On the anomalous asymptotic performance of the regular computer methods for grounding analysis

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Abstract

Grounding systems are designed to guarantee personal security, protection of equipments and continuity of power supply. Hence, engineers must compute the equivalent resistance of the system and the potential distribution on the earth surface when a fault condition occurs [1, 2, 3]. While very crude approximations were available until the 70's, several computer methods have been more recently proposed on the basis of practice, semi-empirical works and intuitive ideas such as superposition of punctual current sources and error averaging [1, 3, 4, 5, 6]. Although these techniques are widely used, several problems have been reported. Namely: large computational requirements, unrealistic results when segmentation of conductors is increased, and uncertainty in the margin of error [2, 5].

A Boundary Element formulation for grounding analysis is presented in this paper. Existing computer methods such as APM are identified as particular cases within this theoretical framework. While linear and quadratic leakage current elements allow to increase accuracy, computing time is reduced by means of new analytical integration techniques. Former intuitive ideas can now be explained as suitable assumptions introduced in the BEM formulation to reduce computational cost. Thus, the anomalous asymptotic behaviour of this kind of methods is mathematically explained, and sources of error are rigorously identified.

1 Introduction

Fault currents dissipation into the earth can be modelled by means of Maxwell's Electromagnetic Theory [7, 8, 9]. Constraining the analysis to the electrokinetic

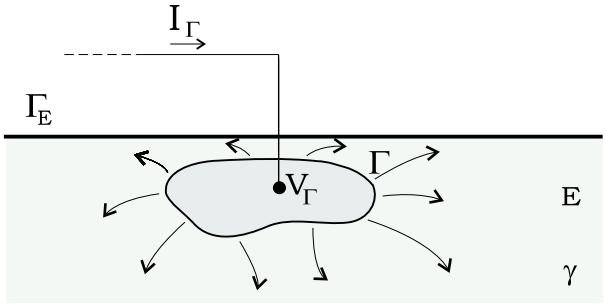

Figure 1: Fault current disipation in a single layer soil model.

steady-state response, and neglecting the resistivity of the earthing electrode, the 3D problem associated to an electrical current derivation to earth can be written in terms of the following equilibrium and constitutive differential equations and boundary conditions

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0 \text{ in } E,\ \boldsymbol{\sigma}=-\boldsymbol{\gamma}\boldsymbol{\nabla}V,\ \boldsymbol{\sigma}^t\boldsymbol{n}_{_E}=0 \text{ in } \Gamma_E,\ V=V_\Gamma \text{ in } \Gamma, V\to 0 \text{ if } |\boldsymbol{x}|\to\infty, \tag{1}$$

where E is the earth and $\boldsymbol{\gamma}$ its conductivity tensor, Γ_E is the earth surface and $\boldsymbol{n}_{_E}$ its normal exterior unit field, and Γ is the earthing electrode surface [10, 11, 12, 13]. The solution to this problem gives the potential $V(\boldsymbol{x})$ and the current density $\boldsymbol{\sigma}(\boldsymbol{x})$ at an arbitrary point $\boldsymbol{x}$ in E when the earthing electrode is energized to the so-called Ground Potential Rise V_Γ relative to remote earth.

In these terms, the leakage current density $\sigma(\boldsymbol{\xi})$ at an arbitrary point $\boldsymbol{\xi}$ on the earthing electrode surface, the ground current I_Γ (total surge current being leaked into the earth) and the equivalent resistance of the earthing system R_{eq}, can be written as

$$\sigma(\boldsymbol{\xi})=\boldsymbol{\sigma}^t(\boldsymbol{\xi})\boldsymbol{n},\quad I_\Gamma=\iint_{\boldsymbol{\xi}\in\Gamma}\sigma(\boldsymbol{\xi})\ d\Gamma,\quad R_{eq}=\frac{V_\Gamma}{I_\Gamma}, \tag{2}$$

being $\boldsymbol{n}$ the normal exterior unit field to Γ. Since V and $\boldsymbol{\sigma}$ are proportional to the GPR, the assumption $V_\Gamma=1$ is not restrictive at all and it will be used from now on.

For most practical purposes, the assumption of homogeneous and isotropic soil can be considered acceptable [1], and the tensor $\boldsymbol{\gamma}$ can be substituted by a measured apparent scalar conductivity γ (see Figure 1). Otherwise, a multi-layer model can be accepted without risking a serious calculation error [14, 15]. Since the kind of techniques described in this paper can be extended to multi-layer soil models [16], further discussion is restricted to uniform soils. Hence, problem (1) reduces to the Laplace equation with mixed boundary conditions [7, 8]. If one further assumes that the earth surface is horizontal, symmetry allows to rewrite (1) in terms of a Dirichlet Exterior Problem [13, 17].

This kind of problems has been rigorously studied [18], and its solution can be obtained in many technical applications by means of standard numerical methods, such as the Finite Diference or the Finite Element methods. However, in most substation grounding systems, the buried earthing electrode (grounding grid) consists of a number of interconnected bare cylindrical conductors, which ratio diameter/lenght is relatively small ($\approx 10^{-3}$). Since domain E is half-infinite and the electrode must be excluded, the adequate discretization of E requires an extremely large number of degrees of freedom. Thus, the prohibitive computing requirements preclude the use of FD or FE methods in practice [19].

On the other hand, since computation of potential is only required on the earth surface Γ_E for obtaining the main grounding safety parameters, and the equivalent resistance can be easily obtained in terms of the leakage current (2), a Boundary Element approach [20] seems to be the right choice [10, 11, 12].

2 Variational Statement of the Problem

The application of Green's Identity [18] to (1) allows to obtain the following expression for the potential V in E, in terms of the unknown leakage current σ [10, 11, 12, 13]

$$V(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \iint_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{x},\boldsymbol{\xi})\, \sigma(\boldsymbol{\xi})\ d\Gamma, \tag{3}$$

being $k(\boldsymbol{x},\boldsymbol{\xi})$ the weakly singular kernel

$$k(\boldsymbol{x},\boldsymbol{\xi}) = \left(\frac{1}{r(\boldsymbol{x},\boldsymbol{\xi})} + \frac{1}{r(\boldsymbol{x},\boldsymbol{\xi}')} \right), \quad r(\boldsymbol{x},\boldsymbol{\xi}) = \left| \boldsymbol{x} - \boldsymbol{\xi} \right|, \tag{4}$$

where $\boldsymbol{\xi}'$ is the symmetric of $\boldsymbol{\xi}$ with respect to the earth surface. Since (3) holds on the earthing electrode surface [11], the boundary condition $V_\Gamma = 1$ leads to the Fredholm integral equation of the first kind on Γ

$$1 - \frac{1}{4\pi\gamma} \iint_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{\chi},\boldsymbol{\xi})\, \sigma(\boldsymbol{\xi})\ d\Gamma = 0 \qquad \forall \boldsymbol{\chi} \in \Gamma, \tag{5}$$

which solution is the unknown leakage current density σ. By application of the weighted residuals method [20, 21], equation (5) can be written in the weaker variational form

$$\iint_{\boldsymbol{\chi}\in\Gamma} w(\boldsymbol{\chi}) \left[1 - \frac{1}{4\pi\gamma} \iint_{\boldsymbol{\xi}\in\Gamma} k(\boldsymbol{\chi},\boldsymbol{\xi})\, \sigma(\boldsymbol{\xi}) d\Gamma \right] d\Gamma = 0, \tag{6}$$

which must hold for all members $w(\boldsymbol{\chi})$ of a suitable class of so-called test (or weighting) functions defined on Γ [10, 11, 12, 13]. Weak form (6) will be our starting point to obtain an approximate solution to the original problem (1) by means of the Boundary Element Method.

3 General Boundary Element Formulation

For a given set $\{N_i(\boldsymbol{\xi})\}$ of N so-called trial (or interpolating) functions [20, 21] defined on Γ, and for a given set $\{\Gamma^\alpha\}$ of M 2D boundary elements (portions of the electrode surface), the unknown leakage current density σ and the electrode surface Γ can be discretized in the form

$$\sigma(\boldsymbol{\xi}) \approx \sigma^h(\boldsymbol{\xi}) = \sum_{i=1}^{N} \sigma_i \, N_i(\boldsymbol{\xi}), \qquad \Gamma = \bigcup_{\alpha=1}^{M} \Gamma^\alpha. \tag{7}$$

Then, a discretized form of (3) can be written as

$$V(\boldsymbol{x}) \approx V^h(\boldsymbol{x}) = \sum_{i=1}^{N} \sigma_i \, V_i(\boldsymbol{x}), \quad V_i(\boldsymbol{x}) = \sum_{\alpha=1}^{M} V_i^\alpha(\boldsymbol{x}), \tag{8}$$

$$V_i^\alpha(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \iint_{\boldsymbol{\xi}\in\Gamma^\alpha} k(\boldsymbol{x},\boldsymbol{\xi}) \, N_i(\boldsymbol{\xi}) \; d\Gamma. \tag{9}$$

Finally, for a given set $\{w_j(\boldsymbol{\xi})\}$ of N test functions defined on Γ, (6) reduces to the linear system [10, 11, 12]

$$\sum_{i=1}^{N} R_{ji}\sigma_i = \nu_j, \qquad j = 1, \ldots, N; \tag{10}$$

$$R_{ji} = \sum_{\beta=1}^{M}\sum_{\alpha=1}^{M} R_{ji}^{\beta\alpha}, \;\; \nu_j = \sum_{\beta=1}^{M} \nu_j^\beta; \quad i = 1, \ldots, N; j = 1, \ldots, N; \tag{11}$$

$$R_{ji}^{\beta\alpha} = \frac{1}{4\pi\gamma} \iint_{\boldsymbol{\chi}\in\Gamma^\beta} w_j(\boldsymbol{\chi}) \left[\iint_{\boldsymbol{\xi}\in\Gamma^\alpha} k(\boldsymbol{\chi},\boldsymbol{\xi}) N_i(\boldsymbol{\xi}) d\Gamma \right] d\Gamma, \tag{12}$$

$$\nu_j^\beta = \iint_{\boldsymbol{\chi}\in\Gamma^\beta} w_j(\boldsymbol{\chi}) \; d\Gamma. \tag{13}$$

It can be easily understood that 2D discretizations required to solve the above stated equations in real cases imply a large number of degrees of freedom. Since the coefficients matrix in (10) is not sparse, and 2D integration in (12) must be performed twice over the electrode surface, it is clear that additional assumptions must be introduced in order to overcome the problem complexity.

4 Approximated Variational Statement

For a given generic point $\boldsymbol{\xi}$ at the surface of a cylindrical bar, let $\widehat{\boldsymbol{\xi}}$ be its orthogonal projection over the bar axis, let $\phi(\widehat{\boldsymbol{\xi}})$ be the diameter (assumed much smaller than the bar length) and let $C(\widehat{\boldsymbol{\xi}})$ be the circumferential perimeter of the cross section at this point. Let L be the whole set of axial lines of the buried conductors. If the

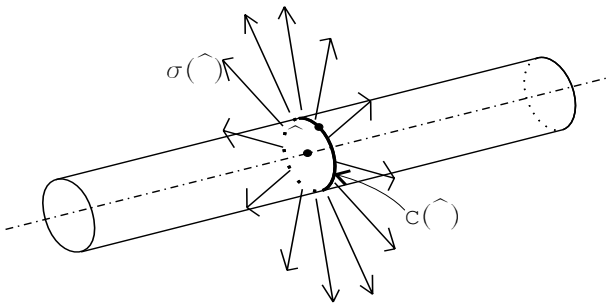

Figure 2: Assumption of circumferential uniformity.

leakage current is assumed uniform around the perimeter of every cross section (see Figure 2), that is $\sigma(\boldsymbol{\xi}) = \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\ \forall \boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})$, expression (3) can be written in the form [10, 11, 12]

$$\widehat{V}(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x}, \boldsymbol{\xi})\, dC \right] \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\, dL. \tag{14}$$

This assumption seems to be quite adequate and not too restrictive, if we take into account the real geometry of grounding grids [1, 2, 5]. Nevertheless, boundary condition $V = 1$ will not be exactly satisfied yet at every point on the electrode surface, since the leakage current is not exactly uniform around the cross section. Therefore, variational equality (6) will not hold anymore (except in particular cases where the leakage current is really uniform around the perimeter). However, if we restrict the class of test functions to those with circumferential uniformity, that is $w(\boldsymbol{\chi}) = \widehat{w}(\widehat{\boldsymbol{\chi}})\ \forall \boldsymbol{\chi} \in C(\widehat{\boldsymbol{\chi}})$, (6) results in

$$\int_{\widehat{\boldsymbol{\chi}} \in L} \widehat{w}(\widehat{\boldsymbol{\chi}}) \left[\pi\phi(\widehat{\boldsymbol{\chi}}) - \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) \widehat{\sigma}(\widehat{\boldsymbol{\xi}}) dL \right] dL = 0 \tag{15}$$

which must hold for all members $\widehat{w}(\widehat{\boldsymbol{\chi}})$ of a suitable class of test functions on L, being the integral kernel

$$K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) = \int_{\boldsymbol{\chi} \in C(\widehat{\boldsymbol{\chi}})} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{\chi}, \boldsymbol{\xi})\, dC \right] dC. \tag{16}$$

In this way, boundary condition $V = 1$ is forced to be satisfied on the average at every cross section. In fact, (15) can be considered as a weaker variational (or weighted residuals) statement of the Fredholm integral equation of the first kind on L

$$\pi\phi(\widehat{\boldsymbol{\chi}}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}})\, \widehat{\sigma}(\widehat{\boldsymbol{\xi}})\, dL \qquad \forall \widehat{\boldsymbol{\chi}} \in L. \tag{17}$$

Since ends and junctions of conductors are not taken into account in this formulation, slightly anomalous local effects can be expected at these points.

5 Approximated Boundary Element Formulation

For a given set $\{\widehat{N}_i(\widehat{\boldsymbol{\xi}})\}$ of n trial functions defined on L, and for a given set $\{L^\alpha\}$ of m 1D boundary elements (segments of the cylindrical conductors), the unknown leakage current $\widehat{\sigma}$, and the whole set of axial lines of the buried conductors L, can be discretized in the form

$$\widehat{\sigma}(\widehat{\boldsymbol{\xi}}) \approx \widehat{\sigma}^h(\widehat{\boldsymbol{\xi}}) = \sum_{i=1}^{n} \widehat{\sigma}_i \, \widehat{N}_i(\widehat{\boldsymbol{\xi}}), \qquad L = \bigcup_{\alpha=1}^{m} L^\alpha. \tag{18}$$

Then, a discretized version of (14) can be written as

$$\widehat{V}(\boldsymbol{x}) \approx \widehat{V}^h(\boldsymbol{x}) = \sum_{i=1}^{n} \widehat{\sigma}_i \, \widehat{V}_i(\boldsymbol{x}), \quad \widehat{V}_i(\boldsymbol{x}) = \sum_{\alpha=1}^{m} \widehat{V}_i^\alpha(\boldsymbol{x}), \tag{19}$$

$$\widehat{V}_i^\alpha(\boldsymbol{x}) = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\xi}} \in L^\alpha} \left[\int_{\boldsymbol{\xi} \in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x}, \boldsymbol{\xi}) \, dC \right] \widehat{N}_i(\widehat{\boldsymbol{\xi}}) \; dL. \tag{20}$$

Finally, for a given set $\{\widehat{w}_j(\widehat{\boldsymbol{\chi}})\}$ of n test (weighting) functions defined on L, (15) reduces to the linear system [10, 11, 12]

$$\sum_{i=1}^{n} \widehat{R}_{ji} \widehat{\sigma}_i = \widehat{\nu}_j, \qquad j = 1, \ldots, n; \tag{21}$$

$$\widehat{R}_{ji} = \sum_{\beta=1}^{m} \sum_{\alpha=1}^{m} \widehat{R}_{ji}^{\beta\alpha}, \;\; \widehat{\nu}_j = \sum_{\beta=1}^{m} \widehat{\nu_j}^{\beta}; \quad i = 1, \ldots, n; j = 1, \ldots, n; \tag{22}$$

$$\widehat{R}_{ji}^{\beta\alpha} = \frac{1}{4\pi\gamma} \int_{\widehat{\boldsymbol{\chi}} \in L^\beta} \widehat{w}_j(\widehat{\boldsymbol{\chi}}) \left[\int_{\widehat{\boldsymbol{\xi}} \in L^\alpha} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\xi}}) \widehat{N}_i(\widehat{\boldsymbol{\xi}}) dL \right] dL, \tag{23}$$

$$\widehat{\nu}_j^\beta = \int_{\widehat{\boldsymbol{\chi}} \in L^\beta} \pi \, \phi(\widehat{\boldsymbol{\chi}}) \, \widehat{w}_j(\widehat{\boldsymbol{\chi}}) \; dL. \tag{24}$$

The size of the linear equations system (21) and the number of contributions (23) that must be calculated are expected to be significantly smaller than those in (10) and (12). Therefore, the computational work required by this approximated 1D formulation should be much lower in practice than the corresponding to the general formulation given in section 3. However, extensive computing is still required, mainly because of circumferential integration in (20) and (16), and further simplifications are necessary to reduce computing time under acceptable levels.

Simplified 1D Boundary Element Formulation

The inner integral of kernel $k(\boldsymbol{x},\boldsymbol{\xi})$ in (20) can be approximated as [10, 11, 12]

$$\int_{\boldsymbol{\xi}\in C(\widehat{\boldsymbol{\xi}})} k(\boldsymbol{x},\boldsymbol{\xi})\, dC \approx \pi\, \phi(\widehat{\boldsymbol{\xi}})\, \widehat{k}(\boldsymbol{x},\widehat{\boldsymbol{\xi}}), \tag{25}$$

being

$$\widehat{k}(\boldsymbol{x},\widehat{\boldsymbol{\xi}}) = \left(\frac{1}{\widehat{r}(\boldsymbol{x},\widehat{\boldsymbol{\xi}})} + \frac{1}{\widehat{r}(\boldsymbol{x},\widehat{\boldsymbol{\xi}}')} \right), \tag{26}$$

and

$$\widehat{r}(\boldsymbol{x},\widehat{\boldsymbol{\xi}}) = \sqrt{\left|\boldsymbol{x}-\widehat{\boldsymbol{\xi}}\right|^2 + \frac{\phi^2(\widehat{\boldsymbol{\xi}})}{4}}, \tag{27}$$

where $\widehat{\boldsymbol{\xi}}'$ is the symmetric of $\widehat{\boldsymbol{\xi}}$ with respect to the earth surface. This approximation is quite accurate, unless the distance between points $\boldsymbol{x}$ and $\widehat{\boldsymbol{\xi}}$ is in the order of magnitude of the diameter $\phi(\widehat{\boldsymbol{\xi}})$. Then, integral kernel (16) can be approximated as

$$K(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}}) \approx \pi\, \phi(\widehat{\boldsymbol{\chi}})\, \pi\, \phi(\widehat{\boldsymbol{\xi}})\ \widehat{\widehat{k}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}}), \tag{28}$$

being

$$\widehat{\widehat{k}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}}) \;=\; \left(\frac{1}{\widehat{\widehat{r}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}})} + \frac{1}{\widehat{\widehat{r}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}}')} \right), \tag{29}$$

and

$$\widehat{\widehat{r}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}}) \;=\; \sqrt{\left|\widehat{\boldsymbol{\chi}}-\widehat{\boldsymbol{\xi}}\right|^2 \;+\; \frac{\phi^2(\widehat{\boldsymbol{\xi}})+\phi^2(\widehat{\boldsymbol{\chi}})}{4}}, \tag{30}$$

where symmetry is preserved in (21) even for different conductor diameters at points $\widehat{\boldsymbol{\chi}}$ and $\widehat{\boldsymbol{\xi}}$.

Now, specific selections of the sets of trial and test functions lead to different formulations. Thus, for constant leakage current elements (current density is assumed constant within each segment), Point Collocation (test functions are Dirac deltas) leads to the very early methods based on the idea that each segment of conductor is substituted by an "imaginary sphere". Similarly, Galerkin type weighting (test functions are identical to trial functions) leads to a kind of more recent methods (such as the "Average Potential Method", APM) based on the idea that each segment of conductor is substituted by a "line of point sources over the lenght of the conductor" [5]. Coefficients (23) correspond to "mutual and self resistances" between "segments of conductor" [5]. For higher order leakage current elements (current density is assumed linear, quadratic, etc., within each segment), more advanced formulations can be derived [11, 12].

Analytical Integration Techniques

Further discussion and examples are restricted to Galerkin type formulations, where the matrix of coefficients in (21) is symmetric and positive definite [20]. Diameter of conductors is assumed constant within each element. Therefore, (20) and (23) can be rewritten as

$$\widehat{V}_i^{\alpha}(\boldsymbol{x}) = \frac{1}{4\pi\gamma}\pi\,\phi^{\alpha}\int_{\widehat{\boldsymbol{\xi}}\in L^{\alpha}} \widehat{k}(\boldsymbol{x},\widehat{\boldsymbol{\xi}})\,\widehat{N}_i(\widehat{\boldsymbol{\xi}})\;dL, \tag{31}$$

$$\widehat{R}_{ji}^{\beta\alpha} = \frac{\pi\phi^{\beta}\,\pi\phi^{\alpha}}{4\pi\gamma}\int_{\widehat{\boldsymbol{\chi}}\in L^{\beta}} \widehat{N}_j(\widehat{\boldsymbol{\chi}})\left[\int_{\widehat{\boldsymbol{\xi}}\in L^{\alpha}} \widehat{\widehat{k}}(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\xi}})\,\widehat{N}_i(\widehat{\boldsymbol{\xi}})\;dL\right]dL, \tag{32}$$

being ϕ^{α} and ϕ^{β} the conductor diameters within elements L^{α} and L^{β}. Obviously, contributions (32) produce a symmetric matrix in (21).

Computation of remaining integrals in (31) and (32) by means of numerical quadratures is very costly due to the undesirable behaviour of the integrands [10, 11]. Therefore, we turn our attention to analytical integration techniques. Explicit formulae were initially derived to compute (31) in the case of constant (1 functional node), linear (2 functional nodes) and quadratic (3 functional nodes) leakage current elements [10, 11, 12]. Explicit expressions were subsequently derived [11, 12] for contributions (32). For the most simple cases, these formulae reduce to those proposed in the literature (i.e. constant leakage current elements in APM [4]). A summary of these explicit formulae can be found in [12].

6 Why Do These Methods Fail To Converge?

We expect that the discretized leakage current density $\widehat{\sigma}^h(\widehat{\boldsymbol{\xi}})$ will converge to the exact solution $\sigma(\boldsymbol{\xi})$ as the number of degrees of freedom n is increased. We also expect that the discretized potential $\widehat{V}^h(\boldsymbol{x})$ will simultaneously converge to the exact solution $V(\boldsymbol{x})$. In general, we can try to obtain these effects in (18) either by increasing the segmentation of the conductors, or by choosing more sophisticated trial functions $\widehat{N}_i(\widehat{\boldsymbol{\xi}})$ (that is, using higher order elements) [20, 21]. In the usual terminology of Finite Elements, the first option is referred to as the *h* method, while the second is known as the *p* method.

However, these formulations fail to converge to the exact solution, since the discretized leakage current density becomes polluted by increasing numerical instabilities when discretization is refined beyond a certain point [5, 17]. In fact, numerical instabilities can extend to the whole length of the conductors when segmentation is increased. This produces unrealistic results in subsequent computation of potentials on the earth surface, although the equivalent resistance R_{eq} seems to converge [11, 13, 19].

These problems were pointed out by Garret and Pruitt in their remarkable and indeed classical paper [5] about the accuracy of the Average Potential Method. In spite of lacking a rigorous derivation for the method, these authors established and

discussed most of the sources of error. However, the origin of the above mentioned instabilities could not be explained in that incomplete theoretical framework.

Problem (1) is a well-posed problem [18]. One can argue that neglecting the resistivity of the earthing electrode is not fully realistic, and thus V_Γ is not exactly constant on the electrode surface. Should this line of reasoning be followed, one would accept the need for more sophisticated models when the resistivity of the electrode must be taken into account. But this idealization seems to be reasonable and accurate enough for most practical purposes [11, 19], and one can not attribute the origin of the observed instabilities to this assumption. On the other hand, derivations of expression (3) and Fredholm integral equation of the first kind (5) have been rigorously established [11]. Furthermore, the problem defined by variational form (6) is well-posed, kernel (4) is weakly singular, and linear system (10) is quite well-conditioned for realistic discretizations of the electrode surface [20]. The latter is in contrast to other similar problems having smooth kernels, which are frequently very ill-conditioned and thus extremely difficult to solve[20].

Therefore, the origin of the convergence failure must be sought for in the assumptions introduced to overcome the computational complexity of the 2D BEM general formulation [10, 11, 12], that is: **A)** the leakage current is assumed uniform around the perimeter of every cylindrical conductor, **B)** the ends and junctions of conductors are not taken into account, and **C)** approximations (25) and (28) are introduced to avoid circumferential integration and reduce computing time.

Several numerical tests have been performed for the single bar in infinite domain problem [11, 19]. The results prove that assumption **A)** is not the origin of the problems encountered with this kind of methods. No specific numerical tests have been performed so far in order to quantify the error due to assumption **B)**. Anyhow, in the authors' experience, slightly anomalous local effects can be expected at the ends and junctions of conductors, but global results should not be noticeably affected. We remark that derivations of expression (14) and Fredholm integral equation of the first kind (17) have been rigorously established [11, 12]. Furthermore, the problem defined by variational form (15) is approximated but well-posed, kernel (16) is weakly singular, and linear system (21) must be quite well-conditioned for realistic segmentations of the electrodes [20].

Therefore, the origin of the instabilities must be attributted to the approximations (25) and (28). The fact is that these approximations are not valid for short distances. When discretization is refined, the size of the segments become comparable to or smaller than the diameter of the conductors. Then, approximation (28) introduces significant errors in the coefficients of the linear system (21), including the diagonal terms. From another point of view, since the approximation error increases as discretization becomes thiner, numerical results for dense discretizations do not trend to the solution of integral equation (17) with kernel (16), but to the solution of an ill-conditioned integral equation with non singular kernel (28). It is a known theoretical result for Fredholm equations of the first kind that the inverse of a completely continuous operator is unbounded [22]. In plain words: if approximations (25) and (28) are used, the exact solution of the ill-conditioned simplified problem can not be found numerically, since one can always come upon

very different leakage current distributions that apparently verify the boundary condition $V = V_\Gamma$ with arbitrarily small errors. This explains why unrealistic results are obtained when discretization is refined [5], and convergence is precluded [13, 17].

7 Accuracy and Overall Efficiency

At this point, we endorse the lucid advices stated in [5]. This kind of methods should be applied in an iterative way, increasing the number of segments of conductors per computer run. A simple strategy could be to start with a low number of segments of similar size, and to bisect all the segments at each run of the program until the results converge within acceptable errors. We recall that segmentation can not be indefinitely increased, for the above stated reasons. As a practical rule, we can say that approximations (25) and (28) are not valid if the size of segments becomes comparable to or smaller than the diameter of the electrode.

Results obtained for low and medium levels of discretization can be considered sufficiently accurate for most of practical purposes [11, 19]. However, it is obvious that more accurate results could be required in special cases, and it has been reported that APM failed to determine satisfactory results in specific instances due to the problems analyzed in this paper. In cases like these, the use of higher order elements (linear or quadratic) could help, at least up to a certain level of precision.

On the other hand, the proposed approach shows the path to remove the annoying instabilities of this kind of methods. We remark that the simplified 1D BEM formulation is ill-conditioned, but the previous approximated 1D BEM formulation is correct. Thus, the obvious solution is to substitute (25) and (28) by better approximations that were valid for short distances too. This is neither obvious nor straightforward, since it should be necessary to adapt most of the analytical work described in section IV. Anyhow, further research in this direction could supply efficient asymptotically stable methods in a close future.

With regard to the overall computational cost, for a given discretization (m elements of p nodes each, and a total number of n degrees of freedom) a linear system (21) of order n must be generated and solved. Since the matrix is symmetric, but not sparse, its resolution by means of a direct method should require $O(n^3/3)$ operations. Matrix generation requires $O(m^2p^2/2)$ operations, since p^2 contributions of type (32) have to be computed for every pair of elements, and approximately half of them are discarded because of symmetry. Once the leakage current has been obtained, the cost of computing the equivalent resistance is negligible. The additional cost of computing potential at any given point (normally on the earth surface) by means of (19) requires only $O(mp)$ operations, since p contributions of type (31) have to be computed for every element. However, if it is necessary to compute potentials at a large number of points (i.e. to draw contours), the corresponding computing time could as well be important.

Hence, most of computing effort is devoted to matrix generation in small/medium problems, while linear system resolution prevails in medium/large ones. In these cases, the use of direct methods for the linear system resolution is out of range.

Therefore iterative or semiiterative techniques will be preferable. The best results have been obtained by a diagonal preconditioned conjugate gradient algorithm with assembly of the global matrix [11, 23]. This technique has turned out to be highly efficient for solving large scale problems, with a very low computational cost. Finally, the first critical time-consuming process is matrix generation, followed by computation of potential at a large number of points. Both accept massive parallelization [24].

Selection of the type of leakage current density elements is an important point in the resolution of a real problem. We recall that obtaining asymptotical solutions by indefinitely increasing the discretization level is precluded. Thus, for a given problem it will be essential to consider the relative advantages and disadvantages of increasing the number of elements (*h* method) or using higher order elements (*p* method) in order to define an adequate discretizacion [11, 12, 13]. In general, higher order elements are advantageous in comparison with constant elements, since better results can be obtained with a lower number of degrees of freedom.

8 Application to Real Cases

The techniques derived by the authors have been implemented in a Computer Aided Design system for earthing grids of electrical substations called TOTBEM [25]. At present, the single-layer code runs in real-time in personal computers, and the size of the largest problem that can be solved is limited by the memory storage required to handle the coefficients matrix. Thus, for a problem with 2000 degrees of freedom, at least 16Mb would be needed, while computing times for matrix generation and system resolution would be in the same order of magnitude (around 15 seconds in what is considered a medium performance single processor personal computer in year 2000). The system has been used by the authors and by several Spanish power companies to analyze several medium/large installations during the last 8 years. Some of these results can be found in [10, 11, 12, 13, 25].

All the cases analyzed have been repeatedly solved for an increasing segmentation of the electrodes. As the theory predicts (and it has been reported) the numerical instabilities pollute the results when the discretization is refined beyond a certain point.

Anyway, it seems that a reasonable (moderate) level of segmentation is sufficient to obtain quite accurate results in practice. In our experience, increasing the number of elements was needless in all the studied cases, since the results (at the scale of the whole grid) were not noticeably improved. It seems that increasing the segmentation is only justified when high accuracy local results are required for a limited part of the whole grounding system.

On the other hand, the use of higher order elements (liner or quadratic) seems to be more advantageous (in general) than increasing intensively the segmentation of constant elements, since the accuracy is higher for a remarkably smaller total number of degrees of freedom [11].

The techniques described in this paper can be extended for multi-layer soil models [26, 27], although computing time becomes not contemptible whatsoever. The

proposed formulation has been implemented in a high-performance parallel computer and the code has been applied to the analysis of several real grounding systems [24, 26, 27]. The results obtained by the authors with the multi-layer code have been noticeably different from those obtained by using a single layer soil model. Thus, it is the authors' belief that the proposed multi-layer BEM formulation will become a real-time design tool in a close future, as high-performance parallel computing becomes a widespread available resource in engineering. The formulation can also be adapted for computing transferred potentials [11, 29].

9 Conclusions

A Boundary Element approach for the analysis of substation earthing systems has been presented. For 3D problems, some reasonable assumptions allow to reduce a general 2D BEM formulation to an approximated less expensive 1D version. Further simplifications reduce computing requirements under acceptable levels. Several widespread methods are identified as particular cases of this approach. In this theoretical framework, problems encountered with the application of these methods have been finally explained from a mathematically rigorous point of view. On the other hand, more efficient and accurate formulations have been derived. New analytical integration techniques allow to obtain accurate results in practical cases with acceptable computing requirements.

The techniques derived by the authors have been implemented in a Computer Aided Design system called TOTBEM. At present, this system runs in real-time in personal computers, and it has been used by the authors and by Spanish power companies to analyze several medium/large installations during the last 8 years. The techniques described in this paper have also been extended for grounding in multi-layer soil models and for the analysis of transferred earth potentials in electrical substations.

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