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AN ENRICHED MESHLESS NUMERICAL APPROACH FOR POTENTIAL THEORY PROBLEMS

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Abstract.
One of the meshless techniques that have been recently proposed for solving Boundary Value Problems is the so-called Weighted Least Square (WLS) Method with a Point Collocation approach. Meshless methods were initially intended for problems in which mesh generation becomes a critical point in the overall process, and/or the subsequent computing effort is extremely high, due to the geometrical complexity of the domain. Obviously, if mesh generation is discarded, the regular element by element kind of interpolation procedures can not be used anymore. Therefore, the essential concepts of meshless methods rely on the availability of simple, efficient and robust interpolation procedures for non-structured distribution of points. In substation grounding analysis we can find a paradigmatic example of the extreme difficulties that mesh generation may involve and/or may produce. The authors have developed a BEM formulation that produces highly accurate results in the earthing analysis of large real grounding systems with uniform and stratified soil models. However, it is not obvious how to extend the BEM formulation in order to incorporate more realistic soil models. For this reason we have turned our attention to meshless methods. In this paper we present a WLS point collocation approach with extrinsic enrichment. In order to point out the performance of the proposed technique we present several 2D application examples. On the other hand, we present two numerical tests that have been designed to explore the possible stabilizing properties of this method when it is applied to the transport equation.

1 INTRODUCTION

For a variety of problems with large deformations, moving boundaries or discontinuities, classical numerical techniques (such as finite elements, boundary elements, finite volumes or finite differences) show significant unexpected difficulties[1]. In general, one must refine the mesh beyond a given level in order to mitigate these undesired effects. This increases the mesh generation cost, and implies a computational effort that can easily become unaffordable. On the other hand, the mesh generation process may become extremely complicated in problems with complex domains.

In recent years, the so-called "meshless methods" have been proposed to overcome these difficulties. The underlying key idea is that discretization becomes a fairly straightforward process when it is not necessary to keep track of the topological and geometrical restrictions of the available elements. In fact, the only target is to obtain a reasonable distribution of nodes within the given domain. Thus, in contrast to other techniques, meshless methods are specially attractive for the above mentioned problems, since no kind of rigid connectivity is imposed a priori.

In order to retain the positive features of the Finite Element Method, one must generate a local approximation to the solution in terms of the nodal values. A certain kind of local interpolation can be constructed by means of the so-called kernel approximation. This technique is used in the Smooth Particle Hydrodynamics method. SPH was originally developed in the computational physics field, but it has been recently applied to solve other problems in solid and fluid mechanics[2]. The Reproducing Kernel Particle Method[3], is similar to SPH, although several correction functions and refinements are introduced in order to assure consistency near boundaries and for nonuniform spacing.

Another possibility is to define the local approximation by means of least squares, as occurs in the Diffuse Element Method[4], proposed by Nayroles in 1992. In order to refine the previous method, Belystchko *et al.* proposed a moving least squares interpolation[5] which can be used in the resolution of partial differential equations with Galerkin schemes (Element Free Galerkin method) or point collocation schemes.

However, the experience with this kind of methods shows that all of them imnply a high computational cost, what reduces their range of applicability. For this reason, some new ideas have been recently proposed, such as combining finite elements and meshless interpolations (enrichment of Finite Elements with the Element Free Galerkin method[6], or with the Reproducing Kernel Particle Method[7]) and developing meshless methods on the basis of the partition of unity concept (*hp*-Clouds method[8], Partition of Unity Finite Element Method[9]), which the aim of providing an efficient way to perform $h-p$ adaptativity.

One of the main advantages of the meshless methods based on the partition of unity concept is that it is possible to include a priori knowledge about the differential equation in the formulation. On the basis on this property, the application of enrichment functions to the Element Free Galerkin Method has been investigated in crack propagation problems[10]. This technique is particularly effective in the presence of high gradients, concentrated forces, and large deformations, because it avoids the remeshing process.

All these techniques are suitable to be applied for solving potential problems in electrical engineering. In particular, a very interesting case is the grounding analysis[11], where the use of standard numerical methods (such as finite elements) is precluded due to the complexity of the domain. Furthermore, the possible existence of high potential gradients in the vicinity of the earthing grid recommend the use of enrichment approaches in the treatment of this kind of problems.

The physical phenomena underlying fault current dissipation into the earth can be described in terms of Maxwell's Electromagnetic Theory. In certain hypothesis, the problem can be written in terms of a Neumann Exterior Problem. In this case, it is necessary to discretize a semi-infinite domain (the soil) and to impose the boundary conditions on the surface of a grid of conductors which length is huge in comparison with its diameter[12]. This specific geometry precludes the use of standard numerical techniques (such as Finite Differences or Finite Elements), since the obtention of sufficiently accurate results would imply unacceptable computing efforts.

In order to solve these problems, the authors have developed a Boundary Element numerical formulation that has proved to produce highly accurate results in the earthing analysis of large real grounding systems with uniform and stratified soil models[13]. However, it is not obvious how to extend this BEM formulation in order to incorporate more realistic soil models.

For this reason, we have turned our attention to meshless methods based on Weighted Least Square approaches combined with point collocation schemes[14]. This kind of methods do not require any kind of mesh, what could enable the computational analysis of grounding grids even for those cases in which mesh generation is unaffordable in practice.

The first results that have been obtained for this kind of problems with meshless methods[15] can be described as very encouraging. Furthermore, we propose the use of enrichment procedures, i.e., techniques which allow to introduce some information about the solution in the numerical formulation, in order to improve the quality of the results while reducing the computational cost.

In this paper we present a WLS point collocation approach with extrinsic enrichment. In order to point out the performance of the proposed technique, we present several 2D application examples. On the other hand, we present two numerical tests that have been designed to explore the possible stabilizing properties of this method when it is applied to the transport equation.

2 FUNCTIONAL APPROXIMATION BASED ON A LOCAL WEIGHTED LEAST SQUARES TECHNIQUE

The Weighted Least Squares methodology is an effective numerical tecnique for the approximation of a certain function in terms of a given set of non-structured data. Essentially, to obtain a WLS approximation at a given point ($\boldsymbol{x}$) we perform a local weighted least square fitting on a certain neighbourhood of the given point.

Let Ω be the whole interpolation domain, and let $u(\boldsymbol{r})$ be a function defined for all $\boldsymbol{r}$ in Ω. First, we construct a local approximation $\widehat{u}(\boldsymbol{r})$ to the function $u(\boldsymbol{r})$ in the vicinity of the given point $\boldsymbol{x}$ as[15]

$$\widehat{u}(\boldsymbol{r}) \equiv \sum_{i=1}^{n_\alpha} p_i(\boldsymbol{r})\alpha_i = \boldsymbol{p}^t(\boldsymbol{r})\boldsymbol{\alpha}, \qquad \boldsymbol{p}(\boldsymbol{r}) = \boldsymbol{\varphi}(\boldsymbol{z})\Big|_{\boldsymbol{z}=(\boldsymbol{r}-\boldsymbol{x})/\rho}, \tag{1}$$

where $\boldsymbol{\varphi}(\boldsymbol{z})$ is a complete base of selected interpolating functions (generally polynomials of a certain order) in $\boldsymbol{z}$, $\boldsymbol{\alpha}$ is the corresponding set of n_α unknown coefficients to be determined, and ρ is the so-called dilation parameter. In expression (1), ρ does not play an essential but a harmless role, that is scaling the values of the coefficients $\boldsymbol{\alpha}$.

For a given scalar product $< \cdot, \cdot >$ in Ω we define the quadratic functional associated to the residual error distribution

$$\begin{aligned} Q(\boldsymbol{\alpha}) &=< u-\widehat{u}, u-\widehat{u} > \\ &=< u, u > -2 < u, \widehat{u} > + < \widehat{u}, \widehat{u} > \\ &=< u, u > -2 < u, \boldsymbol{p}^t > \boldsymbol{\alpha} + \boldsymbol{\alpha}^t < \boldsymbol{p}, \boldsymbol{p}^t > \boldsymbol{\alpha}. \end{aligned} \tag{2}$$

Now, since

$$\left\{\frac{\partial Q}{\partial \boldsymbol{\alpha}}\right\}^t = -2 < \boldsymbol{p}, u > +2 < \boldsymbol{p}, \boldsymbol{p}^t > \boldsymbol{\alpha},$$

we can obtain the least square fitting coefficients $\boldsymbol{\alpha}^*$ that minimize the quadratic functional (2). Thus

$$\left\{\frac{\partial Q}{\partial \boldsymbol{\alpha}}\right\}^t\Bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*} = \mathbf{0} \qquad \Longrightarrow \qquad \boldsymbol{\alpha}^* =< \boldsymbol{p}, \boldsymbol{p}^t >^{-1}< \boldsymbol{p}, u >, \tag{3}$$

where the so-called moment matrix $< \boldsymbol{p}, \boldsymbol{p}^t >$ is a *Gram* matrix, that is positive semidefinite at least. If the scalar product is well defined and the interpolating functions are well selected the moment matrix is positive definite, what guarantees that the unknown vector $\boldsymbol{\alpha}^*$ is uniquely determined. Then, we can write the least squares approximation as

$$\widehat{u}^*(\boldsymbol{r}) = \boldsymbol{p}^t(\boldsymbol{r})\boldsymbol{\alpha}^* = \boldsymbol{p}^t(\boldsymbol{r}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1}< \boldsymbol{p}, u > . \tag{4}$$

The above stated approximation gives a least square fitting that is intended to be valid for all points $\boldsymbol{r}$ in a neighbourhood of the given point $\boldsymbol{x}$. In particular, when $\boldsymbol{r} = \boldsymbol{x}$ the latter expression gives the approximation

$$u(\boldsymbol{x}) \approx \widehat{u}^*(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1} < \boldsymbol{p}, u > . \tag{5}$$

It is easy to prove that this kind of approximation is exact in the whole domain if $u(\boldsymbol{r})$ belongs to the space of interpolating functions defined in (1), since

$$u(\boldsymbol{r}) = \boldsymbol{p}^t(\boldsymbol{r})\boldsymbol{\beta} \qquad \Longrightarrow \qquad \widehat{u}^*(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1} < \boldsymbol{p}, \boldsymbol{p}^t > \boldsymbol{\beta} = u(\boldsymbol{x}). \tag{6}$$

Therefore, the approximation reproduces exactly any polynomial function up to the order considered in (1).

Usual choices for the scalar product are

$$< f, g > = \int_{\boldsymbol{r} \in \Omega} w(\boldsymbol{r}) f(\boldsymbol{r}) g(\boldsymbol{r}) d\Omega \qquad \text{and} \qquad < f, g > = \sum_{i_p=1}^{n_p} w(\boldsymbol{r}_{i_p}) f(\boldsymbol{r}_{i_p}) g(\boldsymbol{r}_{i_p}), \tag{7}$$

where $w(\boldsymbol{r})$ and $w(\boldsymbol{r}_{i_p})$ allow to assign suitable weightings to the different areas of the domain Ω. It is important to notice that the positive definiteness of the moment matrix is not automatically guaranteed in the discrete case, unless the points distribution fulfills certain prerequisites[16].

We wish to construct a FEM type discrete approximation $u^h(\boldsymbol{x})$ to the function $u(\boldsymbol{r})$ at every point $\boldsymbol{x}$ in Ω, in terms of the values $\{u_{i_p}\}$, where $u_{i_p} = u(\boldsymbol{r}_{i_p})$, being $\{\boldsymbol{r}_{i_p}\}$ for $i_p = 1, \ldots, n_p$ the nodal points selected within the domain Ω. The answer is indeed in expression (5). Thus, if we adopt

$$u^h(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1} < \boldsymbol{p}, u >, \tag{8}$$

with the discrete scalar product given in (7) we get

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = \sum_{i_p=1}^{n_p} N_{i_p}(\boldsymbol{x}) u_{i_p}, \tag{9}$$

where

$$N_{i_p}(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1} w(\boldsymbol{r}_{i_p}) \boldsymbol{p}(\boldsymbol{r}_{i_p}). \tag{10}$$

At this point we must remark that the local values of the approximating function do not necessarily fit the nodal unknown values (that is $u^h(\boldsymbol{r}_{j_p}) \neq u_{j_p}$ for $j_p = 1, \ldots, n_p$), due to the least square character of the approximation. However, the assertion (6) guarantees that the partition of unity is fulfilled by the trial functions defined by (10).

In general we seek for a local rather than a global approximation, since the latter should destroy the sparseness of the matrices involved in the numerical formulation. Since we wish to enforce the local character of the approximation, we expect that most of the trial functions $N_{i_p}(\boldsymbol{x})$ vanish at any given point, namely

$$N_{i_p}(\boldsymbol{x}) = 0 \; \forall i_p \;/\; \boldsymbol{x}_{i_p} \notin B(\boldsymbol{x}),$$

being $B(\boldsymbol{x})$ a selected suitable subdomain in the neighbourhood of the given point $\boldsymbol{x}$, for instance

$$B(\boldsymbol{x}) = \{\boldsymbol{r} \in \Omega \;/\; |\boldsymbol{r} - \boldsymbol{x}| \leq R\}\,.$$

The local character of the approximation can be induced by the choice of a truncated weighting function, that must take its maximum value at the given point and must vanish outside a selected surrounding region, for example

$$w(\boldsymbol{r}) = \begin{cases} W(z)\Big|_{z=|(\boldsymbol{r}-\boldsymbol{x})/\rho|} > 0, & \text{if } \boldsymbol{r} \in B(\boldsymbol{x}); \\ = 0, & \text{otherwise}, \end{cases} \tag{11}$$

being $W(z)$ an adequate function, such as a gaussian function or a conoidal. Hence, the dilation parameter ρ plays its more important role in (11), since it contributes to characterize the support of the weighting function[7]. In practice, the parameter R is adjusted for each point in order to ensure that no less than a minimum number n_s of nodes are taken into account in the interpolation. Therefore, the approximated value of the function at any point $\boldsymbol{x}$ is constructed in terms of the information provided by a certain number of its closest (at least n_s) nodal points. On the other hand, for each node $\boldsymbol{r}_{i_p}$ we can consider the set Ω_{i_p} of all the points $\boldsymbol{x} \in \Omega$ which approximated value $u^h(\boldsymbol{x})$ is affected by the nodal term u_{i_p}. The proper definition of the approximation at every point requires that all these subdomains cover the whole interpolation domain Ω. Moreover, these subdomains must overlap, and every point $\boldsymbol{x} \in \Omega$ must belong to the subdomains of as many nodal points (at least n_s) as to ensure the uniqueness of the interpolation ($n_s \geq n_\alpha$) and the convergence of the method. We can also point out that the approximation (9) reduces to the usual type of FEM interpolation (that is, it fits the nodal values) when $n_s = n_\alpha$, since no effect of least squares is presented. On the other hand, the standard least square aproximation is reproduced when the weighting function is constant and equals the unity[19].

3 ENRICHMENT OF WLS APPROXIMATIONS

The enrichment of a certain numerical approach is intended as a better alternative to perform extremely expensive mesh refinements in several problems of computational mechanics. In general, the enrichment process consists of introducing some information about the solution of the problem into the trial functions (e.g. its behaviour near singularities or discontinuities in the domain). The enrichment techniques were developed in the mid-seventies for finite element methods, and have been succesfully applied to different problems since then[10].

This technique has been recently applied to meshless methods. The experience shows that it is simpler and easier to enrich this kind of approaches than the regular finite element formulations. In the early research, the enrichment of meshless methods was carried out both intrinsically and extrinsically[18]. The intrinsic enrichment consists of the inclusion of special functions in the complete polynomial basis of the weighted least squares interpolation. In this case, it is not necessary to increase the number of unknowns, although additional computational effort is required to obtain the trial functions. In contrast to this approach, the extrinsic enrichment is based on the addition of enrichment functions to the set of trial functions. This requires the introduction of new unknowns in the numerical scheme, but the polynomial basis of the interpolation is not modified[10].

On the other hand, its local quality has turned out to be the main advantage of extrinsic enrichment. In fact, special techniques are required to mix nodal points with different basis functions, if intrinsic approaches are used to achieve partial enrichment. The reason is that no functions can be deleted from the basis without introducing discontinuities in the approximation. On the contrary, an extrinsic enrichment can be constructed by means of the partition of unity concept. In this case, since the consistency is assured by the partition of unity (given by the trial functions formed with the basis of WLS interpolants) the enrichment of the approximation may be performed locally, by extrinsically adding functions of a new basis[18]. For these reasons, we focus our attention in the extrinsic enrichment of meshless methods on the basis of a weighted least squares approach.

The basic ideas are quite simple. Let $u(\boldsymbol{r})$ be a function defined for all $\boldsymbol{r}$ in Ω. As we did before, we wish to construct a FEM type discrete approximation $u^h(\boldsymbol{x})$ to the function $u(\boldsymbol{r})$ at every point $\boldsymbol{x}$ in Ω, in terms of the nodal values $\{u_{i_p}\}$, for $i_p = 1, \ldots, n_p$. Let's suppose that we have additional information, such that for a certain kind of functions $u^e(\boldsymbol{r})$, the difference $[u(\boldsymbol{r}) - u^e(\boldsymbol{r})]$ is easier to approximate than the original function $u(\boldsymbol{r})$. Thus, for approximation purposes it seems preferable to split $u(\boldsymbol{r}) = u^e(\boldsymbol{r}) + [u(\boldsymbol{r}) - u^e(\boldsymbol{r})]$, in order to approximate the latter term by means of (5), instead of approximating the original function itself. Thus, if we adopt

$$u^h(\boldsymbol{x}) = u^e(\boldsymbol{x}) + \boldsymbol{p}^t(\boldsymbol{x}) < \boldsymbol{p}, \boldsymbol{p}^t >^{-1} < \boldsymbol{p}, u - u^e > . \tag{12}$$

with the discrete scalar product given in (7) we get

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = u^e(\boldsymbol{x}) + \sum_{i_p=1}^{n_p} N_{i_p}(\boldsymbol{x})\Big(u_{i_p} - u^e(\boldsymbol{r}_{i_p})\Big). \tag{13}$$

where the trial functions $N_{i_p}(\boldsymbol{x})$ are identical to the ones that were defined by (10) for the non-enriched case. The enrichment term $u^e(\boldsymbol{r})$ is normally expressed as

$$u^e(\boldsymbol{r}) = \sum_{j=1}^{n_f} k_j F_j(\boldsymbol{r}) = \boldsymbol{k}^t \boldsymbol{F}(\boldsymbol{r}), \tag{14}$$

where $\boldsymbol{k}$ is the vector of unknowns associated to the basis of enrichment functions $\boldsymbol{F}(\boldsymbol{r})$.

Another type of extrinsic enrichment in meshless methods, that is simpler and computationally faster than (14), can be obtained by using the partition of unity concept[10]. In this case, the approximation is modified by adding extrinsically a basis of enrichment functions to the existing WLS approximation. These new functions can be polynomials of higher order than the WLS interpolants basis, or functions contained in the exact solution of the problem, which are smoothly added to the WLS approximation by multiplying it by a partition of unity. Since trial functions in WLS approximations are partitions of unity, this extrinsic enrichment procedure frequently takes the form

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = \sum_{i_p=1}^{n_p} N_{i_p}(\boldsymbol{x})\Big(u_{i_p} + \sum_{j=1}^{n_f(i_p)} k_{i_p j} F_j(\boldsymbol{r}_{i_p})\Big), \tag{15}$$

where $n_f(i_p)$ is the number of enrichment functions corresponding to the nodal point i_p (n_f may be different for each nodal point), and $k_{i_p j}$ are the unknowns associated to the basis of enrichment functions.

In this paper, we will consider the extrinsic enrichment technique based on the partition of unity approach for WLS meshless methods. Although the number of degrees of freedom increases due to the enrichment (if a nodal point i_p is enriched, then the total number of unknowns to obtain for the node becomes $n_f(i_p) + 1$, instead of one), this enhancement procedure can be applied locally in different parts of the approximation, and it is also quite straightforward to implement in a meshless code.

4 BOUNDARY VALUE PROBLEM APPROXIMATION BASED ON A POINT COLLOCATION SCHEME

In previous sections we have presented the WLS interpolation and the different kind of enrichments that can be performed. In this section we focus on how to obtain the discretized equations of a boundary value problem. Thus, if $\mathcal{A}$ and $\mathcal{B}$ are two differential operators, Ω the domain of our problem and Γ its boundary ($\Gamma = \Gamma_t \cup \Gamma_u$), a scalar BVP can be written as,

$$\mathcal{A}(u) = b \quad \text{in} \quad \Omega, \tag{16}$$

with the boundary conditions,

$$\mathcal{B}(u) = t \quad \text{in} \quad \Gamma_{\rm t}, \qquad {\rm u} - {\rm u_p} = 0 \quad \text{in} \quad \Gamma_{\rm u}, \tag{17}$$

where u is the solution, b and t represent the actions in Ω and Γ_t, and u_p is the prescribed value of u in Γ_u.

The application of the weighted-residuals method leads to the variational form of the above stated problem in terms of the trial approximation u^h to the solution u

$$\begin{aligned} &\int_{\boldsymbol{r}\in\Omega} \omega_{j_p} \, [\mathcal{A}(u^h) - b] \; d\Omega \; + \\ &\int_{\boldsymbol{r}\in\Gamma_t} \omega^t_{j_p} \, [\mathcal{B}(u^h) - t] \; d\Gamma \; + \\ &\quad \int_{\boldsymbol{r}\in\Gamma_u} \omega^u_{j_p} \, [u^h - u_p] \; d\Gamma \; = \; 0, \qquad j_p = 1, \ldots, n_p \end{aligned} \tag{18}$$

which must hold for the test functions $\{\omega_{j_p}\}$, $\{\omega^t_{j_p}\}$ and $\{\omega^u_{j_p}\}$ defined on Ω, Γ_t and Γ_u respectively.

If we do not perform any enrichment, for a given set of n_p trial functions defined on Ω the approximation u^h to the solution u can be discretized as,

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = \sum_{i_p=1}^{n_p} N_{i_p}(\boldsymbol{x}) u_{i_p} \tag{19}$$

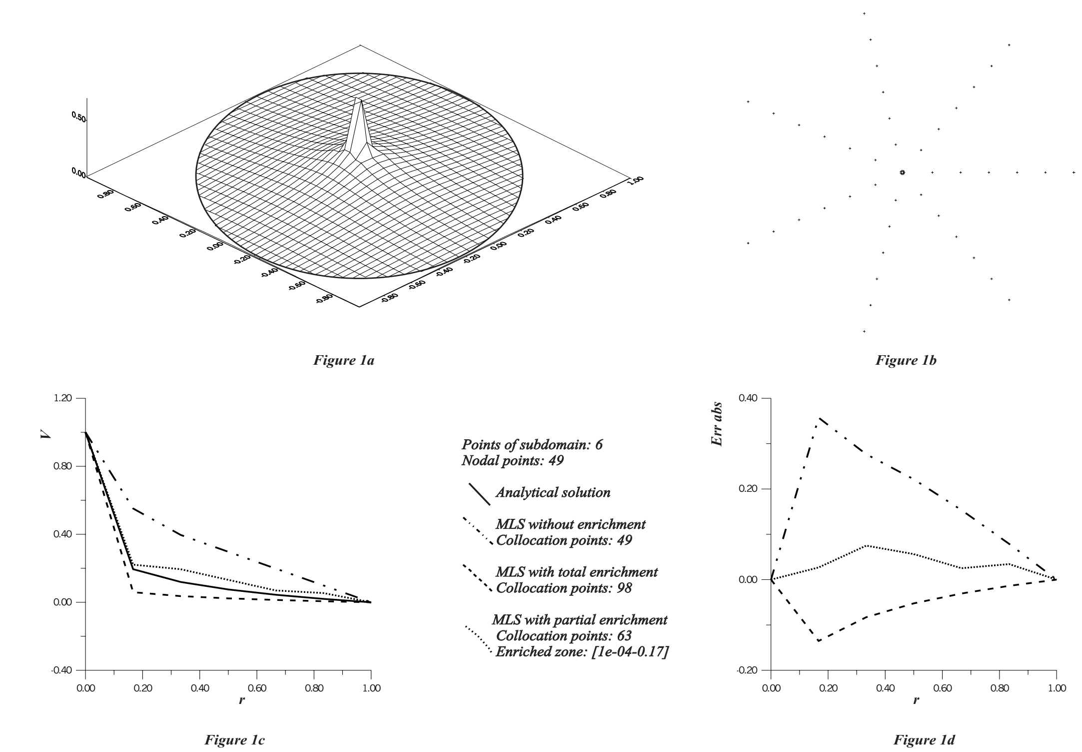
where n_p are the total scattered points of the solution domain, and the trial functions $N_{i_p}(\boldsymbol{x})$ can be constructed by using the previous WLS methodology.

Now, we can derive different numerical formulations from the variational form (18). In order to take advantage of the meshless character of the approximation we can use a point-collocation approach ($\omega_j = \omega^t_j = \omega^u_j = \delta(\boldsymbol{r} - \boldsymbol{r}_{j_p})$, where $\delta(\cdot)$ is the Dirac delta)[19]. Several different approaches have been proposed by other authors[1], but some kind of auxiliar grid is then inexorably required to evaluate the resulting integrals. If the differential operators $\mathcal{A}$ and $\mathcal{B}$ are linear, and we use a point-collocation scheme, the following set of equations is obtained:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{20}$$

where the coefficient matrix $\boldsymbol{K}$ is sparse but not necessarily symmetric, $\boldsymbol{f}$ contains the contribution of b, t and u_p, and $\boldsymbol{u}$ contains the unknown values of the function at the nodal points.

Obviously, if enrichment functions are used to define the WLS interpolation it will be necessary to use more collocation-points in (18) in order to obtain all the unknowns, as we have explained in the previous section.



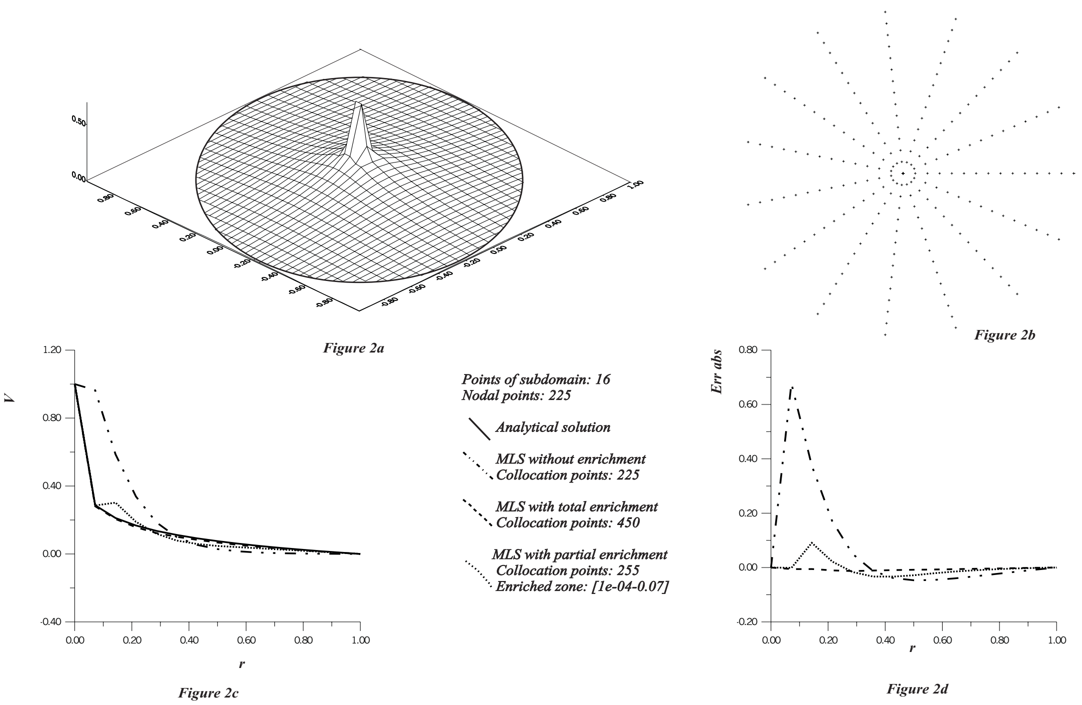


Fig. 1 & 2. 2D numerical example: Comparison of results obtained by using different WLS formulations.

5 NUMERICAL EXAMPLES

5.1 Potencial problems

In previous works[20] we have studied the applicability of the weighted least squares meshless methods to the analysis of potential problems in the electrical engineering field. In this section, we compare the results that are obtained when a standard weighted least-squares approach and an enriched one are applied for the numerical solution of two 2D potential problem tests.

As a first example, we have selected the following 2D axisymmetrical test problem:

$$\Delta V = 0, \qquad L_0 \leq \sqrt{x^2+y^2} \leq L_1 \ ,$$
$$V(x,y)\Big|_{\sqrt{x^2+y^2}=L_0} = 1, \quad V(x,y)\Big|_{\sqrt{x^2+y^2}=L_1} = 0, \tag{21}$$

that has been solved by using 2D trial functions. The analytical solution to the above stated problem is given by $V(x,y) = \left(\ln L_1 - \ln\sqrt{x^2+y^2}\right)/(\ln L_1 - \ln L_0)$.

Thus, for the enrichment function $\ln\sqrt{x^2+y^2}$ and for a given set of n_p trial functions N_i defined on the domain, the approximations $\widehat{V}$ to the solution V can be written as:

$$\begin{aligned}
&\text{Without enrichment functions:} && V^h = \sum_{i_p=1}^{n_p} N_{i_p} u_{i_p}. \\
&\text{With enrichment functions:} && V^h = \sum_{i_p=1}^{n_p} N_{i_p}\Big(u_{i_p} + k_{i_p 1} F_1(r)\Big). \\
&\text{Total enrichment:} && F_1(r) = \ln\Big(\sqrt{x^2+y^2}\Big), \quad L_0 \leq \sqrt{x^2+y^2} \leq L_1. \\
&\text{Local enrichment:} && \begin{cases} F_1(r) = 0, & L_0 \leq \sqrt{x^2+y^2} < r_0; \\ F_1(r) = \ln\Big(\sqrt{x^2+y^2}\Big), & r_0 \leq \sqrt{x^2+y^2} \leq r_1; \\ F_1(r) = 0, & r_1 < \sqrt{x^2+y^2} \leq L_1. \end{cases}
\end{aligned} \tag{22}$$

In the above expression, n_p is the total number of nodal points of the solution domain. The weighting function that has been used is the truncated gaussian with $\alpha = 0.25$ and $k = 1.1$.

Figures 1*a* and 2*a* show the numerical approximation of the surface that has been obtained when $L_0 = 10^{-4}$ and $L_1 = 1$. In figures 1*b* and 2*b* we present the nodal points distribution. In figures 1*c*, 1*d*, 2*c* and 2*d* we present a comparison between the analytical solution and the approximations obtained by using a weighted least squares approach with or without enrichment funtions along a radial line. As we show in this example, the use of enhanced approaches allows to obtain very good approximations. In particular, the use of partial enrichment in selected parts of the domain may be specially worth while, since it opens the possibility to improve the numerical approximation with a low computing effort.

As a second example, we solve the above problem with non symmetric boundary conditions:

$$\Delta V = 0, \quad L_0 \le \sqrt{x^2+y^2} \le L_1 \ ,$$

$$V(x,y)\Big|_{\sqrt{x^2+y^2}=L_0} = 1, \quad V(x,y)\Big|_{\sqrt{x^2+y^2}=L_1} = \sin(\theta), \tag{23}$$

which analytical solution is given by

$$V(x,y) = \frac{\ln L_1 - \ln\left(\sqrt{x^2+y^2}\right)}{\ln L_1 - \ln L_0} + \frac{\dfrac{\sqrt{x^2+y^2}}{L_0} - \dfrac{L_0}{\sqrt{x^2+y^2}}}{\dfrac{L_1}{L_0} - \dfrac{L_0}{L_1}} \sin(\theta) \tag{24}$$

The approximation V^h to the solution V can be written in the same form as in the previous example, using the same enrichment function.

In figure 3 we present the analytical solution ($3a$) and the results obtained when 225 nodal points (see $3b$) are used. In this case, we compare the analytical solution and the approximations obtained by using the weighted least squares approach with or without enrichment functions along five different radial lines ($3c$, $3e$, $3g$, $3i$, $3k$). The relative errors are compared in figures $3d$, $3f$, $3h$, $3j$, $3l$. In figure 4 we present the results obtained for a distribution of 900 nodal points (see $4b$).

In this example we confirm that enhanced approaches allow to obtain very good results. As it can be observed, a much higher number of scattered nodal points would be necessary with a standard formulation to obtain results of the same quality as those obtained with enrichment functions.

5.2 Transport problems

In these examples we show the results obtained when an enriched weighted least squares approach, combined with a point collocation scheme, is applied to a 1D high advective transport problem.

As first example we consider the next 1D numerical test defined by:

$$u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2} \ , \quad 0 \ < \ x \ < \ 1; \quad \phi(0) = 1, \quad \phi(1) = 0. \tag{25}$$

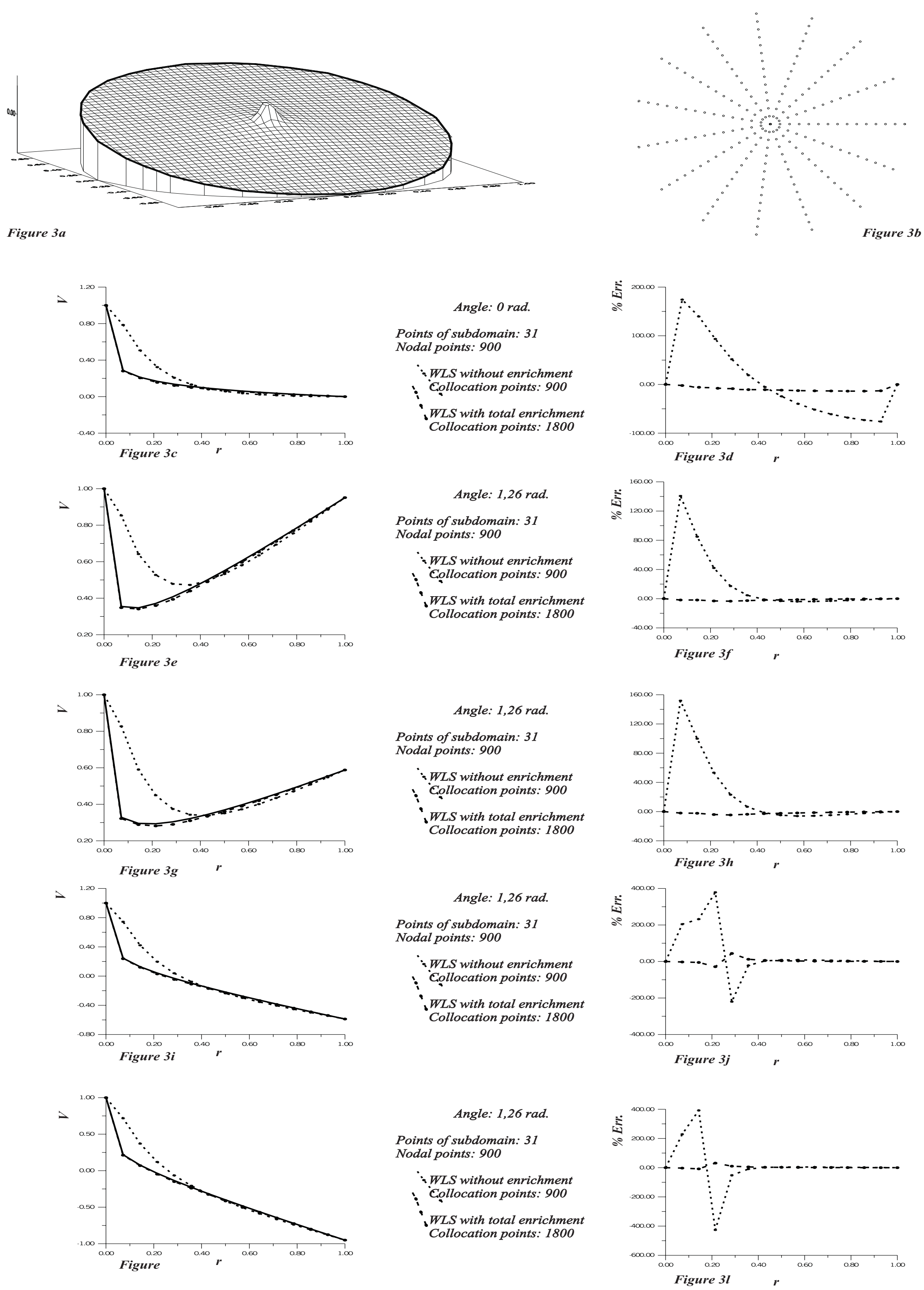


Fig. 3. 2D numerical example: Comparison of results obtained by using different WLS formulations.

In this case, the analytical solution is given by $\phi(x) = \left(e^{u/k} - e^{xu/k}\right)/\left(e^{u/k} - 1\right)$. We use as enrichment functions the normalized eigenfunctions of the boundary problem $\left(e^{xu/k} - 1\right)/\left(e^{u/k} - 1\right)$.

The approximations ϕ^h to the solution ϕ can be written in the form:

$$
\begin{aligned}
&\text{Without enrichment functions:} \qquad \phi^h = \sum_{i=1_p}^{n_p} N_{i_p} u_{i_p}. \\
&\text{With enrichment functions:} \qquad \phi^h = \sum_{i=1_p}^{n_p} N_{i_p}\left(u^h_{i_p} + k_{i_p1} F_1(x)\right). \\
&\text{Total enrichment:} \qquad F_1(x) = \frac{e^{xu/k} - 1}{e^{u/k} - 1}, \quad 0 \le x \le 1. \\
&\text{Local enrichment:} \qquad \begin{cases} F_1(x) = 0, & 0 \le x < x_0; \\ F_1(x) = \dfrac{e^{xu/k} - 1}{e^{u/k} - 1}, & x_0 \le x \le x_1; \\ F_1(x) = 0, & x_1 < x \le 1. \end{cases}
\end{aligned}
\tag{26}
$$

In figure 5 we present the results obtained when the total number of nodal points of the solution domain is $n_p = 10$ and the weighting function is the truncated gaussian with $\alpha = 0.25$ and $k = 1.1$. We also compare the results obtained with $k = 1,\ u = 25$ and $k = 1,\ u = 50$.

We remark the notable stabilizing effect of the enrichment functions. Furthermore, as the enriched zone is increased, the solution obtained is more stable (the relation between the imaginary part and the real part of the eigenvalues decrease[21]). This fact is more evident as the relation between u and k becomes higher.

The second example related to the transport equation consists of the resolution of the following boundary value problem:

$$
k\frac{\partial^2\phi}{\partial x^2} - u\frac{\partial\phi}{\partial x} = f(x)\ , \qquad 0 \ < \ x \ < \ 1; \qquad \phi(0) = 1, \qquad \phi(1) = 0,
$$

$$
f(x) = 2(u/k)\Big[2(u/k)x^2 - (u/k)x - 1\Big]e^{-(1/2-x)^2(u/k)}. \tag{27}
$$

The analytical solution to this problem is given by:

$$
\phi(x) = \frac{2}{1 - e^{u/k}}\left(e^{xu/k} - 1\right)\left(1 - e^{(1/4)u/k}\right) + e^{-(1/2-x)^2(u/k)}
$$

In the following we use the same enhancement functions of the preceding example, that is $\left(e^{xu/k} - 1\right)/\left(e^{u/k} - 1\right)$, and we only enrich the zone $0,9583 \le x \le 1,0000$, where

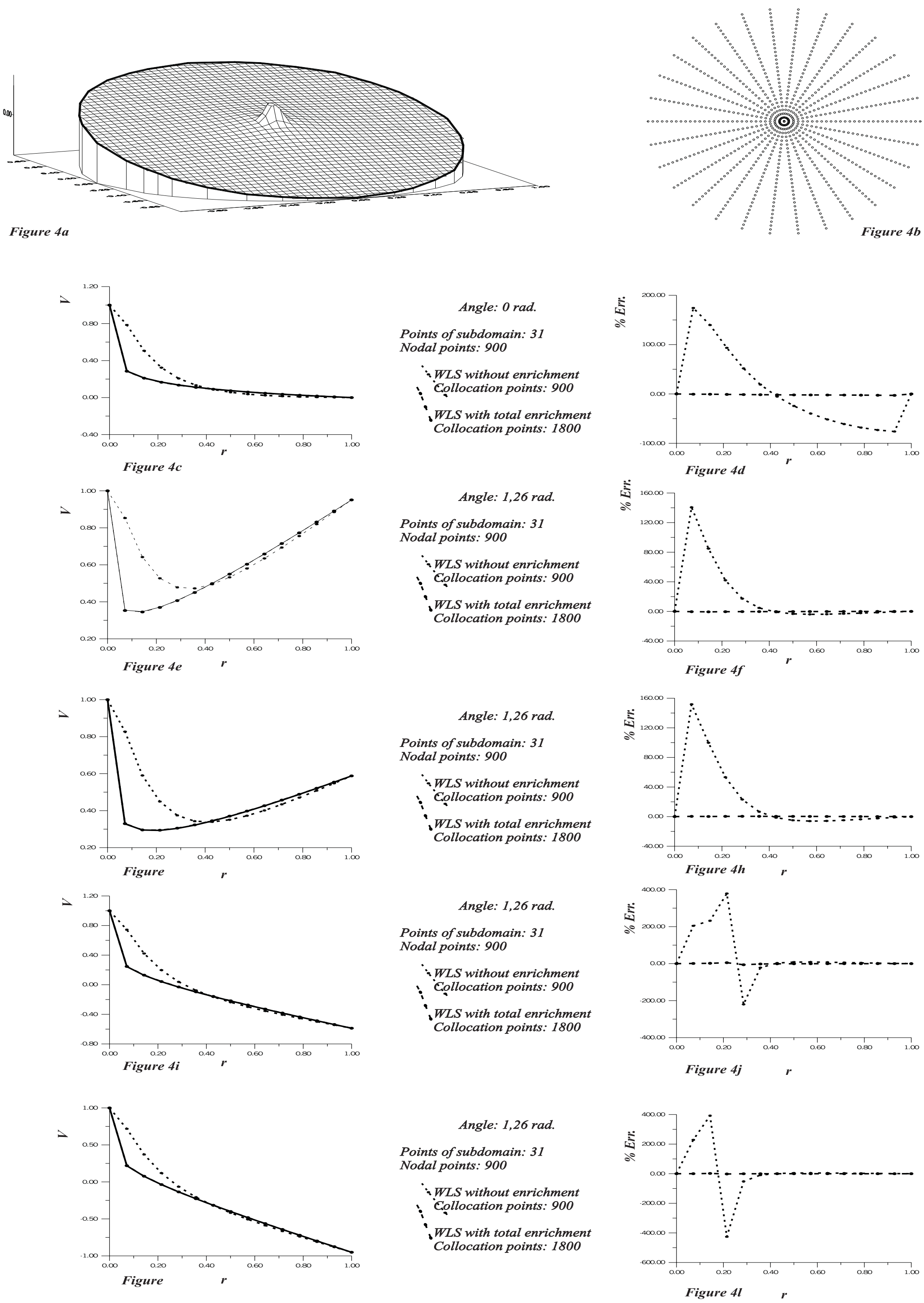


Fig. 4. 2D numerical example: Comparison of results obtained by using different WLS formulations.

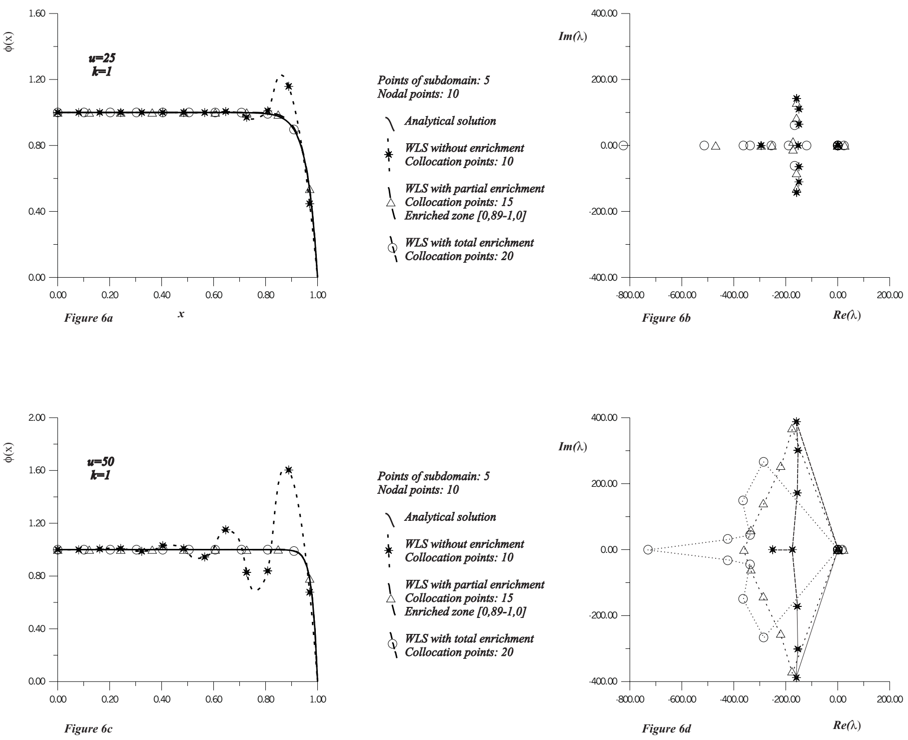


Fig. 5. Transport problem: Comparison of results obtained by using different WLS formulations and eigenvalues representation.

the highest gradients are recorded within the whole domain. Results can be observed in figure 6 for diferent cases ($k = 1,\ u = 50$ and $k = 1,\ u = 500$), when the total number of nodal points in the solution domain is $n_p = 25$ and the weighting function is the truncated gaussian with $\alpha = 0.25$ and $k = 1.1$.

As it happened in the previous example, the enriched formulation is much more stable than the non-enriched one. Furthermore, as more the advetion increases, as the stabilization effect is more clearly appreciated.

6 CONCLUSIONS

In this paper, we have studied the enrichment of Weighted Least Squares interpolations with a point collocation approach. The performance of this kind of enriched formulations has been tested in problems related to the Potential Theory. Furthermore, we have shown the stabilizing features of this technique when it is applied to the transport equation.

The proposed meshless method incorporates enrichment techniques. The meshless character may represent an important improvement. The improvement can be dramatic in cases where the use of standard numerical techniques is precluded, due to the large computational effort required by the discretization process. On the other hand, the enrichment techniques allow to obtain highly accurate results with relatively coarse

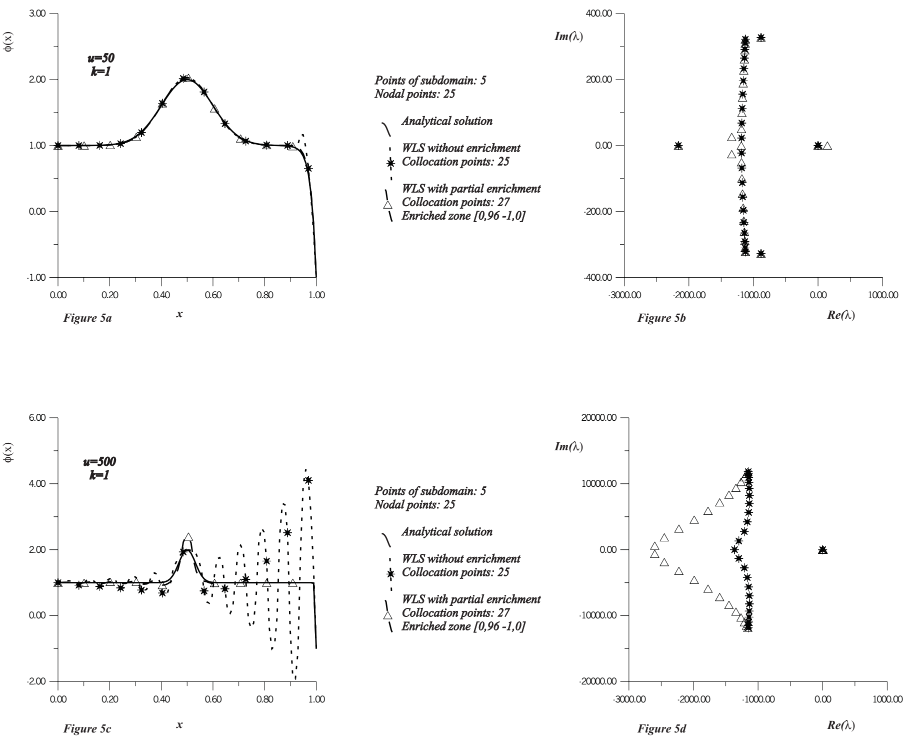


Fig. 6. Transport problem: Comparison of results obtained by using different WLS formulations and eigenvalues representation.

meshes. This feature is particularly valued in some specially difficult problems, since no expensive mesh refinement needs to be performed, even in the presence of high gradients.

As we can observe in the presented examples, the results are very promising, while the required computational cost does not become unreasonable whatsoever. Further analysis must be done in both mathematical and numerical aspects, in order to introduce the enrichment procedure into more practical problems and another meshless approaches. Finally, the stabilization properties of this formulation seem to be really outstanding. Therefore, further research should be performed in this direction too.

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