude number*. This dimensionless number is the quotient between the fluid velocity and the superficial waves celerities, namely,

$$Fr = \frac{a}{\sqrt{gd}} \tag{13}$$

where d is the free surface depht and g is the acceleration of gravity. In the case of sound waves propagation, the problem is governed by the *Mach number*, namely,

$$M = \frac{a}{c_s} \tag{14}$$

where c_s is the sound wave celerity.

Our next step will be to define a dimensionless number that determines how we must state the transport problem. We call T this number and we will define it as

$$T = \frac{a}{\sqrt{\widetilde{k}/\widetilde{\tau}}}. \tag{15}$$

By using the above number we can know the nature of a generic transport problem. Thus, although from a physical point of view the transport problem is always an initial value problem, if $|T| < 1$ an equivalent boundary value problem exists. If $|T| \geq 1$, the fluid velocity is greater than c and a condition in the downstream boundary has no physical sense. We can now define the *critical fluid velocity* as the maximum velocity at which we can state the transport problem as a boundary value problem, namely,

$$a_c = c. \tag{16}$$

The above statement implies that the problem

$$\begin{aligned}
&a\frac{du}{dx} - (k - \tau a^2)\frac{d^2u}{dx^2} = 0; \quad x \in (0, L) \\
&u(0) = u_0 \\
&u(L) = u_L
\end{aligned} \tag{17}$$

is not well–posed when $|a| > c$, because (17) is the steady state of (10).

The final step in this section will be to derive the dimensionless transport equation. We define the dimensionless variables:

$$\widehat{x} = \frac{x}{x_0}, \quad \widehat{t} = \frac{t}{t_0}, \quad \widehat{a} = \frac{a}{a_0}. \tag{18}$$

where x_0 and t_0 are a characteristic length and a characteristic time and $a_0 = x_0/t_0$ is a characteristic velocity. If we substitute (18) into (10) and we make some algebra, the field equation takes the form:

$$\frac{\tau}{t_0}\frac{\partial^2 u}{\partial \widehat{t}^2} + 2\frac{\tau}{t_0}\frac{\partial^2 u}{\partial \widehat{t}\partial \widehat{x}} - \left(\frac{1}{Pe} - \frac{\tau}{t_0}\widehat{a}^2\right)\frac{\partial^2 u}{\partial \widehat{x}^2} + \widehat{a}\frac{\partial u}{\partial \widehat{x}} + \frac{\partial u}{\partial \widehat{t}} = 0 \tag{19}$$

where Pe is the *Péclet number*

$$Pe = \frac{a_0 x_0}{k}. \tag{20}$$

As we can see in (19) the transport problem is governed by two dimensionless numbers: τ/t_0 and the coefficient of the diffusive term. The first one is the quotient between the typical time scales of the transport problem and therefore we attempt not to involve the parameter t_0 in the second dimensionless number. It is easy to prove that

$$\frac{1}{Pe} - \frac{\tau}{t_0}\widehat{a}^2 = \frac{k - \tau a^2}{a_0 x_0}. \tag{21}$$

We call the *Héctor number* (He) as the inverse of the right side of (21), namely

$$He = \frac{a_0 x_0}{k - \tau a^2}. \tag{22}$$

The He *number* is analogous to the Péclet number for the classical transport problem. However, the He *number* has an important physical meaning. If we make some algebra in (21), we can rewrite the He *number* in terms of characteristic velocities, times or lengths of the problem. From (22)

$$He = \frac{a_0 x_0}{k - \tau a^2} = \frac{a_0 x_0}{\tau(c-a)(c+a)} \tag{23}$$

where c is the mass wave celerity defined by (9). In the above expression $c - a$ is the celerity of the wave that advances upstream, an $c + a$ is the celerity of the wave that advances downstream. In the next section we will use the physical meaning of the He *number* to stablish a stability condition for (17).

4 NUMERICAL FORMULATION OF THE TRANSPORT PROBLEM

In this section we will study the steady state transport problem in a one–dimensional, incompressible, homogeneous and isotropic media. If we use Cattaneo's equation as the constitutive equation the physical phenomenon is governed by (17). While if we use Fick's law as the constitutive equation we obtain

$$\begin{aligned} &a\frac{du}{dx} - k\frac{d^2u}{dx^2} = 0; \quad x \in (0, L) \\ &u(0) = u_0 \\ &u(L) = u_L \end{aligned} \tag{24}$$

The above problem is equivalent to (17) with $\tau = 0$. The instability of (24) has been widely studied in the bibliography [1, 3, 9] and hence, we will study only equation (17). We determine an approximation $u^h(x)$ of the solution of (17) by using the Galerkin finite element method. We use linear trial —interpolating— functions. In this case the above formulation is equivalent to the formulation obtained by using the finite difference method with a central difference approach. Both techniques lead to the same difference nodal equation [3]:

$$(1 - \gamma_{He})u_{i+1} - 2u_i + (1 + \gamma_{He})u_{i-1} = 0 \tag{25}$$

In this equation $u_i = u^h(x_i) \approx u(x_i)$, being x_i a generic interior node of the uniform partition $\mathcal{P}$ of $[0, L]$ defined by the nodes $0 = x_0 < x_1 < \cdots < x_n = L$. In addition, we have called γ_{He} the elemental He number related to the partition $\mathcal{P}$, namely

$$\gamma_{He} = \frac{ah}{2(k - \tau a^2)} \tag{26}$$

where h is the distance between two consecutive nodes of $\mathcal{P}$. In the same way, we call γ_{Pe} the elemental Péclet number related to $\mathcal{P}$

$$\gamma_{Pe} = \frac{ah}{2k} \tag{27}$$

Note that if $\tau = 0$ in the elemental He number we obtain the elemental Péclet number.

4.1 Stability conditions

In this section we show that when Cattaneo's law is used, we can stablish a stability condition for (17) because of the wave nature of the transport problem. First we determine the exact solution of the difference equation (25). It is possible to show that the nodal values of $u^h(x)$ are [3]:

$$u_i = C_1 + C_2 \left(\frac{1 + \gamma_{He}}{1 - \gamma_{He}}\right)^i \tag{28}$$

where C_1 y C_2 are constants that depend on boundary conditions. Given (28), the numerical solution of (17) will be stable if

$$|\gamma_{He}| \leq 1. \tag{29}$$

The equation (29) is a stability condition for (17). In the same way, if we take $\tau = 0$ in (29), we obtain

$$|\gamma_{Pe}| \leq 1. \tag{30}$$

and (30) is a stability condition for (24). Relations (29) and (30) seem to be useless because they can only be applied to (17) and (24). Indeed, the above assertion is true in the case of (30). However, the asymtotic behaviour of (29) is equivalent —except for a scale factor— to impose that the grid step size is smaller than typical sizes related to the waves which give the solution of the transport problem. As we said before, the waves which determine the solution propagate with a celerity $c - a$ and $c + a$. Thus, the typical sizes upstream and downstream are $\tau(c-a)$ and $\tau(c+a)$, respectively. Hence, it is possible to show [11] that

$$h < \min\left(\tau(c-a), \tau(c+a)\right) \tag{31}$$

tends to (29) when a tends to the critical velocity a_c, except for a scale factor. We will call λ this scale factor. The stability condition (31) is very important because it can be applied to complex problems by using its physical meaning.

4.2 Numerical examples

In this section we will obtain the numerical solution of (17). We use several set of values for the parameters of the problem. Three groups of numerical examples will be presented. At each group the relaxation time is a constant. In the same way, the grid step size, the diffusivity, the domain length and the boundary values are the same for all the numerical examples. However, at each group we will show three results defined by the fluid velocity a. For all examples in this study we use a 20 element discretization, $L = 1$ —thus, $h = 0.05$— and $k = 1$.

4.2.1 Group 1: Small relaxation time

The first group of results is defined by $\tau = 0.01$. This is a small value for the relaxation time τ, and hence this example is near Fick's law. By using the above values for k and τ we obtain the critical fluid velocity $a_c = \sqrt{k/\tau} = 10$. Thus, if $a \geq 10$, (17) has no physical sense. Our next step will be to calculate the maximum a value to obtain a stable solution of (17) by using the stability condition (31). To obtain the wanted result we should fix a value for the scale factor λ. The value of this scale factor is of no importance, because we can rewrite the stability conditions by using $\lambda(\tau - c)$ and $\lambda(\tau + c)$ as typical sizes. In this case, the new value for λ is one. Anyway, it is possible to prove that taking $\lambda = 4$

the stability conditions (31) tends to the exact stability conditions when a tends to a_c. Taking this into account, the solution of (17) will be unstable if

$$h > \lambda\tau\left(\sqrt{k/\tau} - a\right) \tag{32}$$

because we are using a positive value for a. If we use the above value for λ and we make some algebra in (32) we obtain $a > 8.75$. Therefore this numerical scheme will give unstable solutions when $a > 8.75$, i.e., $a > 0.875a_c$. So, we can say that the numerical solution of (17) is stable for a 87.5% of the domain of a, because (17) is not well–posed if $a > a_c$. In the first example —figure 5— we show the numerical solution and the exact solution for $a = 7$. Note that, although γ_{He} is relatively small, we obtain a stable solution for a large value of a, because the critical velocity is $a_c = 10$.

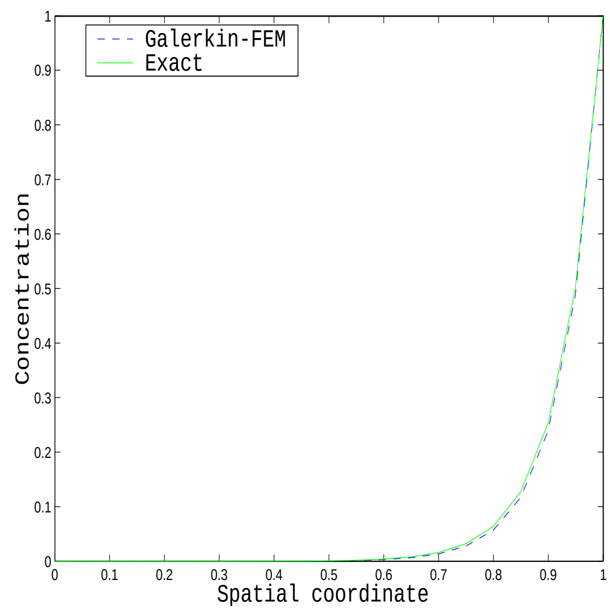

Figure 5: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 0.01$, $a = 0.7a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

In the second example —figure 6—, we show the exact solution and the numerical solution for $a = 8.75$. As a consequence of (32) this is the greatest value for the velocity that gives a stable solution. It is shown in figure 7 that the numerical solution of the transport problem is oscillatory if the advective term is greater than the value obtained before. In this case, we used $a = 9.75$

4.2.2 Group 2: Medium relaxation time

This group of problems is defined by $\tau = 1$. Therefore, the critical velocity is $a_c = \sqrt{k/\tau} = 1$. In addition, according to (32) the largest velocity that gives a stable solution

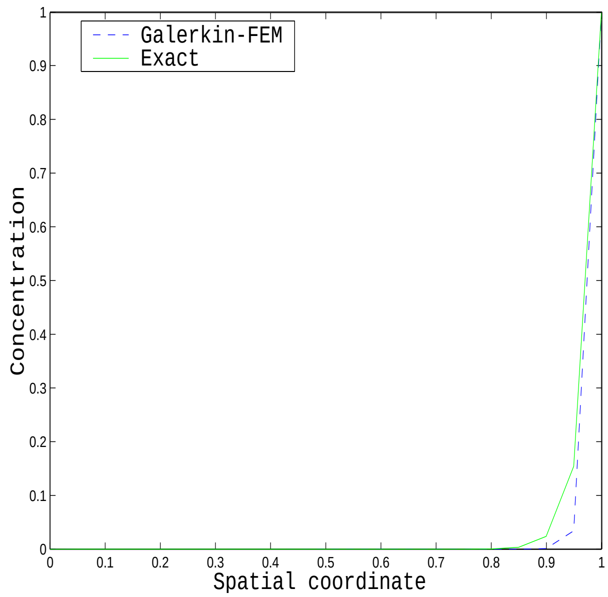

Figure 6: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 0.01$, $a = 0.875a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

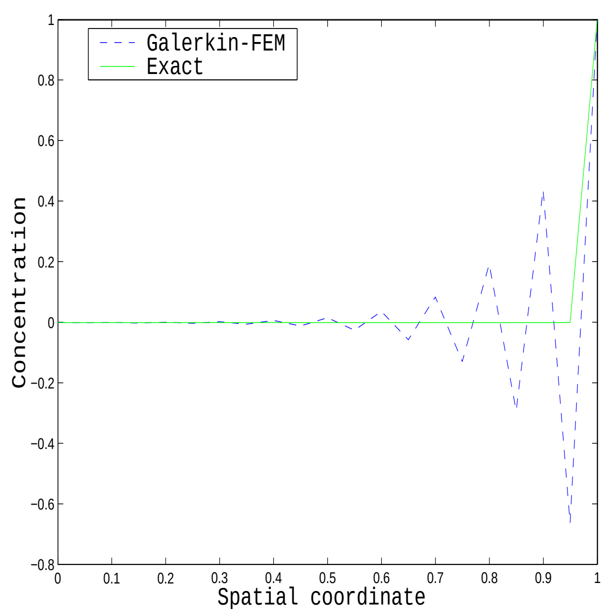

Figure 7: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 0.01$, $a = 0.975a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

is $a = \sqrt{k/\tau} - h/(4\tau) = 0.9875$. Hence, we will obtain stable solutions if $a < 0.9875 a_c$, i.e., we can solve (17) in a stable way for a 98.75% of the domain of a. We show three numerical tests for this relaxation time. In Figure 8 we show the solution for $a = 0.97$. Figure 9 shows the solution for $a = 0.9875$. Because of (32), this is the greatest value of a

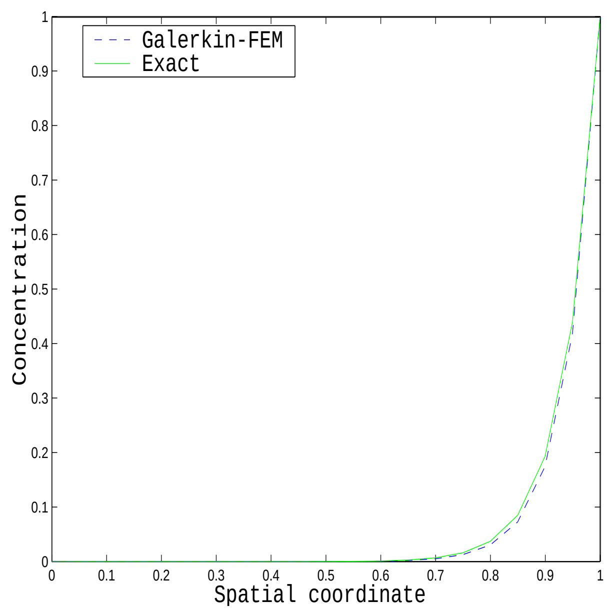

Figure 8: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 1$, $a = 0.97 a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

that gives a stable solution. Finally figure 10 shows the unstable solution of the problem defined by $a = 0.995$

4.2.3 Conclusions from the numerical examples

We conclude first that as τ increases the transport problem stabilizes. We can explain the above assertion as follows: although as τ increases the diffusive term decreases, the velocity a has an upper bound that decreases as τ increases. We have also shown that if we use Cattaneo's equation we obtain stable solutions in a very significative part of the domain of a. This is true even if we use the Galerkin method. Hence, from a practical point of view, we can say that the transport equation by using Cattaneo's law is a stable equation, because the values of a who make (17) unstable are negligible even for small relaxation times. However, the most important consequences about using Cattaneo's law are the following: when we use Cattaneo's law the transport problem has a physical meaning, because the velocity of diffusion is bounded. In addition, when a numerical scheme provides an inaccurate solution we know the causes of these results. This is

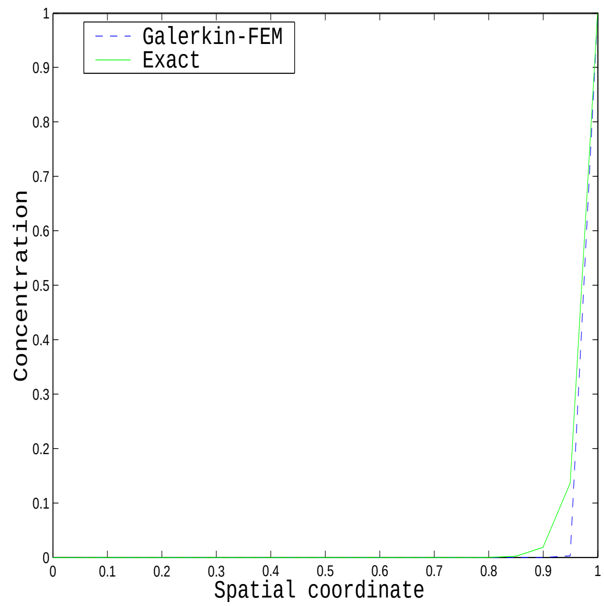

Figure 9: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 1$, $a = 0.9875a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

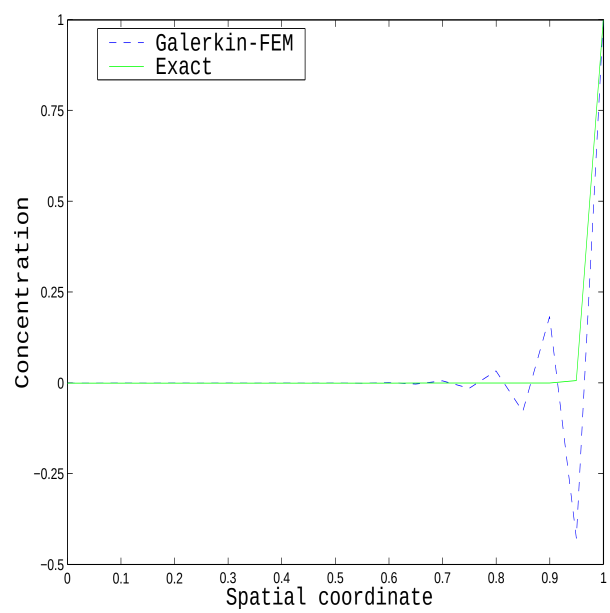

Figure 10: Transport problem by using Cattaneo's law. This problem is defined by $k = 1$, $\tau = 1$, $a = 0.995a_c$. The numerical solution is obtained by using the Galerkin FEM with linear shape functions. A 20 element grid has been used.

related to the wave nature of the transport problem. Rouhgly speaking, we can say that the numerical scheme should “capture” the diffusive waves.

5 CONCLUSIONS

In this paper we propose a stabilization technique for the advective–diffusive transport problem. This stabilization technique is very different to the nowadays widely–used stabilization methods. The basic idea is to use Cattaneo's equation as the constitutive equation, because Fick's law leads to the unreasonable result that mass propagates at an infinite speed. According to Cattaneo's equation the transport problem is a wave problem. In this paper we study the consequences of using Cattaneo's law. First we stablish that a critical fluid velocity exists for each transport problem. The transport problem should be stated as a initial value problem when fluid velocity is greater than a critical speed. This is a good result because it is intuitively clear that the transport problem is an initial value problem. In addition, there exist in nature several physical phenomena similar to the above one. For instante, propagation of superficial waves in a free surface flow or propagation of sound waves in a fluid.

A very important result is the wave nature of the new equation, because a wave problem has a intuitive physical meaning, while a parabolic problem has not. The above assertion is the basic idea for the proposed stability conditions.

In the last section we solve several problems in order to check the stability of the transport problem by using Cattaneo's equation. These examples show the good behaviour of the proposed stability conditions. In addition, we show that the relaxation time τ is a stabilizing parameter.

We can say as a summary that when we use Cattaneo's equation instead of Fick's law we obtain a meaningful stable problem. In addition, very easy to implement stability conditions can be derived.

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