

A DYNAMIC PROBLEM INVOLVING A COUPLED SUSPENSION BRIDGE SYSTEM: NUMERICAL ANALYSIS AND COMPUTATIONAL EXPERIMENTS

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(Communicated by Josef Malek)

ABSTRACT. In this paper we study, from the numerical point of view, a dynamic problem which models a suspension bridge system. This problem is written as a nonlinear system of hyperbolic partial differential equations in terms of the displacements of the bridge and of the cable. By using the respective velocities, its variational formulation leads to a coupled system of parabolic nonlinear variational equations. An existence and uniqueness result, and an exponential energy decay property, are recalled. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme. A discrete stability property is shown and a priori error estimates are proved, from which the linear convergence of the algorithm is deduced under suitable additional regularity conditions. Finally, some numerical results are shown to demonstrate the accuracy of the approximation and the behaviour of the solution.

1. Introduction. During the last decades the study of the so-called suspension bridges has received a large attention because this kind of bridges is a common type of civil engineering structure. It is well-known that these bridges may display certain oscillations under external aerodynamic forces like, for instance, it occurred in the famous Tacoma's bridge (see [2, 5]), in which a strong wind caused the collapse of a narrow and very flexible suspension bridge.

2000 *Mathematics Subject Classification.* Primary: 74B20, 65M60, 65M15; Secondary: 74K10, 74H15.

Key words and phrases. Coupled bridge system, finite elements, a priori error estimates, numerical simulations.

This work has been supported by Ministerio de Economía y Competitividad under the project MTM2015-66640-P (with the participation of FEDER).

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Since the pioneering works by Lazer and McKenna (see, e.g., [14] or [18]), where mathematical models based on nonlinear partial differential equations were introduced to describe oscillations in suspension bridges including connection cables, a large number of papers have been published (see, for instance, [1, 3, 10, 11, 12, 13, 16, 19, 20, 21, 22, 23, 24, 25] and the numerous references cited therein). Most of the above works deal with mathematical aspects as the application of analytical methods to scrutinize stationary solutions, periodic oscillations, longtime global dynamics, existence and uniqueness of solutions and so on. Moreover, we want to point out that in the literature other types of suspension bridges were studied, where the associated dynamic system was written in the vertical and torsional displacements of a cross-section of the bridge (see, e.g., [4, 15]). However, to our knowledge the numerical analyses of the proposed models have not been performed yet.

In this paper, we revisit the problem considered in [7], where a model taking into account the coupling between the road bed and the suspension main cable was studied from the mathematical point of view, proving the existence and uniqueness of weak solutions by using the Faedo-Galerkin approximation procedure and Gronwall's lemma, and the exponential decay of the system energy. Moreover, the existence of a regular global attractor was also shown by using the semigroup theory and defining an adequate Lyapunov functional. Here, we aim to provide the numerical analysis of this dynamic problem, introducing fully discrete approximations, proving a discrete stability result and a priori error estimates, and performing numerical simulations which demonstrate the accuracy of the algorithm and the behaviour of the solution.

The paper is outlined as follows. The mathematical model is described in Section 2 following [7], deriving its variational formulation. An existence and uniqueness result, and an energy decay property, proved in [7] are also stated. Then, in Section 3 a numerical scheme is introduced, based on the finite element method to approximate the spatial domain and the forward Euler scheme to discretize the time derivatives. A discrete stability property is proved and a priori error estimates are deduced for the approximative solutions from which, under suitable regularity assumptions, the linear convergence of the algorithm is obtained. Finally, some numerical simulations are presented in Section 4.

2. The model and its variational formulation. In this section, we present briefly the model, the required assumptions and the variational formulation of the mechanical problem, and we state an existence and uniqueness result. We refer the reader to [7] for details.

Let $[0, \ell]$, $\ell > 0$, be the one-dimensional beam (bridge) or rod (cable) of length ℓ and denote by $[0, T]$, $T > 0$, the time interval of interest. Moreover, let $x \in [0, \ell]$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on x and t , and a subscript under a variable represents its derivative with respect to the prescribed variable.

Let u denote the downward deflection of the deck midline (of unitary length) in the vertical plane with respect to his reference configuration (that is, it represents the bending displacement of the road bed), and let w be the vertical displacement of the vibrating spring standing for the main cable. Thus, the mechanical problem which simulates the coupled system of a vibrating suspension bridge and a cable is written in the following form (see [7]).

Problem P. Find the bending displacement $u : [0, \ell] \times [0, T] \rightarrow \mathbb{R}$ and the vertical displacement $w : [0, \ell] \times [0, T] \rightarrow \mathbb{R}$ such that

$$u_{tt} + u_{xxxx} + u_t + (p - \|u_x\|_{L^2(0,\ell)}^2)u_{xx} + k_*^2(u - w)^+ = f \tag{1}$$

$$\text{in } (0, \ell) \times (0, T), \tag{2}$$

$$w_{tt} - w_{xx} + w_t - k_*^2(u - w)^+ = g \text{ in } (0, \ell) \times (0, T), \tag{3}$$

$$u(0, t) = u(\ell, t) = u_x(0, t) = u_x(\ell, t) = 0 \text{ for a.e. } t \in (0, T), \tag{4}$$

$$w(0, t) = w(\ell, t) = 0 \text{ for a.e. } t \in (0, T), \tag{5}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) \text{ for a.e. } x \in (0, \ell), \tag{6}$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = e_0(x) \text{ for a.e. } x \in (0, \ell). \tag{7}$$

Here, the notation $(F)^+$ stands for the positive part of a function F , i.e. $(F)^+ = \max\{0, F\}$. Moreover, the term $-k_*^2(u - w)^+$ models a restoring force due to the cables, f is the given vertical dead load distribution on the deck, the constant p represents the axial force acting at the ends of the road bed in the reference configuration (being negative when the bridge is stretched and positive when compressed), and g denotes an external source applied in the cable. Finally, following [7] we assumed that all the constants in the model were equal to 1 for the sake of simplicity in the writing. However, in order to simplify the calculations in this work we note that we have modified slightly the boundary conditions employed in [7], where the ends of the vibrating beam were considered pinned, assuming instead that they are rigidly fixed now. The analysis performed there could be adapted with some minor changes.

In order to obtain the variational formulation of Problem P, let $Y = L^2(0, \ell)$ and denote by (\cdot, \cdot) the scalar product in this space, with corresponding norm $\|\cdot\|$. Moreover, let us define the variational spaces V and E as follows,

$$V = \{v \in H^2(0, \ell); v(0) = v(\ell) = 0 \text{ and } v_x(0) = v_x(\ell) = 0\},$$

$$E = \{e \in H^1(0, \ell); e(0) = e(\ell) = 0\},$$

with scalar product $(\cdot, \cdot)_V$ (resp. $(\cdot, \cdot)_E$) and norm $\|\cdot\|_V$ (resp. $\|\cdot\|_E$) defined in V (resp. E).

By using the integration by parts and the Dirichlet boundary conditions at $x = 0, \ell$, we write the variational formulation of Problem P in terms of the bending velocity $v = u_t$ and the vertical velocity $e = w_t$.

Problem VP. Find the bending velocity $v : [0, T] \rightarrow V$ and the vertical velocity $e : [0, T] \rightarrow E$ such that $v(0) = v_0, e(0) = e_0$ and, for a.e. $t \in (0, T)$,

$$(v_t(t), \xi) + (u_{xx}(t), \xi_{xx}) + (v(t), \xi) + \|u_x(t)\|^2(u_x(t), \xi_x) + p(u_{xx}(t), \xi) + k_*^2((u(t) - w(t))^+, \xi) = (f(t), \xi) \quad \forall \xi \in V, \tag{8}$$

$$(e_t(t), \psi) + (w_x(t), \psi_x) + (e(t), \psi) - k_*^2((u(t) - w(t))^+, \psi) = (g(t), \psi) \quad \forall \psi \in E, \tag{9}$$

where the bending displacement and the vertical displacement are then recovered from the relations:

$$u(t) = \int_0^t v(s) ds + u_0, \quad w(t) = \int_0^t e(s) ds + w_0. \tag{10}$$

The following theorem has been proved in [7], providing the existence of a unique solution to Problem VP as well as an energy decay property.

Theorem 2.1. *Let $f, g \in C([0, T]; Y)$ and assume that the initial data have the regularity*

$$u_0 \in H^2(0, \ell), \quad v_0 \in Y, \quad w_0 \in E, \quad e_0 \in Y.$$

Then, there exists a unique solution to Problem VP with the following regularity:

$$\begin{aligned} u &\in C([0, T]; H^2(0, \ell)), & v &\in C([0, T]; Y), \\ w &\in C([0, T]; H^1(0, \ell)), & e &\in C([0, T]; Y). \end{aligned}$$

Moreover, this solution is continuously dependent on the given data.

In addition, if we assume that $f = g = 0$ (i.e. there are not volume forces) and that the axial force p is small enough, denoting by $\mathcal{E}(t)$ the energy of the system given by

$$\mathcal{E}(t) = \|u(t)\|_V^2 + \|v(t)\|^2 + \|w(t)\|_E^2 + \|e(t)\|^2,$$

then it decays exponentially; i.e. there exist two positive constants Q and ω such that

$$\mathcal{E}(t) \leq Q e^{-\omega t}, \quad t \geq 0.$$

3. Numerical analysis: Fully discrete approximations and a priori error estimates. In this section we consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the interval $[0, \ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \dots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$ and so, to approximate the variational spaces V and E , we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$\begin{aligned} V^h &= \{\xi^h \in C^1([0, \ell]) ; \xi^h_{|[a_i, a_{i+1}]} \in P_3([a_i, a_{i+1}]) \quad i = 0, \dots, M-1, \\ &\quad \xi^h(0) = \xi^h(\ell) = 0 \text{ and } \xi^h_x(0) = \xi^h_x(\ell) = 0\}, \end{aligned} \quad (11)$$

$$\begin{aligned} E^h &= \{\psi^h \in C([0, \ell]) ; \psi^h_{|[a_i, a_{i+1}]} \in P_1([a_i, a_{i+1}]) \quad i = 0, \dots, M-1, \\ &\quad \psi^h(0) = \psi^h(\ell) = 0\}, \end{aligned} \quad (12)$$

where $P_r([a_i, a_{i+1}])$ represents the space of polynomials of degree less or equal to r in the subinterval $[a_i, a_{i+1}]$; i.e. V^h is composed of C^1 and piecewise cubic functions and E^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_0^h, v_0^h, w_0^h and e_0^h , are given by

$$u_0^h = \mathcal{P}_1^h u_0, \quad v_0^h = \mathcal{P}_1^h v_0, \quad w_0^h = \mathcal{P}_2^h w_0, \quad e_0^h = \mathcal{P}_2^h e_0. \quad (13)$$

\mathcal{P}_1^h and \mathcal{P}_2^h being the classical finite element interpolation operators over V^h and E^h , respectively (see [9]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we denote by $\delta w_n = (w_n - w_{n-1})/k$ its divided differences.

Therefore, using the implicit Euler scheme, assuming that $f, g \in C([0, T]; Y)$ the fully discrete approximation of Problem VP is as follows.

Problem VP^{hk}. Find the discrete bending velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete vertical velocity $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset E^h$ such that $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$ and, for $n = 1, \dots, N$,

$$(\delta v_n^{hk}, \xi^h) + ((u_n^{hk})_{xx}, \xi_{xx}^h) + (v_n^{hk}, \xi^h) + \|(u_n^{hk})_x\|^2((u_n^{hk})_x, \xi_x^h) + p((u_n^{hk})_{xx}, \xi^h) + k_*^2((u_n^{hk} - w_n^{hk})^+, \xi^h) = (f_n, \xi^h) \quad \forall \xi^h \in V^h, \tag{14}$$

$$(\delta e_n^{hk}, \psi^h) + ((w_n^{hk})_x, \psi_x^h) + (e_n^{hk}, \psi^h) - k_*^2((u_n^{hk} - w_n^{hk})^+, \psi^h) = (g_n, \psi^h) \quad \forall \psi^h \in E^h, \tag{15}$$

where the discrete bending displacement and the discrete vertical displacement are then recovered from the relations:

$$u_n^{hk} = k \sum_{j=1}^n v_j^{hk} + u_0^h, \quad w_n^{hk} = k \sum_{j=1}^n e_j^{hk} + w_0^h. \tag{16}$$

The existence of a unique solution to Problem VP^{hk} can be obtained proceeding as in the continuous case, applying a standard Faedo-Galerkin approximation procedure (see, for instance, [6, 17]).

The aim of this section is to obtain some a priori error estimates on the numerical errors $u_n - u_n^{hk}$, $v_n - v_n^{hk}$, $w_n - w_n^{hk}$ and $e_n - e_n^{hk}$.

Now, we have the following discrete stability property.

Lemma 3.1. *Let the assumptions of Theorem 2.1 hold. Then, the numerical sequences $\{u^{hk}, v^{hk}, w^{hk}, e^{hk}\}$, generated by Problem VP^{hk}, satisfy the stability estimate:*

$$\|v_n^{hk}\|^2 + \|u_n^{hk}\|_V^2 + \|(u_n^{hk})_x\|^4 + \|e_n^{hk}\|^2 + \|w_n^{hk}\|_E^2 \leq C,$$

where C is a positive constant which is independent of the discretization parameters h and k .

Proof. In order to simplify the writing of this proof, we remove the superscripts h and k in all the variables and we assume that $f = g = 0$.

Taking as a test function $\xi^h = v_n$ in equation (14) we have

$$(\delta v_n, v_n) + ((u_n)_{xx}, (v_n)_{xx}) + (v_n, v_n) + \|(u_n)_x\|^2((u_n)_x, (v_n)_x) + p((u_n)_{xx}, v_n) + k_*^2((u_n - w_n)^+, v_n) = 0.$$

Now, keeping in mind that

$$\begin{aligned} (\delta v_n, v_n) &\geq \frac{1}{2k} \{ \|v_n\|^2 - \|v_{n-1}\|^2 \}, \\ k_*^2((u_n - w_n)^+, v_n) &\leq k_*^2(\|u_n\|^2 + \|w_n\|^2 + \|v_n\|^2), \\ \|(u_n)_x\|^2((u_n)_x, (v_n)_x) &= \frac{1}{k} \|(u_n)_x\|^2((u_n)_x, (u_n - u_{n-1})_x) \\ &\geq \frac{1}{4k} \{ \|(u_n)_x\|^4 - \|(u_{n-1})_x\|^4 \}, \\ p((u_n)_{xx}, v_n) &\leq \frac{|p|}{2} (\|(u_n)_{xx}\|^2 + \|v_n\|^2), \\ ((u_n)_{xx}, (v_n)_{xx}) &\geq \frac{1}{2k} \{ \|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 \}, \end{aligned}$$

where we used Cauchy-Schwarz and Young inequalities, it follows that

$$\begin{aligned} &\frac{1}{2k} \{ \|v_n\|^2 - \|v_{n-1}\|^2 \} + \frac{1}{4k} \{ \|(u_n)_x\|^4 - \|(u_{n-1})_x\|^4 \} \\ &\quad + \frac{1}{2k} \{ \|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 \} \\ &\leq C(\|u_n\|^2 + \|w_n\|^2 + \|v_n\|^2 + \|(u_n)_{xx}\|^2). \end{aligned} \tag{17}$$

Now, taking $\psi^h = e_n$ as a test function in equation (15) we find that

$$(\delta e_n, e_n) + ((w_n)_x, (e_n)_x) + (e_n, e_n) - k_*^2((u_n - w_n)^+, e_n) = 0,$$

and so, taking into account that

$$\begin{aligned} (\delta e_n, e_n) &\geq \frac{1}{2k} \{ \|e_n\|^2 - \|e_{n-1}\|^2 \}, \\ ((w_n)_x, (e_n)_x) &\geq \frac{1}{2k} \{ \|(w_n)_x\|^2 - \|(w_{n-1})_x\|^2 \}, \\ k_*^2((u_n - w_n)^+, e_n) &\leq k_*^2(\|u_n\|^2 + \|w_n\|^2 + \|e_n\|^2), \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{2k} \{ \|e_n\|^2 - \|e_{n-1}\|^2 \} + \frac{1}{2k} \{ \|(w_n)_x\|^2 - \|(w_{n-1})_x\|^2 \} \\ \leq C(\|u_n\|^2 + \|w_n\|^2 + \|e_n\|^2). \end{aligned} \quad (18)$$

Combining now (17) and (18) we can conclude that

$$\begin{aligned} \frac{1}{2k} \{ \|v_n\|^2 - \|v_{n-1}\|^2 \} + \frac{1}{4k} \{ \|(u_n)_x\|^4 - \|(u_{n-1})_x\|^4 \} \\ + \frac{1}{2k} \{ \|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 \} + \frac{1}{2k} \{ \|e_n\|^2 - \|e_{n-1}\|^2 \} \\ + \frac{1}{2k} \{ \|(w_n)_x\|^2 - \|(w_{n-1})_x\|^2 \} \\ \leq C(\|u_n\|^2 + \|w_n\|^2 + \|v_n\|^2 + \|e_n\|^2 + \|(u_n)_{xx}\|^2). \end{aligned}$$

Thus, by induction and using Poincaré's inequality we find that

$$\begin{aligned} \|v_n\|^2 + \|(u_n)_x\|^4 + \|u_n\|_V^2 + \|e_n\|^2 + \|w_n\|_E^2 \\ \leq Ck \sum_{j=1}^n (\|u_j\|_V^2 + \|v_j\|^2 + \|w_j\|_E^2 + \|e_j\|^2) \\ + C(\|v_0\|^2 + \|u_0\|_V^2 + \|e_0\|^2 + \|w_0\|_E^2), \end{aligned}$$

and, applying a discrete version of Gronwall's inequality (see, for instance, [8]) we find the desired stability estimates. \square

Remark 1. We note that we could modify Problem VP^{hk} by using a semi-implicit scheme in the following form:

Find the discrete bending velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete vertical velocity $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset E^h$ such that $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$ and, for $n = 1, \dots, N$,

$$\begin{aligned} (\delta v_n^{hk}, \xi^h) + ((u_n^{hk})_{xx}, \xi_{xx}^h) + (v_n^{hk}, \xi^h) + \|(u_{n-1}^{hk})_x\|^2 ((u_n^{hk})_x, \xi_x^h) + p((u_n^{hk})_{xx}, \xi^h) \\ + k_*^2((u_{n-1}^{hk} - w_{n-1}^{hk})^+, \xi^h) = (f_n, \xi^h) \quad \forall \xi^h \in V^h, \\ (\delta e_n^{hk}, \psi^h) + ((w_n^{hk})_x, \psi_x^h) + (e_n^{hk}, \psi^h) - k_*^2((u_{n-1}^{hk} - w_{n-1}^{hk})^+, \psi^h) \\ = (g_n, \psi^h) \quad \forall \psi^h \in E^h, \end{aligned}$$

where the discrete bending displacement and the discrete vertical displacement are again recovered from relations (16).

This new problem is now uncoupled and linear and so, the numerical resolution is easier. However, Lemma 3.1 should be modified and the resulting estimates will vary accordingly.

Now, we are able to prove the following a priori error estimates result.

Theorem 3.2. *Let the assumptions of Theorem 2.1 still hold and denote by (v, e) and (v^{hk}, e^{hk}) the solutions to problems VP and VP^{hk} , respectively. Assume that the following additional regularity is satisfied:*

$$v \in C^1([0, T]; Y) \cap C([0, T]; V), \quad e \in C^1([0, T]; Y) \cap C([0, T]; E).$$

Then we have the following a priori error estimates, for all $\xi^h = \{\xi_j^h\}_{j=0}^N \subset V^h$ and $\psi^h = \{\psi_j^h\}_{j=0}^N \subset E^h$,

$$\begin{aligned} & \max_{0 \leq n \leq N} \left\{ \|v_n - v_n^{hk}\|^2 + \|e_n - e_n^{hk}\|^2 + \|u_n - u_n^{hk}\|_V^2 + \|w_n - w_n^{hk}\|_E^2 \right\} \\ & \leq Ck \sum_{j=1}^N \left(\|(v_t)_j - \delta v_j\|^2 + \|v_j - \xi_j^h\|_V^2 + \|(u_t)_j - \delta u_j\|_V^2 + \|(e_t)_j - \delta e_j\|^2 \right. \\ & \quad \left. + \|e_j - \psi_j^h\|_E^2 + \|(w_t)_j - \delta w_j\|_E^2 \right) + C \left(\|v_0 - v_0^h\|^2 + \|e_0 - e_0^h\|^2 \right. \\ & \quad \left. + \|u_0 - u_0^h\|_V^2 + \|w_0 - w_0^h\|_E^2 \right) + \frac{C}{k} \sum_{j=1}^{N-1} \|v_j - \xi_j^h - (v_{j+1} - \xi_{j+1}^h)\|^2 \\ & \quad + \frac{C}{k} \sum_{j=1}^{N-1} \|e_j - \psi_j^h - (e_{j+1} - \psi_{j+1}^h)\|^2 + C \max_{0 \leq n \leq N} \|v_n - \xi_n^h\|^2 \\ & \quad + C \max_{0 \leq n \leq N} \|e_n - \psi_n^h\|^2. \end{aligned} \tag{19}$$

Proof. Taking as a test function $\xi = \xi^h \in V^h \subset V$ in equation (8) at time $t = t_n$ and subtracting it to equation (14) we have

$$\begin{aligned} & ((v_t)_n - \delta v_n^{hk}, \xi^h) + ((u_n - u_n^{hk})_{xx}, (\xi^h)_{xx}) + (v_n - v_n^{hk}, \xi^h) \\ & \quad + (\|(u_n)_x\|^2 (u_n)_x - \|(u_n^{hk})_x\|^2 (u_n^{hk})_x, (\xi^h)_x) + p((u_n - u_n^{hk})_{xx}, \xi^h) \\ & \quad + k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, \xi^h) = 0, \end{aligned}$$

and therefore,

$$\begin{aligned} & ((v_t)_n - \delta v_n^{hk}, v_n - v_n^{hk}) + ((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) + (v_n - v_n^{hk}, v_n - v_n^{hk}) \\ & \quad + (\|(u_n)_x\|^2 (u_n)_x - \|(u_n^{hk})_x\|^2 (u_n^{hk})_x, (v_n - v_n^{hk})_x) + p((u_n - u_n^{hk})_{xx}, v_n - v_n^{hk}) \\ & \quad + k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, v_n - v_n^{hk}) \\ & = ((v_t)_n - \delta v_n^{hk}, v_n - \xi^h) + ((u_n - u_n^{hk})_{xx}, (v_n - \xi^h)_{xx}) + (v_n - v_n^{hk}, v_n - \xi^h) \\ & \quad + (\|(u_n)_x\|^2 (u_n)_x - \|(u_n^{hk})_x\|^2 (u_n^{hk})_x, (v_n - \xi^h)_x) + p((u_n - u_n^{hk})_{xx}, v_n - \xi^h) \\ & \quad + k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, v_n - \xi^h) \quad \forall \xi^h \in V^h. \end{aligned}$$

Taking into account that

$$\begin{aligned} & (\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}) \geq \frac{1}{2k} \{ \|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2 \}, \\ & ((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) = ((u_n - u_n^{hk})_{xx}, ((u_t)_n - \delta u_n)_{xx}) \\ & \quad + ((u_n - u_n^{hk})_{xx}, (\delta u_n - \delta u_n^{hk})_{xx}) \\ & \geq ((u_n - u_n^{hk})_{xx}, ((u_t)_n - \delta u_n)_{xx}) \\ & \quad + \frac{1}{2k} \{ \|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \}, \\ & (\|(u_n)_x\|^2 (u_n)_{xx} - \|(u_n^{hk})_x\|^2 (u_n^{hk})_{xx}, w) = (\|(u_n)_x\|^2 - \|(u_n^{hk})_x\|^2) (u_n)_{xx}, w) \\ & \quad + (\|(u_n^{hk})_x\|^2 (u_n - u_n^{hk})_{xx}, w), \\ & |(\|(u_n^{hk})_x\|^2 (u_n - u_n^{hk})_{xx}, w)| \leq C(\|(u_n - u_n^{hk})_{xx}\|^2 + \|w\|^2), \end{aligned}$$

where we used Lemma 3.1 and the notation $\delta u_n = (u_n - u_{n-1})/k$, and the following estimates

$$|(\|(u_n)_x\|^2 - \|(u_n^{hk})_x\|^2) (u_n)_{xx}, w)| \leq \|(\|(u_n)_x\|^2 - \|(u_n^{hk})_x\|^2)\| \|(u_n)_{xx}\| \|w\|$$

$$\begin{aligned}
&\leq C\|w\|^2 + C\left(\|(u_n)_x\|^2 - \|(u_n^{hk})_x\|^2\right)^2, \\
\|(u_n)_x\|^2 - \|(u_n^{hk})_x\|^2 &= \int_0^\ell (u_n)_x^2 - (u_n^{hk})_x^2 dx \\
&= \int_0^\ell (u_n + u_n^{hk})_x (u_n - u_n^{hk})_x dx \\
&\leq \|(u_n + u_n^{hk})_x\| \|(u_n - u_n^{hk})_x\| \leq C\|(u_n - u_n^{hk})_x\|,
\end{aligned}$$

where the regularity $u \in C([0, T]; V)$ and Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R}, \epsilon > 0,$$

have been used, we find that

$$\begin{aligned}
&\frac{1}{2k} \left\{ \|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2 \right\} \\
&\quad + \frac{1}{2k} \left\{ \|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \right\} \\
&\leq C \left(\|(v_t)_n - \delta v_n\|^2 + \|v_n - \xi^h\|^2 + \|(v_n - \xi^h)_{xx}\|^2 + \|(u_t)_n - \delta u_n\|_{xx}^2 \right. \\
&\quad \left. + \|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|^2 + \|w_n - w_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 \right. \\
&\quad \left. + \|(u_n - u_n^{hk})_{xx}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - \xi^h) \right) \quad \forall \xi^h \in V^h. \quad (20)
\end{aligned}$$

Now, taking as a test function $\psi = \psi^h \in E^h \subset E$ in equation (9) at time $t = t_n$ and subtracting it from equation (15) we have

$$\begin{aligned}
&((e_t)_n - \delta e_n^{hk}, \psi^h) + ((w_n - w_n^{hk})_x, \psi_x^h) - k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, \psi^h) \\
&\quad + (e_n - e_n^{hk}, \psi^h) = 0 \quad \forall \psi^h \in E^h,
\end{aligned}$$

and so, we obtain, for all $\psi^h \in E^h$,

$$\begin{aligned}
&((e_t)_n - \delta e_n^{hk}, e_n - e_n^{hk}) + ((w_n - w_n^{hk})_x, (e_n - e_n^{hk})_x) + (e_n - e_n^{hk}, e_n - e_n^{hk}) \\
&\quad - k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, e_n - e_n^{hk}) \\
&= ((e_t)_n - \delta e_n^{hk}, e_n - \psi^h) + ((w_n - w_n^{hk})_x, (e_n - \psi^h)_x) + (e_n - e_n^{hk}, e_n - \psi^h) \\
&\quad - k_*^2((u_n - w_n)^+ - (u_n^{hk} - w_n^{hk})^+, e_n - \psi^h).
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&(\delta e_n - \delta e_n^{hk}, e_n - e_n^{hk}) \geq \frac{1}{2k} \left\{ \|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right\}, \\
&((w_n - w_n^{hk})_x, (e_n - e_n^{hk})_x) = ((w_n - w_n^{hk})_x, ((w_t)_n - \delta w_n)_x) \\
&\quad + ((w_n - w_n^{hk})_x, (\delta w_n - \delta w_n^{hk})_x) \\
&\geq ((w_n - w_n^{hk})_x, ((w_t)_n - \delta w_n)_x) + \frac{1}{2k} \left\{ \|(w_n - w_n^{hk})_x\|^2 - \|(w_{n-1} - w_{n-1}^{hk})_x\|^2 \right\},
\end{aligned}$$

where $\delta w_n = (w_n - w_{n-1})/k$, using several times the above Cauchy's inequality it follows that

$$\begin{aligned}
&\frac{1}{2k} \left\{ \|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right\} \\
&\quad + \frac{1}{2k} \left\{ \|(w_n - w_n^{hk})_x\|^2 - \|(w_{n-1} - w_{n-1}^{hk})_x\|^2 \right\} \\
&\leq C \left(\|(e_t)_n - \delta e_n\|^2 + \|e_n - e_n^{hk}\|^2 + \|w_n - w_n^{hk}\|^2 + \|u_n - u_n^{hk}\|^2 \right. \\
&\quad \left. + \|e_n - \psi^h\|^2 + \|(e_n - \psi^h)_x\|^2 + \|(w_t)_n - \delta w_n\|_x^2 \right. \\
&\quad \left. + (\delta e_n - \delta e_n^{hk}, e_n - \psi^h) \right) \quad \forall \psi^h \in E^h. \quad (21)
\end{aligned}$$

Combining estimates (20) and (21) we have, for all $\xi^h \in V^h$ and $\psi^h \in E^h$,

$$\begin{aligned} & \frac{1}{2k} \{ \|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2 \} + \frac{1}{2k} \{ \|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \} \\ & + \frac{1}{2k} \{ \|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \} \\ & + \frac{1}{2k} \{ \|(w_n - w_n^{hk})_x\|^2 - \|(w_{n-1} - w_{n-1}^{hk})_x\|^2 \} \\ & \leq C \left(\|(v_t)_n - \delta v_n\|^2 + \|v_n - \xi^h\|^2 + \|(v_n - \xi^h)_{xx}\|^2 + \|((u_t)_n - \delta u_n)_{xx}\|^2 \right. \\ & \quad + \|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|^2 + \|w_n - w_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 \\ & \quad + \|(u_n - u_n^{hk})_{xx}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - \xi^h) + \|(e_t)_n - \delta e_n\|^2 \\ & \quad + \|e_n - e_n^{hk}\|^2 + \|e_n - \psi^h\|^2 + \|(e_n - \psi^h)_x\|^2 + \|((w_t)_n - \delta w_n)_x\|^2 \\ & \quad \left. + (\delta e_n - \delta e_n^{hk}, e_n - \psi^h) \right). \end{aligned}$$

Thus, by induction we obtain, for all $\xi^h = \{\xi_j^h\}_{j=0}^N \subset V^h$ and $\psi^h = \{\psi_j^h\}_{j=0}^N \subset E^h$,

$$\begin{aligned} & \|v_n - v_n^{hk}\|^2 + \|e_n - e_n^{hk}\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 + \|(w_n - w_n^{hk})_x\|^2 \\ & \leq Ck \sum_{j=1}^n \left(\|(v_t)_j - \delta v_j\|^2 + \|v_j - \xi_j^h\|^2 + \|(v_j - \xi_j^h)_{xx}\|^2 + \|((u_t)_j - \delta u_j)_{xx}\|^2 \right. \\ & \quad + \|v_j - v_j^{hk}\|^2 + \|u_j - u_j^{hk}\|^2 + \|w_j - w_j^{hk}\|^2 + \|(u_j - u_j^{hk})_x\|^2 \\ & \quad + \|(u_j - u_j^{hk})_{xx}\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - \xi_j^h) + \|(e_t)_j - \delta e_j\|^2 \\ & \quad + \|e_j - e_j^{hk}\|^2 + \|e_j - \psi_j^h\|^2 + \|(e_j - \psi_j^h)_x\|^2 + \|((w_t)_j - \delta w_j)_x\|^2 \\ & \quad \left. + (\delta e_j - \delta e_j^{hk}, e_j - \psi_j^h) \right) + C \left(\|v_0 - v_0^h\|^2 + \|e_0 - e_0^h\|^2 \right. \\ & \quad \left. + \|(u_0 - u_0^h)_{xx}\|^2 + \|(w_0 - w_0^h)_x\|^2 \right). \end{aligned}$$

Finally, taking into account that

$$\begin{aligned} & k \sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - \xi_j^h) \\ & = \sum_{j=1}^n (v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - \xi_j^h) \\ & = (v_n - v_n^{hk}, v_n - \xi_n^h) + (v_0^h - v_0, v_1 - \xi_1^h) \\ & \quad + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - \xi_j^h - (v_{j+1} - \xi_{j+1}^h)), \\ & k \sum_{j=1}^n (\delta e_j - \delta e_j^{hk}, e_j - \psi_j^h) \\ & = \sum_{j=1}^n (e_j - e_j^{hk} - (e_{j-1} - e_{j-1}^{hk}), e_j - \psi_j^h) \\ & = (e_n - e_n^{hk}, e_n - \psi_n^h) + (e_0^h - e_0, e_1 - \psi_1^h) \\ & \quad + \sum_{j=1}^{n-1} (e_j - e_j^{hk}, e_j - \psi_j^h - (e_{j+1} - \psi_{j+1}^h)), \end{aligned}$$

applying a discrete version of Gronwall's inequality (see again [8]) and Poincaré's inequality, we conclude estimates (19). \square

We note that estimates (19) can be used to perform an error analysis. Thus, as an example, assume the following additional regularity on the continuous solution:

$$\begin{aligned} u &\in H^3(0, T; Y) \cap C^1([0, T]; H^3(0, \ell)) \cap H^2(0, T; V), \\ w &\in H^3(0, T; Y) \cap C^1([0, T]; H^2(0, \ell)) \cap H^2(0, T; E). \end{aligned} \quad (22)$$

We note that with this regularity we immediately find that (see [9]):

$$\|v_0 - v_0^h\|^2 + \|e_0 - e_0^h\|^2 + \|u_0 - u_0^h\|_V^2 + \|w_0 - w_0^h\|_E^2 \leq Ch^2.$$

Therefore, we have the following.

Corollary 1. *Let the assumptions of Theorem 3.2 and the additional regularity (22) hold. Then, the linear convergence of the algorithm is deduced; i.e. there exists a positive constant C , independent of the discretization parameters h and k , such that*

$$\max_{0 \leq n \leq N} \left\{ \|v_n - v_n^{hk}\| + \|e_n - e_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|w_n - w_n^{hk}\|_E \right\} \leq C(h + k).$$

The proof of the above result is done in several steps. First, we have the following property of approximation by finite elements (see, e.g., [9]):

$$\begin{aligned} k \sum_{j=1}^N \left(\inf_{\xi_j^h \in V^h} \|v_j - \xi_j^h\|_V^2 + \inf_{\psi_j^h \in E^h} \|e_j - \psi_j^h\|_E^2 \right) + \max_{0 \leq n \leq N} \inf_{\xi_n^h \in V^h} \|v_n - \xi_n^h\|^2 \\ + \max_{0 \leq n \leq N} \inf_{\psi_n^h \in E^h} \|e_n - \psi_n^h\|^2 \leq Ch^2 \left(\|u\|_{C^1([0, T]; H^3(0, \ell))}^2 + \|w\|_{C^1([0, T]; H^2(0, \ell))}^2 \right). \end{aligned}$$

Moreover, using again regularity conditions (22), we have

$$\begin{aligned} k \sum_{j=1}^N \left[\|(u_t)_j - \delta u_j\|_V^2 + \|(w_t)_j - \delta w_j\|_E^2 + \|(v_t)_j - \delta v_j\|^2 + \|(e_t)_j - \delta e_j\|^2 \right] \\ \leq Ck^2 (\|u\|_{H^2(0, T; V)}^2 + \|u\|_{H^3(0, T; Y)}^2 + \|w\|_{H^2(0, T; E)}^2 + \|w\|_{H^3(0, T; Y)}^2). \end{aligned}$$

Finally, the remaining terms in estimates (19) can be bounded as follows (see [8] for details),

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^{N-1} \|v_j - \xi_j^h - (v_{j+1} - \xi_{j+1}^h)\|^2 + \frac{1}{k} \sum_{j=1}^{N-1} \|e_j - \psi_j^h - (e_{j+1} - \psi_{j+1}^h)\|^2 \\ \leq Ch^2 \left(\|u\|_{H^2(0, T; V)}^2 + \|w\|_{H^2(0, T; E)}^2 \right). \end{aligned}$$

Combining all these estimates, the linear convergence is deduced.

4. Numerical results. In order to verify the behaviour of the numerical method described in the previous section, some numerical experiments have been performed. Moreover, for the sake of generality in the simulations, we have included coefficients in the definition of the mechanical problem P.

4.1. Numerical scheme. Given the solution $u_{n-1}^{hk}, v_{n-1}^{hk}, w_{n-1}^{hk}$ and e_{n-1}^{hk} at time t_{n-1} , the discrete bending velocity is obtained from the discrete nonlinear variational equation

$$\begin{aligned} \rho_1(v_n^{hk}, \xi^h) + k^2 a_1((v_n^{hk})_{xx}, \xi_{xx}^h) + k c_1(v_n^{hk}, \xi^h) \\ + k^2 r_1((u_n^{hk})_x)^2((u_{n-1}^{hk} + k v_n^{hk})_x, \xi_x^h) \\ - k^2 p((v_n^{hk})_x, (\xi^h)_x) + k k_*^2((u_n^{hk} - w_n^{hk})^+, \xi^h) = \rho_1(v_{n-1}^{hk}, \xi^h) + k(f_n, \xi^h) \\ - k a_1((u_{n-1}^{hk})_{xx}, \xi_{xx}^h) + k p((u_{n-1}^{hk})_x, (\xi^h)_x). \end{aligned}$$

Later, we get the vertical displacement from the discrete nonlinear variational equation

$$\begin{aligned} &\rho_2(e_n^{hk}, \psi^h) + k^2 a_2((e_n^{hk})_x, \psi_x^h) + c_2 k(e_n^{hk}, \psi^h) - k k_*^2((u_n^{hk} - w_n^{hk})^+, \psi^h) \\ &= \rho_2(e_{n-1}^{hk}, \psi^h) + k(g_n, \psi^h) - k a_2((w_{n-1}^{hk})_x, \psi_x^h), \end{aligned}$$

where the discrete bending displacement and the discrete vertical displacement are then recovered from the relations

$$u_n^{hk} = k \sum_{j=1}^n v_j^{hk} + u_0^h, \quad w_n^{hk} = k \sum_{j=1}^n e_j^{hk} + w_0^h.$$

We note that both numerical problems consist of nonlinear symmetric systems, and so a fixed point method was applied for their solution.

4.2. A first example: Numerical convergence. Our aim with this first example is to verify the numerical convergence of the numerical scheme. In this sense, the following problem is considered.

Problem P1. Find a bending velocity field $v : [0, 1] \times [0, 1] \mapsto \mathbb{R}$ and a vertical velocity $e : [0, 1] \times [0, 1] \mapsto \mathbb{R}$ such that

$$\begin{aligned} &\rho_1 u_{tt} + a_1 u_{xxxx} + c_1 u_t + (p - r_1 \|u_x\|_{L^2(0,\ell)}^2) u_{xx} + k_*^2 (u - w)^+ = f \\ &\text{in } (0, 1) \times (0, 1), \end{aligned}$$

$$\rho_2 w_{tt} - a_2 w_{xx} + c_2 w_t - k_*^2 (u - w)^+ = g \quad \text{in } (0, 1) \times (0, 1),$$

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0 \quad \text{for a.e. } t \in (0, 1),$$

$$w(0, t) = w(1, t) = 0 \quad \text{for a.e. } t \in (0, 1),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) \quad \text{for a.e. } x \in (0, 1),$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = e_0(x) \quad \text{for a.e. } x \in (0, 1).$$

where

$$\begin{aligned} f(x, t) &= e^t (-16x^4 + 32x^3 - 12x^2 - 4x - 384) + 16e^{2t} \frac{512}{105} (12x^2 - 12x + 2), \\ g(x, t) &= e^t (16x^4 - 32x^3 + 28x^2 - 12x - 8), \end{aligned}$$

which corresponds with Problem P with the following data:

$$\begin{aligned} &l = 1, \quad T = 1 \text{ s}, \quad \rho_1 = 1 = \rho_2, \quad a_1 = 1 = a_2, \quad c_1 = 1 = c_2, \quad r_1 = 1, \quad p = 0, \\ &k_*^2 = 1, \quad u_0(x) = -16x^2(1 - x)^2 = v_0(x) \quad \text{for all } x \in (0, 1), \\ &w_0(x) = -4x(1 - x) = e_0(x) \quad \text{for all } x \in (0, 1). \end{aligned}$$

The exact solution to Problem P1 is the following one.

$$u(x, t) = -16x^2(1 - x)^2 e^t, \quad w(x, t) = -4x(1 - x) e^t, \quad \text{for } (x, t) \in [0, 1] \times [0, 1].$$

The numerical errors given by

$$E^{hk} = \max_{0 \leq n \leq N} \left\{ \|v_n - v_n^{hk}\| + \|e_n - e_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|w_n - w_n^{hk}\|_E \right\},$$

and obtained for different discretization parameters nd and k , are depicted in Table 1 (being nd the number of finite elements of the discretization and $h = \frac{1}{nd+1}$ the spatial discretization parameter). Moreover, the evolution of the error depending on $h + k$ is plotted in Fig. 1. We observe that the linear convergence stated in Corollary 1 is achieved.

$nd \downarrow k \rightarrow$	10^{-1}	10^{-2}	10^{-3}	10^{-4}
10	0.1435116	0.0868544	0.0923330	0.0940042
10^2	0.1639226	0.0174114	0.0070235	0.0069553
10^3	0.1641941	0.0161108	0.0017435	0.0007232
10^4	0.1646557	0.0163375	0.0015935	0.0001722

TABLE 1. Example 1: Numerical errors for some discretization parameters.

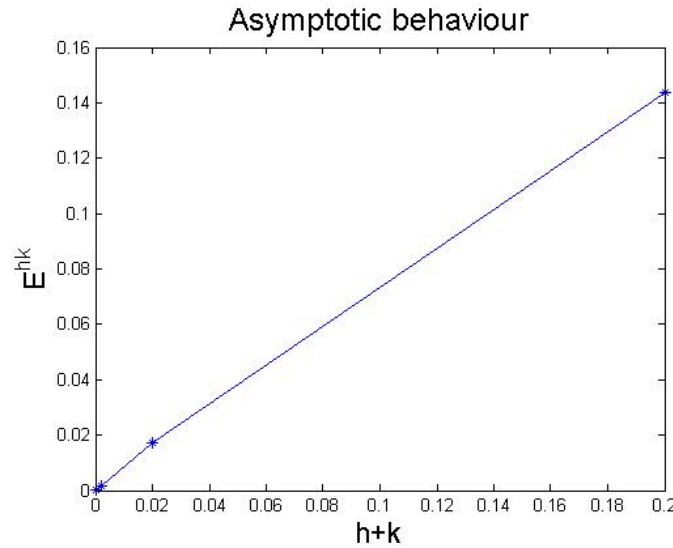


FIGURE 1. Example 1: Asymptotic behaviour of the numerical scheme

4.3. A second example: The effect of the axial force. As a second test several simulations with different values of the axial force p have been performed. With this example we try to show the effect of the axial force on a vibrating situation. Taking a similar forced deformed initial configuration as in the previous example, both forces, over the bridge and the cable, are released at instant $t = 0$, and the evolution of the bending displacements is studied.

The following data have been used in this example.

$$l = 1, \quad T = 1 \text{ s}, \quad \rho_1 = 1 = \rho_2, \quad a_1 = 1 = a_2, \quad c_1 = 10 = c_2,$$

$$r_1 = 1, \quad k_*^2 = 1, \quad f = g = 0,$$

$$u_0(x) = -16x^2(1-x)^2 = w_0(x), \quad v_0(x) = 0 = e_0(x), \quad \text{for all } x \in (0, 1).$$

Using the discretization parameters $k = h = 10^{-3}$, in Fig. 2 the evolution in time of the vertical displacement of the bridge center is shown for several values of the axial force p . As can be observed, the increment in the compression forces makes the oscillating response to be softer when the bridge is released.

4.4. A third example: The coupling term. As a final test we analyze the dependence of the model with respect to the parameter k_*^2 . Taking both the bridge and the cable in their reference configuration as initial conditions, a positive linearly increasing force acts over the bridge for one second.

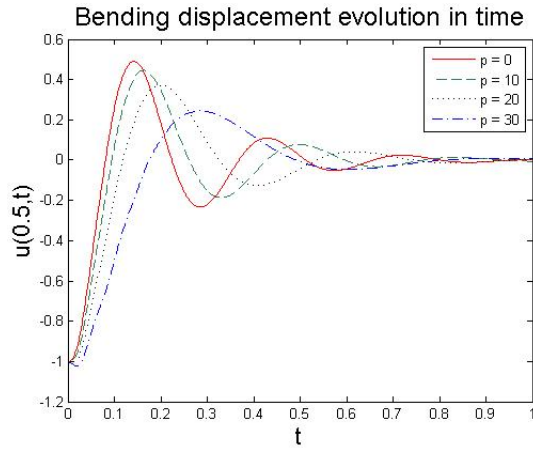


FIGURE 2. Example 2: Oscillations of the bridge for different values of p .

The following data have been used in the simulations:

$$\begin{aligned}
 l &= 1, \quad T = 1 \text{ s}, \quad \rho_1 = 1 = \rho_2, \quad a_1 = 1 = a_2, \quad c_1 = 10 = c_2, \quad r_1 = 1, \\
 p &= 10, \quad f(x, t) = 10t, \quad g(x, t) = 0, \quad \text{for } (x, t) \in [0, 1] \times [0, 1] \\
 u_0(x) &= 0, \quad v_0(x) = 0, \quad w_0(x) = 0, \quad e_0(x) = 0 \quad \text{for all } x \in (0, 1).
 \end{aligned}$$

Taking the discretization parameters $k = h = 10^{-3}$, running several simulations for different values of k_*^2 the effect of the coupling term is easily noticed: when this value raises, the cable restricts the deformation of the bridge and so the displacement decreases, while at the same time the cable deformation increases, as can be seen in Fig. 3.

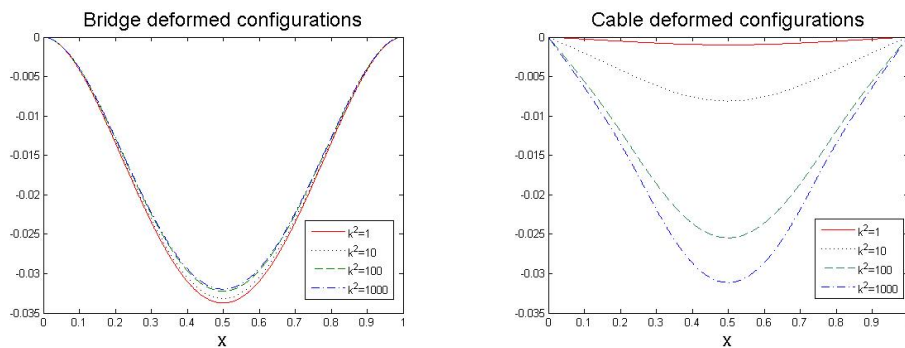


FIGURE 3. Example 3: Bridge and cable deformed configurations at final time for different values of k_*^2 .

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Received April 2018; Revised November 2018.

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