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Pricing pension plans under jump-diffusion models for the salary☆

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Abstract

In this paper we consider the valuation of a defined benefit pension plan in the presence of jumps in the underlying salary and including the possibility of early retirement. We will consider that the salary follows a jump-diffusion model, thus giving rise to a partial integro-differential equation (PIDE). After posing the model, we propose the appropriate numerical methods to solve the PIDE problem. These methods mainly consists of Lagrange-Galerkin discretizations combined with augmented Lagrangian active set techniques and with the explicit treatment of the integral term. Finally, we compare the numerical results with those ones obtained with Monte Carlo techniques.

Keywords: pension plans, jump-diffusion models, option pricing, complementarity problem, numerical methods, Augmented Lagrangian Active Set formulation

1. Introduction

In many financial settings the use of the geometric Brownian motion processes proposed by Black and Scholes in [7] or by Merton in [22] do not fit properly market data or certain situations, such as an abrupt change in the value of the underlying. A list of empirical examples to illustrate this issue and their requirements from the modelling point of view can be found in Cont and Tankov [11], for example. In particular, jump-diffusion models later proposed by Merton in [23] and Kou [19] seem be more appropriate to describe these situations. Both models assume finite jump activity while the CGMY model proposed in [10] allows infinite jump activity. As explained in Pascucci [25], jump-diffusion processes are a particular case of the more general class of Levy processes, which also include stable processes, tempered stable processes (CGMY model), processes obtained by subordination (as Variance-Gamma process) or hyperbolic processes. Analogously to the case of option pricing under geometric Brownian motion, the family of Levy process requires the development of specific stochastic calculus tools, Ito formula and Feynmann-Kac representation leading to partial integro-differential equations (PIDE), see Pascucci [25] and the references therein, for example.

Following this idea, in the present work we mainly extend the previous works [8, 9] by considering the possibility of jumps in the underlying salary. Actually, we assume that the salary dynamics can be modelled by means of a Merton jump-diffusion process. In this new setting, the value of the pension plan can be obtained as the solution of an initial boundary value problem if early retirement is not allowed or as the solution of a complementarity problem when there exists the early retirement opportunity. Due to the presence of jumps, both problems are related to an integro-differential operator of hypoelliptic type.

There are several papers dealing with the numerical valuation of different financial derivatives when the underlying follows a jump-diffusion processes. For example, in [14] an implicit method is developed for the numerical solution of option pricing models where jumps in the underlying following a Merton model

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are assumed and where the integral term is computed using a fast Fourier transform (FFT) method. In [1] a European option pricing problem is addressed, when assuming Merton and Kou jump-diffusion models to account with sudden changes in the price of the underlying asset. A penalization method to handle the American feature is also analyzed in [12]. We notice that when using an implicit scheme, the integral term leads to a dense matrix and efficient algorithms are required to solve the dense system. In this sense, for the Kou model this penalty method is compared with an splitting method in [29] while an alternative front-tracking strategy is proposed in [30]. More recently, under Merton and Kou models an iterative Brennan-Schwartz algorithm is proposed in [28] for treating the complementarity problem related to American options. The application of the semilagrangian scheme for the valuation of American Asian Options under jump diffusion is addressed in [13].

In order to solve the partial integro-differential equation (PIDE) that arises in the presence of jumps in the salary, we propose a Lagrange-Galerkin discretization for the time and space, combined with an Augmented Lagrangian Active Set (ALAS) algorithm for solving the inequalities associated with the free boundary problem and with the explicit treatment of the integral term [11]. This explicit scheme for the integral term maintains the same sparse matrix structure as in the pure diffusion case and modifies the second member of the linear system associated with the discretized problem. The results from the application of these numerical methods are compared with the ones obtained by implementing the Monte Carlo simulation (see [16], for example) for the case without early retirement and the Longstaff-Schwartz technique proposed in [21] for the case with early retirement.

This paper is organized as follows. In Section 2 pension plans pricing models based on a Merton jump-diffusion process for the salary are posed. In Section 3 appropriate numerical methods are applied to find a solution to the PIDE problem that arises in the presence of jumps and Monte Carlo simulation techniques for jumps are proposed. Finally in Section 4, the obtained numerical results by using both procedures, are presented.

2. Mathematical modeling

2.1. A jump-diffusion model for pension plans

As we have pointed out in [8], in some cases of abrupt changes in the salary, Brownian motion is not appropriate enough to describe the evolution of this underlying factor and it is necessary to adopt a jump-diffusion model. More precisely, if we denote by S_t the salary at time t and we define the logarithmic salary $X_t = \ln(S_t)$ then we assume that the process X_t satisfies the following stochastic differential equation:

$$dX_t \;=\; \overline{\alpha}(t, X_t)\, dt \;+\; \overline{\sigma}(t, X_t)\, dZ_t \;+\; d\left(\sum_{i=1}^{N_t} V_i\right)\,,\; X_0 = \ln(S_0), \tag{1}$$

where $\overline{\alpha}$ represents the growth rate of the logarithmic salary and depends on the time t and X_t, $\overline{\sigma}$ is the volatility of the logarithmic salary, $(N_t)_{t\geq 0}$ denotes a Poisson process with parameter $\tilde{\lambda}$ and (V_i) is a sequence of square integrable, independent and identically distributed random variables, so that Z_t, N_t and (V_i) are independent.

We notice that according to [16], [20] and [23], for example, the stochastic differential equation can be formally written in the original variable S_t in the form

$$dS_t \;=\; \alpha(t, S_t)\, dt \;+\; \sigma(t, S_t)\, dZ_t \;+\; S_t\, d\left(\sum_{i=1}^{N_t}(Y_i - 1)\right)\,, \tag{2}$$

where α represents the growth rate of the salary, σ is the volatility of the salary and $Y_i = \exp(V_i)$.

As we are considering that retirement benefits depend on the continuous arithmetic average of the salary, as in [8] and other derivatives on averages of the underlying, we introduce the following process representing the cumulative salary since the last n_y years before T_r:

$$I_t = \int_0^t g(\tau, S_t)\, d\tau, \tag{3}$$

where the function g has the following expression

$$g(t,S) = \begin{cases} 0 & \text{if} \quad 0 \le t < T_r - n_y \\ h(S) & \text{if} \quad T_r - n_y \le t \le T_r, \end{cases} \tag{4}$$

where h is appropriately chosen. Specifically, in this paper we consider the particular case $h(S) = k_1 S$, where the accrual constant k_1 that satisfies $k_1 > 0$.

2.1.1. Partial integral differential equation (PIDE)

In [8], a PDE model for pricing pension plans without early retirement and in the absence of jumps is addressed by using a dynamic hedging technique. In the case of a jump-diffusion model, if we denote the value of the pension plan under jump-diffusion by $V_t = V(t, S_t, I_t)$ then V solves the following partial differential equation (PIDE):

$$\begin{aligned} &\partial_t V + \beta\, \partial_S V + g\, \partial_I V + \tfrac{1}{2}\sigma^2\, \partial_{SS} V - \, (\mu_d + \mu_w + r)\, V \\ &+ \int_{-\infty}^{\infty} \tilde{\lambda}\, [V(t, S\exp(y), I) - V(t,S,I) - S(\exp(y)-1)\partial_S V(t,S,I)]\, \nu(y) dy \\ &= -\mu_d A_d - \mu_w A_w, \end{aligned} \tag{5}$$

posed in the unbounded domain $(0, T_r) \times \mathbb{R}_+^2$. In (5) the value $\beta = \alpha - \lambda\sigma$ is related to the market price of risk associated to the uncertainty of the salary, here denoted by λ. Moreover, r is the deterministic risk-free interest rate, μ_d and μ_w are the transition intensities from active to death or withdrawal respectively, and $A_i = \alpha_i S, \quad \alpha_i \ge 0,\ i = d, w$, denotes the deterministic benefit paid by the fund in case of death (i=d) or withdrawal (i=w). In [11] the statement of the equation for a European vanilla call option under jump-diffusion model is proved.

In order to complete the problem formulation the final condition associated to (5) is defined by [8]:

$$V(T_r, S, I) = \frac{a}{n_y}\, I, \tag{6}$$

where $a \in (0,1)$ is a given accrual constant and I/n_y represents the average salary during last n_y years.

Moreover, in order to completely define the model, we must also specify the distribution of jump sizes $\nu(y)$. For this purpose, we will consider Merton model [23], so that (V_i) are taken from the normal distribution $(N(\mu, \gamma^2))$, with the density

$$\nu(y) = \frac{1}{\gamma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\gamma^2}\right), \tag{7}$$

where μ is the mean jump size and γ is the standard deviation of the jump size. There are other possibilities that could be taken into account, such as Kou model [19].

As we pointed out before, $\nu(y)$ is the probability density function of the jump amplitude V_i, thus

$$\int_{-\infty}^{\infty} \nu(y) dy = 1.$$

Moreover,

$$E[\exp(V_i)] = \int_{-\infty}^{\infty} \exp(y)\nu(y) dy = e^{\mu + \gamma^2/2}.$$

Therefore, the PIDE (5) can be written as

$$\partial_t V + (\beta - \tilde{\lambda}\kappa S)\,\partial_S V + g\,\partial_I V + \tfrac{1}{2}\sigma^2\,\partial_{SS} V - \left(\mu_d + \mu_w + r + \tilde{\lambda}\right) V$$
$$+\tilde{\lambda}\int_{-\infty}^{\infty} V(t, S\exp(y), I)\nu(y)dy = -\mu_d A_d - \mu_w A_w, \tag{8}$$

where $\kappa = e^{\mu+\gamma^2/2} - 1$. There is a new integral term in the equation due to the presence of jumps. This term makes the PIDE more difficult to solve than the corresponding PDE. In a forthcoming section we show how to discretize this integral in order to find a numerical solution of the PIDE problem.

As in the absence of jumps in the salary, in what follows we assume that $\beta(t, S, I) = \theta S$ and $\sigma(t, S) = \sigma S$. Next, we define the integro-differential operator

$$\mathcal{L}V = \partial_t V + (\theta - \tilde{\lambda}\kappa)S\partial_S V + g\partial_I V + \frac{\sigma^2 S^2}{2}\partial_{SS} V - \left(r + \mu_d + \mu_w + \tilde{\lambda}\right) V$$
$$+ \tilde{\lambda}\int_{-\infty}^{\infty} V(t, S\exp(y), I)\nu(y)dy.$$

For the case without early retirement, we consider the PIDE problem

$$\begin{cases} \mathcal{L}V = f, & \text{in } (0, T_r) \times \mathbb{R}_+^2, \\ V(T_r, S, I) = \dfrac{a}{n_y} I, & (S, I) \in \mathbb{R}_+^2. \end{cases} \tag{9}$$

If we incorporate the possibility of early retirement, the model is posed as the following complementarity problem

$$\begin{cases} \max\{\mathcal{L}V - f, \Psi - V\} = 0, & \text{in } (0, T_r) \times \mathbb{R}_+^2, \\ V(T_r, S, I) = \dfrac{a}{n_y} I, & (S, I) \in \mathbb{R}_+^2, \end{cases} \tag{10}$$

where

$$f(t, S, I) = -(\mu_d \alpha_d + \mu_w \alpha_w) S$$

and the expression for the obstacle Ψ, which is the amount received for the member of the plan if he/she decides to retire before the expected date, is given by

$$\Psi(t, S, I) = \begin{cases} 0 & \text{if } t < T_0, \\ \left(1 - \dfrac{T_r - t}{T_r - T_0}\right) \dfrac{aI}{t - (T_r - n_y)} & \text{if } t \geq T_0. \end{cases} \tag{11}$$

In the case without jumps, the mathematical analysis of the model leading to existence and uniqueness of solution has been addressed in [8] without early retirement and in [9] with early retirement option. The analogous analysis for the case with jumps is an interesting mathematical open problem.

3. Numerical solution

In [8], we propose a Lagrange-Galerkin method to discretize the PDE model that appears when the valuation of pension plans based on average salary without early retirement is considered. This Lagrange-Galerkin method has been analyzed in [4, 5] Then, we include the early retirement option and, in [9], this numerical method is combined with an augmented Lagrangian active set technique. In this paper, we extend these numerical techniques including the explicit computation of the integral term in order to solve the PIDE model associated to the jump-diffusion process. The results obtained with these numerical methods are compared with the ones got by using Monte Carlo techniques.

The PIDE is initially posed on an unbounded domain, so that we aproximate it by a bounded domain formulation and also introduce boundary conditions. In the PIDE case, the domain of integration in the integral term also needs to be localized. For this purpose, we introduce again the following change in time variable and notation

$$\tau = T_r - t, \quad x_1 = S, \quad x_2 = I, \quad \bar{x}_1 = log(x_1). \tag{12}$$

Let us consider x_1^∞ and x_2^∞ be large enough real numbers and let $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ be the computational bounded domain. Moreover, let $\partial\Omega = \bigcup_{i=1}^2 (\Gamma_i^- \bigcup \Gamma_i^+)$, with $\Gamma_i^- = \{(x_1, x_2) \in \partial\Omega \mid x_i = 0\}$ and $\Gamma_i^+ = \{(x_1, x_2) \in \partial\Omega \mid x_i = x_i^\infty\}$ for $i = 1, 2$.

The estimation of the error associated to the truncation of the domain is an open problem. In the classical European vanilla option pricing problem the error analysis has been addressed in [17]. However, to our knowledge this analysis has not been extended to other financial problems in the literature. The presence of jumps may require the use of the fundamental solution of the Merton model, which is written in terms of Gaussian functions.

Then, we write problems (9) and (10) in the bounded domain. In the case of the complementarity problem, in order to apply the iterative algorithm we propose, we replace the inequality by an identity and we incorporate the Lagrange multiplier P as a new unknown of the problem, identically to the case with no jumps (see [9]). Thus, the equivalent problem in the bounded domain for the case without early retirement is: Find $V : [0, T_r] \times \Omega \to \mathbb{R}$, such that

$$\begin{gathered} \partial_\tau V - Div(A\nabla V) + \vec{v} \cdot \nabla V + lV - \\ \tilde{\lambda} \int_{ymin}^{ymax} \bar{V}(\tau, \bar{x}_1 + y, x_2)\nu(y)dy = f \quad \text{in } (0, T_r) \times \Omega \end{gathered} \tag{13}$$

and for the case with early retirement:
Find V and $P : [0, T_r] \times \Omega \longrightarrow R$, satisfying the partial differential equation

$$\begin{gathered} \partial_\tau V - Div\,(A\nabla V) + \vec{v} \cdot \nabla V + lV - \\ \tilde{\lambda} \int_{ymin}^{ymax} \bar{V}(\tau, \bar{x}_1 + y, x_2)\nu(y)dy + P = f \quad \text{in } (0, T_r) \times \Omega \end{gathered} \tag{14}$$

the complementarity conditions

$$V \geq \overline{\Psi}, \quad P \leq 0, \quad (V - \overline{\Psi})P = 0 \quad \text{in } (0, T_r) \times \Omega. \tag{15}$$

where $\bar{V}(\tau, \bar{x}_1 + y, x_2) = V(\tau, \exp(\bar{x}_1 + y), x_2)$ the involved data in both cases is defined as follows

$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}(\tau, x_1, x_2) = \begin{pmatrix} (\sigma^2 - \theta + \tilde{\lambda}\kappa)x_1 \\ -g(T_r - \tau, x_1) \end{pmatrix}, \tag{16}$$

$$l(\tau, x_1, x_2) = r + \mu_d + \mu_w + \tilde{\lambda}, \quad f(\tau, x_1, x_2) = (\mu_d \alpha_d + \mu_w \alpha_w)x_1, \tag{17}$$

$$\overline{\Psi}(\tau, x_1, x_2) = \Psi(T_r - \tau, x_1, x_2), \quad g_1(\tau, x_1, x_2) = 0, \tag{18}$$

$$g_2(\tau, x_1, x_2) = \frac{a}{n_y}. \tag{19}$$

Moreover, the initial and boundary conditions are

$$V(0, x_1, x_2) = \frac{a}{n_y} x_2 \quad \text{in} \quad \Omega, \tag{20}$$

$$\frac{\partial V}{\partial x_1} = g_1 \quad \text{on } (0, T_r) \times \Gamma_1^+\,, \tag{21}$$

$$\frac{\partial V}{\partial x_2} = g_2 \quad \text{on } (0, T_r) \times \Gamma_2^+\,. \tag{22}$$

Remark 3.1. *Note that the differential term of the PIDE is computed in the domain* $[0, x_1^\infty] \times [0, x_2^\infty]$, *using the discrete grid* $0 = x_{1_0}, x_{1_1}, \cdots, x_{1_q} = x_1^\infty$.
Since $log(x_{1_0}) = -\infty$, *we choose* $ymin = log(x_{1_1})$ *and* $ymax = log(x_{1_q})$ *as it is proposed in [14].*

3.1. Time discretization

For the time discretization we consider the method of characteristics (also known as semilagrangian scheme) to approximate the material derivative. A first order version of this method has been introduced in [26], [15] and [3], for example. This version has been used in [31] for vanilla options and in [13] for American Asian options with jumps.

In the characteristics (or semilagrangian) method we first define the characteristics curve $X_e(\mathbf{x}, \bar{\tau}; s)$ through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, i.e. the integral path associated with the vector field $\vec{v}$ through $\mathbf{x}$, which verifies

$$\partial_s X_e(\mathbf{x}, \bar{\tau}; s) = \vec{v}(X_e(\mathbf{x}, \bar{\tau}; s)), \quad X_e(\mathbf{x}, \bar{\tau}; \bar{\tau}) = \mathbf{x}. \tag{23}$$

For $N > 1$ let us consider the time step $\Delta\tau = T_r/N$ and the time mesh-points $\tau_n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by the characteristics method is given by

$$\frac{DV}{D\tau} = \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau},$$

where $X_e^n(\mathbf{x}) := X_e(\mathbf{x}, \tau^{n+1}; \tau^n)$. In view of the expression of the velocity field and the continuous function g given by (16) and (4) respectively, the components of $X_e^n(\mathbf{x})$ can be analytically computed. More precisely, we distinguish the following two cases:

- if $(\sigma^2 - \theta + \tilde{\lambda}\kappa) \neq 0$ then $[X^n]_1(\mathbf{x}) = x_1 \exp((\theta - \sigma^2 - \tilde{\lambda}\kappa)\Delta\tau)$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ \dfrac{k_1 x_1}{\sigma^2 - \theta + \tilde{\lambda}\kappa}\Big(1 - \exp((\theta - \sigma^2 - \tilde{\lambda}\kappa)\Delta\tau)\Big) + x_2 & \text{if} \quad n\Delta\tau \leq n_y; \end{cases}$$

- if $(\sigma^2 - \theta + \tilde{\lambda}\kappa) = 0$ then $[X^n]_1(\mathbf{x}) = x_1$ and

$$[X_e^n]_2(\mathbf{x}) = \begin{cases} x_2 & \text{if} \quad n\Delta\tau > n_y, \\ k_1 x_1 \Delta\tau + x_2 & \text{if} \quad n\Delta\tau \leq n_y. \end{cases}$$

More recently, a version of the method that combines characteristics and Crank-Nicolson scheme has been analyzed in [4] and [5] for a general equation, thus extending the work in [27]. More precisely, under certain assumptions, second order convergence in time is obtained in [4], while second order convergence in space when combined with piecewise quadratic Lagrage finite elements is proved in [5].

Using this version, as in the case without jumps (see [8]), we consider a Crank-Nicolson scheme around $\big(X(\mathbf{x}, \tau^{n+1}; \tau), \tau\big)$ for $\tau = \tau^{n+\frac{1}{2}}$ in order to discretize the differential term, while the integral term is computed explicitly using the solution obtained at the preceding iteration. So, for $n = 0, \ldots, N-1$, the time discretized equation for the case without early retirement ($P = 0$) can be written as:

- *Find V^{n+1} such that*

$$\frac{V^{n+1}(\mathbf{x}) - V^n(X_e^n(\mathbf{x}))}{\Delta\tau} - \frac{1}{2}Div(A\nabla V^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla V^n)(X_e^n(\mathbf{x})) + \frac{1}{2}(l\, V^{n+1})(\mathbf{x}) + \frac{1}{2}(l\, V^n)(X^n(\mathbf{x})) - \tilde{\lambda}\int_{ymin}^{ymax} \bar{V}^n(\bar{x}_1 + y, x_2)\nu(y)dy = \frac{1}{2}f^{n+1}(\mathbf{x}) + \frac{1}{2}f^n(X_e^n(\mathbf{x})).$$

In order to apply finite elements, first we obtain the variational formulation of the semidiscretized problem in a similar way to the case under no jump diffusion, we multiply equation (24) by a suitable test function, integrate in Ω, use the classical Green formula and the following one (see Lemma 3.4 in [24]):

$$
\begin{aligned}
\int_{\Omega} Div(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} \quad = \quad & \int_{\Gamma} (\mathbf{F}_e^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} \\
& - \int_{\Omega} (\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\cdot\nabla\Psi(\mathbf{x})d\mathbf{x} \\
& - \int_{\Omega} Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}.
\end{aligned}
\tag{24}
$$

Note that in the present case we have

$$
\int_{\Omega} Div((\mathbf{F}_e^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} = 0.
$$

After these steps, we can write a variational formulation for the semidiscretized problem as follows:

Find $V^{n+1}\in H^1(\Omega)$ such that, $\forall \Psi\in H^1(\Omega)$:

$$
\begin{aligned}
& \int_{\Omega} V^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega}(\mathbf{A}\nabla V^{n+1})(\mathbf{x})\nabla\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega} lV^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} = \\
& \int_{\Omega} V^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_{\Omega}(\mathbf{F}_e^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla V^n)(X_e^n(\mathbf{x}))\nabla\Psi(\mathbf{x})d\mathbf{x} - \\
& \frac{\Delta\tau}{2}\int_{\Omega} lV^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Gamma}\tilde{g}^n(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \frac{\Delta\tau}{2}\int_{\Gamma_{1,+}}\bar{g_1}^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \\
& \frac{\Delta\tau}{2}\int_{\Omega} f^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_{\Omega} f^n(X_e^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \\
& \Delta\tau\tilde{\lambda}\int_{\Omega}\int_{ymin}^{ymax}\bar{V}^n(\bar{x}_1+y,x_2)\nu(y)dy\Psi(\mathbf{x})d\mathbf{x}\,,
\end{aligned}
\tag{25}
$$

where $\mathbf{F}_e^n = \nabla X_e^n$ can be analytically computed, $\bar{g}_1^{n+1}(\mathbf{x}) = g_1^{n+1}(\mathbf{x})a_{11}(\mathbf{x}) = 0$ and

$$
\tilde{g}^n(\mathbf{x}) = \begin{cases}
0 & on \quad \Gamma_{1,-} \\
-\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & on \quad \Gamma_{2,-} \\
\left[(\mathbf{F}_e^n)^{-T}\right]_{12}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))\frac{\partial V}{\partial x_1}(X_e^n(\mathbf{x})) & on \quad \Gamma_{2,+} \\
\left[(\mathbf{F}_e^n)^{-T}\right]_{11}(\mathbf{x})a_{11}(X_e^n(\mathbf{x}))g_1^n(X_e^n(\mathbf{x})) & on \quad \Gamma_{1,+}
\end{cases}
\tag{26}
$$

Remark 3.2. *We approximate the integral term in the PIDE by using the composite trapezoidal rule with $m+1$ points in the following way:*

$$
\begin{aligned}
& \int_{ymin}^{ymax}\bar{V}^n(\bar{x}_1+y,x_2)\nu(y)dy \approx \tfrac{h}{2}\Big[\bar{V}^n(\bar{x}_1+ymin,x_2)\nu(ymin) \\
& +\bar{V}^n(\bar{x}_1+ymax,x_2)\nu(ymax) + 2\textstyle\sum_{j=1}^{m-1}\bar{V}^n(\bar{x}_1+k_j,x_2)\nu(k_j)\Big]
\end{aligned}
\tag{27}
$$

where $k_j = ymin + jh$ for $j = 1,...,m-1$ and $h = \frac{ymax-ymin}{m}$.

3.2. Finite elements discretization

As we have already mention, we use finite elements for space discretization. For this purpose, as in the case without jumps, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T,\mathcal{Q}_2,\Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. More precisely, let us define the finite elements space V_h by

$$V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) : \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{28}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

The numerical analysis of the proposed characteristics method with finite elements (Lagrange-Galerkin) has been carried out in [5].

Note that when using piecewise quadratic finite elements and the treatment of the integral term in the previous time step we maintain the sparse (pentadiagonal) structure of the matrix. Furthermore, we use an appropriate lexicographical ordering of the nodes that allows a very efficient implementation of the ALAS iterative algorithm we propose for solving the complementarity problem (see [6] or [24], for details).

Remark 3.3. *Note that we have a grid in coordinates (x_1, x_2) and at each time step $n+1$, we know the value of V at the previous time step n in the grid points. So, in order to obtain the value of $\bar{V}^n(\bar{x}_1 + y, x_2) = V^n(\exp(\bar{x}_1 + y), x_2)$ we carry out a piecewise quadratic interpolation associated to the finite element discretization.*

3.3. Augmented Lagrangian Active Set (ALAS) algorithm

In the presence of jumps and in order to treat the non-linearities associated with the inequality constraints in the complementarity formulation when early retirement is allowed, we implement, as in the case without jumps, the ALAS algorithm proposed in [18] and explained in [9].

3.4. Monte Carlo

In this section we describe how to simulate the underlying salary with jumps. Once we obtain the salary, Monte Carlo method is applied in a similar way to the case without jumps. For the case with early retirement opportunity a Longstaff-Schwartz [21] algorithm is implemented as in the case without jumps too.

In the absence of jumps, the evolution of the salary under the risk neutral measure Q was given by

$$dS_t = \beta(t, S_t)dt + \sigma(t, S_t)dZ_t^Q, \tag{29}$$

where Z^Q denotes a Wiener process under this measure, $\beta(t, S_t) = \theta S$ and $\sigma(t, S_t) = \sigma S$. We will suppose that the salary at initial time is known.

If $\kappa = e^{\mu + \gamma^2/2} - 1$ then the process

$$\sum_{i=1}^{N_t} V_i - \tilde{\lambda}\kappa t$$

is a martingale. Having the previous property in view, in the case with jumps and assuming Merton model, the stochastic differential equation that describes the risk neutral dynamics of the logarithmic salary ($X_t = \ln(S_t)$) can be written in the form

$$dX_t = \bar{\beta}(t, X_t)dt + \bar{\sigma}(t, X_t)dZ_t^Q + d\left(\sum_{i=1}^{N_t} V_i\right), \tag{30}$$

where Z^Q denotes a Wiener process under this measure, $\bar{\beta} = (\theta - \tilde{\lambda}\kappa)$ and $\bar{\sigma} = \sigma$. In the case without early retirement the value of the plan is given by the expectation

$$V(t, S_t, I_t) = E_Q\left[e^{-(r+\mu_d+\mu_w)(T_r - t)}\frac{a}{n_y}I - \int_t^{T_r} e^{-(r+\mu_d+\mu_w)(u-t)} f(u, S_u)d_u\right], \tag{31}$$

and when early retirement is allowed the value of the plan is written in probabilistic terms as the Snell envelope

$$V(t,S_t,I_t)=\sup_{\tau\in\mathcal{T}(t,T_r)} E_Q\left[e^{-(r+\mu_d+\mu_w)(\tau-t)}\Psi(\tau,S_\tau,I_\tau)-\int_t^\tau e^{-(r+\mu_d+\mu_w)(u-t)}f(u,S_u)d_u\right]. \quad (32)$$

3.5. Simulating at fixed dates

In order to simulate the salary with jumps, we consider the approach in which we simulate the process at a fixed set of dates $0=t_0<t_1<...<t_m=T_r$ without explicitly distinguishing the effect of the jump and diffusion terms. We simulate $X(t)=\ln(S(t))$ at time $t_1,\cdots,t_m$ in the following way:

$$X(t_{j+1})=X(t_j)+(\theta-\frac{1}{2}\sigma^2-\tilde{\lambda}\kappa)(t_{j+1}-t_j)+\sigma\sqrt{t_{j+1}-t_j}W_{j+1})+\sum_{i=N(t_j)+1}^{N(t_{j+1})}V_i, \quad (33)$$

where $W_1,...,W_m$ are independent standard normal random variables. By convention, the product over i is equal to 1 if $N(t_{i+1})=N(t_i)$. Then, we can exponentiate simulated values of the $X(t_i)$ to produce samples of the $S(t_i)$. A general method for simulating (33) from t_j to t_{j+1} comprises the following steps:

1. generate $W\sim N(0,1)$
2. generate $N\sim Poisson(\tilde{\lambda}(t_{j+1}-t_j))$; if $N=0$, set $M=0$ and go to step 4
3. assuming $V_i\sim N(\mu,\gamma^2)$ and $\sum_{i=1}^n V_i\sim N(\mu n,\gamma^2 n)=\mu n+\gamma\sqrt{n}N(0,1)$, then, generate $W_2\sim N(0,1)$; set $M=\mu N+\gamma\sqrt{N}W_2$
4. set
$$X(t_{j+1})=X(t_j)+(\theta-\frac{1}{2}\sigma^2-\tilde{\lambda}\kappa)(t_{j+1}-t_j)+\sigma\sqrt{t_{j+1}-t_j}W+M.$$

4. Numerical results

In this section, some numerical results corresponding to a jump-diffusion model for the salary are shown.

First, we consider a pension plan without the early retirement option. In this case, we present the results obtained by solving the PIDE model and by implementing the Monte Carlo simulation technique based on the techniques described in the previous section.

Next, we incorporate the early retirement option and the hereafter presented results correspond to solving the complementarity problem associated with the PIDE and with the Longstaff-Schwartz technique [21].

The number of nodes and elements of the quadratic finite element meshes are indicated in Table 1. All Monte Carlo simulations presented in this section have been performed by using 7000 time steps per year and 10^6 paths. Moreover, a variance reduction technique based on antithetic variables has been considered.

In all the following examples, the retirement date $T_r=40$, the number of subintervals in the composite trapezoidal rule $m=50$ and the bounded computational domain defined by the values $x_1^\infty=40$ and $x_2^\infty=40$ have been considered. Particularly, for the jump-diffusion model we considered the data estimated in [2] for the underlying associated with European call options:

$$\tilde{\lambda}=0.1,\quad \mu=-0.90\quad and\quad \gamma=0.45. \quad (34)$$

In future work, clearly this data need to be obtained from real data corresponding to salary evolution in some way, being the estimation from historical data the a priori easiest way to obtain these parameters.

4.1. Pension plans without early retirement

In this section we show the results obtained when the early retirement option is not allowed. In Table 2 the computed values of the pension plan at origination and mesh point $(S, I) = (25, 20)$ for different meshes and time steps are indicated in order to illustrate the convergence. Note that the second order convergence for this scheme is proved in [4, 5] under certain assumptions which do not include the presence of a nonlocal term as the one associated to jumps. In fact, we use a composed trapezoidal rule for this term and perhaps an alternative integration formula could improve the convergence.

Next, in Table 3 for different parameters we show the computed values at time $t = 0$ obtained by solving the PIDE problem for the same salaries and average salaries considered in [8]. The corresponding values with the Monte Carlo technique are those ones appearing in Table 4. As in the absence of jumps, note that only the salary at time $t = 0$ is known, so that the salary and cumulative salary values are simulated at the different dates along the path. As expected, the value of the pension plan does not depend on the cumulative salary I at origination because we only take into account the average salary from time $t = T_r - n_y$.

The behaviour of the plan at time $t = 38$ is shown in Tables 5 and 6. The first values are obtained with de PIDE model and the second ones with Monte Carlo simulation. As in the case without jumps, at $t = 38$, the salary and the cumulative salary are known and we simulate the values from this date. In this case the value of the variable I up to this time is considered and influences the value obtained with Monte Carlo. In order to obtain the solution of the PIDE model, Mesh 96 and 100000 time steps have been considered.

	Number of elements	**Number of nodes**
Mesh 12	144	625
Mesh 24	576	2401
Mesh 48	2304	9409
Mesh 96	9216	37249

Table 1: FEM meshes data

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
100	2.603778	2.604122	2.604051	2.604056
1000	2.598576	2.599026	2.599109	2.599123
10000	2.598018	2.598408	2.598447	2.598455
100000	2.598004	2.598389	2.598421	2.598422

Table 2: Retirement benefits under a jump diffusion process without early retirement at time $t = 0$ and at mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$, are considered. The 99% confidence interval with Monte Carlo simulation is [2.597423, 2.599365].

4.2. Pension plans with early retirement option

In this section we show the results obtained when the early retirement option is allowed. For this purpose, we have considered the following model parameters:

$$\sigma = 0.1,\ \theta = 0.025,\ r = 0.025,\ a = 0.75,\ T_r = 40,\ T_0 = 15,$$
$$n_y = 30,\ k_1 = 0.5,\ \mu_d = 0.025,\ \mu_w = 0.2,\ \alpha_d = 1,\ \alpha_w = 0$$

and we set $\beta = 10000$ in the ALAS algorithm.

Table 7 illustrates the convergence of the method in the presence of jumps as soon as the mesh is refined in time and space at $t = 38$ and the mesh point $(S, I) = (25, 20)$. Note that for all times the point

n_y	σ	r	a	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$
30	0.1	0.025	0.75	0.133059	0.133094	0.269216
			0.95	0.133132	0.133169	0.269375
		0.075	0.75	0.108654	0.108681	0.219359
			0.95	0.108859	0.108883	0.219732
	0.2	0.025	0.75	0.133175	0.133192	0.269321
			0.95	0.133393	0.133421	0.269483
		0.075	0.75	0.108771	0.108798	0.219532
			0.95	0.108947	0.108991	0.219941
				$(S,I)=(1.2,7.5)$	$(S,I)=(1.2,11.25)$	$(S,I)=(2.4,15)$
15	0.1	0.025	0.75	0.132617	0.132682	0.266294
			0.95	0.132745	0.132799	0.266357
		0.075	0.75	0.108549	0.108578	0.218003
			0.95	0.108974	0.108997	0.218342
	0.2	0.025	0.75	0.132691	0.132733	0.266344
			0.95	0.132853	0.132894	0.266414
		0.075	0.75	0.108697	0.108725	0.218058
			0.95	0.109021	0.109086	0.218192

Table 3: Retirement benefits under a jump diffusion process without early retirement at time $t = 0$ for different (S, I) points and parameter values

n_y	σ	r	a	$S=1.2$	$S=2.4$
30	0.1	0.025	0.75	[0.132979, 0.133173]	[0.269018, 0.269421]
			0.95	[0.133081, 0.133269]	[0.269104, 0.269485]
		0.075	0.75	[0.108598, 0.108702]	[0.219084, 0.219402]
			0.95	[0.108792, 0.108898]	[0.219536, 0.219853]
	0.2	0.025	0.75	[0.133103, 0.133371]	[0.269119, 0.269432]
			0.95	[0.133311, 0.133609]	[0.269213, 0.269547]
		0.075	0.75	[0.108705, 0.108873]	[0.219253, 0.219598]
			0.95	[0.108897, 0,109067]	[0.219672, 0.219991]
				$S=1.2$	$S=2.4$
15	0.1	0.025	0.75	[0.132546, 0.132744]	[0.266048, 0.266398]
			0.95	[0.132664, 0.132869]	[0.266123, 0.266476]
		0.075	0.75	[0.108499, 0.108601]	[0.217887, 0.218197]
			0.95	[0.108911, 0.109007]	[0.218105, 0.218402]
	0.2	0.025	0.75	[0.132619, 0.132854]	[0.266176, 0.266497]
			0.95	[0.132783, 0.132994]	[0.266219, 0.266531]
		0.075	0.75	[0.108618, 0.108796]	[0.217884, 0.218127]
			0.95	[0.108997, 0.109169]	[0.218099, 0.218352]

Table 4: The 99% confidence intervals with Monte Carlo simulation under a jump diffusion process at time $t = 0$ for different salaries and parameter values

$(S, I) = (25, 20)$ is located at the region where early retirement is not optimal. For the same point, in Tables 8 and 9 the results for $t = T_r - n_y = 10$ and $t = 0$, respectively, also illustrate the convergence with mesh refinement in time and space. Note that the convergence analysis of the characteristics-Crank-Nicolson scheme for the complementarity problem is an open question. However, numerical experiments with different meshes seem to provide a first order convergence.

Moreover, the obtained results by solving the PIDE have been compared with those ones of Monte Carlo simulation techniques. More precisely, we have implemented the Longstaff-Schwartz technique and the corresponding confidence intervals are indicated in the caption.

n_y	σ	r	a	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$
30	0.1	0.025	0.75	0.293409	0.406986	0.586518
			0.95	0.358895	0.502986	0.718043
		0.075	0.75	0.267828	0.370626	0.535824
			0.95	0.327221	0.457491	0.654372
	0.2	0.025	0.75	0.293479	0.407026	0.586697
			0.95	0.358923	0.503005	0.718192
		0.075	0.75	0.267915	0.370697	0.535897
			0.95	0.327251	0.457518	0.654429
				$(S,I)=(1.2,7.5)$	$(S,I)=(1.2,11.25)$	$(S,I)=(2.4,15)$
15	0.1	0.025	0.75	0.311833	0.424796	0.622203
			0.95	0.382331	0.525624	0.763118
		0.075	0.75	0.284386	0.386681	0.567829
			0.95	0.348298	0.478046	0.695367
	0.2	0.025	0.75	0.311902	0.422867	0.622293
			0.95	0.382419	0.525735	0.763386
		0.075	0.75	0.284417	0.386872	0.577897
			0.95	0.348392	0.478139	0.695471

Table 5: Retirement benefits value under a jump diffusion process without early retirement at time $t = 38$ for different (S, I) points and parameter values

n_y	σ	r	a	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$
30	0.1	0.025	0.75	[0.293376, 0.293429]	[0.406964, 0.407012]	[0.586451, 0.586546]
			0.95	[0.358883, 0.358934]	[0.502943, 0.502993]	[0.717976, 0.718074]
		0.075	0.75	[0.267805, 0.267855]	[0.370601, 0.370652]	[0.535803, 0.535892]
			0.95	[0.327194, 0.327241]	[0.457471, 0.457518]	[0.654311, 0.654399]
	0.2	0.025	0.75	[0.293423, 0.293497]	[0.406995, 0.407058]	[0.586603, 0.586725]
			0.95	[0.358911, 0.358991]	[0.502991, 0.503059]	[0.718107, 0.718239]
		0.075	0.75	[0.267903, 0.267969]	[0.370675, 0.370734]	[0.535817, 0.535936]
			0.95	[0.327213, 0.327284]	[0.457501, 0.457568]	[0.654356, 0.654497]
				$(S,I)=(1.2,7.5)$	$(S,I)=(1.2,11.25)$	$(S,I)=(2.4,15)$
15	0.1	0.025	0.75	[0.311821, 0.311891]	[0.424759, 0.424823]	[0.622138, 0.622258]
			0.95	[0.382307, 0.382379]	[0.525603, 0.525672]	[0.763033, 0.763159]
		0.075	0.75	[0.284351, 0.284412]	[0.386654, 0.386717]	[0.567757, 0.567867]
			0.95	[0.348272, 0.348333]	[0.478011, 0.478077]	[0.695321, 0.695441]
	0.2	0.025	0.75	[0.311891, 0.311976]	[0.422815, 0.422899]	[0.622211, 0.622390]
			0.95	[0.382385, 0.382483]	[0.525701, 0.525798]	[0.763303, 0.763464]
		0.075	0.75	[0.284389, 0.284469]	[0.386821, 0.386895]	[0.577779, 0.577935]
			0.95	[0.348351, 0.348437]	[0.478101, 0.478193]	[0.695352, 0.695512]

Table 6: The 99% confidence intervals with Monte Carlo simulation under a jump diffusion process at time $t = 38$ for different salaries and parameter values

Finally, Table 10 shows a comparison between the values obtained with both techniques, solving the PIDE and with Monte Carlo and if early retirement is allowed or not. In the first rows the values at different points with Monte Carlo simulation (MC) and PIDE model (PIDE) when early retirement is not allowed. Then forth and fifth rows indicate the value of V with Longstaff-Schwartz technique (LS) and PIDE model (PIDE) when early retirement is allowed. Last row shows the computed multiplier in the PIDE solution. As in the case without jumps, the value of V is always greater in case of allowing early retirement and also, the point $(S, I) = (4, 10)$ belongs to the region where early retirement is not optimal as indicated by the null value of the multiplier. The other points belong to the early retirement region and the value of the pension plan matches the early retirement value and the value of the multiplier

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	1.695962	1.695863	1.695855	1.696236
2500	1.696683	1.696559	1.696543	1.696944
5000	1.697076	1.696938	1.696919	1.697326
10000	1.697285	1.697128	1.697107	1.697112

Table 7: Retirement benefits under a jump diffusion process with early retirement at time $t = 38$ at the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [1.696812, 1.698176].

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.756395	2.758438	2.758443	2.758448
2500	2.758234	2.759291	2.759294	2.759297
5000	2.758651	2.760128	2.760134	2.760137
10000	2.760221	2.760392	2.760413	2.760414

Table 8: Retirement benefits under a jump diffusion process with early retirement at time $t = 10$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.758998, 2.760943].

time steps	**Mesh 12**	**Mesh 24**	**Mesh 48**	**Mesh 96**
1250	2.598371	2.598801	2.598848	2.598867
2500	2.598153	2.598582	2.598627	2.598671
5000	2.598032	2.598463	2.598507	2.598513
10000	2.598019	2.598408	2.598447	2.598455

Table 9: Retirement benefits under a jump diffusion process with early retirement at time $t = 0$ and the mesh point $(S, I) = (25, 20)$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_c = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. The 99% confidence interval with LS simulation is [2.597937, 2.599538].

is different from zero. In order to obtain these results solving the PIDE model the Mesh 96 and 10000 time steps are considered.

	$(S,I)=(1.2,15)$	$(S,I)=(1.2,22.5)$	$(S,I)=(2.4,30)$	$(S,I)=(4,10)$
V without ER(MC)	(0.293376, 0.293429)	(0.406964, 0.407012)	(0.586451, 0.586546)	(0.374553, 0.374732)
V without ER$(PIDE)$	0.293409	0.406986	0.586518	0.374628
V with ER(LS)	(0.369643, 0.369643)	(0.554464, 0.554464)	(0.739286, 0.739286)	(0.374593, 0.374741)
V with ER$(PIDE)$	0.369643	0.554464	0.739286	0.374635
P with ER$(PIDE)$	-1.645011$\times 10^{-5}$	-2.264008$\times 10^{-5}$	-2.304657$\times 10^{-5}$	0

Table 10: Retirement benefits under a jump diffusion process without and with early retirement (ER) at different points when time $t = 38$ and the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered. Computed values with (PIDE), Monte Carlo (MC) and Longstaff-Schwartz (LS) and Multiplier (P).

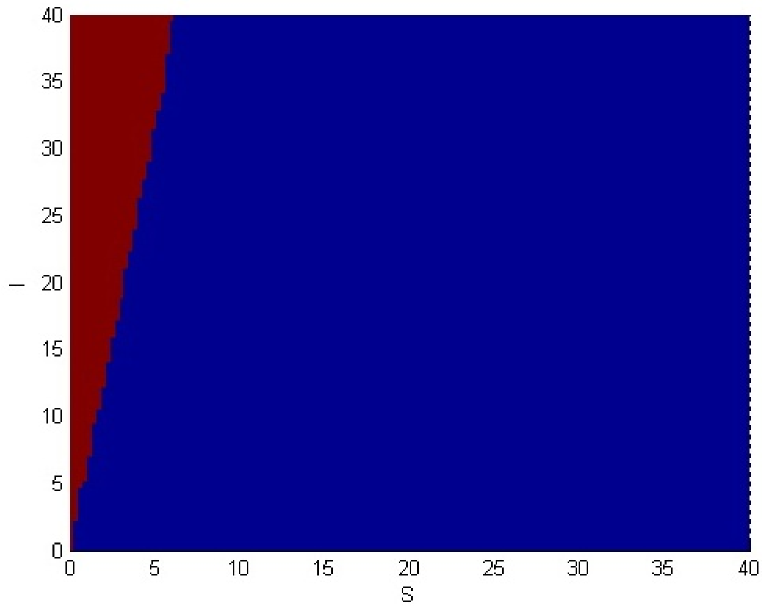

Figure 1: Free boundary at time $t = 38$ when the parameters $\sigma = 0.1$, $\theta = 0.025$, $r = 0.025$, $a = 0.75$, $n_y = 30$, $k_1 = 0.5$, $\mu_d = 0.025$, $\mu_w = 0.2$, $\alpha_d = 1$ and $\alpha_w = 0$ are considered

Moreover, Figure 1 shows the early retirement region in red and the non early retirement region in blue, the boundary separating both regions is the optimal retirement boundary (free boundary). As expected, the early retirement region corresponds to low salaries and large cumulative salaries (therefore large average), while in the region with large salaries and low cumulative salaries it results optimal to continue at work.

Concerning the comparison between PIDE numerical solution and Monte Carlo simulation in terms of computational cost, in the case without early retirement, Monte Carlo simulation takes around 3 times the computing time of PIDE method to arrive to $t = 0$, the las method providing more information: the pension plan values for all mesh and time points of the discretization are obtained. In the case with possibility of early retirement, both Longstaff-Schwartz and PIDE approaches are more time consuming than without early retirement, the cost of the first one being 1,5 times the cost of the second approach. So, we can conclude that the PIDE approach results to be more efficient than the alternative Monte Carlo simulation techniques.

5. Conclusions

In this paper we consider the valuation of pension plans based on average salary where the salary is supposed to follow a jump-diffusion process. In this case a PIDE arises and we posed the model for both cases, with and without early retirement. We proposed appropriate numerical methods based on Lagrange-Galerkin formulation to solve the problems combined with the ALAS algorithm to deal with the non-linearities associated with the free boundary problem that appears when early retirement is allowed. The integral term of the PIDE is treated explicitly. The results obtained by means of these techniques are compared with the ones got by implementing Monte Carlo simulation. In all the examples shown in this paper the results obtained by solving the PIDE are into the Monte Carlo confidence intervals.

Finally, after comparing PIDE and Monte Carlo based numerical techniques in terms of computing time, the PIDE results less time consuming in both cases without and with early retirement opportunity. Moreover, the PIDE numerical solution provides the value of the pension plan for as many pairs salary-average salary as the number of nodes of the finite element mesh and for all discretization times. Indeed, the use of an appropriate interpolation method allows to approximate also the value for any time, salary and average salary. So, we can conclude that the PIDE approach results to be more efficient than the alternative Monte Carlo simulation techniques.

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