A low-order mixed finite element method for a class of quasi-Newtonian Stokes flows. Part II: a posteriori error analysis

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Abstract

This is the second part of a work dealing with a low-order mixed finite element method for a class of nonlinear Stokes models arising in quasi-Newtonian fluids. In the first part we showed that the resulting variational formulation is given by a twofold saddle point operator equation, and that the corresponding Galerkin scheme becomes well posed with piecewise constant functions and Raviart–Thomas spaces of lowest order as the associated finite element subspaces. In this paper we develop a Bank–Weiser type a posteriori error analysis yielding a reliable estimate and propose the corresponding adaptive algorithm to compute the mixed finite element solutions. Several numerical results illustrating the efficiency of the method are also provided.

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1. Introduction

We first recall from [5] the boundary value problem of interest. Indeed, let Ω be a bounded and simply connected domain in $\mathbf{R}^2$ with Lipschitz-continuous boundary Γ. Our goal is to determine the velocity $\mathbf{u} := (u_1, u_2)^t$ and the pressure p of a nonlinear Stokes fluid occupying the region Ω under the action of an external force. More precisely, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for $(\mathbf{u}, p)$ in appropriate spaces such that

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$$
\begin{aligned}
&-\mathbf{div}(\psi(|\nabla \mathbf{u}|)\nabla \mathbf{u} - p\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega, \\
&\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \text{ and } \mathbf{u} = \mathbf{g} \text{ on } \Gamma,
\end{aligned} \tag{1.1}
$$

where $\mathbf{div}$ and div are the usual vector and scalar divergence operators, $\nabla \mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $\mathbf{R}^2$, $\mathbf{I}$ is the identity matrix of $\mathbf{R}^{2\times 2}$, and $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is the nonlinear kinematic viscosity function of the fluid. We remark that $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$ must satisfy the compatibility condition $\int_\Gamma \mathbf{g} \cdot \boldsymbol{v} \, \mathrm{d}s = 0$, where $\boldsymbol{v}$ is the unit outward normal to Γ.

We now let $\boldsymbol{\psi} : \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ be the tensor defined by $\boldsymbol{\psi}(\mathbf{r}) := (\psi(|\mathbf{r}|)r_{ij})$ for all $\mathbf{r} \in \mathbf{R}^{2\times 2}$. Then, the mixed variational formulation of (1.1), as deduced in [5], which introduces $\boldsymbol{\sigma} := \boldsymbol{\psi}(\nabla \mathbf{u}) - p\mathbf{I}$ and $\mathbf{t} := \nabla \mathbf{u}$ as further unknowns, reads as follows: Find $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \xi)) \in X_1 \times M_1 \times M$ such that

$$
\begin{aligned}
&[\mathbf{A}_1(\mathbf{t}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{s}), (\boldsymbol{\sigma}, p)] = 0, \\
&[\mathbf{B}_1(\mathbf{t}), (\boldsymbol{\tau}, q)] + [\mathbf{B}(\boldsymbol{\tau}, q), (\mathbf{u}, \xi)] = [\mathbf{G}, (\boldsymbol{\tau}, q)], \\
&[\mathbf{B}(\boldsymbol{\sigma}, p), (\mathbf{v}, \eta)] = [\mathbf{F}, (\mathbf{v}, \eta)],
\end{aligned} \tag{1.2}
$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \eta)) \in X_1 \times M_1 \times M$, where $X_1 := [L^2(\Omega)]^{2\times 2}$, $M_1 := H(\mathbf{div}; \Omega) \times L^2(\Omega)$, $M := [L^2(\Omega)]^2 \times \mathbf{R}$, and the operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, and $\mathbf{B} : M_1 \to M'$, and the functionals $(\mathbf{G}, \mathbf{F}) \in M_1' \times M'$, are defined as follows:

$$
[\mathbf{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \boldsymbol{\psi}(\mathbf{r}) : \mathbf{s} \, \mathrm{d}x, \quad [\mathbf{B}_1(\mathbf{r}), (\boldsymbol{\tau}, q)] := -\int_\Omega \boldsymbol{\tau} : \mathbf{r} \, \mathrm{d}x - \int_\Omega q \operatorname{tr}(\mathbf{r}) \, \mathrm{d}x, \tag{1.3}
$$

$$
[\mathbf{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \eta)] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau} \, \mathrm{d}x + \eta \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \, \mathrm{d}x, \tag{1.4}
$$

$$
[\mathbf{G}, (\boldsymbol{\tau}, q)] := -\langle \boldsymbol{\tau}\boldsymbol{v}, \mathbf{g} \rangle_\Gamma \quad \text{and} \quad [\mathbf{F}, (\mathbf{v}, \eta)] := \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x, \tag{1.5}
$$

for all $\mathbf{r}, \mathbf{s} \in X_1$, $(\boldsymbol{\tau}, q) \in M_1$, and $(\mathbf{v}, \eta) \in M$.

Hereafter, $[\cdot, \cdot]$ stands for the duality pairing induced by the corresponding operators and functionals, $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing of $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product, and $H(\mathbf{div}; \Omega)$ is the space of tensors $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ satisfying $\mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2$. It is well known that $H(\mathbf{div}; \Omega)$, provided with the inner product $\langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\, \boldsymbol{\zeta}, \mathbf{div}\, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^2}$, is a Hilbert space, where $\langle \cdot, \cdot \rangle_{[L^2(\Omega)]^{2\times 2}}$ and $\langle \cdot, \cdot \rangle_{[L^2(\Omega)]^2}$ stand for the usual inner products of $[L^2(\Omega)]^{2\times 2}$ and $[L^2(\Omega)]^2$, respectively. The other notations to be used in this paper are the same as those employed in [5].

In order to define the corresponding mixed finite element scheme, we now assume for simplicity that Γ is a polygonal curve, and let $\{\mathscr{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ by triangles T of diameter h_T such that $h := \max\{h_T : T \in \mathscr{T}_h\}$ and $\overline{\Omega} = \cup\{T : T \in \mathscr{T}_h\}$. For each $T \in \mathscr{T}_h$ we let $\mathrm{RT}_0(T)$ be the local Raviart–Thomas space of order zero, that is $\mathrm{RT}_0(T) := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$, where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbf{R}^2$. In addition, given a nonnegative integer k and a subset $\mathscr{S}$ of $\mathbf{R}^2$, we let $\mathbf{P}_k(\mathscr{S})$ be the space of polynomials defined on $\mathscr{S}$ of degree $\leqslant k$.

Then we introduce the following finite element subspaces:

$$
X_{1,h} := \left\{ \mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \mathbf{s}|_T \in [\mathbf{P}_0(T)]^{2\times 2} \, \forall T \in \mathscr{T}_h \right\},
$$

$$
M_{1,h}^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau} := (\tau_{ij}) \in H(\mathbf{div}; \Omega) : (\tau_{i1}\tau_{i2})^t|_T \in \mathrm{RT}_0(T) \, \forall i \in \{1, 2\}, \forall T \in \mathscr{T}_h \right\},
$$

$$M_{1,h}^{p} := \{q \in L^2(\Omega) : q|_T \in \mathbf{P}_0(T)\ \forall T \in \mathcal{T}_h\},$$

$$M_{1,h} := M_{1,h}^{\boldsymbol{\sigma}} \times M_{1,h}^{p},$$

$$M_{h}^{\mathbf{u}} := \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_T \in [\mathbf{P}_0(T)]^2\ \forall T \in \mathcal{T}_h\},$$

and

$$M_h := M_h^{\mathbf{u}} \times \mathbf{R}.$$

Hence, the Galerkin scheme associated with (1.2) reads: Find $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \xi_h)) \in X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned} &[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), (\boldsymbol{\sigma}_h, p_h)] = 0,\\ &[\mathbf{B}_1(\mathbf{t}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h, \xi_h)] = [\mathbf{G}, (\boldsymbol{\tau}_h, q_h)],\\ &[\mathbf{B}(\boldsymbol{\sigma}_h, p_h), (\mathbf{v}_h, \eta_h)] = [\mathbf{F}, (\mathbf{v}_h, \eta_h)], \end{aligned} \tag{1.6}$$

for all $(\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)) \in X_{1,h} \times M_{1,h} \times M_h$.

In [5] we proved that, under suitable assumptions on the nonlinear kinematic viscosity function ψ (see Eqs. (1.2) and (1.3) in [5]), the continuous formulation (1.2) and the Galerkin scheme (1.6) are well posed. In addition, we derived there the associated a priori error analysis and the corresponding rate of convergence. We refer to Theorems 2.4, 3.1, and 3.2 in [5] for details.

On the other hand, we recall that the application of adaptive algorithms, based on a posteriori error estimates, usually guarantees the quasi-optimal rate of convergence of the finite element solution to boundary value problems. In addition, this adaptivity is specially necessary for nonlinear problems where no a priori hints on how to build suitable meshes are available. To this respect, we have shown recently that the combination of the usual Bank–Weiser approach from [1] with the analysis from [3,4] allows to derive fully explicit and reliable a posteriori error estimates for the dual-mixed variational formulations (showing a two-fold saddle point structure) of some linear and nonlinear problems (see, e.g. [2,6,7]). However, no a posteriori error analysis has been developed yet for the nonlinear Stokes problems studied in [5]. Therefore, as a natural continuation of our results in [5], in the present paper we apply the Bank–Weiser type a posteriori error analysis mentioned above to derive reliable estimates for the mixed finite element scheme (1.6). The rest of this work is organized as follows. In Section 2 we collect some basic results on Sobolev spaces and state the main result of this paper. The proof of our a posteriori estimate, which makes use of the Ritz projection of the error, is provided in Section 3. In Section 4 we prove the *quasi-efficiency* of the estimator and discuss on suitable choices for the auxiliary functions needed for its computation. Finally, several numerical results illustrating the good performance of the adaptive algorithm are reported in Section 5.

2. The main result

2.1. Preliminaries

Let us first introduce some notations. Given $T \in \mathcal{T}_h$, we let $E(T)$ be the set of its edges, and let E_h be the set of all edges of the triangulation $\mathcal{T}_h$. In particular, we put $E_h(\Gamma) := \{e \in E_h : e \subseteq \Gamma\}$. Also, $\langle \cdot, \cdot \rangle_{H(\mathbf{div};T)}$ denotes the inner product of $H(\mathbf{div}; T)$, and $\boldsymbol{\nu}_T$ stands for the unit outward normal to ∂T.

In addition, given a polygonal domain $\mathcal{S} \subset R^2$ and $s \in (1, \infty)$, the Sobolev space $W^{1,s}(\mathcal{S})$ is the space of functions $v \in L^s(\mathcal{S})$ such that the first order distributional derivatives of v are functions of $L^s(\mathcal{S})$ (see [8]). It is well known that $W^{1,s}(\mathcal{S})$ endowed with the norm

$$\|v\|_{W^{1,s}(\mathscr{S})} := \left(\|v\|^s_{L^s(\mathscr{S})} + \|\nabla v\|^s_{[L^s(\mathscr{S})]^2}\right)^{1/s}$$

is a Banach space. The trace Theorem ensures that there exists a linear continuous map $\gamma : W^{1,s}(\mathscr{S}) \mapsto L^s(\partial\mathscr{S})$ such that $\gamma v = v|_{\partial\mathscr{S}}$ for each $v \in W^{1,s}(\mathscr{S}) \cap C(\overline{\mathscr{S}})$. It is usual to denote $W^{1-1/s,s}(\partial\mathscr{S}) := \gamma(W^{1,s}(\mathscr{S}))$ which is a strict subspace of $L^s(\partial\mathscr{S})$ (see [8]). We also recall, by virtue of a Sobolev imbedding theorem, that $W^{1,s}(\mathscr{S}) \subset C(\overline{\mathscr{S}})$ if $s > 2$.

We now take in particular $\mathscr{S} := T \in \mathscr{T}_h$. Then when $s = 2$ we use the standard notation and write $H^{1/2}(\partial T)$ instead of $W^{1/2,2}(\partial T)$. The fractional Sobolev spaces $H^{1/2}(\partial T)$ may be equivalently defined by the completion of the space of indefinitely differentiable functions in the norm:

$$\|v\|_{H^{1/2}(\partial T)} = \left(\|v\|^2_{L^2(\partial T)} + |v|^2_{H^{1/2}(\partial T)}\right)^{1/2},$$

where

$$|v|^2_{H^{1/2}(\partial T)} := \int_{\partial T}\int_{\partial T} \frac{|v(x) - v(y)|^2}{|x-y|^2}\,\mathrm{d}s_x\,\mathrm{d}s_y.$$

Let us now consider an edge $e \in E(T)$. Then, $H^1_0(e)$ stands for the closure in $H^1(e)$ of the space of indefinitely differentiable functions with compact support in e. Finally, we recall that the interpolation space with index 1/2 between $H^1_0(e)$ and $L^2(e)$ is $H^{1/2}_{00}(e)$ (cf. [8]), and its norm is given by

$$\|v\|_{H^{1/2}_{00}(e)} = \left(|v|^2_{H^{1/2}(e)} + \int_e \frac{v^2(x)}{|x-a_1|}\,\mathrm{d}s_x + \int_e \frac{v^2(x)}{|x-a_2|}\,\mathrm{d}s_x\right)^{1/2},$$

where a_1 and a_2 are the end points of the edge e. The space $H^{1/2}_{00}(e)$ may be alternatively defined as the subspace of functions in $H^{1/2}(e)$ whose extensions by zero to the rest of ∂T belong to $H^{1/2}(\partial T)$.

We will also need in the sequel the dual space of $H^{1/2}(\partial T)$ denoted here $H^{-1/2}(\partial T)$. It is important to retain that the restriction of an element in $H^{-1/2}(\partial T)$ over e does not belong in general to $H^{-1/2}(e)$, but to the dual of $H^{1/2}_{00}(e)$, usually denoted by $H^{-1/2}_{00}(e)$, and which is larger than $H^{-1/2}(e)$. According to this, in what follows we denote by $\langle\cdot,\cdot\rangle_e$ the duality pairing between $[H^{-1/2}_{00}(e)]^2$ and $[H^{1/2}_{00}(e)]^2$ with respect to the $[L^2(e)]^2$-inner product. Further, we also denote by $\langle\cdot,\cdot\rangle_{\partial T}$ the duality pairing between $[H^{-1/2}(\partial T)]^2$ and $[H^{1/2}(\partial T)]^2$ with respect to the $[L^2(\partial T)]^2$-inner product.

2.2. *The a posteriori error estimate*

The main result of this paper is stated as follows.

Theorem 2.1. *Let* $\vec{\mathbf{t}} := (\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \xi)) \in X_1 \times M_1 \times M$ *and* $\vec{\mathbf{t}}_h := (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \xi_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *be the solutions of the continuous and Galerkin formulations* (1.2) *and* (1.6), *respectively. Assume there exists* $s > 2$ *such that* $\mathbf{g} \in [H^{1/2}(\Gamma) \cap W^{1-1/s,s}(\Gamma)]^2$ *and let* $\boldsymbol{\varphi}_h$ *be a function in* $[H^1(\Omega) \cap W^{1,s}(\Omega)]^2$ *such that* $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = \mathbf{g}(\bar{\mathbf{x}})$ *for each vertex* $\bar{\mathbf{x}}$ *of* $\mathscr{T}_h$ *lying on* Γ. *In addition, let* $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div}; T)$ *be the unique solution of the local problem*

$$\langle\hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{H(\mathbf{div};T)} = \mathscr{F}_{h,T}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in H(\mathbf{div}; T), \tag{2.1}$$

where $\mathscr{F}_{h,T} \in H(\mathbf{div}; T)'$ *is defined by*

$$\mathscr{F}_{h,T}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \mathbf{t}_h \mathrm{d}x + \int_T \mathbf{u}_h \cdot \mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x - \xi_h \int_T \mathrm{tr}(\boldsymbol{\tau})\,\mathrm{d}x - \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h - \mathbf{g}\rangle_e.$$

Then, there exists $C > 0$, independent of h, such that

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1 \times M_1 \times M} \leqslant C\boldsymbol{\theta} := C\left\{ \sum_{T \in \mathscr{T}_h} \theta_T^2 \right\}^{1/2}, \tag{2.2}$$

where for each triangle $T \in \mathscr{T}_h$ we define

$$\theta_T^2 := \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}. \tag{2.3}$$

Further, let $\tilde{\boldsymbol{\varphi}}_h$ be a function in $[L^2(\Omega)]^2$ such that $\tilde{\boldsymbol{\varphi}}_{h,T} := \tilde{\boldsymbol{\varphi}}_h|_T \in [H^1(T)]^2$ for each $T \in \mathscr{T}_h$. Then, there exists $\tilde{C} > 0$, independent of h, such that

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1 \times M_1 \times M} \leqslant \tilde{C}\tilde{\boldsymbol{\theta}} := \tilde{C}\left\{ \sum_{T \in \mathscr{T}_h} \tilde{\theta}_T^2 \right\}^{1/2}, \tag{2.4}$$

where

$$\begin{aligned}\tilde{\theta}_T^2 := {} & \|\mathbf{t}_h - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[L^2(T)]^2} + h_T^2|\xi_h|^2 + \sum_{e \in E(T)\cap E_h(\Gamma)} \|\boldsymbol{\varphi}_h - \mathbf{g}\|^2_{[H_{00}^{1/2}(e)]^2} \\ & + \|\boldsymbol{\varphi}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[H^{1/2}(\partial T)]^2} + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}.\end{aligned} \tag{2.5}$$

In particular, if we take $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{\varphi}_h$, then (2.4) becomes

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1 \times M_1 \times M} \leqslant \hat{C}\hat{\boldsymbol{\theta}} := \hat{C}\left\{ \sum_{T \in \mathscr{T}_h} \hat{\theta}_T^2 \right\}^{1/2}, \tag{2.6}$$

where

$$\begin{aligned}\hat{\theta}_T^2 := {} & \|\mathbf{t}_h - \nabla\boldsymbol{\varphi}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \boldsymbol{\varphi}_h\|^2_{[L^2(T)]^2} + h_T^2|\xi_h|^2 + \sum_{e \in E(T)\cap E_h(\Gamma)} \|\boldsymbol{\varphi}_h - \mathbf{g}\|^2_{[H_{00}^{1/2}(e)]^2} \\ & + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}.\end{aligned} \tag{2.7}$$

The proof of Theorem 2.1 is given in the following section. We just remark here that the hypotheses on $\mathbf{g}$ and $\boldsymbol{\varphi}_h$ guarantee, by virtue of the Sobolev imbedding theorems, that $\mathbf{g}$ and $\boldsymbol{\varphi}_h$ are both continuous and that $(\mathbf{g} - \boldsymbol{\varphi}_h)|_e \in [H_{00}^{1/2}(e)]^2$ for each $e \in E_h(\Gamma)$.

3. The proof of the main result

The proof itself is provided below in Section 3.2. For this purpose, we need to introduce first the Ritz projection of the error.

3.1. Ritz projection of the error

Let $X := X_1 \times M_1$ and introduce the nonlinear saddle point operator $\mathbf{A} : X \to X'$ given by the first two rows and two columns of (1.2), that is

$$[\mathbf{A}(\mathbf{t},(\boldsymbol{\sigma},p)),(\mathbf{s},(\boldsymbol{\tau},q))] := [\mathbf{A}_1(\mathbf{t}),\mathbf{s}] + [\mathbf{B}_1(\mathbf{s}),(\boldsymbol{\sigma},p)] + [\mathbf{B}_1(\mathbf{t}),(\boldsymbol{\tau},q)],$$

for all $(\mathbf{t},(\boldsymbol{\sigma},p))$, $(\mathbf{s},(\boldsymbol{\tau},q)) \in X$.

Then we define the Ritz projection of the error, with respect to the inner product of X, as the unique $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}) \in X$ such that

$$\langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_X = [\mathbf{A}(\mathbf{t},(\boldsymbol{\sigma},p)),(\mathbf{s},(\boldsymbol{\tau},q))] - [\mathbf{A}(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h)),(\mathbf{s},(\boldsymbol{\tau},q))] + [\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{u},\xi)-(\mathbf{u}_h,\xi_h)] \quad \forall(\mathbf{s},(\boldsymbol{\tau},q)) \in X, \tag{3.1}$$

where $\langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_X := \langle\bar{\mathbf{t}},\mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle\bar{\boldsymbol{\sigma}},\boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} + \langle\bar{p},q\rangle_{L^2(\Omega)}$.

The following lemma provides a suitable upper bound for $\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X$.

Lemma 3.1. *For each $T \in \mathscr{T}_h$, let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ be the unique solution of the local problem* (2.1). *Then there holds*

$$\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 \leqslant \sum_{T\in\mathscr{T}_h}\left\{\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}\right\}. \tag{3.2}$$

Proof. From the first two equations of (1.2) we have

$$[\mathbf{A}(\mathbf{t},(\boldsymbol{\sigma},p)),(\mathbf{s},(\boldsymbol{\tau},q))] + [\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{u},\xi)] = -\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma,$$

and hence

$$\langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_X = -\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma - [\mathbf{A}(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h)),(\mathbf{s},(\boldsymbol{\tau},q))] - [\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{u}_h,\xi_h)], \tag{3.3}$$

for all $(\mathbf{s},(\boldsymbol{\tau},q)) \in X$.

According to the definitions of the operators $\mathbf{A}$ and $\mathbf{B}$, we deduce from (3.3) that

$$\bar{\mathbf{t}} = \boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}, \quad \bar{p} = \mathrm{tr}(\mathbf{t}_h), \tag{3.4}$$

and

$$\langle\bar{\boldsymbol{\sigma}},\boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} = -\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma + \int_\Omega \boldsymbol{\tau} : \mathbf{t}_h\,\mathrm{d}x + \int_\Omega \mathbf{u}_h\cdot\mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x - \xi_h\int_\Omega \mathrm{tr}(\boldsymbol{\tau})\,\mathrm{d}x, \tag{3.5}$$

for all $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$.

On the other hand, using Gauss's formula on each $T \in \mathscr{T}_h$ and on Ω, we obtain

$$\sum_{T\in\mathscr{T}_h}\langle\boldsymbol{\tau}\boldsymbol{\nu}_T,\boldsymbol{\varphi}_h\rangle_{\partial T} = \sum_{T\in\mathscr{T}_h}\left\{\int_T \nabla\boldsymbol{\varphi}_h : \boldsymbol{\tau}\,\mathrm{d}x + \int_T \boldsymbol{\varphi}_h\cdot\mathrm{div}\boldsymbol{\tau}\,\mathrm{d}x\right\} = \int_\Omega \nabla\boldsymbol{\varphi}_h : \boldsymbol{\tau}\,\mathrm{d}x + \int_\Omega \boldsymbol{\varphi}_h\cdot\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x = \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h\rangle_\Gamma,$$

that is

$$\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h\rangle_\Gamma - \sum_{T\in\mathscr{T}_h}\langle\boldsymbol{\tau}\boldsymbol{\nu}_T,\boldsymbol{\varphi}_h\rangle_{\partial T} = 0. \tag{3.6}$$

In addition, since $(\boldsymbol{\varphi}_h - \mathbf{g})|_e \in [H_{00}^{1/2}(e)]^2$ for each $e \in E_h(\Gamma)$, we can write

$$\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h - \mathbf{g}\rangle_\Gamma = \sum_{e\in E_h(\Gamma)}\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h - \mathbf{g}\rangle_e. \tag{3.7}$$

Then, including (3.6) into the right hand side of (3.5), and using (3.7) and the fact that

$$-\frac{1}{2}\|\bar{\boldsymbol{\sigma}}\|^2_{H(\mathbf{div};\Omega)} = \min_{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)}\left\{\frac{1}{2}\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} - \langle\bar{\boldsymbol{\sigma}},\boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)}\right\},$$

we find that

$$-\frac{1}{2}\|\bar{\boldsymbol{\sigma}}\|^2_{H(\mathbf{div};\Omega)} = \min_{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)} \left\{ \sum_{T\in\mathscr{T}_h} \mathscr{Q}_T(\boldsymbol{\tau}_T) \right\},$$

where $\boldsymbol{\tau}_T$ is the restriction of $\boldsymbol{\tau}$ to the triangle T, and $\mathscr{Q}_T(\boldsymbol{\tau}_T) := \frac{1}{2}\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};T)} - \mathscr{F}_{h,T}(\boldsymbol{\tau}_T)$.

Next, since $H(\mathbf{div};\Omega)$ is contained in $\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \boldsymbol{\tau}_T \in H(\mathbf{div};T)\ \forall T \in \mathscr{T}_h\}$, it follows that

$$-\frac{1}{2}\|\bar{\boldsymbol{\sigma}}\|^2_{H(\mathbf{div};\Omega)} \geqslant \sum_{T\in\mathscr{T}_h} \left\{ \min_{\boldsymbol{\tau}_T\in H(\mathbf{div};T)} \mathscr{Q}_T(\boldsymbol{\tau}_T) \right\} = -\frac{1}{2}\sum_{T\in\mathscr{T}_h} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)}.$$

This inequality and (3.4) yield (3.2) and complete the proof. □

3.2. Proof of Theorem 2.1

We begin with the main a posteriori error estimate.

Lemma 3.2. *There exists $C > 0$, independent of h, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1\times M_1\times M} \leqslant C\boldsymbol{\theta}.$$

Proof. We first recall from the proof of Theorem 2.4 in [5] that $\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ is a uniformly bounded and uniformly elliptic bilinear form on $X_1 \times X_1$, for all $\tilde{\mathbf{r}} \in X_1$, and that the operators $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the corresponding continuous inf–sup conditions. Therefore, the linear operator obtained by adding the three equations of the left hand side of (1.2), after replacing $\mathbf{A}_1$ by the Gâteaux derivative $\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ at any $\tilde{\mathbf{r}} \in X_1$, satisfies a global inf–sup condition with a constant $\tilde{C} > 0$, independent of $\tilde{\mathbf{r}}$.

In particular, we consider $\tilde{\mathbf{r}} \in X_1$ such that $\mathscr{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{t} - \mathbf{t}_h, \mathbf{s}) = [\mathbf{A}_1(\mathbf{t}), \mathbf{s}] - [\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}]$ for all $\mathbf{s} \in X_1$, and apply the above inf–sup condition to the error $\vec{\mathbf{t}} - \vec{\mathbf{t}}_h$, thus obtaining

$$\frac{1}{\tilde{C}}\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1\times M_1\times M} \leqslant \sup_{\|\vec{\mathbf{s}}\|\leqslant 1} \Big\{ [\mathbf{A}(\mathbf{t},(\boldsymbol{\sigma},p)),(\mathbf{s},(\boldsymbol{\tau},q))] - [\mathbf{A}(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h)),(\mathbf{s},(\boldsymbol{\tau},q))] + [\mathbf{B}(\boldsymbol{\tau},q),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)] + [\mathbf{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,p-p_h),(\mathbf{v},\eta)] \Big\},$$

where $\vec{\mathbf{s}} := (\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\eta))$.

Using now the Ritz projection $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}) \in X$ (cf. (3.1)), the definition of the operator $\mathbf{B}$, and the third equations of the continuous and Galerkin formulations (1.2) and (1.6), respectively, the above estimate becomes

$$\frac{1}{\tilde{C}}\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X_1\times M_1\times M} \leqslant \sup_{\|\vec{\mathbf{s}}\|\leqslant 1} \left\{ \langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_X + \int_\Omega (\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h)\cdot\mathbf{v}\,\mathrm{d}x \right\}. \tag{3.8}$$

Finally, (3.8), Lemma 3.1, and Cauchy–Schwarz's inequality, conclude the proof. □

We provide now a priori estimates for the solution of the local problem (2.1).

Lemma 3.3. *Let $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$ be as indicated in Theorem* 2.1. *Then there exists $C > 0$, independent of h and T, such that*

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} \leqslant C\left\{\|\mathbf{t}_h - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[L^2(T)]^2} + h_T^2|\xi_h|^2\right.$$

$$\left. + \sum_{e\in E(T)\cap E_h(\Gamma)} \|\boldsymbol{\varphi}_h - \mathbf{g}\|^2_{[H_{00}^{1/2}(e)]^2} + \|\boldsymbol{\varphi}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[H^{1/2}(\partial T)]^2}\right\}. \qquad (3.9)$$

Furthermore, for any $\mathbf{z} \in [H^1(\Omega)\cap W^{1,s}(\Omega)]^2$, *with* $s > 2$, *such that* $\mathbf{z} = \mathbf{g}$ *on* Γ, *we get*

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} \leqslant C\left\{\|\mathbf{t}_h - \nabla\mathbf{z}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \mathbf{z}\|^2_{[L^2(T)]^2} + h_T^2|\xi_h|^2 + \|\mathbf{J}_{h,T}(\mathbf{z})\|^2_{[H^{1/2}(\partial T)]^2}\right\}, \qquad (3.10)$$

where $\mathbf{J}_{h,T}(\mathbf{z}) := \begin{cases} 0 & on\ \partial T\cap\Gamma, \\ \mathbf{z} - \boldsymbol{\varphi}_h & otherwise. \end{cases}$

Proof. We recall from (2.1) that $\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} = \|\mathscr{F}_{h,T}\|_{H(\mathbf{div};T)'}$, where

$$\mathscr{F}_{h,T}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \mathbf{t}_h\,\mathrm{d}x + \int_T \mathbf{u}_h\cdot\mathbf{div}\,\boldsymbol{\tau}\,\mathrm{d}x - \xi_h\int_T \mathrm{tr}(\boldsymbol{\tau})\,\mathrm{d}x - \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h - \mathbf{g}\rangle_e. \qquad (3.11)$$

Then, using that $\langle\boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\varphi}_h\rangle_{\partial T} = \langle\boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\varphi}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\rangle_{\partial T} + \langle\boldsymbol{\tau}\boldsymbol{\nu}, \tilde{\boldsymbol{\varphi}}_{h,T}\rangle_{\partial T}$, applying Gauss's formula to the term $\langle\boldsymbol{\tau}\boldsymbol{\nu}, \tilde{\boldsymbol{\varphi}}_{h,T}\rangle_{\partial T}$, and replacing back into (3.11), we get (3.9).

The proof of (3.10) is similar. We just need to observe that

$$-\langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h - \mathbf{g}\rangle_e = -\langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \mathbf{z}\rangle_{\partial T} + \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \mathbf{z} - \boldsymbol{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h - \mathbf{z}\rangle_e$$

$$= -\langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \mathbf{z}\rangle_{\partial T} + \langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \mathbf{J}_{h,T}(\mathbf{z})\rangle_{\partial T},$$

and then proceed as before, applying now Gauss's formula to $\langle\boldsymbol{\tau}\boldsymbol{\nu}_T, \mathbf{z}\rangle_{\partial T}$. □

Consequently, the proof of Theorem 2.1 follows straightforwardly from Lemma 3.2 and the estimate (3.9) (cf. Lemma 3.3).

At this point we observe that it would also be desirable to obtain an efficiency result for the a posteriori error estimate. This basically means to be able to prove the existence of a constant $C > 0$ such that $\boldsymbol{\theta} \leqslant C\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|$. As we show next, we do not prove the above inequality but just a related result.

4. Quasi-efficiency and choice of $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$

We remark first that Theorem 2.1 and Lemma 3.1 do not require any further assumptions on the given functions $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$. However, we show now in Section 4.1 that $\boldsymbol{\theta}$ (cf. Theorem 2.1) becomes efficient up to the traces of $(\mathbf{u} - \boldsymbol{\varphi}_h)$ on the edges of $\mathscr{T}_h$. This property of the a posteriori error estimator leads us to the concept of *quasi-efficiency*, which restricts the possible choices of $\boldsymbol{\varphi}_h$. We also notice that the introduction of the second auxiliary function $\tilde{\boldsymbol{\varphi}}_h$ yields an additional degree of freedom for the definition and computation of the local estimator. We refer again to these points in Section 4.2 below.

4.1. Quasi-efficiency

It is well known that the Bank–Weiser type a posteriori error analysis does not yield *efficiency*, and that it is possible to derive an explicit lower bound of the error only through the use of another estimator, usually of residual type. Nevertheless, motivated by the a priori estimate (3.10) (cf. Lemma 3.3), we prove next that the reliable estimate $\boldsymbol{\theta}$ is *quasi-efficient*, which means that it is efficient up to a term depending on the traces $(\mathbf{u} - \boldsymbol{\varphi}_h)$ on the edges e of $\mathscr{T}_h$.

Lemma 4.1. *Let $\boldsymbol{\varphi}_h$ be as stated before, and assume that $\mathbf{u} \in [W^{1,s}(\Omega)]^2$, with $s > 2$. Then there exists $C > 0$, independent of h, such that for all $T \in \mathscr{T}_h$*

$$\theta_T^2 \leqslant C\Big\{ \|\mathbf{t}-\mathbf{t}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{H(\mathbf{div};T)} + \|p-p_h\|^2_{L^2(T)} + \|\mathbf{u}-\mathbf{u}_h\|^2_{[L^2(T)]^2} + h_T^2|\xi-\xi_h|^2 + \|\mathbf{J}_{h,T}(\mathbf{u})\|^2_{[H^{1/2}(\partial T)]^2}\Big\}, \tag{4.1}$$

and hence

$$\boldsymbol{\theta}^2 \leqslant C\left\{ \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|^2_{X_1\times M_1\times M} + \sum_{T\in\mathscr{T}_h} \|\mathbf{J}_{h,T}(\mathbf{u})\|^2_{[H^{1/2}(\partial T)]^2} \right\}. \tag{4.2}$$

Proof. The first equation of (1.2) yields $\boldsymbol{\sigma} = \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I}$ in Ω. In addition, from the second equation of (1.2) we easily get $\xi = 0$ and $\mathrm{tr}(\mathbf{t}) = 0$ in Ω. Then, taking $\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^{2\times 2}$ in this equation, we deduce that $\mathbf{t} = \nabla\mathbf{u}$ in Ω, and $\mathbf{u} = \mathbf{g}$ on Γ, whence $\mathbf{u} \in [H^1(\Omega)]^2$. Also, it follows from the third equation of (1.2) that $\mathrm{div}\boldsymbol{\sigma} = -\mathbf{f}$ in Ω and $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{d}x = 0$.

Then, applying (3.10) (cf. Lemma 3.3) with $\mathbf{z} = \mathbf{u}$, we deduce that

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} \leqslant C\Big\{ \|\mathbf{t}_h - \mathbf{t}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \mathbf{u}\|^2_{[L^2(T)]^2} + h_T^2|\xi - \xi_h|^2 + \|\mathbf{J}_{h,T}(\mathbf{u})\|^2_{[H^{1/2}(\partial T)]^2}\Big\}. \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}
\|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|_{[L^2(T)]^{2\times 2}} &\leqslant \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\sigma} - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|_{[L^2(T)]^{2\times 2}} \\
&\leqslant \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\psi}(\mathbf{t}_h)\|_{[L^2(T)]^{2\times 2}} + \|p_h\mathbf{I} - p\mathbf{I}\|_{[L^2(T)]^{2\times 2}} \\
&\leqslant C\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}} + \|\mathbf{t} - \mathbf{t}_h\|_{[L^2(T)]^{2\times 2}} + \|p - p_h\|_{L^2(T)}\},
\end{aligned} \tag{4.4}$$

where the term $\|\boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\psi}(\mathbf{t}_h)\|_{[L^2(T)]^{2\times 2}}$ has been bounded using the Lipschitz continuity of the nonlinear operator $\mathbf{A}_1$ (restricted to the triangle $T \in \mathscr{T}_h$).

Next, it is easy to see that

$$\|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \leqslant \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathbf{div};T)} \quad \text{and} \quad \|\mathrm{tr}(\mathbf{t}_h)\|_{L^2(T)} \leqslant \|\mathbf{t} - \mathbf{t}_h\|_{[L^2(T)]^{2\times 2}}. \tag{4.5}$$

Therefore, (4.3)–(4.5), and the definition of θ_T (cf. Theorem 2.1), imply (4.1).

Finally, the *quasi-efficiency* of $\boldsymbol{\theta}$ (given by (4.2)) is obtained summing up (4.1) over all the triangles $T \in \mathscr{T}_h$. $\square$

4.2. Further comments and choice of φ_h and $\tilde{\varphi}_h$

We observe that the solution of the local problem (2.1) lives in the infinite dimensional space $H(\mathbf{div};T)$. This implies that (2.1) must be solved approximately by using, for instance, the h or the $h-p$ version of the finite element method, which yields approximations of the local indicators θ_T (and hence of $\boldsymbol{\theta}$). Nevertheless, the main property of $\boldsymbol{\theta}$, as proved by Lemma 4.1, is that it constitutes a *quasi-efficient* and reliable a posteriori error estimate.

Alternatively, the advantage of $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$, which are not necessarily *quasi-efficient*, lies on the fact that they do not require neither the exact nor any approximate solutions of the local problems (2.1), and hence they constitute fully explicit reliable a posteriori error estimates.

Now, concerning the choice of $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$, and because of (4.2) (cf. Lemma 4.1), we first realize that the traces $\boldsymbol{\varphi}_h|_{\partial T}$ have to be as close as possible to the exact traces $\mathbf{u}|_{\partial T}$ for all $T \in \mathscr{T}_h$. Certainly, since the exact solution $\mathbf{u}$ is not known, the above criterion must be understood in an empirical sense. Also, although a priori $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$ are not necessarily related, the terms $\|\boldsymbol{\varphi}_h - \tilde{\boldsymbol{\varphi}}_{h,T}\|^2_{[H^{1/2}(\partial T)]^2}$ appearing in the definition of $\tilde{\theta}_T$ suggest that these functions should be close to each other as well. Further, since the restrictions of $\tilde{\boldsymbol{\varphi}}_h$ on the triangles $T \in \mathscr{T}_h$ can be defined independently, one may choose these local functions so that the computation of $\tilde{\theta}_T$ becomes simpler.

According to the above, for each $T \in \mathscr{T}_h$ we suggest to take $\tilde{\boldsymbol{\varphi}}_{h,T}$ as the function in $[C(T)]^2$ satisfying the following conditions:

1. $\tilde{\boldsymbol{\varphi}}_{h,T} \in [\mathbf{P}_1(T)]^2$.
2. $\nabla \tilde{\boldsymbol{\varphi}}_{h,T} = \mathbf{t}_h|_T$.
3. $\tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_T) = \mathbf{u}_h|_T$, where $\bar{\mathbf{x}}_T$ is the barycenter of the triangle T.

We remark that $\tilde{\boldsymbol{\varphi}}_{h,T}$ is uniquely determined by the above conditions, which yield a straightforward computation of this function. Certainly, the terms $\|\mathbf{t}_h - \nabla \tilde{\boldsymbol{\varphi}}_{h,T}\|_{[L^2(T)]^{2\times 2}}$ now disappear from the definition of the local indicator $\tilde{\theta}_T$ (cf. Theorem 2.1).

Then, we take $\boldsymbol{\varphi}_h$ as the *continuous average* of the local functions $\tilde{\boldsymbol{\varphi}}_{h,T}$. More precisely, $\boldsymbol{\varphi}_h \in [C(\overline{\Omega})]^2$ is the unique function satisfying the following conditions:

1. $\boldsymbol{\varphi}_h|_T \in [\mathbf{P}_1(T)]^2$ for all $T \in \mathscr{T}_h$.
2. $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = \mathbf{g}(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathscr{T}_h$ lying on Γ.
3. For each vertex $\bar{\mathbf{x}}$ of $\mathscr{T}_h$ lying in Ω, $\boldsymbol{\varphi}_h(\bar{\mathbf{x}})$ is the weighted average of the values $\tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}})$ on all the triangles $T \in \mathscr{T}_h$ to which $\bar{\mathbf{x}}$ belongs. The weighting here is either constant or with respect to the areas of those triangles.

We end this section by observing that the $H^{1/2}$-norms appearing in the definition of $\tilde{\theta}_T$ (cf. Theorem 2.1) can be bounded by using the interpolation theorem. In particular, given $T \in \mathscr{T}_h$, $e \in E(T)$, and $\boldsymbol{\rho} \in [H_0^1(e)]^2$, we have

$$\|\boldsymbol{\rho}\|^2_{[H_{00}^{1/2}(e)]^2} \leqslant \|\boldsymbol{\rho}\|_{[L^2(e)]^2} \|\boldsymbol{\rho}\|_{[H_0^1(e)]^2}.$$

5. Numerical results

In this section we provide some numerical examples illustrating the performance of the mixed finite element scheme (1.6) and the explicit a posteriori error estimate given in Theorem 2.1.

In what follows, N is the number of degrees of freedom defining the subspaces $X_{1,h}$, $M_{1,h}$, and M_h, that is $N := 7$ (number of triangles of $\mathscr{T}_h$) + 2 (number of edges of $\mathscr{T}_h$) + 1.

Further, the individual and total errors are defined as follows:

$$\mathbf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{[L^2(\Omega)]^{2\times 2}}, \quad \mathbf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)},$$

$$\mathbf{e}(p) := \|p - p_h\|_{L^2(\Omega)}, \quad \mathbf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{[L^2(\Omega)]^2}, \quad \mathbf{e}(\xi) := |\xi - \xi_h|,$$

and

$$\mathbf{e} := \{[\mathbf{e}(\mathbf{t})]^2 + [\mathbf{e}(\boldsymbol{\sigma})]^2 + [\mathbf{e}(p)]^2 + [\mathbf{e}(\mathbf{u})]^2 + [\mathbf{e}(\xi)]^2\}^{1/2},$$

where $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \xi))$ and $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \xi_h))$ are the unique solutions of the continuous and discrete mixed formulations (1.2) and (1.6), respectively.

In addition, given two consecutive triangulations with degrees of freedom N and N', and corresponding total errors given by $\mathbf{e}$ and $\mathbf{e}'$, the experimental rate of convergence is defined by $\gamma := -2\frac{\log(\mathbf{e}/\mathbf{e}')}{\log(N/N')}$.

Now, the a posteriori error estimate to be used in the mesh refinement process for the computation of the solutions of (1.6) is the reliable one given by $\tilde{\boldsymbol{\theta}}$ (see (2.4) and (2.5)) with the functions $\boldsymbol{\varphi}_h$ and $\tilde{\boldsymbol{\varphi}}_h$ defined in Section 4.2.

Table 1
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the uniform refinement (Example 1)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
89	0.9436	3.4698	0.7774	0.4146	3.8782	0.9546	–
337	0.7135	4.7834	0.4239	0.1900	4.9657	0.9784	–
1313	0.4901	5.2289	0.2252	0.0890	5.3116	0.9898	–
5185	0.2970	4.2814	0.1164	0.0435	4.3211	0.9936	0.2949
20 609	0.1622	2.7426	0.0577	0.0216	2.7622	0.9949	0.6466
82 177	0.0838	1.5084	0.0278	0.0108	1.5181	0.9953	0.8650

Table 2
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the adaptive refinement (Example 1)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
89	0.9436	3.4698	0.7774	0.4146	3.8782	0.9546	–
211	0.8055	4.8320	0.6367	0.2258	5.0337	0.9824	–
333	0.6519	5.2927	0.5994	0.1565	5.3888	0.9962	–
455	0.5517	4.3776	0.5906	0.1420	4.4389	1.0034	1.1960
577	0.5071	2.9355	0.5876	0.1394	2.9931	1.0155	3.2177
699	0.4949	1.9459	0.5870	0.1390	2.0176	1.0391	3.8708
821	0.4927	1.5886	0.5869	0.1390	1.6724	1.0579	2.1119
2681	0.3159	0.8790	0.3271	0.0686	0.9549	1.0388	0.9777
4079	0.2116	0.6971	0.1888	0.0425	0.7565	0.9964	1.3086
10 359	0.1356	0.4149	0.1126	0.0252	0.4619	0.9775	1.1000
16 738	0.1078	0.3373	0.0905	0.0193	0.3727	0.9822	0.8737
40 921	0.0684	0.2080	0.0551	0.0123	0.2324	0.9730	1.0783
69 385	0.0536	0.1674	0.0422	0.0093	0.1866	0.9701	0.8428

Table 3
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the uniform refinement (Example 2)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
89	0.5597	1.2555	0.7527	0.6447	1.4444	1.1732	–
337	0.3630	0.9359	0.3959	0.3214	1.0686	1.0536	0.6140
1313	0.2089	0.8348	0.1932	0.1597	0.8978	0.9983	0.3354
5185	0.1126	0.8025	0.0908	0.0796	0.8253	0.9928	0.1306
20 609	0.0588	0.7208	0.0436	0.0397	0.7283	0.9964	0.1759
82 177	0.0301	0.5608	0.0214	0.0199	0.5634	0.9982	0.3686

Table 4
Individual errors, error estimate $\tilde{\theta}$, effectivity index, and rate of convergence for the adaptive refinement (Example 2)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\theta}$	$\mathbf{e}/\tilde{\theta}$	γ
89	0.5597	1.2555	0.7527	0.6447	1.4444	1.1732	–
326	0.3530	0.9306	0.3916	0.3204	1.0579	1.0555	–
448	0.3020	0.9360	0.3363	0.2679	1.0209	1.0514	–
570	0.2917	0.9455	0.3311	0.2628	1.0177	1.0573	–
692	0.2893	0.8907	0.3306	0.2624	0.9636	1.0660	0.4794
814	0.2887	0.7753	0.3305	0.2623	0.8570	1.0837	1.2414
936	0.2885	0.6586	0.3305	0.2623	0.7528	1.1075	1.5458
1491	0.2309	0.4710	0.2250	0.1946	0.5857	1.0296	1.3913
2645	0.1596	0.3415	0.1491	0.1281	0.4198	1.0127	1.2198
6376	0.1184	0.2209	0.1044	0.0918	0.2912	0.9842	0.8962
12 859	0.0758	0.1564	0.0658	0.0577	0.1975	0.9854	1.1036
20 797	0.0650	0.1184	0.0555	0.0496	0.1580	0.9759	0.9680
49 037	0.0398	0.0802	0.0330	0.0294	0.1033	0.9663	1.0146
78 877	0.0338	0.0607	0.0281	0.0253	0.0820	0.9636	0.9834

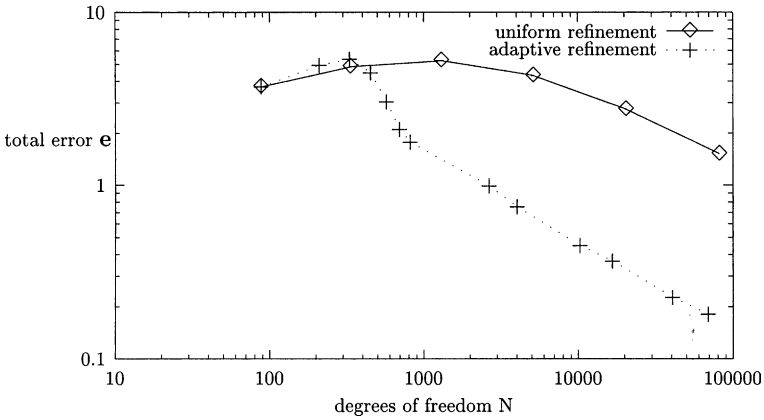

Fig. 1. Total error **e** for uniform and adaptive refinements (Example 1).

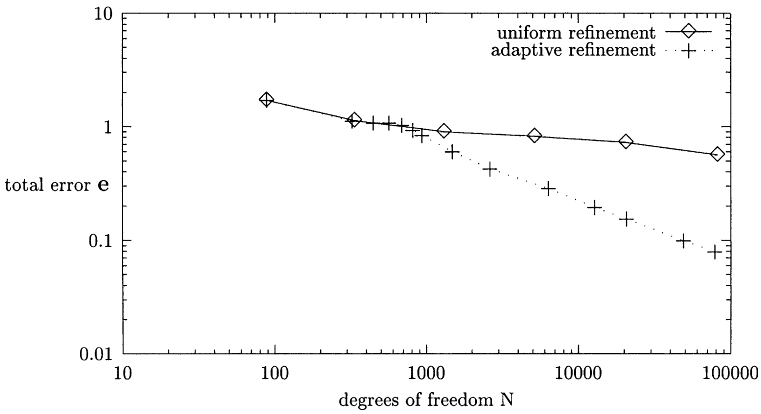

Fig. 2. Total error **e** for uniform and adaptive refinements (Example 2).

The corresponding adaptive algorithm, which applies a usual procedure from [10], reads as follows:

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the discrete problem (1.6) for the actual mesh $\mathcal{T}_h$.
3. Compute $\tilde{\theta}_T$ for each triangle $T \in \mathcal{T}_h$.
4. Evaluate stopping criterion and decide to finish or go to next step.
5. Use *blue–green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\tilde{\theta}_{T'}$ satisfies

   $$\tilde{\theta}_{T'} \geqslant \tfrac{1}{2}\max\{\tilde{\theta}_T : T \in \mathcal{T}_h\}.$$

6. Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

The numerical results presented here were obtained in a *Compaq Alpha ES40 Parallel Computer* using a MATLAB code. Some aspects of this computational implementation and further details on the solution of (1.6) will be reported in a separate work.

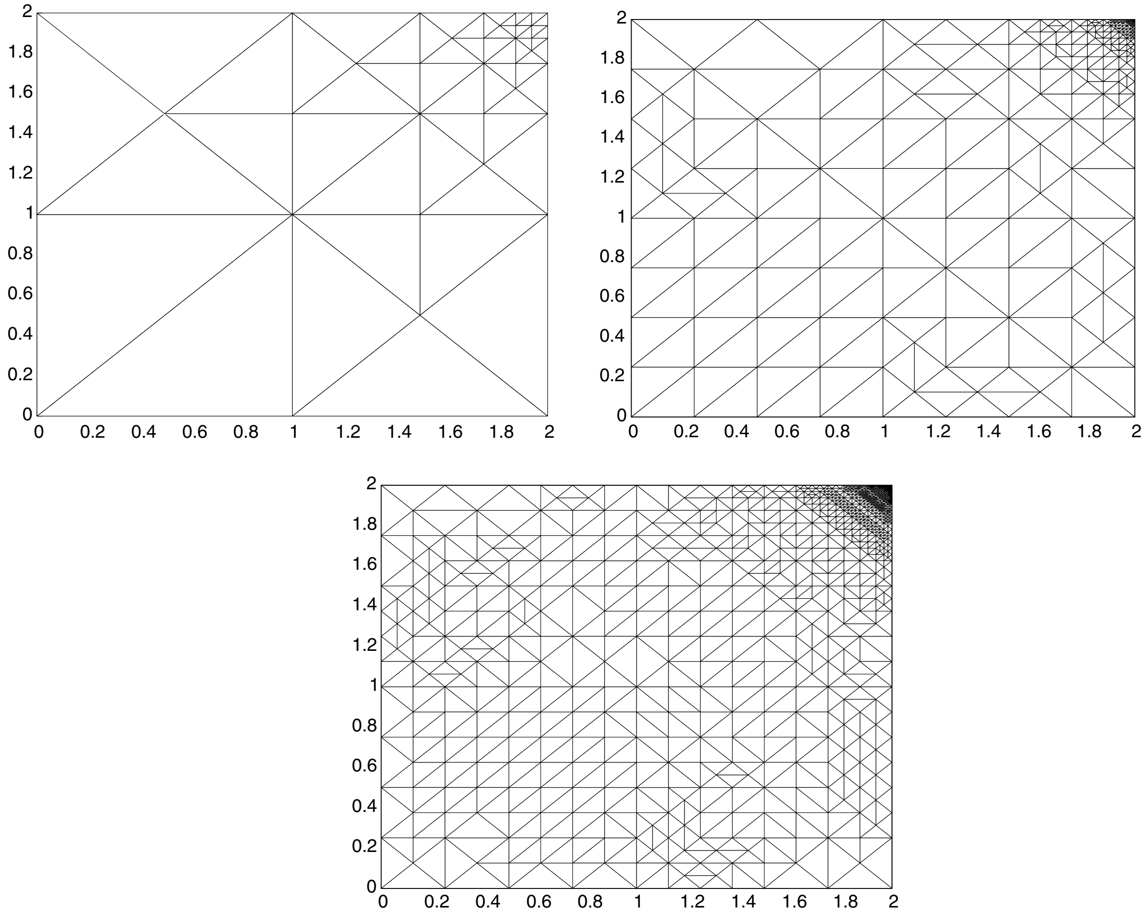

Fig. 3. Adapted intermediate meshes with 577, 4079, and 16 738 degrees of freedom, respectively, for Example 1.

We first consider the linear version of the boundary value problem (1.1) on the square $\Omega := (0,2) \times (0,2)$. We take the kinematic viscosity function $\psi \equiv 1$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (1.1) is, respectively, for Examples 1 and 2,

$$\mathbf{u}_1(x) := (-(4.1 - x_1 - x_2)^{-1/3}, (4.1 - x_1 - x_2)^{-1/3})^t, \quad p_1(x) := x_1 + x_2,$$

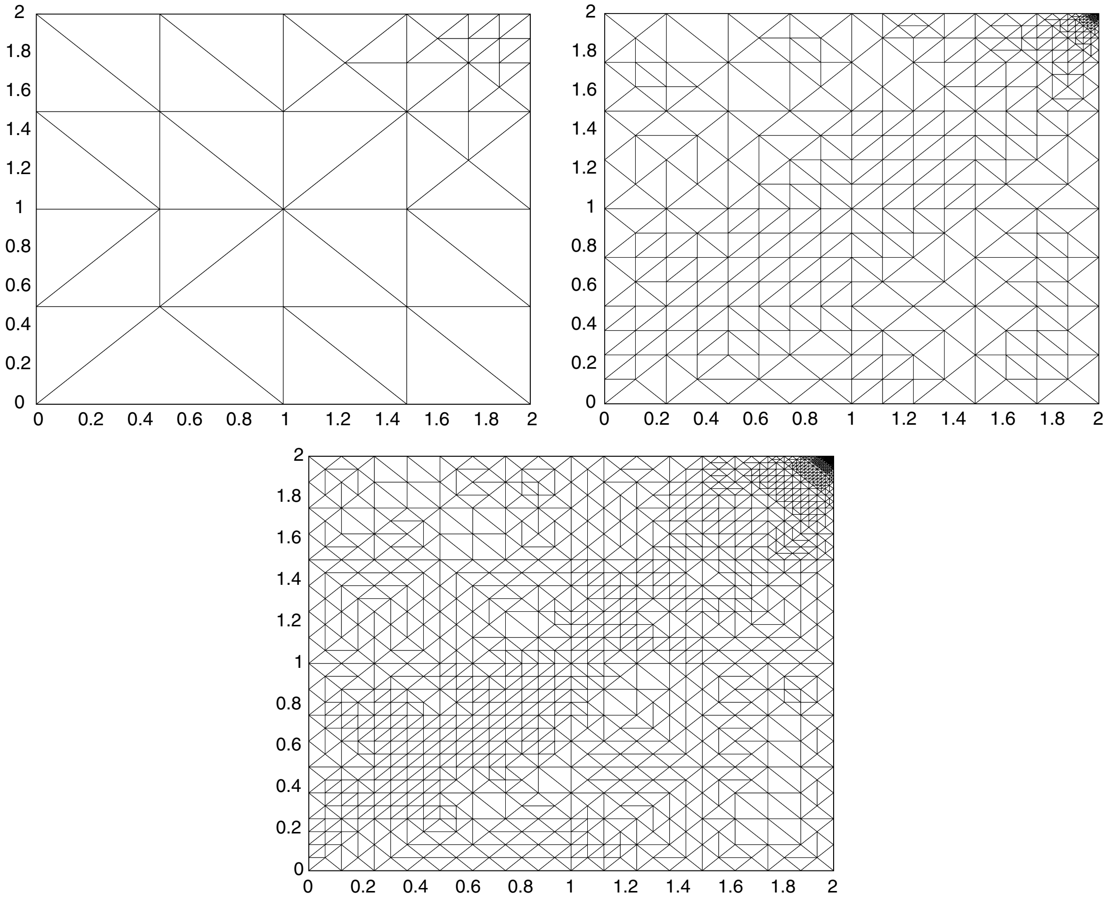

Fig. 4. Adapted intermediate meshes with 570, 6376, and 20 797 degrees of freedom, respectively, for Example 2.

Table 5
Individual errors, error estimate $\tilde{\theta}$, effectivity index, and rate of convergence for the uniform refinement (Example 3)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\theta}$	$\mathbf{e}/\tilde{\theta}$	γ
69	5.2794	10.7222	5.7792	1.7154	12.0815	1.1081	–
257	3.1864	8.3151	3.7115	0.6670	8.3606	1.1566	0.4948
993	2.7048	7.0207	3.3386	0.5004	6.8517	1.2036	0.2356
3905	2.3132	4.7988	2.5155	0.2964	5.1273	1.1504	0.4894
15 489	2.1680	3.4936	1.9672	0.1642	4.3630	1.0454	0.3732

and

$$\mathbf{u}_2(x) := (-(4.01 - x_1 - x_2)^{3/4}, (4.01 - x_1 - x_2)^{3/4})^t, \quad p_2(x) := x_1 + x_2,$$

for all $x := (x_1, x_2) \in \Omega$. We notice that $\mathbf{u}_1$ and $\mathbf{u}_2$ are divergence free in Ω and singular in an exterior neighborhood of the point $(2, 2)$.

In Tables 1–4, we give the errors for each unknown (except $\mathbf{e}(\xi)$, which converges very rapidly to zero), the error estimate $\tilde{\boldsymbol{\theta}}$, the effectivity index $\mathbf{e}/\tilde{\boldsymbol{\theta}}$, and the experimental rate of convergence γ for the uniform and adaptive refinements. The individual and global errors are computed on each triangle using a 7 points Gaussian quadrature rule (see [9]). We observe here that the effectivity indexes are bounded above and below, which confirms the reliability of the a posteriori estimate $\tilde{\boldsymbol{\theta}}$ (cf. Theorem 2.1), and provides

Table 6
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the adaptive refinement (Example 3)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
69	5.2794	10.7222	5.7792	1.7154	12.0815	1.1081	–
202	3.2431	9.6118	4.3531	0.8386	8.1037	1.3661	0.3538
326	1.8141	5.8557	1.1372	0.3873	6.0926	1.0253	2.3911
528	1.1815	3.7349	0.2741	0.2353	4.1806	0.9410	1.9183
730	0.8565	2.3970	0.2993	0.2173	2.8618	0.8988	2.6230
1667	0.5601	1.5001	0.2795	0.1477	1.8178	0.8979	1.1017
3590	0.3975	1.0074	0.1671	0.1134	1.2672	0.8694	1.0246
9034	0.2528	0.6283	0.0978	0.0717	0.8054	0.8543	1.0200
15 492	0.1893	0.4895	0.0690	0.0531	0.6230	0.8539	0.9542
36 185	0.1265	0.3115	0.0468	0.0355	0.4049	0.8427	1.0473

Table 7
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the uniform refinement (Example 4)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
69	5.5291	20.7907	6.9546	1.6773	20.9411	1.0827	–
257	3.7377	14.9927	3.5568	0.7641	15.4037	1.0305	0.5421
993	2.9702	11.6530	3.2730	0.5219	11.8187	1.0554	0.3568
3905	2.4196	7.9531	2.4870	0.2999	8.3101	1.0448	0.5292
15 489	2.2007	5.1378	1.9601	0.1646	5.8258	1.0171	0.5547

Table 8
Individual errors, error estimate $\tilde{\boldsymbol{\theta}}$, effectivity index, and rate of convergence for the adaptive refinement (Example 4)

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\tilde{\boldsymbol{\theta}}$	$\mathbf{e}/\tilde{\boldsymbol{\theta}}$	γ
69	5.5291	20.7907	6.9546	1.6773	20.9411	1.0827	–
111	5.3733	17.1622	6.6382	1.5643	17.6647	1.0890	0.6916
317	4.3304	14.4414	6.3571	1.2294	13.3123	1.2328	0.3026
878	3.0736	10.7921	4.8276	0.7634	9.6218	1.2721	0.5758
1109	1.2380	7.3492	0.6084	0.1824	7.6423	0.9787	4.2166
1537	0.9658	5.1429	0.4443	0.1657	5.4367	0.9664	2.1642
2979	0.6404	3.1625	0.3380	0.1353	3.3701	0.9635	1.4544
4411	0.5447	2.5478	0.3043	0.1328	2.7265	0.9633	1.0810
10 140	0.3987	1.6597	0.2001	0.1017	1.8212	0.9453	1.0148
16 879	0.2809	1.3344	0.1452	0.0658	1.4366	0.9557	0.8881
42 271	0.1941	0.8190	0.0909	0.0478	0.9025	0.9396	1.0498

numerical evidences for it being efficient. Then, Figs. 1 and 2 show **e** versus the degrees of freedom N for Examples 1 and 2. In each case the total error **e** of the adaptive algorithm decreases much faster than that of the uniform one. In particular, the slow convergence observed in the uniform refinement of Example 2 is considerably improved by the corresponding adaptive strategy. These facts are also emphasized by the experimental rates of convergence provided in the tables, which show that the adaptive method recovers the order of convergence guaranteed by Theorem 3.2 in [5], that is $O(h)$. Next, Figs. 3 and 4 display some intermediate meshes obtained with the refinement procedure. We remark, as expected, that the algorithm is able to recognize the neighborhood of the *singular* point $(2, 2)$ in both examples.

We now consider the full nonlinear boundary value problem (1.1) on the L-shaped domain $\Omega := (-1,1)^2 - (0,1)^2$. We take the kinematic viscosity function ψ as given by the Carreau law with $\kappa_0 = \kappa_1 = 1/2$ and $\beta = 3/2$ (see Section 1 in [5]), that is $\psi(t) := \frac{1}{2} + \frac{1}{2}(1+t^2)^{-1/4}$, and choose the data **f** and **g** so that the exact solution of (1.1) is, respectively, for Examples 3 and 4,

$$\mathbf{u}_3(x) := \left[(x_1 - 0.1)^2 + (x_2 - 0.1)^2\right]^{-1/2}(x_2 - 0.1, 0.1 - x_1)^t, \quad p_3(x) := (2 - x_1 - x_2)^{1/2},$$

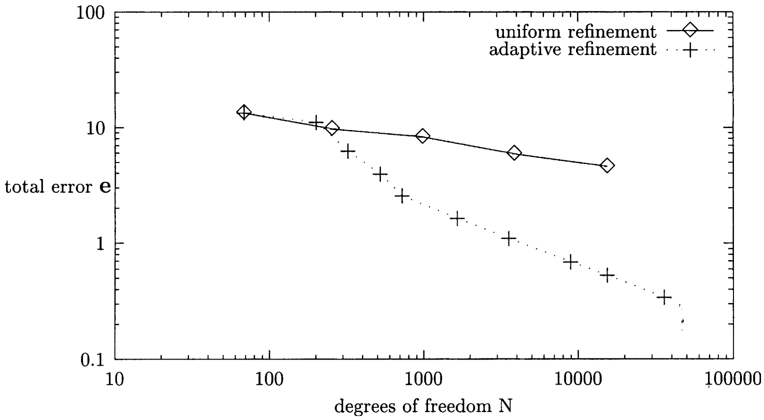

Fig. 5. Total error **e** for uniform and adaptive refinements (Example 3).

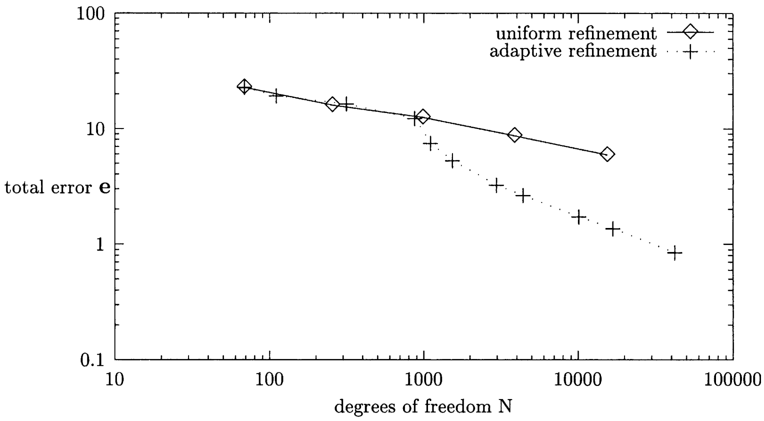

Fig. 6. Total error **e** for uniform and adaptive refinements (Example 4).

and

$$\mathbf{u}_4(x) := \left[(x_1 - 0.1)^2 + (x_2 - 0.1)^2\right]^{-1/2} (x_2 - 0.1, 0.1 - x_1)^t, \quad p_4(x) := 1/(x_1 - 1.1),$$

for all $x := (x_1, x_2) \in \Omega$. We note that $\mathbf{u}_3$ and $\mathbf{u}_4$ are divergence free in Ω and singular in an exterior neighborhood of $(0, 0)$. In addition, the singularity of p_4 runs along the line $x_1 = 1.1$.

Similarly as for the linear case, we present in Tables 5–8 the errors for the main unknowns, the error estimate $\tilde{\boldsymbol{\theta}}$, the effectivity index $\mathbf{e}/\tilde{\boldsymbol{\theta}}$, and the experimental rate of convergence $\boldsymbol{\gamma}$. The discrete scheme (1.6) is solved by Newton's method with an initial guess given by the solution of the linear problem ($\psi \equiv 1$), and a tolerance of 10^{-3} for the relative error. The number of iterations needed in each mesh is 3 (for both examples). Next, Figs. 5 and 6 show $\mathbf{e}$ versus the degrees of freedom N, and Figs. 7 and 8 provide some intermediate meshes obtained with the refinement method.

The remarks and conclusions here are the same of the linear examples. In particular, the effectivity indexes confirm the reliability of $\tilde{\boldsymbol{\theta}}$ and constitute experimental evidences of an eventual efficiency. Further, the adaptive procedure leads again to the quasi-optimal linear rate of convergence, and it is able to identify the singularities of each problem. This means, as observed in Figs. 7 and 8, that the adapted meshes are

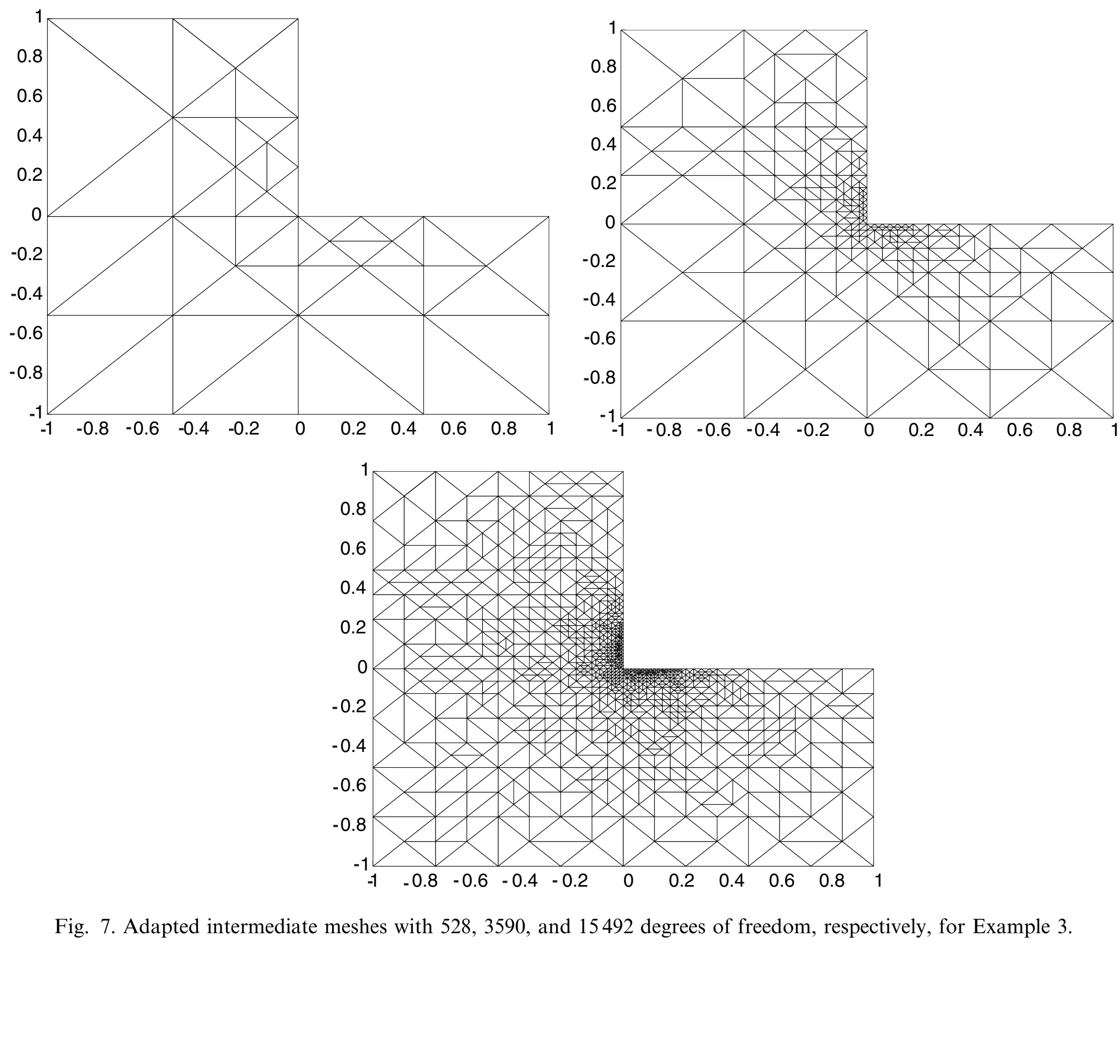

Fig. 7. Adapted intermediate meshes with 528, 3590, and 15 492 degrees of freedom, respectively, for Example 3.

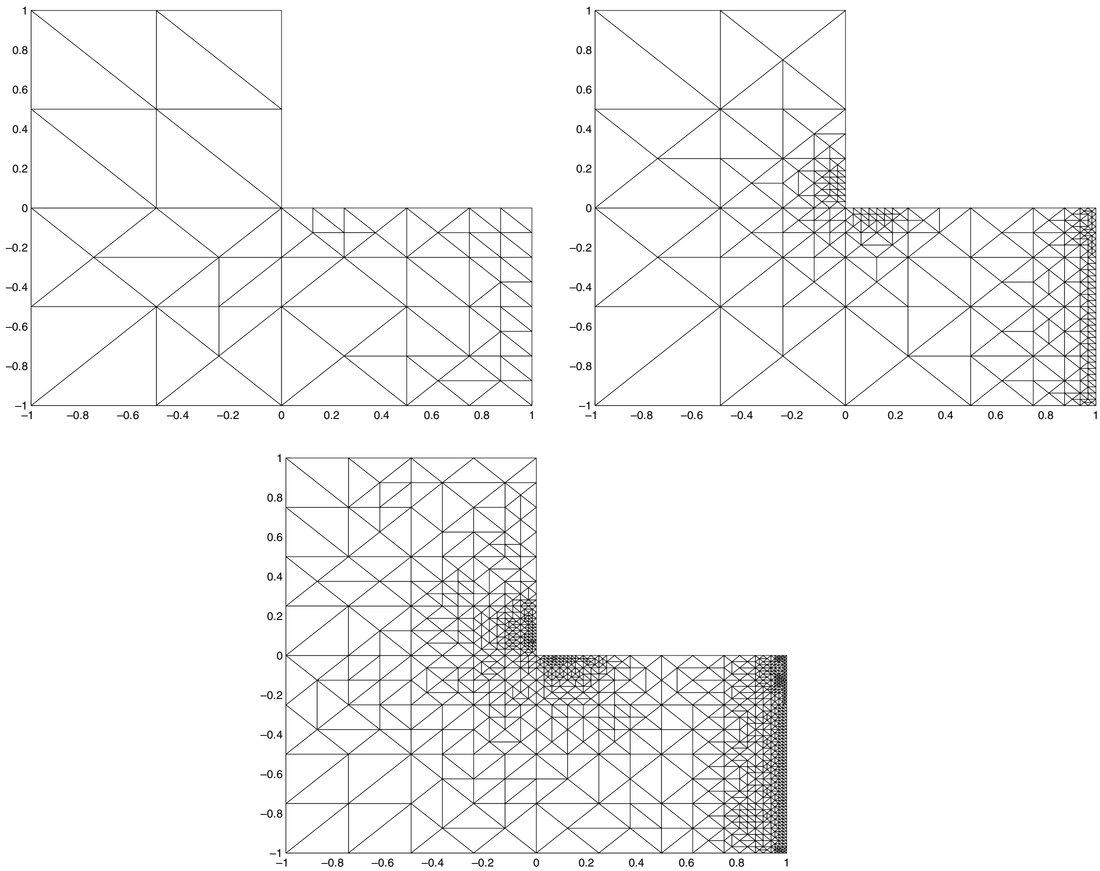

Fig. 8. Adapted intermediate meshes with 878, 4411, and 16 879 degrees of freedom, respectively, for Example 4.

highly refined around the point $(0,0)$ for Examples 3 and 4, and also around the segment $x_1 = 1.0$ for Example 4.

Summarizing, the results presented in this section provide enough support for the adaptive algorithm being much more efficient than a uniform discretization procedure when solving the mixed finite element scheme (1.6).

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