

Preprint of the paper

"Development of integration schemes for meshless numerical approaches based on the SPH method"

G. Mosqueira, I. Colominas, J. Bonet, F. Navarrina, M. Casteleiro (2001)

En "ECCOMAS CFD 2001: Computational Fluid Dynamics Conference Proceedings", (en CDROM, 20 páginas); K. Morgan, N. Weatherill (Editors); ECCOMAS and The Institute of Mathematics and its Applications; Swansea, UK. (ISBN: 0-905-091-124)

<http://caminos.udc.es/gmni>

DEVELOPMENT OF INTEGRATION SCHEMES FOR MESHLESS NUMERICAL APPROACHES BASED ON THE SPH METHOD

Gonzalo Mosqueira*, Ignasi Colominas*, Javier Bonet †, Fermín Navarrina*,
and Manuel Casteleiro*

*Dept. de Métodos Matemáticos y de Representación. Universidad de La Coruña
Campus de Elviña, 15192 La Coruña, SPAIN
e-mail: mosqueira@iccp.udc.es

†Dept. of Civil Engineering. Institute for Numerical Methods in Engineering
University of Wales Swansea. Singleton Park, Swansea SA2 8PP, UK

Key words: Meshless Methods, Corrected Smooth Particle Hydrodynamics, Weighted
Least Squares, Nodal Integration

Abstract. *In this paper we discuss some of the meshless formulations that have been recently proposed for solving Boundary Value Problems. Several functional approximations are studied both, from a continuum and from a discrete point of view, with special emphasis in the verification of the consistency conditions. On the other hand, different integration schemes are approached, considering both the so-called global integration and the so-called local integration schemes. Finally, some examples are developed to show the behaviour of all these numerical techniques when a nodal integration is used.*

1 INTRODUCTION

“Meshless methods” are a set of numerical techniques recently developed to solve partial differential equations. Its main purpose is to avoid the rigid connectivity needed in other usual numerical methods to discretize the integration domain. This feature lets to obtain important advantages in some applications in which the use of standard numerical techniques would imply a high refinement (and consequently an unaffordable computational effort), such as problems with very complicated domains, large deformations, moving boundaries, crack propagation or discontinuities¹.

Such as in Finite Elements, the construction of a meshless numerical approach implies the statement of a variational form of the boundary value problem, the generation of a local approximation in terms of a certain nodal points, and the integration of the variational form by using a suitable scheme.

Different meshless methods solve these steps by following different strategies. In general, the local character can be obtained by means of the so-called kernel approximation or by a weighted least squares interpolation. The first technique is used, for example, in the Smooth Particle Hydrodynamics method² (SPH) or in the Reproducing Kernel Particle Method³ (RKPM). The Diffuse Element Method⁴ and the Element Free Galerkin Method⁵ use the second one. Several alternatives have been studied to deal with the integration scheme needed to solve the boundary value problem⁶. Some of them involve an underlying mesh. In other cases, this mesh is avoided by using a point collocation approach or a nodal integration scheme⁷.

In the first section of the paper, we propose the inclusion of the kernel approximation and the weighted least squares interpolation in a general formulation. From this point we set up the Corrected Smooth Particle Hydrodynamics method (CSPH), which combines both techniques in order to enforce the consistency conditions and improve the accuracy of the standard SPH.

In the second section we deal with the integration of a Boundary Value Problem, analyzing two different alternatives: a global integration and a local integration. In both cases, these integrals can be solved numerically with a nodal integration scheme or a gaussian type.

In the third part, we focus our attention in approaching four different nodal integration techniques, which numerical behaviour is examined in the examples shown in the last part of the paper.

2 BASIS OF MESHLESS APROXIMATIONS

Meshless approximations are numerical techniques that allows to estimate the value of a function at a certain point from the known values of this function at a non-structured set of points.

Its main purpose is to avoid the rigid connectivity between the points where the function value is known, what is needed in other usual numerical methods, such as finite elements or finite differences.

The mathematical approach of these techniques can be done from both a continuum or a discrete point of view. First of all, we focus our attention in the continuum form in order to characterize this kind of approximations.

A meshless approximation can be understood as a smoothing interpolation technique⁹ in which the estimated value $u_h(\mathbf{x})$ of a function $u(\mathbf{x})$ at a point is obtained by using its values at a set of disordered points of a certain domain Ω , weighted with a function $K(\mathbf{x}, \mathbf{r})$, in the following way⁷:

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \int_{\mathbf{r} \in \Omega} K(\mathbf{x}, \mathbf{r}) u(\mathbf{r}) d\Omega \quad (1)$$

It is obvious that the weighting function plays a role of fundamental importance. Furthermore, it is the responsible for the local character of the approximation. One of the options to enforce this, is to define the weighting function so that it takes its maximum value at the point \mathbf{x} , while the information of any other point is weighted according to their distance to \mathbf{x} . Then, if the weighting function vanish outside a certain surrounding region, the approximation will have the desired local character. For example⁷:

$$K(\mathbf{x}, \mathbf{r}) = \begin{cases} H(z) \Big|_{z=|\mathbf{r}-\mathbf{x}|/\rho} > 0, & \text{if } \mathbf{r} \in B(\mathbf{x}); \\ 0, & \text{in other point,} \end{cases} \quad (2)$$

being $H(z)$ an adequate function, such as a gaussian function, a cubic spline or any other function with similar characteristics. $B(\mathbf{x})$ is a selected suitable subdomain in the neighbourhood of the given point \mathbf{x} , for instance

$$B(\mathbf{x}) = \{\mathbf{r} \in \Omega / |\mathbf{r} - \mathbf{x}| \leq R\} \quad (3)$$

The so-called dilation parameter ρ plays a very important role in (2), since it contributes to characterize the support of the weighting function.

On the other hand, the weighting function $K(\mathbf{x}, \mathbf{r})$ must verify the consistency requirements, which depend on the highest order of the polynomial that must be exactly represented by the approximation.

Consistency conditions are closely related to completeness and reproducing conditions. An approximation is complete if it provides a basis which can reproduce any function up to an arbitrary order of accuracy. Since any approximation which can exactly reproduce linear polynomials can reproduce any smooth function and its first derivative with arbitrary accuracy as the approximation is refined, an approximation which has linear consistency also has linear completeness⁶.

Reproducing conditions refer to the ability of an approximation to reproduce a function if the nodal values are set by the function; thus the ability to reproduce n th order polynomials is equivalent to n th order consistency⁶.

In order to achieve this, the weighting function must satisfy that:

$$\int_{\mathbf{r} \in \Omega} K(\mathbf{x}, \mathbf{r}) d\Omega = 1; \quad \int_{\mathbf{r} \in \Omega} K(\mathbf{x}, \mathbf{r})(\mathbf{r} - \mathbf{x})^n d\Omega = 0 \quad (4)$$

An example of weighting function with lineal consistency is the following cubic spline⁹:

$$K(\mathbf{x}, \mathbf{r}) = W(z) \Big|_{z=|\mathbf{r}-\mathbf{x}|/\rho} = p \begin{cases} 1 - \frac{3}{2}z^2 + \frac{3}{4}z^3 & \text{if } z \leq 1 \\ \frac{1}{4}(2-z)^3 & \text{if } 1 < z \leq 2 \\ 0 & \text{if } z > 2 \end{cases} \quad (5)$$

being $p = \frac{2}{3\rho}$ in one-dimensional problems; and $p = \frac{10}{7\pi\rho^2}$ in two-dimensional ones.

Once we have shown the main features that a good approximation must verify, it is necessary to explain how to construct the discrete form, i.e., how to calculate the previous integrals in a numerical way.

$$u^h(\mathbf{x}) = \int_{\mathbf{r} \in \Omega} K(\mathbf{x}, \mathbf{r}) u(\mathbf{r}) d\mathbf{r} \simeq \hat{u}^h(\mathbf{x}) = \sum_{in=1}^{nn} W(\mathbf{x}, \mathbf{r}_{in}) K(\mathbf{x}, \mathbf{r}_{in}) u(\mathbf{r}_{in}) \quad (6)$$

being \mathbf{r}_{in} the integration points, $W(\mathbf{x}, \mathbf{r}_{in})$ the integration weighting functions and nn the total number of non-structured integration points in Ω . In this paper, these integration points are also called nodal points.

The following consistency conditions must also be satisfied:

$$\sum_{in=1}^{nn} W(\mathbf{x}, \mathbf{r}_{in}) K(\mathbf{x}, \mathbf{r}_{in}) = 1; \quad \sum_{in=1}^{nn} W(\mathbf{x}, \mathbf{r}_{in}) K(\mathbf{x}, \mathbf{r}_{in})(\mathbf{x} - \mathbf{r}_{in})^n = 0. \quad (7)$$

So, given a point $\mathbf{x} \in \Omega$, the approximated value of the function at this point, $\hat{u}^h(\mathbf{x})$, is constructed in terms of the information provided by a certain number $n\alpha$ of its closest integration points ($n\alpha < nn$). Depending on the consistency order we are looking for, a minimum number ns of nodes must be taken into account in the interpolation ($n\alpha \geq ns$). On the other hand, for each node \mathbf{r}_{in} we can consider the set of subdomains Ω_{in} of all the points $\mathbf{x} \in \Omega$ which approximated value $\hat{u}^h(\mathbf{x})$ is affected by the nodal term $u(\mathbf{r}_{in})$. The proper definition of the approximation at every point requires that all these subdomains cover the whole interpolation domain Ω . Moreover, these subdomains must overlap, and every point $\mathbf{x} \in \Omega$ must belong to the subdomains of as many integration points (at least ns) as to ensure the uniqueness of the interpolation and the convergence of the method.

The following step is to determine an adequate integration weighting function. For this reason, we construct a local approximation $\hat{u}(\mathbf{r})$ to the function $u(\mathbf{r})$ in the vicinity of the given point \mathbf{x} , taking $n\alpha$ integration points (nodal points), as¹⁰

$$\hat{u}(\mathbf{r}) \equiv \sum_{i=1}^{n\alpha} p_i(\mathbf{r}) \alpha_i = \mathbf{p}^t(\mathbf{r}) \boldsymbol{\alpha}, \quad \mathbf{p}(\mathbf{r}) = \boldsymbol{\varphi}(z) \Big|_{z=|\mathbf{r}-\mathbf{x}|/\rho}, \quad (8)$$

where $\boldsymbol{\varphi}(\mathbf{z})$ is a complete base of selected interpolating functions (generally polynomials of a certain order) in \mathbf{z} , $\boldsymbol{\alpha}$ is the corresponding set of $n\alpha$ unknown coefficients to be determined, and ρ is the so-called dilation parameter. In expression (8), ρ does not play an essential but a harmless role, that is scaling the values of the coefficients $\boldsymbol{\alpha}$. For the following given scalar product $\langle \cdot, \cdot \rangle$ in Ω based on equation (6)

$$\langle f, g \rangle = \sum_{in=1}^{nn} K(\mathbf{x}, \mathbf{r}_{in}) f(\mathbf{r}_{in}) g(\mathbf{r}_{in}), \quad (9)$$

we define the quadratic functional associated to the residual error distribution

$$\begin{aligned} Q(\boldsymbol{\alpha}) &= \langle u - \hat{u}, u - \hat{u} \rangle \\ &= \langle u, u \rangle - 2 \langle u, \hat{u} \rangle + \langle \hat{u}, \hat{u} \rangle \\ &= \langle u, u \rangle - 2 \langle u, \mathbf{p}^t \rangle \boldsymbol{\alpha} + \boldsymbol{\alpha}^t \langle \mathbf{p}, \mathbf{p}^t \rangle \boldsymbol{\alpha}. \end{aligned} \quad (10)$$

Now, since

$$\left\{ \frac{\partial Q}{\partial \boldsymbol{\alpha}} \right\}^t = -2 \langle \mathbf{p}, u \rangle + 2 \langle \mathbf{p}, \mathbf{p}^t \rangle \boldsymbol{\alpha}, \quad (11)$$

we can obtain the least square fitting coefficients $\boldsymbol{\alpha}^*$ that minimize the quadratic functional (10). Thus

$$\left\{ \frac{\partial Q}{\partial \boldsymbol{\alpha}} \right\}^t \Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*} = \mathbf{0} \quad \implies \quad \boldsymbol{\alpha}^* = \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \langle \mathbf{p}, u \rangle, \quad (12)$$

where the so-called moment matrix $\langle \mathbf{p}, \mathbf{p}^t \rangle$ is a *Gram* matrix, that is positive semidefinite at least³. If the scalar product is well defined and the interpolating functions are well selected, the moment matrix is positive definite, what guarantees that the unknown vector $\boldsymbol{\alpha}^*$ is uniquely determined. Then, we can write the least squares approximation as

$$\hat{u}^*(\mathbf{r}) = \mathbf{p}^t(\mathbf{r}) \boldsymbol{\alpha}^* = \mathbf{p}^t(\mathbf{r}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \langle \mathbf{p}, u \rangle. \quad (13)$$

The above stated approximation gives a least square fitting that is intended to be valid for all points \mathbf{r} in a neighbourhood of the given point \mathbf{x} . In particular, when $\mathbf{r} = \mathbf{x}$ the latter expression gives the approximation

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) \approx \hat{u}^h(\mathbf{x}) = \hat{u}^*(\mathbf{x}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \langle \mathbf{p}, u \rangle. \quad (14)$$

It is easy to prove that this kind of approximation is exact in the whole domain if $u(\mathbf{r})$ belongs to the space of interpolating functions defined in (8), since

$$u(\mathbf{r}) = \mathbf{p}^t(\mathbf{r}) \boldsymbol{\beta} \quad \implies \quad \hat{u}^*(\mathbf{x}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \langle \mathbf{p}, \mathbf{p}^t \rangle \boldsymbol{\beta} = u(\mathbf{x}). \quad (15)$$

Therefore, the approximation reproduces exactly any polynomial function up to the order considered in (8).

In short, a way to construct the integration weighting functions, with a certain consistency order is:

$$W(\mathbf{x}, \mathbf{r}_{ip}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \mathbf{p}(\mathbf{r}_{ip}). \quad (16)$$

This approach is used, for instance, in the Element Free Galerkin Method¹¹ (EFGM) or in the Weighted Least Square Meshless Method⁷. Other meshless methods, like the Smooth Particle Hydrodynamics (SPH), use a cheaper integration weighting function which does not verify any consistency conditions⁸:

$$W(\mathbf{x}, \mathbf{r}_{in}) = \frac{1}{\sum_{in=1}^{nn} K(\mathbf{x}, \mathbf{r}_{in})}, \quad (17)$$

A modified version of the SPH is the Corrected Smooth Particle Hydrodynamics⁸ (CSPH). In this method it is proposed a successful integration weighting function which combines both schemes (16) and (17), in the following sense:

$$W(\mathbf{x}, \mathbf{r}_{in}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \mathbf{p}(\mathbf{r}_{in}), \quad (18)$$

where the scalar product $\langle \cdot, \cdot \rangle$ is given by

$$\langle f, g \rangle = \sum_{in=1}^{nn} V(\mathbf{x}) K(\mathbf{x}, \mathbf{r}_{in}) f(\mathbf{r}_{in}) g(\mathbf{r}_{in}); \quad V(\mathbf{x}) = \frac{1}{\sum_{in=1}^{nn} K(\mathbf{x}, \mathbf{r}_{in})}. \quad (19)$$

Thus, the approximated value of a function at a point \mathbf{x} is constructed as:

$$u^h(\mathbf{x}) = \int_{\mathbf{r} \in \Omega} K(\mathbf{x}, \mathbf{r}) u(\mathbf{r}) d\mathbf{r} \approx \hat{u}^h(\mathbf{x}) = \sum_{in=1}^{nn} V(\mathbf{x}) W(\mathbf{x}, \mathbf{r}_{in}) K(\mathbf{x}, \mathbf{r}_{in}) u(\mathbf{r}_{in}) \quad (20)$$

If we focus our attention on the CSPH method, the following step is to construct a FEM type discrete approximation $\hat{u}^h(\mathbf{x})$ to the function $u(\mathbf{r})$ at every point \mathbf{x} in Ω , in terms of the values $\{u_{in}\}$, where $u_{in} = u(\mathbf{r}_{in})$, being $\{\mathbf{r}_{in}\}$ for $in = 1, \dots, nn$ the integration or nodal points selected within the domain Ω . The answer is indeed in expression (14). Thus, if we adopt

$$\hat{u}^h(\mathbf{x}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} \langle \mathbf{p}, u \rangle, \quad (21)$$

with the discrete scalar product given in (19) we get

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) \approx \hat{u}^h(\mathbf{x}) = \sum_{in=1}^{nn} N(\mathbf{x}, \mathbf{r}_{in}) u(\mathbf{r}_{in}), \quad (22)$$

where

$$N(\mathbf{x}, \mathbf{r}_{in}) = \mathbf{p}^t(\mathbf{x}) \langle \mathbf{p}, \mathbf{p}^t \rangle^{-1} V(\mathbf{x}) K(\mathbf{x}, \mathbf{r}_{in}) \mathbf{p}(\mathbf{r}_{in}). \quad (23)$$

At this point we must remark that the local values of the approximating function do not necessarily fit the nodal unknown values (that is $\hat{u}^h(\mathbf{r}_{jn}) \neq u_{jn}$ for $jn = 1, \dots, nn$), due to the least square character of the approximation. However, the assertion (15) guarantees that the partition of unity is fulfilled by the trial functions defined by (23).

3 DIFFERENT NUMERICAL APPROACHES TO SOLVE A BVP WITH MESHLESS METHODS.

In order to present the different numerical approaches proposed to solve partial differential equations, we focus our study in time-independent linear problems like the Poisson equation, with both Dirichlet and Neumann boundary conditions. Thus, we consider the following BVP:

Given $f : \Omega \rightarrow \mathbb{R}$ and constants ϕ_u and t , find $\phi : \Omega \rightarrow \mathbb{R}$, such that

$$\begin{aligned} \Delta\phi + f &= 0 & \text{en } \Omega \\ \phi &= \phi_u & \text{en } \Gamma_u \\ \nabla\phi \cdot \mathbf{n} &= t & \text{en } \Gamma_t \end{aligned} \quad (24)$$

A variational form of the previous BVP can be stated as

$$\int_{\Omega} \omega [\Delta\phi + f] d\Omega = 0, \quad (25)$$

for all members ω of a suitable class of test functions ν ,

$$\nu = \{\omega / \omega(\Gamma_u) = 0\}.$$

If we apply the theorem of the divergence and taking into account that $\omega(\Gamma_u) = 0$, results in:

$$- \int_{\Omega} \nabla\omega \nabla\phi d\Omega + \int_{\Gamma_t} \omega t d\Gamma + \int_{\Omega} \omega f d\Omega = 0. \quad (26)$$

In order to define the weak counterpart of (24) it is necessary to characterize two classes of functions, the trial functions δ and the test functions ν . Both of them must be H^1 - functions, that is, their derivatives are require to be square-integrable¹²:

$$\begin{aligned} \delta &= \{\phi / \phi \in H^1, \phi(\Gamma_u) = \phi_u\} \\ \nu &= \{\omega / \omega \in H^1, \omega(\Gamma_u) = 0\} \end{aligned}$$

Following, we approach an approximation ϕ^h to the exact solution of the problem ϕ . Thus, the trial and the test functions, δ and ν , are approximated by δ^h and ν^h . Now,

we have to assume that ν^h is approximately zero in Γ_ϕ . Moreover, we are going to write each function ϕ^h of δ^h as $\phi^h = v^h + \phi_u^h$, where $v^h \in \nu^h$ and ϕ_u^h satisfy approximately the boundary condition $\phi = \phi_u$ in Γ_ϕ ¹².

Taking all of this into account, the problem we have to solve is:

Given f , ϕ_u and t , find $\phi^h = v^h + \phi_u^h \in \delta^h$ such that $\forall \omega^h \in \nu^h$

$$\int_{\Omega} \nabla v^h \nabla \omega^h d\Omega = \int_{\Omega} \omega^h f d\Omega + \int_{\Gamma_t} \omega^h t d\Gamma - \int_{\Omega} \nabla \phi_u^h \nabla \omega^h d\Omega. \quad (27)$$

Once the problem we want to solve is stated, it is necessary to indicate how to calculate the value of the unknowns, ω^h , v^h y ϕ_u^h . If we use the meshless approximation defined in the previous section:

$$\begin{aligned} \omega^h(\mathbf{x}) &= \int_{\mathbf{r} \in \Omega - \Gamma_\phi} K(\mathbf{x}, \mathbf{r}) \omega(\mathbf{r}) d\mathbf{r} \simeq \hat{w}^h(\mathbf{x}) = \sum_{in=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \\ v^h(\mathbf{x}) &= \int_{\mathbf{r} \in \Omega - \Gamma_\phi} K(\mathbf{x}, \mathbf{r}) v(\mathbf{r}) d\mathbf{r} \simeq \hat{v}^h(\mathbf{x}) = \sum_{jn=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{jn}) v(\mathbf{r}_{jn}) \\ \phi_u^h(\mathbf{x}) &= \int_{\mathbf{r} \in \Gamma_\phi} K(\mathbf{x}, \mathbf{r}) \phi_u(\mathbf{r}) d\mathbf{r} \simeq \hat{\phi}_u^h(\mathbf{x}) = \sum_{iu=1}^{nn_2} N(\mathbf{x}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu}), \end{aligned} \quad (28)$$

being nn_1 the number of nodal points in $\Omega - \Gamma_\phi$ and nn_2 the number of nodal points in Γ_ϕ .

If these expressions are introduced in the formulation, we obtain:

$$\begin{aligned} & \int_{\mathbf{x} \in \Omega} \left[\nabla \sum_{in=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \right] \left[\nabla \sum_{jn=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{jn}) v(\mathbf{r}_{jn}) \right] d\Omega \\ &= \int_{\mathbf{x} \in \Omega} \left[\sum_{in=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \right] f(\mathbf{x}) d\Omega \\ &+ \int_{\mathbf{x} \in \Gamma_t} \left[\nabla \sum_{in=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \right] t(\mathbf{x}) d\Gamma \\ &- \int_{\mathbf{x} \in \Omega} \left[\nabla \sum_{in=1}^{nn_1} N(\mathbf{x}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \right] \left[\nabla \sum_{iu=1}^{nn_2} N(\mathbf{x}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu}) \right] d\Omega \end{aligned} \quad (29)$$

Now it is possible to derive different numerical approaches depending on the type of integration selected. In this paper we consider two kinds of integration: global integration or local integration.

3.1 Global integration.

In this case, the numerical integration is performed over the whole domain. In order to determine a set of suitable integration points and a set of suitable integration weighting functions, we consider the following options:

Nodal type Integration: Consists of taking as integration points the nodal points. In the following section we show some techniques to calculate the integration weighting functions.

Gaussian type integration: If this option is chosen, it is necessary an auxiliary mesh, similar to those used in the finite element method. In each element, the integration points and the weighting functions are calculated using a Gaussian quadrature formula. The advantages of the nodal integration is that a less computational effort is needed because it is not necessary neither to construct a mesh, nor to evaluate the function in the new points created. On the contrary, the biggest problem, appart from the stability questions, is the low accuracy obtained with this technique against the Gaussian integration.

In both cases, if we consider two sets of nc_1 and nc_2 integration points \mathbf{x}_{ic} , and $W_{c1}(\cdot, \cdot)$ and $W_{c2}(\cdot, \cdot)$ weighting functions, expression (29) can be written as:

$$\begin{aligned}
 & \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) \sum_{in=1}^{nn_1} \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \sum_{jn=1}^{nn_1} \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) v(\mathbf{r}_{jn}) \\
 = & \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) \sum_{in=1}^{nn_1} N(\mathbf{x}_{ic}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) f(\mathbf{x}_{ic}) \\
 + & \sum_{ic=1}^{nc_2} W_{c2}(\mathbf{x}_{ic}) \sum_{in=1}^{nn_1} \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\
 - & \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) \sum_{in=1}^{nn_1} \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) w(\mathbf{r}_{in}) \sum_{iu=1}^{nn_2} \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu})
 \end{aligned} \tag{30}$$

If we arrange this expression and we take into account that it must be verify for all $w(\mathbf{r}_{in})$, we obtain

$$\begin{aligned}
 \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \left\{ \sum_{jn=1}^{nn_1} \left[\sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) \right] v(\mathbf{r}_{jn}) \right. \\
 - \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) N(\mathbf{x}_{ic}, \mathbf{r}_{in}) f(\mathbf{x}_{ic}) - \sum_{ic=1}^{nc_2} W_{c2}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\
 \left. + \sum_{ic=1}^{nc_1} \sum_{iu=1}^{nn_2} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu}) \right\} = 0,
 \end{aligned} \tag{31}$$

that it can be written in terms of a linear equations system:

$$\mathbf{A}\mathbf{v} = \mathbf{b},$$

being:

$$\begin{aligned} a_{in,jn} &= \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) \\ b_{in} &= \sum_{ic=1}^{nc_1} W_{c1}(\mathbf{x}_{ic}) N(\mathbf{x}_{ic}, \mathbf{r}_{in}) f(\mathbf{x}_{ic}) + \sum_{ic=1}^{nc_2} W_{c2}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\ &\quad - \sum_{ic=1}^{nc_1} \sum_{iu=1}^{nn_2} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu}). \end{aligned} \quad (32)$$

3.2 Local integration.

Equation (29) can also be numerically solved if the integrals are inserted into the summatories. In this case, taking into account the definition of the weighting function $K(\mathbf{x}, \mathbf{r})$, given by (2), this integral is zero if \mathbf{x} is outside the region defined by the intersection between the subdomains of the points \mathbf{r}_{in} and \mathbf{r}_{jn} . Therefore, equation (29) can be written:

$$\begin{aligned} &\sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{jn=1}^{nn_1} v(\mathbf{r}_{jn}) \int_{x \in \Omega_{in,jn}} \nabla N(\mathbf{x}, \mathbf{r}_{in}) \nabla N(\mathbf{x}, \mathbf{r}_{jn}) d\Omega \\ &= \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \int_{x \in \Omega_{in}} N(\mathbf{x}, \mathbf{r}_{in}) f(\mathbf{x}) d\Omega + \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \int_{x \in \Gamma_{in}} \nabla N(\mathbf{x}, \mathbf{r}_{in}) t(\mathbf{x}) d\Gamma \\ &\quad - \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{iu=1}^{nn_2} \phi_u(\mathbf{r}_{iu}) \int_{x \in \Omega_{in,iu}} \nabla N(\mathbf{x}, \mathbf{r}_{in}) \nabla N(\mathbf{x}, \mathbf{r}_{iu}) d\Omega \end{aligned} \quad (33)$$

where Ω_{in} is the subdomain of the nodal point r_{in} , Γ_{in} is the part of the boundary within the subdomain of the point r_{in} , $\Omega_{in,jn}$ is the intersecting region between the subdomains of the points r_{in} and r_{jn} , and $\Omega_{in,iu}$ is the intersecting region between the subdomains of the points r_{in} and r_{iu} .

As in the previous case, we can perform the integration by using a nodal integration, being the integration points the nodal points which lies in the integration domain, or a gaussian integration, creating new points in this region. Unlike the global integration, this local integration does not require an auxiliary mesh, because the integration domain is sufficiently small.

If we consider the sets of nc_i , nc_t , $nc_{i,j}$, and $nc_{i,u}$ integration points and $W_{c1}(\cdot, \cdot)$ and $W_{c2}(\cdot, \cdot)$ weighting functions, expression (33) can be written as:

$$\begin{aligned}
 & \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{jn=1}^{nn_1} v(\mathbf{r}_{jn}) \sum_{ic=1}^{nc_{i,j}} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) \\
 = & \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{ic=1}^{nc_i} W_{c1}(\mathbf{x}_{ic}) N(\mathbf{x}_{ic}, \mathbf{r}_{in}) f(\mathbf{x}_{ic}) \\
 + & \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{ic=1}^{nc_t} W_{c2}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\
 - & \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \sum_{iu=1}^{nn_2} \phi_u(\mathbf{r}_{iu}) \sum_{ic=1}^{nc_{i,u}} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu})
 \end{aligned} \tag{34}$$

This expression can be arranged in the following way:

$$\begin{aligned}
 \sum_{in=1}^{nn_1} w(\mathbf{r}_{in}) \left\{ \sum_{jn=1}^{nn_1} \left[\sum_{ic=1}^{nc_{i,j}} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) \right] v(\mathbf{r}_{jn}) \right. \\
 - \sum_{ic=1}^{nc_i} W_{c1}(\mathbf{x}_{ic}) N(\mathbf{x}_{ic}, \mathbf{r}_{in}) f(\mathbf{x}_{ic}) - \sum_{ic=1}^{nc_t} W_{c2}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\
 \left. + \sum_{ic=1}^{nc_{i,u}} \sum_{iu=1}^{nn_2} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu}) \right\} = 0
 \end{aligned} \tag{35}$$

Taking into account that this equation must be verified for every $w(\mathbf{r}_{in})$, it can be expressed as the following linear equations system:

$$\mathbf{A}\mathbf{v} = \mathbf{b},$$

being:

$$\begin{aligned}
 a_{in,jn} &= \sum_{ic=1}^{nc_{i,j}} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{jn}) \\
 b_{in} &= \sum_{ic=1}^{nc_i} W_{c1}(\mathbf{x}_{ic}) N(\mathbf{x}_{ic}, \mathbf{r}_{in}) f(\mathbf{x}_{ic}) + \sum_{ic=1}^{nc_t} W_{c2}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) t(\mathbf{x}_{ic}) \\
 & - \sum_{ic=1}^{nc_{i,u}} \sum_{iu=1}^{nn_2} W_{c1}(\mathbf{x}_{ic}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{in}) \nabla N(\mathbf{x}_{ic}, \mathbf{r}_{iu}) \phi_u(\mathbf{r}_{iu})
 \end{aligned} \tag{36}$$

4 GLOBAL AND LOCAL WEIGHTING INTEGRATION FUNCTIONS IN NODAL INTEGRATION

In this section we explore several ways to calculate the integration weighting functions. In all cases, the integration points are the nodal points. For this reason, the integration is called nodal integration⁶.

4.1 SPH Integration

These are the weighting function used in the SPH method⁹. $W_c(\mathbf{x}_{ic})$ is calculated as the tributary or statistical volume associated with the point \mathbf{x}_{ic} . A typical expression for this volume derived from Montecarlo integration theory is given by:

$$W_c(\mathbf{x}_{ic}) = \frac{1}{\sum_{in=1}^{nn_{ic}} K(\mathbf{x}_{ic}, \mathbf{r}_{in})}, \quad (37)$$

being nn_{ic} the total number of nodal points associated to the point \mathbf{x}_{ic} .

4.2 Global Integration - 1

In two-dimensional problems, if the support of the weighting function $K(\mathbf{x}, \mathbf{r})$ is circular, we propose in this paper the following way to calculate the integration weighting functions:

$$W_{c1}(\mathbf{x}_{ic}) = \frac{A_{ic}}{\sum_{jc=1}^{nc_1} A_{jc}} A_{\Omega}, \quad (38)$$

being A_{ic} the area of the subdomain associated to point \mathbf{x}_{ic} , nc_1 the total number of integration points and A_{Ω} the total area of the domain.

If the subdomains are circular, these areas can be calculated easily in a exactly way, since they are circular areas or intersections between the circular subdomains and the lines that defines the boundaries.

The integrals along the boundary Γ_t can be calculated as:

$$W_{c2}(\mathbf{x}_{ic}) = \frac{L_{ic}}{\sum_{jc=1}^{nc_2} L_{jc}} L_{\Gamma}, \quad (39)$$

being, in two-dimensional problems, L_{ic} the length of the part of the boundary Γ_t which is inside the subdomain of the point \mathbf{x}_{ic} , nc_2 the total number of nodal points in the Γ_t boundary and L_{Γ} the total length of the Γ_t boundary.

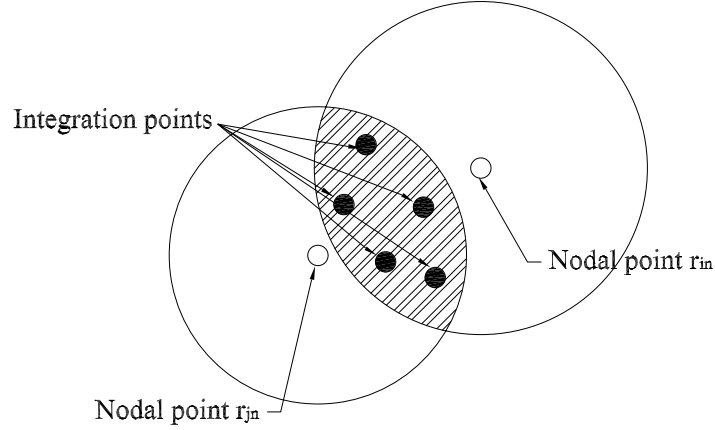


Fig. 1 Intersecting area of the subdomains of the points \mathbf{r}_{in} y \mathbf{r}_{jn} .

4.3 Global Integration - 2

A simple way to calculate the previous integrals, and occasionally more efficient is proposed in reference⁶. In this case, the integration weighting functions are:

$$W_{c1}(\mathbf{x}_{ic}) = \frac{f_{ic}R_{ic}^2}{\sum_{jc=1}^{nc_1} f_{jc}R_{jc}^2} A_{\Omega}, \quad (40)$$

where R_{ic} is the radio of the subdomain associated to the integration point \mathbf{x}_{ic} and f_{ic} indicates which part of the weighting function support belongs to the total domain Ω . For interior points, $f_{ic} = 1$; for nodes on a straight edge, $f_{ic} = 0.5$; for right corner nodes, $f_{ic} = 0.25$.

In the contour integration over the boundary Γ_t , the integration weight is:

$$W_{c2}(\mathbf{x}_{ic}) = \frac{g_{ic}R_{ic}}{\sum_{jc=1}^{nc_2} g_{jc}R_{jc}} L_{\Gamma}, \quad (41)$$

where L_{Γ} is the total length of the boundary Γ_t and g_{ic} the fraction of $2R_{ic}$ which lies in the support of the weighting function.

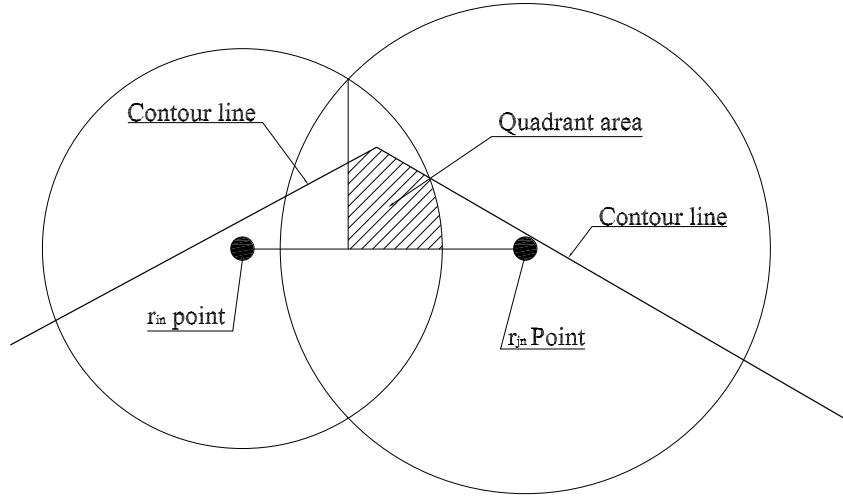


Fig. 2 A quadrant area, calculated as the intersection between the subdomains of two points, r_{in} and r_{jn} , and the contour lines.

4.4 Local Integration

Like in the preceding techniques, in this paper we propose a nodal integration, i.e., the integration points are the nodal points. In figure 1, it is shown a pair of nodal points \mathbf{r}_{in} and \mathbf{r}_{jn} , and its respective subdomains. Inside the region (integration domain) intersection between both subdomains, there are some nodal points which are use as integration points \mathbf{x}_{ic} .

In order to approach a 0 order integration scheme, the integration weighting functions can be calculated as:

$$W_c(\mathbf{x}_{ic}) = \frac{A_{in,jn}}{nn_{in,jn}}, \quad (42)$$

being $A_{in,jn}$ the total area of the integration domain and $nn_{in,jn}$ the total number of nodal points inside this region.

Because of the circular shape of the subdomains, the area $A_{in,jn}$ can be exactly calculated. In this paper, these areas are calculated dividing them in four quadrants, considering up to two intersections between the integration area and the boundaries. In figure 2 it is shown an example of a quadrant. The total area is the sum of the four quadrants.

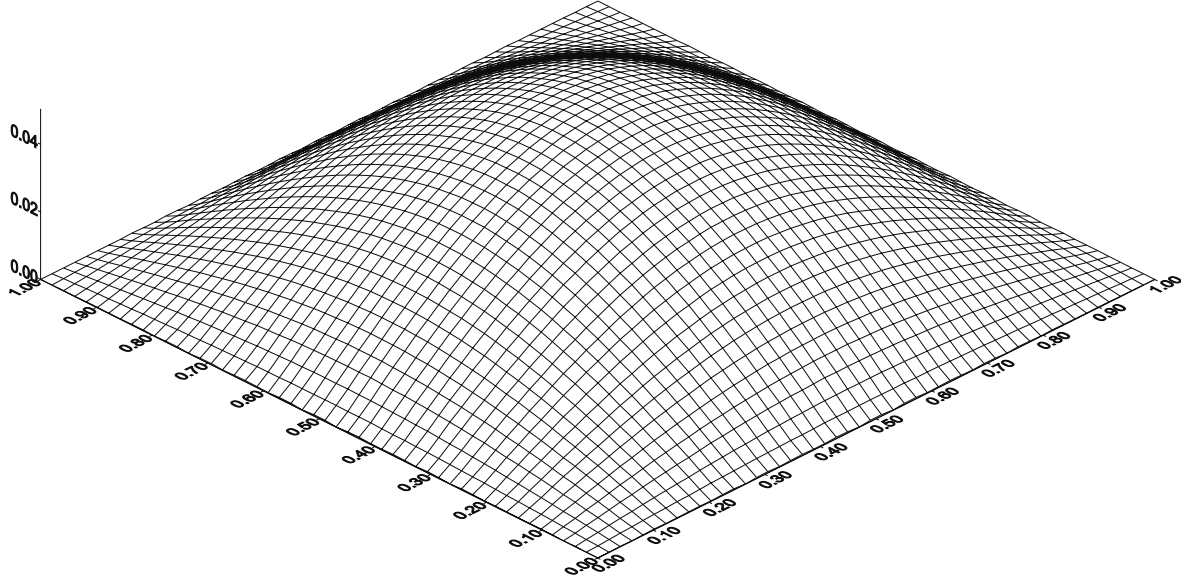


Fig. 3 Analytical solution of the problem considered in (43).

5 APPLICATION EXAMPLES

In order to show the behaviour of the different integration techniques studied in the previous section, we apply them to solve the following example:

$$\begin{aligned} \Delta\phi + \sin(\pi x) \sin(\pi y) &= 0 \quad \text{in } x \in (0, 1) \quad y \in (0, 1) \\ \phi &= 0 \quad \text{in } x = 0; \quad x = 1; \quad y = 0; \quad y = 1 \end{aligned} \quad (43)$$

The solution to this problem is (see figure 3):

$$\phi = \frac{\sin(\pi x) \sin(\pi y)}{2\pi^2} \quad (44)$$

We focus our attention in the CSPH method with the cubic kernels given by (5). In the figure 4 it is shown the results obtained using 6x6 structured points to discretize the domain. In the left side of this figure it is depicted the solution of the problem (43), with the so-called SPH integration, Global Integration - 1, Global Integration - 2, Local Integration and the analytical solution, in four different sections: $y = 0.2$, $y = 0.4$, $y = 0.6$ and $y = 0.8$. In the right side it is shown the relative error between all the numerical solutions and the analytical one. Despite the few points taken into account, the Global Integration - 1 and the Local Integration let to obtain quite good results.

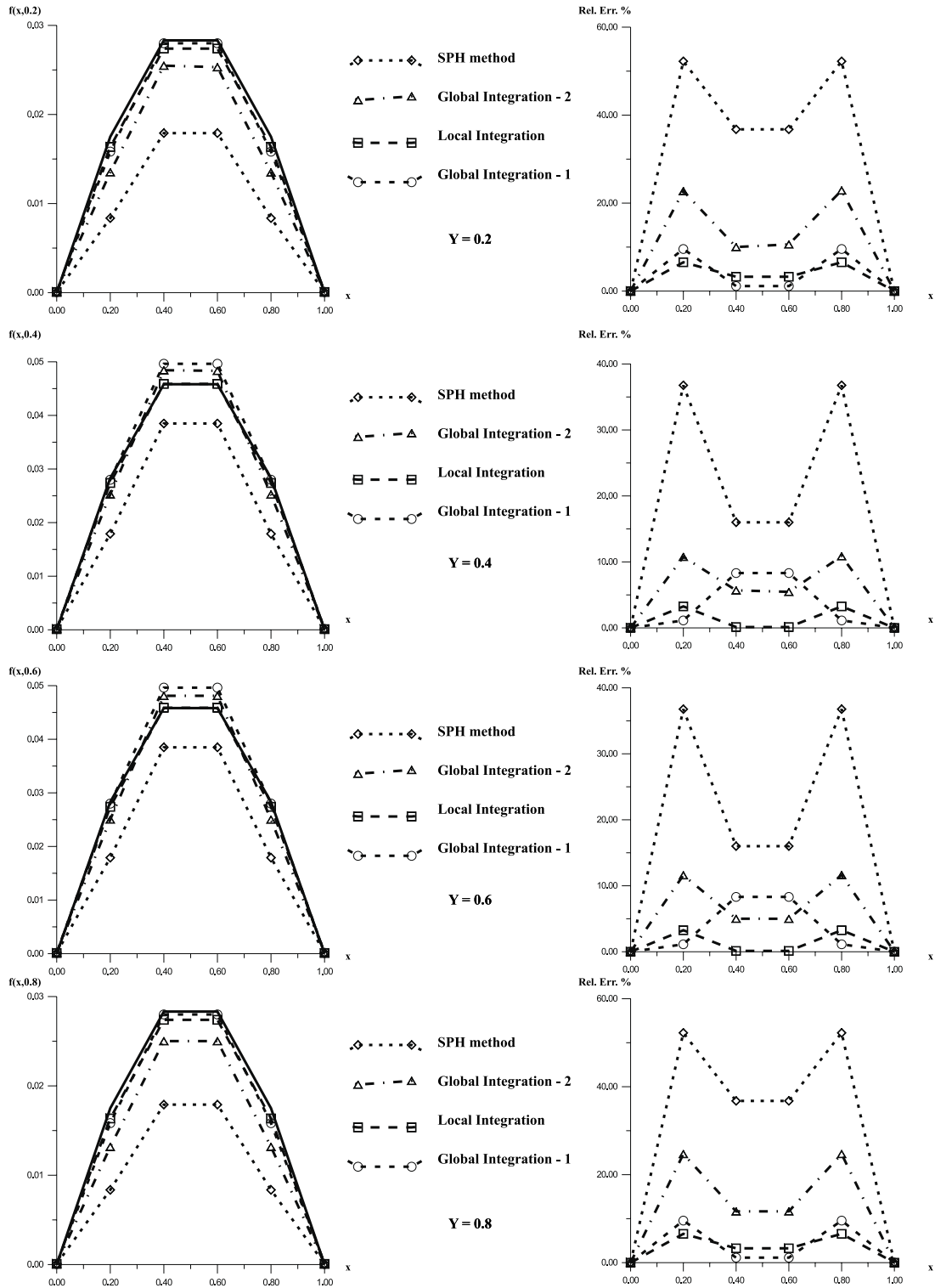


Fig. 4 Results obtained with 6x6 structured points.

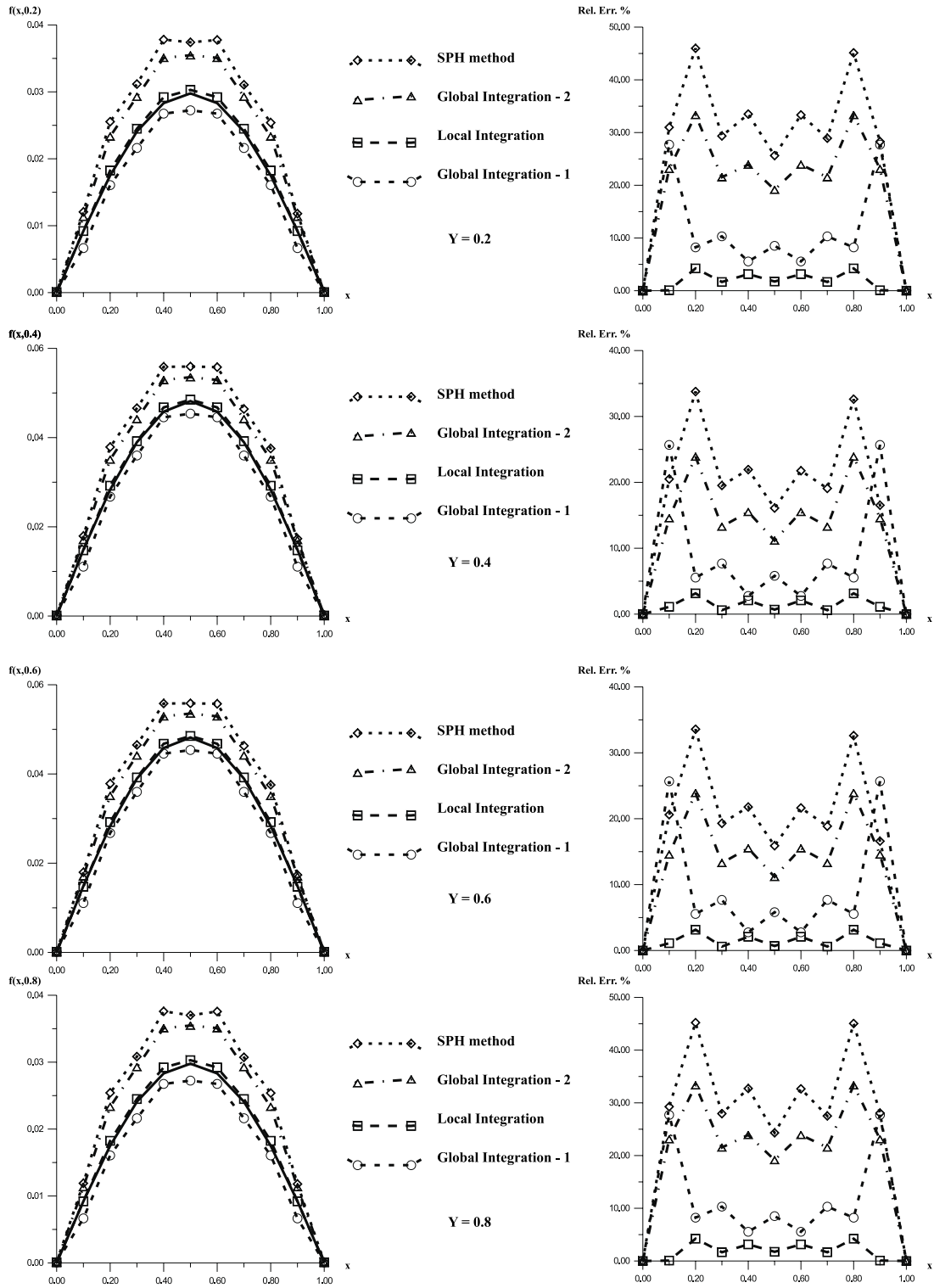


Fig. 5 Results obtained with 11x11 structured points.

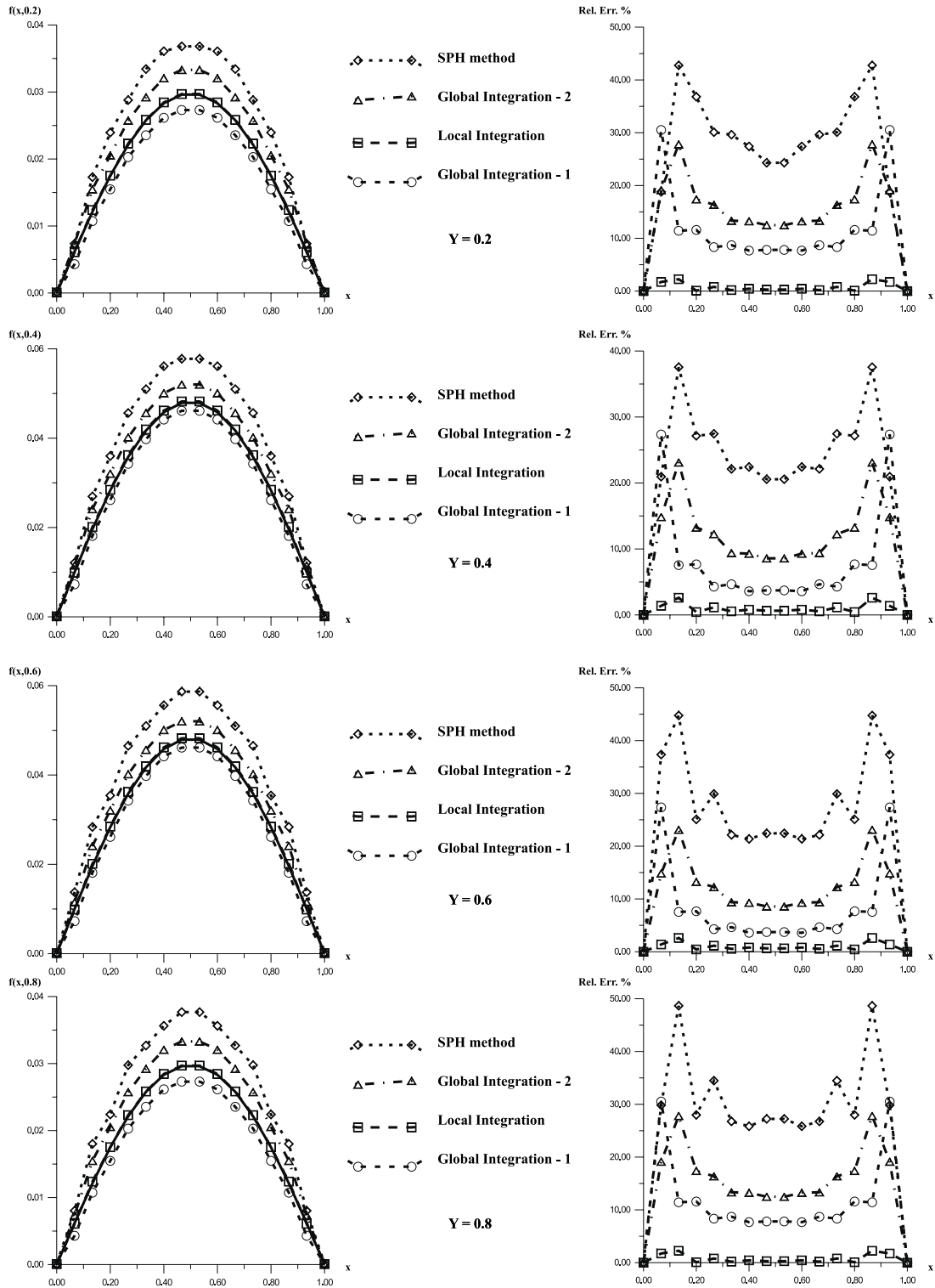


Fig. 6 Results obtained with 16x16 structured points.

In the figures 5 and 6, the same graphs are depicted, with 11x11 structured points (figure 5) and 16x16 structured points (figure 6). In these cases, it can be observed that the relative error decreases using either of the four different methods, and, furthermore, the results are better as the number of nodal points is increased. In any case, the best results are always obtained with the Global Integration - 1 and the Local Integration approaches.

6 CONCLUSIONS

In this paper we have studied different meshless approximations, based on a continuum form and in a discrete form, using a weighted least squares technique. Both of them were constructed in order to satisfy some consistency conditions.

Furthermore, we have used these approximations to solve a Boundary Value Problem. In this case, we have setted up two options: a global and a local integration. The advantages of the local integration is that it avoids the use of a mesh like in the global one although the integration domain is more complicated. The numerical resolution of both integrals can be carried out using a nodal or a gaussian integration. In this paper, we have focused our attention on the nodal integration because of its lower computational cost.

The example solved show good results. Nevertheless, other alternatives could be used to improve these results. It would be interesting to compare the nodal integration with the gaussian one, particularly in the local integration case, or to use the same integration weighting functions to approximate the function and to solve the Boundary Value Problem. Nowadays, we are working in this direction, taking into account that the final objective is to overcome any patch test.

7 ACKNOWLEDGEMENTS

This work has been partially supported by the “Ministerio de Ciencia y Tecnología (project #1FD97-0108)” of the Spanish Government, cofinanced with European Union FEDER funds, and by research fellowships of the R&D Secretary of “Xunta de Galicia” and the “Universidad de La Coruña”. The authours wish also to thank Dr. S. Kulasegaram and Mr. M. Rodríguez for their interest, encouragement and numerous useful discussions.

REFERENCES

- [1] Belystchko T., Krongauz Y., Organ D., Krysl P. (1996): *Meshless methods: An overview and recent developments*, Computer Methods in Applied Mechanics and Engineering, **139**, 3–48.
- [2] Bonet J., Lok T.S.L. (1998): *Variational and momentum preseving aspects of Smooth Particle Hydrodynamics formulations*, Computer Methods in Applied Mechanics and Engineering, **180**,97-115.

- [3] W. K. Liu, S. Li and T. Belytschko (1997): *Moving least square reproducing kernel methods. (I) Methodology and convergence*, Comput. Meth. Appl. Mech. Engrg, **143**, 113-154.
- [4] Nayroles, B., Touzot, G. and Villon, P. (1992): *Generalizing the Finite Element Method: Diffuse Approximation and Diffuse Elements*, Computational Mechanics, **10**, 307-318.
- [5] Belytschko, T., Lu, Y.Y. and Gu, L. (1994): *Element Free Galerkin Methods*, International Journal for Numerical Methods in Engineering, **37**, 229-256.
- [6] S. Beissel and T. Belytschko (1996): *Nodal integration of the element-free Galerkin method*, Comput. Meth. Appl. Mech. Engrg, **139**.
- [7] Mosqueira G., Colominas I., Navarrina F. and Casteleiro M. (2000): *An Enriched Meshless Numerical Approach for Potential Theory Problems.*, Proceedings of the "European Congress on Computational Methods in Applied Sciences and Engineering, ECCOMAS 2000, Barcelona.
- [8] Bonet J. and Kulasegaram S. (2000): *Correction and Stabilization of Smooth Particle Hydrodynamics Methods with Applications in Metal Forming Simulations*, International Journal for Numerical Methods in Engineering, **47**, 1189-1214.
- [9] Kulasegaram S. (1999): *Development of particle based meshless method with applications in metal forming simulations*, Doctoral Thesis. University of Wales, Swansea, UK.
- [10] W. K. Liu, S. Jun and Y. F. Zhang (1995): *Reproducing kernel particle methods*, Int. J. Numer. Meth. Fluids, **20**, 1081-1106.
- [11] Belytschko T., Gu L., Lu Y.Y. (1994): *Element Free Galerkin Methods*, International Journal for Numerical Methods in Engineering, **37**, 229-256.
- [12] Hughes, Thomas J. R. (1987): *The Finite Element Method. Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall Inc., New Jersey.