Modelling Free-Surface Flows by a Galerkin Based SPH Formulation

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Summary

An algorithm based on Moving Least Squares Particle Hydrodynamics (MLSPH) to solve free surface flow is presented. MLS shape functions remarkably improve stability and accuracy of standard SPH algorithms, providing a clear framework for the derivation of the discretized equations. Numerical performance is tested through a free surface flow simulation.

Introduction

Meshless methods in computational mechanics are not simply different interpolation schemes but constitute, indeed, a powerful and ambitious attempt to solve the equations of continuum mechanics without the computational limitations associated to the explicit partition of the domain into certain non-overlapping cells.

The Smoothed Particle Hydrodynamics (SPH) method was developed in late 70's to simulate fluid dynamics in astrophysics and later applied to engineering problems [1]. The extension to solid mechanics was introduced by Libersky, Petschek et al. [2]. Johnson and Beissel proposed a Normalized Smoothing Function (NSF) algorithm [3] and other corrected SPH methods have been developed by Bonet et al. [4], and Chen et al. [5]. More recently, Dilts has introduced Moving Least Squares (MLS) shape functions into SPH computations [6].

The ability of the Smoothed Particle Hydrodynamics method to handle severe distortions allows this technique to be succesfully applied to simulate free surface flows. In this paper we briefly review the MLS approximants, present the discrete model equations for a compressible newtonian fluid and analyze one free-surface flow simulation.

Moving Least Squares Shape functions

Let us consider a function $u(\boldsymbol{x})$ defined in a bounded, or unbounded, domain Ω. The basic idea of the MLS approach is to approximate $u(\boldsymbol{x})$, at a given point $\boldsymbol{x}$, through a polynomial least-squares fitting of $u(\boldsymbol{x})$ in a neighbourhood of a reference node $\boldsymbol{x}'$:

$$u(\boldsymbol{x}) \approx \hat{u}(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{\alpha}(\boldsymbol{x}') \tag{1}$$

where $\boldsymbol{p}^t(\boldsymbol{x}-\boldsymbol{x}')$ is an m-dimensional polynomial basis and $\boldsymbol{\alpha}$ is a set of parameters to be determined, such that minimize the functional:

$$J(\boldsymbol{\alpha}) = \int_\Omega W(\boldsymbol{x}-\boldsymbol{x}',h)\left[u(\boldsymbol{x}) - \boldsymbol{p}^t(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{\alpha}(\boldsymbol{x}')\right]^2 d\Omega \tag{2}$$

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being $W(\boldsymbol{x}-\boldsymbol{x}',h)$ a symmetric kernel with compact support (frequently chosen among the kernels used in standard SPH) and h the smoothing length. If Ω is discretized by a set of n nodes or particles, a nodal integration scheme is used (the integral is evaluated using the nodes as quadrature points), and the interpolation and weighting domains are moved to the point $\boldsymbol{x}$ where the approximation is to be evaluated, the stationary conditions of J with respect to $\boldsymbol{\alpha}$ lead to [7]

$$\hat{u}(\boldsymbol{x}) = \boldsymbol{p}^t(\mathbf{0})\mathbf{M}^{-1}(\boldsymbol{x})\mathbf{P}\mathbf{W_V}(\boldsymbol{x})\boldsymbol{u}_{\Omega_{\boldsymbol{x}}} \tag{3}$$

where $\boldsymbol{u}_{\Omega_{\boldsymbol{x}}}$ contains certain nodal parameters of neighbouring nodes, $\mathbf{M}(\boldsymbol{x}) = \mathbf{P}\mathbf{W_V}(\boldsymbol{x})\mathbf{P}^t$, and matrices $\mathbf{P}$ and $\mathbf{W_V}(\boldsymbol{x})$ can be obtained as:

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{p}(\boldsymbol{x}_1-\boldsymbol{x}) & \boldsymbol{p}(\boldsymbol{x}_2-\boldsymbol{x}) & \cdots & \boldsymbol{p}(\boldsymbol{x}_{n_{\boldsymbol{x}}}-\boldsymbol{x}) \end{pmatrix} \tag{4}$$

$$\mathbf{W_V}(\boldsymbol{x}) = diag\,\{W_i(\boldsymbol{x}-\boldsymbol{x}_i)V_i\}\,, \qquad i = 1,\ldots,n_{\boldsymbol{x}} \tag{5}$$

In the above equations, $n_{\boldsymbol{x}}$ denotes the total number of nodes within the neighbourhood of point $\boldsymbol{x}$ and V_i and $\boldsymbol{x}_i$ are, respectively, the tributary volume and coordinates associated to node i. Note that the tributary volumes of neighbouring nodes are included in matrix $\mathbf{W_V}$, obtaining an MLS version of Reproducing Kernel Particle Method [7]. Otherwise, we can use $\mathbf{W}$ instead of $\mathbf{W_V}$,

$$\mathbf{W}(\boldsymbol{x}) = diag\,\{W_i(\boldsymbol{x}-\boldsymbol{x}_i)\}\,, \qquad i = 1,\ldots,n_{\boldsymbol{x}} \tag{6}$$

which corresponds to the classical MLS approximation (in the nodal integration of the functional (2), the same quadrature weight is associated to all nodes). Expression (3) can be rearranged to identify the interpolation structure [7]:

$$\hat{u}(\boldsymbol{x}) = p^t(\mathbf{0})\mathbf{M}^{-1}(\boldsymbol{x})\mathbf{B}(\boldsymbol{x})\boldsymbol{u}_{\Omega_{\boldsymbol{x}}} = \boldsymbol{N}^t(\boldsymbol{x})\boldsymbol{u}_{\Omega_{\boldsymbol{x}}} \tag{7}$$

In this work a linear polynomial basis $\boldsymbol{p}^t(\boldsymbol{x}-\boldsymbol{x}^*) = \left(1, \frac{x-x^*}{h}, \frac{y-y^*}{h}\right)$ was used, providing linear completeness.

Discrete Equations for Free Surface Flow Analysis

We consider a compressible newtonian fluid. The nodal integration of the Galerkin weak form of the model equations yields [7]:

• Conservation of mass

$$\frac{d\rho_i}{dt} = -\rho_i \sum_{j=1}^{n} \boldsymbol{v}_j \cdot \boldsymbol{\nabla} N_j(\boldsymbol{x}_i) \tag{8}$$

• Linear momentum

$$\frac{d\mathbf{v}_i}{dt} = \frac{1}{m_i} \sum_{j=1}^{n} \boldsymbol{\sigma}_j \boldsymbol{\nabla} N_i(\boldsymbol{x}_j) V_j + \boldsymbol{f}_i \tag{9}$$

• Particle velocities

$$\frac{d\boldsymbol{x}_i}{dt} = \hat{\boldsymbol{v}}_i, \qquad \hat{\boldsymbol{v}}_i = \sum_{j=1}^{n} \boldsymbol{v}_j N_j(\boldsymbol{x}_i) \tag{10}$$

In the above equations, ρ_i, V_i, m_i, $\boldsymbol{v}_i$ and $\boldsymbol{f}_i$ denote density, associated volume, lumped mass, velocity and force per unit mass of particle i, respectively. Preservation of linear and angular momenta is a most important issue to be considered in free surface flow simulations and will be achieved with the proposed algorithm, provided that first order consistency shape functions are employed.

The internal forces are related to the Cauchy stress tensor, $\boldsymbol{\sigma}$, which is calculated using the following constitutive equation:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\left(\boldsymbol{D} - \frac{1}{3}tr(\boldsymbol{D})\boldsymbol{I}\right); \qquad \boldsymbol{D} = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^t), \tag{11}$$

being μ the fluid viscosity, $\boldsymbol{I}$ the second order identity tensor and p a pressure scalar field, evaluated using the thermodynamic expression [1]:

$$\frac{p}{p_o} = (k+1)\left(\frac{\rho}{\rho_o}\right)^{\gamma} - k, \tag{12}$$

where k and γ are adimensional parameters and p_o and ρ_o are the atmospherical standard values. Using these parameters, the sound velocity can be defined as $c = \sqrt{\gamma k/\rho}$ [1].

Field variables are updated following a second order predictor-corrector scheme as exposed in [8].

Numerical Example

The performance of the algorithm proposed is tested through a free-surface flow simulation. Two breaking dams of fluids with densities $1000 kg/m^3$ and $2000 kg/m^3$ are set up in the configuration shown in figure 1. The results obtained (figures 2 and 3) demonstrate the ability of SPH to simulate complex unsteady free-surface flow problems.

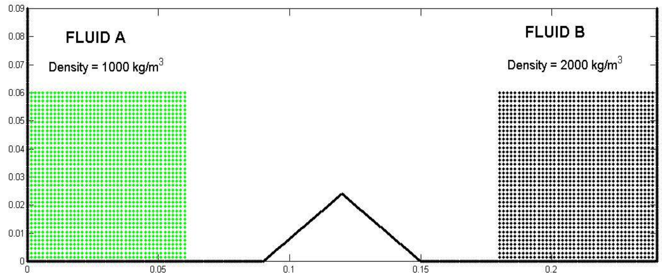

Figure 1: Initial configuration.

Conclusions

We have presented an algorithm based on Moving Least Squares Particle Hydrodynamics (MLSPH) to simulate free surface flows in engineering applications. The Galerkin formulation provides a clear framework to derive the discrete equations and the moving least squares approximation remarkably improves the standard SPH kernel stimates. The numerical results are encouraging and demonstrate that particle methods constitute a very attractive tool in the modelization of complex free-surface flows.

Acknowledgements

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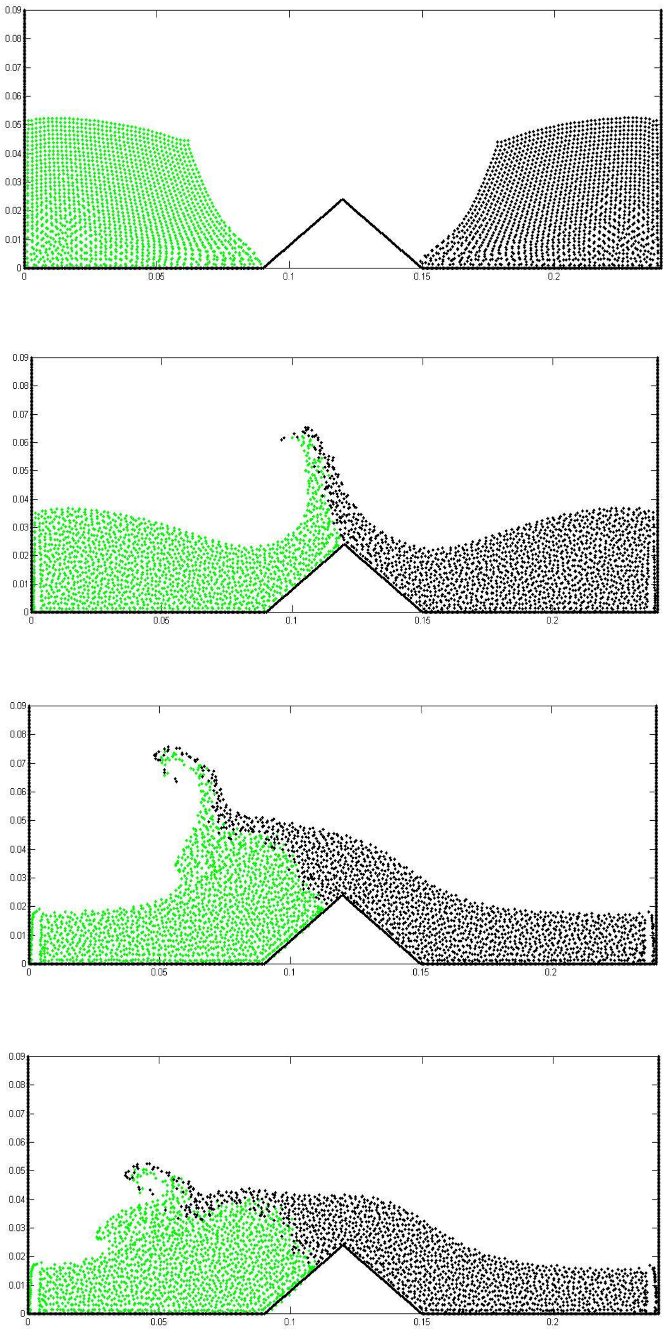

Figure 2: Simulation at various stages.

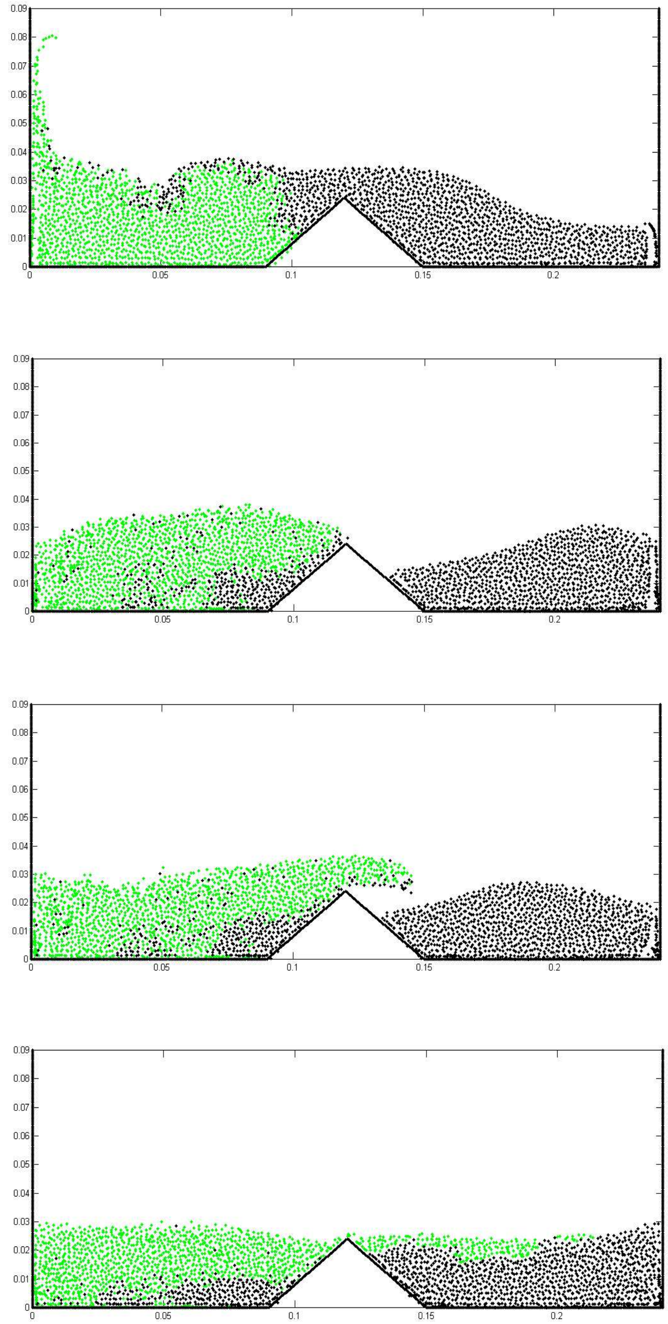

Figure 3: Simulation at various stages.