On g-barrelled groups and their permanence properties

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Abstract

The g-barrelled groups constitute a vast class of abelian topological groups. It might be considered as a natural extension of the class of barrelled topological vector spaces.

In this paper we prove that g-barrelledness is a multiplicative property, thus we obtain new examples of g-barrelled groups. We also prove that direct sums and inductive limits of g-barrelled locally quasi-convex groups are g-barrelled, too. Other permanence properties are considered as well.

Keywords: g-barrelled group; barrelled space; reflexive group; Mackey topology; pointwise convergence topology; equicontinuous set

1 Introduction

Barrelled spaces constitute a well behaved class of locally convex spaces. They were introduced by Bourbaki in 1950 and one of its main features is that they are a class of spaces which satisfy the Closed Graph Theorem in an optimal way. In fact, the following assertion holds: if a locally convex space E has the property that any linear mapping from E into any Banach space is continuous provided it has a closed graph, then E is a barrelled space [24, 4.1.10]. Although one can define barrelled spaces in the broader class of topological vector spaces, the context of locally convex spaces is a more natural ground, in which they have richer properties.

The following characterization of barrelledness paves the way to introduce a similar notion for topological abelian groups: "A locally convex space

X is barrelled if and only if every pointwise bounded subset M of its dual space X^* is equicontinuous" [25, IV, 1.6].

The g-barrelled groups constitute the counterpart of barrelled spaces for the class of abelian topological groups. We need some notation before giving their formal definition. The unit complex circle is denoted by $\mathbb{T}$. For an abelian group G, $\mathrm{Hom}(G, \mathbb{T})$ denotes the set of all homomorphisms from G to $\mathbb{T}$ (also called characters), which endowed with the pointwise operation becomes a topological group. If G is further a topological group, the set of all **continuous** characters of G constitutes a subgroup $\mathrm{CHom}(G, \mathbb{T})$ of $\mathrm{Hom}(G, \mathbb{T})$, denoted by $G^\wedge$. It is called the dual group of G. Let $\sigma(G^\wedge, G)$ denote the weak topology on $G^\wedge$ with respect to the family of all evaluation mappings $\{\tilde{x}, x \in G\}$ where $\tilde{x} : G^\wedge \to \mathbb{T}$ is defined by $\tilde{x}(\chi) := \chi(x)$ for all $\chi \in G^\wedge$. The topology $\sigma(G^\wedge, G)$ presents some similarity with the so called weak* topology, a standard topology for the dual of a topological vector space.

An abelian topological group (G, τ) is *g-barrelled* if any $\sigma(G^\wedge, G)$-compact subset $M \subseteq G^\wedge$ is equicontinuous with respect to τ. This definition was first given in [14], where it is also proved that, among others, locally compact, completely metrizable and Baire separable abelian groups are g-barrelled. Thus, we can say that g-barrelled groups form a big class of abelian topological groups. On the other hand, countable abelian nondiscrete groups with sufficiently many continuous characters are not g-barrelled (see Proposition 2.9 below).

Later on we will recall the notion of locally quasi-convex topology for an abelian group. By the time being, let us loosely say that the locally quasi-convex groups constitute a class of abelian topological groups that contains the locally convex spaces considered just as groups, forgetting the linear operation.

The subclass $\mathcal{G}$ formed by those g-barrelled groups that are locally quasi-convex is best fitted inside the abelian topological groups to extend the Mackey-Arens Theorem. This well known theorem asserts that, for a given topological vector space (X, τ), the family $\mathcal{LC}(X_\tau)$ of all locally convex topologies on X giving rise to the same dual space as τ, has always a maximum μ with respect to inclusion. It is called the Mackey topology for (X, τ). Further, μ can be described as the topology of uniform convergence on the $\sigma(X^*, X)$-compact and absolutely convex subsets of X^*.

Let us now consider for a fixed topological group (G, τ) the family $\mathcal{LQC}(G_\tau)$ of all locally quasi-convex topologies on G giving rise to the same dual group as τ. Only recently (in [4] and [16] independently), it has been proved that the family $\mathcal{LQC}(G_\tau)$ may not have a maximum with respect to the order relation $\subseteq$. When such a maximum exists, we call it the Mackey topology for (G, τ).

The class $\mathcal{G}$ is very satisfactory in the following sense: if $(G, \tau) \in \mathcal{G}$, then (G, τ) is a *Mackey group*, i. e. τ is precisely the Mackey topology for

(G, τ). In other words, the original topology τ is the maximum in $\mathcal{LQC}(G_\tau)$. Further, τ can be described as the topology of uniform convergence on the $\sigma(G^\wedge, G)-$compact quasi-convex subsets of the dual group of G. Thus, τ is the Mackey topology for (G, τ) in the strongest possible sense.

Since topological vector spaces are in particular abelian topological groups with respect to their addition operation, it makes sense to study conditions under which they are g-barrelled groups. It was proved in [14, 5.1] that a tvs X is a g-barrelled group if and only if every $\sigma(X^*, X)$-compact subset $M \subseteq X^*$ is equicontinuous. In other words, the vector space X^* of continuous linear forms on X and the group $X^\wedge$ of continuous characters on X (each endowed with the corresponding weak-star topology) play the same role in proving whether a topological vector space X is a g-barrelled group. Thus, any barrelled topological vector space is a g-barrelled topological group; the converse is not true [14, Remark 16].

In this paper we first collect some properties of g-barrelled groups, either known or easy to obtain. We enumerate the classes of groups which so far are known to be g-barrelled. Our main contribution is to prove that arbitrary products of g-barrelled groups are g-barrelled (Theorem 3.4). We also prove that direct sums and inductive limits of g-barrelled groups are g-barrelled. We show that the property of being g-barrelled is not inherited by closed subgroups. Nevertheless, we prove that if a topological group G contains an open subgroup A, then A is g-barrelled if and only if G is g-barrelled. We also show that completions of g-barrelled groups are g-barrelled.

Notation and terminology

If G is an abelian group, the set of all homomorphisms from G to $\mathbb{T}$ will be denoted by $\mathrm{Hom}(G, \mathbb{T})$, where $\mathbb{T}$ is the unit complex circle. The elements of $\mathrm{Hom}(G, \mathbb{T})$ are called *characters* and $\mathrm{Hom}(G, \mathbb{T})$ has a group structure with respect to the pointwise operation.

A *group duality* is a pair $\langle G, H \rangle$ where G is an abelian group and H is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$. Given a group duality $\langle G, H \rangle$, the weak topology $\sigma(G, H)$ is the initial topology on G with respect to the elements of H.

If (G, τ) is a topological abelian group, its *dual group* $G^\wedge := \mathrm{CHom}(G, \mathbb{T})$ is the set of all continuous characters of G. It is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ and in particular gives rise to the natural duality $\langle G, G^\wedge \rangle$. When $G^\wedge$ separates points of G, we say that G is MAP (a shorthand for "maximally almost periodic"). Let $A \subseteq G$ and $B \subseteq G^\wedge$. The polar set of A is defined by

$$A^\triangleright = \{\chi \in G^\wedge \ : \ \chi(x) \in \mathbb{T}_+ \ \ \forall x \in A\}$$

and the inverse polar of B is defined by

$$B^\triangleleft = \{x \in G \ : \ \chi(x) \in \mathbb{T}_+ \ \ \forall \chi \in B\}$$

where $\mathbb{T}_+ := \{e^{2\pi i t} : t \in [-\frac{1}{4}, \frac{1}{4}]\}$.

A subset $A \subseteq G$ is *quasi-convex* if for every $x \in G \setminus A$ there exists an element $\phi \in A^{\triangleright}$ such that $\phi(x) \notin \mathbb{T}_+$. The *quasi-convex hull* of a subset $M \subseteq G$ is the smallest quasi-convex subset of G that contains M. It is straightforward to prove that it coincides with $M^{\triangleright\triangleleft}$; in particular M is quasi-convex if and only if $M = M^{\triangleright\triangleleft}$. The topological group (G,τ) is said to be *locally quasi-convex* if it admits a basis of neighborhoods of zero formed by quasi-convex subsets.

If H is a subgroup of G, then $H^{\triangleright} = H^{\perp} := \{\chi \in G^{\wedge} : \chi(H) = \{1\}\}$. A subset $S \subseteq G^{\wedge}$ is equicontinuous with respect to τ if and only if $S \subseteq U^{\triangleright}$ for some τ-neighborhood of zero U in G.

Let (G,τ) be a topological abelian group. The quasi-convex neighborhoods of zero in (G,τ) form a basis of neighborhoods of zero for a group topology $\mathcal{Q}\tau$ on G. Explicitly, such a basis can be described as the family $\mathcal{B} = \{\mathcal{U}^{\triangleright\triangleleft},\ \mathcal{U} \in \mathcal{N}\}$, where $\mathcal{N}$ stands for the τ - neighborhood system of the neutral element. The topology $\mathcal{Q}\tau$ is the finest among the locally quasi-convex group topologies weaker than τ (see [9, Lemma 7]). It will be called the *locally quasi-convex modification* of τ. It is easy to prove that $(G,\mathcal{Q}\tau)^{\wedge} = G^{\wedge}$, and that $(G,\mathcal{Q}\tau)$ is Hausdorff if and only if (G,τ) is a MAP group.

All precompact Hausdorff topologies on an abelian group G have the form $\sigma(G,L)$ where L is a subgroup of $\mathrm{Hom}(G,\mathbb{T})$ which separates the points of G. Moreover $L = (G,\sigma(G,L))^{\wedge}$ [15]. This result will be quoted in what follows as "Comfort-Ross Theorem".

Let (G,τ) be a topological abelian group. The dual group of G endowed with the pointwise convergence topology $\sigma(G^{\wedge},G)$ will be abbreviated to $G_p^{\wedge}$ whilst $G_{\mathrm{co}}^{\wedge}$ will stand for $G^{\wedge}$ endowed with the compact-open topology τ_{co}. We call the latter the Pontryagin dual group of G.

A group topology ν on an abstract abelian group G is said to be compatible with the duality $\langle G,L\rangle$ if $(G,\nu)^{\wedge} = L$. By Comfort-Ross Theorem the weak topology $\sigma(G,L)$ is compatible with the duality $\langle G,L\rangle$ and it is in fact the bottom element of the family of all compatible topologies partially ordered by the relation $\subseteq$.

If (G,τ) is an abelian topological group, a topology compatible with the duality $\langle G,G^{\wedge}\rangle$ will be also called compatible with τ.

If G and H are topological abelian groups and $\varphi : G \to H$ is a continuous homomorphism, the adjoint homomorphism $\varphi^{\wedge} : H^{\wedge} \to G^{\wedge}$ is defined by $\varphi(\chi) = \chi \circ \varphi$. Clearly $\varphi^{\wedge}$ is continuous if both dual groups $H^{\wedge}$ and $G^{\wedge}$ are endowed with their respective pointwise convergence topologies, or with their respective compact-open topologies.

For a topological abelian group G, we denote by $\alpha_G : G \to (G_{\mathrm{co}}^{\wedge})^{\wedge}$ and $\beta_G : G \to (G_p^{\wedge})^{\wedge}$ the corresponding evaluation maps. We will use the symbols α_G and β_G regardless of the topologies considered on their ranges. Observe that for every $M \subseteq G$ one has $\alpha_G^{-1}(M^{\triangleright\triangleright}) = \beta_G^{-1}(M^{\triangleright\triangleright}) = M^{\triangleright\triangleleft}$.

A topological abelian group G is said to be *reflexive* if α_G is a topological isomorphism from G to $(G^\wedge_{\text{co}})^\wedge_{\text{co}}$. The celebrated Pontryagin-van Kampen Theorem asserts that every locally compact abelian group is reflexive. After this result appeared, many other classes of groups have been shown to satisfy this important property. (A survey of results in this direction can be found in [11].)

On the other hand, by Comfort-Ross Theorem, $(G^\wedge_p)^\wedge \cong G$ if G is a MAP group, and β_G is always continuous from G to $(G^\wedge_p)^\wedge_p$. It is a topological isomorphism if and only if G is precompact.

If $(G_i)_{i\in I}$ is a family of abelian groups, we will denote by $\bigoplus_{i\in I} G_i$ its direct sum, i.e. the subgroup of the product $\prod_{i\in I} G_i$ formed by those $x = (x_i)_{i\in I}$ such that $x_i = 0$ for almost all $i \in I$.

If $(G_i)_{i\in I}$ is a family of topological abelian groups, the symbol $\prod_{i\in I} G_i$ will stand for the product group endowed with the product topology. On the other hand, whenever we refer to the direct sum $\bigoplus_{i\in I} G_i$ as a topological group, the topology implicitly considered is Kaplan's asterisk topology [19], which is in general finer than the one induced by the product. Taking these conventions into account, we have the following natural isomorphisms [6, Proposition 14.11]:

$$\text{(i)}\ \ (\prod_{i\in I} G_i)^\wedge_{\text{co}} \cong \bigoplus_{i\in I} (G_i)^\wedge_{\text{co}}, \quad \text{(ii)}\ \ (\bigoplus_{i\in I} G_i)^\wedge_{\text{co}} \cong \prod_{i\in I} (G_i)^\wedge_{\text{co}} \tag{1}$$

In particular, the product and the direct sum of a family of reflexive groups are both reflexive.

The classical Glicksberg theorem [17] states that for a locally compact abelian group (G,τ), the family of τ-compact subsets of G coincides with that of $\sigma(G,G^\wedge)$–compact subsets of G. This result was generalized in [8] to the wider class of nuclear groups and later in [3] to the even wider class of Schwartz groups. Nuclear groups constitute a class of topological groups which includes all locally compact abelian groups and all nuclear locally convex spaces, and is closed under formation of subgroups, Hausdorff quotients and products. The notion of a Schwartz group, based upon that of a Schwartz vector space, was introduced in [5]. The class of Schwartz groups contains the nuclear groups and all the free abelian groups over a k_ω-space. It is also closed under formation of subgroups, Hausdorff quotients and products. In particular dual groups of metrizable topological abelian groups are k_ω-spaces and therefore they are Schwartz groups.

The following fact will be used in the sequel:

Proposition 1.1. *[2, Proposition 3.5] Let G be a topological abelian group and U a neighbourhood of zero in G. Then $U^\rhd$ is compact in $G^\wedge_{\text{co}}$. In particular, $\sigma(G^\wedge, G)$ and τ_{co} induce the same topology on any equicontinuous subset of $G^\wedge$.*

2 Definition and first properties of g-barrelled groups

The notion of g-barrelled group was first given in [14], where it is highligthed that g-barrelled groups constitute a vast class of topological groups.

Definition 2.1. A Hausdorff topological abelian group (G, τ) is said to be g-barrelled if every $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ is τ-equicontinuous.

We next give some general properties of g-barrelled groups.

Proposition 2.2. *Let (G, τ) be a g-barrelled group. The following claims hold:*

(a) The canonical mapping $\alpha_G : G \to (G^\wedge_{\text{co}})^\wedge_{\text{co}}$ is continuous.

(b) The group $G^\wedge_{\text{co}}$ satisfies Glicksberg's Theorem.

(c) If $K \subseteq G^\wedge$ is $\sigma(G^\wedge, G)$-compact, then its $\sigma(G^\wedge, G)-$quasi-convex hull $K^{\triangleleft\triangleright}$ is compact in τ_{co}. In particular, it is also $\sigma(G^\wedge, G)$-compact.

Proof. The proof is straightforward using Proposition 1.1. □

Proposition 2.3. *(a) Let (G, τ) be a MAP topological abelian group. Then (G, τ) is g-barrelled if and only if $(G, \mathcal{Q}\tau)$ is g-barrelled.*

(b) Let (G, τ) be a precompact abelian group. Then (G, τ) is g-barrelled if and only if every $\sigma(G^\wedge, G)-$compact subset of $G^\wedge$ is finite.

Proof. (a) follows from the fact that (G, τ) and $(G, \mathcal{Q}\tau)$ have the same dual group and the same equicontinuous subsets. To prove (b), note that the only equicontinuous subsets of the dual group of a precompact group are the finite ones. □

The following result collects the properties which currently are known to imply g-barrelledness.

Theorem 2.4. *Let G be a topological abelian group satisfying any of the following properties:*

(a) G is separable and a Baire space

(b) G is metrizable and all its closed separable subgroups are Baire spaces

(c) G is Čech-complete

(d) G is pseudocompact

(e) G is precompact bounded torsion and a Baire space

Then G is g-barrelled.

Proof. For (a) and (b) see [14, Corollary 1.6]. (c) is a consequence of Theorems 2.3 and 1.1 in [23]. (d) follows from [18, Proposition 4.4]. (e) follows from [13, Theorem 3.3]. □

In [21] it was proved that ω-bounded groups are g-barrelled. This was the first new class of g-barrelled groups after the ones given in [14]. Note that ω-bounded groups are included in (d) of the previous theorem.

Since g-barrelled groups were first defined in the context of Mackey topologies for groups, let us briefly revisit this important connection in a way convenient for our purposes. We start with a general fact:

Lemma 2.5. *Let (G,τ) be a topological abelian group. The evaluation mapping $\beta_G : G \to (G^\wedge, \sigma(G^\wedge, G))^\wedge$ is a homomorphism onto, and it is open from $(G, \mathcal{Q}\tau)$ to $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$.*

Proof. By Comfort-Ross Theorem, β_G is onto.

In order to see that $\beta_G : (G, \mathcal{Q}\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$ is open, take a quasi-convex neighborhood of zero U in G. The set $K := U^\triangleright$ is a $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ by Proposition 1.1, thus $K^\triangleright$ is a neighborhood of zero in $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$. Hence, it is enough to show that $K^\triangleright \subseteq \beta_G(U)$. Indeed, taking into account that β_G is onto, fix $\beta_G(g) \in K^\triangleright = U^{\triangleright\triangleright}$. We have that $g \in U^{\triangleright\triangleleft} = U$.

□

Proposition 2.6. *Let (G,τ) be a topological abelian group. The following properties are equivalent:*

(a) (G,τ) is g-barrelled.

(b) The evaluation mapping $\beta_G : (G,\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$ is continuous.

(c) The evaluation mapping $\beta_G : (G, \mathcal{Q}\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$ is continuous.

If moreover (G,τ) is MAP (resp. locally quasi-convex), these properties are also equivalent to

(d) The evaluation mapping $\beta_G : (G, \mathcal{Q}\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$ (resp. $\beta_G : (G,\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$) is a topological isomorphism.

Proof. (a)⇒(c): Fix a $\sigma(G^\wedge, G)$-compact $K \subseteq G^\wedge$. By hypothesis it is equicontinuous. Hence there exists a τ-neighborhood of zero U such that $K \subseteq U^\triangleright$. Thus $U^{\triangleright\triangleleft} \subseteq K^\triangleleft$, or in other words, $\beta_G(U^{\triangleright\triangleleft}) \subseteq K^\triangleright$. Since K was arbitrary, this proves continuity.

(d)⇒(c)⇒(b) is trivial.

(b)⇒(a): Fix a $\sigma(G^\wedge, G)$-compact subset $K \subseteq G^\wedge$; let us see that K is equicontinuous. Since $K^\triangleright$ is a neighborhood of zero in $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\mathrm{co}}$ and

β_G is continuous, $K^{\triangleleft} = \beta_G^{-1}(K^{\triangleright})$ is a τ-neighborhood of zero. This means that K is equicontinuous.

Assume that (G, τ) is MAP. To prove (c)⇒(d) it suffices to take into account that the evaluation mapping β_G is injective and apply Lemma 2.5. □

Note that (a)⇔(d) in Proposition 2.6 is a slight improvement of Proposition 8.35 in [21].

The next results lead to the proof that whenever there exists a g-barrelled topology in the family of all compatible topologies, it is necessarily the Mackey topology.

Proposition 2.7. *Let (G, τ) be a g-barrelled, MAP topological abelian group. Let ν be a group topology on G which is compatible with τ. Then,*

(a) $\mathcal{Q}\nu \leq \mathcal{Q}\tau$

(b) if moreover $\mathcal{Q}\tau \leq \nu$, then ν is also g-barrelled and has the same locally quasi-convex modification as τ.

Proof. (a) By Proposition 2.6 the evaluation mapping $\beta_\tau : (G, \mathcal{Q}\tau) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\text{co}}$ is continuous. Since ν is compatible with τ, we obtain the same group $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\text{co}}$ if we consider the topology ν on G instead of τ. By Lemma 2.5 the evaluation mapping $\beta_\nu : (G, \mathcal{Q}\nu) \to (G^\wedge, \sigma(G^\wedge, G))^\wedge_{\text{co}}$ is open and onto; since (G, ν) is MAP, it is also injective. We deduce that $\beta_\nu^{-1} \circ \beta_\tau$ is continuous. But this composite mapping is simply the identity from $(G, \mathcal{Q}\tau)$ to $(G, \mathcal{Q}\nu)$. The result follows.

(b) We have $\mathcal{Q}\tau \leq \mathcal{Q}\nu$ by the assumption, and $\mathcal{Q}\nu \leq \mathcal{Q}\tau$ by (a). Thus $\mathcal{Q}\tau = \mathcal{Q}\nu$. Since ν is compatible with τ, they give rise to the same dual group $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\text{co}}$. By (a)⇔(c) in Proposition 2.6, ν is g-barrelled. □

In the class of precompact groups, g-barrelledness is a very restrictive property (see for instance Proposition 2.3 (b)). However it does not imply pseudocompactness, as we next prove.

Example 2.8. A family of precompact g-barrelled groups that are not pseudocompact.

Proof. Denote by $t\mathbb{T}$ the torsion part of $\mathbb{T}$. It is a divisible subgroup, therefore $\mathbb{T}$ can be decomposed as an algebraic direct sum, say $\mathbb{T} = t\mathbb{T} \oplus G$, for some subgroup $G \leq \mathbb{T}$. If G is endowed with the topology induced by that of $\mathbb{T}$, it is of the second category in $\mathbb{T}$. Otherwise, taking into account that $t\mathbb{T}$ is countable and the above decomposition of $\mathbb{T}$, we would contradict the fact that $\mathbb{T}$ is a Baire space. Now, by [20, Proposition 9.8], the topological group G is a Baire space, and by Theorem 2.4 (a), it is g-barrelled.

As G is a proper, infinite subgroup of $\mathbb{T}$, it is precompact and non-compact. On the other hand any metrizable pseudocompact space must be compact. Thus, we obtain that G is not pseudocompact.

Clearly, all possible direct summands G complementing $t\mathbb{T}$ in $\mathbb{T}$ have the same properties.

□

Next we present a class of topological groups which are not g-barrelled.

Proposition 2.9. *Any countable, MAP g-barrelled group is discrete.*

Proof. Let (G, τ) be a countable, MAP g-barrelled group. From the natural embedding $(G^\wedge, \sigma(G^\wedge, G)) \hookrightarrow \mathbb{T}^G$ we deduce that $(G^\wedge, \sigma(G^\wedge, G))$ is metrizable. Since it is also precompact, it has the same Pontryagin dual group as its completion ([2, 4.10] or [10]). This implies that $(G^\wedge, \sigma(G^\wedge, G))^\wedge_{\text{co}}$ is discrete. From (a)⇔(d) in Proposition 2.6 we deduce that $(G, \mathcal{Q}\tau)$ is discrete. Hence (G, τ) is also discrete. □

Since locally quasi-convex, g-barrelled groups are always "dual" groups (Proposition 2.6 (d)), it is natural to study reflexivity in the class $\mathcal{G}$, formed by the latter. We first state a general fact, whose proof is straightforward taking into account that the quasi-convex hull of a finite subset of a MAP group is finite [2, 7.11].

Lemma 2.10. *For a MAP topological group (G, τ) the following facts are equivalent:*

(a) The τ-compact subsets of G are finite.

(b) The compact-open topology on $G^\wedge$ coincides with $\sigma(G^\wedge, G)$.

Proposition 2.11. *Let G be a g-barrelled, locally quasi–convex group. Consider the following properties:*

(a) The τ-compact subsets of G are finite.

(b) $G_p^\wedge$ is reflexive.

(c) (G, τ) is reflexive.

Then (a) ⇔ (b) ⇒ (c).

Proof. By Proposition 2.6(d) and Lemma 2.10 it is clear that $(a) \Rightarrow (c)$. The converse does not hold as $G = \mathbb{R}$ endowed with the euclidean topology shows.

Let us prove now that $(a) \Rightarrow (b)$. From Proposition 2.6(d), we have

$$(G_p^\wedge)_{\text{co}}^\wedge \cong (G, \tau)$$

Taking dual groups on both sides

$$((G_p^\wedge)_{\text{co}}^\wedge)_{\text{co}}^\wedge \cong G_{\text{co}}^\wedge$$

Now by Lema 2.10 applied to the right-hand side of the isomorphism, we get

$$((G_p^\wedge)_{\text{co}}^\wedge)_{\text{co}}^\wedge \cong G_p^\wedge$$

.

For the converse $(b) \Rightarrow (a)$, starting with

$$((G_p^\wedge)_{\text{co}}^\wedge)_{\text{co}}^\wedge \cong G_p^\wedge$$

and taking into account that G is g-barrelled and locally quasi-convex we apply Proposition 2.6(d) to the left-hand side of the isomorphism to obtain

$$(G, \tau)_{\text{co}}^\wedge \cong G_p^\wedge$$

Again by Lemma 2.10 we obtain that the τ-compact subsets of G are finite. □

3 Permanence properties of g-barrelled groups

In this section we obtain new classes of g-barrelled groups by establishing permanence properties.

In order to prove that g-barrelledness is a multiplicative property, we first consider countable products.

Lemma 3.1. *Let H_n $(n \in \mathbb{N})$ be a sequence of g-barrelled groups. Then $\prod H_n$ is a g-barrelled group.*

Proof. Let τ_n be the topology on each H_n and let τ_π denote the product topology. By Proposition 2.6, we need to show that the evaluation map

$$(\prod H_n, \tau_\pi) \to \Big(\bigoplus H_n^\wedge, \sigma(\bigoplus H_n^\wedge, \prod H_n)\Big)_{\text{co}}^\wedge$$

is continuous. We will express it as the composition of three continuous mappings, as follows:

Step 1. Consider the evaluation mapping $\beta_{H_n} : (H_n, \tau_n) \to (H_n^\wedge, \sigma(H_n^\wedge, H_n))_{\text{co}}^\wedge$ for all $n \in \mathbb{N}$ and denote by j the product mapping:

$$j : (\prod H_n, \tau_\pi) \to \prod (H_n^\wedge, \sigma(H_n^\wedge, H_n))_{\text{co}}^\wedge$$

By Proposition 2.6, β_{H_n} is continuous for all $n \in \mathbb{N}$, therefore j is continuous.

Step 2. As mentioned in the formula (1)(ii), there is a natural isomorphism

$$\prod (H_n^\wedge, \sigma(H_n^\wedge, H_n))^\wedge_{\mathrm{co}} \cong \Big(\bigoplus (H_n^\wedge, \sigma(H_n^\wedge, H_n)) \Big)^\wedge_{\mathrm{co}}$$

Step 3. It is easy to see that the topological groups $\bigoplus (H_n^\wedge, \sigma(H_n^\wedge, H_n))$ and $(\bigoplus H_n^\wedge, \sigma(\bigoplus H_n^\wedge, \prod H_n))$ are algebraically the same and have the same dual group $\prod H_n$. Let us see that they also have the same compact subsets.

Indeed, the nontrivial part is to show that every $\sigma(\bigoplus H_n^\wedge, \prod H_n)$-compact set is compact in $\bigoplus (H_n^\wedge, \sigma(H_n^\wedge, H_n))$. But $\bigoplus (H_n^\wedge, \sigma(H_n^\wedge, H_n))$ is a nuclear group, as a countable sum of precompact groups [6, Proposition 7.8]. Since nuclear groups satisfy Glicksberg theorem [8], all weakly compact subsets (i. e. $\sigma(\bigoplus H_n^\wedge, \prod H_n)$-compact sets) of this group are compact.

We conclude that the identity mapping

$$\Big(\bigoplus (H_n^\wedge, \sigma(H_n^\wedge, H_n)) \Big)^\wedge_{\mathrm{co}} \to \Big(\bigoplus H_n^\wedge, \sigma\big(\bigoplus H_n^\wedge, \prod H_n\big) \Big)^\wedge_{\mathrm{co}}$$

is continuous. □

Leaning on Lemma 3.1 we can prove now that an arbitrary product of g-barrelled groups is g-barrelled. To this end we give several lemmas.

For an arbitrary index set I, and for all $i \in I$, let G_i be a topological abelian group and $G := \prod_{i \in I} G_i$. We first prove that equicontinuous subsets of $G^\wedge$ are "contained" in finite products and the same happens to $\sigma(G^\wedge, G)$-compact subsets of $G^\wedge$, provided that each G_i is g-barrelled. In the following Lemmas, we denote by $p_j : \prod_{i \in I} G_i \to G_j$ and $\pi_j : \bigoplus_{i \in I} G_i^\wedge \to G_j^\wedge$ the corresponding projections for every $j \in I$.

Lemma 3.2. *Let $G := \prod_{i \in I} G_i$ where G_i is a topological abelian group for every $i \in I$, and G has the corresponding product topology. Any equicontinuous subset $M \subseteq G^\wedge$ satisfies $\pi_i(M) = \{0\}$ for almost all $i \in I$.*

Proof. Let M be an equicontinuous subset of $G^\wedge$. It must be contained in the polar of a 0-neighborhood V. Since V contains a set of the form $\bigcap_{i \in F} p_i^{-1}(\{0\})$ where F is a finite subset of I, clearly for every $j \in I \setminus F$ we must have $\pi_j(M) = 0$. □

Lemma 3.3. *Let $G = \prod_{i \in I} G_i$, where G_i is a g-barrelled group, and I an arbitrary index set. Every $\sigma(G^\wedge, G)$-compact subset $M \subseteq G^\wedge$ satisfies $\pi_i(M) = \{0\}$ for almost all $i \in I$.*

Proof. Assume by contradiction that $M \subseteq G^\wedge$ is $\sigma(G^\wedge, G)$-compact and $\pi_i(M) \neq \{0\}$ for all $i \in J \subseteq I$ being $|J| = \aleph_0$. Denote by $H := \prod_{i \in J} G_i$ and correspondingly $H^\wedge = \bigoplus_{i \in J} G_i^\wedge$.

Define

$$\Phi : \bigoplus_{i \in I} G_i^\wedge \to \bigoplus_{i \in J} G_i^\wedge$$

as the natural projection.

Clearly Φ is continuous from the $\sigma(G^\wedge, G)$ topology to the $\sigma(H^\wedge, H)$ topology of the range space. Thus $\Phi(M)$ is $\sigma(H^\wedge, H)$-compact.

By Lemma 3.1, the product of countably many g-barrelled groups is g-barrelled and therefore $\Phi(M)$ is equicontinuous. By Lemma 3.2, $\pi_i(\Phi(M)) = \{0\}$ for almost all $i \in J$, which contradicts our choice of $J \subseteq I$. $\square$

Theorem 3.4. *Let $G = \prod_{i \in I} G_i$ be an arbitrary product of abelian topological groups. Then G is g-barrelled if and only if G_i is g-barrelled for all $i \in I$.*

Proof. Assume that each G_i is g-barrelled. Fix a $\sigma(G^\wedge, G)$-compact subset $M \subseteq G^\wedge$. By Lemma 3.3 there exists a finite subset $F \subseteq I$ such that $M \subseteq \prod_{i \in F} G_i^\wedge \times \prod_{i \notin F} \{0\}$. Thus M can be identified with a subset of $\prod_{i \in F} G_i^\wedge$, call it M', and let us work in the finite subproduct corresponding to the indexing set F. Since $\prod_{i \in F} G_i^\wedge$ is the dual group of $\prod_{i \in F} G_i$ which by Lemma 3.1 is g-barrelled, there must exist a zero neighborhood $U = \prod_{i \in F} O_i$ in $\prod_{i \in F} G_i$ such that $M' \subseteq U^\triangleright$. Now clearly $M \subseteq (\prod_{i \in F} O_i \times \prod_{i \in I \setminus F} G_i)^\triangleright$. Therefore M is equicontinuous in $G^\wedge$ with respect to the product topology on G. Thus, G is g-barrelled.

Next we show that each G_i is g-barrelled if G is g-barrelled. Let M be a $\sigma(G_i^\wedge, G_i)$-compact subset of $G_i^\wedge$. Clearly

$$W := M \times \prod_{j \neq i} \{0\} \subseteq \bigoplus_{i \in I} G_i^\wedge$$

is $\sigma(G^\wedge, G)$-compact.

By the assumption on G, W is equicontinuous and hence there exists a neighborhood of zero, V such that $\varphi(V) \subseteq \mathbb{T}_+$ for all $\varphi \in W$. Since all $\varphi \in W$ have the form $\varphi := (\varphi_j)_{j \in I}$ where $\varphi_i \in M$ and $\varphi_j = 0$ for every $j \neq i$, then $\varphi(x) = \varphi_i(x_i) \in \mathbb{T}_+$ for all $x \in V$ and all $\varphi_i \in M$. Hence $M \subseteq (\pi_i(V))^\triangleright$.

$\square$

In [22] several attempts were done in order to prove that products of Mackey groups are again Mackey. Positive results were achieved there for "small products" (less or equal than $\mathfrak{c}$ factors) of separable locally compact abelian groups. Now, by means of Theorem 3.4 we can prove it in greater generality.

Corollary 3.5. *Let $\{(G_i, \tau_i), i \in I\}$ be an arbitrary family of locally compact abelian groups. The product $\prod_{i \in I} G_i$ endowed with the product topology is a Mackey group.*

Proof. Just take into account Theorem 3.4, together with the fact that a g-barrelled group carries the Mackey topology. □

The following question remains open:

Question 3.6. If G_i is a Mackey group for every $i \in J$, is the product $\prod_{i\in J} G_i$ also a Mackey group?

Next we are going to obtain the stability of g-barrelledness for direct sums, quotients and inductive limits.

Proposition 3.7. *Let I be an arbitrary index set. Let G_i be a g-barrelled topological abelian group for each $i \in I$. The direct sum $\bigoplus_{i\in I} G_i$, endowed with Kaplan's asterisk topology, is g-barrelled.*

Proof. Let G be the direct sum $\bigoplus_{i\in I} G_i$. Then $G^\wedge = \prod_{i\in I} G_i^\wedge$.

For each $i \in I$, the projection $\pi_i : (G^\wedge, \sigma(G^\wedge, G)) \to (G_i^\wedge, \sigma(G_i^\wedge, G_i))$ is continuous. Therefore if we take a $\sigma(G^\wedge, G)$-compact set K, its image $\pi_i(K)$ is $\sigma(G_i^\wedge, G_i)$-compact and then by hypothesis equicontinuous for each $i \in I$.

Let us take for each $i \in I$ a neighborhood of zero U_i in G_i such that $\pi_i(K) \subseteq U_i^\triangleright$. It is not difficult to prove that $\prod_{i\in I} U_i^\triangleright \subseteq (\bigoplus_{i\in I}^* U_i)^\triangleright$, where $\bigoplus_{i\in I}^* U_i$ is the zero neighborhood in G associated to the family of neighborhoods $\{U_i\}$ (see [12, Lemma 20]). Therefore $K \subseteq \prod_{i\in I} \pi_i(K) \subset \prod_{i\in I} U_i^\triangleright \subseteq (\bigoplus_{i\in I}^* U_i)^\triangleright$ and consequently G is g–barrelled.

□

Proposition 3.8. *Let G be a g-barrelled group and let H be a closed subgroup of G. Then G/H is g-barrelled.*

Proof. Let $\varphi : G \to G/H$ be the quotient mapping. Its adjoint mapping $\varphi^\wedge : ((G/H)^\wedge, \sigma((G/H)^\wedge, G/H)) \to (G^\wedge, \sigma(G^\wedge, G))$ defined by $\varphi^\wedge(\chi) = \chi \circ \varphi$ is clearly continuous. Choose a $\sigma((G/H)^\wedge, G/H)-$compact set $K \subseteq (G/H)^\wedge$. The set $\varphi^\wedge(K)$ is $\sigma(G^\wedge, G)-$compact. Since G is g-barrelled by hypothesis, there exists a neighborhood of zero U in G with $\varphi^\wedge(K)(U) \subseteq \mathbb{T}_+$. This can be rewritten as $K(\varphi(U)) \subseteq \mathbb{T}_+$ and since $\varphi(U)$ is a neighborhood of zero in G/H, we conclude that K is equicontinuous. □

Before dealing with inductive limits let us first recall the definition of the coproduct topology.

Definition 3.9. Let I be an arbitrary index set, G_k an abelian locally quasi-convex topological group for each $k \in I$ and let $j_k : G_k \to \bigoplus_{i\in I} G_i$ be the inclusion mapping. The coproduct topology in $\bigoplus_{i\in I} G_i$ is the finest group topology making the inclusions $\{j_k\}_{k\in I}$ continuous.

As proved in [12, Corollary 22] the locally quasi-convex modification of the coproduct topology of an arbitrary family of locally quasi-convex groups is the Kaplan asterisk topology. Therefore, using Proposition 3.7 and Proposition 2.3(a) we can now claim the following:

Corollary 3.10. *Let I be an arbitrary index set and let G_i be a g-barrelled locally quasi-convex group for each $i \in I$. Then, the group $\bigoplus_{i\in I} G_i$ with the coproduct topology is g-barrelled.*

Proposition 3.11. *The direct limit of an arbitrary family of locally quasi-convex g-barrelled groups is a g-barrelled group.*

Proof. The standard construction of the inductive limit in the category of Hausdorff abelian groups is the following:

$$\varinjlim G_i \cong (\bigoplus_{i\in I} G_i)/H,$$

where $\bigoplus_{i\in I} G_i$ has the coproduct group topology with respect to the inclusions $i_k : G_k \to \bigoplus_{i\in I} G_i$ and H is a fixed closed subgroup. Therefore by the preceding corollary and Proposition 3.8 we obtain that $\varinjlim G_i$ is g-barrelled. □

The property of being g-barrelled is not inherited by closed subgroups and neither by dense subgroups, as shown next:

Example 3.12. An example of a pseudocompact group with closed subgroups that are not g-barrelled.

Let G be a pseudocompact abelian group with $|G| = \mathfrak{c}$ and such that all its countable subgroups are closed [26, Example 4.5]. Then G is g-barrelled by Theorem 2.4(d). However, any infinite countable subgroup $H \leq G$ is closed and (by Proposition 2.9) not g-barrelled.

The hereditary behavior improves for open subgroups. Let G be a topological group that contains an open subgroup A. As happens with other properties like reflexivity, g-barrelledness of A is equivalent to the same property for G. In order to prove this assertion we need some auxiliary facts.

Facts 3.13.

1. An open subgroup A of a topological group G is dually embedded in G [7, Lemma 2.2(b)]. This means that the dual mapping of the inclusion $j : A \hookrightarrow G$, $j^\wedge : G^\wedge \to A^\wedge$ is a surjective homomorphism.

2. The mapping $j^\wedge : G_p^\wedge \to A_p^\wedge$ is a continuous homomorphism. Its kernel $A^\perp$ is compact by Proposition 1.1. The induced continuous isomorphism $\phi : G_p^\wedge/A^\perp \to A_p^\wedge$ is a topological isomorphism. This derives from the fact that both the initial group and the target group are precompact with the same dual A (just observe that $A^{\perp\perp}$ can be identified with A, since the second orthogonal is taken with respect to $(G_p^\wedge)^\wedge \cong G$).

3. The canonical quotient mapping $q : G_p^\wedge \to G_p^\wedge / A^\perp$ is compact covering by [1, 4.6.22]. Therefore the composition $\phi \circ q : G_p^\wedge \to A_p^\wedge$ is also compact covering. Let us call $r := \phi \circ q$ which clearly is the restriction mapping $j^\wedge$.

Theorem 3.14. *Let G be a topological group that contains an open subgroup A. Then G is g-barrelled if and only if A is g-barrelled.*

Proof. Assume first that G is g-barrelled. Let S be a $\sigma(A^\wedge, A)$-compact subset of $A^\wedge$. We need to prove that it is equicontinuous with respect to A. Since $r : G_p^\wedge \to A_p^\wedge$ is compact covering, there exists a $\sigma(G^\wedge, G)$-compact subset $K \subseteq G^\wedge$ such that $r(K) = S$. By the assumption on G, K is an equicontinuous set relative to G. Thus, there exist a neighborhood of zero U in G such that $K \subseteq U^\triangleright$. If $V := U \cap A$, then V is a neighborhood of zero in A such that $S \subseteq V^\triangleright$. (Observe that an element χ of S has the form $r(\chi_1)$ for $\chi_1 \in K$ and for every $v \in V \cap A$, $\chi(v) = \chi_1(v) \in \mathbb{T}_+$.) Therefore S is equicontinuous with respect to A, which proves that A is g-barrelled.

For the converse implication, assume that A is g-barrelled and pick a $\sigma(G^\wedge, G)-$compact subset $M \subseteq G^\wedge$. Clearly $r(M)$ is $\sigma(A^\wedge, A)$-compact and by the assumption it is equicontinuous with respect to A. Thus, a neighborhood of zero W in A can be found, so that $r(M) \subseteq W^\triangleright$. Since $A \subseteq G$ is open, W is also a 0-neighborhood in G, and it is clear that $M \subseteq W^\triangleright$. This proves that M is equicontinuous with respect to G, therefore G is g-barrelled. □

A dense subgroup of a g-barrelled group may not be g-barrelled, as shown by the group of rational numbers $\mathbb{Q}$ considered as a dense subgroup of $\mathbb{R}$ with respect to the euclidean topology. By Theorem 2.4 (a), $\mathbb{R}$ is g-barrelled and by Proposition 2.9, $\mathbb{Q}$ is not g-barrelled. However, the converse statement holds:

Proposition 3.15. *Let G be a topological group that contains a dense g-barrelled subgroup H. Then G is g-barrelled. In particular, the completion of a g-barrelled group is g-barrelled.*

Proof. Observe that the dual groups of H and G may be identified by means of the adjoint $j^\wedge : (G^\wedge, \sigma(G^\wedge, G)) \to (H^\wedge, \sigma(H^\wedge, H))$ of the inclusion mapping $j : H \to G$, which is a continuous isomorphism. Fix a $\sigma(G^\wedge, G)$-compact $K \subseteq G^\wedge$. Since $\sigma(G^\wedge, H) \leq \sigma(G^\wedge, G)$ we have that K is $\sigma(G^\wedge, H)$-compact too, therefore equicontinuous as a set of characters of H. If U denotes a 0-neighborhood in H such that $K \subseteq U^\triangleright$, we have that the closure of U in G, say $\overline{U}$, is a 0-neighborhood in G such that $K \subseteq \overline{U}^\triangleright$. Thus G is g-barrelled. □

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References

[1] A. V. Arhangel'skii, M. G. Tkachenko, Topological Groups and Related Structures. Atlantis Press/World Scientific, Amsterdam-Paris, 2008.

[2] L. Außenhofer, Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups, Dissertationes Math., vol. 384, Warsaw, 1999.

[3] L. Außenhofer, On the Glicksberg theorem for locally quasi-convex Schwartz groups, Fund. Math. 201, No. 2 (2008), 163–177.

[4] L. Außenhofer, On the non-existence of the Mackey topology for locally quasi-convex groups, Forum Math. Vol. 30, 5 (2018), 1119–1128.

[5] L. Außenhofer, M. J. Chasco, X. Domínguez, V. Tarieladze, On Schwartz groups, Studia Math. 181, No. 3 (1999), 199–210.

[6] W. Banaszczyk, Additive Subgroups of Topological Vector Spaces, Lecture Notes in Mathematics 1466. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[7] W. Banaszczyk, M. J. Chasco, E. Martín-Peinador, Open subgroups and Pontryagin duality, Math. Z. (2) 215 (1994), 195–204.

[8] W. Banaszczyk, E. Martín-Peinador, The Glicksberg theorem on weakly compact sets for nuclear groups, Ann. New York Acad. Sci. 788 (1996), 34–39.

[9] M. Bruguera, Some properties of locally quasi-convex groups, Topology Appl. 77 (1997), no 2, 87–94.

[10] M. J. Chasco, Pontryagin duality for metrizable groups, Arch. Math. (Basel) 70 (1998), 22–28.

[11] M. J. Chasco, D. Dikranjan, E. Martín-Peinador, A survey on reflexivity of abelian topological groups, Topology Appl. 159 (2012), no. 9, 2290–2309.

[12] M. J. Chasco, X. Domínguez, Topologies on the direct sum of topological Abelian groups, Topology Appl. 133 (2003), 209–223.

[13] M. J. Chasco, X. Domínguez, M. Tkachenko, Duality properties of bounded torsion topological abelian groups, J. Math. Anal. Appl. 448 (2017), no. 2, 968–981.

[14] M. J. Chasco, E. Martín-Peinador, V. Tarieladze, On Mackey topology for groups, Studia Math. 132 (1999), no.3, 257–284 .

[15] W. Comfort, K. A. Ross, Topologies induced by groups of characters, Fund. Math. 55 (1964), 283–291.

[16] S. Gabriyelyan, A locally quasi-convex abelian group without a Mackey group topology, Proc. Amer. Math. Soc. 146 (2018), no. 8, 3627–3632.

[17] I. Glicksberg, Uniform boundedness for groups, Canad. J. Math, 14 (1962), 269–276.

[18] S. Hernández, S. Macario, Dual properties in totally bounded Abelian groups, Arch. Math. (Basel) 80 (2003), 271–283.

[19] S. Kaplan, Extensions of the Pontrjagin duality I: infinite products, Duke. Math. J. 15 (1948), 649–658.

[20] A. Kechris, Classical Descriptive Set Theory. Graduate Texts in Mathematics, 156. Springer Verlag, 1995.

[21] L. de Leo, Weak and strong topologies in topological abelian groups, http://eprints.ucm.es/8462/1/T30642.pdf

[22] E. Martín-Peinador, V. Pérez Valdés, A class of topological groups which do not admit normal compatible locally quasi-convex topologies. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 112 (2018), no. 3, 867–876.

[23] I. Namioka, Separate continuity and joint continuity, Pacific J. Math. 51 (1974), 515–531.

[24] P. Pérez Carreras, J. Bonet, Barrelled locally convex spaces, North-Holland Mathematics Studies 131, Amsterdam, 1987.

[25] H. H. Schaefer, Topological vector spaces, The Macmillan Co., New York; Collier-Macmillan Ltd., London 1966.

[26] M. Tkachenko, Compactness type properties in topological groups, Czechoslovak Math. J. 38 (113) (1988), no 2, 324–341.